Counting common perpendicular arcs in negative curvature

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Abstract

We study in this paper the asymptotics of marked ortholength spectra, in the presence of equilibrium states. Let D^- and D^+ be properly immersed closed convex subsets of a Riemannian manifold with pinched negative sectional curvature. Using mixing properties of Gibbs measures, we give an asymptotic formula as $t \to +\infty$ for the number of common perpendiculars of length at most t from D^- to D^+, counted with multiplicities and weights, and we prove the equidistribution in the outer and inner unit normal bundles of D^- and D^+ of the tangent vectors at the endpoints of the common perpendiculars. When the manifold is compact with exponential decay of correlations or arithmetic with finite volume, we give an error term for the asymptotic. We give an asymptotic formula for the number of connected components of the domain of discontinuity of Kleinian groups as their diameter goes to 0. [1]

1 Introduction

Let M be a complete connected Riemannian manifold with pinched sectional curvature at most -1 whose fundamental group is not virtually nilpotent, let $(g^t)_{t\in\mathbb{R}}$ be its geodesic flow, and let $F : T^1M \to \mathbb{R}$ be a *potential*, that is, a bounded Hölder-continuous function. Let D^- and D^+ be proper nonempty properly immersed closed convex subsets of M. A *common perpendicular* from D^- to D^+ is a locally geodesic path in M starting perpendicularly from D^- and arriving perpendicularly to D^+.

In this paper, we give a general asymptotic formula as $t \to +\infty$ for the number of common perpendiculars of length at most t from D^- to D^+, counted with multiplicities and with weights defined by the potential, and we prove the equidistribution of the initial and terminal tangent vectors of the common perpendiculars in the outer and inner unit normal bundles of D^- and D^+, respectively. Common perpendiculars have been studied, in various particular cases, sometimes not explicitly, by Basmajian, Bridgeman, Bridgeman-Kahn, Eskin-McMullen, Herrmann, Huber, Kontorovich-Oh, Margulis, Martin-McKee-Wambach, Meyerhoff, Mirzakhani, Oh-Shah, Pollicott, Roblin, Shah, the authors and many others (see the comments after Theorem 1 below, and the survey [PP6] for references).

We refer to Subsection 2.3 for a precise definition of the common perpendiculars when the boundaries of D^- and D^+ are not smooth, and to Section 3.3 for the definition of the multiplicities, which are equal to 1 if D^- and D^+ are embedded and disjoint. Higher

[1]**Keywords:** counting, geodesic arc, convexity, common perpendicular, ortholength spectrum, equidistribution, mixing, decay of correlation, negative curvature, skinning measure, Gibbs measure, Kleinian groups. **AMS codes:** 37D40, 37A25, 53C22, 30F40

multiplicities can occur for instance when $D^\pm$ are non-simple closed geodesics in dimension at least 3. We denote the length of a common perpendicular α by $\ell(\alpha)$, and its initial and terminal unit tangent vectors by v_α^- and v_α^+. For all $t > 0$, we denote by $\operatorname{Perp}(D^-, D^+, t)$ the set of common perpendiculars from D^- to D^+ with length at most t (considered with multiplicities), and we define the counting function with weights by

$$\mathscr{N}_{D^-,\,D^+,\,F}(t) = \sum_{\alpha\in\operatorname{Perp}(D^-,\,D^+,\,t)} e^{\int_\alpha F}\ ,$$

where $\int_\alpha F = \int_0^{\ell(\alpha)} F(g^t v_\alpha^-)\,dt$.

We prove an asymptotic formula of the form $\mathscr{N}_{D^-,\,D^+,\,F}(t) \sim c\,e^{c' t}$ as $t \to +\infty$, with error term estimates. The constants c, c' that will appear in such asymptotic formulas will be explicit, in terms of ergodic properties of the measures naturally associated to the potential F, that we now describe.

Let $\mathscr{M}$ be the set of probability measures on T^1M invariant under the geodesic flow and let $h_m(g^1)$ be the (metric) entropy of the geodesic flow with respect to $m \in \mathscr{M}$. The *pressure* of the potential F is

$$\delta_F = \sup_{m\in\mathscr{M}} \Big(h_m(g^1) + \int_{T^1M} F\,dm\Big)\ .$$

When $F = 0$, the pressure δ_F coincides with the *critical exponent* of the fundamental group of M, by [OtP]. We assume that $\delta_F > 0$. We will prove that δ_F is the exponential growth rate of $\mathscr{N}_{D^-,\,D^+,\,F}(t)$.

Let m_F be a Gibbs measure on T^1M associated to the potential F (see [PPS] and Section 3.1). When finite and normalised to be a probability measure, it is an equilibrium state: it attains the upper bound defining the pressure δ_F (see [PPS, Theo. 6.1], improving [OtP] when $F = 0$). For instance, m_F is (up to a constant multiple) the Bowen-Margulis measure $m_{\rm BM}$ if $F = 0$, and the Liouville measure if M is compact and F is the strong unstable Jacobian $v \mapsto -\frac{d}{dt}_{|t=0} \log \operatorname{Jac}\big({g^t}_{|W^{\rm su}(v)}\big)(v)$. We will use the construction of m_F by Paulin-Pollicott-Schapira [PPS] (building on work of Hamenstädt, Ledrappier, Coudène, Mohsen) via Patterson densities on the boundary at infinity of a universal cover of M associated to the potential F. We avoid any compactness assumption on M, we only assume that the Gibbs measure m_F of F is finite and that it is mixing for the geodesic flow. We refer to [PPS, Sect. 8.2] for finiteness criteria of m_F (improving on [DOP] when $F = 0$). By Babillot's theorem [Bab], if the length spectrum of M is not contained in a discrete subgroup of $\mathbb{R}$, then m_F is mixing if finite. This condition is satisfied for instance if the limit set of a fundamental group of M is not totally disconnected, see for instance [Dal1, Dal2].

The measures $\sigma^+_{D^-}$ and $\sigma^-_{D^+}$ on the outer and inner unit normal bundles of D^- and D^+ to which the initial and terminal tangent vectors of the common perpendiculars will equidistribute are the skinning measures of D^- and D^+. We construct these measures as appropriate pushforwards of the Patterson(-Sullivan) densities associated with the potential F to the unit normal bundles of the lifts of D^- and D^+ in the universal cover of M. This construction generalises the one in [PP5] when $F = 0$, which itself generalises the one in [OS1, OS2] when M has constant curvature and D^-, D^+ are balls, horoballs or totally geodesic submanifolds. In [PP5], we gave a finiteness criterion for the skinning measures when $F = 0$, generalising the one in [OS2] in the context described above.

We now state our counting and equidistribution results. We denote the total mass of any measure m by $\|m\|$.

Theorem 1 *Assume that the skinning measures $\sigma^+_{D^-}$ and $\sigma^-_{D^+}$ are finite and nonzero. Then, as $s \to +\infty$,*

$$\mathscr{N}_{D^-,\,D^+,\,F}(s) \sim \frac{\|\sigma^+_{D^-}\|\,\|\sigma^-_{D^+}\|}{\|m_F\|}\,\frac{e^{\delta_F\,s}}{\delta_F}\,.$$

When $F = 0$, the counting function $\mathscr{N}_{D^-,\,D^+,\,0}(t)$ has been studied in various special cases since the 1950's and in a number of recent works, sometimes in a different guise, see the survey [PP6] for more details. A number of special cases (all with $F = 0$) were known before our result:

- D^- and D^+ are reduced to points, by for instance [Hub2], [Mar1] and [Rob],
- D^- and D^+ are horoballs, by [BHP], [HP2], [Cos] and [Rob] without an explicit form of the constant in the asymptotic expression,
- D^- is a point and D^+ is a totally geodesic submanifold, by [Her], [EM] and [OS3] in constant curvature,
- D^- is a point and D^+ is a horoball, by [Kon] and [KO] in constant curvature, and [Kim] in rank one symmetric spaces,
- D^- is a horoball and D^+ is a totally geodesic submanifold, by [OS1] and [PP3] in constant curvature, and
- D^- and D^+ are (properly immersed) locally geodesic lines in constant curvature and dimension 3, by [Pol].

When M is a finite volume hyperbolic manifold and the potential F is constant 0, the Gibbs measure is proportional to the Liouville measure and the skinning measures of totally geodesic submanifolds, balls and horoballs are proportional to the induced Riemannian measures of the unit normal bundles of their boundaries. In this situation, we get very explicit forms of some of the counting results in finite-volume hyperbolic manifolds. See Corollary 30 for the cases where both D^- and D^+ are totally geodesic submanifolds or horoballs, which are new even in these special cases. As an example of this result, if D^- and D^+ are closed geodesics of M of lengths ℓ_- and ℓ_+, respectively, then the number $\mathscr{N}(s)$ of common perpendiculars (counted with multiplicity) from D^- to D^+ of length at most s satisfies, as $s \to +\infty$,

$$\mathscr{N}_{D^-,\,D^+,\,0}(s) \sim \frac{\pi^{\frac{n}{2}-1}\Gamma(\frac{n-1}{2})^2}{2^{n-2}(n-1)\Gamma(\frac{n}{2})}\,\frac{\ell_-\ell_+}{\mathrm{Vol}(M)}\,e^{(n-1)s}\,. \tag{1}$$

Let $\mathrm{Perp}(D^-, D^+)$ be the set of common perpendiculars from D^- to D^+ (considered with multiplicities). The family $(\ell(\alpha))_{\alpha\in\mathrm{Perp}(D^-,\,D^+)}$ will be called the *marked ortholength spectrum* from D^- to D^+. The set of lengths (with multiplicities) of elements of $\mathrm{Perp}(D^-, D^+)$ will be called the *ortholength spectrum* of D^-, D^+. This second set has been introduced by Basmajian [Bas] (under the name "full orthogonal spectrum") when M has constant curvature, and D^- and D^+ are disjoint or equal embedded totally geodesic hypersurfaces or embedded horospherical cusp neighbourhoods or embedded balls (see also [BK] when M is a compact hyperbolic manifold with totally geodesic boundary and $D^- = D^+ = \partial M$). When M is a closed hyperbolic surface and $D^- = D^+$, the formula (1) has been obtained by Martin-McKee-Wambach [MMW] by trace formula methods, though obtaining the case $D^- \neq D^+$ seems difficult by these methods.

Theorem 1 is deduced in Section 4 from the following equidistribution result that shows that the initial and terminal unit tangent vectors of common perpendiculars equidistribute to the product measure of skinning measures. We denote the unit Dirac mass at a point z by Δ_z.

Theorem 2 *For the weak-star convergence of measures on $T^1M \times T^1M$, we have*

$$\lim_{t\to+\infty} \delta_F \, \|m_F\| \, e^{-\delta_F t} \sum_{\alpha\in\mathrm{Perp}(D^-,\,D^+,\,t)} e^{\int_\alpha F} \, \Delta_{v_\alpha^-} \otimes \Delta_{v_\alpha^+} \;=\; \sigma^+_{D^-} \otimes \sigma^-_{D^+} \,.$$

Both results are valid when M is a good orbifold instead of a manifold (for the appropriate notion of multiplicities), and when D^-, D^+ are replaced by locally finite families (see Section 3.3).

Besides its generality, this work presents several new features not present in earlier related work: we have very weak finiteness and curvature assumptions on the manifold; non-constant weights using equilibrium states have only been used in the orbit-counting problem in [PPS]; all previous works have regularity assumptions on the convex sets (see Corollary 5 below for a striking application using convex sets with fractal boundary); except in the case of two points (and with a different presentation, as initiated by Roblin, with several subsequent works), no other work (besides Herrmann's) proved a simultaneous equidistribution result of the initial and terminal normal vectors of the common perpendiculars. The ideas from Margulis's thesis to use the mixing of the geodesic flow (in our case, extended to equilibrium states) are certainly present, but major new techniques needed to be developed to treat the problem in the generality considered in this paper. Due to the symmetry of the problem, a one-sided pushing of the geodesic flow, as in all the previous works using Margulis's ideas, is not sufficient, and we need to push simultaneously the outer unit normal vectors to the convex sets in opposite directions. We also need a new study of the geometry and the dynamics of the accidents that occur around midway (see in particular the effective statement of creation of common perpendiculars in Subection 2.3).

The techniques of Section 4 allow to obtain in Section 4.2 the following generalization of [PP5, Theo. 1] which corresponds to the case $F = 0$, which itself generalised the ones in [Mar2, EM, Rob, PP3] when M has constant curvature, $F = 0$ and D^- is a ball, a horoball or a totally geodesic submanifold.

Theorem 3 *Assume that the skinning measure $\sigma^+_{D^-}$ is finite and nonzero. Then, as t tends to $+\infty$, the pushforwards $(g^t)_*\sigma^+_{D^-}$ of the skinning measure of D^- by the geodesic flow equidistribute towards the Gibbs measure m_F.*

In the cases when the geodesic flow is known to be exponentially mixing on T^1M (see the works of Kleinbock-Margulis, Clozel, Dolgopyat, Stoyanov, Liverani, and Giulietti-Liverani-Pollicott, and Section 5 for definitions and precise references), we obtain an exponentially small error term in the equidistribution result of Theorem 3 (generalizing [PP5, Theo. 20] where $F = 0$) and in the approximation of the counting function $\mathscr{N}_{D^-,\,D^+,\,0}$ by the expression introduced in Theorem 1. In particular when M is arithmetic, the error term in Equation (1) is $\mathrm{O}(e^{(n-1-\kappa)t})$ for some $\kappa > 0$.

Theorem 4 *Assume that M is compact and m_F is exponentially mixing under the geodesic flow for the Hölder regularity, or that M is locally symmetric, the boundary of $D^\pm$ is smooth, m_F is finite, smooth, and exponentially mixing under the geodesic flow for the Sobolev regularity. Assume that the strong stable/unstable ball masses by the conditionals of m_F are Hölder-continuous in their radius.*

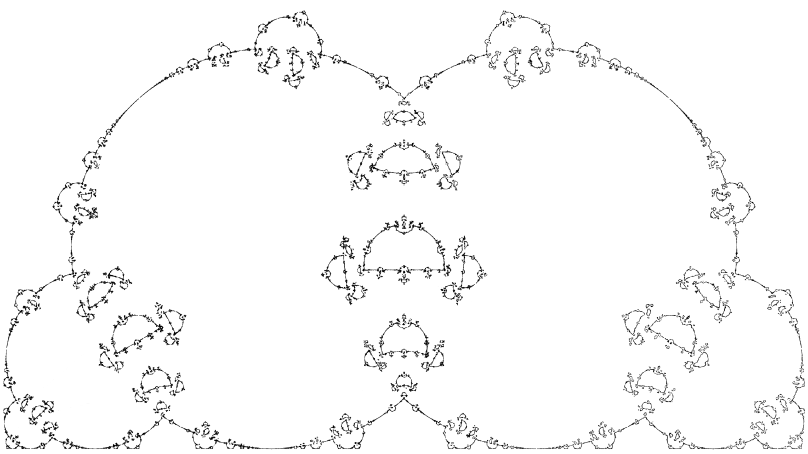
(1) As t tends to $+\infty$, the pushforwards $(g^t)_*\sigma^+_{D^-}$ of the skinning measure of D^- by the geodesic flow equidistribute towards the Gibbs measure m_F with exponential speed (see Theorem 27 for a precise statement).

(2) There exists $\kappa > 0$ such that, as $t \to +\infty$,

$$\mathscr{N}_{D^-,\,D^+,\,F}(t) = \frac{\|\sigma^+_{D^-}\|\;\|\sigma^-_{D^+}\|}{\delta_F\;\|m_F\|}\;e^{\delta_F t}\big(1 + \mathrm{O}(e^{-\kappa t})\big)\;.$$

See Section 5 for a discussion of the assumptions and the dependence of $\mathrm{O}(\cdot)$ on the data. Similar (sometimes more precise) error estimates were known earlier for the counting function in special cases of $D^\pm$ in constant curvature geometrically finite manifolds (often in small dimension) through the work of Huber, Selberg, Patterson, Lax-Phillips [LaP], Cosentino [Cos], Kontorovich-Oh [KO], Lee-Oh [LeO].

We conclude this introduction by stating a simplified version of an application of Theorem 1 to the counting asymptotic of the images by the elements of a Kleinian group of a subset of its limit set when their diameters tend to 0. For instance, consider the picture below produced by D. Wright's program `kleinian`, which is the limit set of a free product $\Gamma = \Gamma_0 * \gamma_0\Gamma_0\gamma_0^{-1}$ of a quasifuchsian group Γ_0 and its conjugate by a big power γ_0 of a loxodromic element whose attracting fixed point is contained in the bounded component of $\mathbb{C} - \Lambda\Gamma$, so that the limit set of Γ is the closure of a countable union of quasi-circles. As we will see in Section 4.4, the number of Jordan curves in $\Lambda\Gamma$ with diameter at least $1/T$ is equivalent to $c\,T^\delta$ where $c > 0$ and $\delta \in \,]1, 2[$ is the Hausdorff dimension of the limit set.



We will also prove in Corollary 25 that the number of connected components of the domain of discontinuity of a geometrically finite discrete group of $\mathrm{PSL}_2(\mathbb{C})$ which is not virtually quasifuchsian but has bounded and not totally disconnected limit set, has such a growth.

Corollary 5 *Let Γ be a geometrically finite discrete group of isometries of the upper halfspace model of $\mathbb{H}^n_{\mathbb{R}}$, with bounded limit set $\Lambda\Gamma$ in $\mathbb{R}^{n-1} = \partial_\infty \mathbb{H}^n_{\mathbb{R}} - \{\infty\}$ (endowed with the usual Euclidean distance). Let δ be the Hausdorff dimension of $\Lambda\Gamma$. Let Γ_0 be a convex-cocompact subgroup of Γ with infinite index. Then, there exists an explicitable $c > 0$ such that, as $T \to +\infty$,*

$$\operatorname{Card}\{\gamma \in \Gamma/\Gamma_0 \;:\; \operatorname{diam}(\gamma\Lambda\Gamma_0) \geq 1/T\} \sim \; c\,T^{\delta}\;.$$

This corollary is due to Oh-Shah [OS3] when the limit set of Γ_0 is a round sphere. We refer to Corollary 24 for a more general version and to Section 4.4 for complements, generalizing results of Oh-Shah [OS3], and for extensions to any rank one symmetric space.

The results of this paper have been announced in the survey [PP6]. Some applications and generalisations will appear in separate papers: In [PP7], we give several arithmetic applications of the results, obtained by considering arithmetically defined manifolds and orbifolds of constant negative curvature. In [PP8], we consider arithmetic applications in the Heisenberg group via complex hyperbolic geometry. In [PP9], we study counting and equidistribution in conjugacy classes, giving a new proof of the main result of [Hub1] and generalising it to parabolic cyclic and more general subgroups, arbitrary dimension, infinite volume and variable curvature.

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2 Geometry, dynamics and convexity

Let $\widetilde{M}$ be a complete simply connected Riemannian manifold with (dimension at least 2 and) pinched negative sectional curvature $-b^2 \leq K \leq -1$, and let $x_0 \in \widetilde{M}$ be a fixed basepoint. Let Γ be a nonelementary (not virtually nilpotent) discrete group of isometries of $\widetilde{M}$, and let M be the quotient Riemannian orbifold $\Gamma\backslash\widetilde{M}$.

In this section, we review the required background on negatively curved Riemannian manifolds seen as locally CAT$(-\kappa)$ spaces (see [BH] for definitions, proofs and complements). We introduce the notation for the outward and inward pointing unit normal bundles of the boundary of a convex subset, and we define dynamical thickenings in the unit tangent bundle of subsets of these submanifolds, expanding on [PP5]. We give a precise definition of common perpendiculars in Subsection 2.3 and we also give a procedure to construct them by dynamical means.

For every $\epsilon > 0$, we denote by $\mathscr{N}_\epsilon A$ the closed ϵ-neighbourhood of a subset A of any metric space, by $\mathscr{N}_{-\epsilon} A$ the set of points $x \in A$ at distance at least ϵ from the complement of A, and by convention $\mathscr{N}_0 A = \overline{A}$.

2.1 Strong stable and unstable foliations, and Hamenstädt's distances

We denote by $\partial_\infty \widetilde{M}$ the boundary at infinity of $\widetilde{M}$ and by $\Lambda\Gamma$ the limit set of Γ.

We identify the unit tangent bundle $T^1 N$ (endowed with Sasaki's Riemannian metric and its Riemannian distance) of a complete Riemannian manifold N with the set of its locally geodesic lines $\ell : \mathbb{R} \to N$, by the inverse of the map sending a (locally) geodesic line ℓ to its (unit) tangent vector $\dot{\ell}(0)$ at time $t = 0$. We denote by $\pi : T^1N \to N$ the *basepoint projection*, given by $\pi(\ell) = \ell(0)$.

The *geodesic flow* on T^1N is the smooth one-parameter group of diffeomorphisms $(g^t)_{t\in\mathbb{R}}$ of $T^1\widetilde{M}$, where $g^t\ell\,(s) = \ell(s+t)$, for all $\ell \in T^1N$ and $s,t \in \mathbb{R}$. The action of the isometry group of N on T^1N by postcomposition (that is, by $(\gamma, \ell) \mapsto \gamma \circ \ell$) commutes with the geodesic flow.

When Γ acts without fixed points on $\widetilde{M}$, we have an identification $\Gamma\backslash T^1\widetilde{M} = T^1M$. More generally, we denote by T^1M the quotient Riemannian orbifold $\Gamma\backslash T^1\widetilde{M}$. We use the notation $(g^t)_{t\in\mathbb{R}}$ also for the (quotient) geodesic flow on T^1M.

For every $v \in T^1\widetilde{M}$, let $v_- \in \partial_\infty\widetilde{M}$ and $v_+ \in \partial_\infty\widetilde{M}$, respectively, be the endpoints at $-\infty$ and $+\infty$ of the geodesic line defined by v. Let $\partial^2_\infty\widetilde{M}$ be the subset of $\partial_\infty\widetilde{M} \times \partial_\infty\widetilde{M}$ which consists of pairs of distinct points at infinity of $\widetilde{M}$. *Hopf's parametrisation* of $T^1\widetilde{M}$ is the homeomorphism which identifies $T^1\widetilde{M}$ with $\partial^2_\infty\widetilde{M} \times \mathbb{R}$, by the map $v \mapsto (v_-, v_+, t)$, where t is the signed distance of the closest point to x_0 on the geodesic line defined by v to $\pi(v)$. We have $g^s(v_-, v_+, t) = (v_-, v_+, t+s)$ for all $s \in \mathbb{R}$, and for all $\gamma \in \Gamma$, we have $\gamma(v_-, v_+, t) = (\gamma v_-, \gamma v_+, t + t_{\gamma,\, v_-,\, v_+})$ where $t_{\gamma,\, v_-,\, v_+} \in \mathbb{R}$ depends only on γ, v_-, v_+.

Let $\iota : T^1\widetilde{M} \to T^1\widetilde{M}$ be the (Hölder-continuous) *antipodal (flip) map* of $T^1\widetilde{M}$ defined by $\iota v = -v$ or, using geodesic lines, by $\iota v : t \mapsto v(-t)$. In Hopf's parametrisation, the antipodal map is the map $(v_-, v_+, t) \mapsto (v_+, v_-, -t)$. We denote the quotient map of ι again by $\iota : T^1M \to T^1M$, and call it the *antipodal map* of T^1M. We have $\iota \circ g^t = g^{-t} \circ \iota$ for all $t \in \mathbb{R}$.

The *strong stable manifold* of $v \in T^1\widetilde{M}$ is

$$W^{\rm ss}(v) = \{v' \in T^1\widetilde{M}\ :\ d(v(t), v'(t)) \to 0 \text{ as } t \to +\infty\},$$

and its *strong unstable manifold* is

$$W^{\rm su}(v) = \{v' \in T^1\widetilde{M}\ :\ d(v(t), v'(t)) \to 0 \text{ as } t \to -\infty\}.$$

The union for $t \in \mathbb{R}$ of the images under g^t of the strong stable manifold of $v \in T^1\widetilde{M}$ is the *stable manifold* $W^{\rm s}(v) = \bigcup_{t\in\mathbb{R}} g^t W^{\rm ss}(v)$ of v, which consists of the elements $v' \in T^1\widetilde{M}$ with $v'_+ = v_+$. Similarly, $W^{\rm u}(v) = \bigcup_{t\in\mathbb{R}} g^t W^{\rm su}(v)$, which consists of the elements $v' \in T^1\widetilde{M}$ with $v'_- = v_-$, is the *unstable manifold* $W^{\rm u}(v)$ of v. The maps from $\mathbb{R} \times W^{\rm ss}(v)$ to $W^{\rm s}(v)$ and from $\mathbb{R} \times W^{\rm su}(v)$ to $W^{\rm u}(v)$ defined by $(s, v') \mapsto g^s v'$ are smooth diffeomorphisms.

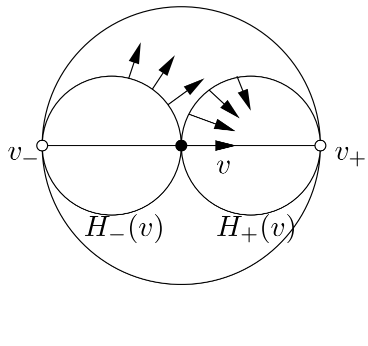
The strong stable manifolds, stable manifolds, strong unstable manifolds and unstable manifolds are the (smooth) leaves of topological foliations that are invariant under the geodesic flow and the group of isometries of $\widetilde{M}$, denoted by $\mathscr{W}^{\rm ss}, \mathscr{W}^{\rm s}, \mathscr{W}^{\rm su}$ and $\mathscr{W}^{\rm u}$ respectively. These foliations are Hölder-continuous when $\widetilde{M}$ has compact quotients by [Ano] or when $\widetilde{M}$ has pinched negative sectional curvature with bounded derivatives (see for instance [Bri], [PPS, Thm. 7.3]) and even smooth when $\widetilde{M}$ is symmetric.

For any point $\xi \in \partial_\infty\widetilde{M}$, let $\rho_\xi : [0, +\infty[\ \to \widetilde{M}$ be the geodesic ray with origin x_0 and point at infinity ξ. The *Busemann cocycle* of $\widetilde{M}$ is the map $\beta : \widetilde{M} \times \widetilde{M} \times \partial_\infty\widetilde{M} \to \mathbb{R}$ defined by

$$(x, y, \xi) \mapsto \beta_\xi(x, y) = \lim_{t\to+\infty} d(\rho_\xi(t), x) - d(\rho_\xi(t), y)\ .$$

The above limit exists and is independent of x_0.

The projections in $\widetilde{M}$ of the strong unstable and strong stable manifolds of $v \in T^1\widetilde{M}$, denoted by $H_-(v) = \pi(W^{\rm su}(v))$ and $H_+(v) = \pi(W^{\rm ss}(v))$, are called, respectively, the *unstable and stable horospheres* of v, and are said to be *centered at* v_- and v_+, respectively. The unstable horosphere of v coincides with the zero set of the map $x \mapsto f_-(x) = \beta_{v_-}(x, \pi(v))$ and the stable horosphere of v coincides with the zero set of $x \mapsto f_+(x) = \beta_{v_+}(x, \pi(v))$. The corresponding sublevel sets $HB_-(v) = f_-^{-1}(]-\infty, 0])$ and $HB_+(v) = f_+^{-1}(]-\infty, 0])$ are called the *horoballs* bounded by $H_-(v)$ and $H_+(v)$. Horoballs are (strictly) convex subsets of $\widetilde{M}$.



For every $v \in T^1\widetilde{M}$, let $d_{W^{\rm su}(v)}$ and $d_{W^{\rm ss}(v)}$ be *Hamenstädt's distances* on the strong unstable and strong stable leaf of v, defined as follows (see [HP1, Appendix] and compare with [Ham]): for all $w, z \in W^{\rm su}(v)$, let

$$d_{W^{\rm su}(v)}(w, z) = \lim_{t \to +\infty} e^{\frac{1}{2}d(w(t),\, z(t)) - t} \;,$$

and for all $w', z' \in W^{\rm ss}(v)$, let

$$d_{W^{\rm ss}(v)}(w', z') = d_{W^{\rm su}(\iota v)}(\iota w', \iota z') = \lim_{t \to +\infty} e^{\frac{1}{2}d(w'(-t),\, z'(-t)) - t} \;.$$

The above limits exist, and Hamenstädt's distances are distances inducing the original topology on $W^{\rm su}(v)$ and $W^{\rm ss}(v)$. For all $w, z \in W^{\rm su}(v)$, all $w', z' \in W^{\rm ss}(v)$, and for every isometry γ of $\widetilde{M}$, we have

$$d_{W^{\rm su}(\gamma v)}(\gamma w, \gamma z) = d_{W^{\rm su}(v)}(w, z) \quad \text{and} \quad d_{W^{\rm ss}(\gamma v)}(\gamma w', \gamma z') = d_{W^{\rm ss}(v)}(w', z') \,.$$

For all $v \in T^1\widetilde{M}$ and $s \in \mathbb{R}$, we have for all $w, z \in W^{\rm su}(v)$

$$d_{W^{\rm su}(g^s v)}(g^s w, g^s z) = e^s d_{W^{\rm su}(v)}(w, z) \;. \tag{2}$$

and for all $w', z' \in W^{\rm ss}(v)$

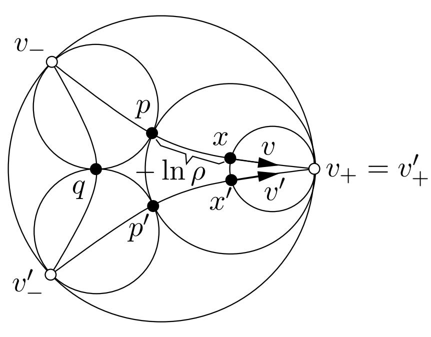
$$d_{W^{\rm ss}(g^s v)}(g^s w', g^s z') = e^{-s} d_{W^{\rm ss}(v)}(w', z') \;. \tag{3}$$

Lemma 6 *For all $v \in T^1\widetilde{M}$, $v' \in W^{\rm ss}(v)$ and $v'' \in W^{\rm su}(v)$, we have*

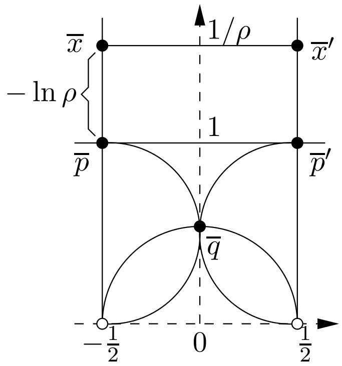
$$d(\pi(v), \pi(v')) \le d_{W^{\rm ss}(v)}(v, v') \quad \text{and} \quad d(\pi(v), \pi(v'')) \le d_{W^{\rm su}(v)}(v, v'') \,.$$

Proof. We prove the first inequality, the second one follows by using the antipodal map.

Let $x = \pi(v)$, $x' = \pi(v')$ and $\rho = d_{W^{\rm ss}(v)}(v, v')$. Consider the ideal triangle Δ with vertices v_-, v'_- and $v_+ = v'_+$. Let $p \in \,]v_-, v_+[\,$, $p' \in \,]v'_-, v'_+[$ and $q \in \,]v_-, v'_-[$ be the pairwise tangency points of horospheres centered at the vertices of Δ. These points are uniquely defined by the equations $\beta_{v_+}(p, p') = \beta_{v_-}(p, q) = \beta_{v'_-}(p', q) = 0$. By the definition of Hamenstädt's distance, the algebraic distance from p to x on the geodesic line $]v_-, v_+[$ (oriented from v_- to v_+) is $-\ln \rho$.



