On Hausdorff dimension in inhomogeneous Diophantine approximation over global function fields

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Abstract

In this paper, we study inhomogeneous Diophantine approximation over the completion K_v of a global function field K (over a finite field) for a discrete valuation v, with affine algebra R_v. We obtain an effective upper bound for the Hausdorff dimension of the set

$$\mathbf{Bad}_A(\epsilon) = \left\{ \boldsymbol{\theta} \in K_v^m : \liminf_{(\mathbf{p},\mathbf{q}) \in R_v^m \times R_v^n, \|\mathbf{q}\| \to \infty} \|\mathbf{q}\|^n \|A\mathbf{q} - \boldsymbol{\theta} - \mathbf{p}\|^m \geqslant \epsilon \right\},$$

of ϵ-badly approximable targets $\boldsymbol{\theta} \in K_v^m$ for a fixed matrix $A \in \mathscr{M}_{m,n}(K_v)$, using an effective version of entropy rigidity in homogeneous dynamics for an appropriate diagonal action on the space of R_v-grids. We further characterize matrices A for which $\mathbf{Bad}_A(\epsilon)$ has full Hausdorff dimension for some $\epsilon > 0$ by a Diophantine condition of singularity on average. Our methods also work for the approximation using weighted ultrametric distances. [1]

1 Introduction

In the theory of inhomogeneous Diophantine approximation of real numbers by rational ones (in several variables), one studies the distribution of the vectors $A\,\mathbf{x} \in \mathbb{R}^m$ modulo $\mathbb{Z}^m$, as $\mathbf{x}$ varies over $\mathbb{Z}^n$, near a vector $b \in \mathbb{R}^m$ for a $m \times n$ real matrix $A \in \mathscr{M}_{m,n}(\mathbb{R})$. For instance, if $m, n \geqslant 1$ and $\langle\, \boldsymbol{\xi} \,\rangle = \inf_{\mathbf{x} \in \mathbb{Z}^m} \|\, \xi - \mathbf{x} \,\|$ denotes the distance from $\xi \in \mathbb{R}^m$ to a nearest integral vector with respect to the Euclidean norm $\|\cdot\|$ on $\mathbb{R}^m$, using the inhomogeneous Khintchine-Groshev theorem of [Sch1, Theorem1], we have

$$\liminf_{\mathbf{x} \in \mathbb{Z}^n,\ \|\,\mathbf{x}\,\| \to \infty} \|\,\mathbf{x}\,\|^n \langle\, A\,\mathbf{x} - b \,\rangle^m = 0$$

for almost every $(A, b) \in \mathscr{M}_{m,n}(\mathbb{R}) \times \mathbb{R}^m$.

Let us consider the exceptional set of solutions (A, b) of the above equation. We call A *badly approximable for* b if

$$\liminf_{\mathbf{x} \in \mathbb{Z}^n,\ \|\,\mathbf{x}\,\| \to \infty} \|\,\mathbf{x}\,\|^n \langle\, A\,\mathbf{x} - b \,\rangle^m > 0\ .$$

If the left hand side is at least ϵ, we say that A is *ϵ-bad* for b. It is known that given any $b \in \mathbb{R}^m$, the set of badly approximable matrices $A \in \mathscr{M}_{m,n}(\mathbb{R})$ has zero Lebesgue

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measure but full Hausdorff dimension, see [Sch2, ET]. In [KKL], the first two authors, with Wooyeon Kim, show that given a matrix $A \in \mathscr{M}_{m,n}(\mathbb{R})$, under a necessary assumption of non-singularity on average, the set of vectors $b \in \mathbb{R}^m$ with respect to which A is ϵ-bad does not have full Hausdorff dimension, and obtain an explicit upper bound: there exist $c(A) > 0$, depending only on m, n and A, such that for every $\epsilon > 0$, the Hausdorff dimension of the set of vectors $b \in \mathbb{R}^m$ that are ϵ-bad for A is bounded from above by $m - c(A)\,\frac{\epsilon}{\ln(1/\epsilon)}$.

In this paper, we prove analogous results for function fields, in the weighted setting. Let us state our main results, refering to Section 2.1 for more precise definitions.

Let K be any global function field over a finite field $\mathbb{F}_q$ of q elements for a prime power q, that is, the function field of a geometrically connected smooth projective curve $\mathbf{C}$ over $\mathbb{F}_q$. The most studied example in Diophantine approximation in positive characteristic is the case of the field $K = \mathbb{F}_q(Z)$ of rational fractions in one variable Z over $\mathbb{F}_q$, where $\mathbf{C} = \mathbb{P}^1$ is the projective line, but we emphasize the fact that our work applies in the general situation above.

We fix a (normalized) discrete valuation v on K. Let K_v and $\mathscr{O}_v$ be the completion of K with respect to v and its valuation ring, respectively. We fix a uniformizer $\pi_v \in K$, which satisfies $v(\pi_v) = 1$. Let $k_v = \mathscr{O}_v/\pi_v\mathscr{O}_v$ be the residual field and let q_v be its cardinality. The (normalized) absolute value $|\cdot|$ associated with v is defined by $|\,x\,| = q_v^{-v(x)}$. For every $\sigma \in \mathbb{Z}_{\geqslant 1}$, let $\|\ \| : K_v^\sigma \to [0, +\infty[$ be the norm $(\xi_1, \dots, \xi_\sigma) \mapsto \max_{1\leqslant i\leqslant \sigma} |\,\xi_i\,|$. We denote by $\dim_{\rm Haus}$ the Hausdorff dimension of the subsets of K_v^σ for this standard norm.

The discrete object analogous to the set of integers $\mathbb{Z}$ in $\mathbb{R}$ is the affine algebra R_v of the curve $\mathbf{C}$ minus the point v. If $K = \mathbb{F}_q(Z)$ and $v = [1:0]$ is the standard point at infinity of $\mathbf{C} = \mathbb{P}^1$, then $R_v = \mathbb{F}_q[Z]$ is the ring of polynomials in Z over $\mathbb{F}_q$.

Let $m, n \in \mathbb{Z}_{\geqslant 1}$. Let us fix, throughout the paper, two *weights* consisting of a m-tuple $\mathbf{r} = (r_1, \cdots, r_m)$ and a n-tuple $\mathbf{s} = (s_1, \cdots, s_n)$ of positive integers such that we have $|\mathbf{r}| = \sum_{1\leqslant i\leqslant m} r_i = \sum_{1\leqslant j\leqslant n} s_j$. The $\mathbf{r}$-*quasinorm* of $\boldsymbol{\xi} \in K_v^m$ and $\boldsymbol{s}$-*quasinorm* of $\boldsymbol{\theta} \in K_v^n$ are given by

$$\|\,\boldsymbol{\xi}\,\|_{\mathbf{r}} = \max_{1\leqslant i\leqslant m} |\,\boldsymbol{\xi}_i\,|^{\frac{1}{r_i}} \quad\text{and}\quad \|\,\boldsymbol{\theta}\,\|_{\mathbf{s}} = \max_{1\leqslant j\leqslant n} |\,\boldsymbol{\theta}_j\,|^{\frac{1}{s_j}}\ .$$

We denote by $\langle\,\boldsymbol{\xi}\,\rangle_{\mathbf{r}} = \inf_{\mathbf{x}\in R_v^m} \|\,\boldsymbol{\xi} - \mathbf{x}\,\|_{\mathbf{r}}$ the (weighted) distance from $\boldsymbol{\xi}$ to the set R_v^m of integral vectors in K_v^m.

Let $\epsilon > 0$. A matrix $A \in \mathscr{M}_{m,n}(K_v)$ is said to be ϵ-*bad* for a vector $\boldsymbol{\theta} \in K_v^m$ if

$$\liminf_{\mathbf{x}\in R_v^n,\ \|\,\mathbf{x}\,\|_{\mathbf{s}}\to\infty} \|\,\mathbf{x}\,\|_{\mathbf{s}}\,\langle\,A\mathbf{x} - \boldsymbol{\theta}\,\rangle_{\mathbf{r}} \geqslant \epsilon\ . \tag{1}$$

Denote by $\mathbf{Bad}_A(\epsilon)$ the set of vectors $\boldsymbol{\theta} \in K_v^m$ such that A is ϵ-bad for $\boldsymbol{\theta}$. Given two subsets U and V of a given set, we denote $U - V = \{x \in U : x \notin V\}$. We say that a matrix $A \in \mathscr{M}_{m,n}(K_v)$ is $(\mathbf{r},\mathbf{s})$-*singular on average* if for every $\epsilon > 0$, we have

$$\lim_{N\to\infty} \frac{1}{N}\ \mathrm{Card}\{\,\ell \in \{1, \dots, N\} : \exists\ \mathbf{y} \in R_v^n - \{0\},\ \langle\,A\,\mathbf{y}\,\rangle_{\mathbf{r}} \leqslant \epsilon\, q_v^{-\ell},\ \|\,\mathbf{y}\,\|_{\mathbf{s}} \leqslant q_v^{\ell}\,\} = 1\ . \tag{2}$$

For the basic example of function field, when $K = \mathbb{F}_q[Z]$ and $v = [1:0]$, Bugeaud and Zhang [BZ] found a sufficient condition (and an equivalent one when $n = m = 1$) on A for the Hausdorff dimension of $\mathbf{Bad}_A(\epsilon)$ to be full. We first strenghten and extend their result to general function fields.

Theorem 1.1 *Let $A \in \mathscr{M}_{m,n}(K_v)$ be a matrix. The following assertions are equivalent:*
(1) *there exists $\epsilon > 0$ such that the set $\mathbf{Bad}_A(\epsilon)$ has full Hausdorff dimension,*
(2) *the matrix A is $(\mathbf{r},\mathbf{s})$-singular on average.*

We also provide an effective upper bound on the Hausdorff dimension in terms of ϵ, which is a new result even in the basic case $K = \mathbb{F}_q[Z]$ and $v = [1:0]$.

Theorem 1.2 *For every $A \in \mathscr{M}_{m,n}(K_v)$ which is not $(\mathbf{r},\mathbf{s})$-singular on average, there exists a constant $c(A) > 0$ depending only on A, $\mathbf{r}$, $\mathbf{s}$, such that for every $\epsilon > 0$, we have* $\dim_{\mathrm{Haus}} \mathbf{Bad}_A(\epsilon) \leqslant m - c(A)\frac{\epsilon^{|\mathbf{r}|}}{\ln(1/\epsilon)}$.

The proofs of the above main theorems of this paper are largely divided into two parts. Firstly, assuming the singular on average property in order to prove the full Hausdorff dimension property, we give a lower bound on the Hausdorff dimension of appropriately chosen subsets of K_v^m, using new function fields versions of classical tools in Diophantine approximation such as geometry of numbers, transference principle and best approximation vectors (see for instance [Cas, Sch3, Kri, KlW, Cheu, Chev, GE, CC, KlST, GG, BZ, Ger, LSST, CGGMS, BuKLR]). Secondly, in order to prove the upper bound in Theorem 1.2, we use technics of homogeneous dynamics of diagonal actions and in particular the entropy method (see for instance [Kle, EL, LSS, ELW]). Let us explain briefly the latter part.

Let $d = m + n$. The dynamical space relevant to inhomogeneous Diophantine approximation is the space $\mathscr{Y}$ of *unimodular grids* $\Lambda + b$ in K_v^d, that is of (Haar-covolume 1) R_v-lattices Λ of K_v^d translated by vectors $b \in K_v^d$, endowed with the affine action of the diagonal subgroup of $\mathrm{SL}_d(K_v)$. This is in higher dimension more convenient than the study of the commuting actions of $\mathrm{SL}_d(R_v)$ and of the diagonal group on the Bruhat-Tits building associated with $\mathrm{SL}_d(K_v)$ (see [BPP, Part III] when $d = 2$). Given the above weights $\mathbf{r}$ and $\mathbf{s}$, we consider the affine action on $\mathscr{Y}$ of the 1-parameter diagonal subgroup $(\mathfrak{a}^k)_{k\in\mathbb{Z}}$ where

$$\mathfrak{a} = \mathrm{diag}(\pi_v^{-r_1}, \cdots, \pi_v^{-r_m}, \pi_v^{s_1}, \cdots, \pi_v^{s_n}) .$$

The space $\mathscr{Y}$ of unimodular grids $\mathfrak{a}$-equivariantly projects onto the space of unimodular R_v-lattices $\mathscr{X} = \mathrm{SL}_d(K_v)/\mathrm{SL}_d(R_v)$ by the map sending $\Lambda + b$ to Λ. We say that a unimodular R_v-lattice Λ *diverges on average* under the action of $\mathfrak{a}$ if for every compact subset Q of $\mathscr{X}$, we have

$$\liminf_{N\to\infty} \frac{1}{N} \,\mathrm{Card}\,\big\{\ell \in \{1, \cdots, N\} : \mathfrak{a}^\ell \Lambda \notin Q\big\} = 1 .$$

Following Dani's path, we prove in Section 5.2 that the lattice $\Lambda_A = \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} R_v^d$ diverges on average under $\mathfrak{a}$ if and only if A is $(\mathbf{r},\mathbf{s})$-singular on average.

As developped in the last Section 6, the main idea of the entropy method in our situation, as in [KKL] for the real case, is that if the point Λ_A does not diverge on average, then the Hausdorff dimension of $\mathbf{Bad}_A(\epsilon)$ provides a lower bound on the conditional entropy of $\mathfrak{a}$ with respect to a measure μ constructed by well-separated sets on the fibers of the projection $\mathscr{Y} \to \mathscr{X}$. An effective control of the maximal conditional entropy by a control of the support of μ on the thin/thick parts of $\mathscr{Y}$ hence gives an effective upper bound.

Before Section 6, our paper is organized as follows. In Section 2, we recall basic facts on the geometry of numbers, define the best approximation sequence and prove transference principle for the weighted case, which generalize previous results of Bugeaud-Zhang [BZ].

In Section 3, we give a characterization of the singular on average property with weights in terms of the best approximation sequence. In Section 4, we establish the lower bound on the Hausdorff dimension by constructing a subsequence with controlled growth of the best approximation sequence for a matrix whose transpose is singular on average. In Section 5, we recall some background on homogeneous dynamics and conditional entropy, and prove an effective and positive characteristic version of the variational principle for conditional entropy of [EL, §7.55] (see [KKL] in the real case).

We remark that in [KKL], the first two authors, with Wooyeon Kim, also show that given any vector $b \in \mathbb{R}^m$, the set of matrices $A \in \mathscr{M}_{m,n}(\mathbb{R})$ that are ϵ-bad for b does not have full Hausdorff dimension, and estimate an explicit upper bound. Thus it seems very interesting to obtain a similar result in the global function field case.

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2 Background material for the lower bound

2.1 On global function fields

We refer for instance to [Gos, Ros], as well as [BPP, §14.2], for the content of this section. Let $\mathbb{F}_q$ be a finite field with q elements, where q is a positive power of a positive prime. Let K be the function field of a geometrically connected smooth projective curve $\mathbf{C}$ over $\mathbb{F}_q$, or equivalently an extension of $\mathbb{F}_q$ with transcendence degree 1, in which $\mathbb{F}_q$ is algebraically closed. We denote by g the genus of $\mathbf{C}$. There is a bijection between the set of closed points of $\mathbf{C}$ and the set of normalized discrete valuations v of K, the valuation of a given element $f \in K$ being the order of the zero or the opposite of the order of the pole of f at the given closed point. We fix such an element v throughout this paper, and use the notation K_v, $\mathscr{O}_v$, π_v, k_v, q_v, $|\cdot|$ defined in the introduction. We furthermore denote by $\deg v$ the degree of v, so that

$$q_v = q^{\deg v} .$$

We denote by vol_v the normalized Haar measure on the locally compact additive group K_v such that $\mathrm{vol}_v(\mathscr{O}_v) = 1$. For any positive integer d, let vol_v^d be the normalized Haar measure on $K_v^{\,d}$ such that $\mathrm{vol}_v^d(\mathscr{O}_v^d) = 1$. Note that for every $g \in \mathrm{GL}_d(K_v)$ we have

$$d\,\mathrm{vol}_v^d(gx) = |\det(g)|\, d\,\mathrm{vol}_v^d(x) ,$$

where det is the determinant of a matrix. For every discrete additive subgroup Λ of $K_v^{\,d}$, we again denote by vol_v^d (and simply vol_v when $d = 1$) the measured induced on $K_v^{\,d}/\Lambda$ by vol_v^d.

Note that the completion K_v of K for v is the field $k_v((\pi_v))$ of Laurent series $x = \sum_{i\in\mathbb{Z}} x_i(\pi_v)^i$ in the variable π_v over k_v, where $x_i \in k_v$ is zero for $i \in \mathbb{Z}$ small enough. We have

$$|x| = q_v^{-\sup\{j\in\mathbb{Z}\,:\,\forall i<j,\; x_i=0\}} ,$$

and $\mathscr{O}_v = k_v[[\pi_v]]$ is the local ring of power series $x = \sum_{i\in\mathbb{Z}_{\geq 0}} x_i(\pi_v)^i$ in the variable π_v over k_v.

Recall that the affine algebra R_v of the affine curve $\mathbf{C} - \{v\}$ consists of the elements of K whose only poles are at the closed point v of $\mathbf{C}$. Its field of fractions is equal to K, hence we can write elements of K as x/y with $x, y \in R_v$ and $y \neq 0$. By for instance [BPP, Eq. (14.2)], we have

$$R_v \cap \mathscr{O}_v = \mathbb{F}_q \,. \tag{3}$$

For every $\xi \in K_v$, we denote by

$$|\langle\, \xi \,\rangle| = \inf_{x \in R_v} \|\, \xi - x \,\|$$

the distance in K_v from ξ to the set R_v of integral points of K_v.

For instance, if $\mathbf{C}$ is the projective line $\mathbb{P}^1$, if $\infty = [1:0]$ is its usual point at infinity and if Z is a variable name, then $g = 0$, $K = \mathbb{F}_q(Z)$, $\pi_\infty = Z^{-1}$, $K_\infty = \mathbb{F}_q((Z^{-1}))$, $\mathscr{O}_\infty = \mathbb{F}_q[[Z^{-1}]]$, $k_\infty = \mathbb{F}_q$, $q_\infty = q$ and $R_\infty = \mathbb{F}_q[Z]$. In this setting, there are numerous results on Diophantine approximation in the fields of formal power series, see for instance [Las], [Bug, Chap. 9]. On the other hand, little is known about Diophantine approximation over general global function fields, see for instance [KlST] (for a single valuation in positive characteristic) for the ground work on the geometry of number for function fields.

2.2 On the geometry of numbers and Dirichlet's theorem

Let d be a positive integer. An R_v-*lattice* Λ in $K_v^{\,d}$ is a discrete R_v-submodule in $K_v^{\,d}$ that generates $K_v^{\,d}$ as a K_v-vector space. The *covolume* of Λ, denoted by $\operatorname{Covol}(\Lambda)$, is defined as the measure of the (compact) quotient space $K_v^{\,d}/\Lambda$:

$$\operatorname{Covol}(\Lambda) = \operatorname{vol}_v^d(K_v^{\,d}/\Lambda) \,.$$

For example, $R_v^{\,d}$ is an R_v-lattice in $K_v^{\,d}$, and by for instance [BPP, Lem. 14.4)], we have

$$\operatorname{Covol}(R_v^{\,d}) = q^{(g-1)d} \,. \tag{4}$$

Let $\overline{B}(0,r)$ be the closed ball of radius r centered at zero in $K_v^{\,d}$ with respect to the norm $\|\cdot\| : (\xi_1, \dots, \xi_d) \mapsto \max_{1 \leqslant i \leqslant d} |\,\xi_i\,|$. For every integer $k \in \{1, \dots, d\}$, the k-*th minimum* of an R_v-lattice Λ is defined by

$$\lambda_k(\Lambda) = \min\{r > 0 : \dim_{K_v}(\operatorname{span}_{K_v}(\overline{B}(0,r) \cap \Lambda)) \geqslant k\},$$

where $\operatorname{span}_{K_v}$ denotes the K_v-linear span of a subset of a K_v-vector space and $\dim_{K_v}$ is the dimension of a K_v-vector space. Note that $\lambda_1(\Lambda), \dots, \lambda_d(\Lambda) \in q_v^{\mathbb{Z}}$. The next result follows from [KlST, Theo. 4.4] and Equation (4).

Theorem 2.1 (Minkovski's theorem) *For every R_v-lattice Λ in $K_v^{\,d}$, we have*

$$q^{-(g-1)d}\ \operatorname{Covol}(\Lambda) \leqslant \lambda_1(\Lambda) \dots \lambda_d(\Lambda) \leqslant q_v^{\,d}\ \operatorname{Covol}(\Lambda) \,. \quad \square$$

Since $\lambda_1(\Lambda) \leqslant \dots \leqslant \lambda_d(\Lambda)$, the following result follows immediately from Minkowski's theorem 2.1.

Corollary 2.2 *For every R_v-lattice Λ in $K_v^{\,d}$, we have*

$$\lambda_1(\Lambda) \leqslant q_v\ \operatorname{Covol}(\Lambda)^{\frac{1}{d}} \,. \quad \square$$

The following result generalizes [GG, Theo. 2.1], which is proved only when $K = \mathbb{F}_q(Z)$ and $v = \infty$, to all function fields K and valuations v. See also [KlW, Theo. 1.3] in the case of the field $\mathbb{Q}$.

Theorem 2.3 (Dirichlet's theorem) *For every matrix $A \in \mathscr{M}_{m,n}(K_v)$ whose rows are denoted by $A_1, \dots, A_m$, for all $(r'_1, \dots, r'_m) \in \mathbb{Z}^m_{\geqslant 0}$ and $(s'_1, \dots, s'_n) \in \mathbb{Z}^n_{\geqslant 0}$ with*

$$r'_i > 1 + \frac{g-1}{\deg v} \quad \text{and} \quad \sum_{i=1}^{m} r'_i = \sum_{j=1}^{n} s'_j \ ,$$

there exists an element $\mathbf{y} = (y_1, \dots, y_n) \in R_v^n - \{0\}$ such that, for all $i = 1, \dots, m$ and $j = 1, \dots, n$, we have

$$|\langle A_i \, \mathbf{y} \rangle| \leqslant q_v \, q^{g-1} \, {q_v}^{-r'_i} \quad \text{and} \quad |\, y_j \,| \leqslant q_v \, q^{g-1} \, {q_v}^{s'_j} \ .$$

Proof. With A, $r'_1, \dots, r'_m$ and $s'_1, \dots, s'_n$ as in the statement, we apply Corollary 2.2 with $d = m + n$ to the R_v-lattice

$$\Lambda = \begin{pmatrix} \pi_v^{-r'_1} & & & & & 0 \\ & \ddots & & & & \\ & & \pi_v^{-r'_m} & & & \\ & & & \pi_v^{s'_1} & & \\ & & & & \ddots & \\ 0 & & & & & \pi_v^{s'_n} \end{pmatrix} \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} R_v^d \ ,$$

where I_k is the $k \times k$ identity matrix. Since the above two matrices have determinant 1 by the assumption $\sum_{i=1}^m r'_i = \sum_{j=1}^n s'_j$, and by Equation (4), we have $\operatorname{Covol}(\Lambda) = q^{(g-1)d}$. Corollary 2.2 hence says that there exists $(\mathbf{x} = (x_1, \dots, x_m), \mathbf{y} = (y_1, \dots, y_n)) \in R_v^d - \{0\}$ such that

$$\max\Big\{ \max_{i=1,\dots,m} \ |\, {\pi_v}^{-r'_i}(x_i + A_i \mathbf{y}) \,|, \ \max_{j=1,\dots,n} |\, \pi_v^{s'_j} y_j \,| \ \Big\} \leqslant q_v \operatorname{Covol}(\Lambda)^{\frac{1}{d}} = q_v \, q^{g-1} \ .$$

Assume for a contradiction that $\mathbf{y} = 0$. Then for all $i = 1, \dots, m$, since $|\, \pi_v \,| = q_v^{-1}$, we have the inequality $|\, x_i \,| \leqslant q_v \, q^{g-1} \, {q_v}^{-r'_i}$. Since $r'_i > 1 + \frac{g-1}{\deg v}$, this would imply that $|\, x_i \,| < 1$. By Equation (3), we have $\{z \in R_v : |\, z \,| < 1\} = \{0\}$. Since $x_i \in R_v$, we would have that $\mathbf{x} = 0$, contradicting the fact that $(\mathbf{x}, \mathbf{y}) \neq 0$. Therefore $\mathbf{y} \neq 0$ and the result follows. $\square$

The following corollary is due to [Kri, Theo. 1.1] (see also [BZ, Theo. 3.2] where the assumption that $c\,m$ is divisible by n is implicit) in the special case when $K = \mathbb{F}_q(Z)$ and $v = \infty$ and without weights.

Let $\min \mathbf{r} = \min_{1 \leqslant i \leqslant m} r_i$ and similarly for $\min \mathbf{s}$, $\max \mathbf{r}$ and $\max \mathbf{s}$.

Corollary 2.4 *For all $A \in \mathscr{M}_{m,n}(K_v)$ and $\alpha \in \mathbb{Z}_{\geqslant 0}$ with $\alpha > \frac{1}{\min \mathbf{r}} + \frac{g-1}{(\min \mathbf{r})(\deg v)}$, there exists $\mathbf{y} \in R_v^n - \{0\}$ such that*

$$\langle \, A \, \mathbf{y} \, \rangle_{\mathbf{r}} \leqslant q^{\frac{\deg v + g - 1}{\min \mathbf{r}}} \ q_v^{-\alpha} \quad \text{and} \quad \|\, \mathbf{y} \,\|_{\mathbf{s}} \leqslant q^{\frac{\deg v + g - 1}{\min \mathbf{s}}} \ q_v^{\alpha} \ .$$

Proof. We apply Theorem 2.3 with $r'_i = \alpha\, r_i > 1 + \frac{g-1}{\deg v}$ for $i = 1, \ldots, m$ and $s'_j = \alpha\, s_j$ for $j = 1, \ldots, n$, noting that $\sum_{i=1}^m r'_i = \sum_{j=1}^n s'_j$ since $\sum_{i=1}^m r_i = \sum_{j=1}^n s_j$. □

Remark. When $\mathbf{r} = (n, n, \ldots, n)$ and $\mathbf{s} = (m, m, \ldots, m)$, the above result says that for every integer $\alpha > \frac{1}{n} + \frac{g-1}{n \deg v}$, there exists $\mathbf{y} \in R_v^n - \{0\}$ such that

$$\min_{\mathbf{x} \in R_v^m} \|A\,\mathbf{y} - \mathbf{x}\| \leqslant q_v\, q^{g-1}\, q_v^{-\alpha\, n} \quad \text{and} \quad \|\,\mathbf{y}\,\| \leqslant q_v\, q^{g-1}\, q_v^{\alpha\, m}\ ,$$

where $\|\cdot\|$ is the sup norm.

2.3 Best approximation sequences with weights

In this subsection, we construct a version with weights, valid for all function fields, of the best approximation sequences associated with a completely irrational matrix by Bugeaud-Zhang [BZ].

A matrix $A \in \mathscr{M}_{m,n}(K_v)$ is said to be *completely irrational* if $\langle\, A\,\mathbf{y}\,\rangle_{\mathbf{r}} \neq 0$ for every $\mathbf{y} \in R_v^n - \{0\}$. Note that this does not depend on the weight $\mathbf{r}$, and that the fact that A is completely irrational might not necessarily imply that ${}^tA$ is completely irrational.

Remark 2.5 *Let $A \in \mathscr{M}_{m,n}(K_v)$ be such that ${}^tA$ is not completely irrational.*
(1) The matrix ${}^tA$ is $(\mathbf{s}, \mathbf{r})$-singular on average.
(2) For every $\epsilon > 0$ small enough, the set $\mathbf{Bad}_A(\epsilon)$ has full Hausdorff dimension.

Proof. By assumption, there exist $\mathbf{x} \in R_v^n$ and $\mathbf{y} = (y_1, \ldots, y_m) \in R_v^m - \{0\}$ such that ${}^tA\,\mathbf{y} - \mathbf{x} = 0$.

(1) For every $\epsilon > 0$, if $\ell_0 = \lceil \log_{q_v} \|\,\mathbf{y}\,\|_{\mathbf{r}} \rceil$ then for all integers $N \geqslant \ell_0$ and $\ell \in \{\ell_0, \ldots, N\}$, we have $\langle\, {}^tA\,\mathbf{y}\,\rangle_{\mathbf{s}} = 0 \leqslant \epsilon\, q_v^{-\,\ell}$ and $\|\,\mathbf{y}\,\|_{\mathbf{r}} \leqslant q_v^{\ell}$, hence ${}^tA$ is $(\mathbf{s}, \mathbf{r})$-singular on average (see Equation (2)).

(2) For every $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_m) \in K_v^m$, let

$$\mathbf{y} \cdot \boldsymbol{\theta} = \sum_{j=1}^{m}\ y_i\,\theta_i\ \in K_v\ .$$

For every $\epsilon \in\,]0, \frac{1}{\|\,\mathbf{y}\,\|_{\mathbf{r}}}]$, let $U_{\mathbf{y},\epsilon} = \big\{\boldsymbol{\theta} \in K_v^m : |\langle\,\mathbf{y}\cdot\boldsymbol{\theta}\,\rangle| \geqslant (\,\epsilon\,\|\,\mathbf{y}\,\|_{\mathbf{r}})^{\min \mathbf{r}}\big\}$. If ϵ is small enough, then the set $U_{\mathbf{y},\epsilon}$ contains a closed ball of positive radius: For instance, let $j_0 \in \{1, \ldots, m\}$ be such that $y_{j_0} \neq 0$; define $\theta_{0,j} = 0$ if $j \neq j_0$, $\theta_{0,j_0} = \frac{\pi_v}{y_{j_0}}$ and $\boldsymbol{\theta}_0 = (\theta_{0,1}, \ldots, \theta_{0,m})$; then it is easy to check using the ultrametric inequality that the closed ball $\overline{B}(\boldsymbol{\theta}_0, \frac{1}{q_v^2\,\|\,\mathbf{y}\,\|})$ is contained in $U_{\mathbf{y},\epsilon}$ if $\epsilon < q_v^{-\frac{1}{\min \mathbf{r}}}\,\|\,\mathbf{y}\,\|_{\mathbf{r}}^{-1}$.

Let us prove that $\mathbf{Bad}_A(\epsilon)$ contains $U_{\mathbf{y},\epsilon}$, which implies that $\dim_{\mathrm{Haus}}\big(\mathbf{Bad}_A(\epsilon)\big) = m$ if ϵ is small enough. Let $\boldsymbol{\theta} \in U_{\mathbf{y},\epsilon}$ and $(\mathbf{y}', \mathbf{x}') \in R_v^m \times (R_v^n - \{0\})$.

If $\|\,\mathbf{y}\,\|_{\mathbf{r}} \|A\mathbf{x}' + \mathbf{y}' - \boldsymbol{\theta}\,\|_{\mathbf{r}} \geqslant 1$, then since $\mathbf{x}' \in R_v^n - \{0\}$ so that $\|\,\mathbf{x}'\,\|_{\mathbf{s}} \geqslant 1$, we have

$$\|\,\mathbf{x}'\,\|_{\mathbf{s}}\ \|A\mathbf{x}' + \mathbf{y}' - \boldsymbol{\theta}\,\|_{\mathbf{r}} \geqslant \frac{1}{\|\,\mathbf{y}\,\|_{\mathbf{r}}}\ \|\,\mathbf{y}\,\|_{\mathbf{r}}\ \|A\mathbf{x}' + \mathbf{y}' - \boldsymbol{\theta}\|_{\mathbf{r}} \geqslant \frac{1}{\|\,\mathbf{y}\,\|_{\mathbf{r}}} \geqslant \epsilon\ .$$

If $\| \mathbf{y} \|_{\mathbf{r}} \, \| A\mathbf{x}' + \mathbf{y}' - \boldsymbol{\theta} \|_{\mathbf{r}} \leqslant 1$, then since $\mathbf{y} \cdot (A\mathbf{x}' + \mathbf{y}') = ({}^t\!A\mathbf{y}) \cdot \mathbf{x}' + \mathbf{y} \cdot \mathbf{y}' = \mathbf{x} \cdot \mathbf{x}' + \mathbf{y} \cdot \mathbf{y}' \in R_v$, and since $\boldsymbol{\theta} \in U_{\mathbf{y},\epsilon}$, we have

$$\begin{aligned} \| \mathbf{x}' \|_{\mathbf{s}} \, \| A\mathbf{x}' + \mathbf{y}' - \boldsymbol{\theta} \|_{\mathbf{r}} &\geqslant \frac{1}{\| \mathbf{y} \|_{\mathbf{r}}} \, \| \mathbf{y} \|_{\mathbf{r}} \, \| A\mathbf{x}' + \mathbf{y}' - \boldsymbol{\theta} \|_{\mathbf{r}} \\ &\geqslant \frac{1}{\| \mathbf{y} \|_{\mathbf{r}}} \, \big(\max_{1 \leqslant j \leqslant m} \| \mathbf{y} \|_{\mathbf{r}}^{r_j} \| A\mathbf{x}' + \mathbf{y}' - \boldsymbol{\theta} \|_{\mathbf{r}}^{r_j} \big)^{\frac{1}{\min \mathbf{r}}} \\ &\geqslant \frac{1}{\| \mathbf{y} \|_{\mathbf{r}}} \, \big| \mathbf{y} \cdot (A\mathbf{x}' + \mathbf{y}' - \boldsymbol{\theta}) \big|^{\frac{1}{\min \mathbf{r}}} \geqslant \frac{1}{\| \mathbf{y} \|_{\mathbf{r}}} \, |\langle \, \mathbf{y} \cdot \boldsymbol{\theta} \, \rangle|^{\frac{1}{\min \mathbf{r}}} \geqslant \epsilon \, . \end{aligned}$$

Therefore $\boldsymbol{\theta} \in \mathbf{Bad}_A(\epsilon)$, as wanted. □

For every matrix $A \in \mathscr{M}_{m,n}(K_v)$, a *best approximation sequence* for A with weights $(\mathbf{r}, \mathbf{s})$ is a sequence $(\mathbf{y}_i)_{i \geqslant 1}$ in R_v^n such that, with $Y_i = \| \mathbf{y}_i \|_{\mathbf{s}}$ and $M_i = \langle \, A \, \mathbf{y}_i \, \rangle_{\mathbf{r}}$,

- the sequence $(Y_i)_{i \geqslant 1}$ is positive and strictly increasing,
- the sequence $(M_i)_{i \geqslant 1}$ is positive and strictly decreasing, and
- for every $\mathbf{y} \in R_v^n - \{0\}$ with $\| \mathbf{y} \|_{\mathbf{s}} < Y_{i+1}$, we have $\langle \, A \, \mathbf{y} \, \rangle_{\mathbf{r}} \geqslant M_i$.

We denote by $\operatorname{lcm} \mathbf{r}$ the least common multiple of $r_1, \ldots, r_m$, and similarly for $\operatorname{lcm} \mathbf{s}$.

Lemma 2.6 *Assume that $A \in \mathscr{M}_{m,n}(K_v)$ is completely irrational.*

(1) There exists a best approximation sequence $(\mathbf{y}_i)_{i \geqslant 1}$ for A with weights $(\mathbf{r}, \mathbf{s})$.

