Geometric group actions on trees

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Abstract : We define *geometric* group actions on $\mathbb{R}$-trees, as dual to a measured foliation on a 2-complex with some finiteness and injectivity properties. We prove that an action is nongeometric if and only if it is a nontrivial strong limit in the sense of Gillet-Shalen. We give a simple new construction of the Bass-Serre tree of a graph of groups, and we show that a simplicial action is geometric if and only if edge groups are finitely generated. We prove that geometric actions with trivial edge stabilizers have finitely many orbits of branch points, and finite rank.

Introduction[1]

An $\mathbb{R}$-tree is an arcwise connected metric space in which every arc is isometric to an interval of $\mathbb{R}$. See for instance [Sha1, Sha2, Mor] for historical remarks, references and motivation.

It is well known that codimension one measured foliations (or laminations) on compact manifolds have strong connections with group actions on $\mathbb{R}$-trees (see [MS1, MO]). This was used for instance by Morgan-Shalen [MS2] to show that (most) surface groups act freely on $\mathbb{R}$-trees. Conversely, a theorem by Skora [Sko1] (see also [Ota]) asserts that every (minimal) action of a surface group $\pi_1\Sigma$ on an $\mathbb{R}$-tree with cyclic arc stabilizers is geometric: it comes from a measured foliation on Σ.

On the other hand, many interesting actions cannot be obtained from foliations. For instance, iteration of (irreducible) automorphisms of finitely generated free groups leads to actions that may fail to be geometric (see [BF2, GL]).

This concept of a *geometric* action on an $\mathbb{R}$-tree has been used by many authors [GSS, BF1, BF2, Mor, GL, Lev3], with various meanings. One of its main usefulnesses is to replace (or approximate) a given action on an $\mathbb{R}$-tree by one which is simpler to analyze while keeping a control over edge stabilizers (see for instance [GL]).

In this paper we offer what we think is the right definition. It is given in terms of measured foliations, and our main result (Theorem 0.2) states that there is a very

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simple equivalent definition: *nongeometric actions are precisely those actions that can be viewed non trivially as strong limits* (in the sense of Gillet-Shalen [GS]). We then illustrate the general theory on several types of examples: simplicial actions, actions with trivial edge stabilizers, abelian actions...

The general idea is that an action of a group G on an $\mathbb{R}$-tree T is geometric if and only if it comes from a measured foliation on a finite complex. One can then try the following definition.

Start with a measured foliation $\mathcal{F}$ (with suitable regularity conditions) on a finite complex Σ with $\pi_1\Sigma = G$. Lift $\mathcal{F}$ to a measured foliation $\widetilde{\mathcal{F}}$ on the universal covering $\widetilde{\Sigma}$. We associate to $\widetilde{\mathcal{F}}$ a metric space $T(\widetilde{\mathcal{F}})$, the "leaf space made Hausdorff" [Lev1], as follows. We consider the pseudodistance $\tilde{d}(x,y)$ on $\widetilde{\Sigma}$ defined as the infimum over all paths γ from x to y of the total mass $|\gamma|_{\widetilde{\mathcal{F}}}$ placed on γ by the transverse measure of $\widetilde{\mathcal{F}}$. We define $T(\widetilde{\mathcal{F}})$ as the associated metric space (obtained by identifying x, y if $\tilde{d}(x,y) = 0$). The space $T(\widetilde{\mathcal{F}})$ is an $\mathbb{R}$-tree [GS] with a natural action of G. We may then say that an action is geometric if it is obtained in this way.

There are two problems with this definition. First, it may happen that $T(\widetilde{\mathcal{F}})$ is very different from the leaf space of $\widetilde{\mathcal{F}}$. For an extreme example, take $\mathcal{F}$ to be a foliation with dense leaves on $\Sigma = \mathbb{S}^2$ (there exists such foliations, with 4 thorns as singular set: see for instance [FLP, page 217].

This leads to imposing an extra condition that guarantees in particular that leaves of $\widetilde{\mathcal{F}}$ are closed: every compact arc transverse to $\widetilde{\mathcal{F}}$ may be subdivided into finitely many subintervals that are mapped isometrically into $T(\widetilde{\mathcal{F}})$ (compare [BF1, BF2]). After subdividing Σ (see Lemma 1.3), one may then assume that every edge of $\widetilde{\Sigma}$ is either tangent to $\widetilde{\mathcal{F}}$, or transverse to $\widetilde{\mathcal{F}}$ and mapped isometrically into $T(\widetilde{\mathcal{F}})$.