Consider the ideal triangle $\overline{\Delta}$ in the upper halfplane model of the real hyperbolic plane $\mathbb{H}^2_{\mathbb{R}}$, with vertices $-\frac{1}{2}$, $\frac{1}{2}$ and ∞. Let $\overline{p} = (-\frac{1}{2}, 1)$, $\overline{p}' = (\frac{1}{2}, 1)$ and $\overline{q} = (0, \frac{1}{2})$ be the pairwise tangency points of horospheres centered at the vertices of $\overline{\Delta}$. Let $\overline{x}$ and $\overline{x}'$ be the point at algebraic (hyperbolic) distance $-\ln\rho$ from $\overline{p}$ and $\overline{p}'$, respectively, on the upwards oriented vertical line through them. By comparison, we have $d(x, x') \leq d(\overline{x}, \overline{x}') \leq 1/e^{-\ln\rho} = \rho$. □



2.2 Dynamical thickening of outer and inner unit normal bundles

Let D be a nonempty *proper* (that is, different from $\widetilde{M}$) closed convex subset in $\widetilde{M}$. We denote by ∂D its boundary in $\widetilde{M}$ and by $\partial_\infty D$ its set of points at infinity. In this subsection, we recall from [PP5] the definition of the outer unit normal bundle of ∂D, the dynamical thickenings of its subsets, and we extend these definitions to the inner unit normal bundle of ∂D.

Let $P_D : \widetilde{M} \cup (\partial_\infty \widetilde{M} - \partial_\infty D) \to D$ be the (continuous) *closest point map* defined on $\xi \in \partial_\infty \widetilde{M} - \partial_\infty D$ by setting $P_D(\xi)$ to be the unique point in D that minimises the function $y \mapsto \beta_\xi(y, x_0)$ from D to $\mathbb{R}$. The *outer unit normal bundle* $\partial^1_+ D$ of the boundary of D is the topological submanifold of $T^1\widetilde{M}$ consisting of the geodesic lines $v : \mathbb{R} \to \widetilde{M}$ with $P_D(v_+) = v(0)$. The *inner unit normal bundle* of the boundary of D is $\partial^1_- D = \iota \partial^1_+ D$. Note that $\pi(\partial^1_\pm D) = \partial D$, that $\partial^1_+ HB_-(v)$ is the strong unstable manifold $W^{\rm su}(v)$ of v and that $W^{\rm ss}(v) = \partial^1_- HB_+(v)$. When D is a totally geodesic submanifold of $\widetilde{M}$, then $\partial^1_+ D = \partial^1_- D$.

The restriction of P_D to $\partial_\infty \widetilde{M} - \partial_\infty D$ is not injective in general, but the inverse P^+_D of the restriction to $\partial^1_+ D$ of the *(positive) endpoint map* $v \mapsto v_+$ is a natural lift of P_D to a homeomorphism from $\partial_\infty \widetilde{M} - \partial_\infty D$ to $\partial^1_+ D$ such that $\pi \circ P^+_D = P_D$. Similarly, $P^-_D = \iota \circ P^+_D : \partial_\infty \widetilde{M} - \partial_\infty D \to \partial^1_- D$ is a homeomorphism such that $\pi \circ P^-_D = P_D$.

For every isometry γ of $\widetilde{M}$, we have $\partial^1_\pm(\gamma D) = \gamma\, \partial^1_\pm D$ and $P^\pm_{\gamma D} \circ \gamma = \gamma \circ P^\pm_D$. In particular, $\partial^1_\pm D$ is invariant under the isometries of $\widetilde{M}$ that preserve D. For all $t \geq 0$, we have $g^{\pm t} \partial^1_\pm D = \partial^1_\pm(\mathscr{N}_t D)$.

We define

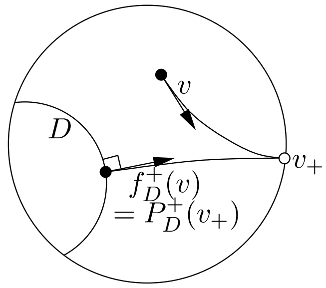
$$\mathscr{U}^+_D = \{v \in T^1\widetilde{M} :\ v_+ \notin \partial_\infty D\} \tag{4}$$

and

$$\mathscr{U}^-_D = \{v \in T^1\widetilde{M} :\ v_- \notin \partial_\infty D\} = \iota \mathscr{U}^+_D\ . \tag{5}$$

Note that $\mathscr{U}^\pm_D$ is an open subset of $T^1\widetilde{M}$, invariant under the geodesic flow. We have $\mathscr{U}^\pm_{\gamma D} = \gamma \mathscr{U}^\pm_D$ for every isometry γ of $\widetilde{M}$ and, in particular, $\mathscr{U}^\pm_D$ is invariant under the isometries of $\widetilde{M}$ preserving D.

Define a map $f_D^+ : \mathscr{U}_D^+ \to \partial_+^1 D$ as the composition of the positive endpoint map from $\mathscr{U}_D^+$ onto $\partial_\infty \widetilde{M} - \partial_\infty D$ and the homeomorphism P_D^+ from $\partial_\infty \widetilde{M} - \partial_\infty D$ to $\partial_+^1 D$. The map f_D^+ is a fibration as the composition of such a map with the homeomorphism P_D^+. The fiber of $w \in \partial_+^1 D$ for f_D^+ is exactly the stable leaf



$$W^{\rm s}(w) = \{v \in T^1\widetilde{M} \ : \ v_+ = w_+\}\,.$$

Analogously, we define a fibration $f_D^- = \iota \circ f_D^+ \circ \iota : \mathscr{U}_D^- \to \partial_-^1 D$ as the composition of the negative endpoint map and the map P_D^-, for which the fiber of $w \in \partial_-^1 D$ is the unstable leaf

$$W^{\rm u}(w) = \{v \in T^1\widetilde{M} \ : \ v_- = w_-\}.$$

For every isometry γ of $\widetilde{M}$, we have

$$f_{\gamma D}^\pm \circ \gamma = \gamma \circ f_D^\pm\,. \tag{6}$$

We have $f_{\mathscr{N}_t D}^\pm = g^{\pm t} \circ f_D^\pm$ for all $t \geq 0$, and $f_D^\pm \circ g^t = f_D^\pm$ for all $t \in \mathbb{R}$. In particular, the fibrations $f_D^\pm$ are invariant under the geodesic flow.

The next result will only be used for the error term estimates in Section 5. Note that if $\widetilde{M}$ is a symmetric space (in which case the strong stable and unstable foliations are smooth, and the sphere at infinity has a smooth structure such that the maps $v \mapsto v_+$ from $W^{\rm su}(w)$ to $\partial_\infty \widetilde{M} - \{w_-\}$ and $v \mapsto v_-$ from $W^{\rm ss}(w)$ to $\partial_\infty \widetilde{M} - \{w_+\}$ are smooth), and if D has smooth boundary, then the fibrations $f_D^\pm$ are smooth.

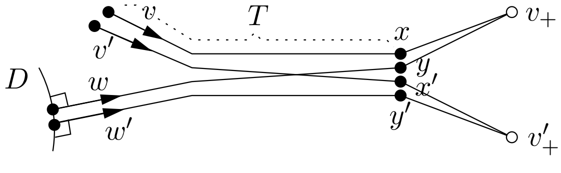
Recall that a map $f : X \to Y$ between two metric spaces is (uniformly locally) *Hölder-continuous* if there exist $c, c' > 0$ and $\alpha \in \,]0,1]$ such that $d(f(x), f(y)) \leq c\, d(x,y)^\alpha$ for all $x, y \in X$ with $d(x,y) \leq c'$.

Lemma 7 *The maps $f_D^\pm$ are Hölder-continuous on the set of elements $v \in \mathscr{U}_D^\pm$ such that $d(\pi(v), \pi(f_D^\pm(v)))$ is bounded.*

Proof. We prove the result for f_D^+, the one for f_D^- follows similarly. For all $u, u' \in T^1\widetilde{M}$, denote the geodesic lines they define by $t \mapsto u_t, u'_t$, and let

$$\delta_1(u,u') = \exp(-\sup\{t \geq 0 \ : \ \sup_{s\in[-t,t]} d(u_s, u'_s) \leq 1\}) \quad \text{and} \quad \delta_2(u,u') = \sup_{t \in [0,1]} d(u_t, u'_t)\,.$$

with the convention $\delta_1(u,u') = 1$ if $d(u_0, u'_0) > 1$ and $\delta_1(u,u') = 0$ if $u = u'$. By for instance [Bal, page 70], the maps δ_1, δ_2 are distances on $T^1\widetilde{M}$ which are Hölder-equivalent to Sasaki's distance.



Let $v, v' \in T^1\widetilde{M}$ be such that $d(v_0, v'_0) \leq 1$, let $w = f_D^+(v)$ and $w' = f_D^+(v')$. Let $T = \sup\{t \geq 0 \ : \ \sup_{s\in[0,t]} d(v_s, v'_s) \leq 1\}$, so that $\delta_1(v,v') \geq e^{-T}$. We may assume that

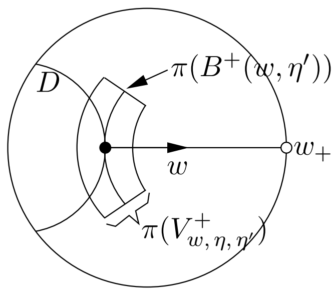
T is finite, otherwise $v_+ = v'_+$, hence $w = w'$. Let $x = v_T$ and $x' = v'_T$, which satisfy $d(x, x') \le 1$. Let y (respectively y') be the closest point to x (respectively x') on the geodesic ray defined by w (respectively w'). By convexity, since $d(v_0, w_0)$ and $d(v'_0, w'_0)$ are bounded by a constant $c > 0$ and since $v_+ = w_+, v'_+ = w'_+$, we have $d(x, y) \le c$ and $d(x', y') \le c$. By the triangle inequality, we have $d(y, y') \le 2c + 1$, $d(y, w_1) \ge T - 2c - 1$ and $d(y', w'_1) \ge T - 2c - 1$. By convexity, and since projection maps exponentially decrease the distances, there exists a constant $c' > 0$ such that

$$\delta_2(w, w') = d(w_1, w'_1) \le c' d(y, y') e^{-(T-2c-1)} \le c'(2c+1)e^{2c+1}\, \delta_1(v, v')\,.$$

The result follows. $\square$

Let $\eta, \eta' > 0$. For all $w \in T^1\widetilde{M}$, let

$$B^+(w, \eta') = \{v' \in W^{\rm ss}(w) \;:\; d_{W^{\rm ss}(w)}(v', w) < \eta'\} \qquad (7)$$



and

$$B^-(w, \eta') = \{v' \in W^{\rm su}(w) \;:\; d_{W^{\rm su}(w)}(v', w) < \eta'\} \qquad (8)$$

be the open balls of radius η' centered at w for Hamenstädt's distance in the strong stable and strong unstable leaves of w. Let

$$V^{\pm}_{w,\,\eta,\,\eta'} = \bigcup_{s \in\,]-\eta,\,\eta\,[} g^s B^{\pm}(w, \eta')\,.$$

We have $B^-(w, \eta') = \iota B^+(\iota w, \eta')$ hence $V^-_{w,\,\eta,\,\eta'} = \iota V^+_{\iota w,\,\eta,\,\eta'}$. We have

$$g^s B^{\pm}(w, \eta') = B^{\pm}(g^s w, e^{\mp s}\eta') \;\text{ hence }\; g^s V^{\pm}_{w,\,\eta,\,\eta'} = V^{\pm}_{g^s w,\,\eta,\,e^{\mp s}\eta'} \qquad (9)$$

for all $s \in \mathbb{R}$. For every isometry γ of $\widetilde{M}$, we have $\gamma B^{\pm}(w, \eta') = B^{\pm}(\gamma w, \eta')$ and $\gamma V^{\pm}_{w,\,\eta,\,\eta'} = V^{\pm}_{\gamma w,\,\eta,\,\eta'}$. The map from $]-\eta, \eta[\, \times B^{\pm}(w, \eta')$ to $V^{\pm}_{w,\,\eta,\,\eta'}$ defined by $(s, v') \mapsto g^s v'$ is a homeomorphism.

For all subsets Ω^- of $\partial^1_+ D$ and Ω^+ of $\partial^1_- D$, let

$$\mathscr{V}^+_{\eta,\,\eta'}(\Omega^-) = \bigcup_{w \in \Omega^-} V^+_{w,\,\eta,\,\eta'} \quad\text{and}\quad \mathscr{V}^-_{\eta,\,\eta'}(\Omega^+) = \bigcup_{w \in \Omega^+} V^-_{w,\,\eta,\,\eta'}\,.$$

For every isometry γ of $\widetilde{M}$, we have $\gamma \mathscr{V}^{\pm}_{\eta,\,\eta'}(\Omega^{\mp}) = \mathscr{V}^{\pm}_{\eta,\,\eta'}(\gamma\Omega^{\mp})$ and for every $t \ge 0$, we have

$$g^{\pm t} \mathscr{V}^{\pm}_{\eta,\,\eta'}(\Omega^{\mp}) = \mathscr{V}^{\pm}_{\eta,\,e^{-t}\eta'}(g^{\pm t}\Omega^{\mp})\,. \qquad (10)$$

The thickenings (or *dynamical neighbourhoods*) $\mathscr{V}^{\pm}_{\eta,\,\eta'}(\Omega^{\mp})$ are nondecreasing in η and in η' and their intersections and unions satisfy

$$\bigcap_{\eta\,,\,\eta' > 0} \mathscr{V}^{\pm}_{\eta,\,\eta'}(\Omega^{\mp}) = \Omega^{\mp} \quad\text{and}\quad \bigcup_{\eta\,,\,\eta' > 0} \mathscr{V}^{\pm}_{\eta,\,\eta'}(\partial^1_{\pm} D) = \mathscr{U}^{\pm}_D\,.$$

The restriction of $f^{\pm}_D$ to $\mathscr{V}^{\pm}_{\eta,\,\eta'}(\Omega^{\mp})$ is a fibration over $\Omega^{\mp}$, whose fiber over $w \in \Omega^{\mp}$ is the open subset $V^{\pm}_{w,\,\eta,\,\eta'}$ of the stable/unstable leaf of w.

2.3 Creating common perpendiculars

We start this subsection by giving a precise definition of the main objects in this paper we are interested in.

For any two closed convex subsets D^- and D^+ of $\widetilde{M}$, we say that a geodesic arc $\alpha : [0, T] \to \widetilde{M}$, where $T > 0$, is a *common perpendicular* from D^- to D^+ if its initial tangent vector $\dot\alpha(0)$ belongs to $\partial^1_+ D^-$ and if its terminal tangent vector $\dot\alpha(T)$ belongs to $\partial^1_- D^+$. It is important to think of common perpendiculars as oriented arcs (from D^- to D^+). Note that there exists a common perpendicular from D^- to D^+ if and only if D^- and D^+ are nonempty and the closures $\overline{D^-}$ and $\overline{D^+}$ of D^- and D^+ in the compactification $\widetilde{M} \cup \partial_\infty \widetilde{M}$ are disjoint. Also note that a common perpendicular from D^- to D^+, if it exists, is unique.

When $\overline{D^-}$ and $\overline{D^+}$ are disjoint, and when ∂D^- and ∂D^+ are $\mathscr{C}^1$-submanifolds (for instance, by [Wal], if $D^\pm$ are closed ϵ-neighbourhoods of nonempty convex subsets of $\widetilde{M}$ for some $\epsilon > 0$), this definition of a common perpendicular corresponds to the usual one. But there are interesting closed convex subsets that do not have this boundary regularity, such as in general the convex hulls of limits sets of nonelementary discrete groups of isometries of $\widetilde{M}$. Although it would be possible to take the closed ϵ-neighbourhood, to count common perpendiculars in the usual sense, and then to take a limit as ϵ goes to 0, it is more natural to work directly in the above generality (see [PP6, Sect. 3.2] for further comments).

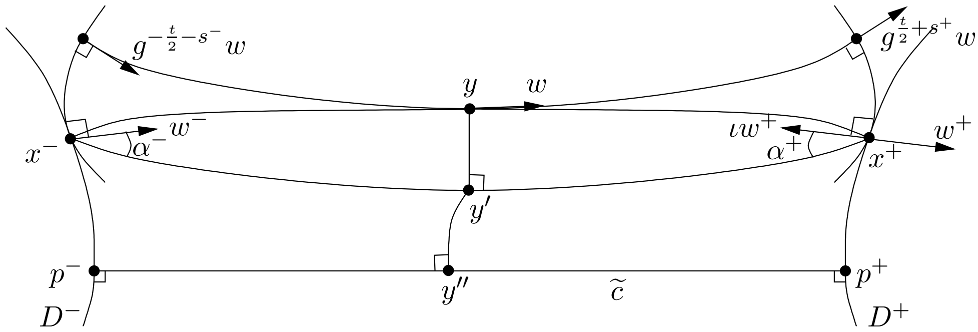
The aim of this paper is to count orbits of common perpendiculars between two equivariant families of closed convex subsets of $\widetilde{M}$. The crucial remark is that two nonempty proper closed convex subsets D^- and D^+ of $\widetilde{M}$ have a common perpendicular α of length a given $t > 0$ if and only if the pushforwards and pullbacks by the geodesic flow at time $\frac{t}{2}$ of the outer and inner normal bundles of D^- and D^+, that is the subsets $g^{\frac{t}{2}} \partial^1_+ D^-$ and $g^{-\frac{t}{2}} \partial^1_- D^+$ of $T^1\widetilde{M}$, intersect. Then their intersection is the singleton consisting of the tangent vector of α at its midpoint.

In this subsection, we relate the existence of a common perpendicular (and its length) from D^- to D^+ with the intersection properties in $T^1\widetilde{M}$ of the dynamical neighbourhoods of the outer and inner normal bundles of D^- and D^+ introduced in the previous subsection, pushed/pulled equal amounts by the geodesic flow.

Lemma 8 *For every $R > 0$, there exist $t_0, c_0 > 0$ such that for all $\eta \in \;]0, 1]$ and all $t \in \;[t_0, +\infty[$, for all nonempty closed convex subsets D^-, D^+ in $\widetilde{M}$, and for all $w \in g^{t/2}\mathscr{V}^+_{\eta, R}(D^-) \cap g^{-t/2}\mathscr{V}^-_{\eta, R}(D^+)$, there exist $s \in \;]-2\eta, 2\eta[$ and a common perpendicular $\widetilde{c}$ from D^- to D^+ such that*

- *the length of $\widetilde{c}$ is contained in $[t + s - c_0\, e^{-\frac{t}{2}}, t + s + c_0\, e^{-\frac{t}{2}}]$,*
- *if $w^\mp = f^\pm_{D^\mp}(w)$ and if $p^\pm$ is the endpoint of $\widetilde{c}$ in $D^\pm$, then $d(\pi(w^\pm), p^\pm) \le c_0\, e^{-\frac{t}{2}}$,*
- *the basepoint $\pi(w)$ of w is at distance at most $c_0\, e^{-\frac{t}{2}}$ from a point of $\widetilde{c}$, and*

$$\max\{\; d(\pi(g^{\frac{t}{2}} w^-), \pi(w)),\; d(\pi(g^{-\frac{t}{2}} w^+), \pi(w)) \;\} \le \eta + c_0\, e^{-\frac{t}{2}}\;.$$

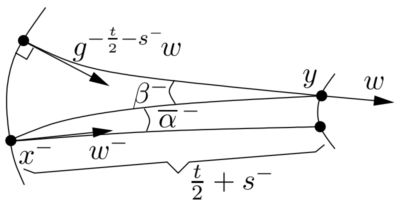


Proof. Let $t \geq 3$ and $\eta \in \,]0,1]$. By definition of the dynamical neighbourhoods $\mathscr{V}^{\mp}_{\eta,\,R}(D^{\pm})$, there exist $w^{\pm} \in \partial^1_{\mp} D^{\pm}$ and $s^{\pm} \in \,]-\eta, +\eta[$ such that

$$d_{W^{\rm ss}(w^-)}(g^{-\frac{t}{2}-s^-}w, w^-) \leq R \quad\text{and}\quad d_{W^{\rm su}(w^+)}(g^{\frac{t}{2}+s^+}w, w^+) \leq R\;.$$

Let $x^{\pm} = \pi(w^{\pm})$, $y = \pi(w)$, and let α^- (respectively α^+) be the angle at x^- (respectively x^+) between w^- (respectively ιw^+) and the geodesic segment $[x^-, x^+]$.

Step 1. Let $\overline{\alpha}^-$ (respectively $\overline{\alpha}^+$) be the angle at x^- (respectively x^+) between the outer normal vector w^- (respectively ιw^+) and the geodesic segment $[x^-, y]$ (respectively $[x^+, y]$). Let $\beta^{\pm}$ be the angle at y between $\pm w$ and the geodesic segment $[y, x^{\pm}]$. Let us prove that there exist two constants $t_1, c_1 > 0$ depending only on R such that if $t \geq t_1$ then $\overline{\alpha}^{\,\pm}, \beta^{\pm} \leq c_1\, e^{-\frac{t}{2}}$.



By Lemma 6 and Equation (3), we have

$$d(\pi(g^{-\frac{t}{2}-s^-}w), x^-) \leq d_{W^{\rm ss}(w^-)}(g^{-\frac{t}{2}-s^-}w, w^-) \leq R\;,$$

$$d(y, \pi(g^{\frac{t}{2}+s^-}w^-)) \leq d_{W^{\rm ss}(w)}(w, g^{\frac{t}{2}+s^-}w^-) \leq R\, e^{-\frac{t}{2}-s^-}\;. \tag{11}$$

In particular,

$$d(\pi(w), \pi(g^{\frac{t}{2}}w^-)) \leq d(y, \pi(g^{\frac{t}{2}+s^-}w^-)) + |s^-| \leq R\, e^{-\frac{t}{2}-s^-} + \eta \leq c_0\; e^{-\frac{t}{2}} + \eta$$

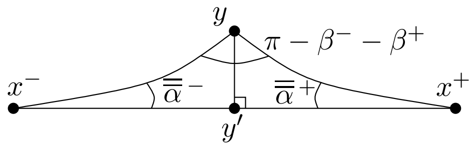
if we assume, as we may, that $c_0 \geq R\, e^{\eta}$. With a similar argument for w^+, this proves the last formula of Lemma 8.

Recall that by a hyperbolic trigonometric formula (see for instance [Bea, page 147]), for any geodesic triangle in the real hyperbolic plane, with angles α, β, γ and opposite side lengths a, b, c, if $\gamma \geq \frac{\pi}{2}$, then $\tan\alpha \leq \frac{1}{\sinh b}$, which is at most $\frac{1}{\sinh(c-a)}$ if $c > a$ by the triangle inequality. By comparison, if $t \geq 2(R+2)$ (which implies that $\frac{t}{2} + s^- - R \geq 1$), we hence have

$$\max\{\tan\overline{\alpha}^{\,-}, \tan\beta^-\} \leq \frac{1}{\sinh(\frac{t}{2}+s^- - R)} \leq 4\, e^{-\frac{t}{2}-s^-+R}$$

With a symmetric argument for $\overline{\alpha}^{\,+}, \beta^+$, the result follows.

Step 2. Let $\overline{\overline{\alpha}}^{\,\pm}$ be the angles at $x^\pm$ of the geodesic triangle with vertices x_-, x_+, y. Let y' be the closest point to y on the side $[x_-, x_+]$. Let us prove that there exist two constants $t_2, c_2 > 0$ depending only on R such that if $t \geq t_2$ then $\overline{\overline{\alpha}}^{\,\pm}, d(y, y') \leq c_2\, e^{-\frac{t}{2}}$.



Since the angle $\angle_y(x^-, x^+)$ is at least $\pi - \beta^- - \beta^+$, at least one of the two angles $\angle_y(y', x^\pm)$ is at least $\frac{\pi-\beta^--\beta^+}{2}$. By a comparison argument applied to one of the two triangles with vertices $(y, y', x^\pm)$, as in the end of the first step, we have $\tan\frac{\pi-\beta^--\beta^+}{2} \leq \frac{1}{\sinh d(y,\, y')}$. Hence

$$d(y, y') \leq \sinh\ d(y, y') \leq \tan\frac{\beta^- + \beta^+}{2}\ ,$$

and the desired majoration of $d(y, y')$ follows from Step 1. By the same argument, we have $\tan\overline{\overline{\alpha}}^{\,\pm} \leq \frac{1}{\sinh(d(x^\pm,\, y) - d(y,\, y'))}$. Since $d(x^\pm,\, y) \geq \frac{t}{2} - s^\pm - R\, e^{-\frac{t}{2} - s^-}$ by the inverse triangle inequality and Equation (11), the desired majoration of $\overline{\overline{\alpha}}^{\,\pm}$ follows.

Step 3. Let us prove that there exist two constants $t_3, c_3 > 0$ depending only on R such that if $t \geq t_3$ then there exists a common perpendicular $\widetilde{c} = [p^-, p^+]$ from D^- to D^+ such that $d(x^-, p^-), d(x^+, p^+) \leq c_3\, e^{-\frac{t}{2}}$. This will prove the second point of Lemma 8 (if $t_0 \geq t_3$ and $c_0 \geq c_3$).

By the first two steps, we have, if $t \geq \min\{t_1, t_2\}$,

$$\alpha^\pm \leq \overline{\alpha}^{\,\pm} + \overline{\overline{\alpha}}^{\,\pm} \leq (c_1 + c_2)e^{-\frac{t}{2}}\ . \tag{12}$$

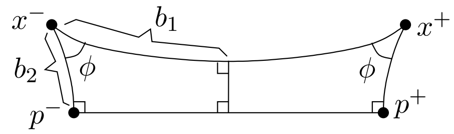
Assume by absurd that the intersection of the closures of D^- and D^+ in $\widetilde{M} \cup \partial_\infty \widetilde{M}$ contains a point z. Then by convexity of $D^\pm$, and since the distance $d(x^-, x^+)$ is big and the angles $\alpha^\pm$ are small if t is big, the angles at $x^\pm$ of the geodesic triangle with vertices z, x^-, x^+ are almost at least $\frac{\pi}{2}$, which is impossible since $\widetilde{M}$ is CAT(-1). Hence the nonempty closed convex subsets D^- and D^+ have a common perpendicular $\widetilde{c} = [p^-, p^+]$, with $p^\pm \in D^\pm$.

Consider the geodesic quadrilateral Q with vertices $x^\pm, p^\pm$. By convexity of $D^\pm$, its angles at $p^\pm$ are at least $\frac{\pi}{2}$ and its angles at $x^\pm$ are at least $\frac{\pi}{2} - \alpha^\pm$. Note that if $t \geq t_2' = 2(R + c_2 + 1 + \operatorname{argsinh} 2)$ then we have, by Step 2,

$$\begin{aligned} d(x^-, x^+) &\geq d(x^-, y) + d(y, x^+) - 2\, d(y, y') \\ &\geq (\frac{t}{2} + s^- - R) + (\frac{t}{2} + s^+ - R) - 2\, c_2\, e^{-\frac{t}{2}} \geq 2 \operatorname{argsinh} 2\ . \end{aligned} \tag{13}$$

Up to replacing Q by a comparison quadrilateral (obtained by gluing two comparison triangles) in the real hyperbolic plane $\mathbb{H}^2_{\mathbb{R}}$, having the same side lengths and bigger angles, we may assume that $\widetilde{M} = \mathbb{H}^2_{\mathbb{R}}$ and that x^- and x^+ are on the same side of the geodesic line through p^-, p^+. Up to replacing Q by a quadrilateral having same distances $d(x^-, x^+)$, $d(x^-, p^-)$, $d(x^+, p^+)$ and bigger angles at x^-, x^+, we may assume that the angles at p^-, p^+ are exactly $\frac{\pi}{2}$. If, say, the angle at x^+ was bigger than the angle at x^-, up to replacing x^+ by a point on the geodesic line through p^+, x^+ on the other side of x^+ than p^+ if $p^+ \neq x^+$, which increases $d(x^-, x^+)$, $d(x^+, p^+)$, decreases the angle at x^+ and increases the angle at x^-, we may assume that the angles at $x^\pm$ are equal, and we denote this common value by $\phi \geq \frac{\pi}{2} - \min\{\alpha^-, \alpha^+\}$.

Let $b_1 = \frac{1}{2}\, d(x^-, x^+)$ and $b_2 = d(x^-, p^-) = d(x^+, p^+)$. By formulas of [Bea, page 157] on Lambert quadrilaterals, we have



$$\cosh b_2 = \frac{\sinh b_1}{\sqrt{\sinh^2 b_1 \sin^2 \phi - \cos^2 \phi}}\;.$$

By Equation (12), with $c_2' = c_1 + c_2$, let $t_2'' > 0$ be a constant, depending only on R, such that if $t \geq t_2''$, then $\sin\phi \geq \max\{\cos\alpha^\pm\} \geq 1 - {c_2'}^2\, e^{-t} \geq 1/2$. By Equation (13), if $t \geq t_2'$, then $b_1 \geq \frac{t}{2} - R - 1 - c_2 \geq \operatorname{argsinh} 2$ (and in particular $1/\sinh b_1 \leq 1/2$). Hence, if $t \geq \max\{t_2', t_2''\}\}$, then

$$\cosh b_2 \leq \frac{1}{\sqrt{\sin^2\phi - \frac{1}{\sinh^2 b_1}}} \leq \Big((1 - {c_2'}^2\, e^{-t})^2 - \frac{1}{\sinh^2(\frac{t}{2} - R - 1 - c_2)}\Big)^{-\frac{1}{2}} = 1 + \mathrm{O}(e^{-t})\;.$$

Since $\cosh u \sim 1 + \frac{u^2}{2}$ as $u \to 0$, Step 3 follows.

Step 4. Let us now conclude the proof of Lemma 8.

Let $t \geq t_0 = \max\{t_2, t_3, 3\}$, $c_0 = \max\{2e^2 R, 2(c_2 + c_3)\}$ and, with the previous notation, let $s = s^- + s^+ \in\,]-2\eta, 2\eta[\,$. By convexity, the triangle inequality and Equation (11), we have

$$\begin{aligned} d(p^-, p^+) &\leq d(x^-, x^+) \leq d(x^-, y) + d(y, x^+) \\ &\leq (\frac{t}{2} + s^- + R\, e^{-\frac{t}{2} - s^-}) + (\frac{t}{2} + s^+ + R\, e^{-\frac{t}{2} - s^+}) \leq t + s + c_0\, e^{-\frac{t}{2}}\;. \end{aligned}$$

Similarly, using Step 3 and Step 2, we have

$$\begin{aligned} d(p^-, p^+) &\geq d(x^-, x^+) - d(p^-, x^-) - d(x^+, p^+) \geq d(x^-, x^+) - 2c_3\, e^{-\frac{t}{2}} \\ &\geq d(x^-, y) + d(y, x^+) - 2d(y, y') - 2\, c_3\, e^{-\frac{t}{2}} \\ &\geq (\frac{t}{2} + s^-) + (\frac{t}{2} + s^+) - 2(c_2 + c_3)\, e^{-\frac{t}{2}} \geq t + s - c_0\, e^{-\frac{t}{2}}\;. \end{aligned}$$

Let y'' be the closest point to y' on the common perpendicular $[p^-, p^+]$ (see the picture before this proof). Then, by Step 2, and by convexity and Step 3, we have

$$d(y, y'') \leq d(y, y') + d(y', y'') \leq c_2\, e^{-\frac{t}{2}} + c_3\, e^{-\frac{t}{2}} \leq c_0\, e^{-\frac{t}{2}}\;.$$

This concludes the proof of Lemma 8. □

2.4 Pushing measures by branched covers

All measures in this paper are nonnegative Borel measures.

In Section 3 and Section 4, we will need to associate to a Γ-invariant measure on $T^1\widetilde{M}$ or to a $\Gamma \times \Gamma$-invariant measure on $T^1\widetilde{M} \times T^1\widetilde{M}$ (which is in general infinite) a measure on T^1M or $T^1M \times T^1M$ (hopefully finite), in a continuous way. For lack of references, we recall here the construction (which is not the standard pushforward of measures), not-so-well-known when Γ has torsion.