(2) If $(\mathbf{y}_i)_{i \geqslant 1}$ is a best approximation sequence for A with weights $(\mathbf{r}, \mathbf{s})$, then

i) we have $M_i \in q_v^{\frac{1}{\operatorname{lcm} \mathbf{r}} \mathbb{Z}}$ and $M_i \in q_v^{\frac{1}{\operatorname{lcm} \mathbf{r}} \mathbb{Z}_{\leqslant 0}}$ if i is large enough,

ii) we have $Y_i \in q_v^{\frac{1}{\operatorname{lcm} \mathbf{s}} \mathbb{Z}_{\geqslant 0}}$ and $Y_i \geqslant q_v^{\frac{i-1}{\operatorname{lcm} \mathbf{s}}}$ for every $i \geqslant 1$,

iii) the sequence $\big(M_i \, Y_{i+1} \big)_{i \geqslant 1}$ is uniformly bounded.

Note that a best approximation sequence might be not unique (and the terminology "best", though traditional, is not very appropriate). When $m = n = r_1 = s_1 = 1$, $K = \mathbb{F}_q(Z)$ and $v = \infty$, then $A \in K_v$ is completely irrational if and only if $A \in K_v - K$, and with $\big(\frac{P_k}{Q_k} \big)_{k \geqslant 0}$ the sequence of convergents of A (see for instance [Las]), we may take $y_i = Q_{i-1}$ for all $i \geqslant 1$.

If $A \in \mathscr{M}_{m,n}(K_v)$ is not completely irrational, a *best approximation sequence* for A with weights $(\mathbf{r}, \mathbf{s})$ is a finite sequence $(\mathbf{y}_i)_{1 \leqslant i \leqslant i_0}$ in R_v^n, such that, with $Y_i = \| \mathbf{y}_i \|_{\mathbf{s}}$ and $M_i = \langle \, A \, \mathbf{y}_i \, \rangle_{\mathbf{r}}$,

- $1 = Y_1 < \cdots < Y_{i_0}$,
- $M_1 > \cdots > M_{i_0} = 0$,
- for all $i \in \{1, \ldots, i_0 - 1\}$ and $\mathbf{y} \in R_v^n - \{0\}$ with $\| \mathbf{y} \|_{\mathbf{s}} < Y_{i+1}$, we have $\langle \, A \, \mathbf{y} \, \rangle_{\mathbf{r}} \geqslant M_i$,

and

- which stops at the first i_0 such that there exists $\mathbf{z} \in R_v^n$ with $0 < \| \mathbf{z} \|_{\mathbf{s}} \leqslant Y_{i_0}$ and $\langle \, A \, \mathbf{z} \, \rangle_{\mathbf{r}} = 0$.

The proof of Lemma 2.6 is similar to the one given after [BZ, Def. 3.3] in the particular case when $K = \mathbb{F}_q(Z)$, $v = \infty$ and without weights.

Proof. (1) Let us prove by induction on $i \geqslant 1$ that there exist $\mathbf{y}_1, \ldots, \mathbf{y}_i$ in R_v^n such that, with $Y_j = \| \mathbf{y}_j \|_{\mathbf{s}}$ and $M_j = \langle \, A \, \mathbf{y}_j \, \rangle_{\mathbf{r}}$ for every $1 \leqslant j \leqslant i$, we have $1 = Y_1 < \cdots < Y_i$, $M_1 > \cdots > M_i > 0$, and (using $M_0 = +\infty$ by convention)

(a_i) we have $\langle \, A \, \mathbf{y} \, \rangle_{\mathbf{r}} \geqslant M_{i-1}$ for every $\mathbf{y} \in R_v^n - \{0\}$ with $\| \mathbf{y} \|_{\mathbf{s}} < Y_i$,

(b_i) we have $\langle A\,\mathbf{y}\rangle_{\mathbf{r}} \geqslant M_i$ for every $\mathbf{y} \in R_v^n - \{0\}$ with $\|\,\mathbf{y}\,\|_{\mathbf{s}} \leqslant Y_i$.

Note that $\{x \in R_v : |\,x\,| \leqslant 1\} = R_v \cap \mathscr{O}_v = \mathbb{F}_q$ by Equation (3). Hence the elements with smallest $\mathbf{s}$-quasinorm in $R_v^n - \{0\}$ are the elements in the finite set $\mathbb{F}_q^n - \{0\}$, which is the set of elements in R_v^n with $\mathbf{s}$-quasinorm 1. Furthermore, the set $\{\|\,\mathbf{y}\,\|_{\mathbf{s}} : \mathbf{y} \in R_v^n - \{0\}\}$ is contained in $q_v^{\bigcup_{j=1}^n \frac{1}{s_j}\mathbb{Z}_{\geqslant 0}} \subset q_v^{\frac{1}{\operatorname{lcm}\mathbf{s}}\mathbb{Z}_{\geqslant 0}}$. Similarly, for every $\mathbf{x} \in K_v^m - \{0\}$, we have $\langle\,\mathbf{x}\,\rangle_{\mathbf{r}} \in q_v^{\frac{1}{\operatorname{lcm}\mathbf{r}}\mathbb{Z}}$.

Therefore there exists an element $\mathbf{y}_1 \in R_v^n$ with $\|\,\mathbf{y}_1\,\|_{\mathbf{s}} = 1$ such that

$$\langle A\,\mathbf{y}_1\rangle_{\mathbf{r}} = \min\{\,\langle A\,\mathbf{y}\rangle_{\mathbf{r}} : \mathbf{y} \in R_v^n,\ \ \|\,\mathbf{y}\,\|_{\mathbf{s}} = 1\,\}\,.$$

We thus have $Y_1 = \|\,\mathbf{y}_1\,\|_{\mathbf{s}} = 1$ and $M_1 = \langle A\,\mathbf{y}_1\rangle_{\mathbf{r}} > 0$ since A is completely irrational. There is no $\mathbf{y} \in R_v^n - \{0\}$ with $\|\,\mathbf{y}\,\|_{\mathbf{s}} < Y_1$, and if $\|\,\mathbf{y}\,\|_{\mathbf{s}} = Y_1$, then $\langle A\,\mathbf{y}\rangle_{\mathbf{r}} \geqslant M_1$, hence the claims (a_1) and (b_1) are satisfied.

Assume by induction that $\mathbf{y}_1, \dots, \mathbf{y}_i$ as above are constructed. Let

$$S = \{\,\mathbf{y} \in R_v^n :\ \|\,\mathbf{y}\,\|_{\mathbf{s}} > Y_i,\ \ \langle A\,\mathbf{y}\rangle_{\mathbf{r}} < M_i\,\}\,.$$

Note that the set $\{\mathbf{z} \in R_v^n,\ \ 0 < \|\,\mathbf{z}\,\|_{\mathbf{s}} \leqslant Y_i\}$ is finite by the discreteness of R_v^n, and $\epsilon_i = \min\{\,\langle A\,\mathbf{z}\rangle_{\mathbf{r}} : \mathbf{z} \in R_v^n,\ \ 0 < \|\,\mathbf{z}\,\|_{\mathbf{s}} \leqslant Y_i\,\}$ is positive, since A is completely irrational. Corollary 2.4 of Dirichlet's theorem implies in particular, by taking in its statement α large enough, that for every $\epsilon > 0$, there exists $\mathbf{y} \in R_v^n - \{0\}$ such that $\langle A\,\mathbf{y}\rangle_{\mathbf{r}} < \epsilon$. Applying this with $\epsilon = \min\{M_i, \epsilon_i\} > 0$ proves that the set S is nonempty. Hence the set $S_{\min}$ of elements of S with minimal $\mathbf{s}$-quasinorm, which is finite again by the discreteness of R_v^n, is nonempty. Therefore there exists $\mathbf{y}_{i+1} \in S_{\min}$ such that

$$\langle A\,\mathbf{y}_{i+1}\rangle_{\mathbf{r}} = \min\{\,\langle A\,\mathbf{z}\rangle_{\mathbf{r}} : \mathbf{z} \in S_{\min}\,\}\,.$$

Then $Y_{i+1} = \|\,\mathbf{y}_{i+1}\,\|_{\mathbf{s}} = \min \|\,S\,\|_{\mathbf{s}} > Y_i$ by the definition of the set S. We also have that $M_{i+1} = \langle A\,\mathbf{y}_{i+1}\rangle_{\mathbf{r}} < M_i$ since $\mathbf{y}_{i+1} \in S_{\min} \subset S$, and again by the definition of S.

Let us now prove that $\mathbf{y}_{i+1}$ satisfies the properties (a_{i+1}) and (b_{i+1}).

- Let $\mathbf{y} \in R_v^n - \{0\}$ be such that $\|\,\mathbf{y}\,\|_{\mathbf{s}} < Y_{i+1}$. If $\|\,\mathbf{y}\,\|_{\mathbf{s}} \leqslant Y_i$, then by the induction hypothesis (b_i), we have $\langle A\,\mathbf{y}\rangle_{\mathbf{r}} \geqslant M_i$, as wanted for Property (a_{i+1}). If $\|\,\mathbf{y}\,\|_{\mathbf{s}} > Y_i$, then by the definition of S, we have $\langle A\,\mathbf{y}\rangle_{\mathbf{r}} \geqslant M_i$ as wanted for Property (a_{i+1}), otherwise $\mathbf{y}$ would be an element of S with $\mathbf{s}$-quasinorm strictly less than the minimum $\mathbf{s}$-quasinorm of the elements of S, a contradiction.
- Let $\mathbf{y} \in R_v^n - \{0\}$ be such that $\|\,\mathbf{y}\,\|_{\mathbf{s}} \leqslant Y_{i+1}$. Either $\|\,\mathbf{y}\,\|_{\mathbf{s}} < Y_{i+1}$, in which case, as just seen, $\langle A\,\mathbf{y}\rangle_{\mathbf{r}} \geqslant M_i \geqslant M_{i+1}$, as wanted for Property (b_{i+1}). Or $\|\,\mathbf{y}\,\|_{\mathbf{s}} = Y_{i+1} > Y_i$, in which case either $\langle A\,\mathbf{y}\rangle_{\mathbf{r}} \geqslant M_i \geqslant M_{i+1}$, as wanted for Property (b_{i+1}), or $\langle A\,\mathbf{y}\rangle_{\mathbf{r}} < M_i$, so that $\mathbf{y}$ belongs to $S_{\min}$, hence $\langle A\,\mathbf{y}\rangle_{\mathbf{r}} \geqslant \min\{\,\langle A\,\mathbf{z}\rangle_{\mathbf{r}} : \mathbf{z} \in S_{\min}\,\} = M_{i+1}$.

By induction, this proves Assertion (1) of Lemma 2.6.

(2) i) This follows from the facts that $M_i \in q_v^{\frac{1}{\operatorname{lcm}\mathbf{r}}\mathbb{Z}}$ and that $M_{i+1} < M_i$.

ii) Since $Y_1 = 1$, this follows by induction from the facts that $Y_i \in q_v^{\frac{1}{\operatorname{lcm}\mathbf{s}}\mathbb{Z}}$ and that $Y_{i+1} > Y_i$.

iii) Let $\alpha = \big\lfloor \log_{q_v}(q^{-\frac{\deg v + g - 1}{\min \mathbf{s}}}\ Y_{i+1})\big\rfloor - 1$, which satisfies $\alpha > \frac{1}{\min \mathbf{r}} + \frac{g-1}{(\min \mathbf{r})(\deg v)}$ if i is large enough, by Assertion (2) ii). By Corollary 2.4, there exists $\mathbf{y} \in R_v^n - \{0\}$ such that

$$\|\,\mathbf{y}\,\|_{\mathbf{s}} \leqslant q^{\frac{\deg v + g - 1}{\min \mathbf{s}}}\ q_v^{\alpha} < q^{\frac{\deg v + g - 1}{\min \mathbf{s}}}\ q_v^{\log_{q_v}(q^{-\frac{\deg v + g - 1}{\min \mathbf{s}}}\ Y_{i+1})} = Y_{i+1}$$

and

$$\langle A\,\mathbf{y}\rangle_{\mathbf{r}} \leqslant q^{\frac{\deg v+g-1}{\min \mathbf{r}}}\, q_v^{-\alpha}$$

$$\leqslant q^{\frac{\deg v+g-1}{\min \mathbf{r}}}\, q_v^{-\left(\log_{q_v}(q^{-\frac{\deg v+g-1}{\min \mathbf{s}}}\, Y_{i+1})-2\right)} = q^{(\deg v+g-1)\left(\frac{1}{\min \mathbf{r}}+\frac{1}{\min \mathbf{s}}\right)+2\deg v}\,(Y_{i+1})^{-1}\ .$$

Since $M_i \leqslant \min\{\ \langle A\,\mathbf{y}\rangle_{\mathbf{r}}\ :\ \mathbf{y}\in R_v^n,\ 0<\|\,\mathbf{y}\,\|_{\mathbf{s}}<Y_{i+1}\}$ by the definition of a best approximation sequence, the result follows. □

2.4 Transference theorems with weights

In this section, we will show that a matrix $A \in \mathscr{M}_{m,n}(K_v)$ is singular on average if and only if its transpose ${}^tA$ is singular on average. To do this, following [Cas, Chap. V], we prove a transference principle between two problems of homogeneous approximations with weights. See also [GE, Ger] in the disjoint case of the field $\mathbb{Q}$.

Let $d \in \mathbb{Z}_{\geqslant 2}$ be a positive integer at least 2. For all $\boldsymbol{\xi} = (\xi_1,\dots,\xi_d)$ and $\boldsymbol{\theta} = (\theta_1,\dots,\theta_d)$ in $K_v^{\,d}$, we denote

$$\boldsymbol{\xi}\cdot\boldsymbol{\theta} = \sum_{k=1}^{d} \xi_k\,\theta_k\ .$$

Let $\alpha_1,\dots,\alpha_d \in \mathbb{Z}$ be integers and let $\alpha = \sum_{k=1}^{d} \alpha_k$. We consider the parallelepiped

$$\mathscr{P} = \left\{\boldsymbol{\xi} = (\xi_1,\dots,\xi_d) \in K_v^{\,d} : \forall\ k = 1,\dots,d,\ \ |\,\xi_k\,| \leqslant q_v^{\alpha_k}\right\}\ .$$

Following Schmidt's terminology [Sch3, page 109] in the case of the field $\mathbb{Q}$ (building on Mahler's compound one), we call the parallelepiped

$$\mathscr{P}^* = \Big\{\boldsymbol{\xi} = (\xi_1,\dots,\xi_d) \in K_v^{\,d} : \forall\ k = 1,\dots,d,\ \ |\,\xi_k\,| \leqslant \frac{1}{q_v^{\alpha_k}} \prod_{i=1}^{d} q_v^{\alpha_i} = q_v^{\alpha-\alpha_k}\Big\}$$

the *pseudocompound* of $\mathscr{P}$. Note that $\mathscr{P}$ and $\mathscr{P}^*$ are preserved by the multiplication of the components of their elements by elements of $\mathscr{O}_v$.

Theorem 2.7 *With $\mathscr{P}$ and $\mathscr{P}^*$ as above, for every $F \in \mathrm{SL}_d(K_v)$,*

$$\textit{if}\ \ \mathscr{P}^* \cap {}^tF^{-1}(R_v^d) \neq \{0\}, \quad \textit{then}\ \ \pi_v^{-\beta_d}\mathscr{P} \cap F(R_v^d) \neq \{0\},$$

where

$$\beta_d = \left\lceil \frac{1}{d-1}\Big(d+1+\frac{(g-1)d}{\deg v}\Big)\right\rceil .$$

Remark. The R_v-lattice ${}^tF^{-1}(R_v^d)$ is called the *dual lattice* of the R_v-lattice $F(R_v^d)$ since we have $\mathbf{z}\cdot\mathbf{w} \in R_v$ for all $\mathbf{z} \in {}^tF^{-1}(R_v^d)$ and $\mathbf{w} \in F(R_v^d)$. They have the same covolume as R_v^d, since $\det(F) = 1$.

Proof. Let $\mathbf{z} = (z_1,\dots,z_d) \in \mathscr{P}^* \cap {}^tF^{-1}(R_v^d) - \{0\}$ and $\kappa_0 = \max\{k \in \mathbb{Z}_{\geqslant 0} : \mathbf{z} \in \pi_v^k \mathscr{P}^*\}$. Up to permuting the coordinates, we may assume that, for all $k = 2,\dots,d$, we have

$$|\,z_1\,| = q_v^{\alpha-\alpha_1-\kappa_0} \quad \text{and} \quad |\,z_k\,| \leqslant q_v^{\alpha-\alpha_k-\kappa_0}\ . \tag{5}$$

With F_k the k-th row of F, let us consider the R_v-lattice $\Lambda = M(R_v^d)$ where

$$M = \begin{pmatrix} \pi_v^{-1}\sum_{k=1}^d z_k F_k \\ \pi_v^{\beta_d+\alpha_2} F_2 \\ \vdots \\ \pi_v^{\beta_d+\alpha_d} F_d \end{pmatrix} .$$

By substracting to the first row a linear combination of the other rows, and since $\det F = 1$, the determinant of the above matrix M is equal to $\pi_v^{(d-1)\beta_d+\alpha-\alpha_1-1}\, z_1$. By Equations (5) and (4), we thus have

$$\operatorname{Covol}(\Lambda) = \det(M)\operatorname{Covol}(R_v^d) = q_v^{1-\kappa_0-(d-1)\beta_d}\, q^{(g-1)d} .$$

Since $d \geqslant 2$ and $\beta_d \geqslant \frac{1}{d-1}\big(d+1+\frac{(g-1)d}{\deg v}\big)$, Corollary 2.2 applied to the R_v-lattice Λ gives that

$$\lambda_1(\Lambda) \leqslant q_v \operatorname{Covol}(\Lambda)^{\frac{1}{d}} \leqslant 1.$$

Hence, by the definition of the first minimum $\lambda_1(\Lambda)$, there exists $\mathbf{w} \in R_v^d - \{0\}$ such that for every $k = 2, \dots, d$, we have

$$|\,\mathbf{z}\cdot F(\mathbf{w})\,| \leqslant q_v^{-1} < 1 \quad \text{and} \quad |\,F_k(\mathbf{w})\,| \leqslant q_v^{\beta_d+\alpha_k} . \tag{6}$$

Since $\mathbf{z} \in {}^tF^{-1}(R_v^d)$ and $\mathbf{w} \in R_v^d$, we have $\mathbf{z}\cdot F(\mathbf{w}) \in R_v$ by the above Remark. The first inequality of Equation (6) hence implies that $\mathbf{z}\cdot F(\mathbf{w}) = 0$, which means that

$$z_1 F_1(\mathbf{w}) = -\sum_{k=2}^d z_k F_k(\mathbf{w}) .$$

By the ultrametric property of $|\cdot|$, by Equations (5) and (6), we have

$$\begin{aligned} q_v^{\alpha-\alpha_1-\kappa_0}|\,F_1(\mathbf{w})\,| = |\,z_1F_1(\mathbf{w})\,| &\leqslant \max_{2\leqslant k\leqslant d} |\,z_kF_k(\mathbf{w})\,| \\ &\leqslant \max_{2\leqslant k\leqslant d} q_v^{\alpha-\alpha_k-\kappa_0} q_v^{\beta_d+\alpha_k} = q_v^{\alpha+\beta_d-\kappa_0} . \end{aligned}$$

Therefore $|\,F_1(\mathbf{w})\,| \leqslant q_v^{\beta_d+\alpha_1}$ and with the second inequality of Equation (6), we conclude that $F(\mathbf{w}) \in \pi_v^{\beta_d}\mathscr{P}$. □

Corollary 2.8 *There exist $\kappa_1,\kappa_2,\kappa_3,\kappa_4 \geqslant 0$ with $\kappa_2 > 0$, depending only on m, n, g, $\deg v$, $\mathbf{r}$ and $\mathbf{s}$, such that for all $A \in \mathscr{M}_{m,n}(K_v)$ and $\epsilon \in q_v^{\mathbb{Z}_{\leqslant -1}}$, for every large enough $Y \in q_v^{\mathbb{Z}_{\geqslant 1}}$, if there exists $\mathbf{y} \in R_v^n - \{0\}$ such that*

$$\langle\, A\mathbf{y}\,\rangle_{\mathbf{r}} \leqslant \epsilon\, Y^{-1} \quad \textit{and} \quad \|\,\mathbf{y}\,\|_{\mathbf{s}} \leqslant Y , \tag{7}$$

then there exists $\mathbf{x} \in R_v^m - \{0\}$ such that

$$\langle\, {}^tA\mathbf{x}\,\rangle_{\mathbf{s}} \leqslant q_v^{\kappa_1}\,\epsilon^{\kappa_2}\, X^{-1} \quad \textit{and} \quad \|\,\mathbf{x}\,\|_{\mathbf{r}} \leqslant X , \tag{8}$$

where $X = q_v^{\kappa_3}\,\epsilon^{-\kappa_4}\, Y$.

Proof. Let $|\mathbf{s}| = \sum_{j=1}^{n} s_j$. Denoting $\alpha_\epsilon = -\log_{q_v} \epsilon \in \mathbb{Z}_{\geqslant 1}$ and $\alpha_Y = \log_{q_v} Y \in \mathbb{Z}_{\geqslant 1}$, we define $\delta = q_v^{-\alpha_\delta}$ and $Z = q_v^{\alpha_Z}\, Y$ where

$$\alpha_\delta = \Big\lfloor \frac{\alpha_\epsilon - 1}{|\mathbf{s}|\big(\frac{1}{\min \mathbf{r}} + \frac{1}{\min \mathbf{s}}\big) - 1} \Big\rfloor \quad \text{and} \quad \alpha_Z = \Big\lceil \Big(\frac{|\mathbf{s}|}{\min \mathbf{s}} - 1\Big)\alpha_\delta \Big\rceil \;. \tag{9}$$

Note that α_δ is well defined since $\frac{|\mathbf{s}|}{\min \mathbf{s}} \geqslant 1$, and that α_δ and α_Z are nonnegative. We have

$$\begin{aligned} \big(|\mathbf{s}|\big(\frac{1}{\min \mathbf{r}} + \frac{1}{\min \mathbf{s}}\big) - 1\big)\, \alpha_\delta &\leqslant \alpha_\epsilon - 1 \;, \\ \text{hence} \;\; \big(\frac{|\mathbf{s}|}{\min \mathbf{s}} - 1\big)\, \alpha_\delta + 1 &\leqslant \alpha_\epsilon - \frac{|\mathbf{s}|}{\min \mathbf{r}}\, \alpha_\delta \;, \\ \text{therefore} \;\; \big(\frac{|\mathbf{s}|}{\min \mathbf{s}} - 1\big)\, \alpha_\delta \leqslant \alpha_Z &\leqslant \alpha_\epsilon - \frac{|\mathbf{s}|}{\min \mathbf{r}}\, \alpha_\delta \;. \end{aligned} \tag{10}$$

Let $d = m + n \geqslant 2$. Let us consider the following parallelepipeds

$$\mathscr{Q} = \left\{ \xi = (\xi_1, \dots, \xi_d) \in K_v^{\,d} : \begin{array}{l} \forall\, i = 1, \dots, m, \;\; |\,\xi_i\,| \leqslant \epsilon^{r_i}\, Y^{-r_i} \\ \forall\, j = 1, \dots, n, \;\; |\,\xi_{m+j}\,| \leqslant Y^{s_j} \end{array} \right\} ,$$

$$\mathscr{P} = \left\{ \xi = (\xi_1, \dots, \xi_d) \in K_v^{\,d} : \begin{array}{l} \forall\, i = 1, \dots, m, \;\; |\,\xi_i\,| \leqslant Z^{r_i} \\ \forall\, j = 1, \dots, n, \;\; |\,\xi_{m+j}\,| \leqslant \delta^{s_j}\, Z^{-s_j} \end{array} \right\} .$$

Since $\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} s_j$, the pseudocompound $\mathscr{P}^*$ of $\mathscr{P}$ is equal to

$$\mathscr{P}^* = \left\{ \xi = (\xi_1, \dots, \xi_d) \in K_v^{\,d} : \begin{array}{l} \forall\, i = 1, \dots, m, \;\; |\,\xi_i\,| \leqslant \delta^{|\mathbf{s}|}\, Z^{-r_i} \\ \forall\, j = 1, \dots, n, \;\; |\,\xi_{m+j}\,| \leqslant \delta^{|\mathbf{s}| - s_j} Z^{s_j} \end{array} \right\} .$$

By the right inequality of Equation (10), for every $i = 1, \dots, m$, we have

$$\delta^{|\mathbf{s}|} Z^{-r_i} = q_v^{-|\mathbf{s}|\alpha_\delta - r_i \alpha_Z}\; Y^{-r_i} \geqslant q_v^{-r_i(\alpha_Z + \frac{|\mathbf{s}|}{\min \mathbf{r}}\alpha_\delta)}\; Y^{-r_i} \geqslant \epsilon^{r_i}\, Y^{-r_i} \;.$$

By the left inequality of Equation (10), for every $j = 1, \dots, n$, we have

$$\delta^{|\mathbf{s}| - s_j} Z^{s_j} = q_v^{-(|\mathbf{s}| - s_j)\alpha_\delta + s_j \alpha_Z}\; Y^{s_j} \geqslant q_v^{s_j(\alpha_Z - (\frac{|\mathbf{s}|}{\min \mathbf{s}} - 1)\alpha_\delta)}\; Y^{s_j} \geqslant Y^{s_j} \;.$$

Therefore $\mathscr{Q}$ is contained in $\mathscr{P}^*$.

Now, by the assumption of Corollary 2.8, let $\mathbf{y} \in R_v^{\,n} - \{0\}$ be such that the inequalities (7) are satisfied. Then there exists $(\mathbf{x}', \mathbf{y}) \in R_v^{\,m} \times (R_v^{\,n} - \{0\})$ such that

$$\|A\mathbf{y} - \mathbf{x}'\|_{\mathbf{r}} \leqslant \epsilon\, Y^{-1} \quad \text{and} \quad \|\, \mathbf{y}\,\|_{\mathbf{s}} \leqslant Y \;.$$

Therefore

$$\mathscr{Q} \cap \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} R_v^{\,d} \neq \{0\} \;.$$

Since $\mathscr{Q} \subset \mathscr{P}^*$, this implies that

$$\mathscr{P}^* \cap \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} R_v^{\,d} \neq \{0\} \;.$$

By Theorem 2.7, we have

$$\pi_v^{-\beta_d}\mathscr{P}\cap\begin{pmatrix} I_m & 0\\ -\,{}^tA & I_n\end{pmatrix} R_v^{\,d}\neq\{0\}\;.$$

Then there exists $(\mathbf{x},\mathbf{y}')\in(R_v^{\,m}\times R_v^{\,n})-\{0\}$ such that

$$\|\pi_v^{\beta_d}\,\mathbf{x}\|_{\mathbf{r}}\leqslant Z\quad\text{and}\quad\|\pi_v^{\beta_d}(-\,{}^tA\mathbf{x}-\mathbf{y}')\|_{\mathbf{s}}\leqslant\delta\,Z^{-1}\;. \tag{11}$$

The above inequality on the left-hand side and the two equalities of Equation (9) give

$$\begin{aligned}\|\,\mathbf{x}\,\|_{\mathbf{r}}&\leqslant q_v^{\frac{\beta_d}{\min\mathbf{r}}}\,Z=q_v^{\frac{\beta_d}{\min\mathbf{r}}+\alpha_Z}\,Y\leqslant q_v^{\frac{\beta_d}{\min\mathbf{r}}+1+(\frac{|\mathbf{s}|}{\min\mathbf{s}}-1)\frac{\alpha_\epsilon-1}{|\mathbf{s}|(\frac{1}{\min\mathbf{r}}+\frac{1}{\min\mathbf{s}})-1}}\;Y\\ &\leqslant q_v^{\frac{\beta_d}{\min\mathbf{r}}+1}\;\epsilon^{-\frac{\frac{|\mathbf{s}|}{\min\mathbf{s}}-1}{|\mathbf{s}|(\frac{1}{\min\mathbf{r}}+\frac{1}{\min\mathbf{s}})-1}}\;Y\;.\end{aligned}$$

If $\kappa_3=\frac{\beta_d}{\min\mathbf{r}}+1>0$ and $\kappa_4=\frac{\frac{|\mathbf{s}|}{\min\mathbf{s}}-1}{|\mathbf{s}|(\frac{1}{\min\mathbf{r}}+\frac{1}{\min\mathbf{s}})-1}\geqslant 0$, this proves the right inequality in Equation (8) with $X=q_v^{\kappa_3}\,\epsilon^{-\kappa_4}\,Y$.

The right inequality in Equation (11), since $\beta_d\geqslant 0$ and by using the left inequality in Equation (10) and the definition (9) of α_δ, gives

$$\begin{aligned}\langle\,{}^tA\mathbf{x}\,\rangle_{\mathbf{s}}&\leqslant q_v^{\frac{\beta_d}{\min\mathbf{s}}}\,\delta\,Z^{-1}=q_v^{\frac{\beta_d}{\min\mathbf{s}}-\alpha_\delta-\alpha_Z}\,Y^{-1}\leqslant q_v^{\frac{\beta_d}{\min\mathbf{s}}-\frac{|\mathbf{s}|}{\min\mathbf{s}}\alpha_\delta+\kappa_3}\;\epsilon^{-\kappa_4}\,X^{-1}\\ &\leqslant q_v^{\frac{\beta_d}{\min\mathbf{s}}-\frac{|\mathbf{s}|}{\min\mathbf{s}}\Big(\frac{\alpha_\epsilon-1}{|\mathbf{s}|\left(\frac{1}{\min\mathbf{r}}+\frac{1}{\min\mathbf{s}}\right)-1}-1\Big)+\kappa_3+\frac{\frac{|\mathbf{s}|}{\min\mathbf{s}}-1}{|\mathbf{s}|(\frac{1}{\min\mathbf{r}}+\frac{1}{\min\mathbf{s}})-1}\alpha_\epsilon}\;X^{-1}\\ &=q_v^{\frac{\beta_d}{\min\mathbf{s}}+\frac{|\mathbf{s}|}{\min\mathbf{s}}\Big(\frac{1}{|\mathbf{s}|\left(\frac{1}{\min\mathbf{r}}+\frac{1}{\min\mathbf{s}}\right)-1}+1\Big)+\kappa_3}\;\epsilon^{\frac{1}{|\mathbf{s}|(\frac{1}{\min\mathbf{r}}+\frac{1}{\min\mathbf{s}})-1}}\;X^{-1}\;.\end{aligned}$$

This proves the left inequality in Equation (8) for appropriate positive constants κ_1 and κ_2.

If $\mathbf{x}=0$, then we have $\mathbf{y}'\neq 0$ and $\|\,\mathbf{y}'\,\|_{\mathbf{s}}\leqslant q_v^{\kappa_1-\kappa_3}\,\epsilon^{\kappa_2+\kappa_4}\,Y^{-1}$, which contradicts the fact that $\mathbf{y}'\in R_v^{\,n}$ if Y is large enough. This concludes the proof of Corollary 2.8. □

Corollary 2.9 *Let m,n be positive integers and $A\in\mathscr{M}_{m,n}(K_v)$. Then A is $(\mathbf{r},\mathbf{s})$-singular on average if and only if ${}^tA$ is $(\mathbf{s},\mathbf{r})$-singular on average.*

Proof. This follows from Corollary 2.8. □

It follows from this corollary and from Remark 2.5 that if $A\in\mathscr{M}_{m,n}(K_v)$ is such that ${}^tA$ is not completely irrational, then A is $(\mathbf{r},\mathbf{s})$-singular on average.

3 Characterisation of singular on average property

In this section, we give a characterisation of the singular on average property with weights in terms of an asymptotic property in average of the best approximation sequence with weights. In the real case, the relation between the singular property and the best approximation sequence has been studied in [Cheu, Chev, CC, LSST]. Also in the real case, and with weights, the relation (similar to the one below) between the singular on average property and the best approximation sequence has been studied in [KKL, Prop. 6.7].

For the sake of later applications, we work with transposes of matrices.

Theorem 3.1 *Let $A \in \mathscr{M}_{m,n}(K_v)$ and let $(\mathbf{y}_i)_{i\geqslant 1}$ be a best approximation sequence in K_v^m for ${}^tA$ with weights $(\mathbf{s},\mathbf{r})$. The following statements are equivalent.*

(1) *For all $a > 1$ and $\epsilon > 0$, we have*

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{Card}\{\, \ell \in \{1,\dots,N\} : \exists\, \mathbf{y} \in R_v^m - \{0\},\ \langle\, {}^tA\,\mathbf{y}\,\rangle_{\mathbf{s}} \leqslant \epsilon\, a^{-\ell},\ \|\,\mathbf{y}\,\|_{\mathbf{r}} \leqslant a^{\ell}\,\} = 1 \,.$$

(2) *The matrix ${}^tA$ is $(\mathbf{s},\mathbf{r})$-singular on average.*

(3) *There exists $a > 1$ such that for every $\epsilon > 0$, we have*

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{Card}\{\, \ell \in \{1,\dots,N\} : \exists\, \mathbf{y} \in R_v^m - \{0\},\ \langle\, {}^tA\,\mathbf{y}\,\rangle_{\mathbf{s}} \leqslant \epsilon\, a^{-\ell},\ \|\,\mathbf{y}\,\|_{\mathbf{r}} \leqslant a^{\ell}\,\} = 1 \,.$$

(4) *For every $\epsilon' > 0$, we have*

$$\lim_{k\to\infty} \frac{1}{\log_{q_v} Y_k} \operatorname{Card}\big\{ i \leqslant k : M_i\, Y_{i+1} > \epsilon' \big\} = 0 \,.$$

Proof. Since Assertion (2) is Assertion (1) for $a = q_v > 1$, it is immediate that (1) implies (2) implies (3).

Let us first prove that Assertion (3) implies Assertion (4). Let $a > 1$ be as in Assertion (3) and let $\epsilon' \in \,]0,1[\,$. Let $\epsilon = \frac{\epsilon'}{a} > 0$.