Second, the above definition only applies to finitely presented groups, since G has to be the fundamental group of a finite complex. On the other hand, one is interested in studying actions of *finitely generated* groups. we get around this difficulty by considering normal coverings $\theta : \overline{\Sigma} \to \Sigma$ with transformation group G, where $\overline{\Sigma}$ may fail to be simply connected (compare [BNS]).

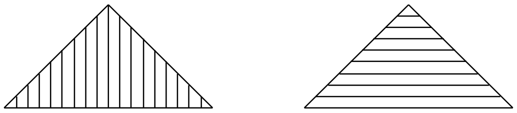

Figure 1 : Foliated 2-simplices.

Our definition then goes as follows. Consider a triple $(\Sigma, \rho, \mathcal{F})$, where:

- Σ is a connected finite simplicial 2-complex, and ρ is an epimorphism $\pi_1\Sigma \to G$.
- $\mathcal{F}$ is a measured foliation on Σ, i. e. a foliation equipped with a nonatomic transverse measure with full support. Each edge of Σ is either contained in a leaf or transverse to $\mathcal{F}$. The foliation induced on any 2-simplex of Σ is

topologically conjugate to one of the two models pictured on Figure 1, with leaves either parallel or perpendicular to one side.

Definition 0.1 *An action of G on an $\mathbb{R}$-tree T is geometric if there exists $(\Sigma, \rho, \mathcal{F})$ as above such that:*

- *T is G-equivariantly isometric to $T(\overline{\mathcal{F}})$, where $\overline{\mathcal{F}}$ is the pullback of $\mathcal{F}$ to the covering $\overline{\Sigma}$ of Σ associated to ρ and $T(\overline{\mathcal{F}})$ is the "leaf space made Hausdorff" defined above.*
- *every edge of $\overline{\Sigma}$ that is transverse to $\overline{\mathcal{F}}$ is mapped isometrically into $T(\overline{\mathcal{F}})$ by the canonical map $\pi : \overline{\Sigma} \to T(\overline{\mathcal{F}})$.*

Recall that an action is *minimal* if there is no proper invariant subteree. Assuming for simplicity that G is finitely presented (see Section 2 for the general case), we then have:

Theorem 0.2 *Let T be an $\mathbb{R}$-tree with a minimal action of G.*

1. *If the action is not geometric, then it is a strong limit of geometric actions of G on $\mathbb{R}$-trees T_n.*
2. *If the action is geometric, then it may only be a strong limit in a trivial (i.e. stationary) way.*

Corollary 0.3 *An action is nongeometric if and only if it is a nontrivial strong limit.*

Roughly speaking, saying that the trees T_n converge *strongly* towards T (in the sense of [GS]) means that they approximate T in such a way that any finite subtree of T may be lifted isometrically (and equivariantly with respect to a finite set in G) to T_n for n large enough. See Section 1.1 for the precise definition. In particular we have (see Theorem 3.7 for the general case of finitely generated groups):

Corollary 0.4 *If a finitely presented group G acts on an $\mathbb{R}$-tree T, then G has a geometric action on an $\mathbb{R}$-tree T' such that edge stabilizers for T' are subgroups of edge stabilizers for T.*

If G is finitely generated, then either there is a point $x \in T$ fixed by all elements of G, or there is a smallest nonempty invariant subtree $T_{\min} \subset T$ ([AB, CM]). In Section 4 we prove:

Theorem 0.5 *Let T be an $\mathbb{R}$-tree with a geometric action of a finitely generated group G. If there is no global fixed point, then $T_{\min}$ is closed in T and the restriction of the action to $T_{\min}$ is geometric.*

This was proved in [GL] when G is free. By analogy with [GL], we may ask the following question (in the situation of Theorem 0.5): if A is a free factor of G, acting with no global fixed point, is the minimal subtree of A closed, and is the restriction of the action of A to this subtree still geometric?