Let $\widetilde{X}$ be a locally compact metrisable space, endowed with a proper (but not necessarily free) action of a discrete group G. Let $p : \widetilde{X} \to X = G\backslash\widetilde{X}$ be the canonical projection. Let $\widetilde{\mu}$ be a locally finite G-invariant measure on $\widetilde{X}$.

Note that the map N from $\widetilde{X}$ to $\mathbb{N} - \{0\}$ sending a point $x \in X$ to the order of its stabiliser in G is upper semi-continuous. In particular, for every $n \geq 1$, the G-invariant subset $\widetilde{X}_n = N^{-1}(\{n\})$ is locally closed, hence locally compact metrisable and $\widetilde{\mu}_{|\widetilde{X}_n}$ is a locally finite G-invariant measure on X_n. With $X_n = p(\widetilde{X}_n)$, the restriction $p_{|\widetilde{X}_n} : \widetilde{X}_n \to X_n$ is a local homeomorphism. Since $\widetilde{\mu}$ is G-invariant, there exists a unique measure μ_n on X_n such that the map $p_{|\widetilde{X}_n}$ locally preserves the measure. Now, considering a measure on X_n as a measure on X with support in X_n, define

$$\mu = \sum_{n \geq 1} \frac{1}{n} \, \mu_n \ ,$$

which is a locally finite measure on X, called the measure *induced by* $\widetilde{\mu}$ on X.

Note that if $\widetilde{\mu}$ gives measure 0 to the set $N^{-1}([2, +\infty[)$ of fixed points of non-trivial elements of G, then $\mu = \mu_1$, and the above construction is not needed.

If $\widetilde{X}'$ is another locally compact metrisable space, endowed with a proper action of a discrete group G', if $\widetilde{\mu}'$ is a locally finite G'-invariant measure on $\widetilde{X}'$, with induced measure μ' on $G'\backslash X'$, then the measure, on the product of the quotient spaces $G\backslash X \times G'\backslash X'$, induced by the product measure $\widetilde{\mu} \otimes \widetilde{\mu}'$, is the product $\mu \otimes \mu'$ of the induced measures.

It is easy to check that the map $\widetilde{\mu} \mapsto \mu$ from the space of locally finite measures on $\widetilde{X}$ to the space of locally finite measures on X, both endowed with their weak-star topologies, is continuous.

Recall (see for instance [Bil, Part]) that the *narrow topology* on the set $\mathscr{M}_f(Y)$ of finite measures on a locally compact metrisable space Y is the smallest topology on $\mathscr{M}_f(Y)$ such that, for every bounded continuous map $g : Y \to \mathbb{R}$, the map from $\mathscr{M}_f(Y)$ to $\mathbb{R}$ defined by $\mu \mapsto \mu(g)$ is continuous.

It is easy to check that if $\widetilde{\mu}_k$ for $k \in \mathbb{N}$ and $\widetilde{\mu}$ are G-invariant locally finite measures on $\widetilde{X}$, with finite induced measures on X, such that for every Borel subset B of $\widetilde{X}$ such that $\widetilde{\mu}(B)$ is finite and $\widetilde{\mu}(\partial B) = 0$, we have $\lim_{k\to\infty} \widetilde{\mu}_k(B) = \widetilde{\mu}(B)$, then the sequence $(\mu_k)_{k\in\mathbb{N}}$ narrowly converges to μ.

3 Potentials, cocycles and measures

Let $\widetilde{M}$, x_0, Γ and M be as in the beginning of Section 2. In our main result, we will count common perpendicular arcs between two locally convex subsets of M using weights defined by a fixed potential. We introduce in this section, in addition to the potentials themselves, the basic properties of the measure-theoretic structure induced by a potential on $T^1\widetilde{M}$ and T^1M, in particular in connection with (families of) convex subsets of $\widetilde{M}$.

For any Riemannian orbifold N, we denote by $\mathscr{C}_c(N)$ the space of continuous real-valued functions on N with compact support.

3.1 Potentials, Patterson densities and Gibbs measures

The content of this subsection is extracted from [PPS], to which we refer for the proofs of the claims and for more details.

Let $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ be a fixed bounded Hölder-continuous Γ-invariant function, called a *potential* on $T^1\widetilde{M}$. The potential $\widetilde{F}$ induces a bounded Hölder-continuous function

$F : T^1M \to \mathbb{R}$, called a *potential* on T^1M. We will also consider the potential $\widetilde{F}\circ\iota$ on $T^1\widetilde{M}$ and its induced potential $F\circ\iota$ on T^1M.

For any two distinct points $x, y \in \widetilde{M}$, let $v_{xy} \in T^1_x\widetilde{M}$ be the initial tangent vector of the oriented geodesic segment $[x, y]$ in $\widetilde{M}$ that connects x to y. The $\widetilde{F}$-*weighted length* of the segment $\alpha = [x, y]$ is

$$\int_\alpha \widetilde{F} = \int_x^y \widetilde{F} = \int_0^{d(x,y)} \widetilde{F}(g^t v_{xy})\, dt$$

and we set $\int_x^x \widetilde{F} = 0$ for all $x \in \widetilde{M}$. Note that $\int_x^y \widetilde{F} = \int_y^x \widetilde{F}\circ\iota$.

Since $\widetilde{F}$ is Hölder-continuous (see [PPS, Lem. 3.2]), there exist two constants $c_1 > 0$, $c_2 \in\,]0, 1]$ such that for all $x, x', y, y' \in \widetilde{M}$ with $d(x, x') \le 1$ and $d(y, y') \le 1$, we have

$$\Big|\int_{x'}^{y'} \widetilde{F} - \int_x^y \widetilde{F}\,\Big| \le c_1\, d(x, x')^{c_2} + c_1\, d(y, y')^{c_2} + \|F\|_\infty\, (d(x, x') + d(y, y'))\ . \tag{14}$$

The *critical exponent* of F is

$$\delta_F = \limsup_{n\to+\infty} \frac{1}{n} \log \sum_{\gamma\in\Gamma,\, n-1<d(x,\gamma y)\le n} e^{\int_x^{\gamma y} \widetilde{F}}\ .$$

The above upper limit is finite since F is bounded, is independent of x, y, and in particular

$$\delta_F = \delta_{F\circ\iota}\ .$$

In what follows, we assume that δ_F is positive (which is the case up to adding a constant to F, since $\delta_{F+\sigma} = \delta_F + \sigma$). By [PPS, Theo. 6.1], the critical exponent δ_F is equal to the pressure of F on T^1M, see the Introduction for the definition of the pressure.

The *(normalised) Gibbs cocycle* associated with the group Γ and the potential $\widetilde{F}$ is the function $C^+ : \partial_\infty\widetilde{M} \times \widetilde{M} \times \widetilde{M} \to \mathbb{R}$ defined by

$$(\xi, x, y) \mapsto C^+_\xi(x, y) = \lim_{t\to+\infty} \int_y^{\xi_t} (\widetilde{F} - \delta_F) - \int_x^{\xi_t} (\widetilde{F} - \delta_F)\ ,$$

where $t \mapsto \xi_t$ is any geodesic ray with endpoint $\xi \in \partial_\infty\widetilde{M}$. (Note the order of y and x on the right-hand side.) The above limit exists. We denote by C^- the Gibbs cocycle of the potential $\widetilde{F}\circ\iota$. If $\widetilde{F} = 0$, then $C^- = C^+ = \delta_\Gamma\beta$, where β is the Busemann cocycle defined in Section 2 and $\delta_F = \delta_\Gamma$ is the usual critical exponent of Γ.

The Gibbs cocycles satisfy the following equivariance and cocycle properties: For all $\xi \in \partial_\infty\widetilde{M}$ and $x, y, z \in \widetilde{M}$, and for every isometry γ of $\widetilde{M}$, we have

$$C^\pm_{\gamma\xi}(\gamma x, \gamma y) = C^\pm_\xi(x, y) \;\text{ and }\; C^\pm_\xi(x, z) + C^\pm_\xi(z, y) = C^\pm_\xi(x, y)\,. \tag{15}$$

If x is a point in the (image of the) geodesic ray from y to ξ, then $C^+_\xi(x, y) = \int_y^x (\widetilde{F} - \delta_F)$. Hence for every $w \in T^1\widetilde{M}$, for all x and y on the image of the geodesic line defined by w, with w_-, x, y, w_+ in this order, we have

$$C^-_{w_-}(x, y) = C^+_{w_+}(y, x) = -C^+_{w_+}(x, y) = \int_x^y (\widetilde{F} - \delta_F)\ . \tag{16}$$

By taking limits in Equation (14), there exist two constants $c_1 > 0$, $c_2 \in]0,1]$ such that for all $\xi \in \partial_\infty \widetilde{M}$ and $z, z' \in \widetilde{M}$ with $d(z,z') \leq 1$, we have

$$\Big| \, C^{\pm}_{\xi}(z,z') \, \Big| \leq c_1 \, d(z,z')^{c_2} + \|F\|_\infty \, d(z,z') \, . \tag{17}$$

A family $(\mu^+_x)_{x\in\widetilde{M}}$ (respectively $(\mu^-_x)_{x\in\widetilde{M}}$) of finite measures on $\partial_\infty\widetilde{M}$, whose support is the limit set $\Lambda\Gamma$ of Γ, is a *Patterson density of dimension* δ_F for the pair $(\Gamma, \widetilde{F})$ (respectively $(\Gamma, \widetilde{F}\circ\iota)$) if

$$\gamma_* \mu^{\pm}_x = \mu^{\pm}_{\gamma x}$$

for all $\gamma \in \Gamma$ and $x \in \widetilde{M}$, and if the following Radon-Nikodym derivatives exist for all $x, y \in \widetilde{M}$ and satisfy for all $\xi \in \partial_\infty \widetilde{M}$

$$\frac{d\mu^{\pm}_x}{d\mu^{\pm}_y}(\xi) = e^{-C^{\pm}_{\xi}(x,\,y)} \, .$$

We fix two Patterson densities $(\mu^+_x)_{x\in\widetilde{M}}$ and $(\mu^-_x)_{x\in\widetilde{M}}$ of dimension δ_F for the pairs $(\Gamma, \widetilde{F})$ and $(\Gamma, \widetilde{F}\circ\iota)$, respectively, which exist. The *Gibbs measure* on $T^1\widetilde{M}$ for $\widetilde{F}$ (associated to this ordered pair of Patterson densities) is the measure $\widetilde{m}_F$ on $T^1\widetilde{M}$ given by the density

$$d\widetilde{m}_F(v) = e^{C^-_{v_-}(x_0,\,\pi(v)) \,+\, C^+_{v_+}(x_0,\,\pi(v))} \; d\mu^-_{x_0}(v_-)\, d\mu^+_{x_0}(v_+)\, dt \tag{18}$$

in Hopf's parametrisation. The Gibbs measure $\widetilde{m}_F$ is independent of x_0, and it is invariant under the actions of the group Γ and of the geodesic flow. Thus (see Section 2.4), it defines a measure m_F on T^1M which is invariant under the quotient geodesic flow, called the *Gibbs measure* on T^1M. If m_F is finite, then the Patterson densities are unique up to a multiplicative constant; hence the Gibbs measure of $\widetilde{F}$ is uniquely defined, up to a multiplicative constant, and, when normalised to be a probability measure, it is the unique measure of maximal pressure of the geodesic flow for the potential F. If $F = 0$ and the Patterson densities coincide, the Gibbs measure coincides with the Bowen-Margulis measure (associated to this Patterson density) which, when finite and normalised to be a probability measure, is the unique measure of maximal entropy of the geodesic flow. If M is compact and $\widetilde{F}(v) = -\frac{d}{dt}|_{t=0} \log \mathrm{Jac}(g^t|_{W^{\rm su}(v)})(v)$, then the Gibbs measure coincides with the Liouville measure (see [PPS, §7] for extensions of this result).

Babillot [Bab, Thm. 1] showed that if the Gibbs measure is finite, then it is mixing for the geodesic flow of M if the length spectrum of M is not contained in a discrete subgroup of $\mathbb{R}$. This condition is satisfied, for example, if Γ has a parabolic element, if $\Lambda\Gamma$ is not totally disconnected (hence if M is compact), or if $\widetilde{M}$ is a surface or a (rank-one) symmetric space, see for instance [Dal1, Dal2].

We refer to [PPS, §8] for finiteness criteria of m_F. In particular, if Γ is geometrically finite (see for instance [Bow] for a definition), if $\|F\|_\infty < \frac{1}{2}(\delta_\Gamma - \delta_{\Gamma_p})$ for every parabolic fixed point p of Γ in $\partial_\infty\widetilde{M}$, where δ_{Γ_p} is the critical exponent of the stabiliser Γ_p of p in Γ, then m_F is finite (see [PPS, Coro. 8.6]).

3.2 Skinning measures

Let D be a nonempty proper closed convex subset of $\widetilde{M}$. The *(outer) skinning measure* on $\partial^1_+ D$ (associated to the Patterson density $(\mu^+_x)_{x\in\widetilde{M}}$ for $(\Gamma, \widetilde{F})$) is the measure $\widetilde{\sigma}^+_D =$

$\widetilde{\sigma}^+_{D,\widetilde{F}}$ on $\partial^1_+ D$ defined, using the positive endpoint homeomorphism $v \mapsto v_+$ from $\partial^1_+ D$ to $\partial_\infty \widetilde{M} - \partial_\infty D$, by

$$d\widetilde{\sigma}^+_D(v) = e^{C^+_{v_+}(x_0,\, P_D(v_+))}\, d\mu^+_{x_0}(v_+)\ ,$$

and the *(inner) skinning measure* on $\partial^1_- D = \iota \partial^1_+ D$ (associated to the Patterson density $(\mu^-_x)_{x\in\widetilde{M}}$ for $(\Gamma, \widetilde{F}\circ\iota)$) is the measure $\widetilde{\sigma}^-_D = \iota_* \widetilde{\sigma}^+_{D,\,\widetilde{F}\circ\iota}$ defined, using the negative endpoint homeomorphism $v \mapsto v_-$ from $\partial^1_- D$ to $\partial_\infty \widetilde{M} - \partial_\infty D$, by

$$d\widetilde{\sigma}^-_D(v) = e^{C^-_{v_-}(x_0,\, P_D(v_-))}\, d\mu^-_{x_0}(v_-)\,.$$

Since $P_D(v_\pm) = \pi(v)$ for every $v \in \partial^1_\pm D$, we will often replace $P_D(v_\pm)$ by $\pi(v)$ in the above formulas when there is no doubt on what v is.

When $\widetilde{F} = 0$, the skinning measure has been defined by Oh and Shah [OS2] for the outer unit normal bundles of spheres, horospheres and totally geodesic subspaces in real hyperbolic spaces, and the definition was generalised in [PP5] to the outer unit normal bundles of nonempty proper closed convex sets in variable negative curvature. We refer to [PP5] for more background.

Remarks (1) A potential $\widetilde{F}$ is said to be *reversible* if there exists a Hölder-continuous Γ-invariant function $\widetilde{G} : T^1\widetilde{M} \to \mathbb{R}$ which is differentiable along the flow lines and satisfies

$$\widetilde{F}(v) - \widetilde{F}\circ\iota(v) = \frac{d}{dt}_{|t=0}\ \widetilde{G}(g^t v)$$

for all $v \in T^1\widetilde{M}$. When $\widetilde{F}$ is reversible (and in particular when $F = 0$), we have $C^- = C^+$, we may (and will) take $\mu^-_x = \mu^+_x$ for all $x \in \widetilde{M}$, hence $\iota_*\widetilde{m}_F = \widetilde{m}_F$, and $\widetilde{\sigma}^-_D = \iota_*\widetilde{\sigma}^+_D$.
(2) If $D = \{x\}$ is a singleton, we have $\partial^1_\pm D = T^1_x\widetilde{M}$ and

$$d\widetilde{\sigma}^\pm_D(v) = \ d\mu^\pm_x(v_\pm)\ .$$

(3) The (normalised) Gibbs cocycle being unchanged under replacing the potential F by the potential $F+\sigma$ for any constant σ, we may (and will) take the Patterson densities, hence the Gibbs measure and the skinning measures, to be unchanged by such a replacement.

The following results give the basic properties of the skinning measures analogous to those in [PP5, Sect. 3] when the potential is zero.

Proposition 9 *Let D be a nonempty proper closed convex subset of $\widetilde{M}$, and let $\widetilde{\sigma}^\pm_D$ be the skinning measures on $\partial^1_\pm D$ for the potential $\widetilde{F}$.*
(i) The skinning measures $\widetilde{\sigma}^\pm_D$ are independent of x_0.
(ii) For all $\gamma \in \Gamma$, we have $\gamma_*\widetilde{\sigma}^\pm_D = \widetilde{\sigma}^\pm_{\gamma D}$. In particular, the measures $\widetilde{\sigma}^\pm_D$ are invariant under the stabiliser of D in Γ.
(iii) For all $s \geq 0$ and $w \in \partial^1_\pm D$, we have

$$\frac{d(g^{\pm s})_*\widetilde{\sigma}^\pm_D}{d\,\widetilde{\sigma}^\pm_{\mathscr{N}_s D}}(g^{\pm s}w) = e^{-C^\pm_{w_\pm}(\pi(w),\,\pi(g^{\pm s}w))} = \begin{cases} e^{\int_{\pi(w)}^{\pi(g^s w)}(\widetilde{F}-\delta_F)} & \text{if } \pm = + \\ e^{\int_{\pi(g^{-s}w)}^{\pi(w)}(\widetilde{F}-\delta_F)} & \text{otherwise}\,. \end{cases}$$

(iv) The support of $\widetilde{\sigma}^\pm_D$ is $\{v \in \partial^1_\pm D : v_\pm \in \Lambda\Gamma\} = P^\pm_D(\Lambda\Gamma - (\Lambda\Gamma \cap \partial_\infty D))$. In particular, $\widetilde{\sigma}^\pm_D$ is the zero measure if and only if $\Lambda\Gamma$ is contained in $\partial_\infty D$.

Proof. We give details only for the proof of claim (iii) for the measure $\widetilde{\sigma}^+_D$, the case of $\widetilde{\sigma}^-_D$ being similar, and the proofs of the other claims being straightforward modifications of those in [PP5, Prop. 4]. Since $(g^s w)_+ = w_+$ and $w \in \partial^1_+ D$ if and only if $g^s w \in \partial^1_+ \mathscr{N}_s D$, we have, using the definition of the skinning measure and the cocycle property (15), for all $s \geq 0$,

$$d\,\widetilde{\sigma}^+_{\mathscr{N}_s D}(g^s w) = e^{C^+_{w_+}(x_0,\,\pi(g^s w))}\,d\mu^+_{x_0}(w_+) = e^{C^+_{w_+}(\pi(w),\,\pi(g^s w))}\,d\,\widetilde{\sigma}^+_D(w)\,,$$

which proves the claim (iii) for $\widetilde{\sigma}^+_D$, using Equation (16). $\Box$

Given two nonempty closed convex subsets D and D' of $\widetilde{M}$, let $A_{D,D'} = \partial_\infty \widetilde{M} - (\partial_\infty D \cup \partial_\infty D')$ and let $h^\pm_{D,D'} : P^\pm_D(A_{D,D'}) \to P^\pm_{D'}(A_{D,D'})$ be the restriction of $P^\pm_{D'} \circ (P^\pm_D)^{-1}$ to $P^\pm_D(A_{D,D'})$. It is a homeomorphism between open subsets of $\partial^1_\pm D$ and $\partial^1_\pm D'$, associating to the element w in the domain the unique element w' in the range with $w'_\pm = w_\pm$. The simple proof of Proposition 5 of [PP5] generalises immediately to give the following result.

Proposition 10 *Let D and D' be nonempty closed convex subsets of $\widetilde{M}$ and $h^\pm = h^\pm_{D,D'}$. The measures $(h^\pm)_*\,\widetilde{\sigma}^\pm_D$ and $\widetilde{\sigma}^\pm_{D'}$ on $P^\pm_{D'}(A_{D,D'})$ are absolutely continuous one with respect to the other, with*

$$\frac{d(h^\pm)_*\,\widetilde{\sigma}^\pm_D}{d\widetilde{\sigma}^\pm_{D'}}(w') = e^{-C^\pm_{w_\pm}(\pi(w),\,\pi(w'))},$$

for all $w \in P^\pm_D(A_{D,D'})$ and $w' = h^\pm(w)$. $\Box$

The skinning measures associated with horoballs are of particular importance in this paper. Let $w \in T^1\widetilde{M}$. We denote the skinning measures on the strong stable and strong unstable leaves $W^{\rm ss}(w)$ and $W^{\rm su}(w)$ of w by

$$\mu^-_{W^{\rm ss}(w)} = \widetilde{\sigma}^-_{HB_+(w)} \qquad \text{and} \qquad \mu^+_{W^{\rm su}(w)} = \widetilde{\sigma}^+_{HB_-(w)}\,.$$

For future use, using the homeomorphisms $v \mapsto v_+$ from $W^{\rm su}(w)$ to $\partial_\infty\widetilde{M} - \{w_-\}$ and $v \mapsto v_-$ from $W^{\rm ss}(w)$ to $\partial_\infty\widetilde{M} - \{w_+\}$, we have

$$d\mu^+_{W^{\rm su}(w)}(v) = e^{C^+_{v_+}(x_0,\,\pi(v))}\,d\mu^+_{x_0}(v_+) \tag{19}$$

and

$$d\mu^-_{W^{\rm ss}(w)}(v) = e^{C^-_{v_-}(x_0,\,\pi(v))}\,d\mu^-_{x_0}(v_-)\,. \tag{20}$$

It follows from (ii) of Proposition 9 that, for all $\gamma \in \Gamma$, we have

$$\gamma_*\mu^-_{W^{\rm ss}(w)} = \mu^-_{W^{\rm ss}(\gamma w)} \quad \text{and} \quad \gamma_*\mu^+_{W^{\rm su}(w)} = \mu^+_{W^{\rm su}(\gamma w)}\,.$$

It follows from (iii) that for all $t \geq 0$ and $w \in T^1\widetilde{M}$, we have for all $v \in W^{\rm ss}(w)$,

$$\frac{d(g^{-t})_*\mu^-_{W^{\rm ss}(w)}}{d\,\mu^-_{W^{\rm ss}(g^{-t}w)}}(g^{-t}v) = e^{-C^-_{v_-}(\pi(v),\,\pi(g^{-t}v))} = e^{C^+_{w_+}(\pi(v),\,\pi(g^{-t}v))}\,, \tag{21}$$

and for all $v \in W^{\rm su}(w)$,

$$\frac{d(g^t)_*\mu^+_{W^{\rm su}(w)}}{d\mu^+_{W^{\rm su}(g^t w)}}(g^t v) = e^{-C^+_{v_+}(\pi(v),\,\pi(g^t v))} = e^{C^-_{w_-}(\pi(v),\,\pi(g^t v))}\;. \tag{22}$$

The homeomorphisms $(s, v') \mapsto v = g^s v'$ from $\mathbb{R} \times W^{\rm ss}(w)$ to $W^{\rm s}(w)$ and from $\mathbb{R} \times W^{\rm su}(w)$ to $W^{\rm u}(w)$ conjugate the action of $\mathbb{R}$ by translation on the first factor to the geodesic flow. Using them, we define, for all $w \in T^1\widetilde{M}$, the measures ν^-_w on $W^{\rm s}(w)$ and ν^+_w on $W^{\rm u}(w)$ by the densities

$$d\nu^-_w(v) = e^{C^+_{w_+}(\pi(w),\,\pi(v'))}\; ds\; d\mu^-_{W^{\rm ss}(w)}(v') \tag{23}$$

and

$$d\nu^+_w(v) = e^{C^-_{w_-}(\pi(w),\,\pi(v'))}\; ds\; d\mu^+_{W^{\rm su}(w)}(v')\,. \tag{24}$$

The importance of these measures will be seen in Proposition 13, where they will be shown to be the conditional measures on the pointed stable/unstable leaves of the Gibbs measure $\widetilde{m}_F$. Note that the measures $\nu^{\pm}_w$ depend in general on w (and not only on the stable and unstable leaves of w as when $\widetilde{F} = 0$). They are invariant under the geodesic flow: for all $t \in \mathbb{R}$, we have

$$(g^t)_*\nu^{\pm}_w = \nu^{\pm}_w\,. \tag{25}$$

For all $\gamma \in \Gamma$, we have $\gamma_*\nu^{\pm}_w = \nu^{\pm}_{\gamma w}$. The supports of ν^-_w and ν^+_w are respectively $\{v \in W^{\rm s}(w) \;:\; v_- \in \Lambda\Gamma\}$ and $\{v \in W^{\rm u}(w) \;:\; v_+ \in \Lambda\Gamma\}$. We give two other properties of the measures $\nu^{\pm}_w$ in the next two lemmas.

Lemma 11 *For all $w \in T^1\widetilde{M}$ and $t \geq 0$, we have*

$$\nu^{\pm}_{g^{\pm t}w} = e^{C^{\pm}_{w_{\pm}}(\pi(w),\,\pi(g^{\pm t}w))}\;\nu^{\pm}_w\,.$$

Note that this result proves that the measures $\nu^{\pm}_{g^{\pm t}w}$ and $\nu^{\pm}_w$ are proportional (and not only absolutely continuous one with respect to the other).

Proof. Let us prove the claim for ν^+_w. Note that if $(s, v') \mapsto v = g^s v'$ is the considered homeomorphism from $\mathbb{R} \times W^{\rm su}(w)$ to $W^{\rm u}(w)$, then $(s, g^t v') \mapsto v = g^s v'$ is the considered homeomorphism from $\mathbb{R} \times W^{\rm su}(g^t w)$ to $W^{\rm u}(g^t w) = W^{\rm u}(w)$. Using the definition of ν^+_w, Equation (22), the cocycle property of C^-, again the definition of ν^+_w, and finally Equation (16), we have, for all $v \in W^{\rm u}(w)$ and $t \geq 0$,

$$\begin{aligned}
d\nu^+_{g^t w}(v) &= e^{C^-_{w_-}(\pi(g^t w),\,\pi(g^t v'))}\; ds\, d\mu^+_{W^{\rm su}(g^t w)}(g^t v') \\
&= e^{C^-_{w_-}(\pi(g^t w),\,\pi(g^t v')) - C^-_{w_-}(\pi(v'),\,\pi(g^t v'))}\; ds\, d\mu^+_{W^{\rm su}(w)}(v') \\
&= e^{C^-_{w_-}(\pi(g^t w),\,\pi(w)) + C^-_{w_-}(\pi(w),\,\pi(v'))}\; ds\, d\mu^+_{W^{\rm su}(w)}(v') \\
&= e^{C^-_{w_-}(\pi(g^t w),\,\pi(w))}\; d\nu^+_w(v) = e^{C^+_{w_+}(\pi(w),\,\pi(g^t w))}\; d\nu^+_w(v)\,,
\end{aligned}$$

which proves the claim. The proof of the other claim is similar, using Equation (21). □

Lemma 12 *For every nonempty proper closed convex subset D' in $\widetilde{M}$, there exists $R_0 > 0$ such that for all $R \geq R_0$ and $\eta > 0$, for every $w \in \partial^1_{\pm}D'$, we have $\nu^{\mp}_w(V^{\pm}_{w,\,\eta,\,R}) > 0$.*

Proof. By [PP5, Lem. 7], there exists $R_0 > 0$ (depending only on D' and on the Patterson densities) such that for all $R \geq R_0$, $w \in \partial^1_+ D'$ and $w' \in \partial^1_- D'$, we have $\mu^-_{W^{\rm ss}(w)}(B^+(w,R)) > 0$ and $\mu^+_{W^{\rm su}(w')}(B^-(w',R)) > 0$. The result hence follows by the definitions of $\nu^\mp_w$ and $V^\pm_{w,\eta,R}$. □

The following disintegration result of the Gibbs measure over the skinning measures of any closed convex subset is a crucial tool for the counting result in Section 4. Recall the definitions (4), (5) of the flow-invariant open sets $\mathscr{U}^\pm_D$ and the fibrations $f^\pm_D : \mathscr{U}^\pm_D \to \partial^1_\pm D$ from Subsection 2.2.

Proposition 13 *Let D be a nonempty proper closed convex subset of $\widetilde{M}$. The restriction to $\mathscr{U}^\pm_D$ of the Gibbs measure $\widetilde{m}_F$ disintegrates by the fibration $f^\pm_D : \mathscr{U}^\pm_D \to \partial^1_\pm D$, over the skinning measure $\widetilde{\sigma}^\pm_D$ of D, with conditional measure $\nu^\mp_w$ on the fiber $(f^\pm_D)^{-1}(w)$ of $w \in \partial^1_\pm D$:*

$$d(\widetilde{m}_F)|_{\mathscr{U}^\pm_D}(v) = \int_{w\in\partial^1_\pm D} d\nu^\mp_w(v)\, d\widetilde{\sigma}^\pm_D(w)\ .$$

Proof. To prove the claim for the fibration f^+_D, let $\phi \in \mathscr{C}_c(\mathscr{U}^+_D)$. Using in the various steps below:

- Hopf's parametrisation with time parameter t and the definition of $\widetilde{m}_F$,
- the positive endpoint homeomorphism $w \mapsto w_+$ from $\partial^1_+ D$ to $\partial_\infty \widetilde{M} - \partial_\infty D$, and the negative endpoint homeomorphism $v' \mapsto v'_-$ from $W^{\rm ss}(w)$ to $\partial_\infty \widetilde{M} - \{w_+\}$, with $s \in \mathbb{R}$ the real parameter such that $v' = g^{-s} v \in W^{\rm ss}(w)$, noting that $t - s$ depends only on $v_+ = w_+$ and $v_- = v'_-$,
- the definitions of the measures $\mu^-_{W^{\rm ss}(w)}$ and $\widetilde{\sigma}^+_D$ and the cocycle properties of $C^\pm$,
- Equation (16) and the cocycle properties of C^+,

we have

$$\begin{aligned}
&\int_{\mathscr{U}^+_D} \phi(v)\, d\widetilde{m}_F(v)\\
&= \int_{v_+\in\partial_\infty\widetilde{M}-\partial_\infty D}\int_{v_-\in\partial_\infty\widetilde{M}-\{v_+\}}\int_{t\in\mathbb{R}} \phi(v)\, e^{C^-_{v_-}(x_0,\,\pi(v))+C^+_{v_+}(x_0,\,\pi(v))}\, dt\, d\mu^-_{x_0}(v_-)\, d\mu^+_{x_0}(v_+)\\
&= \int_{w\in\partial^1_+ D}\int_{v'\in W^{\rm ss}(w)}\int_{s\in\mathbb{R}} \phi(g^s v')\, e^{C^-_{v'_-}(x_0,\,\pi(g^s v'))+C^+_{w_+}(x_0,\,\pi(g^s v'))}\, ds\, d\mu^-_{x_0}(v'_-)\, d\mu^+_{x_0}(w_+)\\
&= \int_{w\in\partial^1_+ D}\int_{v'\in W^{\rm ss}(w)}\int_{s\in\mathbb{R}} \phi(g^s v')\, e^{C^-_{v'_-}(\pi(v'),\,\pi(g^s v'))+C^+_{w_+}(\pi(w),\,\pi(g^s v'))}\, ds\, d\mu^-_{W^{\rm ss}(w)}(v')\, d\widetilde{\sigma}^+_D(w)\\
&= \int_{w\in\partial^1_+ D}\int_{v'\in W^{\rm ss}(w)}\int_{s\in\mathbb{R}} \phi(g^s v')\, e^{C^+_{w_+}(\pi(w),\,\pi(v'))}\, ds\, d\mu^-_{W^{\rm ss}(w)}(v')\, d\widetilde{\sigma}^+_D(w)\,,
\end{aligned}$$

which implies the claim. The proof for the fibration f^-_D is similar. □

In particular, for every $u \in T^1\widetilde{M}$, applying the above proposition to $D = HB_-(u)$ for which $\partial^1_+ D = W^{\rm su}(u)$ and

$$\mathscr{U}^+_D = T^1\widetilde{M} - W^{\rm s}(\iota u) = \bigcup_{w\in W^{\rm su}(u)} W^{\rm s}(w)\ ,$$

the restriction to $T^1\widetilde{M} - W^{\rm s}(\iota u)$ of the Gibbs measure $\widetilde{m}_F$ disintegrates over the strong unstable measure $\mu^+_{W^{\rm su}(u)}$, with conditional measure on the fiber $W^{\rm s}(w)$ of $w \in W^{\rm su}(u)$

the measure ν_w^-: for every $\phi \in \mathscr{C}_c(T^1\widetilde{M} - W^{\rm s}(\iota u))$, we have

$$\int_{T^1\widetilde{M}-W^{\rm s}(\iota u)} \phi(v)\, d\widetilde{m}_F(v)$$
$$= \int_{w\in W^{\rm su}(u)} \int_{v'\in W^{\rm ss}(w)} \int_{s\in\mathbb{R}} \phi(g^s v')\, e^{C^+_{w_+}(\pi(w),\,\pi(v'))}\, ds\, d\mu^-_{W^{\rm ss}(w)}(v')\, d\mu^+_{W^{\rm su}(u)}(w)\,. \quad (26)$$

Note that if the Patterson densities are atomless (for instance if the Gibbs measure m_F is finite, see [PPS, §5.3]), then the stable and unstable leaves have measure zero for the associated Gibbs measure.