We may assume that the set $I = \{i \in \mathbb{Z}_{\geqslant 1} : M_i Y_{i+1} > \epsilon'\}$ is infinite, otherwise Assertion (4) is clear since $\lim_{k\to\infty} Y_k = +\infty$. We consider the increasing sequence $(i_j)_{j\in\mathbb{Z}_{\geqslant 1}}$ of positive integers such that $I = \{i_j : j \geqslant 1\}$. For every $j \geqslant 1$, by taking the logarithm in base a, we thus have $\log_a \epsilon' - \log_a M_{i_j} < \log_a Y_{i_j+1}$, hence

$$\log_a \epsilon - \log_a M_{i_j} < \log_a Y_{i_j+1} - 1 \,. \tag{12}$$

Note that for every $i \geqslant 1$ and $X \in [Y_i, Y_{i+1}[\,$, the system of inequalities

$$\langle\, {}^tA\,\mathbf{y}\,\rangle_{\mathbf{s}} \leqslant \epsilon\, X^{-1} \quad \text{and} \quad 0 < \|\,\mathbf{y}\,\|_{\mathbf{r}} \leqslant X \tag{13}$$

has a solution $\mathbf{y} \in R_v^m$ if and only if $M_i \leqslant \epsilon\, X^{-1}$. Indeed, if the later inequality is satisfied, then $\mathbf{y}_i$ is a solution of the system (13) since $M_i = \langle\, {}^tA\,\mathbf{y}_i\,\rangle_{\mathbf{s}}$ and $X \geqslant Y_i = \|\,\mathbf{y}_i\,\|_{\mathbf{r}}$. Conversely, if this system has a solution, then since

$$M_i \leqslant \min\{\, \langle\, {}^tA\,\mathbf{y}\,\rangle_{\mathbf{s}} : \mathbf{y} \in R_v^m,\ 0 < \|\,\mathbf{y}\,\|_{\mathbf{r}} < Y_{i+1}\,\}$$

by the definition of a best approximation sequence, the inequality $M_i \leqslant \epsilon\, X^{-1}$ holds since $X < Y_{i+1}$. Hence, for every integer $\ell \in [\log_a Y_i, \log_a Y_{i+1}[\,$, the system of inequalities (13) has no integral solutions for $X = a^{\ell}$ if and only if

$$\log_a \epsilon - \log_a M_i < \ell < \log_a Y_{i+1} \,. \tag{14}$$

There exists an integer $j_0 \geqslant 1$ such that for every integer $j \geqslant j_0$, we have $\log_a Y_{i_j+1} \geqslant 2$ by Lemma 2.6 (2) ii). If ℓ is the integer in the interval $[\log_a Y_{i_j+1} - 1, \log_a Y_{i_j+1}[$ (which is half-open and has length 1, hence does contain one and only one integer), then $\ell \geqslant 1$ and by Equations (12) and (14), the system (13) has no integral solutions for $X = a^{\ell}$.

Let $u = \lceil (\operatorname{lcm} \mathbf{r})(\log_{q_v} a) \rceil$, which belongs to $\mathbb{Z}_{\geqslant 1}$. By Lemma 2.6 (2) ii), for every $k \in \mathbb{Z}_{\geqslant 1}$, since the sequence $(i_j)_{j\in\mathbb{Z}_{\geqslant 1}}$ is increasing, we have

$$Y_{i_{k+u}+1} \geqslant q_v^{\frac{u}{\operatorname{lcm} \mathbf{r}}}\; Y_{i_k+1} \geqslant a\, Y_{i_k+1} \,.$$

The intervals $[\log_a Y_{i_{uj}+1} - 1, \log_a Y_{i_{uj}+1}[$ and $[\log_a Y_{i_{u(j+1)}+1} - 1, \log_a Y_{i_{u(j+1)}+1}[$ are hence disjoint for every $j \in \mathbb{Z}_{\geqslant 1}$. Thus, if j is large enough, with $N_j = \lceil \log_a Y_{i_{uj}+1} \rceil$, the number $n(N_j)$ of integers $\ell \in \{1, \dots, N_j\}$ such that the system of inequalities (13) has no integral solutions for $X = a^\ell$ is at least $j - j_0$. Therefore $\frac{j - j_0}{\lceil \log_a Y_{i_{uj}+1} \rceil} \leqslant \frac{n(N_j)}{N_j}$ tends to 0 as $j \to +\infty$, by Assertion (3). This implies that $\frac{j}{\log_a Y_{i_j}}$ tends to 0 as $j \to +\infty$.

For every integer $k \geqslant 1$, let $j(k) \geqslant 1$ be the unique positive integer such that we have $i_{j(k)} \leqslant k < i_{j(k)+1}$, so that $j(k) = \mathrm{Card}\{i \leqslant k : M_i Y_{i+1} > \epsilon'\}$. Hence, since $(Y_i)_{i \geqslant 1}$ is increasing, we have

$$\lim_{k \to \infty} \frac{1}{\log_{q_v} Y_k} \mathrm{Card}\big\{i \leqslant k : M_i Y_{i+1} > \epsilon'\big\} \leqslant \frac{\ln q_v}{\ln a} \lim_{k \to \infty} \frac{j(k)}{\log_a Y_{i_{j(k)}}} = 0 \;,$$

which proves Assertion (4).

Let us now prove that Assertion (4) implies Assertion (1). Let $a > 1$ and $\epsilon \in \,]0, 1[$. By Lemma 2.6 (2) iii), let $c \geqslant 1$ be such that for every $i \geqslant 1$, we have $M_i Y_{i+1} \leqslant a^c$. By Equation (14), since the number of integer points in an open interval is at most equal to its length, for every $i \geqslant 1$, the number of integers $\ell \in [\log_a Y_i, \log_a Y_{i+1}[$ such that the system of inequalities (13) has no integral solutions for $X = a^\ell$ is at most

$$\log_a Y_{i+1} - (\log_a \epsilon - \log_a M_i) = \big(\log_a M_i Y_{i+1} - \log_a \epsilon\big) \;.$$

For every $N \geqslant 1$ large enough, let $k_N \geqslant 1$ be such that $N \in [\log_a Y_{k_N}, \log_a Y_{k_N+1}[$ and let $n'(N)$ be the number of integers $\ell \in \{1, \dots, N\}$ such that the system of inequalities (13) has no integral solutions for $X = a^\ell$. Then

$$\begin{aligned}\frac{n'(N)}{N} &\leqslant \frac{1}{N} \sum_{i=1}^{k_N} \max\big\{0, \log_a M_i Y_{i+1} - \log_a \epsilon\big\} \\ &\leqslant \big(c - \log_a \epsilon\big) \frac{1}{\log_a Y_{k_N}} \mathrm{Card}\big\{i \leqslant k_N : M_i Y_{i+1} > \epsilon\big\} \;.\end{aligned}$$

This last term tends to 0 as $N \to +\infty$ by Assertion (4) applied with $\epsilon' = \epsilon$. Therefore $\lim_{N \to +\infty} \frac{n'(N)}{N} = 0$, thus proving Assertion (1). □

4 Full Hausdorff dimension for singular on average matrices

4.1 Modified Bugeaud-Zhang sequences

In this subsection, we construct a subsequence with controlled growth of the best approximation sequence with weights of a matrix, assuming that its transpose is singular on average for those weights. We use as inspiration [BZ, page 470] in the special case of $K = \mathbb{F}_q(Z)$ and $v = v_\infty$, and the first claim of the proof of [BuKLR, Theo. 2.2] in the case of the field $\mathbb{Q}$ (with characteristic zero).

Proposition 4.1 *Let $A \in \mathscr{M}_{m,n}(K_v)$ be such that ${}^tA$ is completely irrational and $(\mathbf{s}, \mathbf{r})$-singular on average. Let $(\mathbf{y}_i)_{i \in \mathbb{Z}_{\geqslant 1}}$ be a best approximation sequence in K_v^m for ${}^tA$ with weights $(\mathbf{s}, \mathbf{r})$, and let $c > 0$ be such that $M_i Y_{i+1} \leqslant q_v^c$ for every $i \in \mathbb{Z}_{\geqslant 1}$. For all $a > b > 0$, there exists an increasing map $\varphi : \mathbb{Z}_{\geqslant 1} \to \mathbb{Z}_{\geqslant 1}$ such that*

(1) for every $i \in \mathbb{Z}_{\geqslant 1}$, we have

$$Y_{\varphi(i+1)} \geqslant q_v^b \, Y_{\varphi(i)} \quad \text{and} \quad M_{\varphi(i)} \, Y_{\varphi(i+1)} \leqslant q_v^{b+c} \,, \tag{15}$$

(2) we have

$$\limsup_{k\to\infty} \frac{k}{\log_{q_v} Y_{\varphi(k)}} \leqslant \frac{1}{a} \,. \tag{16}$$

Proof. Let A, $(\mathbf{y}_i)_{i\in\mathbb{Z}_{\geqslant 1}}$ and a, b be as in the statement. We start by proving a particular case, that will be useful in two of the four cases below.

Lemma 4.2 *If furthermore we have $\lim_{k\to\infty} Y_k^{\frac{1}{k}} = +\infty$, then there exists an increasing map $\varphi : \mathbb{Z}_{\geqslant 1} \to \mathbb{Z}_{\geqslant 1}$ such that Equations* (15) *and* (16) *are satisfied.*

Proof. The fact that $\lim_{k\to\infty} Y_k^{\frac{1}{k}} = +\infty$ implies that the set

$$\mathscr{J}_0 = \{j \in \mathbb{Z}_{\geqslant 1} : Y_{j+1} \geqslant q_v^b \, Y_j\}$$

is infinite. We construct the increasing sequence $(\varphi(i))_{i\in\mathbb{Z}_{\geqslant 1}}$ of positive integers by stacks $\{\varphi(i_k+1), \dots, \varphi(i_{k+1})\}$ with $i_{k+1} > i_k$, by induction on $k \in \mathbb{Z}_{\geqslant 0}$. For $k = 0$, let $i_0 = 0$, let $i_1 = 1$ and let $\varphi(1)$ be the smallest element of $\mathscr{J}_0$.

For $k \in \mathbb{Z}_{\geqslant 0}$, assume that i_k and $\varphi(i_k)$ are constructed such that $\varphi(i_k) \in \mathscr{J}_0$ and Equation (15) holds for every $i \leqslant i_k - 1$. Let us construct i_{k+1} and $\varphi(i_k+1), \dots, \varphi(i_{k+1})$ such that $\varphi(i_{k+1}) \in \mathscr{J}_0$ and Equation (15) holds for every $i \leqslant i_{k+1} - 1$. Let j_0 be the smallest element of $\mathscr{J}_0$ greater than $\varphi(i_k)$. Let $r' = 0$ if the set $\{j > \varphi(i_k) : Y_{j_0} \geqslant q_v^b \, Y_j\}$ is empty. Otherwise, let $r' \in \mathbb{Z}_{\geqslant 1}$ be the maximal integer such that by induction there exist $j_1, j_2, \dots, j_{r'} \in \mathbb{Z}_{\geqslant 1}$ such that for $\ell = 1, \dots, r'$, the set $\{j > \varphi(i_k) : Y_{j_{\ell-1}} \geqslant q_v^b \, Y_j\}$ is nonempty and for $\ell = 1, \dots, r'+1$ the integer j_ℓ is its largest element. Since the sequence $(Y_i)_{i\in\mathbb{Z}_{\geqslant 1}}$ is increasing, this in particular implies that $j_{\ell-1} > j_\ell$ for $\ell = 1, \dots, r'+1$, which itself ensures the finiteness of r'. Now we define $i_{k+1} = i_k + r' + 1$ and

$$\varphi(i_k+1) = j_{r'}, \ \varphi(i_k+2) = j_{r'-1}, \ \dots, \ \varphi(i_k+r') = j_1, \ \varphi(i_{k+1}) = j_0 \,.$$

By construction, for $\ell = 1, \dots, r'$, we have

$$Y_{\varphi(i_k+\ell+1)} = Y_{j_{r'-\ell}} \geqslant q_v^b \, Y_{j_{r'-\ell+1}} = q_v^b \, Y_{\varphi(i_k+\ell)} \,.$$

As $\varphi(i_k+1) = j_{r'} > \varphi(i_k)$, we have $Y_{\varphi(i_k+1)} \geqslant Y_{\varphi(i_k)+1} \geqslant q_v^b \, Y_{\varphi(i_k)}$ since $\varphi(i_k) \in \mathscr{J}_0$. Note that $\varphi(i_{k+1}) = j_0 \in \mathscr{J}_0$. This proves the claim on the left hand side of Equation (15) for $i \leqslant i_{k+1} - 1$.

By the maximality property of $j_{r'-\ell}$ in the above construction, for every $\ell = 1, \dots, r'$, we have $Y_{\varphi(i_k+\ell+1)} = Y_{j_{r'-\ell}} < q_v^b \, Y_{j_{r'-\ell+1}+1} = q_v^b \, Y_{\varphi(i_k+\ell)+1}$. By the maximality of r' in the above construction, we have $Y_{\varphi(i_k+1)} < q_v^b \, Y_{\varphi(i_k)+1}$. Hence, by the definition of c, for every $\ell = 0, \dots, r'$, we have

$$M_{\varphi(i_k+\ell)} Y_{\varphi(i_k+\ell+1)} \leqslant M_{\varphi(i_k+\ell)} \, Y_{\varphi(i_k+\ell)+1} \, q_v^b \leqslant q_v^{b+c} \,.$$

This proves the claim on the right hand side of Equation (15) for $i \leqslant i_{k+1} - 1$.

Since $\lim_{k\to\infty} \frac{k}{\log_{q_v} Y_k} = 0$, Equation (16) is satisfied for φ, and this concludes the proof of Lemma 4.2. □

Now in what follows, we will discuss in four cases on the configuration in $\mathbb{Z}_{\geqslant 1}$ of the set

$$\mathscr{J} = \{j \in \mathbb{Z}_{\geqslant 1} : M_j \, Y_{j+1} \leqslant q_v^{b+c-3a}\} \ .$$

By Theorem 3.1 (4) applied with $\epsilon' = q_v^{b+c-3a}$, we have

$$\lim_{k\to\infty} \frac{1}{\log_{q_v} Y_k} \operatorname{Card}\big\{i \leqslant k : i \in {}^c\mathscr{J}\big\} = 0 \ . \tag{17}$$

Case 1. Assume first that $\mathscr{J}$ is finite.

By Equation (17), we then have $\lim_{k\to\infty} \frac{k}{\log_{q_v} Y_k} = 0$, hence Proposition 4.1 follows from Lemma 4.2.

Case 2. Let us now assume that there exists $j_* \in \mathbb{Z}_{\geqslant 1}$ such that $j \in \mathscr{J}$ for every $j \geqslant j_*$.

Let us consider the auxiliary increasing sequence $(\psi(i))_{i\in\mathbb{Z}_{\geqslant 1}}$ of positive integers defined by induction by setting $\psi(1) = \min\{j_* \in \mathbb{Z}_{\geqslant 1} : \ \forall\, j \geqslant j_*, \ j \in \mathscr{J}\}$ and, for every $i \geqslant 1$,

$$\psi(i+1) = \min\{j \in \mathbb{Z}_{\geqslant 1} : q_v^a \, Y_{\psi(i)} \leqslant Y_j\} \ .$$

Since the sequence $(Y_i)_{i\in\mathbb{Z}_{\geqslant 1}}$ is increasing and converges to $+\infty$, this is well defined, and ψ is increasing, hence takes value in $\mathscr{J}$ by the assumption of Case 2. Let us now define the sequence $(\varphi(i))_{i\in\mathbb{Z}_{\geqslant 1}}$ by, for every $i \in \mathbb{Z}_{\geqslant 1}$,

$$\varphi(i) = \begin{cases} \psi(i) & \text{if} \quad M_{\psi(i)} \, Y_{\psi(i+1)} \leqslant q_v^{b+c-a} \ , \\ \psi(i+1) - 1 & \text{otherwise.} \end{cases}$$

Note that the sequence $(\varphi(i))_{i\in\mathbb{Z}_{\geqslant 1}}$ is increasing with $\varphi \geqslant \psi$.

Let $i \in \mathbb{Z}_{\geqslant 1}$. Let us prove that

$$Y_{\varphi(i+1)} \geqslant q_v^a \, Y_{\varphi(i)} \quad \text{and} \quad M_{\varphi(i)} Y_{\varphi(i+1)} \leqslant q_v^{b+c} \ , \tag{18}$$

by discussing on the values of $\varphi(i)$ and $\varphi(i+1)$. This implies that Equation (15) is satisfied since $a \geqslant b$, and that Equation (16) is satisfied since by induction $Y_{\varphi(k)} \geqslant q_v^{a\,(k-1)} \, Y_{\varphi(1)}$ for every $k \in \mathbb{Z}_{\geqslant 1}$.

- Assume that $\varphi(i) = \psi(i)$ and $\varphi(i+1) = \psi(i+1)$. By the definition of $\psi(i+1)$, we have

$$Y_{\varphi(i+1)} = Y_{\psi(i+1)} \geqslant q_v^a \, Y_{\psi(i)} = q_v^a \, Y_{\varphi(i)} \ .$$

If $\psi(i) \neq \psi(i+1) - 1$, then by the definition of $\varphi(i)$, we have

$$M_{\varphi(i)} \, Y_{\varphi(i+1)} = M_{\psi(i)} \, Y_{\psi(i+1)} \leqslant q_v^{b+c-a} \leqslant q_v^{b+c} \ .$$

If $\psi(i) = \psi(i+1) - 1$, then $\varphi(i+1) = \varphi(i) + 1$ and by the definition of c, we have

$$M_{\varphi(i)} \, Y_{\varphi(i+1)} = M_{\varphi(i)} \, Y_{\varphi(i)+1} \leqslant q_v^c \leqslant q_v^{b+c} \ .$$

This proves Equation (18).

- Assume that $\varphi(i) = \psi(i)$ and $\varphi(i+1) = \psi(i+2) - 1$. Since the sequence $(Y_i)_{i \in \mathbb{Z}_{\geqslant 1}}$ is increasing and by the definition of $\psi(i+1)$, we have

$$Y_{\varphi(i+1)} = Y_{\psi(i+2)-1} \geqslant Y_{\psi(i+1)} \geqslant q_v^a\, Y_{\psi(i)} = q_v^a\, Y_{\varphi(i)} \;.$$

We have $q_v^a\, Y_{\psi(i+1)} > Y_{\psi(i+2)-1}$ by the minimality property of $\psi(i+2)$. If $\psi(i+1) > \psi(i)+1$, then $M_{\psi(i)}\, Y_{\psi(i+1)} \leqslant q_v^{b+c-a}$ by the dichotomy in the definition of $\varphi(i)$. Hence

$$M_{\varphi(i)}\, Y_{\varphi(i+1)} = M_{\psi(i)}\, Y_{\psi(i+2)-1} \leqslant M_{\psi(i)}\, Y_{\psi(i+1)}\, q_v^a \leqslant q_v^{b+c-a} q_v^a = q_v^{b+c} \;.$$

If $\psi(i+1) = \psi(i)+1$, then $M_{\psi(i)}\, Y_{\psi(i)+1} \leqslant q_v^{b+c-3a}$ since $\psi(i) \in \mathscr{J}$. Hence

$$M_{\varphi(i)}\, Y_{\varphi(i+1)} = M_{\psi(i)}\, Y_{\psi(i+2)-1} \leqslant M_{\psi(i)}\, Y_{\psi(i)+1}\, q_v^a \leqslant q_v^{b+c-3a} q_v^a \leqslant q_v^{b+c} \;.$$

This proves Equation (18).

- Assume that $\varphi(i) = \psi(i+1) - 1$ and $\varphi(i+1) = \psi(i+1)$. Since $\psi(i+1) - 1 \in \mathscr{J}$, we have

$$M_{\varphi(i)} Y_{\varphi(i+1)} = M_{\psi(i+1)-1} Y_{\psi(i+1)} \leqslant q_v^{b+c-3a} \leqslant q_v^{b+c} \;.$$

If $\psi(i+1) - 1 = \psi(i)$, then by the definition of $\psi(i+1)$, we have

$$\frac{Y_{\varphi(i+1)}}{Y_{\varphi(i)}} = \frac{Y_{\psi(i+1)}}{Y_{\psi(i+1)-1}} = \frac{Y_{\psi(i+1)}}{Y_{\psi(i)}} \geqslant q_v^a \;.$$

If $\psi(i+1) - 1 > \psi(i)$, then we have $M_{\psi(i)}\, Y_{\psi(i+1)} > q_v^{b+c-a}$ by the dichotomy in the definition of $\varphi(i)$, we have $Y_{\psi(i+1)-1} < q_v^a\, Y_{\psi(i)} \leqslant q_v^a\, Y_{\psi(i)+1}$ by the minimality property of $\psi(i+1)$, and we have $M_{\psi(i)}\, Y_{\psi(i)+1} \leqslant q_v^{b+c-3a}$ since $\psi(i) \in \mathscr{J}$. Therefore

$$\frac{Y_{\varphi(i+1)}}{Y_{\varphi(i)}} = \frac{Y_{\psi(i+1)}}{Y_{\psi(i+1)-1}} = \frac{M_{\psi(i)}\, Y_{\psi(i+1)}}{M_{\psi(i)}\, Y_{\psi(i+1)-1}} \geqslant \frac{q_v^{b+c-a}}{M_{\psi(i)}\, Y_{\psi(i)+1}\, q_v^a} \geqslant \frac{q_v^{b+c-a}}{q_v^{b+c-3a}\, q_v^a} = q_v^a \;.$$

This proves Equation (18).

- Assume that $\varphi(i) = \psi(i+1) - 1$ and $\varphi(i+1) = \psi(i+2) - 1$. By the previous case computations, we have

$$\frac{Y_{\varphi(i+1)}}{Y_{\varphi(i)}} = \frac{Y_{\psi(i+2)-1}}{Y_{\psi(i+1)-1}} \geqslant \frac{Y_{\psi(i+1)}}{Y_{\psi(i+1)-1}} \geqslant q_v^a \;.$$

We have $q_v^a\, Y_{\psi(i+1)} > Y_{\psi(i+2)-1}$ by the minimality property of $\psi(i+2)$. Hence since $\psi(i+1) - 1 \in \mathscr{J}$, we have

$$\begin{aligned} M_{\varphi(i)} Y_{\varphi(i+1)} &= M_{\psi(i+1)-1} Y_{\psi(i+2)-1} = M_{\psi(i+1)-1} Y_{\psi(i+1)} \Big(\frac{Y_{\psi(i+2)-1}}{Y_{\psi(i+1)}} \Big) \\ &\leqslant q_v^{b+c-3a}\, q_v^a \leqslant q_v^{b+c} \;. \end{aligned}$$

This proves Equation (18) and concludes the proof of Case 2.

Case 3. Let us now assume that $\mathscr{J}$ and ${}^c\mathscr{J}$ are both infinite, and that the number of sequences of consecutive elements of $\mathscr{J}$ with length at least $3a$ is finite.

Let $j_0 = \min \mathscr{J}$. Let us write the set $\mathbb{Z}_{\geqslant j_0} = \bigcup_{i \in \mathbb{Z}_{\geqslant 1}} C_i \cup D_i$ as the disjoint union of nonempty finite sequences C_i of consecutive integers in $\mathscr{J}$ and finite nonempty sequences D_i of consecutive integers in ${}^c\mathscr{J}$ with $\max C_i < \min D_i \leqslant \max D_i < \min C_{i+1}$ for all $i \in \mathbb{Z}_{\geqslant 1}$. Under the assumption of Case 3, let $i_0 \in \mathbb{Z}_{\geqslant 1}$ be such that $\operatorname{Card} C_i < 3a$ for every $i \geqslant i_0$. Let $k_0 = \min C_{i_0}$.

Then there exists an element of ${}^c\mathscr{J}$ in any finite sequence of $3\lceil a\rceil+1$ consecutive integers at least k_0, so that for every $k \in \mathbb{Z}_{\geqslant 1}$ we have

$$\frac{k}{\log_{q_v} Y_k} \leqslant \frac{k_0 + (3\lceil a\rceil + 1) \operatorname{Card}\{i \leqslant k : i \in {}^c\mathscr{J}\}}{\log_{q_v} Y_k} ,$$

which converges to 0 as $k \to +\infty$ by Equation (17) and since $\lim_{k\to\infty} Y_k = +\infty$. Therefore $\lim_{k\to\infty} Y_k^{\frac{1}{k}} = +\infty$, and Lemma 4.2 implies Proposition 4.1.

Case 4. Let us finally assume that $\mathscr{J}$ and ${}^c\mathscr{J}$ are both infinite, and that there are infinitely many sequences of consecutive elements of $\mathscr{J}$ with length at least $3a$.

With the notation $(C_i)_{i\in\mathbb{Z}_{\geqslant 1}}$ and $(D_i)_{i\in\mathbb{Z}_{\geqslant 1}}$ of the beginning of Case 3, let $(i_k)_{k\in\mathbb{Z}_{\geqslant 1}}$ be the increasing sequence of positive integers such that $\{i \in \mathbb{Z}_{\geqslant 1} : \operatorname{Card} C_i \geqslant 3a\} = \{i_k : k \in \mathbb{Z}_{\geqslant 1}\}$.

For every $k \in \mathbb{Z}_{\geqslant 1}$, let us define an increasing finite sequence $(\psi_k(i))_{1\leqslant i\leqslant m_k+1}$ of positive integers by setting $\psi_k(1) = \min C_{i_k}$ and by induction

$$\psi_k(i+1) = \min\{\, j \in C_{i_k} : q_v^a\, Y_{\psi_k(i)} \leqslant Y_j \,\} ,$$

as long as this set is nonempty. Since C_{i_k} is a finite sequence of consecutive positive integers with length at least $3a$ and $Y_{i+1} \geqslant q_v^{\frac{1}{\min \mathbf{r}}}\, Y_i$ for every $i \in \mathbb{Z}_{\geqslant 1}$, there exists $m_k \in \mathbb{Z}_{\geqslant 2}$ such that $\psi_k(i)$ is defined for $i = 1, \dots, m_k+1$. Note that $\psi_k(i)$ belongs to $\mathscr{J}$ for $i = 1, \dots, m_{k+1}$ since $C_{i_k} \subset \mathscr{J}$.

As in Case 2, let us define an increasing finite sequence $(\varphi_k(i))_{1\leqslant i\leqslant m_k}$ of positive integers by

$$\varphi_k(i) = \begin{cases} \psi_k(i) & \text{if } \; M_{\psi_k(i)}\, Y_{\psi_k(i+1)} \leqslant q_v^{b+c-a} , \\ \psi_k(i+1) - 1 & \text{otherwise.} \end{cases}$$

As in the proof of Case 2, since for $i = 1, \dots, m_k$, the integers $\psi_k(i)$, $\psi_k(i+1)$ as well as $\psi_k(i+1) - 1$ belong to $\mathscr{J}$, we have, for every $i = 1, \dots, m_k - 1$,

$$Y_{\varphi_k(i+1)} \geqslant q_v^a\, Y_{\varphi_k(i)} \quad \text{and} \quad M_{\varphi_k(i)}\, Y_{\varphi_k(i+1)} \leqslant q_v^{b+c} . \tag{19}$$

Since $\varphi_k(m_k) \in C_{i_k}$ and $\varphi_{k+1}(1) \in C_{i_{k+1}}$, we have $\varphi_k(m_k) < \varphi_{k+1}(1)$. Let us define an increasing finite sequence $(\varphi'_k(i))_{1\leqslant i\leqslant r_k+1}$ of positive integers that will allow us to interpolate between $\varphi_k(m_k)$ and $\varphi_{k+1}(1)$. Let $j_0 = \varphi_{k+1}(1)$. If $\{j \in \mathbb{Z}_{\geqslant \varphi_k(m_k)} : Y_{j_0} \geqslant q_v^b\, Y_j\}$ is empty, let $r'_k = 0$ and $\varphi'_k(1) = j_0 = \varphi_{k+1}(1)$. Otherwise, by decreasing induction, let $r'_k \in \mathbb{Z}_{\geqslant 1}$ be the maximal positive integer such that there exist $j_1, \dots, j_{r'_k} \in \mathbb{Z}_{\geqslant 1}$ such that for $\ell = 1, \dots, r'_k$, the set $\{j \in \mathbb{Z}_{\geqslant \varphi_k(m_k)} : Y_{j_{\ell-1}} \geqslant q_v^b\; Y_j\}$ is nonempty and for $\ell = 1, \dots, r'_k + 1$, the integer j_ℓ is its largest element. As in the part of the proof of Case 1 that does not need some belonging to $\mathscr{J}_0$, the sequence $(\varphi'_k(i) = j_{r'_k+1-i})_{1\leqslant i\leqslant r'_k+1}$ is well defined, it is contained in $[\varphi_k(m_k), \varphi_{k+1}(1)]$, and for $i = 1, \dots, r'_k$, we have

$$Y_{\varphi'_k(i+1)} \geqslant q_v^b\, Y_{\varphi'_k(i)} \quad \text{and} \quad M_{\varphi'_k(i)}\, Y_{\varphi'_k(i+1)} \leqslant q_v^{b+c} . \tag{20}$$

Putting alternatively together the sequences $(\varphi_k(i))_{1\leqslant i\leqslant m_k-1}$ and $(\varphi'_k(i))_{1\leqslant i\leqslant r'_k}$ as k ranges over $\mathbb{Z}_{\geqslant 1}$, we now define (with the standard convention that an empty sum is zero) $N_k=\sum_{\ell=1}^{k-1}(m_\ell-1+r'_\ell)$ and

$$\varphi(i)=\begin{cases}\varphi_k\big(i-N_k\big) & \text{if } \ 1+N_k\leqslant i\leqslant m_k-1+N_k\\ \varphi'_k\big(i+1-m_k-N_k\big) & \text{if } \ m_k+N_k\leqslant i\leqslant r'_k-1+m_k+N_k\ .\end{cases}$$

By Equation (19) for $i=1,\dots,m_k-2$, by Equation (20) for $i=1,\dots,r'_k$, and since $\varphi'_k(r'_k+1)=\varphi_{k+1}(1)$, in order to prove that the map φ satisfies Equation (15), hence Assertion (1) of Proposition 4.1, we only have to prove the following lemma.

Lemma 4.3 *For every $k\in\mathbb{Z}_{\geqslant 1}$, we have*

$$Y_{\varphi'_k(1)}\geqslant q_v^{\,b}\,Y_{\varphi_k(m_k-1)}\quad\text{and}\quad M_{\varphi_k(m_k-1)}\,Y_{\varphi'_k(1)}\leqslant q_v^{\,b+c}\ . \tag{21}$$

Proof. Since $\varphi'_k(1)\geqslant\varphi_k(m_k)$, hence $Y_{\varphi'_k(1)}\geqslant Y_{\varphi_k(m_k)}$, the left hand side of Equation (21) follows from the left hand side of Equation (19) with $i=m_k-1$. If $\varphi'_k(1)=\varphi_k(m_k)$, then the right hand side of Equation (21) follows from the right hand side of Equation (19) with $i=m_k-1$.

Let us hence assume that $\varphi'_k(1)>\varphi_k(m_k)$, so that

$$Y_{\varphi'_k(1)}\leqslant q_v^{\,b}\,Y_{\varphi_k(m_k)}\leqslant q_v^{\,a}\,Y_{\varphi_k(m_k)} \tag{22}$$

by the maximality of r'_k. Let us prove that $\varphi_k(m_k)=\psi_k(m_k)$. For a contradiction, assume otherwise that $\varphi_k(m_k)=\psi_k(m_k+1)-1>\psi_k(m_k)$. As in the third subcase of Case 2, we have $M_{\psi_k(m_k)}\,Y_{\psi_k(m_k+1)}>q_v^{\,b+c-a}$ by the dichotomy in the definition of $\varphi_k(m_k)$, we have $Y_{\psi_k(m_k+1)-1}<q_v^{\,a}\,Y_{\psi_k(m_k)}\leqslant q_v^{\,a}\,Y_{\psi_k(m_k)+1}$ by the minimality property of $\psi_k(m_k+1)$, and we have $M_{\psi_k(m_k)}\,Y_{\psi_k(m_k)+1}\leqslant q_v^{\,b+c-3a}$ since $\psi_k(m_k)\in\mathscr{J}$. Therefore, as in the third subcase of Case 2, we have

$$\frac{Y_{\psi_k(m_k+1)}}{Y_{\psi_k(m_k+1)-1}}=\frac{M_{\psi_k(m_k)}\,Y_{\psi_k(m_k+1)}}{M_{\psi_k(m_k)}\,Y_{\psi_k(m_k+1)-1}}\geqslant q_v^{\,a}\ .$$

Hence by the construction of $\varphi'_k(1)$, we have $\varphi'_k(1)=\varphi_k(m_k)$, a contradiction to our assumption that $\varphi'_k(1)>\varphi_k(m_k)$. We now discuss on the two possible values of $\varphi_k(m_k-1)$.

First assume that $\varphi_k(m_k-1)=\psi_k(m_k-1)$. If $\psi_k(m_k-1)\neq\psi_k(m_k)-1$ then $M_{\psi_k(m_k-1)}\,Y_{\psi_k(m_k)}\leqslant q_v^{\,b+c-a}$ by the dichotomy in the definition of $\varphi_k(m_k-1)$. If on the contrary $\psi_k(m_k-1)=\psi_k(m_k)-1$ then $M_{\psi_k(m_k-1)}\,Y_{\psi_k(m_k)}\leqslant q_v^{\,b+c-3a}\leqslant q_v^{\,b+c-a}$ since the integer $\psi_k(m_k)-1$ belong to $\mathscr{J}$ as $m_k\geqslant 2$. Since $\varphi_k(m_k)=\psi_k(m_k)$ by Equation (22), we have

$$M_{\varphi_k(m_k-1)}\,Y_{\varphi'_k(1)}=M_{\psi_k(m_k-1)}\,Y_{\psi_k(m_k)}\Big(\frac{Y_{\varphi'_k(1)}}{Y_{\varphi_k(m_k)}}\Big)\leqslant q_v^{\,b+c-a}\,q_v^{\,a}=q_v^{\,b+c}\ .$$

This proves the right hand side of Equation (21).

Now assume that $\varphi_k(m_k-1)=\psi_k(m_k)-1$. Again since $\varphi_k(m_k)=\psi_k(m_k)$, since the integer $\psi_k(m_k)-1$ belongs to $\mathscr{J}$ as $m_k\geqslant 2$, and by Equation (22), we have

$$M_{\varphi_k(m_k-1)}\,Y_{\varphi'_k(1)}=M_{\psi_k(m_k)-1}\,Y_{\psi_k(m_k)}\Big(\frac{Y_{\varphi'_k(1)}}{Y_{\varphi_k(m_k)}}\Big)\leqslant q_v^{\,b+c-3a}\,q_v^{\,a}\leqslant q_v^{\,b+c}\ .$$

This proves the right hand side of Equation (21), and concludes the proof of Lemma 4.3. □

Finally, let us prove Assertion (2) of Proposition 4.1. Since there exists an element of ${}^c\mathscr{J}$ in any finite sequence of $3\lceil a\rceil+1$ consecutive integers in the complement of $\bigcup_{k\in\mathbb{Z}_{\geqslant 1}} C_{i_k}$, there exists $c_0 \geqslant 0$ such that, for every $k \in \mathbb{Z}_{\geqslant 1}$, we have

$$\frac{\operatorname{Card}\{j \leqslant \varphi(k) : j \notin \bigcup_{k\in\mathbb{Z}_{\geqslant 1}} C_{i_k}\}}{\log_{q_v} Y_{\varphi(k)}} \leqslant \frac{c_0 + (3\lceil a\rceil + 1)\operatorname{Card}\{j \leqslant \varphi(k) : j \in {}^c\mathscr{J}\}}{\log_{q_v} Y_{\varphi(k)}} ,$$

which converges to 0 as $k \to +\infty$ as seen at the end of the proof of Case 3. Let us define $n(k) = \operatorname{Card}\{i \leqslant k : Y_{\varphi(i)} \geqslant q_v^a\, Y_{\varphi(i+1)}\}$. For every $\ell \in \mathbb{Z}_{\geqslant 1}$, since $Y_{j+1} \geqslant q_v^{\frac{1}{\min \mathbf{r}}}\, Y_j$ for every $j \in \mathbb{Z}_{\geqslant 1}$, and by the maximality of m_ℓ in the construction of $\big(\varphi_\ell(i)\big)_{1\leqslant i\leqslant m_\ell}$, we have $\operatorname{Card}\{j \in C_{i_\ell} : j \geqslant \varphi_\ell(m_\ell)\} \leqslant 2\lceil a\rceil \min \mathbf{r}$. If $\varphi(i)$ belongs to C_{i_ℓ} but $\varphi(i+1)$ does not, then $\varphi(i) \geqslant \varphi_\ell(m_\ell)$. Since when $\varphi(i)$ and $\varphi(i+1)$ belong to C_{i_ℓ} for some $\ell \in \mathbb{Z}_{\geqslant 1}$, then φ and φ_ℓ coincide on i and $i+1$, and since Equation (19) holds, we hence have

$$k - n(k) = \operatorname{Card}\{i \leqslant k : Y_{\varphi(i)} < q_v^a\, Y_{\varphi(i+1)}\} \leqslant 2\lceil a\rceil\, \min \mathbf{r}\, \operatorname{Card}\{j \leqslant \varphi(k) : j \notin \bigcup_{k\in\mathbb{Z}_{\geqslant 1}} C_{i_k}\} .$$

Hence

$$\begin{aligned}\limsup_{k\to+\infty} \frac{k}{\log_{q_v} Y_{\varphi(k)}} &= \limsup_{k\to+\infty} \frac{n(k) + k - n(k)}{\log_{q_v} Y_{\varphi(k)}} = \limsup_{k\to+\infty} \frac{n(k)}{\log_{q_v} Y_{\varphi(k)}} \\ &\leqslant \limsup_{k\to+\infty} \frac{n(k)}{\log_{q_v} q_v^{a(n(k)-1)} Y_{\varphi(1)}} = \frac{1}{a} .\end{aligned}$$

This proves Equation (16) and concludes the proof of Proposition 4.1. □

4.2 Lower bound on the Hausdorff dimension of $\mathbf{Bad}_A(\epsilon)$

In this subsection, we use the scheme of proof in the real case of [CGGMS, Theo. 6.1], which is a weighted version of [BuKLR, Theo. 5.1], in order to estimate the lower bound on the Hausdorff dimension of the ϵ-bad sets of $(\mathbf{r}, \mathbf{s})$-singular in average matrices.