Specializing to simplicial actions, we give a new construction of the Bass-Serre tree associated to a graph of groups (Section 5) and we prove:

Theorem 0.6 *A minimal simplicial action of a finitely generated group is geometric if and only if all edge groups are finitely generated.*

Geometric actions with abelian length function have been studied in [Lev3] (it is easily checked that the above definition of a geometric action may be used in the proof of Theorem 3.1 of [Lev3]). Using work by Bieri-Neumann-Strebel [BNS], one has for instance:

Theorem 0.7 *[Lev3] Let G be finitely generated. The following conditions are equivalent:*

- *Every action of G on $\mathbb{R}$ by translations is geometric.*
- *The commutator subgroup $[G, G]$ is finitely generated.*

Since they are associated to finite complexes, geometric actions may be expected to have strong finiteness properties: the number of orbits of branch points should be finite (bounded by the number of vertices of Σ), the action should have finite $\mathbb{Z}$-rank (its length function should take its values in a finitely generated subgroup of $\mathbb{R}$).

These finiteness properties hold if $\mathcal{F}$ has the property that the pseudodistance $d(x, y)$ between any two points $x, y \in \overline{\Sigma}$ is realized: $d(x, y) = |\gamma|_{\mathcal{F}}$ for some path γ from x to y. Unfortunately we can prove this only when the action has trivial edge stabilizers (using arguments from [GLP2]). We get (see Section 3):

Theorem 0.8 *Let T be an $\mathbb{R}$-tree with a geometric action of a finitely presented group G. If the action has trivial edge stabilizers, there are finitely many orbits of branch points and the action has finite $\mathbb{Z}$-rank.*

This was proved in [GL] when G is free. Inspired by [GL], Corollary III.3, we ask:

Question. Suppose G is generated by k elements. Consider a geometric action of G with trivial edge stabilizers. Is the number of orbits of branch points bounded by $2k - 2$? (this is proved in [GL] for G a free group).

By the same arguments as in [GL], a positive answer would imply that any minimal action with trivial edge stabilizers of a finitely presented group has at most $2k - 2$ orbits of branch points, and has $\mathbb{Q}$-rank $\leq 3k - 3$.

Though we cannot prove Theorem 0.8 for a general geometric action, we show (Theorem 3.1) that geometric actions are J-actions in the sense of [Lev2]: there are only finitely many distinct closures of orbits of branch points. See [Lev2], Theorems 1 and 2, for properties of J-actions: topological finiteness of the quotient space, absence of exceptional minimal set.

1 Preliminaries

We define an action of a group on a metric space to be a left isometric action. First we say a few words about quotient spaces. Let N be a countable group acting on an $\mathbb{R}$-tree T. The space of orbits T/N has a natural pseudometric, induced by the metric of T. Identifying points at pseudo-distance 0 in T/N gives a metric space $\widehat{T/N}$.

An N-equivariant Lipschitz map $T \to T'$ induces a continuous map $\widehat{T/N} \to \widehat{T'/N}$. If N is a normal subgroup of a group H acting on T, there is a natural action of H/N on $\widehat{T/N}$.

We shall use the following fact:

Theorem 1.1 *[Lev1] Let N be a countable group acting on an $\mathbb{R}$-tree T. If N is generated by its elliptic elements (i.e. elements acting with a fixed point), then $\widehat{T/N}$ is an $\mathbb{R}$-tree.*

1.1 Morphisms, strong limits

We define morphisms as in [MO]. A *morphism* from a segment I to an $\mathbb{R}$-tree T is a continuous map $f : I \to T$ such that I may be subdivided into finitely many subsegments that f injects isometrically into T. Let T, T' be $\mathbb{R}$-trees endowed with actions of groups G, G' respectively. Let $\varphi : G \to G'$ be a homomorphism. A *morphism* from T to T' is a map $f : T \to T'$, equivariant relative to φ (in the sense that $f(gx) = \varphi(g)f(x)$), which induces a morphism on every segment $I \subset T$. Note that a morphism obviously does not increase distances.

Here is a very useful way to prove that an equivariant map $f : T \to T'$ is a morphism. Suppose $K \subset T$ is a subtree such that every segment $I \subset T$ is contained in a finite union of images of K under G. If $f|K$ is an isometric embedding, then f is a morphism.

Let $\{G_n\}_{n\in\mathbb{N}}$ be a sequence of finitely generated groups with epimorphisms $G_n \to G_{n+1}$ and

$$G = \varinjlim G_n$$

Note that one has epimorphisms $\tau_n : G_n \to G$, and that if G is finitely presented, then the sequence is stationary.