3.3 Equivariant families and multiplicities

Equivariant families. Let I be an index set endowed with a left action $(\gamma, i) \mapsto \gamma i$ of Γ. A family $\mathscr{D} = (D_i)_{i\in I}$ of subsets of $\widetilde{M}$ or $T^1\widetilde{M}$ indexed by I is Γ-*equivariant* if $\gamma D_i = D_{\gamma i}$ for all $\gamma \in \Gamma$ and $i \in I$. We equip the index set I with the Γ-equivariant equivalence relation $\sim$ (to shorten the notation, we do not indicate that $\sim$ depends on $\mathscr{D}$) defined by setting $i \sim j$ if and only if there exists $\gamma \in \operatorname{Stab}_\Gamma D_i$ such that $j = \gamma i$ (or equivalently if $D_j = D_i$ and $j = \gamma i$ for some $\gamma \in \Gamma$). Note that Γ acts on the left on the set of equivalence classes $I/_\sim$.

An example of such a family is given by fixing a subset D of $\widetilde{M}$ or $T^1\widetilde{M}$, by setting $I = \Gamma$ with the left action by translations $(\gamma, i) \mapsto \gamma i$, and by setting $D_i = iD$ for every $i \in \Gamma$. In this case, we have $i \sim j$ if and only if $i^{-1}j$ belongs to the stabiliser Γ_D of D in Γ, and $I/_\sim\ = \Gamma/\Gamma_D$. More general examples include Γ-orbits of (usually finite) collections of subsets of $\widetilde{M}$ or $T^1\widetilde{M}$ with (usually finite) multiplicities.

A Γ-equivariant family $(A_i)_{i\in I}$ of closed subsets of $\widetilde{M}$ or $T^1\widetilde{M}$ is said to be *locally finite* if for every compact subset K in $\widetilde{M}$ or $T^1\widetilde{M}$, the quotient set $\{i \in I : A_i \cap K \neq \emptyset\}/_\sim$ is finite. In particular, the union of the images of the sets A_i by the map $\widetilde{M} \to M$ or $T^1\widetilde{M} \to T^1M$ is closed. When $\Gamma\backslash I$ is finite, $(A_i)_{i\in I}$ is locally finite if and only if, for all $i \in I$, the canonical map from $\Gamma_{A_i}\backslash A_i$ to M or T^1M is proper, where Γ_{A_i} is the stabiliser of A_i in Γ.

Skinning measures of equivariant families. Let $\mathscr{D} = (D_i)_{i\in I}$ be a locally finite Γ-equivariant family of nonempty proper closed convex subsets of $\widetilde{M}$. Let $\Omega = (\Omega_i)_{i\in I}$ be a Γ-equivariant family of subsets of $T^1\widetilde{M}$, where Ω_i is a measurable subset of $\partial^1_\pm D_i$ for all $i \in I$ (the sign $\pm$ being constant). For instance $\partial^1_\pm \mathscr{D} = (\partial^1_\pm D_i)_{i\in I}$ is such a family.

Then

$$\widetilde{\sigma}^\pm_\Omega = \sum_{i\in I/_\sim} \widetilde{\sigma}^\pm_{D_i}|_{\Omega_i}\ ,$$

is a well-defined (independent of the choice of representatives in $I/_\sim$), Γ-invariant by Proposition 9 (ii), locally finite measure on $T^1\widetilde{M}$ whose support is contained in $\bigcup_{i\in I/_\sim} \Omega_i$. Hence by Subsection 2.4, the measure $\widetilde{\sigma}^\pm_\Omega$ induces a locally finite measure on T^1M, denoted by $\sigma^\pm_\Omega$.

In the important special cases when $\Omega = \partial^1_+\mathscr{D}$ and $\Omega = \partial^1_-\mathscr{D}$, the measures $\widetilde{\sigma}^+_\Omega$ and $\widetilde{\sigma}^-_\Omega$ are denoted respectively by

$$\widetilde{\sigma}^+_{\mathscr{D}} = \sum_{i\in I/_\sim} \widetilde{\sigma}^+_{D_i} \quad \text{and} \quad \widetilde{\sigma}^-_{\mathscr{D}} = \sum_{i\in I/_\sim} \widetilde{\sigma}^-_{D_i}\,,$$

and are called the *inner and outer skinning measures* of $\mathscr{D}$ on $T^1\widetilde{M}$. Their induced measures on $T^1M = \Gamma\backslash T^1\widetilde{M}$ are called the *inner and outer skinning measures* of $\mathscr{D}$ on T^1M. For instance, if $\widetilde{x}_0$ has stabiliser $\Gamma_{\widetilde{x}_0}$ and maps to $x_0 \in M$, if $\mathscr{D} = (\gamma\widetilde{x}_0)_{\gamma\in\Gamma}$, if $F = 0$, if M has dimension n, constant curvature and finite volume, then, normalising the Patterson density so that $\|\mu_x\| = \operatorname{Vol}(\mathbb{S}^{n-1})$, we have $\sigma^{\pm}_{\mathscr{D}} = \operatorname{Vol}_{T^1_{\widetilde{x}_0}M}$ and $\|\sigma^{\pm}_{\mathscr{D}}\| = \frac{\operatorname{Vol}(\mathbb{S}^{n-1})}{\operatorname{Card}(\Gamma_{\widetilde{x}_0})}$. See Section 6 for other examples.

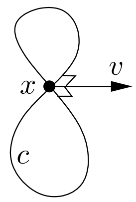
Multiplicity of unit tangent vectors. Given $v \in T^1M$, we define the natural *multiplicity* of v with respect to a family Ω as above by

$$m_\Omega(v) = \frac{\operatorname{Card}\{i \in I/_\sim \;:\; \widetilde{v} \in \Omega_i\}}{\operatorname{Card}(\operatorname{Stab}_\Gamma \widetilde{v})}\;,$$

for any preimage $\widetilde{v}$ of v in $T^1\widetilde{M}$. The numerator and the denominator are finite, by the local finiteness of the family $\mathscr{D}$ and the discreteness of Γ, and they depend only on the orbit of $\widetilde{v}$ under Γ.

This multiplicity is indeed natural. Concerning the denominator, in any counting problem of objects possibly having symmetries, the appropriate counting function consists in taking as the multiplicity of an object the inverse of the cardinality of its symmetry group. The numerator is here in order to take into account the multiplicities of the images of the elements of $\mathscr{D}$ in T^1M. Note that if Γ is torsion-free, if $\Omega = \partial^1_\pm\mathscr{D}$, if for every $i \in I$ the quotient $\Gamma_{D_i}\backslash D_i$ of D_i by its stabiliser Γ_{D_i} maps injectively in $M = \Gamma\backslash\widetilde{M}$ (by the map induced by the inclusion of D_i in M), and if for every $i, j \in I$ such that $j \notin \Gamma i$, the intersection $D_i \cap D_j$ is empty, then the nonzero multiplicities $m_\Omega(v)$ are all equal to 1.

Here is a simple example of a multiplicity different from 0 or 1. Assume Γ is torsion-free. Let c be a closed geodesic in M, let $\widetilde{c}$ be a geodesic line in $\widetilde{M}$ mapping to c in M, let $\mathscr{D} = (\gamma\,\widetilde{c})_{\gamma\in\Gamma}$, let x be a double point of c, let $v \in T^1_xM$ be orthogonal to the two tangent lines to c at x (this requires the dimension of $\widetilde{M}$ to be at least 3). Then $m_{\partial^1_\pm\mathscr{D}}(v) = 2$.



Weighted number of geodesic paths with given initial/terminal vectors. Given $t > 0$ and two unit tangent vectors $v, w \in T^1M$, we define the number $n_t(v, w)$ of locally geodesic paths having v and w as initial and terminal tangent vectors respectively, weighted by the potential F, with length at most t, by

$$n_t(v, w) = \sum_\alpha \operatorname{Card}(\Gamma_\alpha)\; e^{\int_\alpha F}\;,$$

where the sum ranges over the locally geodesic paths $\alpha : [0, s] \to M$ such that $\dot\alpha(0) = v$, $\dot\alpha(s) = w$ and $s \in\,]0, t]$, and Γ_α is the stabiliser in Γ of any geodesic path $\widetilde{\alpha}$ in $\widetilde{M}$ mapping to α by the quotient map $\widetilde{M} \to M$. If $F = 0$ and Γ is torsion free, then $n_t(v, w)$ is precisely the number of locally geodesic paths having v and w as initial and terminal tangent vectors respectively, with length at most t.

4 Counting and equidistribution of common perpendiculars

Let $\widetilde{M}$, x_0, Γ and M be as in the beginning of Section 2. Let $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ be a bounded Γ-invariant Hölder-continuous function, and let $F : T^1M = \Gamma\backslash T^1\widetilde{M} \to \mathbb{R}$ be its quotient map. Let $\mathscr{D}^- = (D_i^-)_{i\in I^-}$ and $\mathscr{D}^+ = (D_j^+)_{j\in I^+}$ be locally finite Γ-equivariant families of nonempty proper closed convex subsets of $\widetilde{M}$.

The main counting function of this paper is defined as follows. Let $\Omega^- = (\Omega_i^-)_{i\in I^-}$ and $\Omega^+ = (\Omega_j^+)_{j\in I^+}$ be Γ-equivariant families of subsets of $T^1\widetilde{M}$, where $\Omega_k^\mp$ is a measurable subset of $\partial^1_\pm D_k^\mp$ for all $k \in I^\mp$. We will denote by $\mathscr{N}_{\Omega^-,\,\Omega^+,\,F}(t)$ the number of common perpendiculars whose initial vectors belong to the images in M of the elements of Ω^- and terminal vectors to the images in M of the elements of Ω^+, counted with multiplicities and weighted by the potential F, that is:

$$\mathscr{N}_{\Omega^-,\,\Omega^+,\,F}(t) = \sum_{v,\,w\in T^1M} m_{\Omega^-}(v)\; m_{\Omega^+}(w)\; n_t(v,w)\;.$$

When $\Omega^\pm = \partial^1_\mp \mathscr{D}^\pm$, we denote $\mathscr{N}_{\Omega^-,\,\Omega^+,\,F}$ by $\mathscr{N}_{\mathscr{D}^-,\,\mathscr{D}^+,\,F}$. If Γ has no torsion, if $\mathscr{D}^\pm = (\gamma\widetilde{D}^\pm)_{\gamma\in\Gamma}$ where $\widetilde{D}^\pm$ is a nonempty proper closed convex subset of $\widetilde{M}$ (such that the family $\mathscr{D}^\pm$ is locally finite), and if $D^\pm$ is the image of $\widetilde{D}^\pm$ by the covering map $\widetilde{M} \to M = \Gamma\backslash\widetilde{M}$ (which is a nonempty proper properly immersed closed convex subset of M), then $\mathscr{N}_{\mathscr{D}^-,\,\mathscr{D}^+,\,F}$ is the counting function $\mathscr{N}_{D^-,\,D^+,\,F}$ given in the introduction.

In order to give a precise asymptotic to these counting functions (see Corollary 20), we will start by proving a result of independent interest on the equidistribution of the initial/terminal vectors of the common perpendiculars in the outer/inner normal bundles of the convex sets, first in $T^1\widetilde{M}$ (see Subsection 4.1), then in T^1M (see Subsection 4.3).

Let us continue fixing the notation used throughout this Section 4. For every (i,j) in $I^-\times I^+$ such that D_i^- and D_j^+ have a common perpendicular (that is, whose closures $\overline{D_i^-}$ and $\overline{D_j^+}$ in $M\cup\widetilde{\partial}_\infty\widetilde{M}$ have empty intersection), we denote by $\alpha_{i,\,j}$ this common perpendicular, by $\ell(\alpha_{i,\,j})$ its length, by $v^-_{i,\,j} \in \partial^1_+ D_i^-$ its initial tangent vector and by $v^+_{i,\,j} \in \partial^1_- D_i^+$ its terminal tangent vector. Note that if $i' \sim i$, $j' \sim j$ and $\gamma\in\Gamma$, then

$$\gamma\,\alpha_{i',\,j'} = \alpha_{\gamma i,\,\gamma j},\quad \ell(\alpha_{i',\,j'}) = \ell(\alpha_{\gamma i,\,\gamma j}) \quad\text{and}\quad \gamma\, v^\pm_{i',\,j'} = v^\pm_{\gamma i,\,\gamma j}\;. \tag{27}$$

When Γ is torsion free, we have, for the diagonal action of Γ on $I^-\times I^+$,

$$\mathscr{N}_{\mathscr{D}^-,\,\mathscr{D}^+,\,F}(t) = \sum_{(i,j)\in\Gamma\backslash((I^-/\sim)\times(I^+/\sim))\,:\,\overline{D_i^-}\cap\overline{D_j^+}=\emptyset,\;\ell(\alpha_{i,\,j})\le t} e^{\int_{\alpha_{i,\,j}} F}\;.$$

We denote by Δ_x the unit Dirac mass at a point x in any measurable space.

4.1 Equidistribution of endvectors of common perpendiculars in $T^1\widetilde{M}$

The following core theorem shows that the ordered pairs of initial and terminal tangent vectors of common perpendiculars of two locally finite equivariant families of convex sets in the universal cover $\widetilde{M}$ equidistribute towards the product of the skinning measures of the families.

Theorem 14 *Let $\widetilde{M}$ be a complete simply connected Riemannian manifold with pinched sectional curvature at most -1. Let Γ be a nonelementary discrete group of isometries* *of $\widetilde{M}$. Let $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ be a bounded Γ-invariant Hölder-continuous function with positive critical exponent δ_F. Let $\mathscr{D}^- = (D_i^-)_{i\in I^-}$ and $\mathscr{D}^+ = (D_j^+)_{j\in I^+}$ be locally finite Γ-equivariant families of nonempty proper closed convex subsets of $\widetilde{M}$. Assume that the Gibbs measure m_F is finite and mixing for the geodesic flow. Then*

$$\lim_{t\to+\infty} \delta_F\, \|m_F\|\, e^{-\delta_F\, t} \sum_{\substack{i\in I^-/_\sim,\ j\in I^+/_\sim,\ \gamma\in\Gamma \\ \overline{D_i^-}\cap\gamma\overline{D_j^+}=\emptyset,\ \ell(\alpha_{i,\,\gamma j})\le t}} e^{\int_{\alpha_{i,\,\gamma j}} \widetilde{F}}\ \Delta_{v^-_{i,\,\gamma j}} \otimes \Delta_{v^+_{\gamma^{-1} i,\, j}} \;=\; \widetilde{\sigma}^+_{\mathscr{D}^-} \otimes \widetilde{\sigma}^-_{\mathscr{D}^+}$$

for the weak-star convergence of measures on the locally compact space $T^1\widetilde{M} \times T^1\widetilde{M}$.

In the special case of $\mathscr{D}^- = (\gamma x)_{\gamma\in\Gamma}$ and $\mathscr{D}^+ = (\gamma y)_{\gamma\in\Gamma}$ for some $x, y \in \widetilde{M}$, this statement may be proved to be a consequence of the proof of [Rob, Theo. 4.1.1] if $F = 0$, and of [PPS, Theo. 9.1] for general F. Although we use the same technical initial trick as in the proof of [Rob, Theo. 4.1.1], we will immediately after that use a functional approach, better suited to obtain error terms in Section 5. We will give a reformulation in $T^1M \times T^1M$ of this result after its proof, and some applications to particular geometric situations in Section 6.

Proof. We first give a scheme of the proof (see [PP6, §8] for a more elaborate one). The crucial observation is that two nonempty proper closed convex subsets D^- and D^+ of $\widetilde{M}$ have a common perpendicular of length a given $t > 0$ if and only if $g^{\frac{t}{2}}\partial^1_+ D^-$ and $g^{-\frac{t}{2}}\partial^1_- D^+$ intersect. After some reduction of the statement, we will introduce, for η small enough, two test functions $\phi^\mp_\eta$ vanishing outside a small dynamical neighbourhood of $\partial^1_\pm D^\mp$, so that the support of the product function $\phi^-_\eta \circ g^{-\frac{t}{2}}\ \phi^+_\eta \circ g^{\frac{t}{2}}$ detects the intersection of $g^{\frac{t}{2}}\partial^1_+ D^-$ and $g^{-\frac{t}{2}}\partial^1_- D^+$ (using Subsection 2.3). We will then use the mixing property of the geodesic flow for the Gibbs measure to obtain the equidistribution result of Theorem 14.

The estimation of the small terms occuring in the following steps 2, 4 and 5 is much more precise than what is needed to prove Theorem 14. These estimates will be useful to give a speed of equidistribution of the initial and terminal vectors, and an error term in the asymptotic of the counting function $\mathscr{N}_{\mathscr{D}^-,\,\mathscr{D}^+,\,F}(t)$ in Theorem 28.

To shorten the notation, we will fix for the rest of the paper the convention that the sums as in the statement of Theorem 14 are for (i, j, γ) such that D_i^- and γD_j^+ have a common perpendicular (a necessary condition in order for $\alpha_{i,\,\gamma j}$ to exist). The fact that this sum is independent of the choice of representatives of i in $I^-/_\sim$ and j in $I^+/_\sim$ follows from Equation (27).

Step 1: Reduction of the statement. By additivity, by the local finiteness of the families $\mathscr{D}^\pm$, and by the definition of $\widetilde{\sigma}^\pm_{\mathscr{D}^\mp} = \sum_{k\in I^\mp/_\sim} \widetilde{\sigma}^\pm_{D_k^\mp}$, we only have to prove, for all fixed $i \in I^-$ and $j \in I^+$, that

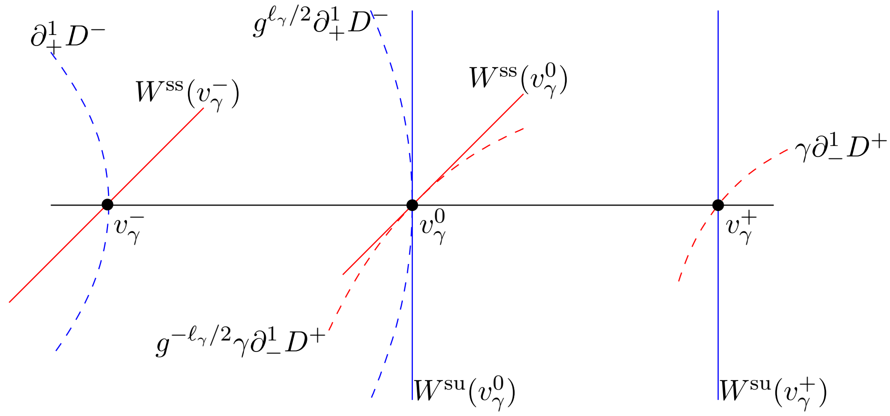
$$\lim_{t\to+\infty} \delta_F\, \|m_F\|\, e^{-\delta_F\, t} \sum_{\gamma\in\Gamma\,:\, 0<\ell(\alpha_{i,\,\gamma j})\le t} e^{\int_{\alpha_{i,\,\gamma j}} \widetilde{F}}\ \Delta_{v^-_{i,\,\gamma j}} \otimes \Delta_{v^+_{\gamma^{-1} i,\, j}} \;=\; \widetilde{\sigma}^+_{D_i^-} \otimes \widetilde{\sigma}^-_{D_j^+} \qquad (28)$$

for the weak-star convergence of measures on $T^1\widetilde{M} \times T^1\widetilde{M}$.

Let Ω^- be a Borel subset of $\partial^1_+ D^-_i$ and let Ω^+ be a Borel subset of $\partial^1_- D^+_j$. To simplify the notation, let

$$D^- = D^-_i\,,\quad D^+ = D^+_j\,,\quad \alpha_\gamma = \alpha_{i,\gamma j}\,,\quad \ell_\gamma = \ell(\alpha_\gamma)\,,\quad v^\pm_\gamma = v^\pm_{i,\gamma j}\ \text{ and }\ \widetilde{\sigma}^\pm = \widetilde{\sigma}^\pm_{D^\mp}\,. \tag{29}$$

Let v^0_γ be the tangent vector at the midpoint of α_γ (see the picture below, sitting in $T^1\widetilde{M}$).



Assume that Ω^- and Ω^+ have positive finite skinning measures:

$$\widetilde{\sigma}^+(\Omega^-), \widetilde{\sigma}^-(\Omega^+) \in\,]0,+\infty[\,,$$

and that their boundaries in $\partial^1_+ D^-$ and $\partial^1_- D^+$ have zero skinning measures:

$$\widetilde{\sigma}^+(\partial\Omega^-) = \widetilde{\sigma}^-(\partial\Omega^+) = 0\,.$$

Let

$$I_{\Omega^-,\,\Omega^+}(t) = \delta_F\,\|m_F\|\,e^{-\delta_F\,t}\sum_{\substack{\gamma\in\Gamma\,:\,0<\ell_\gamma\le t\\ v^-_\gamma\in\Omega^-,\,v^+_\gamma\in\gamma\Omega^+}} e^{\int_{\alpha_\gamma}\widetilde{F}}\,.$$

Let us prove the stronger statement that, for every such $\Omega^\pm$, we have

$$\lim_{t\to+\infty} I_{\Omega^-,\,\Omega^+}(t) \;=\; \widetilde{\sigma}^+(\Omega^-)\,\widetilde{\sigma}^-(\Omega^+)\,. \tag{30}$$

Step 2: Construction of the bump functions. In order to prove Equation (30), we start by defining the test functions $\phi^\pm_\eta$ mentioned in the scheme of the proof.

We consider the measurable families of measures $(\nu^+_w)_{w\in T^1\widetilde{M}}$ and $(\nu^-_w)_{w\in T^1\widetilde{M}}$ defined using Equation (24) and Equation (23) respectively. We fix from now on $R>0$ such that $\nu^\pm_w(V^\mp_{w,\,\eta,\,R})>0$ for all $\eta>0$ and all $w\in\partial^1_\mp D^\pm$, hence for all $w\in\gamma\,\partial^1_\mp D^\pm$ for all $\gamma\in\Gamma$. Such an R exists by Lemma 12.

For all $\eta,\eta'>0$, let $h^\pm_{\eta,\,\eta'}: T^1\widetilde{M}\to[0,+\infty[$ be the measurable maps defined by

$$h^\pm_{\eta,\,\eta'}(w) = \frac{1}{\nu^\pm_w(V^\mp_{w,\,\eta,\,\eta'})}$$

if $\nu^\pm_w(V^\mp_{w,\,\eta,\,\eta'})>0$ (which is for instance satisfied if $w_\pm\in\Lambda\Gamma$), and $h^\pm_{\eta,\,\eta'}(w)=0$ otherwise.

We give a few properties of this technical map. Since $\gamma_*\nu^\pm_w = \nu^\pm_{\gamma w}$ and $\gamma V^\mp_{w,\eta,\eta'} = V^\mp_{\gamma w,\eta,\eta'}$ for all $\gamma \in \Gamma$, the functions $h^\pm_{\eta,\eta'}$ are Γ-invariant.

By Lemma 11, by Equation (9) and by the invariance of $\nu^\pm_w$ under the geodesic flow (see Equation (25)), for all $t \geq 0$ and $w \in T^1\widetilde{M}$, we have

$$h^\mp_{\eta,\eta'}(g^{\mp t}w) = \frac{1}{\nu^\mp_{g^{\mp t}w}(V^\pm_{g^{\mp t}w,\eta,\eta'})} = \frac{1}{e^{C^\mp_{w_\mp}(\pi(w),\pi(g^{\mp t}w))}\nu^\mp_w(g^{\mp t}V^\pm_{w,\eta,e^{-t}\eta'})}$$
$$= e^{C^\mp_{w_\mp}(\pi(g^{\mp t}w),\pi(w))}\ h^\mp_{\eta,e^{-t}\eta'}(w)\,.$$

By Equation (16), we hence have

$$h^-_{\eta,R}(g^{-t}w) = e^{\int_{\pi(g^{-t}w)}^{\pi(w)}(\widetilde{F}-\delta_F)}\ h^-_{\eta,e^{-t}R}(w)\,, \tag{31}$$

$$h^+_{\eta,R}(g^{t}w) = e^{\int_{\pi(w)}^{\pi(g^{t}w)}(\widetilde{F}-\delta_F)}\ h^+_{\eta,e^{-t}R}(w)\,. \tag{32}$$

The last property of $h^\pm_{\eta,\eta'}$ that we will need is its behaviour as η' is very small. Let $\eta' \in\,]0,1]$. For all $w \in T^1\widetilde{M}$ and all $v' \in B^+(w,\eta')$, by Lemma 6, we have

$$d(\pi(w),\pi(v')) \leq d_{W^{\rm ss}(w)}(w,v') < \eta' \leq 1\,.$$

Hence by taking $c_1 > 0$ and $c_2 \in\,]0,1]$ such that Equation (17) is satisfied, by taking $c_3 = c_1 + \|F\|_\infty$, we have by Equation (17)

$$|\ C^+_{w_-}(\pi(w),\pi(v'))\ | \leq c_3\ (\eta')^{c_2}\,.$$

Using the above, the analogous estimate for $C^-_{w_+}$ and the defining equations (23) and (24) of $\nu^\mp_w$, for all $s \in \mathbb{R}$, $w \in T^1\widetilde{M}$ and $v' \in B^\pm(w,\eta')$, we have

$$e^{-c_3\,(\eta')^{c_2}}\ ds\,d\mu^-_{W^{\rm ss}(w)}(v') \leq d\nu^-_w(g^s v') \leq e^{c_3\,(\eta')^{c_2}}\ ds\,d\mu^-_{W^{\rm ss}(w)}(v')\,,$$

and similarly

$$e^{-c_3\,(\eta')^{c_2}}\ ds\,d\mu^+_{W^{\rm su}(w)}(v') \leq d\nu^+_w(g^s v') \leq e^{c_3\,(\eta')^{c_2}}\ ds\,d\mu^+_{W^{\rm su}(w)}(v')\,.$$

It follows that for all $\eta,\eta' \in\,]0,1]$ and $w \in T^1\widetilde{M}$ such that $w_-, w_+ \in \Lambda\Gamma$, we have

$$\frac{e^{-c_3\,(\eta')^{c_2}}}{2\eta\ \mu^-_{W^{\rm ss}(w)}(B^+(w,\eta'))} \leq h^-_{\eta,\eta'}(w) \leq \frac{e^{c_3\,(\eta')^{c_2}}}{2\eta\ \mu^-_{W^{\rm ss}(w)}(B^+(w,\eta'))}\,, \tag{33}$$

$$\frac{e^{-c_3\,(\eta')^{c_2}}}{2\eta\ \mu^+_{W^{\rm su}(w)}(B^-(w,\eta'))} \leq h^+_{\eta,\eta'}(w) \leq \frac{e^{c_3\,(\eta')^{c_2}}}{2\eta\ \mu^+_{W^{\rm su}(w)}(B^-(w,\eta'))}\,. \tag{34}$$

Let us denote by $\mathbb{1}_A$ the characteristic function of a subset A. We finally define the test functions $\phi^\mp_\eta = \phi^\mp_{\eta,R,\Omega^\pm} : T^1\widetilde{M} \to [0,+\infty[$ by

$$\phi^\mp_\eta = \ h^\mp_{\eta,R}\circ f^\pm_{D^\mp}\ \ \mathbb{1}_{\mathcal{V}^\pm_{\eta,R}(\Omega^\mp)}\,, \tag{35}$$

where $\mathscr{V}^{\pm}_{\eta,R}(\Omega^{\mp})$ and $f^{\pm}_{D^{\mp}}$ are as in Subsection 2.2. Note that if $v \in \mathscr{V}^{\pm}_{\eta,R}(\Omega^{\mp})$, then $v_{\pm} \notin \partial_\infty D^{\mp}$ by convexity, that is, v belongs to the domain of definition $\mathscr{U}^{\pm}_{D^{\mp}}$ of $f^{\pm}_{D^{\mp}}$; hence $\phi^{\mp}_{\eta}(v) = h^{\mp}_{\eta,R} \circ f^{\pm}_{D^{\mp}}(v)$ is well defined. By convention, $\phi^{\mp}_{\eta}(v) = 0$ if $v \notin \mathscr{V}^{\pm}_{\eta,R}(\Omega^{\mp})$. For all $v \in T^1\widetilde{M}$ and $t \geq 0$, we have, by Equations (31) and (10),

$$\begin{aligned}\phi^{-}_{\eta,R,\Omega^+}(g^{-t}v) &= h^{-}_{\eta,R} \circ g^{-t} \circ g^{t} \circ f^{+}_{D^-}(v)\ \mathbb{1}_{g^t\mathscr{V}^{+}_{\eta,R}(\Omega^-)}(v)\\ &= e^{\int_{\pi(g^{-t}f^{+}_{\mathscr{N}_tD^-}(v))}^{\pi(f^{+}_{\mathscr{N}_tD^-}(v))}(\widetilde{F}-\delta_F)}\ \phi^{-}_{\eta,e^{-t}R,g^t\Omega^+}(v)\,. \end{aligned}\tag{36}$$

Lemma 15 *For every $\eta > 0$, the functions $\phi^{\mp}_{\eta}$ are measurable, nonnegative and satisfy*

$$\int_{T^1\widetilde{M}} \phi^{\mp}_{\eta}\ d\widetilde{m}_F = \widetilde{\sigma}^{\pm}(\Omega^{\mp})\,.$$

Proof. The proof is similar to that of [PP5, Prop. 18]. By the disintegration result of Proposition 13, by the last two lines of Subsection 2.2, by the definition of $h^{\mp}_{\eta,R}$ and by the choice of R, we have

$$\begin{aligned}\int_{T^1\widetilde{M}} \phi^{\mp}_{\eta}\ d\widetilde{m}_F &= \int_{v\in\mathscr{V}^{\pm}_{\eta,R}(\Omega^{\mp})} h^{\mp}_{\eta,R} \circ f^{\pm}_{D^{\mp}}(v)\ d\widetilde{m}_F(v)\\ &= \int_{w\in\Omega^{\mp}} h^{\mp}_{\eta,R}(w) \int_{v\in V^{\pm}_{w,\eta,R}} d\nu^{\mp}_{w}(v)\ d\widetilde{\sigma}^{\pm}_{D^{\mp}}(w) = \widetilde{\sigma}^{\pm}(\Omega^{\mp})\,. \quad \Box\end{aligned}$$