For a given sequence $(\mathbf{y}_i)_{i\geqslant 1}$ in $R_v^m - \{0\}$ and for every $\delta > 0$, let

$$\mathbf{Bad}^{\delta}_{(\mathbf{y}_i)_{i\geqslant 1}} = \{\boldsymbol{\theta} \in (\pi_v \mathscr{O}_v)^m :\ \forall\, i \geqslant 1,\ \ |\langle\, \boldsymbol{\theta} \cdot \mathbf{y}_i \,\rangle| \geqslant \delta\,\} .$$

Proposition 4.4 *Let $A \in \mathscr{M}_{m,n}(K_v)$ be such that ${}^tA$ is completely irrational and let $(\mathbf{y}_i)_{i\geqslant 1}$ be a best approximation sequence in K_v^m for ${}^tA$ with weights $(\mathbf{s}, \mathbf{r})$. Suppose that there exist $b, c > 0$ and an increasing function $\varphi : \mathbb{Z}_{\geqslant 1} \to \mathbb{Z}_{\geqslant 1}$ such that*

$$\forall\, i \in \mathbb{Z}_{\geqslant 1}, \quad M_{\varphi(i)} Y_{\varphi(i+1)} \leqslant q_v^{b+c} .$$

Then for every $\delta \in\,]0, 1]$, if $\epsilon = \delta^{\frac{1}{\min \mathbf{r}} + \frac{1}{\min \mathbf{s}}}\ q_v^{-b-c}$, then the set $\mathbf{Bad}^{\delta}_{(\mathbf{y}_{\varphi(i)})_{i\geqslant 1}}$ is contained in the set $\mathbf{Bad}_A(\epsilon)$.

Proof. Fix $\delta \in\,]0, 1]$ and $\boldsymbol{\theta} \in \mathbf{Bad}^{\delta}_{(\mathbf{y}_{\varphi(i)})_{i\geqslant 1}}$. Let $\epsilon_1 = \delta^{\frac{1}{\min \mathbf{s}}}\ q_v^{-b-c}$. For every $(\mathbf{y}', \mathbf{x}')$ in $R_v^m \times R_v^n$ such that $\|\, \mathbf{x}' \,\|_{\mathbf{s}} \geqslant \epsilon_1 Y_{\varphi(1)}$, let k be the unique element of $\mathbb{Z}_{\geqslant 1}$ for which

$$Y_{\varphi(k)} \leqslant \epsilon_1^{-1} \|\, \mathbf{x}' \,\|_{\mathbf{s}} < Y_{\varphi(k+1)} ,$$

which exists since $\| \mathbf{x}' \|_{\mathbf{s}} \geqslant \epsilon_1 Y_{\varphi(1)}$ and since the sequence $(Y_{\varphi(i)})_{i\geqslant 1}$ is increasing, converging to $+\infty$. Let $\mathbf{x}_{\varphi(k)} \in R_v^n$ be such that $M_{\varphi(k)} = \| {}^tA\mathbf{y}_{\varphi(k)} - \mathbf{x}_{\varphi(k)} \|_{\mathbf{s}}$. Then by the ultrametric inequality, the assumption of the proposition, the fact that $\epsilon_1 \, q_v^{b+c} = \delta^{\frac{1}{\min \mathbf{s}}} \leqslant 1$ and the definition of $\mathbf{Bad}^{\delta}_{(\mathbf{y}_{\varphi(i)})_{i\geqslant 1}}$, we have

$$|({}^tA\mathbf{y}_{\varphi(k)} - \mathbf{x}_{\varphi(k)}) \cdot \mathbf{x}'| \leqslant \max_{1\leqslant i\leqslant n} M_{\varphi(k)}^{s_i} \| \mathbf{x}' \|_{\mathbf{s}}^{s_i} < \max_{1\leqslant i\leqslant n} (\epsilon_1 M_{\varphi(k)} Y_{\varphi(k+1)})^{s_i}$$
$$\leqslant (\epsilon_1 \, q_v^{b+c})^{\min \mathbf{s}} = \delta \leqslant \min_{\ell' \in R_v} | \mathbf{y}_{\varphi(k)} \cdot \boldsymbol{\theta} - \ell' | \, . \tag{23}$$

Observe that

$$\begin{aligned} \mathbf{y}_{\varphi(k)} \cdot \boldsymbol{\theta} &= \mathbf{y}_{\varphi(k)} \cdot (A\,\mathbf{x}') + \mathbf{y}_{\varphi(k)} \cdot \mathbf{y}' - \mathbf{y}_{\varphi(k)} \cdot (A\,\mathbf{x}' + \mathbf{y}' - \boldsymbol{\theta}) \\ &= ({}^tA\,\mathbf{y}_{\varphi(k)}) \cdot \mathbf{x}' - \mathbf{x}_{\varphi(k)} \cdot \mathbf{x}' + \ell - \mathbf{y}_{\varphi(k)} \cdot (A\mathbf{x}' + \mathbf{y}' - \boldsymbol{\theta}), \end{aligned}$$

where $\ell = \mathbf{x}_{\varphi(k)} \cdot \mathbf{x}' + \mathbf{y}_{\varphi(k)} \cdot \mathbf{y}' \in R_v$. Thus we have, using the equality case of the ultrametric inequality for the second equality below with the strict inequality in Equation (23), and again the definition of $\mathbf{Bad}^{\delta}_{(\mathbf{y}_{\varphi(i)})_{i\geqslant 1}}$ for the last inequality below,

$$\begin{aligned} |\mathbf{y}_{\varphi(k)} \cdot (A\mathbf{x}' + \mathbf{y}' - \boldsymbol{\theta})| &= |({}^tA\mathbf{y}_{\varphi(k)} - \mathbf{x}_{\varphi(k)}) \cdot \mathbf{x}' - \mathbf{y}_{\varphi(k)} \cdot \boldsymbol{\theta} + \ell \,| \\ &= \max \big\{ |({}^tA\mathbf{y}_{\varphi(k)} - \mathbf{x}_{\varphi(k)}) \cdot \mathbf{x}'|, | \, \mathbf{y}_{\varphi(k)} \cdot \boldsymbol{\theta} - \ell \,| \big\} \\ &= | \, \mathbf{y}_{\varphi(k)} \cdot \boldsymbol{\theta} - \ell \,| \geqslant |\langle \, \mathbf{y}_{\varphi(k)} \cdot \boldsymbol{\theta} \, \rangle| \geqslant \delta. \end{aligned}$$

Hence, we have

$$\delta \leqslant |\mathbf{y}_{\varphi(k)} \cdot (A\mathbf{x}' + \mathbf{y}' - \boldsymbol{\theta})| \leqslant \max_{1\leqslant j\leqslant m} Y_{\varphi(k)}^{r_j} \, \|A\mathbf{x}' + \mathbf{y}' - \boldsymbol{\theta}\|_{\mathbf{r}}^{r_j},$$

which implies, since $\delta \leqslant 1$, that

$$Y_{\varphi(k)} \|A\mathbf{x}' + \mathbf{y}' - \boldsymbol{\theta}\|_{\mathbf{r}} \geqslant \min_{1\leqslant j\leqslant m} \delta^{\frac{1}{r_j}} = \delta^{\frac{1}{\min \mathbf{r}}}.$$

Finally, for every $(\mathbf{y}', \mathbf{x}')$ in $R_v^m \times R_v^n$ such that $\| \mathbf{x}' \|_{\mathbf{s}} \geqslant \epsilon_1 Y_{\varphi(1)}$, we have

$$\| \mathbf{x}' \|_{\mathbf{s}} \, \|A\mathbf{x}' + \mathbf{y}' - \boldsymbol{\theta}\|_{\mathbf{r}} \geqslant \epsilon_1 \, Y_{\varphi(k)} \, \|A\mathbf{x}' + \mathbf{y}' - \boldsymbol{\theta}\|_{\mathbf{r}} \geqslant \delta^{\frac{1}{\min \mathbf{r}} + \frac{1}{\min \mathbf{s}}} \, q_v^{-b-c} \, .$$

By Equation (1), this implies that $\boldsymbol{\theta} \in \mathbf{Bad}_A(\epsilon)$ for $\epsilon = \delta^{\frac{1}{\min \mathbf{r}} + \frac{1}{\min \mathbf{s}}} \, q_v^{-b-c}$. □

Proposition 4.5 *For every $\delta \in \,]0, \frac{1}{q_v^{3m}}[$, there exist $b = b(\delta) > 0$ and $C = C(\delta) > 0$ such that for every sequence $(\mathbf{y}_i)_{i \in \mathbb{Z}_{\geqslant 1}}$ in $R_v^m - \{0\}$ satisfying $\| \mathbf{y}_{i+1} \|_{\mathbf{r}} \geqslant q_v^b \, \| \mathbf{y}_i \|_{\mathbf{r}}$ for all $i \in \mathbb{Z}_{\geqslant 1}$, we have*

$$\dim_{\mathrm{Haus}} \mathbf{Bad}^{\delta}_{(\mathbf{y}_i)_{i\geqslant 1}} \geqslant m - C \limsup_{k\to\infty} \frac{k}{\log_{q_v} \| \mathbf{y}_k \|_{\mathbf{r}}} \, .$$

Proof. Fix $\delta \in \,]0, \frac{1}{q_v^{3m}}[$. Let

$$b = b(\delta) = \frac{-\log_{q_v} \delta}{\min \mathbf{r}} \, , \tag{24}$$

which is positive since $\delta < 1$. By the mass distribution principle (see for instance [Fal, page 60]), it is enough to prove that there exist a (Borel, positive) measure μ, supported on $\mathbf{Bad}^{\delta}_{(\mathbf{y}_i)_{i\geqslant 1}}$, and constants $C, C_0, r_0 > 0$, with C depending only on δ, such that, for every closed ball B of radius $r < r_0$, we have

$$\mu(B) \leqslant C_0\, r^{m - C \limsup_{k\to\infty} \frac{k}{\log_{q_v} \|\,\mathbf{y}_k\,\|_{\mathbf{r}}}}\ .$$

We adapt by modifying it quite a lot the measure construction in the proof of [CGGMS, Theo. 6.1].

By convention, let $Y_0 = 1$ and $n_{0,j} = 0$ for $j = 1, \dots, m$. For every $k \in \mathbb{Z}_{\geqslant 1}$, define $Y_k = \|\,\mathbf{y}_k\,\|_{\mathbf{r}}$, which is at least 1 since $\mathbf{y}_k \in R_v^m - \{0\}$, and for every $j = 1, \dots, m$, let $n_{k,j} \in \mathbb{Z}_{\geqslant 0}$ be such that

$$q_v^{-n_{k,j}} \leqslant Y_k^{-r_j} < q_v^{-n_{k,j}+1}\ . \tag{25}$$

Note that the sequence $(n_{k,j})_{k\in\mathbb{Z}_{\geqslant 0}}$ is nondecreasing, for all $j = 1, \dots, m$.

For every $k \in \mathbb{Z}_{\geqslant 0}$, let us consider the polydisc

$$\Pi(Y_k) = \overline{B}(0, \frac{1}{q_v}\, Y_k^{-r_1}) \times \cdots \times \overline{B}(0, \frac{1}{q_v}\, Y_k^{-r_m}) = \overline{B}(0, q_v^{-n_{k,1}-1}) \times \cdots \times \overline{B}(0, q_v^{-n_{k,m}-1})\ ,$$

where $\overline{B}(0, r')$ is the closed ball of radius $r' > 0$ and center 0 in K_v. Note that $\Pi(Y_0) = (\pi_v \mathscr{O}_v)^m$ is the open unit ball of K_v^m and that $\Pi(Y_k)$ is an additive subgroup of K_v^m. Since the residual field $k_v = \mathscr{O}_v/\pi_v\mathscr{O}_v$ lifts as a subfield of order q_v of K_v, for every $\ell \in \mathbb{Z}_{\geqslant 0}$, we have a disjoint union

$$\overline{B}(0, q_v^{-\ell}) = \bigsqcup_{a\in k_v} \big(\, a\, \pi_v^{\ell} + \overline{B}(0, q_v^{-\ell-1})\,\big)\ .$$

Hence by induction, the polydisc $\Pi(Y_k)$ is the disjoint union of

$$\Delta_{k+1} = \prod_{1\leqslant j\leqslant m} q_v^{n_{k+1,j}-n_{k,j}}$$

translates of the polydisc $\Pi(Y_{k+1})$. Note that

$$\Delta_{k+1} \geqslant \prod_{1\leqslant j\leqslant m} Y_{k+1}^{r_j} Y_k^{-r_j} q_v^{-1} = q_v^{-m}\, \big(Y_{k+1} Y_k^{-1}\big)^{|\mathbf{r}|}\ . \tag{26}$$

For every $k \in \mathbb{Z}_{\geqslant 0}$, let us fix some elements $\theta_{1,k+1}, \dots, \theta_{\Delta_{k+1},k+1}$ in $(\pi_v\mathscr{O}_v)^m$ (which are not unique in the ultrametric space K_v^m) such that

$$\Pi(Y_k) = \bigsqcup_{i=1}^{\Delta_{k+1}} \big(\, \theta_{i,k+1} + \Pi(Y_{k+1})\,\big)\ .$$

By convention, let us define $Z_{0,\delta} = \varnothing$ and $I_0 = \{\Pi(Y_0)\}$. For every $k \in \mathbb{Z}_{\geqslant 1}$, let us define

$$Z_{k,\delta} = \{\boldsymbol{\theta} \in (\pi_v\mathscr{O}_v)^m : |\langle\, \mathbf{y}_k \cdot \boldsymbol{\theta}\,\rangle| < \delta\}$$

and

$$I_k = \big\{\theta_{i_1,1} + \cdots + \theta_{i_k,k} + \Pi(Y_k) : \forall\, j \in \{1, \dots, k\},\ \ 1 \leqslant i_j \leqslant \Delta_j\big\}\ .$$

Lemma 4.6 *For every $k \in \mathbb{Z}_{\geqslant 1}$, we have*

(1) for every $I' \in I_{k+1}$, if $I' \cap Z_{k,\delta} \neq \varnothing$ then $I' \subset Z_{k,\delta}$,

(2) for every $I \in I_k$, we have $\mathrm{vol}_v^m(I \cap Z_{k,\delta}) \leqslant \delta\, Y_k^{-|\mathbf{r}|}$.

Proof. (1) If $I' \in I_{k+1}$ and $I' \cap Z_{k,\delta} \neq \varnothing$, let $\boldsymbol{\theta} \in I' \cap Z_{k,\delta}$. Then for every $\boldsymbol{\theta}' \in I'$, if $x, x' \in R_v$ are such that $|\langle\, \mathbf{y}_k \cdot \boldsymbol{\theta}\,\rangle| = |(\mathbf{y}_k \cdot \boldsymbol{\theta}) - x|$ and $|\langle\, \mathbf{y}_k \cdot (\boldsymbol{\theta}' - \boldsymbol{\theta})\,\rangle| = |(\mathbf{y}_k \cdot (\boldsymbol{\theta}' - \boldsymbol{\theta})) - x'|$, then by the ultrametric inequality, since $\boldsymbol{\theta} \in Z_{k,\delta}$ and $\boldsymbol{\theta}' - \boldsymbol{\theta} \in \Pi(Y_{k+1})$, by the assumption of Proposition 4.5, and by the definition of b, we have

$$\begin{aligned}
|\langle\, \mathbf{y}_k \cdot \boldsymbol{\theta}'\,\rangle| &\leqslant |\mathbf{y}_k \cdot (\boldsymbol{\theta} + (\boldsymbol{\theta}' - \boldsymbol{\theta})) - (x + x')| \leqslant \max\big\{|(\mathbf{y}_k \cdot \boldsymbol{\theta}) - x|,\ |(\mathbf{y}_k \cdot (\boldsymbol{\theta}' - \boldsymbol{\theta})) - x'|\big\} \\
&= \max\big\{|\langle\, \mathbf{y}_k \cdot \boldsymbol{\theta}\,\rangle|,\ |\langle\, \mathbf{y}_k \cdot (\boldsymbol{\theta}' - \boldsymbol{\theta})\,\rangle|\big\} \\
&\leqslant \max\big\{\delta, \max_{1 \leqslant j \leqslant m} Y_k^{r_j} \frac{1}{q_v}\, Y_{k+1}^{-r_j}\big\} \leqslant \max\big\{\delta, q_v^{-1-b\min \mathbf{r}}\big\} = \delta\,.
\end{aligned}$$

This inequality $|\langle\, \mathbf{y}_k \cdot \boldsymbol{\theta}'\,\rangle| \leqslant \delta$ is actually strict, since $|\langle\, \mathbf{y}_k \cdot \boldsymbol{\theta}\,\rangle| < \delta$ and by Equation (24), we have $q_v^{-1-b\min \mathbf{r}} = q_v^{-1}\delta < \delta$. Since I' is contained in $\Pi(Y_0) = (\pi_v \mathscr{O}_v)^m$, we thus have that $\boldsymbol{\theta}' \in Z_{k,\delta}$ and this proves Assertion (1).

(2) Let $j_0 \in \{1, \dots, m\}$ be such that $Y_k = |y_{k,j_0}|^{1/r_{j_0}}$ where $\mathbf{y}_k = (y_{k,1}, \dots, y_{k,m})$. In particular, y_{k,j_0} is nonzero. For every $z \in R_v$, let

$$L_k(z) = \{\boldsymbol{\theta} \in K_v^m : \mathbf{y}_k \cdot \boldsymbol{\theta} = z\}\ ,$$

which is an affine hyperplane of K_v^m transverse to the j_0-axis, and let

$$\mathscr{N}(k,z) = \{\boldsymbol{\theta}' \in (\pi_v \mathscr{O}_v)^m : \exists\ \mathbf{u}' \in L_k(z),\ \ |\theta'_{j_0} - u'_{j_0}| \leqslant \delta\, Y_k^{-r_{j_0}} \ \text{ and }\ \forall\ j \neq j_0,\ \theta'_j = u'_j\}\ ,$$

which is the intersection with the open unit ball in K_v^m of the $(\delta\, Y_k^{-r_{j_0}})$-thickening along the j_0-axis of the affine hyperplane $L_k(z)$.

Fix $I \in I_k$. Since $\mathrm{vol}_v(\overline{B})(0, r')) = q_v^{[\log_{q_v} r']} \leqslant r'$ for all $r' > 0$, and by Fubini's theorem, we have

$$\mathrm{vol}_v^m(I \cap \mathscr{N}(k,z)) \leqslant \delta\, Y_k^{-r_{j_0}} \prod_{j \neq j_0} Y_k^{-r_j} = \delta\, Y_k^{-|\mathbf{r}|}\ . \tag{27}$$

Claim 1. Let us prove that the set $Z_{k,\delta}$ is contained in the union of the sets $\mathscr{N}(k,z)$ for $z \in R_v$.

Proof. Let $\boldsymbol{\theta} = (\theta_1, \dots, \theta_m) \in Z_{k,\delta}$ and let $z \in R_v$ be such that $|\langle\, \mathbf{y}_k \cdot \boldsymbol{\theta}\,\rangle| = |\,\mathbf{y}_k \cdot \boldsymbol{\theta} - z\,|$. Let us define $u_j = \theta_j$ if $j \neq j_0$,

$$u_{j_0} = \frac{z - \sum_{j \neq j_0} y_{k,j}\theta_j}{y_{k,j_0}}$$

and $\mathbf{u} = (u_1, \dots, u_m)$, which is the projection of $\boldsymbol{\theta}$ on the affine hyperplane $L_k(z)$ along the j_0-axis. Then, since $\boldsymbol{\theta} \in Z_{k,\delta}$, we have

$$|\theta_{j_0} - u_{j_0}| = \frac{|\,\mathbf{y}_k \cdot \boldsymbol{\theta} - z\,|}{|y_{k,j_0}|} = \frac{|\langle\, \mathbf{y}_k \cdot \boldsymbol{\theta}\,\rangle|}{|y_{k,j_0}|} \leqslant \delta\, Y_k^{-r_{j_0}}\ .$$

Since $Z_{k,\delta}$ is contained in $(\pi_v \mathscr{O}_v)^m$, this proves Claim 1. □

Claim 2. Let us prove that there exists a unique $z \in R_v$ such that $I \cap Z_{k,\delta}$ is contained in $I \cap \mathscr{N}(k,z)$.

Proof. By Claim 1, the set $I \cap Z_{k,\delta}$ is contained in $\bigcup_{z\in R_v} I \cap \mathscr{N}(k,z)$. Assume for a contradiction that there exist two distinct elements z, z' in R_v such that there exist $\boldsymbol{\theta} \in I \cap \mathscr{N}(k,z)$ and $\boldsymbol{\theta}' \in I \cap \mathscr{N}(k,z')$. Let $\mathbf{u} \in L_k(z)$ and $\mathbf{u}' \in L_k(z')$ be the projections of $\boldsymbol{\theta}$ and $\boldsymbol{\theta}'$ along the j_0-axis on $L_k(z)$ and $L_k(z')$ respectively.

Let $j \in \{1, \dots, m\}$. Note that $\boldsymbol{\theta} - \boldsymbol{\theta}' \in \Pi(Y_k)$ since $I \in I_k$. If $j \neq j_0$, then

$$|u_j - u'_j| = |\theta_j - \theta'_j| \leqslant \frac{1}{q_v}\, Y_k^{-r_j} .$$

Furthermore, by the ultrametric inequality, since $\boldsymbol{\theta}$ (respectively $\boldsymbol{\theta}'$) is contained in the $(\delta\, Y_k^{-r_{j_0}})$-thickening along the j_0-axis of $L_k(z)$ (respectively $L_k(z')$), and since $\delta \leqslant \frac{1}{q_v}$, we have

$$\begin{aligned} |u_{j_0} - u'_{j_0}| &= |(u_{j_0} - \theta_{j_0}) + (\theta_{j_0} - \theta'_{j_0}) + (\theta'_{j_0} - u'_{j_0})| \\ &\leqslant \max\{|u_{j_0} - \theta_{j_0}|,\ |\theta_{j_0} - \theta'_{j_0}|,\ |\theta'_{j_0} - u'_{j_0}|\} \\ &\leqslant \max\{\delta\, Y_k^{-r_{j_0}},\ \frac{1}{q_v}\, Y_k^{-r_{j_0}}\} = \frac{1}{q_v}\, Y_k^{-r_{j_0}} . \end{aligned}$$

This implies since $\mathbf{u} \in L_k(z)$ and $u' \in L_k(z)$ that

$$1 \leqslant |z - z'| = |\mathbf{y}_k \cdot \mathbf{u} - \mathbf{y}_k \cdot \mathbf{u}'| \leqslant \max_{1\leqslant j\leqslant m} |y_{k,j}|\, |u_j - u'_j| \leqslant \max_{1\leqslant j\leqslant m} Y_k^{r_j}\, \frac{1}{q_v}\, Y_k^{-r_j} = \frac{1}{q_v} ,$$

which is a contradiction since $q_v > 1$. This proves Claim 2. □

By Equation (27), Claim 2 concludes the proof of Assertion (2) of Lemma 4.6. □

Since every element I' of I_{k+1} is a translate of $\Pi(Y_{k+1})$, and by Equation (25), we have

$$\mathrm{vol}_v^m(I') = \mathrm{vol}_v^m(\Pi(Y_{k+1})) = \prod_{j=1}^m q_v^{-n_{k+1,\,j}-1} \geqslant q_v^{-2m}\, Y_{k+1}^{-|\mathbf{r}|} .$$

For every $I \in I_k$, there are Δ_{k+1} elements $I' \in I_{k+1}$ contained in I, they are pairwise disjoint and they have the same volume $\mathrm{vol}_v^m(\Pi(Y_{k+1}))$. Among them, those who meet $Z_{k,\delta}$ are actually contained in $I \cap Z_{k,\delta}$ by Lemma 4.6 (1), thus their number is at most $\frac{\mathrm{vol}_v^m(I\cap Z_{k,\delta})}{\mathrm{vol}_v^m(\Pi(Y_{k+1}))}$. Therefore, by Equation (26) and Lemma 4.6 (2), we have

$$\begin{aligned} \mathrm{Card}\,\{I' \in I_{k+1} : I' \subset I,\ \ I' \cap Z_{k,\delta} = \varnothing\} &\geqslant \Delta_{k+1} - \frac{\mathrm{vol}_v^m(I \cap Z_{k,\delta})}{\mathrm{vol}_v^m(\Pi(Y_{k+1}))} \\ &\geqslant q_v^{-m}(Y_{k+1}Y_k^{-1})^{|\mathbf{r}|} - \frac{\delta\, Y_k^{-|\mathbf{r}|}}{q_v^{-2m} Y_{k+1}^{-|\mathbf{r}|}} \\ &= c_1\, (Y_{k+1}Y_k^{-1})^{|\mathbf{r}|} , \end{aligned} \tag{28}$$

where $c_1 = q_v^{-m} - q_v^{2m}\delta$ belongs to $]0,1[$ by the assumption on δ.

Now, let us define by induction $J_0 = I_0$ and for every $k \in \mathbb{Z}_{\geqslant 0}$,

$$J_{k+1} = \bigcup_{J\in J_k} \{I \in I_{k+1} : I \subset J,\ \ I \cap Z_{k,\delta} = \varnothing\}.$$

By Equation (28) and by induction, we have

$$\text{Card } J_{k+1} \geqslant \prod_{j=1}^{k} c_1 \,(Y_{j+1}Y_j^{-1})^{|\mathbf{r}|} = c_1^k \,(Y_{k+1}Y_1^{-1})^{|\mathbf{r}|} \,. \tag{29}$$

By Lemma 4.6 (1) and by induction, we have

$$J_{k+1} = \{J \in I_{k+1} : \forall\, j \in \{1, \dots, k\},\;\; J \cap Z_{j,\delta} = \varnothing\} = \{J \in I_{k+1} : J \subset \bigcap_{j=1}^{k} {}^{c}Z_{j,\delta}\;\} \,,$$

where ^{c} denotes the complement in $(\pi_v \mathscr{O}_v)^m$. Hence $\big(\bigcup J_k\big)_{k\geqslant 1}$ is a decreasing sequence of compact subsets of $(\pi_v \mathscr{O}_v)^m$, whose intersection is contained in $\bigcap_{k\geqslant 1} {}^{c}Z_{k,\delta} = \mathbf{Bad}^{\delta}_{(\mathbf{y}_i)_{i\geqslant 1}}$.

For every $k \in \mathbb{Z}_{\geqslant 0}$, let us define a measure

$$\mu_k = \big(\operatorname{vol}_v^m(\Pi(Y_k)) \text{ Card } J_k\big)^{-1} \sum_{J \in J_k} \operatorname{vol}_v^m|_J \,,$$

which is a probability measure with support $\bigcup J_k$. By the compactness of $(\pi_v \mathscr{O}_v)^m$, any weakstar accumulation point μ of the sequence $(\mu_k)_{k\geqslant 1}$ is a probability measure with support in $\mathbf{Bad}^{\delta}_{(\mathbf{y}_i)_{i\geqslant 1}}$.

For every closed ball B in $(\pi_v \mathscr{O}_v)^m$ with radius $r' \in \,]0, r_0 = Y_1^{-\min \mathbf{r}}]$, let $k \in \mathbb{Z}_{\geqslant 1}$ be such that

$$Y_{k+1}^{-\min \mathbf{r}} < r' \leqslant Y_k^{-\min \mathbf{r}} \,. \tag{30}$$

Note that $\lceil t \rceil \leqslant t + 1 \leqslant q_v\, t$ if $t \geqslant 1$, and that $r'\, q_v^{n_{k+1,j}+1} \geqslant Y_{k+1}^{-\min \mathbf{r}}\; Y_{k+1}^{r_j}\; q_v \geqslant 1$ for every $j = 1, \dots, m$, by Equation (25). Then B can be covered by a subset of I_{k+1} with cardinality at most

$$\prod_{j=1}^{m} \big\lceil r'\, q_v^{n_{k+1,j}+1} \big\rceil \leqslant (r')^m\, q_v^{3m}\, Y_{k+1}^{|\mathbf{r}|} \,.$$

Let $C = \frac{-\log_{q_v} c_1}{\min \mathbf{r}} > 0$, which depends (besides on m, q_v and $\mathbf{r}$) only on δ. Defining $C_0 = q_v^{3m}\, Y_1^{|\mathbf{r}|}$, by Equations (29) and (30), we thus have

$$\begin{aligned}\mu_{k+1}(B) &\leqslant q_v^{3m}\,(r')^m\, Y_{k+1}^{|\mathbf{r}|}\,(\text{Card } J_{k+1})^{-1} \leqslant q_v^{3m}\,(r')^m\, c_1^{-k}\, Y_1^{|\mathbf{r}|}\\ &\leqslant C_0\,(r')^{m - C\frac{k}{\log_{q_v} Y_k}} \,.\end{aligned}$$

Therefore, since the ball B is closed and open and since $r' \leqslant r_0 \leqslant 1$, we have

$$\mu(B) \leqslant \limsup_{k\to\infty} C_0\,(r')^{m - C\frac{k}{\log_{q_v} Y_k}} = C_0\,(r')^{m - C \limsup_{k\to\infty} \frac{k}{\log_{q_v} Y_k}} ,$$

which concludes the proof of Proposition 4.5. □

4.3 Proof that Assertion (2) implies Assertion (1) in Theorem 1.1

Suppose that A is $(\mathbf{r},\mathbf{s})$-singular on average. Then by Corollary 2.9, the matrix ${}^tA$ is also $(\mathbf{s},\mathbf{r})$-singular on average. By Remark 2.5 (2), in order to prove that there exists $\epsilon > 0$ such that $\mathbf{Bad}_A(\epsilon)$ has full Hausdorff dimension, we may assume that the matrix ${}^tA$ is completely irrational.

By Lemma 2.6, let $(\mathbf{y}_k)_{k\in\mathbb{Z}_{\geqslant 1}}$ be a best approximation sequence in K_v^m for the matrix ${}^tA$ with weights $(\mathbf{s},\mathbf{r})$, and let $c > 0$ be such that $M_iY_{i+1} \leqslant q_v^c$ for every $i \in \mathbb{Z}_{\geqslant 1}$. Fix some $\delta \in \left]0, \frac{1}{q_v^{3m}}\right[$ and let $b = b(\delta) > 0$ and $C = C(\delta) > 0$ as in Proposition 4.5. By Proposition 4.1, for every $a > b$, we have a subsequence $(\mathbf{y}_{\varphi(k)})_{k\geqslant 1}$ such that the properties (15) and (16) are satisfied. Proposition 4.4, whose assumption is satisfied by the second inequality in Equation (15) and where $\epsilon = \delta^{\frac{1}{\min \mathbf{r}}+\frac{1}{\min \mathbf{s}}}\, q_v^{-b-c}$, gives that $\mathbf{Bad}_A(\epsilon)$ contains $\mathbf{Bad}^{\delta}_{(\mathbf{y}_{\varphi(i)})_{i\geqslant 1}}$. Therefore, using Proposition 4.5 applied to the sequence $(\mathbf{y}_{\varphi(i)})_{i\geqslant 1}$, whose assumption is satisfied by the first inequality in Equation (15), and using Equation (16) for the last inequality, we have

$$\dim_{\mathrm{Haus}} \mathbf{Bad}_A(\epsilon) \geqslant \dim_{\mathrm{Haus}} \mathbf{Bad}^{\delta}_{(\mathbf{y}_{\varphi(i)})_{i\geqslant 1}} \geqslant m - C \limsup_{k\to\infty} \frac{k}{\log_{q_v} Y_{\varphi(k)}} \geqslant m - \frac{C}{a}\ .$$

Letting a tends to $+\infty$, this concludes the proof that Assertion (2) implies Assertion (1) in Theorem 1.1. □

5 Background material for the upper bound

5.1 Homogeneous dynamics

Let $K_v, \mathscr{O}_v, \pi_v, R_v, q_v$ be as in Subsection 2.1. Let $m, n \in \mathbb{N} - \{0\}$ and $d = m + n$. We fix some weights $\mathbf{r} = (r_1, \ldots, r_m)$ and $\mathbf{s} = (s_1, \ldots, s_n)$ as in the introduction. In this subsection, we introduce the space of unimodular grids $\mathscr{Y}$ in K_v^d and the diagonal flow $(\mathfrak{a}^\ell)_{\ell\in\mathbb{Z}}$ acting on this space. Let

$$G_0 = \mathrm{SL}_d(K_v) \quad \text{and} \quad G = \mathrm{ASL}_d(K_v) = \mathrm{SL}_d(K_v) \ltimes K_v^d,$$

and let

$$\Gamma_0 = \mathrm{SL}_d(R_v) \quad \text{and} \quad \Gamma = \mathrm{ASL}_d(R_v) = \mathrm{SL}_d(R_v) \ltimes R_v^d.$$

The product in G is given by

$$(g, u) \cdot (g', u') = (gg', u + gu') \tag{31}$$

for all $g, g' \in G_0$ and $u, u' \in K_v^d$. We also view G as a subgroup of $\mathrm{SL}_{d+1}(K_v)$ by

$$G = \left\{ \begin{pmatrix} g & u \\ 0 & 1 \end{pmatrix} : g \in \mathrm{SL}_d(K_v),\ u \in K_v^d \right\}\ .$$

We shall identify G_0 with the corresponding subgroup of G. We consider the one-parameter diagonal subgroup $(\mathfrak{a}^\ell)_{\ell\in\mathbb{Z}}$ of G_0, where $\mathfrak{a} = \mathrm{diag}(\mathfrak{a}_-, \mathfrak{a}_+)$ and

$$\mathfrak{a}_- = \mathrm{diag}(\pi_v^{-r_1}, \ldots, \pi_v^{-r_m}) \in \mathrm{GL}_m(K_v) \quad \text{and} \quad \mathfrak{a}_+ = \mathrm{diag}(\pi_v^{s_1}, \ldots, \pi_v^{s_n}) \in \mathrm{GL}_n(K_v)\ .$$

Note that for all $\boldsymbol{\theta} \in K_v^m$, $\boldsymbol{\xi} \in K_v^n$ and $\ell \in \mathbb{Z}$, we have

$$\| \mathfrak{a}_-^{\ell} \, \boldsymbol{\theta} \|_{\mathbf{r}} = q_v^{\ell} \, \| \boldsymbol{\theta} \|_{\mathbf{r}} \quad \text{and} \quad \| \mathfrak{a}_+^{\ell} \, \boldsymbol{\xi} \|_{\mathbf{s}} = q_v^{-\ell} \, \| \boldsymbol{\xi} \|_{\mathbf{s}} \, . \tag{32}$$

We denote by G^+ the unstable horospherical subgroup for $\mathfrak{a}$ in G and by U the unipotent radical of G, that is,

$$G^+ = \{ g \in G : \lim_{\ell \to -\infty} \mathfrak{a}^{\ell} g \, \mathfrak{a}^{-\ell} = I_{d+1} \} \quad \text{and} \quad U = \left\{ \begin{pmatrix} I_d & u \\ 0 & 1 \end{pmatrix} : u \in K_v^d \right\} .$$

Let $U^+ = G^+ \cap U = \left\{ \begin{pmatrix} I_m & 0 & w \\ 0 & I_n & 0 \\ 0 & 0 & 1 \end{pmatrix} : w \in K_v^m \right\}$, which is a closed subgroup in G^+ normalized by $\mathfrak{a}$.