Now, the heart of the proof is to give two pairs of upper and lower bounds, as $T \geq 0$ is big enough and $\eta \in\,]0,1]$ is small enough, of the quantity

$$i_\eta(T) = \int_0^T e^{\delta_F\, t} \sum_{\gamma\in\Gamma} \int_{T^1\widetilde{M}} (\phi^{-}_{\eta} \circ g^{-t/2})\ (\phi^{+}_{\eta} \circ g^{t/2} \circ \gamma^{-1})\ d\widetilde{m}_F\ dt\,. \tag{37}$$

Step 3: First upper and lower bounds. For all $t \geq 0$, let

$$a_\eta(t) = \sum_{\gamma\in\Gamma} \int_{v\in T^1\widetilde{M}} \phi^{-}_{\eta}(g^{-t/2}v)\ \phi^{+}_{\eta}(g^{t/2}\gamma^{-1}v)\ d\widetilde{m}_F(v)\,.$$

Note that by Lemma 15 just above, we have that $\int_{T^1\widetilde{M}} \phi^{\mp}_{\eta}\ d\widetilde{m}_F = \widetilde{\sigma}^{\pm}(\Omega^{\mp})$ is finite and positive. By passing to the universal cover the mixing property of the geodesic flow on T^1M for the Gibbs measure m_F, for every $\epsilon > 0$, there hence exists $T_\epsilon \geq 0$ such that for all $t \geq T_\epsilon$, we have

$$\frac{e^{-\epsilon}}{\|m_F\|} \int_{T^1\widetilde{M}} \phi^{-}_{\eta}\ d\widetilde{m}_F \int_{T^1\widetilde{M}} \phi^{+}_{\eta}\ d\widetilde{m}_F \leq a_\eta(t) \leq \frac{e^{\epsilon}}{\|m_F\|} \int_{T^1\widetilde{M}} \phi^{-}_{\eta}\ d\widetilde{m}_F \int_{T^1\widetilde{M}} \phi^{+}_{\eta}\ d\widetilde{m}_F\,.$$

Hence for every $\epsilon > 0$, there exists $c_\epsilon > 0$ such that for all $T \geq 0$ and $\eta \in\,]0,1]$, we have

$$i_\eta(T) \leq e^{\epsilon}\ \frac{e^{\delta_F\, T}}{\delta_F\, \|m_F\|}\ \widetilde{\sigma}^{+}(\Omega^-)\ \widetilde{\sigma}^{-}(\Omega^+) + c_\epsilon\,, \tag{38}$$

and similarly

$$i_\eta(T) \geq e^{-\epsilon}\, \frac{e^{\delta_F\, T}}{\delta_F\, \|m_F\|}\, \widetilde{\sigma}^+(\Omega^-)\, \widetilde{\sigma}^-(\Omega^+) - c_\epsilon\,. \tag{39}$$

Step 4: Second upper and lower bounds. Let $T \geq 0$ and $\eta \in\,]0,1]$. By Fubini's theorem for nonnegative measurable maps and the definition of the test functions $\phi_\eta^\pm$, the quantity $i_\eta(T)$ is equal to

$$\sum_{\gamma\in\Gamma} \int_0^T e^{\delta_F\, t} \int_{T^1\widetilde{M}} h^-_{\eta,\,R} \circ f^+_{D^-}(g^{-t/2}v)\; h^+_{\eta,\,R} \circ f^-_{D^+}(\gamma^{-1}g^{t/2}v)$$
$$\mathbb{1}_{\mathscr{V}^+_{\eta,\,R}(\Omega^-)}(g^{-t/2}v)\; \mathbb{1}_{\mathscr{V}^-_{\eta,\,R}(\Omega^+)}(\gamma^{-1}g^{t/2}v)\; d\widetilde{m}_F\; dt\,. \tag{40}$$

We start the computations by rewriting the product term involving the technical maps $h^\pm_{\eta,\,R}$. For all $\gamma \in \Gamma$ and $v \in \mathscr{U}^+_{D^-} \cap \mathscr{U}^-_{\gamma D^+}$, define (using Equation (6))

$$w^- = f^+_{D^-}(v) \quad \text{and} \quad w^+ = f^-_{\gamma D^+}(v) = \gamma f^-_{D^+}(\gamma^{-1}v)\,. \tag{41}$$

This notation is ambiguous (w^- depends on v, and w^+ depends on V and γ), but will make the computations less heavy. By the invariance of $f^\pm_{D^\mp}$ by precomposition by the geodesic flow, $w^\mp$ is unchanged if v is replaced by $g^s v$ for any $s \in \mathbb{R}$, and by Equation (31), we have

$$\begin{aligned} h^-_{\eta,\,R} \circ f^+_{D^-}(g^{-t/2}v) &= h^-_{\eta,\,R} \circ f^+_{D^-}(v) = h^-_{\eta,\,R} \circ g^{-t/2}(g^{t/2}w^-)\,. \\ &= e^{\int_{\pi(w^-)}^{\pi(g^{t/2}w^-)}(\widetilde{F}-\delta_F)}\; h^-_{\eta,\,e^{-t/2}R}(g^{t/2}w^-)\,. \end{aligned}$$

Similarly, by Equation (6), by the Γ-invariance of $h^+_{\eta,\,R}$ and by Equation (32), we have

$$h^+_{\eta,\,R} \circ f^-_{D^+}(\gamma^{-1}g^{t/2}v) = e^{\int_{\pi(g^{-t/2}w^+)}^{\pi(w^+)}(\widetilde{F}-\delta_F)}\; h^+_{\eta,\,e^{-t/2}R}(g^{-t/2}w^+)\,.$$

Hence,

$$\begin{aligned} & h^-_{\eta,\,R} \circ f^+_{D^-}(g^{-t/2}v)\; h^+_{\eta,\,R} \circ f^-_{D^+}(\gamma^{-1}g^{t/2}v) \\ &= e^{-\delta_F\, t}\, e^{\int_{\pi(w^-)}^{\pi(g^{t/2}w^-)}\widetilde{F} + \int_{\pi(g^{-t/2}w^+)}^{\pi(w^+)}\widetilde{F}}\; h^-_{\eta,\,e^{-t/2}R}(g^{t/2}w^-)\, h^+_{\eta,\,e^{-t/2}R}(g^{-t/2}w^+)\,. \end{aligned}$$

Now, we consider the product term in Equation (40) involving the characteristic functions. Note that (see Section 2.2 and in particular Equation (10)) the quantity $\mathbb{1}_{\mathscr{V}^+_{\eta,\,R}(\Omega^-)}(g^{-t/2}v)\; \mathbb{1}_{\mathscr{V}^-_{\eta,\,R}(\Omega^+)}(\gamma^{-1}g^{t/2}v)$ is different from 0 (hence equal to 1) if and only if

$$v \in g^{t/2}\mathscr{V}^+_{\eta,\,R}(\Omega^-) \cap \gamma g^{-t/2}\mathscr{V}^-_{\eta,\,R}(\Omega^+) = \mathscr{V}^+_{\eta,\,e^{-t/2}R}(g^{t/2}\Omega^-) \cap \mathscr{V}^-_{\eta,\,e^{-t/2}R}(\gamma g^{-t/2}\Omega^+)\,.$$

By Lemma 8 (applied by replacing D^+ by γD^+ and w by v), there exists $t_0, c_0 > 0$ such that for all $\eta \in\,]0,1]$ and $t \geq t_0$, for all $v \in T^1\widetilde{M}$, if $\mathbb{1}_{\mathscr{V}^+_{\eta,\,R}(\Omega^-)}(g^{-t/2}v)\; \mathbb{1}_{\mathscr{V}^-_{\eta,\,R}(\Omega^+)}(\gamma^{-1}g^{t/2}v) \neq 0$, then the following facts hold:

(i) by the convexity of $D^\pm$, we have $v \in \mathscr{U}^+_{D^-} \cap \mathscr{U}^-_{\gamma D^+}$,

(ii) by the definition of $w^\pm$ (see Equation (41)), we have $w^- \in \Omega^-$ and $w^+ \in \gamma\Omega^+$ (The notation (w^-, w^+) here coincides with the notation (w^-, w^+) in Lemma 8),

(iii) there exists a common perpendicular α_γ from D^- to γD^+, whose length ℓ_γ satisfies

$$|\,\ell_\gamma - t\,| \leq 2\eta + c_0\, e^{-t/2}\ ,$$

whose origin $\pi(v_\gamma^-)$ is at distance at most $c_0\, e^{-t/2}$ from $\pi(w^-)$, whose endpoint $\pi(v_\gamma^+)$ is at distance at most $c_0\, e^{-t/2}$ from $\pi(w^+)$, such that the points $\pi(g^{t/2}w^-)$ and $\pi(g^{-t/2}w^+)$ are at distance at most $\eta + c_0\, e^{-t/2}$ from $\pi(v)$, which is at distance at most $c_0\, e^{-t/2}$ from some point p_v of α_γ.

In particular, using (iii) and the uniform continuity property of the $\widetilde{F}$-weighted length (see Inequality (14) which introduces a constant $c_2 \in\,]0,1]$), and since $\widetilde{F}$ is bounded, for all $\eta \in\,]0,1]$, $t \geq t_0$ and $v \in T^1\widetilde{M}$ for which $\mathbb{1}_{\mathscr{V}^+_{\eta,\,R}(\Omega^-)}(g^{-t/2}v)\; \mathbb{1}_{\mathscr{V}^-_{\eta,\,R}(\Omega^+)}(\gamma^{-1}g^{t/2}v) \neq 0$, we have

$$\begin{aligned}
e^{\int_{\pi(w^-)}^{\pi(g^{t/2}w^-)} \widetilde{F} + \int_{\pi(g^{-t/2}w^+)}^{\pi(w^+)} \widetilde{F}} &= e^{\int_{\pi(v_\gamma^-)}^{p_v} \widetilde{F} + \int_{p_v}^{\pi(v_\gamma^+)} \widetilde{F} + \mathrm{O}((\eta + e^{-t/2})^{c_2})} \\
&= e^{\int_{\alpha_\gamma} \widetilde{F}}\; e^{\mathrm{O}((\eta + e^{-\ell_\gamma/2})^{c_2})}\ .
\end{aligned} \tag{42}$$

For all $\eta \in\,]0,1]$, $\gamma \in \Gamma$ and $T \geq t_0$, define $\mathscr{A}_{\eta,\gamma}(T)$ as the set of $(t,v) \in [t_0, T] \times T^1\widetilde{M}$ such that $v \in \mathscr{V}^+_{\eta,\,e^{-t/2}R}(g^{t/2}\Omega^-) \cap \mathscr{V}^-_{\eta,\,e^{-t/2}R}(\gamma g^{-t/2}\Omega^+)\}$, and

$$j_{\eta,\,\gamma}(T) = \iint_{(t,\,v)\in\mathscr{A}_{\eta,\gamma}(T)} h^-_{\eta,\,e^{-t/2}R}(g^{t/2}w^-)\; h^+_{\eta,\,e^{-t/2}R}(g^{-t/2}w^+)\; dt\; d\widetilde{m}_F(v)\,.$$

By the above, since the integral of a function is equal to the integral on any Borel set containing its support, and since the integral of a nonnegative function is nondecreasing in the integration domain, there hence exists $c_4 > 0$ such that for all $T \geq 0$ and $\eta \in\,]0,1]$, we have

$$i_\eta(T) \geq -\,c_4 + \sum_{\substack{\gamma\in\Gamma\,:\; t_0+2+c_0 \leq \ell_\gamma \leq T - \mathrm{O}(\eta+e^{-\ell_\gamma/2}) \\ v_\gamma^- \in \mathscr{N}_{-\,\mathrm{O}(\eta+e^{-\ell_\gamma/2})}\Omega^-,\; v_\gamma^+ \in \gamma\mathscr{N}_{-\,\mathrm{O}(\eta+e^{-\ell_\gamma/2})}\Omega^+}} e^{\int_{\alpha_\gamma}\widetilde{F}}\; j_{\eta,\,\gamma}(T)\; e^{-\,\mathrm{O}((\eta+e^{-\ell_\gamma/2})^{c_2})}\ ,$$

and similarly, for every $T' \geq T$,

$$i_\eta(T) \leq c_4 + \sum_{\substack{\gamma\in\Gamma\,:\; t_0+2+c_0 \leq \ell_\gamma \leq T + \mathrm{O}(\eta+e^{-\ell_\gamma/2}) \\ v_\gamma^- \in \mathscr{N}_{\mathrm{O}(\eta+e^{-\ell_\gamma/2})}\Omega^-,\; v_\gamma^+ \in \gamma\mathscr{N}_{\mathrm{O}(\eta+e^{-\ell_\gamma/2})}\Omega^+}} e^{\int_{\alpha_\gamma}\widetilde{F}}\; j_{\eta,\,\gamma}(T')\; e^{\mathrm{O}((\eta+e^{-\ell_\gamma/2})^{c_2})}\ .$$

We will take T' to be of the form $T + \mathrm{O}(\eta + e^{-\ell_\gamma/2})$, for a bigger $\mathrm{O}(\cdot)$ than the one appearing in the index of the above summation.

Step 5: Conclusion. Let $\gamma \in \Gamma$ be such that D^- and γD^+ have a common perpendicular with length $\ell_\gamma \geq t_0 + 2 + c_0$. Let us prove that for all $\epsilon > 0$, if η is small enough and ℓ_γ is large enough, then for every $T \geq \ell_\gamma + \mathrm{O}(\eta + e^{-\ell_\gamma/2})$ (with the enough's and $\mathrm{O}(\cdot)$ independent of γ), we have

$$1 - \epsilon \leq j_{\eta,\,\gamma}(T) \leq 1 + \epsilon\,. \tag{43}$$

Note that $\widetilde{\sigma}^{\pm}(\mathscr{N}_\varepsilon(\Omega^{\mp}))$ and $\widetilde{\sigma}^{\pm}(\mathscr{N}_{-\varepsilon}(\Omega^{\mp}))$ tend to $\widetilde{\sigma}^{\pm}(\Omega^{\mp})$ as $\varepsilon \to 0$ (since $\widetilde{\sigma}^{\pm}(\partial\Omega^{\mp}) = 0$ as required in Step 1). Using Step 3 and Step 4, this will prove Equation (30), hence will complete the proof of Theorem 14.

We say that $(\widetilde{M}, \Gamma, \widetilde{F})$ *has radius-continuous strong stable/unstable ball masses* if for every $\epsilon > 0$, if $r > 1$ is close enough to 1, then for every $v \in T^1\widetilde{M}$, if $B^-(v, 1)$ meets the support of $\mu^+_{W^{\rm su}(v)}$, then

$$\mu^+_{W^{\rm su}(v)}(B^-(v,r)) \le e^{\epsilon}\mu^+_{W^{\rm su}(v)}(B^-(v,1))$$

and if $B^+(v, 1)$ meets the support of $\mu^-_{W^{\rm ss}(v)}$, then

$$\mu^-_{W^{\rm ss}(v)}(B^+(v,r)) \le e^{\epsilon}\mu^-_{W^{\rm ss}(v)}(B^+(v,1))\,.$$

We say that $(\widetilde{M}, \Gamma, \widetilde{F})$ *has radius-Hölder-continuous strong stable/unstable ball masses* if there exists $c \in \,]0, 1]$ and $c' > 0$ such that for every $\epsilon \in \,]0, 1]$, if $B^-(v, 1)$ meets the support of $\mu^+_{W^{\rm su}(v)}$, then

$$\mu^+_{W^{\rm su}(v)}(B^-(v,r)) \le e^{c'\epsilon^c}\mu^+_{W^{\rm su}(v)}(B^-(v,1))$$

and if $B^+(v, 1)$ meets the support of $\mu^-_{W^{\rm ss}(v)}$, then

$$\mu^-_{W^{\rm ss}(v)}(B^+(v,r)) \le e^{c'\epsilon^c}\mu^-_{W^{\rm ss}(v)}(B^+(v,1))\,.$$

When the sectional curvature has bounded derivatives and when $(\widetilde{M}, \Gamma, \widetilde{F})$ has Hölder strong stable/unstable ball masses, we will prove the following stronger statement: with a constant $c_7 > 0$ and functions $\mathrm{O}(\cdot)$ independent of γ, for all $\eta \in \,]0, 1]$ and $T \ge \ell_\gamma + \mathrm{O}(\eta + e^{-\ell_\gamma/2})$, we have

$$j_{\eta,\gamma}(T) = \Big(1 + \mathrm{O}\Big(\frac{e^{-\ell_\gamma/2}}{2\eta}\Big)\Big)^2 e^{\mathrm{O}((\eta+e^{-\ell_\gamma/2})^{c_7})}\,. \tag{44}$$

This stronger version will be needed for the error term estimate in Section 5. In order to obtain Theorem 14, only the fact that $j_{\eta,\gamma}(T)$ tends to 1 as firstly ℓ_γ tends to $+\infty$, secondly η tends to 0 is needed. A reader not interested in the error term may skip many technical details below.

Given $a, b > 0$ and a point x in a metric space X (with a, b, x depending on parameters), we will denote by $B(x, a\, e^{\mathrm{O}(b)})$ any subset Y of X such that there exists a constant $c > 0$ (independent of the parameters) with

$$B(x, a\, e^{-c\, b}) \subset Y \subset B(x, a\, e^{c\, b})\,.$$

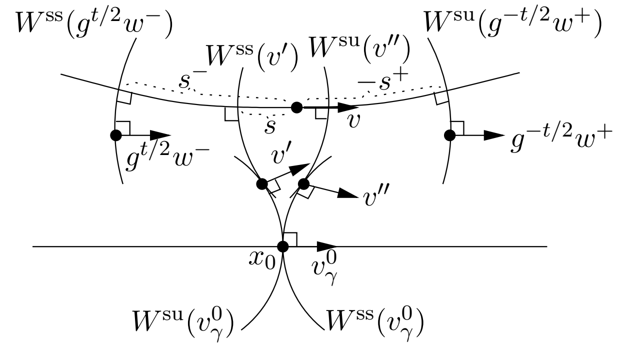
Let $\eta \in \,]0, 1]$ and $T \ge \ell_\gamma + \mathrm{O}(\eta + e^{-\ell_\gamma/2})$. To simplify the notation, let

$$r_t = e^{-t/2}R,\;\; w_t^- = g^{t/2}w^- \;\;\text{ and }\;\; w_t^+ = g^{-t/2}w^+\,.$$

By the definition of $j_{\eta,\gamma}$, using Equations (33) and (34), we hence have

$$\begin{aligned} j_{\eta,\gamma}(T) &= \iint_{(t,v)\in\mathscr{A}_{\eta,\gamma}(T)} h^-_{\eta,\,r_t}(w_t^-)\; h^+_{\eta,\,r_t}(w_t^+)\; dt\; d\widetilde{m}_F(v) \\ &= \frac{e^{\mathrm{O}(e^{-c_2\ell_\gamma/2})}}{(2\eta)^2}\iint_{(t,v)\in\mathscr{A}_{\eta,\gamma}(T)} \frac{dt\; d\widetilde{m}_F(v)}{\mu^-_{W^{\rm ss}(w_t^-)}(B^+(w_t^-, r_t))\;\mu^+_{W^{\rm su}(w_t^+)}(B^-(w_t^+, r_t))}\,. \end{aligned} \tag{45}$$

We start the proof of Equation (43) by defining parameters s^+, s^-, s, v', v'' associated to $(t, v) \in \mathscr{A}_{\eta,\gamma}(T)$.



We have $(t, v) \in \mathscr{A}_{\eta,\gamma}(T)$ if and only if there exist $s^\pm \in\,]-\eta, \eta[$ such that

$$g^{\mp s^\mp} v \in B^\pm(g^{\pm t/2} w^\mp, e^{-t/2} R)\,.$$

The notation $s^\pm$ coincides with the one in the proof of Lemma 8 (where (D^+, w) has been replaced by $(\gamma D^+, v)$).

In order to define the parameters s, v', v'', we use the well known local product structure of the unit tangent bundle in negative curvature. If $v \in T^1M$ is close enough to v^0_γ (in particular, $v_- \neq (v^0_\gamma)_+$ and $v_+ \neq (v^0_\gamma)_-$), then let $v' = f^+_{HB_-(v^0_\gamma)}(v)$ be the unique element of $W^{su}(v^0_\gamma)$ such that $v'_+ = v_+$, let $v'' = f^-_{HB_+(v^0_\gamma)}(v)$ be the unique element of $W^{ss}(v^0_\gamma)$ such that $v''_- = v_-$, and let s be the unique element of $\mathbb{R}$ such that $g^{-s}v \in W^{ss}(v')$. The map $v \mapsto (s, v', v'')$ is a homeomorphism from a neighbourhood of v^0_γ in $T^1\widetilde{M}$ to a neighbourhood of $(0, v^0_\gamma, v^0_\gamma)$ in $\mathbb{R} \times W^{su}(v^0_\gamma) \times W^{ss}(v^0_\gamma)$. Note that if $v = g^r v^0_\gamma$ for some $r \in \mathbb{R}$ close to 0, then

$$w^- = v^-_\gamma,\ w^+ = v^+_\gamma,\ s = r,\ v' = v'' = v^0_\gamma,\ s^- = \frac{\ell_\gamma - t}{2} + s,\ s^+ = \frac{\ell_\gamma - t}{2} - s\,.$$

Up to increasing t_0 (which does not change Step 4, up to increasing c_4), we may assume that for every $(t, v) \in \mathscr{A}_{\eta,\gamma}(T)$, the vector v belongs to the domain of this local product structure of $T^1\widetilde{M}$ at v^0_γ.

The vectors v, v', v'' are close to v^0_γ if t is big and η small, as the following result shows. We denote (also) by d the Riemannian distance induced by Sasaki's metric on $T^1\widetilde{M}$.

Lemma 16 *For every $(t, v) \in \mathscr{A}_{\eta,\gamma}(T)$, we have $d(v, v^0_\gamma), d(v', v^0_\gamma), d(v'', v^0_\gamma) = \mathrm{O}(\eta + e^{-t/2})$.*

Proof. Consider the distance d' on $T^1\widetilde{M}$, defined by

$$\forall\, v_1, v_2 \in T^1\widetilde{M}, \quad d'(v_1, v_2) = \max_{r \in [-1,0]} d\big(\pi(g^r v_1), \pi(g^r v_2)\big)\,.$$

As already seen in Step 4, we have $d(\pi(w^\pm), \pi(v^\pm_\gamma))$, $d(\pi(v), \alpha_\gamma) = \mathrm{O}(e^{-t/2})$, and besides $d(\pi(g^{t/2} w^-), \pi(v)), \frac{\ell_\gamma}{2} - \frac{t}{2} = \mathrm{O}(\eta + e^{-t/2})$. Hence $d(\pi(v), \pi(v^0_\gamma)) = \mathrm{O}(\eta + e^{-t/2})$. By Lemma 6, we have

$$d(\pi(g^{-\frac{t}{2} - s^-} v), \pi(v^-_\gamma)) \le d(\pi(g^{-\frac{t}{2} - s^-} v), \pi(w^-)) + d(\pi(w^-), \pi(v^-_\gamma)) \le R + c_0\, e^{-t/2}\,.$$

By an exponential pinching argument, we hence have $d'(v, v^0_\gamma) = \mathrm{O}(\eta + e^{-\ell_\gamma/2})$. Since d and d' are equivalent (see [Bal, page 70]), we therefore have $d(v, v^0_\gamma) = \mathrm{O}(\eta + e^{-\ell_\gamma/2})$.

For all $w \in T^1\widetilde{M}$ and $V \in T_w T^1\widetilde{M}$, we may uniquely write $V = V^{\rm su} + V^0 + V^{\rm ss}$ with $V^{\rm su} \in T_w W^{\rm su}(w)$, $V^0 \in \mathbb{R}\frac{d}{dt}_{|t_0} g^t w$ and $V^{\rm ss} \in T_w W^{\rm ss}(w)$. By [PPS, §7.2] (building on [Bri] whose compactness assumption on M and torsion free assumption on Γ are not necessary for this, the pinched negative curvature assumption is sufficient), Sasaki's metric (with norm $\|\cdot\|$) is equivalent to the Riemannian metric with (product) norm

$$\|V\|' = \sqrt{\|\, V^{\rm su}\,\|^2 + \|\, V^0\,\|^2 + \|\, V^{\rm ss}\,\|^2}\,.$$

By the dynamical local product structure of $T^1\widetilde{M}$ in the neighbourhood of v^0_γ and by the definition of v', v'', the result follows, since the exponential map of $T^1\widetilde{M}$ at v^0_γ is almost isometric close to 0 and the projection to a factor of a product norm is 1-Lipschitz. □

We now use the local product structure of the Gibbs measure to prove the following result.

Lemma 17 *For every $(t, v) \in \mathscr{A}_{\eta,\gamma}(T)$, we have*

$$dt\, d\widetilde{m}_F(v) = e^{{\rm O}((\eta + e^{-\ell_\gamma/2})^{c_2})}\; dt\; ds\; d\mu^+_{W^{\rm su}(v^0_\gamma)}(v')\; d\mu^-_{W^{\rm ss}(v^0_\gamma)}(v'')\,.$$

Proof. By the definition of the measures (see Equation (18), (19), (20)), since the above parameter s differs, when v_-, v_+ are fixed, only up to a constant from the time parameter in Hopf's parametrisation, we have

$$d\widetilde{m}_F(v) = e^{C^-_{v_-}(x_0,\,\pi(v)) + C^+_{v_+}(x_0,\,\pi(v))}\; d\mu^-_{x_0}(v_-)\, d\mu^+_{x_0}(v_+)\, ds$$

$$d\mu^+_{W^{\rm su}(v^0_\gamma)}(v') = e^{C^+_{v'_+}(x_0,\,\pi(v'))}\; d\mu^+_{x_0}(v'_+)\,,$$

$$d\mu^-_{W^{\rm ss}(v^0_\gamma)}(v'') = e^{C^-_{v''_-}(x_0,\,\pi(v''))}\; d\mu^-_{x_0}(v''_-)\;.$$

By Equation (17), since F is bounded, we have $|\, C^\pm_\xi(z, z')\,| = {\rm O}(d(z, z')^{c_2})$ for all $\xi \in \partial_\infty\widetilde{M}$ and $z, z' \in \widetilde{M}$ with $d(z, z')$ bounded. Since the map $\pi : T^1\widetilde{M} \to \widetilde{M}$ is 1-Lipschitz, and since $v_+ = v'_+$ and $v_- = v''_-$, the result follows from Lemma 16 and the cocycle property (15). □

When ℓ_γ is big, the submanifold $\partial^1_+(g^{\ell_\gamma/2}\Omega^-)$ has a second order contact at v^0_γ with $W^{\rm su}(v^0_\gamma)$ and similarly, $\partial^1_-(g^{-\ell_\gamma/2}\Omega^+)$ has a second order contact at v^0_γ with $W^{\rm ss}(v^0_\gamma)$. Let P_γ be a plane domain of $(t, s) \in \mathbb{R}^2$ such that there exist $s^\pm \in\,]-\eta, \eta[$ with $s^\mp = \frac{\ell_\gamma - t}{2} \pm s + {\rm O}(e^{-\ell_\gamma/2})$. Note that its area is $(2\eta + {\rm O}(e^{-\ell_\gamma/2}))^2$. By the above, we have (with the obvious meaning of a double inclusion)

$$\mathscr{A}_{\eta,\gamma}(T) = P_\gamma \times B^-(v^0_\gamma, r_t\, e^{{\rm O}(\eta + e^{-\ell_\gamma/2})}) \times B^+(v^0_\gamma, r_t\, e^{{\rm O}(\eta + e^{-\ell_\gamma/2})})\,.$$

By Lemma 17, we hence have

$$\int_{\mathscr{A}_{\eta,\gamma}(T)} dt\, d\widetilde{m}_F(v) = e^{{\rm O}((\eta + e^{-\ell_\gamma/2})^{c_2})}\; (2\eta + {\rm O}(e^{-\ell_\gamma/2}))^2\; \times$$
$$\mu^+_{W^{\rm su}(v^0_\gamma)}(B^-(v^0_\gamma, r_t\, e^{{\rm O}(\eta + e^{-\ell_\gamma/2})}))\; \mu^-_{W^{\rm ss}(v^0_\gamma)}(B^+(v^0_\gamma, r_t\, e^{{\rm O}(\eta + e^{-\ell_\gamma/2})}))\,. \tag{46}$$

The last ingredient of the proof of Step 5 is the following continuity property of strong stable and strong unstable ball volumes as their center varies (see [Rob, Lem. 1.16], [PPS, Prop. 10.16] for related properties, though we need a more precise control for the error term in Section 5).

Lemma 18 *Assume that $(\widetilde{M}, \Gamma, \widetilde{F})$ has radius-continuous strong stable/unstable ball masses. There exists $c_5 > 0$ such that for every $\epsilon > 0$, if η is small enough and ℓ_γ large enough, then for every $(t, v) \in \mathscr{A}_{\eta,\gamma}(T)$, we have*

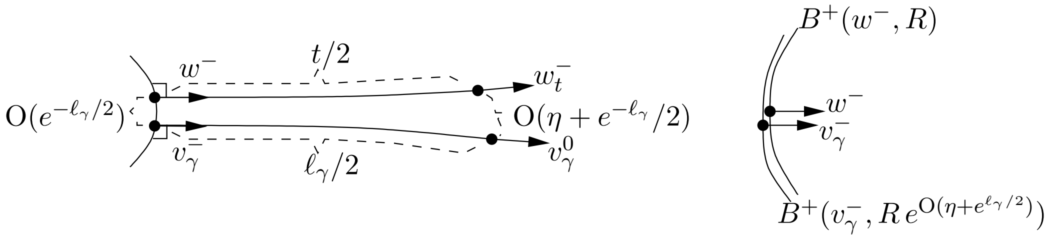
$$\mu^+_{W^{\rm su}(w_t^+)}(B^-(w_t^+, r_t)) = e^{\mathrm{O}(\epsilon^{c_5})}\ \mu^+_{W^{\rm su}(v_\gamma^0)}(B^-(v_\gamma^0, r_t))$$

and

$$\mu^-_{W^{\rm ss}(w_t^-)}(B^+(w_t^-, r_t)) = e^{\mathrm{O}(\epsilon^{c_5})}\ \mu^-_{W^{\rm ss}(v_\gamma^0)}(B^+(v_\gamma^0, r_t))\,.$$

If we furthermore assume that the sectional curvature of $\widetilde{M}$ has bounded derivative and that $(\widetilde{M}, \Gamma, \widetilde{F})$ has radius-Hölder-continuous strong stable/unstable ball masses, then we may replace ϵ by $(\eta + e^{-\ell_\gamma/2})^{c_6}$ for some constant $c_6 > 0$.

Proof. We prove the (second) claim for $\mathscr{W}^{\rm ss}$, the (first) one for $\mathscr{W}^{\rm su}$ follows similarly. The final statement is only used for the error estimates in Section 5.



Using respectively Equation (9) since $w_t^- = g^{t/2} w^-$ and $r_t = e^{-t/2} R$, Equation (21) where (v, t, w) is replaced by $(g^{t/2} v, t/2, g^{t/2} w^-)$, and Equation (16), we have

$$\begin{aligned}
\mu^-_{W^{\rm ss}(w_t^-)}(B^+(w_t^-, r_t)) &= \int_{v \in B^+(w^-,\, R)} d\mu^-_{W^{\rm ss}(g^{t/2} w^-)}(g^{t/2} v) \\
&= \int_{v \in B^+(w^-,\, R)} e^{C^-_{v_-}(\pi(v),\, \pi(g^{t/2} v))}\ d\mu^-_{W^{\rm ss}(w^-)}(v) \\
&= \int_{v \in B^+(w^-,\, R)} e^{\int_{\pi(v)}^{\pi(g^{t/2} v)} (\widetilde{F} - \delta_F)}\ d\mu^-_{W^{\rm ss}(w^-)}(v)\,. \qquad (47)
\end{aligned}$$

Similarly, for every $a > 0$, we have

$$\mu^-_{W^{\rm ss}(v_\gamma^0)}(B^+(v_\gamma^0, a r_t)) = \int_{v \in B^+(v_\gamma^0,\, aR)} e^{\int_{\pi(v)}^{\pi(g^{t/2} v)} (\widetilde{F} - \delta_F)}\ d\mu^-_{W^{\rm ss}(v_\gamma^0)}(v) \qquad (48)$$

Let $h^- : B^+(w^-, R) \to W^{\rm ss}(v_\gamma^-)$ be the map such that $(h^-(v))_- = v_-$, which is well defined and a homeomorphism onto its image if ℓ_γ is big enough (since R is fixed). By Proposition 10 applied with $D = HB_+(w^-)$ and $D' = HB_+(v_\gamma^-)$, we have, for every $v \in B^+(w^-, R)$,

$$d\mu^-_{W^{\rm ss}(w^-)}(v) = e^{-C^-_{v_-}(\pi(v),\, \pi(h^-(v)))}\ d\mu^-_{W^{\rm ss}(v_\gamma^-)}(h^-(v))\,.$$

Let us fix $\epsilon > 0$. The strong stable balls of radius R centered at w^- and v_γ^- are very close (see the above picture). More precisely, recall that R is fixed, and that, as seen above, $d(\pi(w^-), \pi(v_\gamma^-)) = \mathrm{O}(e^{-\ell_\gamma/2})$ and $d(\pi(g^{t/2}w^-), \pi(g^{\ell_\gamma/2}v_\gamma^-)) = \mathrm{O}(\eta + e^{-\ell_\gamma/2})$. Therefore we have $d(\pi(v), \pi(h^-(v))) \leq \epsilon$ for every $v \in B^+(w^-, R)$ if η is small enough and ℓ_γ large enough. If furthermore the sectional curvature has bounded derivatives, then by Anosov's arguments (see for instance [PPS, Theo. 7.3]) the strong stable foliation is Hölder-continuous. Hence we have $d(\pi(v), \pi(h^-(v))) = \mathrm{O}((\eta + e^{-\ell_\gamma/2})^{c_5})$ for every $v \in B^+(w^-, R)$, for some constant $c_5 > 0$, under the additional hypothesis on the curvature. We also have $h^-(B^+(w^-, R)) = B^+(v_\gamma^-, R\, e^{\mathrm{O}(\epsilon)})$ and, under the additional hypothesis on the curvature, $h^-(B^+(w^-, R)) = B^+(v_\gamma^-, R\, e^{\mathrm{O}((\eta + e^{-\ell_\gamma/2})^{c_5})})$. Assume in what follows that $\epsilon = (\eta + e^{-\ell_\gamma/2})^{c_5}$ under the additional hypothesis on the curvature. By Equation (17), we hence have, for every $v \in B^+(w^-, R)$,

$$d\mu^-_{W^{\mathrm{ss}}(w^-)}(v) = e^{\mathrm{O}(\epsilon^{c_2})}\, d\mu^-_{W^{\mathrm{ss}}(v_\gamma^-)}(h^-(v))$$

and, using Equation (14),

$$\int_{\pi(v)}^{\pi(g^{t/2}v)} (\widetilde{F} - \delta_F) - \int_{\pi(h^-(v))}^{\pi(g^{t/2}h^-(v))} (\widetilde{F} - \delta_F) = \mathrm{O}(\epsilon^{c_2})\ .$$

The result follows, by Equation (47) and (48) and the continuity property in the radius. □

Now Lemma 18 (with ϵ as in its statement, and when its hypotheses are satisfied) implies that

$$\iint_{(t,v)\in\mathscr{A}_{\eta,\gamma}(T)} \frac{dt\ d\widetilde{m}_F(v)}{\mu^-_{W^{\mathrm{ss}}(w_t^-)}(B^+(w_t^-, r_t))\ \mu^+_{W^{\mathrm{su}}(w_t^+)}(B^-(w_t^+, r_t))}$$
$$= \frac{e^{\mathrm{O}(\epsilon^{c_5})} \iint_{(t,v)\in\mathscr{A}_{\eta,\gamma}(T)} dt\ d\widetilde{m}_F(v)}{\mu^-_{W^{\mathrm{ss}}(v_\gamma^0)}(B^+(v_\gamma^0, r_t))\ \mu^+_{W^{\mathrm{su}}(v_\gamma^0)}(B^-(v_\gamma^0, r_t))}\ .$$

By Equation (45) and Equation (46), we hence have

$$j_{\eta,\gamma}(T) = e^{\mathrm{O}((\eta + e^{-\ell_\gamma/2})^{c_2})}\ e^{\epsilon^{c_5}}\ \frac{(2\eta + \mathrm{O}(e^{-\ell_\gamma/2}))^2}{(2\eta)^2}$$

under the technical assumptions of Lemma 18. The assumption on radius-continuity of strong stable/unstable ball masses can be bypassed using bump functions, as explained in [Rob, page 81]. □

4.2 Equidistribution of equidistant submanifolds

In this subsection (which is not needed for the counting and equidistribution results of common perpendiculars), we generalize the main theorem of [PP5] from Bowen-Margulis measures to Gibbs measures, to prove that the skinning measure on (any nontrivial piece of) the outer unit normal bundle of any proper nonempty properly immersed closed convex subset, pushed a long time by the geodesic flow, equidistributes towards the Gibbs measure, under finiteness and mixing assumptions.

Theorem 19 *Let $\widetilde{M}$ be a complete simply connected Riemannian manifold with pinched sectional curvature bounded above by -1. Let $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ be a bounded Γ-invariant Hölder-continuous function with positive critical exponent δ_F. Assume that the Gibbs measure m_F is finite and mixing for the geodesic flow. Let $\mathscr{D} = (D_i)$ be a locally finite Γ-equivariant family of nonempty proper closed convex subsets of $\widetilde{M}$. Let $\Omega = (\Omega_i)_{i\in I}$ be a locally finite Γ-equivariant family of measurable subsets of $T^1\widetilde{M}$, with $\Omega_i \subset \partial^1_+ D_i$ for all $i \in I$. Assume that σ^+_Ω is finite and nonzero. Then, as $t \to +\infty$,*

$$\frac{1}{\|(g^t)_*\sigma^+_\Omega\|}\,(g^t)_*\sigma^+_\Omega \;\overset{*}{\rightharpoonup}\; \frac{1}{\|m_F\|}\,m_F\ .$$

Similarly, under the assumptions of this theorem, if $\Omega = (\Omega_i)_{i\in I}$ is a locally finite Γ-equivariant family of measurable subsets of $T^1\widetilde{M}$, with $\Omega_i \subset \partial^1_- D_i$ for all $i \in I$, if σ^-_Ω is finite and nonzero, then, as $t \to +\infty$,

$$\frac{1}{\|(g^{-t})_*\sigma^-_\Omega\|}\,(g^{-t})_*\sigma^-_\Omega \;\overset{*}{\rightharpoonup}\; \frac{1}{\|m_F\|}\,m_F\ .$$

Since pushforwards of measures are weak-star continuous and preserve total mass, we have, under the assumptions of Theorem 19, the following equidistribution result in M of the immersed t-neighbourhood of a proper closed properly immersed convex subset of M: as $t \to +\infty$,

$$\frac{1}{\|\sigma^+_\Omega\|}\,\pi_*(g^t)_*\sigma^+_\Omega \;\overset{*}{\rightharpoonup}\; \frac{1}{\|m_F\|}\,\pi_* m_F\ .$$

When $F = 0$, $\widetilde{M}$ is a symmetric space and Γ has finite covolume, then $\pi_* m_F$ is, up to a constant multiple, the Riemannian volume measure of M. In particular, the boundary of the immersed t-neighbourhood of the convex hull of any infinite index loxodromic cyclic or convex-cocompact nonelementary subgroup of Γ equidistributes in M towards the Riemannian volume.

Proof. The proof is analogous with that of Theorem 19 of [PP5]. Given three numbers a, b, c (depending on some parameters), we write $a = b \pm c$ if $|a - b| \le c$.

Let $\eta \in \,]0, 1]$. We may assume that $\Gamma\backslash I$ is finite, since for every $\epsilon > 0$, there exists a Γ-invariant partition $I = I' \cup I''$ with $\Gamma\backslash I'$ finite such that if $\Omega' = (\Omega_i)_{i\in I'}$ and $\Omega'' = (\Omega_i)_{i\in I''}$, then $\sigma^+_\Omega = \sigma^+_{\Omega'} + \sigma^+_{\Omega''}$ with $\|(g^{-t})_*\sigma^-_{\Omega''}\| = \|\sigma^+_{\Omega''}\| < \epsilon$. Hence, using Lemma 12, we may fix $R > 0$ such that $\nu^-_w(V^+_{w,\eta,R}) > 0$ for all $w \in \partial^1_+ D_i$ and $i \in I$. We will use the global test functions $\widetilde{\phi}_\eta : T^1\widetilde{M} \to [0, +\infty[$ now defined by (using the conventions of Step 2 of Theorem 14)

$$\widetilde{\phi}_\eta(v) = \sum_{i\in I/\sim} \phi^-_{\eta,R,\Omega_i} = \sum_{i\in I/\sim} h^-_{\eta,R} \circ f^+_{D_i}\, \mathbb{1}_{\mathcal{V}^+_{\eta,R}(\Omega_i)}\,.$$

As in [PP5, Lem. 17], the map $\widetilde{\phi}_\eta : T^1\widetilde{M} \to [0, +\infty[$ is well defined (independent of the representatives of i), measurable and Γ-equivariant. Hence it defines, by passing to the quotient, a measurable function $\phi_\eta : T^1M \to [0, +\infty[\,$. By Lemma 15, the function ϕ_η is integrable and satisfies

$$\int_{T^1M} \phi_\eta\, dm_F = \|\sigma^+_\Omega\|\,. \tag{49}$$

Fix $\psi \in \mathscr{C}_c(T^1M)$. Let us prove that

$$\lim_{t \to +\infty} \frac{1}{\|(g^t)_*\sigma^+_\Omega\|} \int_{T^1M} \psi \; d(g^t)_*\sigma^+_\Omega = \frac{1}{\|m_F\|} \int_{T^1M} \psi \; dm_F \;.$$

Consider a fundamental domain Δ_Γ for the action of Γ on $T^1\widetilde{M}$ as in [Rob, page 13] (or in the proof of [PP5, Prop. 18]). By a standard argument of finite partition of unity and up to modifying Δ_Γ, we may assume that there exists a map $\widetilde{\psi} : T^1\widetilde{M} \to \mathbb{R}$ whose support has a small neighbourhood contained in Δ_Γ such that $\widetilde{\psi} = \psi \circ p$, where $p : T^1\widetilde{M} \to T^1M = \Gamma\backslash T^1\widetilde{M}$ is the canonical projection (which is 1-Lipschitz). Fix $\epsilon > 0$. Since $\widetilde{\psi}$ is uniformly continuous, for every $\eta > 0$ small enough and for every $t \geq 0$ large enough, for all $w \in T^1\widetilde{M}$ and $v \in V^+_{w,\,\eta,\,e^{-t}R}$, we have

$$\widetilde{\psi}(v) = \widetilde{\psi}(w) \pm \frac{\epsilon}{2} \;. \tag{50}$$

If t is big enough and η small enough, we have, using respectively

- Proposition 9 (iii) for the second equality,
- the definition of $h_{\eta,\,e^{-t}R}$ for the third equality,
- the disintegration Proposition 13 for the fibration $f^+_{\mathscr{N}_t D_i}$ for the fourth equality,
- the fact that a small neighbourhood of the support of $\widetilde{\psi}$ is contained in Δ_Γ, the definition of the test function $\phi^-_{\eta,\,e^{-t}R,\,g^t\Omega_i}$ and Equation (50), for the fifth equality,
- Equation (36) for the sixth equality,
- the definition of the global test function $\widetilde{\phi}_\eta$ for the seventh equality,
- the invariance of the Gibbs measure under the geodesic flow the last equality.

$$
\begin{aligned}
&\int \psi \; d(g^t)_*\sigma^+_{\Omega} \\
&= \sum_{i\in I/\sim} \int_{w\in g^t\Omega_i} \widetilde{\psi}(w)\; d(g^t)_*\widetilde{\sigma}^+_{D_i}(w) \\
&= \sum_{i\in I/\sim} \int_{w\in g^t\Omega_i} \widetilde{\psi}(w)\; e^{-\int_{\pi(g^{-t}w)}^{\pi(w)}(\widetilde{F}-\delta_F)}\; d\widetilde{\sigma}^+_{\mathscr{N}_t D_i}(w) \\
&= \sum_{i\in I/\sim} \int_{w\in g^t\Omega_i} e^{-\int_{\pi(g^{-t}w)}^{\pi(w)}(\widetilde{F}-\delta_F)}\; h^-_{\eta,\,e^{-t}R}(w)\; \nu^-_w(V^+_{w,\,\eta,\,e^{-t}R})\; \widetilde{\psi}(w)\; d\widetilde{\sigma}^+_{\mathscr{N}_t D_i}(w) \\
&= \sum_{i\in I/\sim} \int_{v\in \mathscr{V}_{\eta,\,e^{-t}R}(g^t\Omega_i)} e^{-\int_{\pi(g^{-t}f^+_{\mathscr{N}_t D_i}(v))}^{\pi(f^+_{\mathscr{N}_t D_i}(v))}(\widetilde{F}-\delta_F)}\; h^-_{\eta,\,e^{-t}R}\circ f^+_{\mathscr{N}_t D_i}(v)\; \widetilde{\psi}(f^+_{\mathscr{N}_t D_i}(v))\; d\widetilde{m}_F(v) \\
&= \sum_{i\in I/\sim} \int_{v\in \Delta_\Gamma} e^{-\int_{\pi(g^{-t}f^+_{\mathscr{N}_t D_i}(v))}^{\pi(f^+_{\mathscr{N}_t D_i}(v))}(\widetilde{F}-\delta_F)}\; \phi^-_{\eta,\,e^{-t}R,\,g^t\Omega_i}(v)\; \big(\widetilde{\psi}(v)\pm\frac{\epsilon}{2}\big)\; d\widetilde{m}_F(v) \\
&= \sum_{i\in I/\sim} \int_{v\in \Delta_\Gamma} \phi^-_{\eta,\,R,\,\Omega_i}(g^{-t}v)\; \big(\widetilde{\psi}(v)\pm\frac{\epsilon}{2}\big)\; d\widetilde{m}_F(v) \\
&= \int_{\Delta_\Gamma} \widetilde{\phi}_\eta\circ g^{-t}\; \widetilde{\psi}\; d\widetilde{m}_F \pm \frac{\epsilon}{2}\int_{\Delta_\Gamma} \widetilde{\phi}_\eta\circ g^{-t}\; d\widetilde{m}_F \\
&= \int_{T^1M} \phi_\eta\circ g^{-t}\; \psi\; dm_F \pm \frac{\epsilon}{2}\int_{T^1M} \phi_\eta\; dm_F\,.
\end{aligned}
$$

By Equation (49), we have $\|(g^t)_*\sigma^+_\Omega\| = \|\sigma^+_\Omega\| = \int_{T^1M}\phi_\eta\; dm_F$. By the mixing property of the geodesic flow on T^1M for the Gibbs measure, for $t\geq 0$ big enough (while η is fixed), we hence have

$$
\frac{\int \psi\; d(g^t)_*\sigma^+_\Omega}{\|(g^t)_*\sigma^+_\Omega\|} = \frac{\int_{T^1M}\phi_\eta\circ g^{-t}\;\psi\; dm_F}{\int_{T^1M}\phi_\eta\; dm_F} \pm \frac{\epsilon}{2} = \frac{\int_{T^1M}\psi\; dm_F}{\|m_F\|}\pm\epsilon\,.
$$

This proves the result. □

4.3 Equidistribution of endvectors of common perpendiculars in T^1M

Using Subsection 2.4, we now deduce from Theorem 14, which is an equidistribution result in the space $T^1\widetilde{M}\times T^1\widetilde{M}$, an equidistribution result in its quotient $T^1M\times T^1M$ by the action of $\Gamma\times\Gamma$.

The following Corollary is the main result of this paper on the counting of common perpendiculars and on the equidistribution of their initial and terminal tangent vectors in T^1M.

Corollary 20 *Let $\widetilde{M},\Gamma,\widetilde{F},\mathscr{D}^-,\mathscr{D}^+$ be as in Theorem 14. Then,*

$$
\lim_{t\to+\infty}\; \delta_F\;\|m_F\|\; e^{-\delta_F t} \sum_{v,\,w\in T^1M} m_{\partial^1_+\mathscr{D}^-}(v)\; m_{\partial^1_-\mathscr{D}^+}(w)\; n_t(v,w)\;\Delta_v\otimes\Delta_w \;=\; \sigma^+_{\mathscr{D}^-}\otimes\sigma^-_{\mathscr{D}^+} \tag{51}
$$

for the weak-star convergence of measures on the locally compact space $T^1M \times T^1M$. If $\sigma^+_{\mathscr{D}^-}$ and $\sigma^-_{\mathscr{D}^+}$ are finite, the result also holds for the narrow convergence.

Furthermore, for all Γ-equivariant families $\Omega^\pm = (\Omega^\pm_k)_{k\in I^\pm}$ of subsets of $T^1\widetilde{M}$ with $\Omega^\mp_k$ a Borel subset of $\partial^1_\pm D^\mp_k$ for all $k \in I^\mp$, with nonzero finite skinning measure and with boundary in $\partial^1_\pm D^\mp_k$ of zero skinning measure, we have

$$\mathscr{N}_{\Omega^-,\,\Omega^+,\,F}(t) \;\sim\; \frac{\|\sigma^+_{\Omega^-}\|\;\|\sigma^-_{\Omega^+}\|}{\delta_F\;\|m_F\|}\;e^{\delta_F t}\;,$$

as $t \to +\infty$.

In particular, if the skinning measures $\sigma^+_{\mathscr{D}^-}$ and $\sigma^-_{\mathscr{D}^+}$ are positive and finite, as $t \to +\infty$, we have

$$\mathscr{N}_{\mathscr{D}^-,\,\mathscr{D}^+,\,F}(t) \;\sim\; \frac{\|\sigma^+_{\mathscr{D}^-}\|\;\|\sigma^-_{\mathscr{D}^+}\|}{\delta_F\;\|m_F\|}\;e^{\delta_F t}\;.$$

Proof. Note that the sum in Equation (51) is locally finite, hence it defines a locally finite measure on $T^1M \times T^1M$. We are going to rewrite the sum in the statement of Theorem 14 in a way which makes it easier to push it down from $T^1\widetilde{M} \times T^1\widetilde{M}$ to $T^1M \times T^1M$.

For every $\widetilde{v} \in T^1\widetilde{M}$, let

$$m^\mp(\widetilde{v}) = \operatorname{Card}\{k \in I^\mp/_\sim \;:\; \widetilde{v} \in \partial^1_\pm D^\mp_k\}\;,$$

so that for every $v \in T^1M$, the multiplicity of v with respect to the family $\partial^1_\pm\mathscr{D}^\mp$ is (see Subsection 3.3)

$$m_{\partial^1_\pm\mathscr{D}^\mp}(v) = \frac{m^\mp(\widetilde{v})}{\operatorname{Card}(\operatorname{Stab}_\Gamma \widetilde{v})}\;,$$

for any preimage $\widetilde{v}$ of v in $T^1\widetilde{M}$.

For all $\gamma \in \Gamma$ and $\widetilde{v}, \widetilde{w} \in T^1\widetilde{M}$, there exists $(i,j) \in (I^-/_\sim) \times (I^+/_\sim)$ such that $\widetilde{v} = v^-_{i,\gamma j}$ and $\widetilde{w} = v^+_{\gamma^{-1}i,j} = \gamma^{-1}v^+_{i,\gamma j}$ if and only if $\gamma\widetilde{w} \in g^{\mathbb{R}}\,\widetilde{v}$, there exists $i' \in I^-/_\sim$ such that $\widetilde{v} \in \partial^1_+ D^-_{i'}$ and there exists $j' \in I^+/_\sim$ such that $\gamma\widetilde{w} \in \partial^1_- D^+_{j'}$. Then the choice of such elements (i,j), as well as i' and j', is free. We hence have

$$\begin{aligned}
&\sum_{\substack{i\in I^-/_\sim,\; j\in I^+/_\sim,\;\gamma\in\Gamma\\ 0<\ell(\alpha_{i,\,\gamma j})\le t\,,\; v^-_{i,\,\gamma j}=\widetilde{v}\,,\; v^+_{\gamma^{-1}i,\,j}=\widetilde{w}}} e^{\int_{\alpha_{i,\,\gamma j}}\widetilde{F}}\;\Delta_{v^-_{i,\,\gamma j}}\otimes\Delta_{v^+_{\gamma^{-1}i,\,j}}\\
&=\sum_{\substack{\gamma\in\Gamma,\;0<s\le t\\ \gamma\widetilde{w}=g^s\widetilde{v}}} e^{\int_{\pi(\widetilde{v})}^{\gamma\pi(\widetilde{w})}\widetilde{F}}\;\operatorname{Card}\big\{(i,j)\in(I^-/_\sim)\times(I^+/_\sim)\;:\; v^-_{i,\,\gamma j}=\widetilde{v}\,,\; v^+_{\gamma^{-1}i,\,j}=\widetilde{w}\big\}\;\Delta_{\widetilde{v}}\otimes\Delta_{\widetilde{w}}\\
&=\sum_{\substack{\gamma\in\Gamma,\;0<s\le t\\ \gamma\widetilde{w}=g^s\widetilde{v}}} e^{\int_{\pi(\widetilde{v})}^{\gamma\pi(\widetilde{w})}\widetilde{F}}\;m^-(\widetilde{v})\;m^+(\gamma\widetilde{w})\;\Delta_{\widetilde{v}}\otimes\Delta_{\widetilde{w}}\;.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sum_{\substack{i\in I^-/_\sim,\; j\in I^+/_\sim,\;\gamma\in\Gamma\\ 0<\ell(\alpha_{i,\,\gamma j})\le t}} e^{\int_{\alpha_{i,\,\gamma j}}\widetilde{F}}\;\Delta_{v^-_{i,\,\gamma j}}\otimes\Delta_{v^+_{\gamma^{-1}i,\,j}}\\
&=\sum_{\widetilde{v},\,\widetilde{w}\,\in\,T^1\widetilde{M}}\Big(\sum_{\substack{\gamma\in\Gamma,\;0<s\le t\\ \gamma\widetilde{w}=g^s\widetilde{v}}} e^{\int_{\pi(\widetilde{v})}^{\gamma\pi(\widetilde{w})}\widetilde{F}}\Big)\;m^-(\widetilde{v})\;m^+(\widetilde{w})\;\Delta_{\widetilde{v}}\otimes\Delta_{\widetilde{w}}\;.
\end{aligned}$$

By definition, $\sigma^{\pm}_{\mathscr{D}^{\mp}}$ is the measure on T^1M induced by the Γ-invariant measure $\widetilde{\sigma}^{\pm}_{\mathscr{D}^{\mp}}$. Thus Corollary 20 follows from Theorem 14 and Equation (30) (after a similar reduction as in Step 1 of the proof of Theorem 14, and since no compactness assumptions were made on $\Omega^{\pm}$ to get this equation), by Subsection 2.4. □

Remark. Under the assumptions of Corollary 20 except that we now assume that $\delta_F < 0$, by considering a big enough constant σ such that $\delta_{F+\sigma} = \delta_F + \sigma > 0$, by applying Corollary 20 with the potential $F + \sigma$ (see Remark (3) before Proposition 9), and by an easy subdivision and geometric series argument, we have the following asymptotic result as $t \to +\infty$ for the growth of the weighted number of common perpendiculars with lengths in $]t-c, t]$ for every fixed $c > 0$:

$$\mathscr{N}_{\mathscr{D}^-,\,\mathscr{D}^+,\,F}(t) - \mathscr{N}_{\mathscr{D}^-,\,\mathscr{D}^+,\,F}(t-c) \sim \frac{(1-e^{-\delta_F\, c})\,\|\sigma^+_{\mathscr{D}^-}\|\,\|\sigma^-_{\mathscr{D}^+}\|}{\delta_F\,\|m_F\|}\, e^{\delta_F t}\,.$$

Using the continuity of the pushforwards of measures for the weak-star and the narrow topologies, applied to the basepoint maps $\pi\times\pi$ from $T^1\widetilde{M}\times T^1\widetilde{M}$ to $\widetilde{M}\times\widetilde{M}$, and from $T^1M\times T^1M$ to $M\times M$, we have the following result of equidistribution of the ordered pairs of endpoints of common perpendiculars between two equivariant families of convex sets in $\widetilde{M}$ or two families of locally convex sets in M. When M has constant curvature and finite volume, $\mathscr{D}^-$ is the Γ-orbit of a point and $\mathscr{D}^+$ is the Γ-orbit of a totally geodesic cocompact submanifold, this result is due to Herrmann [Her].

Corollary 21 *Let $\widetilde{M},\Gamma,\widetilde{F},\mathscr{D}^-,\mathscr{D}^+$ be as in Theorem 14. Then*

$$\lim_{t\to+\infty}\ \delta_F\,\|m_F\|\,e^{-\delta_F\,t}\sum_{\substack{i\in I^-/_\sim,\ j\in I^+/_\sim,\ \gamma\in\Gamma\\ 0<\ell(\alpha_{i,\,\gamma j})\le t}} e^{\int_{\alpha_{i,\,\gamma j}}\widetilde{F}}\ \Delta_{\pi(v^-_{i,\,\gamma j})}\otimes\Delta_{\pi(v^+_{\gamma^{-1}i,\,j})} = \pi_*\widetilde{\sigma}^+_{\mathscr{D}^-}\otimes\pi_*\widetilde{\sigma}^-_{\mathscr{D}^+}\,,$$

for the weak-star convergence of measures on the locally compact space $\widetilde{M}\times\widetilde{M}$, and

$$\lim_{t\to+\infty}\ \delta_F\,\|m_F\|\,e^{-\delta_F\,t}\sum_{v,\,w\in T^1M} m_{\partial^1_+\mathscr{D}^-}(v)\ m_{\partial^1_-\mathscr{D}^+}(w)\ n_t(v,w)\ \Delta_{\pi(v)}\otimes\Delta_{\pi(w)}$$
$$= \pi_*\sigma^+_{\mathscr{D}^-}\otimes\pi_*\sigma^-_{\mathscr{D}^+}\,,$$

for the weak-star convergence of measures on $M\times M$. If the measures $\sigma^{\pm}_{\mathscr{D}^{\mp}}$ are finite, then the above claim holds for the narrow convergence of measures on $M\times M$. □

Before proving the theorems numbered 1, 2 and 3 in the introduction, we recall the precise definition of a proper nonempty properly immersed closed convex subset $D^{\pm}$ in a negatively curved complete connected Riemannian manifold M: it is a locally geodesic (not necessarily connected) metric space $D^{\pm}$ endowed with a continuous map $f^{\pm}: D^{\pm}\to M$ such that, if $\widetilde{M}\to M$ is a universal covering of M with covering group Γ, if $\widetilde{D}^{\pm}\to D^{\pm}$ is a locally isometric covering map which is a universal covering over each component of $D^{\pm}$, if $\widetilde{f}^{\pm}:\widetilde{D}^{\pm}\to\widetilde{M}$ is a lift of $f^{\pm}$, then $\widetilde{f}^{\pm}$ is, on each connected component of $\widetilde{D}^{\pm}$, an isometric embedding whose image is a proper nonempty closed convex subset of $\widetilde{M}$, and the family of images under Γ of the images by $\widetilde{f}^{\pm}$ of the connected component of $\widetilde{D}^{\pm}$ is locally finite.

Proof of Theorems 1, 2 and 3. Let $I^\pm = \Gamma \times \pi_0(\widetilde{D}^\pm)$ with the action of Γ defined by $\gamma \cdot (\alpha, c) = (\gamma\alpha, c)$ for all $\gamma, \alpha \in \Gamma$ and every component c of $\widetilde{D}^\pm$. Consider the families $\mathscr{D}^\pm = (D_k^\pm)_{k \in I^\pm}$ where $D_k^\pm = \alpha\ \widetilde{f}^\pm(c)$ if $k = (\alpha, c)$. Then $\mathscr{D}^\pm$ are Γ-equivariant families of nonempty proper closed convex subsets of $\widetilde{M}$, which are locally finite since $D^\pm$ are properly immersed in M. The theorems 1 and 2 then follow from Corollary 20. Theorem 3 follows by taking $\Omega = \partial^1_+ D^-$ in Theorem 19. □

Corollary 22 *Let $\widetilde{M}, \Gamma, \widetilde{F}, \mathscr{D}^-, \mathscr{D}^+$ be as in Theorem 14. Assume that $\sigma^\pm_{\mathscr{D}^\mp}$ are finite and nonzero. Then*

$$\lim_{s \to +\infty} \lim_{t \to +\infty} \frac{\delta_F \, \|m_F\|^2 \, e^{-\delta_F t}}{\|\sigma^+_{\mathscr{D}^-}\| \|\sigma^-_{\mathscr{D}^+}\|} \sum_{v \in T^1 M} m_{\partial^1_+ \mathscr{D}^-}(v) \; n_{t,\,\mathscr{D}^+}(v) \; \Delta_{g^s v} = m_F \,,$$

where

$$n_{t,\mathscr{D}^+}(v) = \sum_{w \in T^1 M} m_{\partial^1_- \mathscr{D}^+}(w) \; n_t(v, w)$$

is the number (counted with multiplicities) of locally geodesic paths in M of length at most t, with initial vector v, arriving perpendicularly to $\mathscr{D}^+$.

Proof. For every $s \in \mathbb{R}$, by Corollary 20, using the continuity of the pushforwards of measures by the first projection $(v, w) \mapsto v$ from $T^1M \times T^1M$ to T^1M, and by the geodesic flow on T^1M at time s, since $(g^s)_* \Delta_v = \Delta_{g^s v}$, we have

$$\lim_{t \to +\infty} \delta_F \, \|m_F\| \, e^{-\delta_F t} \sum_{v \in T^1 M} m_{\partial^1_+ \mathscr{D}^-}(v) \; n_{t,\mathscr{D}^+}(v) \; \Delta_{g^s v} \;=\; (g^s)_* \sigma^+_{\mathscr{D}^-} \|\sigma^-_{\mathscr{D}^+}\| \,.$$

The result then follows from Theorem 19 with $\Omega = \partial^1_+ \mathscr{D}^-$. □

4.4 Counting closed subsets of limit sets

In this section, we give counting asymptotics on very general equivariant families of subsets of the limit sets of discrete groups of isometries of rank one symmetric spaces, generalising works of Oh-Shah.

Recall (see for instance [Mos, Par]) that the rank one symmetric spaces are the hyperbolic spaces $\mathbb{H}^n_{\mathbb{F}}$ where $\mathbb{F}$ is the set $\mathbb{R}$ of real numbers, $\mathbb{C}$ of complex numbers, $\mathbb{H}$ of Hamilton's quaternions, or $\mathbb{O}$ of octonions, and $n \geq 2$, with $n = 2$ if $\mathbb{K} = \mathbb{O}$. We will normalise them so that their maximal sectional curvature is -1. We denote the convex hull in $\mathbb{H}^n_{\mathbb{F}}$ of any subset A of $\mathbb{H}^n_{\mathbb{F}} \cup \partial_\infty \mathbb{H}^n_{\mathbb{F}}$ by $\mathscr{C} A$.