Let us define

$$\mathscr{X} = G_0/\Gamma_0 \quad \text{and} \quad \mathscr{Y} = G/\Gamma \, .$$

Even though we have $\operatorname{Covol}(R_v^d) = q^{(g-1)d}$ by Equation (4), we say that an R_v-lattice Λ in K_v^d is *unimodular* if $\operatorname{Covol}(\Lambda) = \operatorname{Covol}(R_v^d)$. A translate in the affine space K_v^d of an unimodular lattice is called an *unimodular grid*. We identify the homogeneous space $\mathscr{X} = \operatorname{SL}_d(K_v)/\operatorname{SL}_d(R_v)$ with the space of unimodular lattices in K_v^d by the equivariant homeomorphism

$$x = g\,\Gamma_0 \mapsto \Lambda_x = g\, R_v^d \, ,$$

and the homogeneous space $\mathscr{Y} = \operatorname{ASL}_d(K_v)/\operatorname{ASL}_d(R_v)$ with the space of unimodular grids by the equivariant homeomorphism

$$y = \begin{pmatrix} g & u \\ 0 & 1 \end{pmatrix} \Gamma \mapsto \widetilde{\Lambda}_y = g\, R_v^d + u \, . \tag{33}$$

We denote by $\pi : \mathscr{Y} \to \mathscr{X}$ the natural projection map (forgetting the translation factor), which is a proper map. Note that the fibers of π are exactly the orbits of U in $\mathscr{Y}$, and in particular each orbit under U^+ in $\mathscr{Y}$ is contained in some fiber of π (see Lemma 5.3 for a precise understanding of the U^+-orbits).

For every $N \in \mathbb{N} - \{0\}$, we denote by $d_{\operatorname{SL}_N(K_v)}$ the right-invariant distance on $\operatorname{SL}_N(K_v)$ defined by

$$\forall\, g, h \in \operatorname{SL}_N(K_v), \quad d_{\operatorname{SL}_N(K_v)}(g, h) = \max\{ \ln(1 + |||\, gh^{-1} - \mathrm{id} \,|||\,),\ \ln(1 + |||\, hg^{-1} - \mathrm{id} \,|||\,) \} \, ,$$

where $|||\ \ |||$ is the operator norm on $\mathscr{M}_N(K_v)$ defined by the sup norm $\|\ \|$ on K_v^N. We endow every closed subgroup H of G with the right-invariant distance d_H on H, which is the restriction to H of the distance $d_{\operatorname{SL}_{d+1}(K_v)}$. For instance, identifying the additive group K_v^m with U^+ by the map $w \mapsto \widehat{w} = \begin{pmatrix} I_m & 0 & w \\ 0 & I_n & 0 \\ 0 & 0 & 1 \end{pmatrix}$, we have

$$\forall\, w, w' \in K_v^m, \quad d_{U^+}(\widehat{w}, \widehat{w'}) = \| w - w' \| \, . \tag{34}$$

We endow $\mathscr{Y} = G/\Gamma$ with the quotient distance $d_{\mathscr{Y}}$ of the distance d_G on G, defined by

$$\forall\, y, y' \in \mathscr{Y}, \quad d_{\mathscr{Y}}(y, y') = \min_{\gamma \in \Gamma} d_G(\,\widetilde{y}\,\gamma,\, \widetilde{y}'\gamma)$$

for any representative $\widetilde{y}$ and $\widetilde{y}'$ of the classes y and y' in G/Γ respectively. This is a well defined distance since the canonical projection $G \to \mathscr{Y}$ is a covering map and the distance d_G on G is right-invariant. Given any closed subgroup H of G, we denote by $B_H(x,r)$ (respectively $B_{\mathscr{Y}}(x,r)$) the open ball of center x and radius $r > 0$ for the distance d_H (respectively $d_{\mathscr{Y}}$), and by B_r^H the open ball $B_H(\mathrm{id}, r)$. Note that for all $y \in \mathscr{Y}$ and $r > 0$, we have (for the left action of subsets of G on $\mathscr{Y}$)

$$B_{\mathscr{Y}}(y,r) = B_r^G y\ .$$

Lemma 5.1 *For all $\epsilon > 0$ and $k \in \mathbb{Z}_{\geqslant 0}$, we have $\mathfrak{a}^{-k} B_\epsilon^{U^+} \mathfrak{a}^k \subset B^{U^+}_{\epsilon\, q_v^{-k \min \mathbf{r}}}$ and similarly $\mathfrak{a}^k B_\epsilon^{U^+} \mathfrak{a}^{-k} \subset B^{U^+}_{\epsilon\, q_v^{k \max \mathbf{r}}}$.*

Proof. The proof of the second claim being similar, we only prove the first one. For every $w = (w_1, \dots, w_m) \in K_v^m$, we have $\mathfrak{a}^{-k} \widehat{w}\ \mathfrak{a}^k = \widehat{\mathfrak{a}_-^{-k} w}$ and

$$\|\, \mathfrak{a}_-^{-k} w\,\| = \max_{1 \leqslant i \leqslant m} |\, \pi_v^{r_i k} w_i\,| \leqslant q_v^{-k \min \mathbf{r}}\ \|\, w\,\|\ .$$

The result hence follows from Equation (34). □

Given a point x in $\mathscr{Y}$ (and similarly for x in $\mathscr{X}$), we define the *injectivity radius* of $\mathscr{Y}$ at x to be

$$\mathrm{inj}(x) = \sup \big\{ r > 0 : \forall\, \gamma \in \Gamma - \{\mathrm{id}\},\ \ B_G(\widetilde{x}, r) \cap B_G(\widetilde{x}\,\gamma, r) = \varnothing \big\}\ ,$$

which does not depend on the choice of $\widetilde{x} \in G$ such that $x = \widetilde{x}\,\Gamma$, and is positive and finite since the canonical projection $G \to \mathscr{Y}$ is a nontrivial covering map. For every $r > 0$, we denote the *r-thick part* of $\mathscr{Y}$ by

$$\mathscr{Y}(r) = \{x \in \mathscr{Y} : \mathrm{inj}(x) \geqslant r\}\ .$$

It follows from the finiteness of a (quotient) Haar measure of $\mathscr{Y}$ that $\mathscr{Y}(r)$ is a compact subset of $\mathscr{Y}$ for every $r > 0$, and that the Haar measure of the *r-thin part* $\mathscr{Y} - \mathscr{Y}(r)$ tends to 0 as r goes to 0. For every compact subset K of $\mathscr{Y}$, there exists $r > 0$ such that $K \subset \mathscr{Y}(r)$.

5.2 Dani correspondence

In this subsection, we give an interpretation of the property for a matrix $A \in \mathscr{M}_{m,n}(K_v)$ to be $(\mathbf{r},\mathbf{s})$-singular on average in terms of dynamical properties of the action of the one-parameter diagonal subgroup $(\mathfrak{a}^\ell)_{\ell \in \mathbb{Z}}$ on the space of unimodular lattices, as originally developped by Dani (see for instance [Kle, §4]). For every $A \in \mathscr{M}_{m,n}(K_v)$, let $u_A = \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} \in G_0$.

Proposition 5.2 *A matrix $A \in \mathscr{M}_{m,n}(K_v)$ is $(\mathbf{r},\mathbf{s})$-singular on average if and only if the forward orbit $\{\mathfrak{a}^\ell u_A R_v^d : \ell \in \mathbb{Z}_{\geqslant 0}\}$ in $\mathscr{X}$ of the lattice $u_A R_v^d$ under $\mathfrak{a}$* diverges on average *in $\mathscr{X}$, that is, if and only if for any compact subset Q of $\mathscr{X}$, we have*

$$\lim_{N \to \infty} \frac{1}{N}\ \mathrm{Card}\{\ell \in \{1, \cdots, N\} : \mathfrak{a}^\ell\, u_A\, \Gamma_0 \in Q\} = 0\ .$$

Proof. Let Q be a compact subset of $\mathscr{X}$. By Mahler's compactness criterion (see for instance [KlST, Theo. 1.1]), there exists $\varepsilon \in \,]0,1[$ such that Q is contained in

$$\mathscr{X}_{>\epsilon} = \{g\,R_v^d \in \mathscr{X} : \forall\,(\boldsymbol{\theta},\boldsymbol{\xi}) \in g\,R_v^d - \{0\} \subset K_v^m \times K_v^n,\ \ \max\{\|\,\boldsymbol{\theta}\,\|_{\mathbf{r}}, \|\,\boldsymbol{\xi}\,\|_{\mathbf{s}}\} > \varepsilon\}\ ,$$

which is the subset of $\mathscr{X}$ consisting of the unimodular lattices with systole (for an appropriate quasinorm) larger than ϵ. Observe that by Equation (32), for all sufficiently large $\ell \in \mathbb{Z}_{\geqslant 1}$, there exists an element $\mathbf{y} \in R_v^n - \{0\}$ such that $\langle\, A\,\mathbf{y}\,\rangle_{\mathbf{r}} \leqslant \varepsilon q_v^{-\ell}$ and $\|\,\mathbf{y}\,\|_{\mathbf{s}} \leqslant \varepsilon q_v^{\ell}$ if and only if we have $\mathfrak{a}^\ell u_A R_v^d = \begin{pmatrix}\mathfrak{a}_-^\ell & 0\\ 0 & \mathfrak{a}_+^\ell\end{pmatrix}\begin{pmatrix}I_m & A\\ 0 & I_n\end{pmatrix} R_v^d \in \mathscr{X} - \mathscr{X}_{>\varepsilon}$.

With $\ell_\epsilon = \lfloor -\log_{q_v}\varepsilon\rfloor$, it follows that

$$\begin{aligned}
0 &\leqslant \mathrm{Card}\{\,\ell \in \{1,\cdots,N\} : \mathfrak{a}^\ell\, u_A\, R_v^d \in Q\,\}\\
&\leqslant \mathrm{Card}\{\,\ell \in \{1,\cdots,N\} : \mathfrak{a}^\ell\, u_A\, R_v^d \in \mathscr{X}_{>\varepsilon}\,\}\\
&= \mathrm{Card}\{\ell \in \{1,\cdots,N\} : \nexists\ \mathbf{y} \in R_v^n - \{0\},\ \ \langle\, A\,\mathbf{y}\,\rangle_{\mathbf{r}} \leqslant \varepsilon q_v^{-\ell},\ \ \|\,\mathbf{y}\,\|_{\mathbf{s}} \leqslant \varepsilon q_v^{\ell}\}\\
&\leqslant \mathrm{Card}\{\ell \in \{1,\cdots,N\} : \nexists\ \mathbf{y} \in R_v^n - \{0\},\ \ \langle\, A\,\mathbf{y}\,\rangle_{\mathbf{r}} \leqslant \frac{\varepsilon^2}{q_v} q_v^{-(\ell-\ell_\epsilon)},\ \ \|\,\mathbf{y}\,\|_{\mathbf{s}} \leqslant q_v^{\ell-\ell_\epsilon}\}\\
&\leqslant \ell_\epsilon + \mathrm{Card}\{\ell \in \{1,\cdots,N-\ell_\epsilon\} : \nexists\ \mathbf{y} \in R_v^n - \{0\},\ \ \langle\, A\,\mathbf{y}\,\rangle_{\mathbf{r}} \leqslant \frac{\varepsilon^2}{q_v} q_v^{-\ell},\ \ \|\,\mathbf{y}\,\|_{\mathbf{s}} \leqslant q_v^{\ell}\}\ .
\end{aligned}$$

After dividing by N (or equivalently by $N-\ell_\epsilon$) this last expression, its limit as N tends to 0 exists and is equal to 0 if A is $(\mathbf{r},\mathbf{s})$-singular on average (see Equation (2)). Hence we have $\lim_{N\to\infty} \frac{1}{N}\,\mathrm{Card}\{\ell \in \{1,\cdots,N\} : \mathfrak{a}^\ell\, u_A\,\Gamma_0 \in Q\} = 0$ by the above string of (in)equalities.

The converse implication follows similarly by taking for the compact set Q the subset $\mathscr{X}_{>\varepsilon}$. □

We denote by $\|\ \|_{\mathbf{s},\mathbf{r}}$ the quasi-norm on $K_v^d = K_v^m \times K_v^n$ defined by

$$\|\,(\boldsymbol{\theta},\boldsymbol{\xi})\,\|_{\mathbf{r},\mathbf{s}} = \max\Big\{\|\,\boldsymbol{\theta}\,\|_{\mathbf{r}}^{\frac{d}{m}},\ \|\,\boldsymbol{\xi}\,\|_{\mathbf{s}}^{\frac{d}{n}}\Big\}\ .$$

Let $\varepsilon > 0$. We define

$$\mathscr{L}_\varepsilon = \{y \in \mathscr{Y} : \forall\ u \in \widetilde{\Lambda}_y,\ \ \|\,u\,\|_{\mathbf{r},\mathbf{s}} \geqslant \varepsilon\}\ . \tag{35}$$

By Mahler's compactness criterion (see for instance [KlST, Theo. 1.1]) and since the natural projection $\pi : \mathscr{Y} \to \mathscr{X}$ is proper, the subset $\mathscr{L}_\varepsilon$ is compact.

For every $\boldsymbol{\theta} \in K_v^m$, we denote by $y_{A,\boldsymbol{\theta}}$ the unimodular grid $u_A R_v^d - \binom{\boldsymbol{\theta}}{0}$.

Lemma 5.3 *For every $A \in \mathscr{M}_{m,n}(K_v)$, the map $K_v^m \to \mathscr{Y}$ defined by $\boldsymbol{\theta} \mapsto y_{A,\boldsymbol{\theta}}$ induces an isometry ϕ_A from $\mathbb{T}^m = K_v^m/R_v^m$ endowed with the quotient distance $d_{\mathbb{T}^m}$ of the distance on K_v^m defined by the standard norm $\|\ \|$, and the U^+-orbit $U^+ y_{A,0}$ endowed with the restriction of the distance $d_{\mathscr{Y}}$ of $\mathscr{Y}$.*

Proof. The map $K_v^m \to \mathscr{Y}$ defined by $\boldsymbol{\theta} \mapsto y_{A,\boldsymbol{\theta}}$ is clearly invariant under translations by R_v^m, and induces a bijection

$$\phi_A : \boldsymbol{\theta} \mod R_v^m \mapsto y_{A,\boldsymbol{\theta}} \tag{36}$$

from $\mathbb{T}^m = K_v^m/R_v^m$ to the orbit $U^+ y_{A,0}$. This orbit is contained in the fiber $\pi^{-1}(x_A)$ of $x_A = u_A\, R_v^m$ for the natural projection $\pi : \mathscr{Y} \to \mathscr{X}$, as already seen.

For all $A \in \mathscr{M}_{m,n}(K_v)$ and $\boldsymbol{\theta} \in K_v^m$, let $u_{A,\boldsymbol{\theta}} = \begin{pmatrix} I_m & A & \boldsymbol{\theta} \\ 0 & I_n & 0 \\ 0 & 0 & 1 \end{pmatrix} \in G$, so that we have $y_{A,\boldsymbol{\theta}} = u_{A,-\boldsymbol{\theta}}\Gamma$. For all $\boldsymbol{\theta}, \boldsymbol{\theta}' \in \mathbb{T}^m$, denoting lifts of them to K_v^m by $\widetilde{\boldsymbol{\theta}}, \widetilde{\boldsymbol{\theta}}'$ respectively, identifying K_v^d with $K_v^m \times K_v^n$, and using Equations (31) and (34), we have

$$
\begin{aligned}
d_{\mathscr{Y}}(\phi_A(\boldsymbol{\theta}), \phi_A(\boldsymbol{\theta}')) &= \inf_{\gamma,\gamma' \in \Gamma} d_G\big(u_{A,-\widetilde{\boldsymbol{\theta}}}\,\gamma, u_{A,-\widetilde{\boldsymbol{\theta}}'}\,\gamma'\big) \\
&= \inf_{\substack{g,g' \in \Gamma_0 \\ x,x' \in R_v^m,\; y,y' \in R_v^n}} d_G\big((u_A, (-\widetilde{\boldsymbol{\theta}}, 0))(g, (x,y)), (u_A, (-\widetilde{\boldsymbol{\theta}}', 0))(g', (x', y'))\big) \\
&= \inf_{\substack{g,g' \in \Gamma_0 \\ x,x' \in R_v^m,\; y,y' \in R_v^n}} d_G\big((u_A g, (x + Ay - \widetilde{\boldsymbol{\theta}}, y)), (u_A g', (x' + Ay' - \widetilde{\boldsymbol{\theta}}', y'))\big) \\
&= \inf_{x,x' \in R_v^m} d_{U^+}\big((\mathrm{id}, (x - \widetilde{\boldsymbol{\theta}}, 0)), (\mathrm{id}, (x' - \widetilde{\boldsymbol{\theta}}', 0))\big) \\
&= \inf_{x,x' \in R_v^m} \|\,(x - \widetilde{\boldsymbol{\theta}}) - (x' - \widetilde{\boldsymbol{\theta}}')\,\| = d_{\mathbb{T}^m}(\boldsymbol{\theta}, \boldsymbol{\theta}')\ . \quad \Box
\end{aligned}
$$

Proposition 5.4 *Let $\varepsilon > 0$. For every $(A, \boldsymbol{\theta}) \in \mathscr{M}_{m,n}(K_v) \times K_v^m$ such that $\boldsymbol{\theta} \in \mathbf{Bad}_A(\varepsilon)$, one of the following statements holds.*

(1) *There exists $\mathbf{y} \in R_v^n$ such that $\langle\, A\,\mathbf{y} - \boldsymbol{\theta}\,\rangle_{\mathbf{r}} = 0$. Note that given A, there are only countably many $\boldsymbol{\theta}$ satisfying this statement.*

(2) *The forward $\mathfrak{a}$-orbit of the point $y_{A,\boldsymbol{\theta}}$ is eventually in $\mathscr{L}_\varepsilon$, that is, there exists $T \geqslant 0$ such that for every $\ell \geqslant T$, we have $\mathfrak{a}^\ell\, y_{A,\boldsymbol{\theta}} \in \mathscr{L}_\varepsilon$.*

Proof. Assume for a contradiction that both statements do not hold. Then there exist infinitely many $\ell \in \mathbb{Z}_{\geqslant 1}$ such that $\mathfrak{a}^\ell y_{A,\boldsymbol{\theta}} \notin \mathscr{L}_\varepsilon$, hence such that there exists $\mathbf{y}_\ell \in R_v^n$ with $\langle\, A\,\mathbf{y}_\ell - \boldsymbol{\theta}\,\rangle_{\mathbf{r}} < q_v^{-\ell}\varepsilon^{\frac{m}{d}}$ and $\|\,\mathbf{y}_\ell\,\|_{\mathbf{s}} < q_v^{\ell}\varepsilon^{\frac{n}{d}}$. Since the statement (1) does not hold, the inequality

$$\|\,\mathbf{y}\,\|_s \langle\, A\,\mathbf{y} - \boldsymbol{\theta}\,\rangle_{\mathbf{r}} < \varepsilon$$

has infinitely many solutions $\mathbf{y} \in R_v^n$, which contradicts the assumption $\boldsymbol{\theta} \in \mathbf{Bad}_A(\varepsilon)$. $\Box$

5.3 Entropy, partition construction, and effective variational principle

In this subsection, after recalling the basic definitions and properties about entropy (using [ELW] as a general reference, and in particular its Chapter 2), we give the preliminary constructions of σ-algebras and results on entropy that will be needed in Section 6. In particular, we give an effective and positive characteristic version of the variational principle for conditional entropy of [EL, §7.55], adapting to the function field case the result of [KKL].

Let $(X, \mathscr{B}, \mu)$ be a standard Borel probability space. For every set E of subsets of X, we denote by $\sigma(E)$ the σ-algebra of subsets of X generated by E. Let $\mathscr{P}$ be a (finite or) countable $\mathscr{B}$-measurable partition of X. Let $\mathscr{A}$, $\mathscr{C}$ and $\mathscr{C}'$ be sub-σ-algebras of $\mathscr{B}$. Suppose that $\mathscr{C}$ and $\mathscr{C}'$ are countably generated.

For every $x \in X$, we denote by $[x]_{\mathscr{P}}$ the *atom* of x for $\mathscr{P}$, which is the element of the partition $\mathscr{P}$ containing x. We denote by $[x]_{\mathscr{C}}$ the *atom* of x for $\mathscr{C}$, which is the intersection of all elements of $\mathscr{C}$ containing x. Note that $[x]_{\sigma(\mathscr{P})} = [x]_{\mathscr{P}}$. We denote by $(\mu_x^{\mathscr{A}})_{x \in X}$ an $\mathscr{A}$-mesurable family of (Borel probability) *conditional measures* of μ with respect to $\mathscr{A}$ on X, given for instance by [EL, Theo. 5.9].

Using the standard convention $0\log_{q_v} 0 = 0$ and using $\log_{q_v}$ instead of log for computational purposes in the field K_v, the *entropy* of the partition $\mathscr{P}$ with respect to μ is defined by

$$H_\mu(\mathscr{P}) = -\sum_{P\in\mathscr{P}} \mu(P)\log_{q_v}\mu(P) \;\in [0,\infty]\;.$$

Recall the (logarithmic) cardinality majoration

$$H_\mu(\mathscr{P}) \leqslant \log_{q_v}(\operatorname{Card}\mathscr{P})\;. \tag{37}$$

The *information function* of $\mathscr{C}$ given $\mathscr{A}$ with respect to μ is the measurable map $I_\mu(\mathscr{C}|\mathscr{A}) : X \to [0,\infty]$ defined by

$$\forall\, x\in X,\quad I_\mu(\mathscr{C}|\mathscr{A})(x) = -\log_{q_v}\mu_x^{\mathscr{A}}([x]_{\mathscr{C}})\;.$$

The *conditional entropy* of $\mathscr{C}$ given $\mathscr{A}$ with respect to μ is defined by

$$H_\mu(\mathscr{C}|\mathscr{A}) = \int_X I_\mu(\mathscr{C}|\mathscr{A})\,d\mu\;. \tag{38}$$

Recall the additivity property $H_\mu(\mathscr{C}\vee\mathscr{C}'\,|\,\mathscr{A}) = H_\mu(\mathscr{C}\,|\,\mathscr{C}'\vee\mathscr{A}) + H_\mu(\mathscr{C}'\,|\,\mathscr{A})$ (see for instance [ELW, Prop. 2.13]) so that if $\mathscr{A}\subset\mathscr{C}'\subset\mathscr{C}$, we have

$$H_\mu(\mathscr{C}\,|\,\mathscr{A}) = H_\mu(\mathscr{C}\,|\,\mathscr{C}') + H_\mu(\mathscr{C}'\,|\,\mathscr{A})\;. \tag{39}$$

Let $T : (X,\mathscr{B},\mu)\to(X,\mathscr{B},\mu)$ be a measure-preserving transformation. We denote by $\mathscr{E} = \{B\in\mathscr{B} : \mu(T^{-1}B\Delta B) = 0\}$ the sub-σ-algebra of T-invariant elements of $\mathscr{B}$, and by $(\mu_x^{\mathscr{E}})_{x\in X}$ the associated family of conditional measures. Assume that the σ-algebra $\mathscr{A}$ satisfies the property $T^{-1}\mathscr{A}\subset\mathscr{A}$. If the partition $\mathscr{P}$ has finite entropy with respect to μ, let

$$h_\mu(T,\mathscr{P}|\mathscr{A}) = \lim_{n\to\infty}\frac{1}{n}H_\mu\Big(\bigvee_{i=0}^{n-1}T^{-i}\mathscr{P}|\mathscr{A}\Big) = \inf_{n\geqslant 1}\frac{1}{n}H_\mu\Big(\bigvee_{i=0}^{n-1}T^{-i}\mathscr{P}|\mathscr{A}\Big)\;.$$

The *conditional (dynamical) entropy* of T given $\mathscr{A}$ is

$$h_\mu(T|\mathscr{A}) = \sup_{\mathscr{P}}\; h_\mu(T,\mathscr{P}|\mathscr{A})\;,$$

where the upper bound is taken on all countable $\mathscr{B}$-measurable partitions $\mathscr{P}$ of X with finite entropy with respect to μ.

With the above notations, the following result is proven in [KKL, Prop. 2.2 and Appendix A].

Proposition 5.5 (Entropy and ergodic decomposition) *If $T^{-1}\mathscr{A}\subset\mathscr{A}$, then for every countable $\mathscr{B}$-measurable partition $\mathscr{P}$ with finite entropy with respect to μ, we have*

$$h_\mu(T,\mathscr{P}|\mathscr{A}) = \int_X h_{\mu_x^{\mathscr{E}}}(T,\mathscr{P}|\mathscr{A})\,d\mu(x) \quad\text{and}\quad h_\mu(T|\mathscr{A}) = \int_X h_{\mu_x^{\mathscr{E}}}(T|\mathscr{A})\,d\mu(x)\;.\quad\Box$$

We now work in the standard Borel space $\mathscr{Y}$ of unimodular grids, endowed with the distance $d_{\mathscr{Y}}$ (see Section 5.1). Let $\delta>0$. For every subset B of $\mathscr{Y}$, we define the *δ-boundary* $\partial_\delta B$ of B by

$$\partial_\delta B = \big\{\, y\in\mathscr{Y} : \inf_{y'\,\in B}\; d_{\mathscr{Y}}(y,y') + \inf_{y''\,\in\mathscr{Y}-B}\; d_{\mathscr{Y}}(y,y'') < \delta\,\big\}$$

if B and $\mathscr{Y} - B$ are nonempty, and $\partial_\delta B = \varnothing$ otherwise. Note that for all subsets B and B' of $\mathscr{Y}$, we have

$$\partial_\delta(B \cup B') \subset \partial_\delta B \cup \partial_\delta B' \quad \text{and} \quad \partial_\delta(B - B' \cap B) \subset \partial_\delta B \cup \partial_\delta B' \ . \tag{40}$$

We also have $\partial_\delta B \subset \partial_{\delta'} B$ if $\delta \leqslant \delta'$. Given any set $\mathscr{P}$ of subsets of $\mathscr{Y}$, we define the *δ-boundary* $\partial_\delta \mathscr{P}$ of $\mathscr{P}$ by

$$\partial_\delta \mathscr{P} = \bigcup_{B \in \mathscr{P}} \partial_\delta B \ .$$

Lemma 5.6 *For every $r > 0$, there exist $\delta_r \in \,]0, r]$ and a finite measurable partition $\mathscr{P} = \{P_1, \dots, P_N, P_\infty\}$ by closed and open subsets of $\mathscr{Y}$ such that*
(1) *the subset P_∞ is contained in the r-thin part $\mathscr{Y} - \mathscr{Y}(r)$,*
(2) *for every $i \in \{1, \dots, N\}$, there exists $y_i \in \mathscr{Y}(r)$ such that $B^G_{\frac{r}{2}} y_i \subset P_i \subset B^G_r y_i$,*
(3) *the set $\partial_{\delta_r} \mathscr{P}$ is empty.*

Proof. Choose a finite maximal r-separated subset $\{y_1, \dots, y_N\}$ of $\mathscr{Y}(r)$ for the distance $d_{\mathscr{Y}}$, which exists by the compactness of $\mathscr{Y}(r)$. By induction on $i = 1, \dots, N$, we define a Borel subset P_i of $\mathscr{Y}$ by

$$P_i = B^G_r y_i - \Big(\bigcup_{j=1}^{i-1} P_j \cup \bigcup_{j=i+1}^{N} B^G_{\frac{r}{2}} y_j \Big) \ .$$

Define $P_\infty = \mathscr{Y} - \bigcup_{j=1}^N P_j$, which is also a Borel subset of $\mathscr{Y}$.

By construction, we have $P_i \subset B^G_r y_i$. Since the set $\{y_1, \dots, y_N\}$ is ϵ-separated, the intersection of open balls $B^G_{\frac{r}{2}} y_i \cap B^G_{\frac{r}{2}} y_j = B_{\mathscr{Y}}(y_i, \frac{r}{2}) \cap B_{\mathscr{Y}}(y_j, \frac{r}{2})$ is empty if $j > i$. By construction, the intersection $B^G_{\frac{r}{2}} y_i \cap P_j$ is empty if $j < i$. Therefore P_i contains $B^G_{\frac{r}{2}} y_i$, and Assertion (ii) follows.

By construction, we have $\bigcup_{j=1}^N P_j \subset \bigcup_{j=1}^N B^G_r y_j = \bigcup_{j=1}^N B_{\mathscr{Y}}(y_j, r)$, and the later union contains $\mathscr{Y}(r)$, since the ϵ-separated set $\{y_1, \dots, y_N\}$ is maximal. Assertion (i) follows.

For every $s > 0$, let $n_s = \left\lceil \frac{\ln(e^s - 1)}{\ln q_v} \right\rceil \in \mathbb{Z}$ and $\delta'_s = \ln\big(\frac{1 + q_v^{n_s}}{1 + q_v^{n_s - 1}}\big) > 0$. For all $\delta > 0$ and $y \in \mathscr{Y}$, assume that there exists a point $z \in \partial_\delta B_{\mathscr{Y}}(y, s)$. Let $z' \in B_{\mathscr{Y}}(y, s)$ and $z'' \notin B_{\mathscr{Y}}(y, s)$ be such that $d_{\mathscr{Y}}(z, z') + d_{\mathscr{Y}}(z, z'') < \delta$. Since the operator norm on $\mathscr{M}_{d+1}(K_v)$ has values in $\{0\} \cup q_v^{\mathbb{Z}}$, the set $\{d_{\mathscr{Y}}(y, y') : y, y' \in \mathscr{Y}\}$ of values of the distance function $d_{\mathscr{Y}}$ on $\mathscr{Y}$ is contained in $\{0\} \cup \{\ln(1 + q_v^n) : n \in \mathbb{Z}\}$. Since $s \in \,]\ln(1 + q_v^{n_s - 1}), \ln(1 + q_v^{n_s})]$, we hence have $d_{\mathscr{Y}}(y, z') \leqslant \ln(1 + q_v^{n_s - 1})$ since $z' \in B_{\mathscr{Y}}(y, s)$ and $d_{\mathscr{Y}}(y, z'') \geqslant \ln(1 + q_v^{n_s})$ since $z'' \notin B_{\mathscr{Y}}(y, s)$. Therefore by the triangle inequality and the inverse triangle inequality, we have

$$\begin{aligned} \delta > d_{\mathscr{Y}}(z, z') + d_{\mathscr{Y}}(z, z'') &\geqslant d_{\mathscr{Y}}(z', z'') \geqslant d_{\mathscr{Y}}(y, z'') - d_{\mathscr{Y}}(y, z') \\ &\geqslant \ln(1 + q_v^{n_s}) - \ln(1 + q_v^{n_s - 1}) = \delta'_s \ . \end{aligned}$$

Hence $\partial_\delta B_{\mathscr{Y}}(y, s)$ is empty for every $\delta \in \,]0, \delta'_s]$.

By Equation (40), for every $\delta > 0$, we have

$$\partial_\delta \mathscr{P} \subset \bigcup_{j=1}^N \partial_\delta(B^G_r y_j) \cup \bigcup_{j=1}^N \partial_\delta(B^G_{\frac{r}{2}} y_j) \ .$$

Hence Assertion (iii) follows with $\delta_r = \min\{\delta'_{\frac{r}{2}}, r\}$.

Note that since the distance d_G has values in $\{0\} \cup \{\ln(1 + q_v^n) : n \in \mathbb{Z}\}$, the open balls in G are open and compact, and since the canonical projection $G \to \mathscr{Y}$ is open and continuous, the subsets P_i of $\mathscr{Y}$ are by construction open and compact, and P_∞ is closed and open. □

Let $\mathscr{C}$ be a countably generated σ-algebra of subsets of $\mathscr{Y}$. Note that for every $j \in \mathbb{Z}$, the σ-algebra $\mathfrak{a}^j\mathscr{C}$ is also countably generated and

$$[y]_{a^j\mathscr{C}} = \mathfrak{a}^j\,[\mathfrak{a}^{-j}y]_{\mathscr{C}}\ .$$

We say that $\mathscr{C}$ is $\mathfrak{a}^{-1}$-*descending* if $\mathfrak{a}\mathscr{C}$ is contained in $\mathscr{C}$. In particular, for all $y \in \mathscr{Y}$ and $j \in \mathbb{Z}_{\geqslant 0}$, we have

$$[y]_{\mathscr{C}} \subset [y]_{\mathfrak{a}^j\mathscr{C}}\ .$$

Given a Borel probability measure μ on $\mathscr{Y}$ and a closed subgroup H of G, we say that $\mathscr{C}$ is H-*subordinated modulo* μ if for μ-almost every $y \in \mathscr{Y}$, there exists $r = r_y \in\ ]0,1]$ such that we have

$$B_r^H y \subset [y]_{\mathscr{C}} \subset B_{1/r}^H y\ .$$

If $\mathscr{C}$ is U^+-subordinated modulo μ and if furthermore μ is $\mathfrak{a}$-invariant, since $\mathfrak{a}$ normalises U^+ and by Lemma 5.1, for every $j \in \mathbb{Z}$, the σ-algebra $\mathfrak{a}^j\mathscr{C}$ is also U^+-subordinated modulo μ.

For every σ-algebra $\mathscr{A}$ of subsets of $\mathscr{Y}$, for all a, b in $\mathbb{Z} \cup \{\pm\infty\}$ with $a < b$, we define a σ-algebra $\mathscr{A}_a^b$ of subsets of $\mathscr{Y}$ by

$$\mathscr{A}_a^b = \bigvee_{i=a}^{b} \mathfrak{a}^i\mathscr{A} = \sigma\Big(\bigcup_{a\leqslant i\leqslant b} \mathfrak{a}^i\mathscr{A}\Big)\ .$$

Note that if $\mathscr{A}$ is countably generated, then so is $\mathscr{A}_a^b$.

Proposition 5.7 *For every $r \in\ ]0,1[$, there exists a countably generated sub-σ-algebra $\mathscr{A}^{U^+}$ of the Borel σ-algebra of $\mathscr{Y}$ such that*

(1) *the countably generated σ-algebra $\mathscr{A}^{U^+}$ is $\mathfrak{a}^{-1}$-descending,*

(2) *for every $y \in \mathscr{Y}(r)$, we have $[y]_{\mathscr{A}^{U^+}} \subset B_r^{U^+}y$,*

(3) *for every $y \in \mathscr{Y}$, we have $B_{\delta_r}^{U^+}y \subset [y]_{\mathscr{A}^{U^+}}$, where $\delta_r \in\ ]0,r]$ is as in Lemma 5.6.*

Let μ be a Borel $\mathfrak{a}$-invariant ergodic probability measure on $\mathscr{Y}$ with $\mu(\mathscr{Y}(r)) > 0$. Then $\mathscr{A}^{U^+}$ is U^+-subordinated modulo μ.

Proof. Fix $r \in\ ]0,1[$. Let $\mathscr{P} = \{P_1, \dots, P_N, P_\infty\}$ be a partition given by Lemma 5.6 for this r. We prove a preliminary result on the countably generated sub-σ-algebra $\sigma(\mathscr{P})_0^\infty$.

Lemma 5.8 *For every $y \in \mathscr{Y}$, we have $B_{\delta_r}^{U^+}y \subset [y]_{\sigma(\mathscr{P})_0^\infty}$.*

Proof. Let $h \in B_{\delta_r}^{U^+}$. Assume for a contradiction that $hy \notin [y]_{\sigma(\mathscr{P})_0^\infty}$. Then there exists $k \in \mathbb{Z}_{\geqslant 0}$ such that $\mathfrak{a}^{-k}hy$ and $\mathfrak{a}^{-k}y$ belong to different atoms of the partition $\mathscr{P}$. Let $\alpha = \min \mathbf{r} > 0$. By Lemma 5.1, we have

$$d_{\mathscr{Y}}(\mathfrak{a}^{-k}hy, \mathfrak{a}^{-k}y) \leqslant d_G(\mathfrak{a}^{-k}h\mathfrak{a}^k, \mathrm{id}) = d_{U^+}(\mathfrak{a}^{-k}h\mathfrak{a}^k, \mathrm{id}) < q_v^{-k\alpha}\delta_r \leqslant \delta_r \leqslant r\ . \tag{41}$$

It follows that both $\mathfrak{a}^{-k}hy$ and $\mathfrak{a}^{-k}y$ belong to the δ_r-boundary $\partial_{\delta_r}\mathscr{P}$ of $\mathscr{P}$. But the set $\partial_{\delta_r}\mathscr{P}$ is empty by Lemma 5.6 (3), which gives a contradiction. $\square$

By Lemma 5.6, for every $i \in \{1, \dots, N\}$, there exist $y_i \in \mathscr{Y}(r)$ and a Borel subset V_i of $\mathscr{Y}$ contained in B_r^G such that $P_i = V_i y_i$. Let $\mathscr{P}^{U^+}$ be the sub-σ-algebra of the Borel σ-algebra of $\mathscr{Y}$ generated by the subsets $P_\infty \cap \pi^{-1}(W)$, where W is a Borel subset of $\mathscr{X}$, and the subsets $((U^+B) \cap V_i)y_i$, where $i \in \{1, \dots, N\}$ and B is a Borel subset of G. Then $\mathscr{P}^{U^+}$ is countably generated, since the Borel σ-algebra of $\mathscr{X}$ is countably generated and U^+ is a closed subgroup of G. For every $y \in \mathscr{Y}$, the atom of y for $\mathscr{P}^{U^+}$ is equal to

$$[y]_{\mathscr{P}^{U^+}} = \begin{cases} Uy & \text{if } \ y \in P_\infty \\ P_i \cap (B_r^{U^+} y) & \text{if } \ \exists\, i \in \{1, \dots, N\}, \ y \in P_i \ . \end{cases} \tag{42}$$

Let us now define $\mathscr{A}^{U^+} = (\mathscr{P}^{U^+})_0^\infty$, which is a countably generated sub-σ-algebra of the Borel σ-algebra of $\mathscr{Y}$, since so is $\mathscr{P}^{U^+}$. Note that $\mathfrak{a}\,\mathscr{A}^{U^+} = (\mathscr{P}^{U^+})_1^\infty \subset \mathscr{A}^{U^+}$, which proves Assertion (1).

For every $y \in \mathscr{Y}(r)$, since $P_\infty \subset \mathscr{Y} - \mathscr{Y}(r)$ by Lemma 5.6 (1) and by Equation (42), we have $[y]_{\mathscr{A}^{U^+}} \subset [y]_{\mathscr{P}^{U^+}} \subset B_r^{U^+} y$, which proves Assertion (2).

In order to prove the last Assertion (3), let us take $y \in \mathscr{Y}$ and $h \in B_{\delta_r}^{U^+}$ and let us prove that $hy \in [y]_{\mathscr{A}^{U^+}}$. Since we have $hy \in [y]_{\sigma(\mathscr{P})_0^\infty}$ by Lemma 5.8, for every $k \geqslant 0$, there exists $i \in \{1, \dots, N, \infty\}$ such that the points $\mathfrak{a}^{-k}y$ and $\mathfrak{a}^{-k}hy = \mathfrak{a}^{-k}h\mathfrak{a}^k(\mathfrak{a}^{-k}y)$ both belong to $P_i \in \mathscr{P}$. If $i = \infty$, then by Equation (42), the points $\mathfrak{a}^{-k}y$ and $\mathfrak{a}^{-k}hy$ lie in the same atom $[\mathfrak{a}^{-k}y]_{\mathscr{P}^{U^+}} = U\mathfrak{a}^{-k}y$ since $\mathfrak{a}^{-k}h\mathfrak{a}^k \in U^+$. Assume that $1 \leqslant i \leqslant N$. Since $h \in B_{\delta_r}^{U^+}$, it follows from Equation (41) that $\mathfrak{a}^{-k}h\mathfrak{a}^k \in B_r^{U^+}$. Hence by Equation (42), the points $\mathfrak{a}^{-k}y$ and $\mathfrak{a}^{-k}hy$ lie in the same atom $[\mathfrak{a}^{-k}y]_{\mathscr{P}^{U^+}} = P_i \cap (B_r^{U^+}\mathfrak{a}^{-k}y)$ of $\mathscr{P}^{U^+}$. This proves Assertion (3).

Now let μ be an $\mathfrak{a}$-invariant ergodic probability measure on $\mathscr{Y}$ with $\mu(\mathscr{Y}(r)) > 0$. By ergodicity, for μ-almost every $y \in \mathscr{Y}$, there exists $k \in \mathbb{Z}_{\geqslant 1}$ such that $\mathfrak{a}^{-k}y \in \mathscr{Y}(r)$. Since $\mathfrak{a}^k\mathscr{A}^{U^+} \subset \mathscr{A}^{U^+}$, by Assertion (1) and by Lemma 5.1, we have

$$[y]_{\mathscr{A}^{U^+}} \subset [y]_{\mathfrak{a}^k\mathscr{A}^{U^+}} = \mathfrak{a}^k[\mathfrak{a}^{-k}y]_{\mathscr{A}^{U^+}} \subset \mathfrak{a}^k B_r^{U^+}\mathfrak{a}^{-k}y \subset B_{q_v^k \max \mathbf{r}}^{U^+} y \ .$$

With Assertion (3), this proves that $\mathscr{A}^{U^+}$ is U^+-subordinated modulo μ. $\square$

Let us introduce some material before stating and proving our next Lemma 5.9. The map $d_{K_v^m,\mathbf{r}} : K_v^m \times K_v^m \to [0, +\infty[$ defined by

$$\forall\, \boldsymbol{\xi}, \boldsymbol{\xi}' \in K_v^m, \quad d_{K_v^m,\mathbf{r}}(\boldsymbol{\xi}, \boldsymbol{\xi}') = \|\, \boldsymbol{\xi} - \boldsymbol{\xi}' \,\|_{\mathbf{r}} \tag{43}$$

is an ultrametric distance on K_v^m, since the $\mathbf{r}$-pseudonorm $\|\ \|_{\mathbf{r}}$ satisfies the ultrametric inequality : for all $\boldsymbol{\xi}, \boldsymbol{\xi}' \in K_v^m$, we have

$$\|\, \boldsymbol{\xi} + \boldsymbol{\xi}' \,\|_{\mathbf{r}} \leqslant \max\{\|\, \boldsymbol{\xi} \,\|_{\mathbf{r}},\ \|\, \boldsymbol{\xi}' \,\|_{\mathbf{r}}\} \ , \tag{44}$$

with equality if $\|\, \boldsymbol{\xi} \,\|_{\mathbf{r}} \neq \|\, \boldsymbol{\xi}' \,\|_{\mathbf{r}}$. Note that the map similar to $d_{K_v^m,\mathbf{r}}$ in the real case of [KKL] is not a distance if $m \geqslant 2$ for general $\mathbf{r}$. For every $\epsilon > 0$, we denote by $B_\epsilon^{K_v^m,\mathbf{r}}$ the open ball of center 0 and radius ϵ in K_v^m for $d_{K_v^m,\mathbf{r}}$. Note that the distance $d_{K_v^m,\mathbf{r}}$ is bihölder equivalent to the standard one: For all $\boldsymbol{\xi}, \boldsymbol{\xi}' \in K_v^m$ such that $\|\boldsymbol{\xi} - \boldsymbol{\xi}'\| \leqslant 1$, we have

$$\|\boldsymbol{\xi} - \boldsymbol{\xi}'\|^{\frac{1}{\min \mathbf{r}}} \leqslant d_{K_v^m, \mathbf{r}}(\boldsymbol{\xi}, \boldsymbol{\xi}') \leqslant \|\boldsymbol{\xi} - \boldsymbol{\xi}'\|^{\frac{1}{\max \mathbf{r}}} . \tag{45}$$

We also endow the quotient space $\mathbb{T}^m = K_v^m / R_v^m$ with the quotient distance $d_{\mathbb{T}^m, \mathbf{r}}$ of the distance $d_{K_v^m, \mathbf{r}}$ on K_v^m defined by Equation (43). For every $A \in \mathscr{M}_{m,n}(K_v)$, we denote by $d_{U^+ y_{A,0}, \mathbf{r}}$ the distance on the orbit $U^+ y_{A,0} = \phi_A(\mathbb{T}^m)$ such that the homeomorphism ϕ_A defined in Lemma 5.3 is also an isometry for the distances $d_{\mathbb{T}^m, \mathbf{r}}$ and $d_{U^+ y_{A,0}, \mathbf{r}}$.

Using the identification $w \mapsto \widehat{w}$ between K_v^m and U^+ (see Subsection 5.1), for every $\epsilon > 0$, we denote by $B_\epsilon^{U^+, \mathbf{r}}$ the open ball of radius ϵ in U^+ centered at the identity element for the distance $d_{U^+, \mathbf{r}}$ on U^+ isometric to $d_{K_v^m, \mathbf{r}}$. The map $u \mapsto u\, y_{A,0}$ from U^+ onto $U^+ y_{A,0}$ is 1-Lipschitz and locally isometric for the distances $d_{U^+, \mathbf{r}}$ and $d_{U^+ y_{A,0}, \mathbf{r}}$. Improving Lemma 5.1, for all $\epsilon > 0$ and $k \in \mathbb{Z}$, we have

$$\mathfrak{a}^{-k} B_\epsilon^{U^+, \mathbf{r}} \mathfrak{a}^k = B_{\epsilon\, q_v^{-k}}^{U^+, \mathbf{r}} . \tag{46}$$

Again using the (locally compact) topological group identification $w \mapsto \widehat{w}$ between $(K_v^m, +)$ and U^+, we endow U^+ with the Haar measure m_{U^+} which corresponds to the normalized Haar measure vol_v^m of K_v^m (see Section 2.1). For every $j \in \mathbb{Z}$, the Jacobian Jac_j with respect to the measure m_{U^+} of the homeomorphism $\varphi_j : u \mapsto \mathfrak{a}^j\, u\, \mathfrak{a}^{-j}$ from U^+ to U^+ (which is constant since φ_j is a group automorphism and m_{U^+} is bi-invariant) is easy to compute: we have

$$\mathrm{Jac}_j = q_v^{\,j\,|\mathbf{r}|} . \tag{47}$$

Lemma 5.9 *For every $r \in \left]0,1\right[$, let $\mathscr{A}^{U^+}$ be as in Proposition 5.7. Let μ be an $\mathfrak{a}$-invariant ergodic probability measure on $\mathscr{Y}$. Then*

$$h_\mu(\mathfrak{a}^{-1} | \mathscr{A}^{U^+}) \leqslant |\mathbf{r}| .$$

Furthermore, if $\mu(\mathscr{Y}(r)) > 0$, then

$$h_\mu(\mathfrak{a}^{-1} | \mathscr{A}^{U^+}) \leqslant H_\mu(\mathscr{A}^{U^+} | \,\mathfrak{a}\, \mathscr{A}^{U^+}) .$$

Proof. The proof of the latter assertion is formally the same one as for the real case in [KKL, Lemma 2.8] by replacing a, L, $\mu_{y_0}^{\mathscr{E}}$ therein by $\mathfrak{a}$, U^+, μ herein.

Let us prove the first assertion. By [EL, Prop. 7.44], there exists a countable Borel-measurable partition $\mathscr{G}$ with finite entropy which is a generator for $\mathfrak{a}$ modulo μ, such that $\sigma(\mathscr{G})_0^\infty$ is $\mathfrak{a}^{-1}$-descending and G^+-subordinated modulo μ. We first claim that

$$h_\mu(\mathfrak{a}^{-1} | \mathscr{A}^{U^+}) \leqslant H_\mu(\sigma(\mathscr{G})_0^\infty \vee \mathscr{A}^{U^+} | \,\mathfrak{a}\, (\sigma(\mathscr{G})_0^\infty \vee \mathscr{A}^{U^+})). \tag{48}$$

Indeed, for every countable Borel-measurable partition ξ with $H_\mu(\xi) < \infty$, using Assertions (3), (4), and (6) in [KKL, Prop. A.2], the fact that $\mathscr{A}^{U^+}$ is $\mathfrak{a}^{-1}$-descending and the continuity of entropy [ELW, Prop. 2.14], we have

$$
\begin{aligned}
& h_\mu(\mathfrak{a}^{-1}, \xi \,|\, \mathscr{A}^{U^+}) \\
\underset{(3)}{\leqslant}\ & h_\mu(\mathfrak{a}^{-1}, \bigvee_{i=-k}^{k} \mathfrak{a}^i \mathscr{G} \,|\, \mathscr{A}^{U^+}) + H_\mu(\sigma(\xi) \,|\, \sigma(\mathscr{G})_{-k}^{k} \vee \mathscr{A}^{U^+}) \\
\underset{(4)}{=}\ & h_\mu(\mathfrak{a}^{-1}, \mathscr{G} \,|\, \mathfrak{a}^k \mathscr{A}^{U^+}) + H_\mu(\sigma(\xi) \,|\, \sigma(\mathscr{G})_{-k}^{k} \vee \mathscr{A}^{U^+}) \\
\underset{(6)}{=}\ & H_\mu(\sigma(\mathscr{G}) \,|\, \sigma(\mathscr{G})_1^\infty \vee (\mathfrak{a}^k \mathscr{A}^{U^+})_0^\infty) + H_\mu(\sigma(\xi) \,|\, \sigma(\mathscr{G})_{-k}^{k} \vee \mathscr{A}^{U^+}) \\
=\ & H_\mu(\sigma(\mathscr{G}) \,|\, \sigma(\mathscr{G})_1^\infty \vee (\mathscr{A}^{U^+})_0^\infty) + H_\mu(\sigma(\xi) \,|\, \sigma(\mathscr{G})_{-k}^{k} \vee \mathscr{A}^{U^+}) \\
\xrightarrow[k\to\infty]{}\ & H_\mu(\sigma(\mathscr{G}) \,|\, \sigma(\mathscr{G})_1^\infty \vee (\mathscr{A}^{U^+})_0^\infty) + H_\mu(\sigma(\xi) \,|\, \sigma(\mathscr{G})_{-\infty}^{\infty} \vee \mathscr{A}^{U^+}) \\
\underset{\mathscr{G}\text{generator}}{=}\ & H_\mu(\sigma(\mathscr{G}) \,|\, \sigma(\mathscr{G})_1^\infty \vee (\mathscr{A}^{U^+})_0^\infty) \leqslant H_\mu(\sigma(\mathscr{G}) \,|\, \sigma(\mathscr{G})_1^\infty \vee \mathscr{A}^{U^+}) \\
=\ & H_\mu(\sigma(\mathscr{G})_0^\infty \vee \mathscr{A}^{U^+} \,|\, \sigma(\mathscr{G})_1^\infty \vee \mathscr{A}^{U^+}) \leqslant H_\mu(\sigma(\mathscr{G})_0^\infty \vee \mathscr{A}^{U^+} \,|\, \mathfrak{a}(\sigma(\mathscr{G})_0^\infty \vee \mathscr{A}^{U^+})).
\end{aligned}
$$

This proves the claim (48).

As in the proof of [LSS, Prop. 3.1], the σ-algebra $\sigma(\mathscr{G})_0^\infty \vee \mathscr{A}^{U^+}$ is countably generated, $\mathfrak{a}^{-1}$-descending, and U^+-subordinated since $[y]_{\mathscr{A}^{U^+}} \subset Uy$ for all $y \in \mathscr{Y}$ and since $\sigma(\mathscr{G})_0^\infty$ is G^+-subordinated. Thus by [EL, Prop. 7.34] (recalling that we are using logarithms with base q_v), we have

$$
H_\mu(\sigma(\mathscr{G})_0^\infty \vee \mathscr{A}^{U^+} \,|\, \mathfrak{a}\,(\sigma(\mathscr{G})_0^\infty \vee \mathscr{A}^{U^+})) = \lim_{k\to\infty} \frac{\log_{q_v} \mu_x^{U^+}(\mathfrak{a}^k B_1^{U^+} \mathfrak{a}^{-k})}{k} \;,
$$

where $\mu_x^{U^+}$ is the leaf-wise measure of μ at $x \in \mathscr{Y}$ with respect to U^+ as defined in [EL, Theo. 6.3]. By [EL, Theo. 6.30] (which applies since U^+ is abelian, hence unimodular) and by Equation (47) (see also [EL, §7.42]), we have

$$
\limsup_{k\to\infty} \frac{\mu_x^{U^+}(\mathfrak{a}^k B_1^{U^+} \mathfrak{a}^{-k})}{k^2\, q_v^{k|\mathbf{r}|}} = 0 \;,
$$

hence we have

$$
\lim_{k\to\infty} \frac{\log_{q_v} \mu_x^{U^+}(\mathfrak{a}^k B_1^{U^+} \mathfrak{a}^{-k})}{k} \leqslant |\mathbf{r}| \;.
$$

This proves the first assertion of the lemma. □

Let us introduce some more material before stating and proving our final Proposition 5.10 of Subsection 5.3. Let $\mathscr{A}$ be a countably generated sub-σ-algebra of the Borel σ-algebra of $\mathscr{Y}$. For all $j \in \mathbb{Z}_{\geqslant 0}$ and $y \in \mathscr{Y}$, let

$$
V_y^{\mathfrak{a}^j \mathscr{A}} = \{u \in U^+ : u\, y \in [y]_{\mathfrak{a}^j \mathscr{A}}\} \;, \tag{49}
$$

which is a Borel subset of U^+, called the *U^+-shape of the atom* $[y]_{\mathfrak{a}^j \mathscr{A}}$. Note that for every $j \in \mathbb{Z}_{\geqslant 0}$, we have

$$
V_y^{\mathfrak{a}^j \mathscr{A}} = \mathfrak{a}^j\, V_{\mathfrak{a}^{-j} y}^{\mathscr{A}}\, \mathfrak{a}^{-j} \;.
$$

Let us define a Borel-measurable family $\big(\tau_y^{\mathfrak{a}^j \mathscr{A}}\big)_{y\in\mathscr{Y}}$ of Borel measures on $\mathscr{Y}$, that we call the *U^+-subordinated Haar measure of* $\mathfrak{a}^j \mathscr{A}$, as follows:

- if $m_{U^+}(V_y^{\mathfrak{a}^j\mathscr{A}})$ is equal to 0 or ∞, we set $\tau_y^{\mathfrak{a}^j\mathscr{A}} = 0$,
- otherwise, $\tau_y^{\mathfrak{a}^j\mathscr{A}}$ is the push-forward of the normalized measure $\frac{1}{m_{U^+}(V_y^{\mathfrak{a}^j\mathscr{A}})}\, m_{U^+}|_{V_y^{\mathfrak{a}^j\mathscr{A}}}$

by the map $u \mapsto u\,y$.

Now let μ be a Borel $\mathfrak{a}$-invariant probability measure on $\mathscr{Y}$, such that $\mathscr{A}$ is U^+-subordinated modulo μ. In particular, for μ-almost every $y \in \mathscr{Y}$, the atom $V_y^{\mathfrak{a}^j\mathscr{A}}$ has positive and finite m_{U^+}-measure, hence the measure $\tau_y^{\mathfrak{a}^j\mathscr{A}}$ is a probability measure with support in $[y]_{\mathfrak{a}^j\mathscr{A}}$. Furthermore, if $z \in [y]_{\mathfrak{a}^j\mathscr{A}}$ then there exists $u \in U^+$ such that $z = u\,y$, $V_z^{\mathfrak{a}^j\mathscr{A}} = V_y^{\mathfrak{a}^j\mathscr{A}}\, u^{-1}$, and $\tau_z^{\mathfrak{a}^j\mathscr{A}} = \tau_y^{\mathfrak{a}^j\mathscr{A}}$, by the right-invariance of m_{U^+}.

The following proposition is a function field analog of the effective real case version [KKL, Prop. 2.10, §2.4] of [EL, §7.55].

Proposition 5.10 *Let μ be a Borel $\mathfrak{a}$-invariant ergodic probability measure on $\mathscr{Y}$ and let $\mathscr{A}$ be a countably generated sub-σ-algebra of the Borel σ-algebra of $\mathscr{Y}$ which is $\mathfrak{a}^{-1}$-descending and U^+-subordinated modulo μ. Fix $j \in \mathbb{Z}_{\geqslant 1}$ and a U^+-saturated Borel subset K' of $\mathscr{Y}$. Suppose that there exists $\epsilon > 0$ such that $[z]_{\mathscr{A}} \subset B_\epsilon^{U^+,\mathbf{r}} z$ for every $z \in K'$. Then we have*

$$H_\mu(\mathscr{A}|\mathfrak{a}^j\mathscr{A}) \leqslant j\,|\mathbf{r}| + \int_{\mathscr{Y}} \log \tau_y^{\mathfrak{a}^j\mathscr{A}}((\mathscr{Y} - K') \cup B_\epsilon^{U^+,\mathbf{r}} \operatorname{Supp} \mu)\, d\mu(y).$$

Proof. We fix μ, $\mathscr{A}$, j, K' and ϵ as in the statement. By for instance [EL, Theo. 5.9], let $\big(\mu_y^{\mathfrak{a}^j\mathscr{A}}\big)_{y\in\mathscr{Y}}$ be a measurable family of conditional measures of μ with respect to $\mathfrak{a}^j\mathscr{A}$, so that for μ-almost every $y \in \mathscr{Y}$, the measure $\mu_y^{\mathfrak{a}^j\mathscr{A}}$ is a probability measure on $\mathscr{Y}$ giving full measure to the atom $[y]_{\mathfrak{a}^j\mathscr{A}}$, with $\mu_z^{\mathfrak{a}^j\mathscr{A}} = \mu_y^{\mathfrak{a}^j\mathscr{A}}$ if $z \in [y]_{\mathfrak{a}^j\mathscr{A}}$, and such that the following *disintegration formula* holds true:

$$\mu = \int_{y\in\mathscr{Y}} \mu_y^{\mathfrak{a}^j\mathscr{A}}\; d\mu(y)\ . \tag{50}$$

Let $p_\mu : y \mapsto \mu_y^{\mathfrak{a}^j\mathscr{A}}([y]_{\mathscr{A}})$ and $p_\tau : y \mapsto \tau_y^{\mathfrak{a}^j\mathscr{A}}([y]_{\mathscr{A}})$, which are nonnegative and measurable functions on $\mathscr{Y}$. Since $\mathscr{A}$ is $\mathfrak{a}^{-1}$-descending and U^+-subordinated modulo μ, the atom $[y]_{\mathscr{A}}$ contains an open neighborhood of y in the atom $[y]_{\mathfrak{a}^j\mathscr{A}}$ for μ-almost every $y \in \mathscr{Y}$. In particular, the function p_τ is μ-almost everywhere positive.

Since $\mathscr{A}$ is countably generated and $\mathfrak{a}^{-1}$-descending, for every $y \in \mathscr{Y}$, the atom of y for $\mathfrak{a}^j\mathscr{A}$ is countably partitioned into atoms for $\mathscr{A}$ up to measure 0, that is, there exist a finite or countable subset I_y of $[y]_{\mathfrak{a}^j\mathscr{A}}$ and a $\mu_y^{\mathfrak{a}^j\mathscr{A}}$-measure zero subset N_y of $[y]_{\mathfrak{a}^j\mathscr{A}}$ such that

$$[y]_{\mathfrak{a}^j\mathscr{A}} = N_y \sqcup \bigsqcup_{x\in I_y} [x]_{\mathscr{A}}\ . \tag{51}$$

Let $I'_y = \{x \in I_x : [x]_{\mathscr{A}} \cap \operatorname{Supp} \mu \neq \varnothing\}$.

Lemma 5.11 *Let $x \in I_y$*

(1) If $x \notin I'_y$, then $\mu_y^{\mathfrak{a}^j\mathscr{A}}([x]_{\mathscr{A}}) = 0$.

(2) If $x \in I'_y$, then $[x]_{\mathscr{A}}$ is contained in $(\mathscr{Y} - K') \cup B_\epsilon^{U^+,\mathbf{r}} \operatorname{Supp} \mu$.

Proof. (1) This follows since $\operatorname{Supp}\mu_y^{\mathfrak{a}^j\mathscr{A}}$ is contained in $\operatorname{Supp}\mu$.

(2) If $x \in I'_y$, there exists $z \in [x]_{\mathscr{A}} \cap \operatorname{Supp}\mu$. For every $z' \in [x]_{\mathscr{A}}$, we have either $z' \in \mathscr{Y} - K'$ or $z' \in K'$. In the second case, since $\mathscr{A}$ is U^+-subordinated and K' is U^+-saturated, we have $z \in [x]_{\mathscr{A}} = [z']_{\mathscr{A}} \subset U^+z' \subset K'$. Hence by the assumption of Proposition 5.10, we have $z' \in [x]_{\mathscr{A}} = [z]_{\mathscr{A}} \subset B_\epsilon^{U^+,\mathbf{r}} z \subset B_\epsilon^{U^+,\mathbf{r}} \operatorname{Supp}\mu$, which proves the result. □

By the definition of the U^+-subordinated Haar measure of $\mathfrak{a}^j\mathscr{A}$, for μ-almost every $y \in \mathscr{Y}$, we have

$$p_\tau(y) = \frac{m_{U^+}(V_y^{\mathscr{A}})}{m_{U^+}(V_y^{\mathfrak{a}^j\mathscr{A}})} = \frac{m_{U^+}(V_y^{\mathscr{A}})}{m_{U^+}(\mathfrak{a}^j\, V_{\mathfrak{a}^{-j}y}^{\mathscr{A}}\, \mathfrak{a}^{-j})} = \frac{m_{U^+}(V_y^{\mathscr{A}})}{\operatorname{Jac}_j\ m_{U^+}(V_{\mathfrak{a}^{-j}y}^{\mathscr{A}})}\ .$$

Hence, by the $\mathfrak{a}$-invariance of μ and by Equation (47), we have

$$\int_{z\in\mathscr{Y}} \log_{q_v} p_\tau(z)\, d\mu(z) = -\log_{q_v} \operatorname{Jac}_j = -\,j\,|\mathbf{r}|\ .$$

Respectively

- by the definition of the conditional entropy in Equation (38),
- by the disintegration formula (50),
- since $\mu_y^{\mathfrak{a}^j\mathscr{A}}$ gives full measure to $[y]_{\mathfrak{a}^j\mathscr{A}}$ which is partitionned as in Equation (51), and by Lemma 5.11 (1),
- since when z varies in $[x]_{\mathscr{A}} \subset [y]_{\mathfrak{a}^j\mathscr{A}}$, the values $p_\mu(z) = \mu_z^{\mathfrak{a}^j\mathscr{A}}([z]_{\mathscr{A}}) = \mu_y^{\mathfrak{a}^j\mathscr{A}}([x]_{\mathscr{A}})$ and $p_\tau(z) = \tau_z^{\mathfrak{a}^j\mathscr{A}}([z]_{\mathscr{A}}) = \tau_y^{\mathfrak{a}^j\mathscr{A}}([x]_{\mathscr{A}})$ are constant,
- by the concavity property of the logarithm,
- by Lemma 5.11 (2),

we hence have

$$\begin{aligned}
&H_\mu(\mathscr{A}\,|\,\mathfrak{a}^j\mathscr{A}) \ -j\ |\mathbf{r}|\\
&= -\int_{z\in\mathscr{Y}} \big(\log_{q_v} p_\mu(z) - \log_{q_v} p_\tau(z)\big)\, d\mu(z)\\
&= \int_{y\in\mathscr{Y}} \int_{z\in\mathscr{Y}} \big(\log_{q_v} p_\tau(z) - \log_{q_v} p_\mu(z)\big)\, d\mu_y^{\mathfrak{a}^j\mathscr{A}}(z)\, d\mu(y)\\
&= \int_{y\in\mathscr{Y}} \sum_{x\in I'_y} \int_{z\in[x]_{\mathscr{A}}} \big(\log_{q_v} p_\tau(z) - \log_{q_v} p_\mu(z)\big)\, d\mu_y^{\mathfrak{a}^j\mathscr{A}}(z)\, d\mu(y)\\
&= \int_{y\in\mathscr{Y}} \sum_{x\in I'_y} \log_{q_v} \frac{\tau_y^{\mathfrak{a}^j\mathscr{A}}([x]_{\mathscr{A}})}{\mu_y^{\mathfrak{a}^j\mathscr{A}}([x]_{\mathscr{A}})}\ \mu_y^{\mathfrak{a}^j\mathscr{A}}([x]_{\mathscr{A}})\, d\mu(y)\\
&\leqslant \int_{y\in\mathscr{Y}} \log_{q_v} \Big(\sum_{x\in I'_y} \tau_y^{\mathfrak{a}^j\mathscr{A}}([x]_{\mathscr{A}})\Big)\, d\mu(y)\\
&\leqslant \int_{y\in\mathscr{Y}} \log_{q_v} \big(\tau_y^{\mathfrak{a}^j\mathscr{A}}((\mathscr{Y} - K') \cup B_\epsilon^{U^+,\mathbf{r}} \operatorname{Supp}\mu)\big)\, d\mu(y)\ .
\end{aligned}$$

This proves the result. □

6 Upper bound on the Hausdorff dimension of $\mathbf{Bad}_A(\epsilon)$

6.1 Constructing measures with large entropy

In this subsection, we construct, as in [KKL, Prop. 4.1] in the real case, an $\mathfrak{a}$-invariant probability measure on $\mathscr{Y}$ giving an appropriate lower bound on the conditional entropy of $\mathfrak{a}$ relative to the σ-algebra $\mathscr{A}^{U^+}$ constructed in Proposition 5.7.

For any point x in a measurable space, we denote by Δ_x the unit Dirac measure at x. We denote by $\overset{*}{\rightharpoonup}$ the weak-star convergence of Borel measures on any locally compact space.

Let us denote by $\overline{\mathscr{X}} = \mathscr{X} \cup \{\infty_{\mathscr{X}}\}$ and $\overline{\mathscr{Y}} = \mathscr{Y} \cup \{\infty_{\mathscr{Y}}\}$ the one-point compactifications of $\mathscr{X}$ and $\mathscr{Y}$, respectively. We denote by $\overline{\pi} : \overline{\mathscr{Y}} \to \overline{\mathscr{X}}$ the unique continuous extension of the natural projection $\pi : \mathscr{Y} \to \mathscr{X}$, mapping $\infty_{\mathscr{Y}}$ to $\infty_{\mathscr{X}}$. The left actions of $\mathfrak{a}$ on $\mathscr{X}$ and $\mathscr{Y}$ continuously extend to actions on $\overline{\mathscr{X}}$ and $\overline{\mathscr{Y}}$ fixing the points at infinity $\infty_{\mathscr{X}}$ and $\infty_{\mathscr{Y}}$. For every countably generated σ-algebra $\mathscr{A}$ of subsets of $\mathscr{X}$ or $\mathscr{Y}$, we denote by $\overline{\mathscr{A}}$ the countably generated σ-algebra of subsets of $\overline{\mathscr{X}}$ or $\overline{\mathscr{Y}}$ generated by $\mathscr{A}$ and its point at infinity. For a finite partition $\mathscr{Q} = \{Q_1, \dots, Q_N, Q_\infty\}$ of $\mathscr{Y}$ with only one unbounded atom Q_∞, we denote by $\overline{\mathscr{Q}}$ the finite partition $\{Q_1, \dots, Q_N, \overline{Q}_\infty = Q_\infty \cup \{\infty_{\mathscr{Y}}\}\}$ of $\overline{\mathscr{Y}}$. Note that $\overline{\bigvee_{i=a}^{b} \mathfrak{a}^{-i}\mathscr{Q}} = \bigvee_{i=a}^{b} \mathfrak{a}^{-i}\,\overline{\mathscr{Q}}$ for all a, b in $\mathbb{Z}$ with $a < b$.

For every $\eta \in [0, 1]$, we say that an element $x \in \mathscr{X}$ has *η-escape of mass on average* under the action of $\mathfrak{a}$ if for every compact subset Q of $\mathscr{X}$,

$$\liminf_{N\to\infty} \frac{1}{N} \operatorname{Card}\big\{\ell \in \{1, \cdots, N\} : \mathfrak{a}^{\ell} x \notin Q\big\} \geqslant \eta\ .$$

When $\eta = 1$, as defined in the Introduction and in Proposition 5.2, we say that x diverges on average in $\mathscr{X}$ under the action of $\mathfrak{a}$. For every $A \in \mathscr{M}_{m,n}(K_v)$, we denote by $x_A = u_A R_v^m \in \mathscr{X}$ its associated unimodular lattice (see Section 5.2), and by $\eta_A \in [0, 1]$ the upper bound of the elements $\eta \in [0, 1]$ such that x_A has η-escape of mass on average. Note that this upper bound is actually a maximum.