We start with $\mathbb{H}^n_{\mathbb{R}}$. The Euclidean diameter of a subset A of the Euclidean space $\mathbb{R}^{n-1}$ is denoted by $\operatorname{diam} A$. For any nonempty subset B of the standard sphere $\mathbb{S}^{n-1}$, we denote by $\theta(B)$ the least upper bound of half the visual angle over pairs of points in B seen from the center of the sphere. Let $\mathscr{H}_\infty$ be the horoball in $\mathbb{H}^n_{\mathbb{R}}$ centred at ∞, consisting of the points with vertical coordinates at least 1. For every Patterson density $(\mu_x)_{x \in \widetilde{M}}$ for a discrete nonelementary group of isometries Γ of $\widetilde{M}$ (with critical exponent δ_Γ) and the potential $F = 0$, for every horoball $\mathscr{H}$ in $\widetilde{M}$, and for every geodesic ray ρ starting from a point of $\partial \mathscr{H}$ and converging to the point at infinity ξ of $\mathscr{H}$, the measure $e^{\delta_\Gamma t} \mu_{\rho(t)}$ converges as t tends to $+\infty$ to a measure $\mu_{\mathscr{H}}$ on $\partial_\infty \widetilde{M} - \{\xi\}$, independent on the choice of

ρ (see [HP2, §2]). Since we consider the potential $F = 0$, we have $\delta_F = \delta_\Gamma$ and $m_F = m_{\rm BM}$ as already seen. We take, as we may, $(\mu^-_x)_{x\in\widetilde{M}} = (\mu^+_x)_{x\in\widetilde{M}}$, which is the reason there is no exponent $\pm$ on the skinning and Patterson measures in the following statement.

Corollary 23 *Let Γ be a discrete nonelementary group of isometries of $\mathbb{H}^n_\mathbb{R}$, with finite Bowen-Margulis measure $m_{\rm BM}$. Let $(F_i)_{i\in I}$ be a Γ-equivariant family of nonempty closed subsets in the limit set $\Lambda\Gamma$, whose family $\mathscr{D}^+ = (\mathscr{C} F_i)_{i\in I}$ of convex hulls in $\mathbb{H}^n_\mathbb{R}$ is locally finite, with finite nonzero skinning measure.*

(1) In the upper halfspace model of $\mathbb{H}^n_\mathbb{R}$, assume that $\Lambda\Gamma$ is bounded in $\mathbb{R}^{n-1} = \partial_\infty\mathbb{H}^n_\mathbb{R} - \{\infty\}$, and that ∞ is not the fixed point of an elliptic element of Γ. Let $\mathscr{D}^-$ be the Γ-equivariant family $(\gamma\mathscr{H}_\infty)_{\gamma\in\Gamma}$. Then, as $T \to +\infty$,

$$\mathrm{Card}\{i \in I/_\sim \;:\; \mathrm{diam}(F_i) \geq 1/T\} \sim \frac{\|\sigma_{\mathscr{D}^-}\|\,\|\sigma_{\mathscr{D}^+}\|}{\delta_\Gamma\|m_{\rm BM}\|}\,(2T)^{\delta_\Gamma}\;.$$

(2) In the unit ball model of $\mathbb{H}^n_\mathbb{R}$, assume that no nontrivial element of Γ fixes 0. As $T \to +\infty$, we have

$$\mathrm{Card}\{i \in I/_\sim \;:\; \cot\theta(F_i) < T\} \sim \frac{\|\mu_0\|\,\|\sigma_{\mathscr{D}^+}\|}{\delta_\Gamma\|m_{\rm BM}\|}\,(2T)^{\delta_\Gamma}\;.$$

(3) In the upper halfspace model of $\mathbb{H}^n_\mathbb{R}$, assume that ∞ is not the fixed point of an elliptic element of Γ. Let Ω be a Borel subset of $\mathbb{R}^{n-1} = \partial_\infty\mathbb{H}^n_\mathbb{R} - \{\infty\}$ such that $\mu_{\mathscr{H}_\infty}(\Omega)$ is finite and positive and $\mu_{\mathscr{H}_\infty}(\partial\Omega) = 0$. Then, as $T \to +\infty$,

$$\mathrm{Card}\{i \in I/_\sim \;:\; \mathrm{diam}(F_i) \geq 1/T,\ F_i \cap \Omega \neq \emptyset\} \sim \frac{\mu_{\mathscr{H}_\infty}(\Omega)\,\|\sigma_{\mathscr{D}^+}\|}{\delta_\Gamma\|m_{\rm BM}\|}\,(2T)^{\delta_\Gamma}\;.$$

This corollary generalises results of Oh-Shah (Theorem 1.4 of [OS1] and Theorem 1.2 of [OS3]) when the subsets F_i are round spheres.

When Γ is an arithmetic lattice, the error term in the claims (1) and (2) is

$$\mathrm{O}(T^{\delta_\Gamma-\kappa})$$

for some $\kappa > 0$, as it follows from Theorem 27 (ii) (using the Riemannian convolution smoothing process of Green and Wu as in [PP4, §3] to smooth by a very small perturbation the boundary of $\mathscr{C} F_i$, so that the perturbation of the lengths of the common perpendiculars and the integrals of the potential along them are uniformly small).

Proof. Note that the Bowen-Margulis measure $m_{\rm BM}$, since finite in a locally symmetric space, is mixing (see for instance [Dal2, page 982]).

(1) Note that the skinning measure $\sigma^+_{\mathscr{D}^-}$ is nonzero since Γ is nonelementary, and finite since the support of $\widetilde{\sigma}^+_{\mathscr{H}_\infty}$, consisting of the points $v \in \partial^1_+\mathscr{H}_\infty$ such that $v_+ \in \Lambda\Gamma$, is compact.

For each $i \in I$, let $x_i, y_i \in F_i$ be points that realise the diameter, that is, $\mathrm{diam}\, F_i = \|x_i - y_i\|$, where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^{n-1}$. The (signed) length $\ell(\alpha_{e,\,i})$ of the common perpendicular $\alpha_{e,\,i}$ from $\mathscr{H}_\infty$ to the geodesic line in $\mathbb{H}^n_\mathbb{R}$ with endpoints x_i and y_i (which is also the common perpendicular from $\mathscr{H}_\infty$ to $\mathscr{C} F_i$) is $\log \frac{2}{\|x_i-y_i\|}$. Thus, since the stabiliser in Γ of an element of $\partial^1_+\mathscr{H}_\infty$ is trivial, and since Γ acts transitively on the index set of the family $\mathscr{D}^-$,

$$\begin{aligned}\mathrm{Card}\{i\in I/_\sim\;:\;\mathrm{diam}(F_i)\geq 1/T\} &= \mathrm{Card}\big\{i\in I/_\sim\;:\;\ell(\alpha_{e,i})\leq\log(2T)\big\}\\ &= \mathscr{N}_{\mathscr{D}^-,\,\mathscr{D}^+,\,0}(\log(2T))\end{aligned}$$

which implies the claim (1) by Corollary 20.

(2) Let $\mathscr{D}^-$ be the Γ-equivariant family $(\{\gamma 0\})_{\gamma\in\Gamma}$, whose skinning measure in $T^1\widetilde{M}$ is $\widetilde{\sigma}^+_{\mathscr{D}^-}=\sum_{\gamma\in\Gamma}\mu^+_{\gamma 0}$, so that $\|\sigma^+_{\mathscr{D}^-}\|$ is equal to the (finite and nonzero) total mass $\|\mu^+_0\|$ of the Patterson measure at 0, since the stabiliser of 0 in Γ is trivial.

For each $i\in I$, let $x_i,y_i\in F_i$ be such that $\theta(F_i)=\theta(\{x_i,y_i\})$. The angle of parallelism formula (see for instance [Bea, p. 147]) implies that $\cot\theta(F_i)=\sinh d(0,\mathscr{C}F_i)$, and the rest of the proof is analogous to that of (1).

(3) Note that we do not assume in (3) that the Γ-equivariant family $\mathscr{D}^-=(\gamma\mathscr{H}_\infty)_{\gamma\in\Gamma}$ is locally finite, and we will only use Equation (30) (and not Corollary 20) to prove the claim (3). One can check that the proof of Equation (30) does not use the local finiteness property of $\mathscr{D}^-$. By applying the definition of the skinning measure $\widetilde{\sigma}^+_{\mathscr{H}_\infty}$ with the base point $x_0=\rho(t)$ where ρ is a geodesic ray starting from a point of $\partial\mathscr{H}_\infty$ and converging to ∞, and letting $t\to+\infty$, we see that the pushforward of the measure $\mu_{\mathscr{H}_\infty}$ by the map $x\mapsto(0,-1)\in T^1_{(x,1)}\mathbb{H}^n_\mathbb{R}$ from $\mathbb{R}^{n-1}$ to $\partial^1_+\mathscr{H}_\infty$ is exactly the skinning measure $\widetilde{\sigma}^+_{\mathscr{H}_\infty}$. If $\mathrm{diam}\,F_i$ is small and F_i meets Ω, then F_i is contained in $\mathscr{N}_\epsilon\Omega$ for some small $\epsilon>0$, and $\mu_{\mathscr{H}_\infty}(\mathscr{N}_\epsilon\Omega)$ converges to $\mu_{\mathscr{H}_\infty}(\Omega)$ as $\epsilon\to 0$. We hence apply Equation (30) with Ω^-_e the image of Ω by this map $x\mapsto(0,-1)$. □

Corollary 5 in the Introduction is a special case of the following corollary. For every parabolic fixed point p of a discrete isometry group Γ of $\mathbb{H}^n_\mathbb{R}$, recall that, by Bieberbach's theorem, the stabiliser of p in Γ contains a subgroup isomorphic to $\mathbb{Z}^k$ with finite index, and $k=\mathrm{rk}_\Gamma(p)\geq 1$ is called the *rank* of p in Γ.

Corollary 24 *Let Γ be a geometrically finite discrete group of isometries of the upper halfspace model of $\mathbb{H}^n_\mathbb{R}$, whose limit set $\Lambda\Gamma$ is bounded in $\mathbb{R}^{n-1}=\partial_\infty\mathbb{H}^n_\mathbb{R}-\{\infty\}$ (endowed with the usual Euclidean distance). Let Γ_0 be a geometrically finite subgroup of Γ with infinite index. Assume that the Hausdorff dimension δ of $\Lambda\Gamma$ is bigger than $\mathrm{rk}_\Gamma(p)-\mathrm{rk}_{\Gamma_0}(p)$ for every parabolic fixed point p of Γ_0. Then, there exists an explicitable $c>0$ such that, as $T\to+\infty$,*

$$\mathrm{Card}\{\gamma\in\Gamma/\Gamma_0\;:\;\mathrm{diam}(\gamma\Lambda\Gamma_0)\geq 1/T\}\sim\;c\,T^\delta\;.$$

The assumption on the ranks of parabolic groups (needed to apply [PP5, Theo. 10]) is in particular satisfied if every maximal parabolic sugbroup of Γ_0 has finite index in the maximal parabolic subgroup of Γ containing it, as well as when $n=3$ and $\delta>1$ (or equivalently if Γ does not contain a Fuchsian group with index at most 2, when $\Lambda\Gamma$ is not totally disconnected, see [CaT, Theo. 3 (3)]).

Proof. First assume that ∞ is not fixed by an elliptic element of Γ. Since Γ is geometrically finite, its Bowen-Margulis measure is finite (see for instance [DOP]). The critical exponent δ_Γ of Γ is equal to the Hausdorff dimension δ of $\Lambda\Gamma$. Let Γ'_0 be the stabiliser of the limit set $\Lambda\Gamma_0$ of Γ_0, and recall that Γ_0 has finite index in Γ'_0 (see for instance [Kap, Coro. 4.136]). Let us consider $I=\Gamma$, the family $(F_i=i\Lambda\Gamma_0)_{i\in I}$ (which consists of nonempty closed subsets of $\Lambda\Gamma$), and $\mathscr{D}^+ = (\mathscr{C}F_i)_{i\in I}$ (which is locally finite), so that $I/_\sim = \Gamma/\Gamma'_0$. Since Γ_0 is geometrically finite, the convex set $\mathscr{C}\Lambda\Gamma_0$ is almost cone-like in cusps and any parabolic subgroup of Γ has regular growth (see the definitions in [PP5, Sect. 4]). Hence, under the hypothesis on the ranks of parabolic groups, by [PP5, Theo. 10], the skinning measure $\sigma^-_{\mathscr{D}^+}$ is finite. It is nonzero by Proposition 9 (iv), since $\Lambda\Gamma_0 \neq \Lambda\Gamma$ as Γ_0 has infinite index (as seen above). Note that

$$\operatorname{Card}\{\gamma \in \Gamma/\Gamma_0 \;:\; \operatorname{diam}(\gamma\Lambda\Gamma_0) \geq 1/T\}$$
$$= [\Gamma'_0 : \Gamma_0]\operatorname{Card}\{\gamma \in \Gamma/\Gamma'_0 \;:\; \operatorname{diam}(\gamma\Lambda\Gamma_0) \geq 1/T\}\,.$$

The result then follows from Corollary 23 (1).

Now, if ∞ is fixed by an elliptic element of Γ, let Γ' be a finite-index torsion-free subgroup of Γ (in particular Γ' is geometrically finite, and $\Lambda\Gamma' = \Lambda\Gamma$ is bounded, with Hausdorff dimension δ). The action by left translations of Γ' on Γ/Γ_0 has only finitely many (pairwise distinct) orbits, say $\alpha_1\Gamma_0, \dots, \alpha_k\Gamma_0$. For $i = 1, \dots, k$, the group $\Gamma'_i = \alpha_i\Gamma_0\alpha_i^{-1} \cap \Gamma'$ is geometrically finite with infinite index in Γ'. Let $A(T) = \{\gamma\Gamma_0 \in \Gamma/\Gamma_0 \;:\; \operatorname{diam}(\gamma\Lambda\Gamma_0) \geq 1/T\}$ and $A_i(T) = \{\gamma\Gamma_0 \in A(T) \;:\; \Gamma'\gamma\Gamma_0 = \Gamma'\alpha_i\Gamma_0\}$ for $i = 1, \dots, k$, so that $\operatorname{Card} A(T) = \sum_{i=1}^k \operatorname{Card} A_i(T)$. The map $\gamma\Gamma_0 \to \gamma'\Gamma'_i$ from $A_i(T)$ to $\{\gamma'\Gamma'_i \in \Gamma'/\Gamma'_i \;:\; \operatorname{diam}(\gamma'\Lambda\Gamma'_i) \geq 1/T\}$ where $\gamma' \in \Gamma'$ satisfies $\gamma\Gamma_0 = \gamma'\alpha_i\Gamma_0$ is easily seen to be well-defined and a bijection. Note that the Hausdorff dimension δ of $\Lambda\Gamma' = \Lambda\Gamma$ is bigger than $\operatorname{rk}_{\Gamma'}(p) - \operatorname{rk}_{\Gamma'_i}(p) = \operatorname{rk}_{\Gamma}(\alpha_i^{-1}p) - \operatorname{rk}_{\Gamma_0}(\alpha_i^{-1}p)$ for every parabolic fixed point p of Γ'_i, since $\alpha_i^{-1}p$ is a parabolic fixed point of Γ_0. By the above torsion-free case, for $i = 1, \dots, k$, there exists $c_i > 0$ such that $\operatorname{Card} A_i(T) \sim c_i\, T^\delta$ as $T \to +\infty$. The result then follows with $c = \sum_{i=1}^k c_i$. □

Corollary 25 *Let Γ be a geometrically finite discrete group of $\operatorname{PSL}_2(\mathbb{C})$ with bounded and not totally disconnected limit set in $\mathbb{C}$, which does not contain a quasifuchsian subgroup with index at most 2. Then there exists $c > 0$ such that the number of connected components of the domain of discontinuity $\Omega\Gamma$ of Γ with diameter at least $1/T$ is equivalent, as $T \to +\infty$, to $c\,T^\delta$ where δ is the Hausdorff dimension of the limit set of Γ.*

When ∞ is not the fixed point of an elliptic element of Γ (for instance if Γ is torsion free), we have

$$c = \frac{2^\delta\, \|\sigma_{\mathscr{D}^-}\|}{\delta\, \|m_{\rm BM}\|} \sum_{\Omega} \|\sigma_{\widehat{\Omega}}\|$$

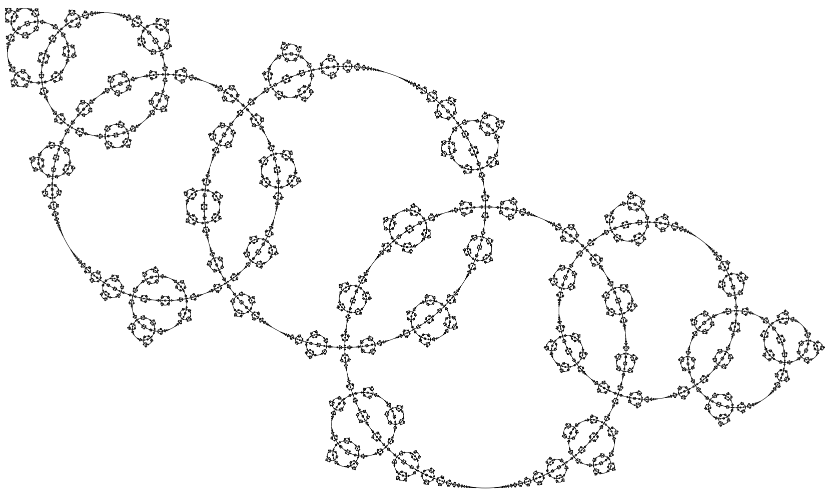
with $\mathscr{D}^- = (\gamma\mathscr{H}_\infty)_{\gamma\in\Gamma}$ and $\widehat{\Omega} = (\gamma\mathscr{C}\Omega)_{\gamma\in\Gamma}$, where Ω ranges over a set of representatives of the orbits under Γ of the connected components of $\Omega\Gamma$ whose stabiliser has infinite index in Γ.

Proof. As mentionned after Corollary 24, we have $\delta > 1$, hence the assumption of this corollary on the ranks of parabolic groups is satisfied. By Ahlfors's finiteness theorem, the domain of discontinuity $\Omega\Gamma$ of Γ (which is a finitely generated Kleinian group) has only finitely many orbits of connected components (see for instance [Kap, Coro. 4.108]). Since Γ is geometrically finite, the stabiliser of a component of $\Omega\Gamma$ is again geometrically finite (see for instance [Kap, Coro. 4.112]). The components of $\Omega\Gamma$ which are stabilised by a finite index subgroup of Γ do not contribute to the asymptotics.

The assumptions on Γ imply that there exists at least one other component of $\Omega\Gamma$. Otherwise indeed, the stabiliser of every component Ω of $\Omega\Gamma$ has finite index in Γ, and in particular $\partial\Omega = \Lambda\Gamma$. Up to taking a finite index subgroup, we may assume that Γ is a function group (that is, leaves invariant a component of $\Omega\Gamma$). By [MaT, Theo. 4.36], Γ is a Klein combination of B-groups (that is, preserving a simply connected component of their domain of discontinuity) and elementary groups. Since $\Lambda\Gamma$ is not totally disconnected and since $\partial\Omega = \Lambda\Gamma$ for all components Ω of $\Omega\Gamma$, this implies that Γ is a B-group. By the structure theorem of geometrically finite B-groups (see [Abi, Theo. 8]), this implies that Γ is quasifuchsian, a contradiction.

The stabilisers of these other components of $\Omega\Gamma$ have infinite index in Γ. Hence the result follows from Corollary 24, by a finite summation. $\square$

For example, Corollary 25 gives an expression for the asymptotic number of the components of the domain of discontinuity with diameter less than $\frac{1}{T}$ as $T \to \infty$ of the crossed Fuchsian group generated by two Fuchsian groups, using the terminology of Chapter VIII §E.8 of [Mas], as in the figure below, produced using McMullen's program `lim`. See for example Maskit's combination theorem in loc. cit. for a proof that crossed Fuchsian groups are geometrically finite.



We now consider $\mathbb{H}^n_{\mathbb{C}}$, leaving to the reader the extension to the other rank one symmetric spaces. We denote by $(w', w) \mapsto w' \cdot w = \sum_{i=1}^{n-1} w'_i \, \overline{w_i}$ the usual Hermitian product on $\mathbb{C}^{n-1}$, and $|w| = \sqrt{w \cdot w}$. Let

$$\mathbb{H}^n_{\mathbb{C}} = \big\{(w_0, w) \in \mathbb{C} \times \mathbb{C}^{n-1} \;:\; 2\,\mathrm{Re}\, w_0 - |w|^2 > 0\big\}\,,$$

endowed with the Riemannian metric (normalised as in the beginning of Section 4.4)

$$ds^2 = \frac{1}{(2\,\mathrm{Re}\, w_0 - |w|^2)^2}\big((dw_0 - dw \cdot w)((\overline{dw_0} - w \cdot dw) + (2\,\mathrm{Re}\, w_0 - |w|^2)\, dw \cdot dw\big)\,,$$

be the Siegel domain model of the complex hyperbolic n-space (see [Gol, Sect. 4.1]). Let

$$\mathscr{H}_\infty = \{(w_0, w) \in \mathbb{C} \times \mathbb{C}^{n-1} \ : \ 2 \operatorname{Re} w_0 - |w|^2 \geq 2\}\,,$$

which is a horoball centred at ∞. The manifold

$$\mathrm{Heis}_{2n-1} = \partial_\infty \mathbb{H}^n_{\mathbb{C}} - \{\infty\} = \{(w_0, w) \in \mathbb{C} \times \mathbb{C}^{n-1} \ : \ 2 \operatorname{Re} w_0 - |w|^2 = 0\}$$

is a Lie group (isomorphic to the $(2n-1)$-dimensional Heisenberg group) for the law

$$(w_0, w) \cdot (w'_0, w') = (w_0 + w'_0 + w \cdot w', w + w')\,.$$

The *Cygan distance* d_{Cyg} (see [Gol, page 160]) and the *modified Cygan distance* d'_{Cyg} (introduced in [PP1, Lem. 6.1]) are the unique left-invariant distances on Heis_{2n-1} with

$$d_{\mathrm{Cyg}}((w_0, w), (0, 0)) = \sqrt{2|w_0|}\,, \quad d'_{\mathrm{Cyg}}((w_0, w), (0, 0)) = \sqrt{2|w_0| + |w|^2}\,.$$

Let $d''_{\mathrm{Cyg}} = \frac{{d_{\mathrm{Cyg}}}^2}{d'_{\mathrm{Cyg}}}$, which, since $d_{\mathrm{Cyg}} \leq d'_{\mathrm{Cyg}} \leq \sqrt{2}\, d_{\mathrm{Cyg}}$, is almost a distance on Heis_{2n-1}. For every nonempty subset A of Heis_{2n-1}, we denote by

$$\mathrm{diam}_{d''_{\mathrm{Cyg}}}(A) = \max_{x,y \in A\,:\, x \neq y} d''_{\mathrm{Cyg}}(x, y)$$

the diameter of A for this almost distance.

Corollary 26 *Let Γ be a discrete nonelementary group of isometries of the Siegel domain model of $\mathbb{H}^n_{\mathbb{C}}$, with finite Bowen-Margulis measure m_{BM}. Assume that $\Lambda\Gamma$ is bounded in* Heis_{2n-1}*, and that ∞ is not the fixed point of an elliptic element of Γ. Let $\mathscr{D}^-$ be the Γ-equivariant family $(\gamma\mathscr{H}_\infty)_{\gamma \in \Gamma}$. Let $(F_i)_{i \in I}$ be a Γ-equivariant family of nonempty closed subsets in $\Lambda\Gamma$, whose family $\mathscr{D}^+ = (\mathscr{C} F_i)_{i \in I}$ of convex hulls in $\mathbb{H}^n_{\mathbb{C}}$ is locally finite, with finite nonzero skinning measure. Then, as $T \to +\infty$,*

$$\mathrm{Card}\{i \in I/_{\sim} \ : \ \mathrm{diam}_{d''_{\mathrm{Cyg}}}(F_i) \geq 1/T\} \sim \frac{\|\sigma_{\mathscr{D}^-}\|\,\|\sigma_{\mathscr{D}^+}\|}{\delta_\Gamma \|m_{\mathrm{BM}}\|}\,(2T)^{\delta_\Gamma}\,.$$

The proof of this corollary is similar to the one of Corollary 23 (1), and has a similar corollary as Corollary 24 (replacing the rank of a parabolic fixed point by twice the critical exponent of its stabiliser), since

- the (signed) length in $\mathbb{H}^n_{\mathbb{C}}$ of the common perpendicular from $\mathscr{H}_\infty$ to a geodesic in $\mathbb{H}^n_{\mathbb{C}}$ with endpoints $x, y \in \mathrm{Heis}_{2n-1}$ is $\log \frac{2}{d''_{\mathrm{Cyg}}(x,y)}$ by [PP2, Lem. 3.4];
- the critical exponent of a geometrically finite group Γ of isometries of $\mathbb{H}^n_{\mathbb{C}}$ is the Hausdorff dimension of $\Lambda\Gamma$ for anyone of the (almost) distance $d_{\mathrm{Cyg}}, d'_{\mathrm{Cyg}}, d''_{\mathrm{Cyg}}$.

5 Error terms

Let $\widetilde{M}$, x_0, Γ and M be as in the beginning of Section 2. Let $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ be a Γ-invariant Hölder-continuous function, and let $F : T^1M = \Gamma\backslash T^1\widetilde{M} \to \mathbb{R}$ be its quotient map. We assume that the Gibbs measure m_F is finite, and we define $\overline{m_F} = \frac{m_F}{\|m_F\|}$.

In this section, we give bounds for the error term in the equidistribution and counting results of the previous section when the geodesic flow is exponentially mixing and the (strong) stable and unstable foliations are assumed to be at least Hölder-continuous.

There are two types of exponential mixing results available in this context. Firstly, when $\widetilde{M}$ is a symmetric space, then the boundary at infinity of $\widetilde{M}$, the strong unstable, unstable, stable, and strong stable foliations of $T^1\widetilde{M}$ are smooth. Hence talking about leafwise $\mathscr{C}^\ell$-smooth functions on T^1M makes sense. We will denote by $\mathscr{C}_c^\ell(T^1M)$ the vector space of $\mathscr{C}^\ell$-smooth functions on T^1M with compact support and by $\|\psi\|_\ell$ the Sobolev $W^{\ell,2}$-norm of any $\psi \in \mathscr{C}_c^\ell(T^1M)$.

Given $\ell \in \mathbb{N}$, we will say that the geodesic flow on T^1M is *exponentially mixing for the Sobolev regularity* ℓ (or that it has *exponential decay of ℓ-Sobolev correlations*) for the potential F if there exist $c, \kappa > 0$ such that for all $\phi, \psi \in \mathscr{C}_c^\ell(T^1M)$ and all $t \in \mathbb{R}$, we have

$$\Big|\int_{T^1M} \phi \circ g^{-t}\, \psi\; d\overline{m}_F - \int_{T^1M} \phi\; d\overline{m}_F \int_{T^1M} \psi\; d\overline{m}_F\Big| \leq c\, e^{-\kappa|t|}\, \|\psi\|_\ell\, \|\phi\|_\ell\,.$$

When $F = 0$ and Γ is an arithmetic lattice in the isometry group of $\widetilde{M}$ (the Gibbs measure then coincides, up to a multiplicative constant, with the Liouville measure), this property, for some $\ell \in \mathbb{N}$, follows from [KM1, Theorem 2.4.5], with the help of [Clo, Theorem 3.1] to check its spectral gap property, and of [KM2, Lemma 3.1] to deal with finite cover problems. Also note that when $F = 0$ and M has finite volume, the conditional measures on the strong stable/unstable leaves are homogeneous, hence $(\widetilde{M}, \Gamma, \widetilde{F})$ has radius-Hölder-continuous strong stable/unstable ball masses.

Secondly, when $\widetilde{M}$ has pinched negative sectional curvature with bounded derivatives, then the boundary at infinity of $\widetilde{M}$, the strong unstable, unstable, stable, and strong stable foliations of $T^1\widetilde{M}$ are only Hölder-smooth (see for instance [Bri] when $\widetilde{M}$ has a compact quotient (a result first proved by Anosov), and [PPS, Theo. 7.3]). Hence the appropriate regularity on functions on $T^1\widetilde{M}$ is the Hölder one. For every $\alpha \in \,]0,1[\,$, we denote by $\mathscr{C}_c^\alpha(X)$ the space of α-Hölder-continuous real-valued functions with compact support on a metric space (X, d), endowed with the Hölder norm

$$\|f\|_\alpha = \|f\|_\infty + \sup_{x,\, y \in X,\; x \neq y} \frac{|f(x) - f(y)|}{d(x,y)^\alpha}\,.$$

Given $\alpha \in \,]0,1[$, we will say that the geodesic flow on T^1M is *exponentially mixing for the Hölder regularity* α (or that it has *exponential decay of α-Hölder correlations*) for the potential F if there exist $c, \kappa > 0$ such that for all $\phi, \psi \in \mathscr{C}_c^\alpha(T^1M)$ and all $t \in \mathbb{R}$, we have

$$\Big|\int_{T^1M} \phi \circ g^{-t}\, \psi\; d\overline{m}_F - \int_{T^1M} \phi\; d\overline{m}_F \int_{T^1M} \psi\; d\overline{m}_F\Big| \leq c\, e^{-\kappa|t|}\, \|\phi\|_\alpha\, \|\psi\|_\alpha\,.$$

This holds for compact manifolds M when M is two-dimensional and F is any Hölder potential by [Dol], when M is 1/9-pinched and $F = 0$ by [GLP, Coro. 2.7], when m_F is the Liouville measure by [Liv], and when M is locally symmetric and F is any Hölder potential by [Sto]. See also [MO] for the first important case of infinite covolume.

Theorem 27 *Let $\widetilde{M}$ be a complete simply connected Riemannian manifold with negative sectional curvature. Let Γ be a nonelementary discrete group of isometries of $\widetilde{M}$. Let $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ be a bounded Γ-invariant Hölder-continuous function with positive critical exponent δ_F. Let $\mathscr{D} = (D_i)_{i \in I}$ be a locally finite Γ-equivariant family of nonempty proper closed convex subsets of $\widetilde{M}$, with finite nonzero skinning measure $\sigma_{\mathscr{D}}$. Let $M = \Gamma\backslash\widetilde{M}$ and let $F : T^1M \to \mathbb{R}$ be the potential induced by $\widetilde{F}$.*

(i) If M is compact and if the geodesic flow on T^1M is mixing with exponential speed for the Hölder regularity for the potential F, then there exist $\alpha \in \,]0,1[$ and $\kappa'' > 0$ such that for all $\psi \in \mathscr{C}_{\rm c}^{\alpha}(T^1M)$, we have, as $t \to +\infty$,

$$\frac{1}{\|\sigma_{\mathscr{D}}\|} \int \psi \; d(g^t)_*\sigma_{\mathscr{D}} = \frac{1}{\|m_F\|} \int \psi \; dm_F + \mathrm{O}(e^{-\kappa'' t} \; \|\psi\|_\alpha) \; .$$

(ii) If $\widetilde{M}$ is a symmetric space, if D_i has smooth boundary for every $i \in I$, if m_F is finite and smooth, and if the geodesic flow on T^1M is mixing with exponential speed for the Sobolev regularity for the potential F, then there exists $\ell \in \mathbb{N}$ and $\kappa'' > 0$ such that for all $\psi \in \mathscr{C}_{\rm c}^{\ell}(T^1M)$, we have, as $t \to +\infty$,

$$\frac{1}{\|\sigma_{\mathscr{D}}\|} \int \psi \; d(g^t)_*\sigma_{\mathscr{D}} = \frac{1}{\|m_F\|} \int \psi \; dm_F + \mathrm{O}(e^{-\kappa'' t} \; \|\psi\|_\ell) \; .$$

Note that if $\widetilde{M}$ is a symmetric space, if M has finite volume and if F is small enough, then m_F is finite, as seen at the end of Section 3.1. The maps $\mathrm{O}(\cdot)$ depend on $\widetilde{M}, \Gamma, F, \mathscr{D}$, and the speeds of mixing.

Proof. Up to rescaling, we may assume that the sectional curvature is bounded from above by -1. The critical exponent δ_F and the Gibbs measure m_F are finite in all the considered cases.

The deduction of this result from the proof of Theorem 19 by regularisations of the global test function ϕ_η introduced in the proof of Theorem 19 is analogous to the deduction of [PP5, Theo. 20] from [PP5, Theo. 19] when $F = 0$. The doubling property of the Patterson densities and the Gibbs measure for general F, required by this deduction in the Hölder regularity case, is given by [PPS, Prop. 3.12]. For the assertion (ii), the required smoothness of m_F (that is, the fact that m_F is absolutely continuous with respect to the Lebesgue measure with smooth Radon-Nikodym derivative) allows to use the convolution approximation. $\square$

Theorem 28 *Let $\widetilde{M}$ be a complete simply connected Riemannian manifold with negative sectional curvature at most -1. Let Γ be a nonelementary discrete group of isometries of $\widetilde{M}$. Let $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ be a bounded Γ-invariant Hölder-continuous function with positive critical exponent δ_F. Assume that $(\widetilde{M}, \Gamma, \widetilde{F})$ has radius-Hölder-continuous strong stable/unstable ball masses. Let $\mathscr{D}^- = (D_i^-)_{i \in I^-}$ and $\mathscr{D}^+ = (D_j^+)_{j \in I^+}$ be locally finite Γ-equivariant families of nonempty proper closed convex subsets of $\widetilde{M}$, with finite nonzero skinning measure $\sigma_{\mathscr{D}^-}$ and $\sigma_{\mathscr{D}^+}$. Let $M = \Gamma \backslash \widetilde{M}$ and let $F : T^1M \to \mathbb{R}$ be the potential induced by $\widetilde{F}$.*

(1) Assume that M is compact and that the geodesic flow on T^1M is mixing with exponential speed for the Hölder regularity for the potential F. Then there exist $\alpha \in \,]0,1[$ and $\kappa' > 0$ such that for all nonnegative $\psi^\pm \in \mathscr{C}_{\rm c}^{\alpha}(T^1M)$, we have, as $t \to +\infty$,

$$\frac{\delta_F \; \|m_F\|}{e^{\delta_F t}} \sum_{v,\, w \in T^1M} m_{\partial^1_+ \mathscr{D}^-}(v) \; m_{\partial^1_- \mathscr{D}^+}(w) \; n_t(v,w) \; \psi^-(v) \, \psi^+(w)$$
$$= \int_{T^1M} \psi^- d\sigma^+_{\mathscr{D}^-} \int_{T^1M} \psi^+ d\sigma^-_{\mathscr{D}^+} + \mathrm{O}(e^{-\kappa' t} \|\psi^-\|_\alpha \, \|\psi^+\|_\alpha) \, .$$

(2) Assume that $\widetilde{M}$ is a symmetric space, that $D_k^\pm$ has smooth boundary for every $k \in I^\pm$, that m_F is finite and smooth, and that the geodesic flow on T^1M is mixing with exponential speed for the Sobolev regularity for the potential F. Then there exist $\ell \in \mathbb{N}$ and $\kappa' > 0$ such that for all nonnegative maps $\psi^\pm \in \mathscr{C}_c^\ell(T^1M)$, we have, as $t \to +\infty$,

$$\frac{\delta_F\,\|m_F\|}{e^{\delta_F t}} \sum_{v,\,w \in T^1M} m_{\partial^1_+\mathscr{D}^-}(v)\; m_{\partial^1_-\mathscr{D}^+}(w)\; n_t(v,w)\; \psi^-(v)\,\psi^+(w)$$
$$= \int_{T^1M} \psi^-\, d\sigma^+_{\mathscr{D}^-} \int_{T^1M} \psi^+ d\sigma^-_{\mathscr{D}^+} + \mathrm{O}(e^{-\kappa' t}\|\psi^-\|_\ell\,\|\psi^+\|_\ell)\,.$$

Furthermore, if $\mathscr{D}^-$ and $\mathscr{D}^+$ respectively have nonzero finite outer and inner skinning measures, if $(\widetilde{M},\Gamma,\widetilde{F})$ satisfies the conditions of (1) or (2) above, then there exists $\kappa'' > 0$ such that, as $t \to +\infty$,

$$\mathscr{N}_{\mathscr{D}^-,\,\mathscr{D}^+,\,F}(t) = \frac{\|\sigma^+_{\mathscr{D}^-}\|\;\|\sigma^-_{\mathscr{D}^+}\|}{\delta_F\;\|m_F\|}\; e^{\delta_F t}\big(1 + \mathrm{O}(e^{-\kappa'' t})\big)\,.$$

The maps $\mathrm{O}(\cdot)$ depend on $\widetilde{M}, \Gamma, F, \mathscr{D}$, and the speeds of mixing.

Proof. We will follow the proofs of Theorem 14 and Corollary 20 to prove generalizations of the assertions (1) and (2) by adding to these proofs a regularisation process of the test functions $\phi^\pm_\eta$ as for the deduction of [PP5, Theo. 20] from [PP5, Theo. 19]. We will then deduce the last statement of Theorem 28 from these generalisations, again by using this regularisation process.

Let β be either $\alpha \in \,]0,1]$ in the Hölder regularity case or $\ell \in \mathbb{N}$ in the Sobolev regularity case. We fix $i \in I^-$, $j \in I^+$, and we use the notation $D^\pm, \alpha_\gamma, \ell_\gamma, v^\pm_\gamma$ and $\widetilde{\sigma}^\pm$ of Equation (29). Let $\widetilde{\psi}^\pm \in \mathscr{C}^\beta(\partial^1_\mp D^\pm)$ be such that $\int_{T^1\widetilde{M}} \widetilde{\psi}^\pm\, d\widetilde{\sigma}^\mp_{\mathscr{D}^\pm}$ is finite. Under the assumptions of Assertion (1) or (2), we first prove the following avatar of Equation (30), indicating only the required changes in its proof: there exists $\kappa_0 > 0$ (independent of $\widetilde{\psi}^\pm$) such that, as $T \to +\infty$,

$$\delta_F\;\|m_F\|\; e^{-\delta_F\,T} \sum_{\gamma \in \Gamma,\, 0 < \ell_\gamma \le T} e^{\int_{\alpha_\gamma} \widetilde{F}}\; \widetilde{\psi}^-(v^-_\gamma)\,\widetilde{\psi}^+(v^+_\gamma)$$
$$= \int_{\partial^1_+ D^-} \widetilde{\psi}^-\; d\widetilde{\sigma}^+ \int_{\partial^1_- D^+} \widetilde{\psi}^+\; d\widetilde{\sigma}^- + \mathrm{O}(e^{-\kappa_0 T}\|\widetilde{\psi}^-\|_\beta\,\|\widetilde{\psi}^+\|_\beta)\,. \tag{52}$$

By Lemma 7 and the Hölder regularity of the strong stable and unstable foliations under the assumptions of Assertion (1), or by the smoothness of the boundary of $D^\pm$ under the assumptions of Assertion (2), the maps $f^\pm_{D^\mp} : \mathscr{V}^\pm_{\eta,R}(\partial^1_\pm D^\mp) \to \partial^1_\pm D^\mp$ are respectively Hölder-continuous or smooth fibrations, whose fiber over $w \in \partial^1_\pm D^\mp$ is exactly $V^\pm_{w,\,\eta,\,R}$. By applying leafwise the regularisation process described in the proof of [PP5, Theo. 20] to characteristic functions, there exists a constant $\kappa_1 > 0$ and $\chi^\pm_{\eta,\,R} \in \mathscr{C}^\beta(T^1\widetilde{M})$ such that

- $\|\chi^\pm_{\eta,\,R}\|_\beta = \mathrm{O}(\eta^{-\kappa_1})$,
- $\mathbb{1}_{\mathscr{V}^\pm_{\eta\, e^{-\mathrm{O}(\eta)},\, R\, e^{-\mathrm{O}(\eta)}}(\partial^1_\mp D^\pm)} \le \chi^\pm_{\eta,\,R} \le \mathbb{1}_{\mathscr{V}^\pm_{\eta,\,R}(\partial^1_\mp D^\pm)}$,
- for every $w \in \partial^1_\mp D^\pm$, we have

$$\int_{\mathscr{V}^\mp_{w,\,\eta,\,R}} \chi^\pm_{\eta,\,R}\; d\nu^\pm_w = \nu^\pm_w(\mathscr{V}^\mp_{w,\eta,R})\, e^{-\,\mathrm{O}(\eta)} = \nu^\pm_w(\mathscr{V}^\mp_{w,\,\eta\, e^{-\,\mathrm{O}(\eta)},\, R\, e^{-\,\mathrm{O}(\eta)}})\; e^{\mathrm{O}(\eta)}\,.$$

We now define the new test functions (compare with the second step of the proof of Theorem 14). For every $w \in \partial^1_\mp D^\pm$, let

$$H^\pm_{\eta,\,R}(w) = \frac{1}{\int_{\mathscr{V}^\mp_{w,\,\eta,\,R}} \chi^\pm_{\eta,\,R}\, d\nu^\pm_w}\,.$$

Let $\Phi^\pm_\eta : T^1\widetilde{M} \to \mathbb{R}$ be the map defined by

$$\Phi^\pm_\eta = (H^\pm_{\eta,\,R}\;\widetilde{\psi}^\pm) \circ f^\mp_{D^\pm}\;\;\chi^\pm_{\eta,\,R}\,.$$

The support of this map is contained in $\mathscr{V}^\pm_{\eta,\,R}(\partial^1_\mp D^\pm)$. Since M is compact in Assertion (1) and by homogeneity in Assertion (2), if R is big enough, by the definitions of the measures $\nu^\pm_w$, the denominator of $H^\pm_{\eta,\,R}(w)$ is a least $c\,\eta$ where $c > 0$. The map $H^\pm_{\eta,\,R}$ is hence Hölder continuous under the assumptions of Assertion (1), and is smooth under the assumptions of Assertion (2). Therefore $\Phi^\pm_\eta \in \mathscr{C}^\beta(T^1\widetilde{M})$ and there exists a constant $\kappa_2 > 0$ such that

$$\|\Phi^\pm_\eta\|_\beta = \operatorname{O}(\eta^{-\kappa_2}\|\widetilde{\psi}^\pm\|_\beta)\,.$$

As in Lemma 15, the functions $\Phi^\mp_\eta$ are measurable, nonnegative and satisfy

$$\int_{T^1\widetilde{M}} \Phi^\mp_\eta \; d\widetilde{m}_F = \int_{\partial^1_\mp D^\pm} \widetilde{\psi}^\pm \, d\widetilde{\sigma}^\mp\,.$$

As in the conclusion of the second step of the proof of Theorem 14, we will estimate in two ways the quantity

$$I_\eta(T) = \int_0^T e^{\delta_F\, t} \sum_{\gamma\in\Gamma} \int_{T^1\widetilde{M}} (\Phi^-_\eta \circ g^{-t/2})\;(\Phi^+_\eta \circ g^{t/2} \circ \gamma^{-1})\; d\widetilde{m}_F\; dt\,. \tag{53}$$

We first apply the mixing property, now with exponential decay of correlations, as in the third step of the proof of Theorem 14. For all $t \geq 0$, let

$$A_\eta(t) = \sum_{\gamma\in\Gamma} \int_{v\in T^1\widetilde{M}} \Phi^-_\eta(g^{-t/2}v)\;\Phi^+_\eta(g^{t/2}\gamma^{-1}v)\; d\widetilde{m}_F(v)\,.$$

Then with $\kappa > 0$ as in the definitions of the exponential mixing for the Hölder or Sobolev regularity, we have

$$\begin{aligned} A_\eta(t) &= \frac{1}{\|m_F\|} \int_{T^1\widetilde{M}} \Phi^-_\eta \, d\widetilde{m}_F \int_{T^1\widetilde{M}} \Phi^+_\eta \, d\widetilde{m}_F \;+\; \operatorname{O}(e^{-\kappa\, t}\|\Phi^-_\eta\|_\beta\|\Phi^+_\eta\|_\beta)\\ &= \frac{1}{\|m_F\|} \int_{\partial^1_+ D^-} \widetilde{\psi}^- \, d\widetilde{\sigma}^+ \int_{\partial^1_- D^+} \widetilde{\psi}^+ \, d\widetilde{\sigma}^- \;+\; \operatorname{O}(e^{-\kappa\, t}\eta^{-2\kappa_2}\|\widetilde{\psi}^-\|_\beta\|\widetilde{\psi}^+\|_\beta)\,. \end{aligned}$$

Hence by integrating,

$$I_\eta(T) = \frac{e^{\delta_F\, T}}{\delta_F\,\|m_F\|}\Big(\int_{\partial^1_+ D^-} \widetilde{\psi}^- \, d\widetilde{\sigma}^+ \int_{\partial^1_- D^+} \widetilde{\psi}^+ \, d\widetilde{\sigma}^- \;+\; \operatorname{O}(e^{-\kappa\, T}\eta^{-2\kappa_2}\|\widetilde{\psi}^-\|_\beta\|\widetilde{\psi}^+\|_\beta)\Big)\,. \tag{54}$$

Now, as in the fourth step of the proof of Theorem 14, we exchange the integral over t and the summation over γ in the definition of $I_\eta(T)$, and we estimate the integral term independently of γ:

$$I_\eta(T)=\sum_{\gamma\in\Gamma}\int_0^T e^{\delta_F\, t}\int_{T^1\widetilde{M}}(\Phi_\eta^-\circ g^{-t/2})\,(\Phi_\eta^+\circ g^{t/2}\circ\gamma^{-1})\;d\widetilde{m}_F\;dt\,.$$

Let $\widehat{\Phi}_\eta^\pm=H^\pm_{\eta,\,R}\circ f^\mp_{D^\pm}\,\chi^\pm_{\eta,\,R}$, so that $\Phi^\pm_\eta=\widetilde{\psi}^\pm\circ f^\mp_{D^\pm}\,\widehat{\Phi}^\pm_\eta$. By the last two properties of the regularised maps $\chi^\pm_{\eta,\,R}$, we have, with $\phi^\mp_\eta$ defined as in Equation (35),

$$\phi^\pm_{\eta\, e^{-\operatorname{O}(\eta)},\,R\,e^{-\operatorname{O}(\eta)},\,\partial^1_\mp D^\pm}\;e^{-\operatorname{O}(\eta)}\le\widehat{\Phi}^\pm_\eta\le\phi^\pm_\eta\,e^{\operatorname{O}(\eta)}\;.\tag{55}$$

If $v\in T^1\widetilde{M}$ belongs to the support of $(\Phi_\eta^-\circ g^{-t/2})\,(\Phi_\eta^+\circ g^{t/2}\circ\gamma^{-1})$, then we have $v\in g^{t/2}\mathscr{V}^+_{\eta,R}(\partial^1_+D^-)\cap g^{-t/2}\mathscr{V}^-_{\eta,R}(\gamma\partial^1_-D^+)$. Hence the properties (i), (ii) and (iii) of the fourth step of the proof of Theorem 14 still hold (with $\Omega_-=\partial^1_+D^-$ and $\Omega_+=\partial^1_-(\gamma D^+)$). In particular, if $w^-=f^+_{D^-}(v)$ and $w^+=f^-_{\gamma D^+}(v)$, we have, as in the fifth step of the proof of Theorem 14, that

$$d(w^\pm,v^\pm_\gamma)=\operatorname{O}(\eta+e^{-\ell_\gamma/2})\,.$$

Hence, with $\kappa_3=\alpha$ in the Hölder case and $\kappa_3=1$ in the Sobolev case (we may assume that $\ell\ge1$), we have

$$|\,\widetilde{\psi}^\pm(w^\pm)-\widetilde{\psi}^\pm(v^\pm_\gamma)\,|=\operatorname{O}((\eta+e^{-\ell_\gamma/2})^{\kappa_3}\|\widetilde{\psi}^\pm\|_\beta)\,.$$

Therefore there exists a constant $\kappa_4>0$ such that

$$I_\eta(T)=\sum_{\gamma\in\Gamma}(\widetilde{\psi}^-(v^-_\gamma)\widetilde{\psi}^+(v^+_\gamma)+\operatorname{O}((\eta+e^{-\ell_\gamma/2})^{\kappa_4}\|\widetilde{\psi}^-\|_\beta\|\widetilde{\psi}^+\|_\beta))\times$$
$$\int_0^T e^{\delta_F\,t}\int_{v\in T^1\widetilde{M}}\widehat{\Phi}^-_\eta(g^{-t/2}v)\;\widehat{\Phi}^+_\eta(\gamma^{-1}g^{t/2}v)\;d\widetilde{m}_F(v)\;dt\,.$$

Now, using the inequalities (55), Equation (52) follows as in the last two steps of the proof of Theorem 14, by taking $\eta=e^{-\kappa_5 T}$ for some $\kappa_5>0$.

In the same way that Corollary 20 is deduced from Theorem 14, the following result can be deduced from Equation (52) under the assumptions of assertions (1) or (2). Let $(\widetilde{\psi}^\pm_k)_{k\in I^\pm}$ be a Γ-equivariant family of nonnegative maps $\widetilde{\psi}^\pm_k\in\mathscr{C}^\beta(\partial^1_\mp D^\pm_k)$ such that $\sup_{k\in I^\pm}\|\widetilde{\psi}^\pm_k\|_\beta$ is finite. Extend $\widetilde{\psi}^\pm_k$ by 0 outside $\partial^1_\mp D^\pm_k$ to define a function on $T^1\widetilde{M}$. The measurable Γ-invariant function $\widetilde{\Psi}^\pm=\sum_{k\in I^\pm/\sim}\widetilde{\psi}^\pm_k$ on $T^1\widetilde{M}$ defines a measurable function $\Psi^\pm$ on T^1M. Assume that $\int_{T^1M}\Psi^\pm\,d\sigma^\mp_{\mathscr{D}^\pm}$ is finite. Then there exists $\kappa_0'>0$ (independent of $(\widetilde{\psi}^\pm_k)_{k\in I^\pm}$) such that, as $t\to+\infty$,

$$\delta_F\;\|m_F\|\;e^{-\delta_F\,t}\sum_{v,\,w\in T^1M}m_{\partial^1_+\mathscr{D}^-}(v)\;m_{\partial^1_-\mathscr{D}^+}(w)\;n_t(v,w)\;\Psi^-(v)\,\Psi^+(w)$$
$$=\int_{T^1M}\Psi^-\,d\sigma^+_{\mathscr{D}^-}\int_{T^1M}\Psi^+\,d\sigma^-_{\mathscr{D}^+}+\operatorname{O}(e^{-\kappa_0'T}\sup_{i\in I^-}\|\widetilde{\psi}^-_i\|_\beta\sup_{j\in I^+}\|\widetilde{\psi}^+_j\|_\beta)\,.\tag{56}$$

Now to prove the assertions (1) and (2), we proceed as follows. If $\psi^{\pm} \in \mathscr{C}_c^{\beta}(T^1M)$, for every $k \in I^{\pm}$, we denote by $\widetilde{\psi}_k^{\pm}$ the restriction to $\partial_{\mp}^1 D_k^{\pm}$ of $\psi^{\pm} \circ Tp$ where $p : \widetilde{M} \to M$ is the universal cover. Note that the map $\Psi^{\pm}$ defined above coincides with $\psi^{\pm}$ on the elements $u \in T^1M$ such that $m_{\mathscr{D}^{\pm}}(u) \neq 0$, and that $\sup_{k\in I^{\pm}} \|\widetilde{\psi}_k^{\pm}\|_{\beta} \leq \|\psi^{\pm}\|_{\beta}$. Hence the assertions (1) and (2) follow from Equation (56).

The last statement of Theorem 28 follows by taking as the functions $\widetilde{\psi}_k^{\pm}$ the constant functions 1 in Equation (56). □

6 Counting arcs in finite volume hyperbolic manifolds

In this subsection, we consider the special case when M is a finite volume complete connected hyperbolic good orbifold and the potential F is zero. Under these assumptions, taking $\widetilde{M} = \mathbb{H}^n_{\mathbb{R}}$ to be the ball model of the real hyperbolic space of dimension n and Γ to be a discrete group of isometries of $\mathbb{H}^n_{\mathbb{R}}$ such that M is isometric to (hence from now on identified with) $\Gamma\backslash\mathbb{H}^n_{\mathbb{R}}$, the limit set of the group Γ is $\mathbb{S}^{n-1}$ and the Patterson density $(\mu_x = \mu_x^+ = \mu_x^-)_{x\in\mathbb{H}^n_{\mathbb{R}}}$ of the pair $(\Gamma, 0)$ can be normalised such that $\|\mu_x\| = \operatorname{Vol}(\mathbb{S}^{n-1})$ for all $x \in \mathbb{H}^n_{\mathbb{R}}$.

The Gibbs measure m_F with $F = 0$ is, by definition, the Bowen-Margulis measure $m_{\rm BM}$, which is known, by homogeneity in this special case, to be a constant multiple of the Liouville measure $\operatorname{Vol}_{T^1M}$ of T^1M. This measure disintegrates as

$$d\operatorname{Vol}_{T^1M} = \int_{x\in M} d\operatorname{Vol}_{T^1_xM}\; d\operatorname{Vol}_M(x)\;.$$

Note that $\operatorname{Vol}(T^1_xM) = \frac{\operatorname{Vol}(\mathbb{S}^{n-1})}{\operatorname{Card}(\Gamma_{\widetilde{x}})}$ where $\Gamma_{\widetilde{x}}$ is the stabiliser in Γ of any lift $\widetilde{x}$ of x in $\mathbb{H}^n_{\mathbb{R}}$, and that $\Gamma_{\widetilde{x}} = \{e\}$ for Vol_M-almost every $x \in M$. Furthermore, if D is a totally geodesic subspace or a horoball in $\mathbb{H}^n_{\mathbb{R}}$, then the skinning measures $\sigma^{\pm}_D$ are, again by homogeneity, constant multiples of the induced Riemannian measures $\operatorname{Vol}_{\partial^1_{\pm}D}$. These measures disintegrate with respect to the basepoint fibration $\partial^1_{\pm}D \to \partial D$ over the Riemannian measure of the boundary ∂D of D in $\mathbb{H}^n_{\mathbb{R}}$ (with $\partial D = D$ if D is totally geodesic of dimension less than n), with measure on the fiber of $x \in \partial D$ the spherical measure on the outer/inner unit normal vectors to D at x:

$$d\operatorname{Vol}_{\partial^1_{\pm}D} = \int_{x\in\partial D} d\operatorname{Vol}_{\partial^1_{+}D\cap T^1_xM}\; d\operatorname{Vol}_{\partial D}(x)\;.$$

The following result gives the proportionality constants of the various measures explicitly. For later use, we also give the pushforward images of these measures under the basepoint map $\pi : T^1M \to M$.

Proposition 29 *Let $M = \Gamma\backslash\mathbb{H}^n_{\mathbb{R}}$ be a finite volume hyperbolic (good) orbifold of dimension $n \geq 2$ and assume that the Patterson density $(\mu_x)_{x\in\mathbb{H}^n_{\mathbb{R}}}$ of $(\Gamma, 0)$ is normalised such that $\|\mu_x\| = \operatorname{Vol}(\mathbb{S}^{n-1})$ for all $x \in \mathbb{H}^n_{\mathbb{R}}$.*

(1) *We have $m_{\rm BM} = 2^{n-1}\operatorname{Vol}_{T^1M}$. In particular,*

$$\|m_{\rm BM}\| = 2^{n-1}\operatorname{Vol}(\mathbb{S}^{n-1})\operatorname{Vol}(M)$$

and

$$\pi_*m_{\rm BM} = 2^{n-1}\operatorname{Vol}(\mathbb{S}^{n-1})\operatorname{Vol}_M\;.$$

(2) *If D is a horoball in $\mathbb{H}^n_\mathbb{R}$, then $\widetilde{\sigma}^\pm_D = 2^{n-1}\operatorname{Vol}_{\partial^1_\pm D}$ and $\pi_*\widetilde{\sigma}^\pm_D = 2^{n-1}\operatorname{Vol}_{\partial D}$. In particular, if D is centered at a parabolic fixed point of Γ with stabiliser Γ_D and if $\mathscr{D} = (\gamma D)_{\gamma\in\Gamma}$, then*

$$\|\sigma^\pm_{\mathscr{D}}\| = 2^{n-1}\operatorname{Vol}(\Gamma_D\backslash\partial^1_\pm D) = 2^{n-1}(n-1)\operatorname{Vol}(\Gamma_D\backslash D)\ .$$

(3) *If D is a totally geodesic submanifold of $\mathbb{H}^n_\mathbb{R}$ with dimension $k \in \{1,\dots,n-1\}$, then $\widetilde{\sigma}^+_D = \widetilde{\sigma}^-_D = \operatorname{Vol}_{\partial^1_\pm D}$ and $\pi_*\widetilde{\sigma}^\pm_D = \operatorname{Vol}(\mathbb{S}^{n-k-1})\operatorname{Vol}_D$. In particular, with Γ_D the stabiliser in Γ of D, if $\Gamma_D\backslash D$ is a properly immersed finite volume suborbifold of M and if $\mathscr{D} = (\gamma D)_{\gamma\in\Gamma}$, then*

$$\|\sigma^\pm_{\mathscr{D}}\| = \operatorname{Vol}(\Gamma_D\backslash\partial^1_\pm D)\ .$$

If m is the number of elements of Γ that pointwise fix D, then

$$\|\sigma^\pm_{\mathscr{D}}\| = \frac{1}{m}\operatorname{Vol}(\mathbb{S}^{n-k-1})\operatorname{Vol}(\Gamma_D\backslash D)\,.$$

Proof. Claims (1) and (3) are proven for the outer skinning measure assuming that Γ has no torsion in Proposition 10 and Claim (1) of Proposition 11 in [PP6], respectively. Note that $\partial^1_+ D = \partial^1_- D$ if D is a totally geodesic submanifold, and $C^+ = C^-$, $\mu^+_{x_0} = \mu^-_{x_0}$ if $F = 0$, hence $\widetilde{\sigma}^-_D = \iota_*\widetilde{\sigma}^+_D$. Note that ι preserves the Riemannian metric of $T^1\widetilde{M}$, hence $\widetilde{\sigma}^+_D = \operatorname{Vol}_{\partial^1_\pm D}$ implies $\widetilde{\sigma}^-_D$ is also equal to $\operatorname{Vol}_{\partial^1_\pm D}$. If Γ has torsion, Claim (1) follows by restricting to the complement of the points in $\widetilde{M}$ with nontrivial stabiliser, this set has zero Riemannian measure in $\widetilde{M}$, and Claim (3) follows from the fact that the fixed point set on D of an isometry which preserves D, but does not pointwise fix D, has measure 0 for the Riemannian measure of D.

The first part of Claim (2) is proved in Claim (1) of [PP6, Prop. 10] for the outer skinning measure. For the second part, note that if the horoball D is precisely invariant (that is, the interiors of D and γD intersect for $\gamma \in \Gamma$ only if $\gamma \in \Gamma_D$), then $\Gamma_D\backslash D$ embeds in M and the image is, by definition, a Margulis cusp neighbourhood. In the general case, there is a precisely embedded horoball D' contained in D such that $D = \mathscr{N}_t D'$ for some $t \geq 0$. Let $\mathscr{D}' = (\gamma D')_{\gamma\in\Gamma}$. As $\Gamma_{D'} = \Gamma_D$, we have

$$\|\sigma^\pm_{\mathscr{D}}\| = e^{(n-1)t}\|\sigma^\pm_{\mathscr{D}'}\| = e^{(n-1)t}2^{n-1}(n-1)\operatorname{Vol}(\Gamma_{D'}\backslash D') = 2^{n-1}(n-1)\operatorname{Vol}(\Gamma_D\backslash D)\,,$$

by Proposition 9 (iii), by [PP6, Prop. 10] and by the scaling of hyperbolic volume. The case with torsion follows as in Claims (1) and (3). □

Proposition 29 allows us to obtain very explicit versions of Theorems 1 and 4 in the case when M is a finite volume hyperbolic manifold (or good orbifold) and the properly immersed closed convex subsets are assumed to be points, totally geodesic orbifolds or Margulis neighbourhoods of cusps. The following result gives these explicit asymptotics of the counting function in the cases that we have not found in the literature. The corresponding result holds for the remaining three combinations when A^- and A^+ are both points, when one of them is a point and the other is totally geodesic, and the case when one of them is totally geodesic and the other is a Margulis cusp neighbourhood. We refer to the Introduction as well as to our survey article [PP6] for more details and references.

Corollary 30 *Let $\widetilde{M} = \mathbb{H}^n_{\mathbb{R}}$, $\widetilde{F} = 0$, let Γ be a discrete group of isometries of $\mathbb{H}^n_{\mathbb{R}}$, and assume that $M = \Gamma\backslash\mathbb{H}^n_{\mathbb{R}}$ has finite volume. If A^- and A^+ are properly immersed finite volume totally geodesic suborbifolds in M of dimensions k^- and k^+ in $\{1, \dots, n-1\}$, respectively, let*

$$c(A^-,\, A^+) = \frac{\operatorname{Vol}(\mathbb{S}^{n-k^--1})\operatorname{Vol}(\mathbb{S}^{n-k^+-1})}{2^{n-1}\,(n-1)\,\operatorname{Vol}(\mathbb{S}^{n-1})}\frac{\operatorname{Vol}(A^-)\operatorname{Vol}(A^+)}{\operatorname{Vol}(M)}\,,$$

as $s \to +\infty$. If A^- and A^+ are Margulis cusp neighbourhoods in M, let

$$c(A^-, A^+) = \frac{2^{n-1}(n-1)\operatorname{Vol}(A^-)\operatorname{Vol}(A^+)}{\operatorname{Vol}(\mathbb{S}^{n-1})\,\operatorname{Vol}(M)}\,.$$

If A^- is a point and A^+ is a Margulis cusp neighbourhood, let

$$c(A^-, A^+) = \frac{\operatorname{Vol}(A^+)}{\operatorname{Vol}(M)}\,.$$

If A^- is a Margulis cusp neighbourhood and A^+ is a properly immersed finite volume totally geodesic suborbifold in M of dimension k in $\{1, \dots, n-1\}$, let

$$c(A^-, A^+) = \frac{\operatorname{Vol}(\mathbb{S}^{n-1-k})\,\operatorname{Vol}(A^-)\operatorname{Vol}(A^+)}{\operatorname{Vol}(\mathbb{S}^{n-1})\,\operatorname{Vol}(M)}\,.$$

In each of these cases, if $m^\pm$ is the cardinality of the intersection of the isotropy groups in the orbifold M of the points of $A^\pm$, then

$$\mathscr{N}_{A^-,\,A^+}(t) = \mathscr{N}_{A^-,\,A^+,\,0}(t) \sim \frac{c(A^-, A^+)}{m^-\,m^+}\;e^{(n-1)t}\,.$$

Furthermore, if Γ is arithmetic or if M is compact, then there is some $\kappa'' > 0$ such that, as $t \to +\infty$,

$$\mathscr{N}_{A^-,\,A^+}(t) = \frac{c(A^-, A^+)}{m^-\,m^+}\;e^{(n-1)t}\big(1 + \operatorname{O}(e^{-\kappa'' t})\big)\,. \quad \Box$$

We refer to [PP7, PP8] for several new arithmetic applications of these results.

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