Proposition 6.1 *For every $A \in \mathscr{M}_{m,n}(K_v)$, there exists a Borel probability measure μ_A on $\overline{\mathscr{X}}$ with $\mu_A(\mathscr{X}) = 1 - \eta_A$ such that for every $\epsilon > 0$, there exists an $\mathfrak{a}$-invariant Borel probability measure $\overline{\mu}$ on $\overline{\mathscr{Y}}$ satisfying the following properties.*

(1) *The support of $\overline{\mu}$ is contained in $\mathscr{L}_\epsilon \cup \{\infty_{\mathscr{Y}}\}$, where $\mathscr{L}_\epsilon$ is defined in Equation* (35).

(2) *We have $\overline{\pi}_* \overline{\mu} = \mu_A$. In particular, there exists an $\mathfrak{a}$-invariant Borel probability measure μ on $\mathscr{Y}$ such that*

$$\overline{\mu} = (1 - \eta_A)\mu + \eta_A \Delta_{\infty_{\mathscr{Y}}}\,.$$

(3) *For every $r \in \,]0, 1[\,$, let $\mathscr{A}^{U^+}$ be the σ-algebra of subsets of $\mathscr{Y}$ constructed in the proof of Proposition 5.7. Then*

$$h_{\overline{\mu}}\big(\mathfrak{a}^{-1} | \,\overline{\mathscr{A}^{U^+}}\,\big) \geqslant |\mathbf{r}|(1 - \eta_A) - \max \mathbf{r}\ (m - \dim_{\mathrm{Haus}} \mathbf{Bad}_A(\epsilon))\ .$$

Proof. Since x_A has η_A-escape of mass on average but does not have $(\eta_A + \delta)$-escape of mass on average for any $\delta > 0$, there exists an increasing sequence of positive integers $(k_i)_{i\in\mathbb{Z}_{\geqslant 1}}$ such that, for the weak-star convergence of Borel probability measures on the compact space $\overline{\mathscr{X}}$, as $i \to +\infty$, we have

$$\frac{1}{k_i} \sum_{k=0}^{k_i - 1} \Delta_{\,\mathfrak{a}^k x_A} \ \overset{*}{\rightharpoonup}\ \mu_A\ , \tag{52}$$

and μ_A is a Borel probability measure on $\overline{\mathscr{X}}$ with $\mu_A(\mathscr{X}) = 1 - \eta_A$. This is equivalent to $\mu_A(\{\infty_{\mathscr{X}}\}) = \eta_A$.

Let $\epsilon > 0$. For every $T \in \mathbb{Z}_{\geqslant 0}$, with the notation of Subsection 5.2 (see in particular Equations (35) and (36)), let

$$R_T = \{\boldsymbol{\theta} \in \mathbb{T}^m : \forall k \geqslant T, \mathfrak{a}^k \phi_A(\boldsymbol{\theta}) \in \mathscr{L}_\epsilon\} \cap \mathbf{Bad}_A(\epsilon) \ .$$

By Proposition 5.4, since a countable subset of K_ν^m has Hausdorff dimension 0, we have $\dim_{\text{Haus}} \big(\bigcup_{T=1}^{\infty} R_T\big) = \dim_{\text{Haus}} \mathbf{Bad}_A(\epsilon)$. Thus, for every $j \in \mathbb{Z}_{\geqslant 1}$, there exists $T_j \in \mathbb{Z}_{\geqslant 0}$ satisfying

$$\dim_{\text{Haus}} R_{T_j} \geqslant \dim_{\text{Haus}} \mathbf{Bad}_A(\epsilon) - \frac{1}{j} \ .$$

For all $i, j \in \mathbb{Z}_{\geqslant 1}$ such that $k_i \geqslant T_j$, let $S_{i,j}$ be a maximal $q_v^{-k_i}$-separated subset of R_{T_j} for the distance $d_{\mathbb{T}^m,\mathbf{r}}$ defined after Equation (45). Then R_{T_j} can be covered by Card $S_{i,j}$ open balls of radius $q_v^{-k_i}$ for $d_{\mathbb{T}^m,\mathbf{r}}$. Each open ball of radius $q_v^{-k_i}$ for $d_{\mathbb{T}^m,\mathbf{r}}$ can be covered by $\prod_{j=1}^m q_v^{-k_i r_j}/q_v^{-k_i \max \mathbf{r}} = q_v^{k_i(m \max \mathbf{r} - |\mathbf{r}|)}$ open balls of radius $q_v^{-k_i \max \mathbf{r}}$ with respect to the standard distance $d_{\mathbb{T}^m}$ (defining the Hausdorff dimension of subsets of $\mathbb{T}^m$). Since the lower Minskowski dimension is at least equal to the Hausdorff dimension, we have

$$\liminf_{i \to \infty} \frac{\log_{q_v} \big(q_v^{k_i(m \max \mathbf{r} - |\mathbf{r}|)} \text{Card } S_{i,j}\big)}{- \log_{q_v} \big(q_v^{-k_i \max \mathbf{r}}\big)} \geqslant \dim_{\text{Haus}} R_{T_j} \geqslant \dim_{\text{Haus}} \mathbf{Bad}_A(\epsilon) - \frac{1}{j} \ ,$$

which implies that

$$\liminf_{i \to \infty} \frac{\log_{q_v} \text{Card } S_{i,j}}{k_i} \geqslant |\mathbf{r}| - \max \mathbf{r} \ \Big(m + \frac{1}{j} - \dim_{\text{Haus}} \mathbf{Bad}_A(\epsilon)\Big) \ . \tag{53}$$

Let us define the Borel probability measures

$$\nu_{i,j} = \frac{1}{\text{Card } S_{i,j}} \sum_{\boldsymbol{\theta} \in S_{i,j}} \Delta_{\phi_A(\boldsymbol{\theta})} \ ,$$

which is the normalized counting measure on the finite subset $\phi_A(S_{i,j})$ of the U^+-orbit $\phi_A(\mathbb{T}^m) = U^+ y_{A,0} \subset \pi^{-1}(x_A)$, and

$$\widetilde{\nu}_{i,j} = \frac{1}{k_i} \sum_{0 \leqslant k \leqslant k_i - 1} \mathfrak{a}_*^k \nu_{i,j} \ ,$$

which is the average of the previous one on the first k_i points of the $\mathfrak{a}$-orbit. Since $\overline{\mathscr{Y}}$ is compact, extracting diagonally a subsequence if necessary, we may assume that $\widetilde{\nu}_{i,j}$ weak-star converges as $i \to +\infty$ towards an $\mathfrak{a}$-invariant Borel probability measure $\widetilde{\mu}_j$, and that $\widetilde{\mu}_j$ weak-star converges as $j \to +\infty$ towards an $\mathfrak{a}$-invariant Borel probability measure $\overline{\mu}$. Let us prove that $\overline{\mu}$ satisfies the three assertions of Proposition 6.1.

(1) For all $k \geqslant T_j$ and $\boldsymbol{\theta} \in S_{i,j} \subset R_{T_j}$, we have $\mathfrak{a}^k \phi_A(\boldsymbol{\theta}) \in \mathscr{L}_\epsilon$ by the definition of R_{T_j}. Since $\mathfrak{a}_*^k \nu_{i,j}$ is a probability measure, we hence have

$$\widetilde{\nu}_{i,j}(\mathscr{Y} - \mathscr{L}_\epsilon) = \frac{1}{k_i} \sum_{k=0}^{k_i - 1} \mathfrak{a}_*^k \nu_{i,j}(\mathscr{Y} - \mathscr{L}_\epsilon) = \frac{1}{k_i} \sum_{k=0}^{T_j} \mathfrak{a}_*^k \nu_{i,j}(\mathscr{Y} - \mathscr{L}_\epsilon) \leqslant \frac{T_j}{k_i} \ .$$

Since $\mathscr{L}_\epsilon \cup \{\infty_{\mathscr{Y}}\}$ is closed in $\overline{\mathscr{Y}}$ and by taking limits first as $i \to +\infty$ then as $j \to +\infty$, we therefore have $\overline{\mu}(\mathscr{Y} - \mathscr{L}_\epsilon) = 0$. This proves Assertion (1).

(2) Since $\phi_A(S_{i,j})$ is contained in the fiber above x_A of $\overline{\pi}$ and since $\nu_{i,j}$ is a probability measure, we have $\overline{\pi}_*\nu_{i,j} = \Delta_{x_A}$. By the linearity and equivariance of $\overline{\pi}_*$, we hence have

$$\overline{\pi}_*\widetilde{\nu}_{i,j} = \frac{1}{k_i} \sum_{0 \leqslant k \leqslant k_i - 1} \mathfrak{a}^k_* \, \overline{\pi}_* \, \nu_{i,j} = \frac{1}{k_i} \sum_{0 \leqslant k \leqslant k_i - 1} \Delta_{\mathfrak{a}^k x_A} \, .$$

By the weak-star continuity of $\overline{\pi}_*$ and Equation (52), we thus have

$$\overline{\pi}_*\overline{\mu} = \lim_{j \to +\infty} \lim_{i \to +\infty} \overline{\pi}_*\widetilde{\nu}_{i,j} = \lim_{j \to +\infty} \mu_A = \mu_A \, .$$

Note that the point at infinity $\infty_{\mathscr{Y}}$ is an isolated point in the support of $\overline{\mu}$ by Assertion (1), since $\mathscr{L}_\epsilon$ is compact. We hence have

$$\overline{\mu}(\{\infty_{\mathscr{Y}}\}) = \overline{\mu}(\overline{\pi}^{-1}(\{\infty_{\mathscr{X}}\})) = \mu_A(\{\infty_{\mathscr{X}}\}) = \eta_A \, . \tag{54}$$

(3) Suppose that $\mathscr{Q}$ is any finite Borel-measurable partition of $\mathscr{Y}$ satisfying

(i) the partition $\mathscr{Q}$ contains an atom Q_∞ of the form $\pi^{-1}(Q^*_\infty)$, where $\mathscr{X} - Q^*_\infty$ has compact closure,

(ii) there exists $\ell_0 \in \mathbb{Z}_{\geqslant 1}$ such that for every atom $Q \in \mathscr{Q}$ different from Q_∞, we have $\operatorname{diam} Q < q_v^{-\ell_0 \max \mathbf{r}}$ for the distance $d_{\mathscr{Y}}$.

(iii) for all $Q \in \mathscr{Q}$ and $j \in \mathbb{Z}_{\geqslant 1}$, we have $\widetilde{\mu}_j(\partial Q) = 0$ and $\overline{\mu}(\partial Q) = 0$.

We first prove the following entropy bound: For every $M \in \mathbb{Z}_{\geqslant 1}$,

$$\frac{1}{M} H_{\overline{\mu}}\big(\,\overline{\sigma(\mathscr{Q})}\,{}_0^{M-1} | \,\overline{\mathscr{A}^{U^+}}\,\big) \geqslant |\mathbf{r}|(1 - \overline{\mu}(\,\overline{Q}_\infty)) - \max \mathbf{r}\; (m - \dim_{\mathrm{Haus}} \mathbf{Bad}_A(\epsilon)) \, . \tag{55}$$

Since Equation (55) is clear if $\overline{\mu}(\,\overline{Q}_\infty) = 1$, we may assume that $\overline{\mu}(\,\overline{Q}_\infty) < 1$, hence that $\widetilde{\mu}_j(\,\overline{Q}_\infty) < 1$ for all large enough $j \geqslant 1$. Now, we fix such a $j \geqslant 1$.

Take $\rho > 0$ small enough so that $\widetilde{\mu}_j(\,\overline{Q}_\infty) + \rho < 1$ and let

$$\beta = \widetilde{\mu}_j(\,\overline{Q}_\infty) + \rho \, . \tag{56}$$

Then for all large enough $i \in \mathbb{Z}_{\geqslant 1}$, since $\phi_A(S_{i,j}) \subset \pi^{-1}(x_A)$ and $Q_\infty = \pi^{-1}(Q^*_\infty)$ by Property (i) of $\mathscr{Q}$, we have

$$\begin{aligned}\beta = \widetilde{\mu}_j(\,\overline{Q}_\infty) + \rho > \widetilde{\nu}_{i,j}(Q_\infty) &= \frac{1}{k_i \mathrm{Card}\; S_{i,j}} \sum_{k=0}^{k_i-1} \sum_{\boldsymbol{\theta} \in S_{i,j}} \Delta_{\mathfrak{a}^k \phi_A(\boldsymbol{\theta})}(Q_\infty) \\ &= \frac{1}{k_i} \sum_{k=0}^{k_i-1} \Delta_{\mathfrak{a}^k x_A}(Q^*_\infty) \, .\end{aligned}$$

Thus, for every $\boldsymbol{\theta} \in \mathbb{T}^m$, since $\mathfrak{a}^k \phi_A(\boldsymbol{\theta}) \in Q_\infty$ implies that $\mathfrak{a}^k x_A \in Q^*_\infty$ by Property (i) of $\mathscr{Q}$, we have

$$\mathrm{Card}\{k \in \{0, \ldots, k_i - 1\} : \mathfrak{a}^k \phi_A(\boldsymbol{\theta}) \in Q_\infty\} < \beta\, k_i \, . \tag{57}$$

Let us prove the following counting lemma inspired by [ELMW, Lem. 4.5] and [LSS, Lem. 2.4], where ℓ_0 is given by Property (ii) of $\mathscr{Q}$.

Lemma 6.2 *There exists a constant $C > 0$ depending only on $\mathbf{r}$ and ℓ_0 such that for all $A \in \mathscr{M}_{m,n}(K_v)$, $\boldsymbol{\theta} \in \mathbb{T}^m$ and $T \in \mathbb{Z}_{\geqslant 0}$, defining $y = \phi_A(\boldsymbol{\theta})$, $I = \{k \in \mathbb{Z}_{\geqslant 0} : \mathfrak{a}^k y \in Q_\infty\}$, and*

$$E_{y,T} = \{z \in U^+ y : \forall\, k \in \{0, \dots, T\} - I,\ \ d_{\mathscr{Y}}(\mathfrak{a}^k y, \mathfrak{a}^k z) < q_v^{-\ell_0 \max \mathbf{r}}\ \}\ ,$$

the set $E_{y,T}$ can be covered by $C\, q_v^{|\mathbf{r}|\mathrm{Card}(I \cap \{0,\dots,T\})}$ closed balls of radius $q_v^{-(\ell_0+T)}$ for the distance $d_{U^+y,\,\mathbf{r}}$.

Proof. As in the proof of [LSS, Lemma 2.4], we proceed by induction on T.

By the compactness of $\mathbb{T}^m$, there exists a constant $C \in \mathbb{Z}_{\geqslant 1}$ depending only on $\mathbf{r}$ and ℓ_0 such that the metric space $(\mathbb{T}^m, d_{\mathbb{T}^m,\mathbf{r}})$ can be covered by C closed balls of radius $q_v^{-\ell_0}$. Since $\phi_A : \mathbb{T}^m \to U^+ y$ is an isometry for the distances $d_{\mathbb{T}^m,\mathbf{r}}$ and $d_{U^+y,\,\mathbf{r}}$, the orbit U^+y can be covered by C closed balls for $d_{U^+y,\,\mathbf{r}}$ of radius $q_v^{-\ell_0}$. Thus the lemma holds for $T = 0$. Let $N_T = C\, q_v^{|\mathbf{r}|\mathrm{Card}(I \cap \{0,\dots,T\})}$.

Assume by induction that $E_{y,T-1}$ can be covered by N_{T-1} balls for $d_{U^+y,\,\mathbf{r}}$ of radius $q_v^{-(\ell_0+T-1)}$. Note that for every $k \in \mathbb{Z}$, since $\pi_v^k \mathscr{O}_v/(\pi_v^{k+1}\mathscr{O}_v)$ has order q_v, every closed ball in K_v of radius q_v^{-k} is the disjoint union of q_v closed ball of radius q_v^{-k-1}. Hence every closed ball for $d_{U^+y,\,\mathbf{r}}$ of radius $q_v^{-(\ell_0+T-1)}$ in U^+y can be covered by $q_v^{|\mathbf{r}|}$ closed balls for $d_{U^+y,\,\mathbf{r}}$ of radius $q_v^{-(\ell_0+T)}$. Therefore, if $T \in I$, then $E_{y,T} = E_{y,T-1}$ can be covered by $N_T = q_v^{|\mathbf{r}|} N_{T-1}$ closed balls for $d_{U^+y,\,\mathbf{r}}$ of radius $q_v^{-(\ell_0+T)}$.

Suppose conversely that $T \notin I$, so that in particular $N_T = N_{T-1}$. Denote the above covering of $E_{y,T-1}$ by $\{B_i : i = 1, \dots, N_{T-1}\}$. Since we have $E_{y,T} \subset E_{y,T-1}$, the set $\{E_{y,T} \cap B_i : i = 1, \dots, N_{T-1}\}$ is a covering of $E_{y,T}$.

Claim. For all $i = 1, \dots, N_{T-1}$ and $z_1, z_2 \in E_{y,T} \cap B_i$, we have $d_{U^+y,\mathbf{r}}(z_1, z_2) \leqslant q_v^{-(\ell_0+T)}$.

Proof. Since $T \notin I$, we have $d_{\mathscr{Y}}(\mathfrak{a}^T y, \mathfrak{a}^T z_j) < q_v^{-\ell_0 \max \mathbf{r}}$ for each $j = 1, 2$. Thus we have $d_{\mathscr{Y}}(\mathfrak{a}^T z_1, \mathfrak{a}^T z_2) < q_v^{-\ell_0 \max \mathbf{r}}$ by the ultrametric inequality property of $\|\cdot\|$. Note that since $z_1, z_2 \in U^+y = U^+y_{A,\boldsymbol{\theta}}$, there exist $\boldsymbol{\theta}_1 = (\theta_{1,1}, \dots, \theta_{1,m})$ and $\boldsymbol{\theta}_2 = (\theta_{2,1}, \dots, \theta_{2,m})$ in $\mathbb{T}^m$ such that (denoting in the same way lifts of $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ to K_v^m) we have $z_1 = y_{A,\boldsymbol{\theta}_1}$ and $z_2 = y_{A,\boldsymbol{\theta}_2}$. With $|\langle\ \rangle|$ the map defined after Equation (3), it follows that we have

$$\begin{aligned}\max_{1 \leqslant i \leqslant m} q_v^{r_i T}\, |\langle\, \theta_{1,i} - \theta_{2,i}\, \rangle| = d_{\mathbb{T}^m}(\mathfrak{a}_-^T \boldsymbol{\theta}_1, \mathfrak{a}_-^T \boldsymbol{\theta}_2) &= d_{\mathscr{Y}}(\mathfrak{a}^T y_{A,\boldsymbol{\theta}_1}, \mathfrak{a}^T y_{A,\boldsymbol{\theta}_2}) \\ &= d_{\mathscr{Y}}(\mathfrak{a}^T z_1, \mathfrak{a}^T z_2) < q_v^{-\ell_0 \max \mathbf{r}}\ .\end{aligned}$$

Hence, we have

$$d_{U^+y,\,\mathbf{r}}(z_1, z_2) = d_{\mathbb{T}^m,\mathbf{r}}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) = \max_{1 \leqslant i \leqslant m} |\langle\, \theta_{1,i} - \theta_{2,i}\, \rangle|^{\frac{1}{r_i}} < q_v^{-(\ell_0+T)}\ ,$$

which concludes the claim. □

By the above claim, the intersection $E_{y,T} \cap B_i$ is contained in a single ball for $d_{U^+y,\,\mathbf{r}}$ of radius $q_v^{-(\ell_0+T)}$ for each $i = 1, \dots, N_{T-1}$. Thus $E_{y,T}$ can be covered by $N_T = N_{T-1}$ balls for $d_{U^+y,\,\mathbf{r}}$ of radius $q_v^{-(\ell_0+T)}$. □

Recall that as constructed in the proof of Proposition 5.7, there exist a Borel-measurable partition $\mathscr{P} = \{P_1, \dots, P_N, P_\infty\}$ of $\mathscr{Y}$ with $N + 1$ elements, and a countably generated

Borel-measurable σ-algebra $\mathscr{P}^{U^+}$ of subsets of $\mathscr{Y}$, with $[y]_{\mathscr{P}^{U^+}} = [y]_{\mathscr{P}} \cap B_r^{U^+} y$ for every $y \in \mathscr{Y}(r)$ by Equation (42), such that we have $\mathscr{A}^{U^+} = (\mathscr{P}^{U^+})_0^{\infty}$.

For all $\ell \in \mathbb{Z}_{\geqslant 1}$ and $y \in \mathscr{Y}(r)$, we have $[y]_{(\mathscr{P}^{U^+})_0^\ell} \subset [y]_{\sigma(\mathscr{P})_0^\ell} \cap B_r^{U^+} y$. Since the support of $\nu_{i,j}$ is a finite set of points on a single U^+-orbit $\phi_A(\mathbb{T}^m) = U^+ y_{A,0}$, the measure $\mathfrak{a}_*^{-k}\nu_{i,j}$ is also supported on a single U^+-orbit $\mathfrak{a}^k U^+ y_{A,0} = U^+ \mathfrak{a}^{-k} y_{A,0}$ for every $k \in \mathbb{Z}_{\geqslant 0}$. Recall (see for instance [EL, Def. 5.7]) that two sub-σ-algebras $\mathscr{A}$ and $\mathscr{A}'$ of a measured space $(X, \mathscr{B})$ are *equivalent modulo* a probability measure ν if for every $A \in \mathscr{A}$ there exists $A' \in \mathscr{A}'$ such that $\nu(A \Delta A') = 0$ (where Δ is the symmetric difference) and vice versa.

Claim. For every $k \in \mathbb{Z}_{\geqslant 0}$, the sub-$\sigma$-algebras $\mathscr{P}^{U^+}$ and $\sigma(\mathscr{P})$ are equivalent modulo $\mathfrak{a}_*^{-k}\nu_{i,j}$.

Proof. By the construction of $\mathscr{P}^{U^+}$ above Equation (42), we have $[y]_{\mathscr{P}^{U^+}} \subset [y]_{\mathscr{P}}$ for every $y \in \mathscr{Y}$ and for every $y \notin P_\infty$, we have

$$U^+ y \cap [y]_{\mathscr{P}} = [y]_{\mathscr{P}^{U^+}} . \tag{58}$$

We first consider any point $y \in \mathscr{Y}$ such that $[y]_{\mathscr{P}^{U^+}}$ intersects $\operatorname{Supp} \mathfrak{a}_*^{-k}\nu_{i,j}$. If $y \notin P_\infty$, by Equation (58), the symmetric difference $[y]_{\mathscr{P}^{U^+}} \Delta [y]_{\mathscr{P}} = [y]_{\mathscr{P}} - [y]_{\mathscr{P}^{U^+}}$ does not intersect $U^+ y$. But $\operatorname{Supp} \mathfrak{a}_*^{-k}\nu_{i,j} \subset U^+ y$ by the single orbit support property, hence we have $\mathfrak{a}_*^{-k}\nu_{i,j}([y]_{\mathscr{P}^{U^+}} \Delta [y]_{\mathscr{P}}) = 0$. If $y \in P_\infty$, since $[y]_{\mathscr{P}^{U^+}} = Uy$ by Equation (42) and $\operatorname{Supp} \mathfrak{a}_*^{-k}\nu_{i,j} \subset Uy$, we have $\mathfrak{a}_*^{-k}\nu_{i,j}([y]_{\mathscr{P}^{U^+}} \Delta [y]_{\mathscr{P}}) = \mathfrak{a}_*^{-k}\nu_{i,j}(P_\infty - Uy) = 0$.

For every point $y \in \mathscr{Y}$ such that $[y]_{\mathscr{P}^{U^+}}$ does not intersect $\operatorname{Supp} \mathfrak{a}_*^{-k}\nu_{i,j}$, we can take $\varnothing \in \sigma(\mathscr{P})$ so that $\mathfrak{a}_*^{-k}\nu_{i,j}([y]_{\mathscr{P}^{U^+}} \Delta \varnothing) = \mathfrak{a}_*^{-k}\nu_{i,j}([y]_{\mathscr{P}^{U^+}}) = 0$.

Conversely, consider any point $y \in \mathscr{Y}$ such that $[y]_{\mathscr{P}}$ intersects $\operatorname{Supp} \mathfrak{a}_*^{-k}\nu_{i,j}$. Let us fix $y' \in [y]_{\mathscr{P}} \cap \operatorname{Supp} \mathfrak{a}_*^{-k}\nu_{i,j}$, so that in particular $\operatorname{Supp} \mathfrak{a}_*^{-k}\nu_{i,j} \subset U^+ y' \subset Uy'$. If $y \in P_\infty$, then $y' \in P_\infty = [y]_{\mathscr{P}}$ and $[y']_{\mathscr{P}^{U^+}} = Uy'$ by Equation (42), hence $\mathfrak{a}_*^{-k}\nu_{i,j}([y]_{\mathscr{P}} \Delta [y']_{\mathscr{P}^{U^+}}) = \mathfrak{a}_*^{-k}\nu_{i,j}(P_\infty - Uy') = 0$. If $y \notin P_\infty$, then $y' \notin P_\infty$ and by Equation (58), the difference $U^+ y' - [y']_{\mathscr{P}^{U^+}}$ does not intersect $[y']_{\mathscr{P}} = [y]_{\mathscr{P}}$, hence $\mathfrak{a}_*^{-k}\nu_{i,j}([y]_{\mathscr{P}} \Delta [y']_{\mathscr{P}^{U^+}}) = 0$.

For every point $y \in \mathscr{Y}$ such that $[y]_{\mathscr{P}}$ does not intersect $\operatorname{Supp} \mathfrak{a}_*^{-k}\nu_{i,j}$, we take $\varnothing \in \mathscr{P}^{U^+}$ so that $\mathfrak{a}_*^{-k}\nu_{i,j}([y]_{\mathscr{P}} \Delta \varnothing) = \mathfrak{a}_*^{-k}\nu_{i,j}([y]_{\mathscr{P}}) = 0$. This proves the claim. □

Using the above claim, we have that $(\mathscr{P}^{U^+})_0^\ell$ is equivalent to $\sigma(\mathscr{P})_0^\ell$ modulo $\nu_{i,j}$. Hence, it follows from Equation (37) that

$$H_{\nu_{i,j}}\big((\mathscr{P}^{U^+})_0^\ell\big) \leqslant H_{\nu_{i,j}}\Big(\bigvee_{k=0}^{\ell} \mathfrak{a}^k \mathscr{P}\Big) \leqslant (\ell+1)\log_{q_v}(N+1) . \tag{59}$$

If Q is any atom of the finite partition $\mathscr{Q}_{i,\ell} = \bigvee_{k=0}^{k_i-1} \mathfrak{a}^k \mathscr{Q} \vee \bigvee_{k=0}^{\ell} \mathfrak{a}^k \mathscr{P}$ of $\mathscr{Y}$, then fixing any $y \in Q$, by Property (ii) of $\mathscr{Q}$, the intersection $\phi_A(S_{i,j}) \cap Q$ is contained in E_{y,k_i-1} with the notation of Lemma 6.2. It follows from Lemma 6.2 and Equation (57) that $\phi_A(S_{i,j}) \cap Q$ can be covered by $C\, q_v^{|\mathbf{r}|\beta k_i}$ closed balls for $d_{U^+ y_{A,0},\mathbf{r}}$ of radius $q_v^{-(\ell_0+k_i-1)} = q_v^{-\ell_0+1} q_v^{-k_i}$, where C depends only on $\mathbf{r}$ and ℓ_0. Since $S_{i,j}$ is $q_v^{-k_i}$-separated (hence $q_v^{-\ell_0+1}q_v^{-k_i}$-separated since $\ell_0 \geqslant 1$) with respect to $d_{\mathbb{T}^m,\mathbf{r}}$, and since $\phi_A : (\mathbb{T}^n, d_{\mathbb{T}^m,\mathbf{r}}) \to (U^+ y_{A,0}, d_{U^+ y_{A,0},\mathbf{r}})$ is an isometry, we have

$$\operatorname{Card}(\phi_A(S_{i,j}) \cap Q) \leqslant C\, q_v^{|\mathbf{r}|\beta k_i} .$$

Since we have $(\mathscr{P}^{U^+})_0^\ell = \sigma(\mathscr{P})_0^\ell$ modulo $\nu_{i,j}$, since $\nu_{i,j}$ is the normalised counting measure on $\phi_A(S_{i,j})$, and since the map $\Psi = -\log_{q_v}$ is nonincreasing, it hence follows that

$$H_{\nu_{i,j}}(\sigma(\mathscr{Q})_0^{k_i-1} \vee (\mathscr{P}^{U^+})_0^\ell) = H_{\nu_{i,j}}(\mathscr{Q}_{i,\ell}) = \sum_{Q \in \mathscr{Q}_{i,\ell}} \nu_{i,j}(Q)\Psi\big(\nu_{i,j}(Q)\big)$$
$$= \sum_{Q \in \mathscr{Q}_{i,\ell}} \nu_{i,j}(Q)\Psi\Big(\frac{\mathrm{Card}(\phi_A(S_{i,j}) \cap Q)}{\mathrm{Card}\; S_{i,j}}\Big) \geqslant \Psi\Big(\frac{C\, q_v^{|\mathbf{r}|\beta k_i}}{\mathrm{Card}\; S_{i,j}}\Big) \sum_{Q \in \mathscr{Q}_{i,\ell}} \nu_{i,j}(Q)$$
$$= \log_{q_v}(\mathrm{Card}\; S_{i,j}) - |\mathbf{r}|\,\beta\, k_i - \log_{q_v} C\,. \tag{60}$$

Combining Equations (59) and (60), we have

$$H_{\nu_{i,j}}(\sigma(\mathscr{Q})_0^{k_i-1} | (\mathscr{P}^{U^+})_0^\ell) = H_{\nu_{i,j}}(\sigma(\mathscr{Q})_0^{k_i-1} \vee (\mathscr{P}^{U^+})_0^\ell) - H_{\nu_{i,j}}((\mathscr{P}^{U^+})_0^\ell)$$
$$\geqslant \log_{q_v}(\mathrm{Card}\; S_{i,j}) - |\mathbf{r}|\beta k_i - \log_{q_v} C - (\ell+1)\log_{q_v}(N+1). \tag{61}$$

By the subadditivity and concavity properties of the entropy as in the proof of [LSS, Eq. (2.9)], for every $M \in \mathbb{Z}_{\geqslant 1}$, we have

$$\frac{1}{M} H_{\widetilde{\nu}_{i,j}}(\sigma(\mathscr{Q})_0^{M-1} | (\mathscr{P}^{U^+})_0^\ell) \geqslant \frac{1}{k_i} H_{\nu_{i,j}}(\sigma(\mathscr{Q})_0^{k_i-1} | (\mathscr{P}^{U^+})_0^\ell) - \frac{2\,M\,\log_{q_v}(\mathrm{Card}\;\mathscr{Q})}{k_i}. \tag{62}$$

Therefore, since $\nu_{i,j}(\infty_{\mathscr{Y}}) = 0$, it follows from Equations (62) and (61) that

$$\frac{1}{M} H_{\widetilde{\nu}_{i,j}}\big(\,\overline{\sigma(\mathscr{Q})}\,_0^{M-1} | (\,\overline{\mathscr{P}^{U^+}}\,)_0^\ell\big) = \frac{1}{M} H_{\widetilde{\nu}_{i,j}}\big(\sigma(\mathscr{Q})_0^{M-1} | (\mathscr{P}^{U^+})_0^\ell\big)$$
$$\geqslant \frac{1}{k_i}\big(\log_{q_v}(\mathrm{Card}\; S_{i,j}) - |\mathbf{r}|\beta k_i - \log_{q_v} C - (\ell+1)\log_{q_v}(N+1) - 2\,M\,\log_{q_v}(\mathrm{Card}\;\mathscr{Q})\big)\,.$$

Now we can take $i \to \infty$ since the atoms Q of the partition $\overline{\mathscr{Q}}$ and hence of the partition $\bigvee_{k=0}^{M-1} \mathfrak{a}^k\, \overline{\mathscr{Q}}$, satisfy $\widetilde{\mu}_j(\partial Q) = 0$ by the property (iii) of $\mathscr{Q}$. Also, the constants C, N, ℓ, and Card $\mathscr{Q}$ are independent of k_i. Thus it follows from Equation (53) that

$$\frac{1}{M} H_{\widetilde{\mu}_j}\big((\,\overline{\sigma(\mathscr{Q})}\,)_0^{M-1} | \,(\,\overline{\mathscr{P}^{U^+}}\,)_0^\ell\big) \geqslant |\mathbf{r}|(1-\beta) - \max \mathbf{r}\ (m + \frac{1}{j} - \dim_{\mathrm{Haus}} \mathbf{Bad}_A(\epsilon))\,.$$

By taking $\rho \to 0$ in Equation (56), we have

$$\frac{1}{M} H_{\widetilde{\mu}_j}\big((\,\overline{\sigma(\mathscr{Q})}\,)_0^{M-1} | \,(\,\overline{\mathscr{P}^{U^+}}\,)_0^\ell\big) \geqslant |\mathbf{r}|(1-\widetilde{\mu}_j(\,\overline{Q}_\infty)) - \max \mathbf{r}\ (m + \frac{1}{j} - \dim_{\mathrm{Haus}} \mathbf{Bad}_A(\epsilon))\,.$$

Hence, it follows by taking $j \to \infty$ and by using the property (iii) of $\mathscr{Q}$ that

$$\frac{1}{M} H_{\overline{\mu}}\big((\,\overline{\sigma(\mathscr{Q})}\,)_0^{M-1} | \,(\,\overline{\mathscr{P}^{U^+}}\,)_0^\ell\big) \geqslant |\mathbf{r}|(1-\overline{\mu}(\,\overline{Q}_\infty)) - \max \mathbf{r}\ (m - \dim_{\mathrm{Haus}} \mathbf{Bad}_A(\epsilon))\,.$$

Since $(\,\overline{\mathscr{P}^{U^+}}\,)_0^\ell \nearrow \,\overline{\mathscr{A}^{U^+}}$ as $\ell \to \infty$, by the continuity of entropy, we finally have

$$\frac{1}{M} H_{\overline{\mu}}\big((\,\overline{\sigma(\mathscr{Q})}\,)_0^{M-1} | \,\overline{\mathscr{A}^{U^+}}\big) \geqslant |\mathbf{r}|(1-\overline{\mu}(\overline{Q}_\infty)) - \max \mathbf{r}\ (m - \dim_{\mathrm{Haus}} \mathbf{Bad}_A(\epsilon))\,,$$

which proves Equation (55).

Hence, by taking $M \to \infty$, we have

$$h_{\overline{\mu}}(\mathfrak{a}^{-1} | \overline{\mathscr{A}^{U^+}}) \geqslant |\mathbf{r}|(1 - \overline{\mu}(\overline{Q}_\infty)) - \max \mathbf{r}\ (m - \dim_{\mathrm{Haus}} \mathbf{Bad}_A(\epsilon)) ,$$

provided that we have a partition $\mathscr{Q}$ satisfying the above requirements (i), (ii) and (iii). After taking a sufficiently small neighborhood of infinity Q^*_∞ in $\mathscr{X}$, so that if $Q_\infty = \pi^{-1}(Q^*_\infty)$, then $\overline{\mu}(\overline{Q}_\infty)$ is sufficiently close to $\overline{\mu}(\infty_{\mathscr{Y}}) = \eta_A$, we can indeed construct a finite Borel-measurable partition $\mathscr{Q}$ of $\mathscr{Y}$ satisfying Properties (i), (ii) and (iii), by following the procedure in [LSS, Proof of Theorem 4.2, Claim 2]. This proves Assertion (3). □

6.2 Effective upper bound on $\dim_{\mathrm{Haus}} \mathbf{Bad}_A(\epsilon)$

For every $\ell \in \mathbb{Z}_{\leqslant 1}$, with λ_1 the shortest length function of a nonzero vector of an R_v-lattice (see Subsection 2.2), we define

$$\mathscr{X}^{\geqslant q_v^\ell} = \{x \in \mathscr{X} : \lambda_1(x) \geqslant q_v^\ell\} \quad \text{and} \quad \mathscr{Y}^{\geqslant q_v^\ell} = \pi^{-1}(\mathscr{X}^{\geqslant q_v^\ell}) .$$

Note that by Corollary 2.2, we have $\lambda_1(x) \leqslant q_v$ for all $x \in \mathscr{X}$, thus $\mathscr{X} = \bigcup_{\ell=-\infty}^{1} \mathscr{X}^{\geqslant q_v^\ell}$. By Mahler's compactness criterion (see for instance [KlST, Theo. 1.1]), the subsets $\mathscr{X}^{\geqslant q_v^\ell}$ and $\mathscr{Y}^{\geqslant q_v^\ell}$ are compact.

Lemma 6.3 *Let μ' be an $\mathfrak{a}$-invariant Borel probability measure on $\mathscr{Y}$ and let $\mathscr{A}$ be a countably generated sub-σ-algebra of the Borel σ-algebra of $\mathscr{Y}$ which is $\mathfrak{a}^{-1}$-descending and U^+-subordinated modulo μ'. For all $r' \geqslant \delta' > 0$, $\epsilon \in \,]0, 1]$ and $\ell \in \mathbb{Z}_{\leqslant 0}$, let j_1, j_2 be integers satisfying*

$$j_1 > \frac{d-(d-1)\ell}{\min \mathbf{r}} - \log_{q_v} \delta' \quad \textit{and} \quad j_2 > \frac{d-(d-1)\ell}{\min \mathbf{s}} - \frac{n}{d} \log_{q_v} \epsilon .$$

If $y \in \mathscr{Y}^{\geqslant q_v^\ell}$ satisfies $B^{U^+,\mathbf{r}}_{\delta'} \mathfrak{a}^{-j_1} y \subset [\mathfrak{a}^{-j_1} y]_{\mathscr{A}} \subset B^{U^+,\mathbf{r}}_{r'} \mathfrak{a}^{-j_1} y$, then we have

$$\tau_y^{\mathfrak{a}^{j_1}\mathscr{A}}(\mathfrak{a}^{-j_2} \mathscr{L}_\epsilon) \leqslant 1 - \left(q_v^{-(j_1+j_2)} (r')^{-1} \epsilon^{\frac{m}{d}} \right)^{|\mathbf{r}|} .$$

Proof. Let $x = \pi(y)$, which belongs to $\mathscr{X}^{\geqslant q_v^\ell}$. Since x is a unimodular R_v-lattice, by Minkowski's theorem 2.1, we hence have

$$q_v^{(d-1)\ell} \lambda_d(x) \leqslant (\lambda_1(x))^{d-1} \lambda_d(x) \leqslant \lambda_1(x)\lambda_2(x) \cdots \lambda_d(x) \leqslant q_v^d ,$$

therefore $\lambda_d(x) \leqslant q_v^{d-(d-1)\ell}$. There are linearly independent vectors $v_1, \ldots, v_d$ in the R_v-lattice x such that $\|v_i\| \leqslant q_v^{d-(d-1)\ell}$. Let Δ be the parallelepiped in K_v^d generated by $v_1, \ldots, v_d$, that is,

$$\Delta = \{t_1 v_1 + \cdots + t_d v_d \in K_v^d : \ \forall\, i = 1, \ldots, d, \ \ |\, t_i\,| \leqslant 1\} .$$

We identify K_v^d with $K_v^m \times K_v^n$. Then for every $\mathbf{b} = (\mathbf{b}^-, \mathbf{b}^+) \in \Delta$ with $\mathbf{b}^- \in K_v^m$ and $\mathbf{b}^+ \in K_v^n$, we have $\|\,\mathbf{b}\,\| \leqslant q_v^{d-(d-1)\ell}$, hence $\|\,\mathbf{b}^-\,\|_{\mathbf{r}} \leqslant q_v^{\frac{d-(d-1)\ell}{\min \mathbf{r}}}$ and $\|\,\mathbf{b}^+\,\|_{\mathbf{s}} \leqslant q_v^{\frac{d-(d-1)\ell}{\min \mathbf{s}}}$ since $\ell \leqslant 0$. Note that the fiber $\pi^{-1}(x)$ can be parametrized as follows: Fixing $g \in G_0$ with $x = g\Gamma_0$, since Δ is a fondamental domain for the action of R_v^d on K_v^d, we have

$$\pi^{-1}(x) = \{w(\mathbf{b}) g \Gamma : \mathbf{b} \in \Delta\}, \ \text{where } w(\mathbf{b}) = \begin{pmatrix} I_d & \mathbf{b} \\ 0 & 1 \end{pmatrix} .$$

In particular, there exists $\mathbf{b}_0 = (\mathbf{b}_0^-, \mathbf{b}_0^+) \in \Delta$ such that $y = w(\mathbf{b}_0)g\Gamma$.

With a slightly simplified notation, let V_y be the U^+-shape of the atom $[y]_{\mathfrak{a}^{j_1}\mathscr{A}}$ (see Equation (49)), so that we have $V_y y = [y]_{\mathfrak{a}^{j_1}\mathscr{A}}$. Let $\Xi = \{\boldsymbol{\theta} \in K_v^m : w(\boldsymbol{\theta}, 0) \in V_y\}$ be the Borel set corresponding to V_y by the canonical bijection $\boldsymbol{\theta} \mapsto w(\boldsymbol{\theta}, 0)$ (see above Equation (34)) between K_v^m and U^+. Note that $0 \in \Xi$ as $I_{d+1} \in V_y$. Since $\mathfrak{a}_-^{j_1}$ expands the $\mathbf{r}$-quasinorm on K_v^m with ratio exactly $q_v^{j_1}$ (see Equation (32)), and by the assumption on y in the statement of Lemma 6.3, we have $B^{U^+,\mathbf{r}}_{q_v^{j_1}\delta'} y \subset [y]_{\mathfrak{a}^{j_1}\mathscr{A}} \subset B^{U^+,\mathbf{r}}_{q_v^{j_1} r'} y$, hence

$$B^{K_v^m,\mathbf{r}}_{q_v^{j_1}\delta'} \subset \Xi \subset B^{K_v^m,\mathbf{r}}_{q_v^{j_1} r'} \ . \tag{63}$$

The atom $[y]_{\mathfrak{a}^{j_1}\mathscr{A}}$ can be parametrized by

$$[y]_{\mathfrak{a}^{j_1}\mathscr{A}} = \big\{ w(\mathbf{b})g\Gamma : \ \exists\ \mathbf{b}^- \in \mathbf{b}_0^- + \Xi, \ \ \mathbf{b} = (\mathbf{b}^-, \mathbf{b}_0^+) \big\} \ ,$$

and $\tau_y^{\mathfrak{a}^{j_1}\mathscr{A}}$ is the pushforward measure of the normalized Haar measure on the Borel set (with positive measure) $\mathbf{b}_0^- + \Xi$ of K_v^m.

Let us consider the sets

$$\Theta^- = \{\mathbf{b}^- \in K_v^m : \|\, \mathbf{b}^- \|_{\mathbf{r}} < q_v^{-j_2} \epsilon^{\frac{m}{d}}\} \ \text{ and } \ \Theta^+ = \{\mathbf{b}^+ \in K_v^n : \|\, \mathbf{b}^+ \|_{\mathbf{s}} < q_v^{j_2} \epsilon^{\frac{n}{d}}\} \ .$$

If $\mathbf{b} = (\mathbf{b}^-, \mathbf{b}^+) \in \Theta^- \times \Theta^+$, then $\|\mathfrak{a}_-^{j_2}\mathbf{b}^-\|_{\mathbf{r}} < \epsilon^{\frac{m}{d}}$ and $\|\mathfrak{a}_+^{j_2}\mathbf{b}^+\|_{\mathbf{s}} < \epsilon^{\frac{n}{d}}$ by Equation (32). By the definition of $\mathscr{L}_\epsilon$ in Equation (35), and since the grid $\mathfrak{a}^{j_2} g R_v^m + (\mathfrak{a}_-^{j_2}\mathbf{b}^-, \mathfrak{a}_+^{j_2}\mathbf{b}^+)$ contains the vector $(\mathfrak{a}_-^{j_2}\mathbf{b}^-, \mathfrak{a}_+^{j_2}\mathbf{b}^+)$, we have

$$\mathfrak{a}^{j_2} w(\mathbf{b}) g\Gamma = w(\mathfrak{a}_-^{j_2}\mathbf{b}^-, \mathfrak{a}_+^{j_2}\mathbf{b}^+)\mathfrak{a}^{j_2} g\Gamma \notin \mathscr{L}_\epsilon \ .$$

Hence we have $w(\mathbf{b})g\Gamma \notin \mathfrak{a}^{-j_2}\mathscr{L}_\epsilon$, so that

$$[y]_{\mathfrak{a}^{j_1}\mathscr{A}} - \mathfrak{a}^{-j_2}\mathscr{L}_\epsilon \supset w\big(\big((\mathbf{b}_0^- + \Xi) \times \{\mathbf{b}_0^+\}\big) \cap (\Theta^- \times \Theta^+)\big) g\Gamma \ . \tag{64}$$

Claim. We have the inclusion $\Theta^- \times \{\mathbf{b}_0^+\} \subset \big((\mathbf{b}_0^- + \Xi) \times \{\mathbf{b}_0^+\}\big) \cap (\Theta^- \times \Theta^+)$.

Proof. We only have to prove that $\mathbf{b}_0^+ \in \Theta^+$ and that $\Theta^- \subset \mathbf{b}_0^- + \Xi$. Since $(\mathbf{b}_0^-, \mathbf{b}_0^+) \in \Delta$, we have $\|\, \mathbf{b}_0^+ \|_{\mathbf{s}} \leqslant q_v^{\frac{d-(d-1)\ell}{\min \mathbf{s}}}$, hence the former assertion follows from the assumption that $j_2 > \frac{d-(d-1)\ell}{\min \mathbf{s}} - \frac{n}{d}\log_{q_v} \epsilon$.

In order to prove the latter assertion, let us fix $\mathbf{b}^- \in \Theta^-$. Recall that the $\mathbf{r}$-quasinorm $\|\cdot\|_{\mathbf{r}}$ satisfies the ultrametric inequality property, see Equation (44). Hence, it follows from the assumptions $j_2 > \frac{d-(d-1)\ell}{\min \mathbf{s}} - \frac{n}{d}\log_{q_v} \epsilon$ and $j_1 > \frac{d-(d-1)\ell}{\min \mathbf{r}} - \log_{q_v} \delta'$, since $\epsilon \leqslant 1$, that

$$\begin{aligned}\|\, \mathbf{b}^- - \mathbf{b}_0^- \|_{\mathbf{r}} &\leqslant \max\{\|\, \mathbf{b}^- \|_{\mathbf{r}}, \|\, \mathbf{b}_0^- \|_{\mathbf{r}}\} \leqslant \max\big\{q_v^{-j_2}\epsilon^{\frac{m}{d}}, q_v^{\frac{d-(d-1)\ell}{\min \mathbf{r}}}\big\} \\ &\leqslant \max\big\{q_v^{-\frac{d-(d-1)\ell}{\min \mathbf{s}}}\epsilon, q_v^{\frac{d-(d-1)\ell}{\min \mathbf{r}}}\big\} = q_v^{\frac{d-(d-1)\ell}{\min \mathbf{r}}} < q_v^{j_1}\delta' \ .\end{aligned}$$

Hence by the left inclusion in Equation (63), we have $\mathbf{b}^- \in \mathbf{b}_0^- + B^{K_v^m,\mathbf{r}}_{q_v^{j_1}\delta'} \subset \mathbf{b}_0^- + \Xi$, which concludes the latter assertion. □

Now by Equation (64), by the above claim and by the right inclusion in Equation (63), we have

$$1-\tau_y^{\mathfrak{a}^{j_1}\mathscr{A}}(\mathfrak{a}^{-j_2}\mathscr{L}_\epsilon)=\tau_y^{\mathfrak{a}^{j_1}\mathscr{A}}([y]_{\mathfrak{a}^{j_1}\mathscr{A}}-\mathfrak{a}^{-j_2}\mathscr{L}_\epsilon)\geqslant\frac{m_{K_v^m}(\Theta^-)}{m_{K_v^m}(\mathbf{b}_0^-+\Xi)}\geqslant\frac{m_{K_v^m}\big(B^{K_v^m,\mathbf{r}}_{q_v^{-j_2}\epsilon^{\frac{m}{d}}}\big)}{m_{K_v^m}\big(B^{K_v^m,\mathbf{r}}_{q_v^{j_1}r'}\big)}$$

$$=\Big(\frac{q_v^{-j_2}\epsilon^{\frac{m}{d}}}{q_v^{j_1}r'}\Big)^{|\mathbf{r}|}=\big(q_v^{-(j_1+j_2)}(r')^{-1}\epsilon^{\frac{m}{d}}\big)^{|\mathbf{r}|}\,.$$

This proves the lemma. □

Proof of Theorem 1.2. We fix a matrix $A\in\mathscr{M}_{m,n}(K_v)$ which is not $(\mathbf{r},\mathbf{s})$-singular on average, or equivalently by Proposition 5.2 and the definition of η_A just before Lemma 6.1, that $\eta_A<1$. We also fix $\epsilon\in\,]0,1]$.

By Proposition 6.1, there exist an $\mathfrak{a}$-invariant Borel probability measure $\overline{\mu}$ on $\overline{\mathscr{Y}}$ (depending on ϵ) and an $\mathfrak{a}$-invariant Borel probability measure μ on $\mathscr{Y}$ (unique since $\eta_A<1$) such that

$$\operatorname{Supp}\overline{\mu}\subset\mathscr{L}_\epsilon\cup\{\infty_{\mathscr{Y}}\},\quad\overline{\pi}_*\overline{\mu}=\mu_A,\quad\text{and}\quad\overline{\mu}=(1-\eta_A)\mu+\eta_A\Delta_{\infty_{\mathscr{Y}}}\,.$$

Take a compact subset K_0 of $\mathscr{X}$ such that $\mu_A(K_0)>0.99\,\mu_A(\mathscr{X})=0.99\,(1-\eta_A)$. Write $K=\pi^{-1}(K_0)$ and choose $r\in\,]0,1[$ such that $K\subset\mathscr{Y}(r)$. Then $\mu(\mathscr{Y}(r))\geqslant\mu(K)>0.99$ since $\eta_A<1$. Note that the choices of K and r are independent of ϵ since the measure μ_A depends only on A (see Proposition 6.1 and Equation (52)).

For such an $r>0$, let $\mathscr{A}^{U^+}$ be the σ-algebra of subsets of $\mathscr{Y}$ constructed in Proposition 5.7. Proposition 6.1 (3) gives the inequality

$$h_{\overline{\mu}}(\mathfrak{a}^{-1}|\,\overline{\mathscr{A}^{U^+}})\geqslant|\mathbf{r}|(1-\eta_A)-\max\mathbf{r}\,\left(m-\dim_{\mathrm{Haus}}\mathbf{Bad}_A(\epsilon)\right)\,.$$

By the linearity of entropy (and since the entropy of $\mathfrak{a}^{-1}$ vanishes on the fixed set $\{\infty_{\mathscr{Y}}\}$), we have

$$h_\mu(\mathfrak{a}^{-1}|\,\mathscr{A}^{U^+})=\frac{1}{1-\eta_A}h_{\overline{\mu}}(\mathfrak{a}^{-1}|\,\overline{\mathscr{A}^{U^+}})\geqslant|\mathbf{r}|-\frac{\max\mathbf{r}}{1-\eta_A}\,\left(m-\dim_{\mathrm{Haus}}\mathbf{Bad}_A(\epsilon)\right)\,.\tag{65}$$

In order to use Lemma 5.9 and Proposition 5.10, we need an ergodicity assumption on the measures that appear in these statements. We will choose an appropriate ergodic component of μ. Let us denote the ergodic decomposition of μ by

$$\mu=\int_{y\in\mathscr{Y}}\mu_y^{\mathscr{E}}\,d\mu(y)$$

as in the second equality of Proposition 5.5 with $T=\mathfrak{a}^{-1}$ and $\mathscr{A}=\mathscr{A}^{U^+}$. Let $E=\{y\in\mathscr{Y}:\mu_y^{\mathscr{E}}(K)>0.9\}$. It follows from $\mu(K)>0.99$ that

$$0.99<\int_{\mathscr{Y}}\mu_y^{\mathscr{E}}(K)\,d\mu(y)\leqslant\mu(E)+0.9\,\mu(\mathscr{Y}-E)=0.9+0.1\,\mu(E)\,,$$

hence $\mu(E)>0.9$. By Proposition 5.5 and Equation (65), we have

$$\int_{\mathscr{Y}}h_{\mu_y^{\mathscr{E}}}(\mathfrak{a}^{-1}|\,\mathscr{A}^{U^+})\,d\mu(y)=h_\mu(\mathfrak{a}^{-1}|\,\mathscr{A}^{U^+})\geqslant|\mathbf{r}|-\frac{\max\mathbf{r}}{1-\eta_A}\,\left(m-\dim_{\mathrm{Haus}}\mathbf{Bad}_A(\epsilon)\right)\,.$$

Since $h_{\mu_y^{\mathscr{E}}}(\mathfrak{a}^{-1}|\,\mathscr{A}^{U^+}) \leqslant |\mathbf{r}|$ for every $y \in \mathscr{Y}$ by Lemma 5.9, we have

$$\int_{\mathscr{Y}-E} h_{\mu_y^{\mathscr{E}}}(\mathfrak{a}^{-1}|\,\mathscr{A}^{U^+})\,d\mu(y) \leqslant |\mathbf{r}|\,\mu(\mathscr{Y}-E)\,.$$

Hence

$$\begin{aligned}\int_E h_{\mu_y^{\mathscr{E}}}(\mathfrak{a}^{-1}|\,\mathscr{A}^{U^+})\,d\mu(y) &\geqslant |\mathbf{r}|\,\mu(E) - \frac{\max\mathbf{r}}{1-\eta_A}\,(m - \dim_{\mathrm{Haus}}\mathbf{Bad}_A(\epsilon))\\ &\geqslant \mu(E)\Big(|\mathbf{r}| - \frac{\max\mathbf{r}}{0.9\,(1-\eta_A)}\,(m - \dim_{\mathrm{Haus}}\mathbf{Bad}_A(\epsilon))\Big)\,.\end{aligned}$$

Therefore, there exists $z \in \mathscr{Y}$ such that $\mu_z^{\mathscr{E}}(K) > 0.9$ and

$$h_{\mu_z^{\mathscr{E}}}(\mathfrak{a}^{-1}|\,\mathscr{A}^{U^+}) \geqslant |\mathbf{r}| - \frac{\max\mathbf{r}}{0.9\,(1-\eta_A)}\,(m - \dim_{\mathrm{Haus}}\mathbf{Bad}_A(\epsilon))\,.$$

We denote $\lambda = \mu_z^{\mathscr{E}}$ for such a $z \in \mathscr{Y}$. Then λ is an $\mathfrak{a}$-invariant ergodic Borel probability measure on $\mathscr{Y}$ and $\operatorname{Supp}\lambda \subset \operatorname{Supp}\mu \subset \mathscr{L}_\epsilon$. By Lemma 5.9, we have

$$H_\lambda(\mathscr{A}^{U^+}|\,\mathfrak{a}\,\mathscr{A}^{U^+}) \geqslant |\mathbf{r}| - \frac{\max\mathbf{r}}{0.9\,(1-\eta_A)}\,(m - \dim_{\mathrm{Haus}}\mathbf{Bad}_A(\epsilon))\,. \tag{66}$$

We will apply Lemma 6.3 with $\mu' = \lambda$ and $\mathscr{A} = \mathfrak{a}^{-k}\mathscr{A}^{U^+}$ for some $k \geqslant 1$. Take an integer $\ell \leqslant 0$ such that $K \subset \mathscr{Y}^{\geqslant q_v^\ell}$, which depends only on A. Set

$$j_1 = \Big\lceil \frac{d-(d-1)\ell}{\min\mathbf{r}} - \log_{q_v}\delta' \Big\rceil + 1 \quad\text{and}\quad j_2 = \Big\lceil \frac{d-(d-1)\ell}{\min\mathbf{s}} - \frac{n}{d}\log_{q_v}\epsilon \Big\rceil + 1\,,$$

where δ' will be determined later on.

Let $k = \big\lceil \log_{q_v}\big(r^{\frac{1}{\max\mathbf{r}}}\,\epsilon^{-\frac{m}{d}}\big)\big\rceil + j_2 + 1$ and $\mathscr{A} = \mathfrak{a}^{-k}\mathscr{A}^{U^+}$. By the properties of $\mathscr{A}^{U^+}$ given in Proposition 5.7 and since $K \subset \mathscr{Y}(r)$, for every $y \in K$, we have

$$B_{\delta_r}^{U^+} y \subset [y]_{\mathscr{A}^{U^+}} \subset B_r^{U^+} y\,.$$

It follows from Equation (45) that for any $y \in K$,

$$B^{U^+,\mathbf{r}}_{\delta_r^{\frac{1}{\min\mathbf{r}}}} y \subset [y]_{\mathscr{A}^{U^+}} \subset B^{U^+,\mathbf{r}}_{r^{\frac{1}{\max\mathbf{r}}}} y\,.$$

Hence, by Equation (46), we have

$$B^{U^+,\mathbf{r}}_{q_v^{-k}\delta_r^{\frac{1}{\min\mathbf{r}}}}\mathfrak{a}^{-k}y \subset [\mathfrak{a}^{-k}y]_{\mathfrak{a}^{-k}\mathscr{A}^{U^+}} = [\mathfrak{a}^{-k}y]_{\mathscr{A}} \subset B^{U^+,\mathbf{r}}_{q_v^{-k}r^{\frac{1}{\max\mathbf{r}}}}\mathfrak{a}^{-k}y\,.$$

Thus for every $y \in \mathfrak{a}^k K$, we have

$$B^{U^+,\mathbf{r}}_{\delta'} y \subset [y]_{\mathscr{A}} \subset B^{U^+,\mathbf{r}}_{r'} y\,, \tag{67}$$

where, by the definition of k, we take

$$r' = q_v^{-j_2-1}\epsilon^{\frac{m}{d}} \quad\text{and}\quad \delta' = q_v^{-1}\,r^{-\frac{1}{\max\mathbf{r}}}\,\delta_r^{\frac{1}{\min\mathbf{r}}}\,r'\,.$$

Equation (67) implies that for every $y \in \mathfrak{a}^{j_1+k}K$, we have

$$B^{U^+,\mathbf{r}}_{\delta'}\mathfrak{a}^{-j_1}y \subset [\mathfrak{a}^{-j_1}y]_{\mathscr{A}} \subset B^{U^+,\mathbf{r}}_{r'}\mathfrak{a}^{-j_1}y\ . \tag{68}$$

Now, we will use Proposition 5.10 with $j = j_1$, $K' = \mathfrak{a}^k K$ (which is U^+-saturated since so is K and as $\mathfrak{a}$ normalizes U^+), and $\epsilon = r'$ (which satisfies the assumption of Proposition 5.10 by Equation (67)). We claim that

$$B^{U^+,\mathbf{r}}_{q_v^{-1}\epsilon^{\frac{m}{d}}}\mathscr{L}_\epsilon \subset \mathscr{L}_\epsilon\ . \tag{69}$$

Indeed, for all $y \in \mathscr{L}_\epsilon$ and $\boldsymbol{\theta} \in K_v^m$ such that $\|\,\boldsymbol{\theta}\,\|_{\mathbf{r}} \leqslant q_v^{-1}\epsilon^{\frac{m}{d}}$, for every vector $u = (u^-, u^+)$ in the grid $w(\boldsymbol{\theta}, \mathbf{0})y$, we can write $u = v + (\boldsymbol{\theta}, \mathbf{0})$ for some $v = (v^-, v^+)$ in the grid $\widetilde{\Lambda}_y$ associated with y (see Equation (33)). Since $y \in \mathscr{L}_\varepsilon$, we have (see Equation (35)) $\|\,v\,\|_{\mathbf{r},\mathbf{s}} = \max\{\|\,v^-\,\|_{\mathbf{r}}^{\frac{d}{m}}, \|\,v^+\,\|_{\mathbf{s}}^{\frac{d}{n}}\} \geqslant \epsilon$. Since $u^+ = v^+$, if $\|\,v^+\,\|_{\mathbf{s}}^{\frac{d}{n}} \geqslant \epsilon$, then $w(\boldsymbol{\theta}, \mathbf{0})y \in \mathscr{L}_\epsilon$. Otherwise $\|\,v^-\,\|_{\mathbf{r}}^{\frac{d}{m}} \geqslant \epsilon$. We then have $\|\,\boldsymbol{\theta}\,\|_{\mathbf{r}} \leqslant q_v^{-1}\epsilon^{\frac{m}{d}} < \epsilon^{\frac{m}{d}} \leqslant \|\,v^-\,\|_{\mathbf{r}}$. It follows from the equality case of the ultrametric inequality property of $\|\ \|_{\mathbf{r}}$ that

$$\|\,u^-\,\|_{\mathbf{r}} = \|\,\boldsymbol{\theta} + v^-\,\|_{\mathbf{r}} = \max\big\{\|\,\boldsymbol{\theta}\,\|_{\mathbf{r}},\ \|\,v^-\,\|_{\mathbf{r}}\big\} = \|\,v^-\,\|_{\mathbf{r}} \geqslant \epsilon^{\frac{m}{d}}\ .$$

Hence $w(\boldsymbol{\theta}, \mathbf{0})y \in \mathscr{L}_\epsilon$, which proves Equation (69).

By Proposition 5.7, the σ-algebra $\mathscr{A}^{U^+}$ is $\mathfrak{a}^{-1}$-descending and U^+-subordinated modulo λ, and so does $\mathscr{A} = \mathfrak{a}^{-k}\mathscr{A}^{U^+}$ since $\mathfrak{a}$ normalizes U^+. Note that $\operatorname{Supp}\lambda \subset \mathfrak{a}^{-j_2}\mathscr{L}_\epsilon$ since λ is $\mathfrak{a}$-invariant. By Equations (46) and (69), we have

$$B^{U^+,\mathbf{r}}_{r'}\mathfrak{a}^{-j_2}\mathscr{L}_\epsilon = \mathfrak{a}^{-j_2}B^{U^+,\mathbf{r}}_{q_v^{j_2}r'}\mathscr{L}_\epsilon = \mathfrak{a}^{-j_2}B^{U^+,\mathbf{r}}_{q_v^{-1}\epsilon^{\frac{m}{d}}}\mathscr{L}_\epsilon \subset \mathfrak{a}^{-j_2}\mathscr{L}_\epsilon\ .$$

Note that we have

$$\tau_y^{\mathfrak{a}^{j_1}\mathscr{A}}(\mathscr{Y} - \mathfrak{a}^k K) = 0$$

for λ-almost every $y \in \mathfrak{a}^k K$, since then (see just above Proposition 5.10) the support $\operatorname{Supp}\tau_y^{\mathfrak{a}^{j_1}\mathscr{A}}$ is contained in $[y]_{\mathfrak{a}^{j_1}\mathscr{A}}$, which is contained in U^+y, hence in $\mathfrak{a}^k K$ since $\mathfrak{a}$ normalizes U^+ and $K = \pi^{-1}(K_0)$ is U^+-saturated. Therefore, it follows from Proposition 5.10 for the first line, from the fact that the integrated function is nonpositive (hence its integral on a smaller domain is larger) for the third line, that

$$\begin{aligned}
H_\lambda(\mathscr{A}|\mathfrak{a}^{j_1}\mathscr{A}) &\leqslant j_1|\mathbf{r}| + \int_{\mathscr{Y}} \log_{q_v} \tau_y^{\mathfrak{a}^{j_1}\mathscr{A}}((\mathscr{Y} - \mathfrak{a}^k K) \cup B^{U^+,\mathbf{r}}_{r'}\operatorname{Supp}\lambda)\, d\lambda(y)\\
&\leqslant j_1|\mathbf{r}| + \int_{\mathscr{Y}} \log_{q_v} \tau_y^{\mathfrak{a}^{j_1}\mathscr{A}}((\mathscr{Y} - \mathfrak{a}^k K) \cup \mathfrak{a}^{-j_2}\mathscr{L}_\epsilon)\, d\lambda(y)\\
&\leqslant j_1|\mathbf{r}| + \int_{\mathfrak{a}^k K \cap \mathfrak{a}^{j_1+k}K \cap \mathscr{Y}^{\geqslant q_v^\ell}} \log_{q_v} \tau_y^{\mathfrak{a}^{j_1}\mathscr{A}}((\mathscr{Y} - \mathfrak{a}^k K) \cup \mathfrak{a}^{-j_2}\mathscr{L}_\epsilon)\, d\lambda(y)\\
&= j_1|\mathbf{r}| + \int_{\mathfrak{a}^k K \cap \mathfrak{a}^{j_1+k}K \cap \mathscr{Y}^{\geqslant q_v^\ell}} \log_{q_v} \tau_y^{\mathfrak{a}^{j_1}\mathscr{A}}(\mathfrak{a}^{-j_2}\mathscr{L}_\epsilon)\, d\lambda(y)\ .
\end{aligned}$$

We now apply Lemma 6.3 with as said above $\mu' = \lambda$ and $\mathscr{A} = \mathfrak{a}^{-k}\mathscr{A}^{U^+}$, and with $y \in \mathfrak{a}^{j_1+k}K \cap \mathscr{Y}^{\geqslant q_v^\ell}$ which satisfies the assumption of Lemma 6.3 by Equation (68). Thus

$$\tau_y^{\mathfrak{a}^{j_1}\mathscr{A}}(\mathfrak{a}^{-j_2}\mathscr{L}_\epsilon) \leqslant 1 - \Big(q_v^{-(j_1+j_2)}{r'}^{-1}\epsilon^{\frac{m}{d}}\Big)^{|\mathbf{r}|} = 1 - q_v^{-(j_1-1)|\mathbf{r}|}\ .$$

Hence

$$-\log_{q_v} \tau_y^{\mathfrak{a}^{j_1}\mathscr{A}}(\mathfrak{a}^{-j_2}\mathscr{L}_\epsilon) \geqslant -\log_{q_v}\big(1-q_v^{-(j_1-1)|\mathbf{r}|}\big) \geqslant \frac{q_v^{-(j_1-1)|\mathbf{r}|}}{\ln q_v}\;.$$

Note that $\lambda(\mathfrak{a}^k K \cap \mathfrak{a}^{j_1+k}K \cap \mathscr{Y}^{\geqslant q_v^\ell}) \geqslant \frac{1}{2}$ since λ is $\mathfrak{a}$-invariant, $K \subset \mathscr{Y}^{\geqslant q_v^\ell}$ and $\lambda(K) > 0.9$, so that the three sets $\mathfrak{a}^k K$, $\mathfrak{a}^{j_1+k}K$ and $\mathscr{Y}^{\geqslant q_v^\ell}$ have λ-measure > 0.9, hence their pairwise intersections have λ-measure $> 2\times 0.9-1 = 0.8$, and their triple intersection has λ-measure $> 2\times 0.8 - 1 = 0.6$. It follows from Equation (39) and the invariance under $\mathfrak{a}$ of λ, hence of the conditional entropy, that

$$\begin{aligned}|\mathbf{r}| - H_\lambda(\mathscr{A}^{U^+}\,|\,\mathfrak{a}\,\mathscr{A}^{U^+}) &= |\mathbf{r}| - \frac{1}{j_1}H_\lambda(\mathscr{A}^{U^+}\,|\,\mathfrak{a}^{j_1}\mathscr{A}^{U^+}) = |\mathbf{r}| - \frac{1}{j_1}H_\lambda(\mathscr{A}\,|\,\mathfrak{a}^{j_1}\mathscr{A})\\ &\geqslant -\frac{1}{j_1}\int_{\mathfrak{a}^k K\cap\mathfrak{a}^{j_1+k}K\cap\mathscr{Y}^{\geqslant q_v^\ell}} \log_{q_v}\tau_y^{\mathfrak{a}^{j_1}\mathscr{A}}(\mathfrak{a}^{-j_2}\mathscr{L}_\epsilon)\,d\lambda(y)\\ &\geqslant \frac{q_v^{|\mathbf{r}|}}{2\ln q_v}\,\frac{q_v^{-j_1|\mathbf{r}|}}{j_1}\;.\end{aligned}$$

Therefore, by Equation (66), we have

$$\frac{\max\mathbf{r}}{0.9\,(1-\eta_A)}\,(m - \dim_{\mathrm{Haus}}\mathbf{Bad}_A(\epsilon)) \geqslant |\mathbf{r}| - H_\lambda(\mathscr{A}^{U^+}\,|\,\mathfrak{a}\,\mathscr{A}^{U^+}) \geqslant \frac{q_v^{|\mathbf{r}|}}{2\ln q_v}\,\frac{q_v^{-j_1|\mathbf{r}|}}{j_1}\;.$$

Observe that

$$\begin{aligned}j_1 &= \Big\lceil\,\frac{d-(d-1)\ell}{\min\mathbf{r}} - \log_{q_v}\delta'\,\Big\rceil + 1 = \Big\lceil\,\frac{d-(d-1)\ell}{\min\mathbf{r}} - \log_{q_v}\Big(\frac{\delta_r^{\frac{1}{\min\mathbf{r}}}}{q_v^2\,r^{\frac{1}{\max\mathbf{r}}}}\,q_v^{-j_2}\epsilon^{\frac{m}{d}}\Big)\Big\rceil + 1\\ &= \Big\lceil\,\frac{d-(d-1)\ell}{\min\mathbf{r}} + \Big\lceil\,\frac{d-(d-1)\ell}{\min\mathbf{s}} - \frac{n}{d}\log_{q_v}\epsilon\,\Big\rceil + 1 - \frac{m}{d}\log_{q_v}\epsilon - \log_{q_v}\frac{\delta_r^{\frac{1}{\min\mathbf{r}}}}{q_v^2\,r^{\frac{1}{\max\mathbf{r}}}}\,\Big\rceil + 1\\ &\leqslant (\,d-(d-1)\ell\,)\Big(\frac{1}{\min\mathbf{r}} + \frac{1}{\max\mathbf{s}}\Big) - \log_{q_v}\frac{\delta_r^{\frac{1}{\min\mathbf{r}}}}{q_v^2 r^{\frac{1}{\max\mathbf{r}}}} + 4 - \log_{q_v}\epsilon\;.\end{aligned}$$

The constants η_A, ℓ, δ_r, and r depend only on the fixed matrix $A \in \mathscr{M}_{m,n}(K_v)$. Hence there exists a constant $c(A) > 0$ depending only on d, $\mathbf{r}$, $\mathbf{s}$ and A such that

$$m - \dim_{\mathrm{Haus}}\mathbf{Bad}_A(\epsilon) \geqslant c(A)\,\frac{\epsilon^{|\mathbf{r}|}}{\log_{q_v}(1/\epsilon)}\;.$$

This proves Theorem 1.2. □

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