We say that a sequence of $\mathbb{R}$-trees T_n with actions of G_n *converges strongly* (see [GS]) to an $\mathbb{R}$-tree T with an action of $G = \varinjlim G_n$ if there exist surjective morphisms f_{np} from T_n to T_p (for $n < p$), and f_n from T_n to T, such that $f_p \circ f_{np} = f_n$, and furthermore for every $n \in \mathbb{N}$ and $x, y \in T_n$ there exists $p \geq n$ such that the distance between $f_{np}(x)$ and $f_{np}(y)$ in T_p equals the distance between $f_n(x)$ and $f_n(y)$ in T. (The definition in [GS] is more general, but the one just given is sufficient for us).

If G is finitely presented, we say that the strong convergence of T_n to T is *trivial* if $f_n : T_n \to T$ is an isometry for n large.

If G is not finitely presented, we have to be more careful; in particular, the convergence of (T, G_n) towards (T, G) should be considered trivial.

Let H_n be the kernel of the epimorphism $\tau_n : G_n \to G$. Abstract nonsense gives an isometric action of $G \simeq G_n/H_n$ on the metric space $\widehat{T_n/H_n}$ and a natural G-equivariant map $\widehat{T_n/H_n} \to T$. We say that the strong convergence from T_n to T is *trivial* if this map is an isometry for n large.

1.2 Foliated 2-complexes, systems of isometries, Imanishi's theorem

Let $\Delta_2 = \{(x, y, z) \in \mathbb{R}^3 \,/\, x + y + z = 1; x, y, z \geq 0\}$ be the standard 2-simplex.

Definition 1.2 *A foliated 2-complex $(\Sigma, \mathcal{F}, \mu)$ (or $(\Sigma, \mathcal{F})$ or Σ for short) is*

- *a connected finite simplicial complex Σ of dimension lesser than or equal to 2 (maybe a point or a graph),*
- *a foliation $\mathcal{F}$, i.e. a decomposition of Σ into disjoint subsets, called leaves (whose embedding in Σ may fail to be proper), such that every edge is either contained in a leaf or transverse to $\mathcal{F}$, and such that the foliation induced on any 2-simplex is topologically conjugate to one of the following two types: $\Delta_2 \cap \{z = C\}$ or $\Delta_2 \cap \{x - y = C\}$ (see Figure 1),*
- *a transverse measure μ: on every transverse edge, there is a positive regular Borel measure, with finite total mass, absolutely continuous with respect to Lebesgue measure, with full support (every open subset has positive measure); this collection should be invariant under holonomy along the leaves.*

A leaf is called *singular* if it contains a vertex of Σ, *regular* otherwise.

Lemma 1.3 *Let $(\Sigma, \mathcal{F})$ be a foliated 2-complex. Let A be a finite subset of the 1-skeleton. There is a subdivision Σ' of Σ such that each $a \in A$ is a vertex of Σ', and Σ' (equipped with the induced measured foliation) is a foliated 2-complex.*

Proof.

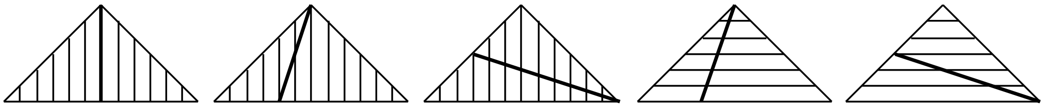

Figure 2 : Subdivision of a foliated 2-simplex.

By induction on the cardinality of A. If A is a point, join it to the opposite vertex in each 2-simplex containing it, and check the foliation on the 2-simplices thus created (see Figure 2). $\square$

Corollary 1.4 *In the second condition of Definition 0.1, it suffices to require that restrictions of* π *to transverse edges of* $\overline{\Sigma}$ *be morphisms.* $\square$

As another application, we observe that a surface endowed with a measured foliation in the sense of [FLP] may be triangulated so as to give a foliated 2-complex.

As E. Rips pointed out, actions of finitely generated groups on $\mathbb{R}$-trees naturally yield foliated 2-complexes.

A *finite* $\mathbb{R}$*-tree* is a compact $\mathbb{R}$-tree which is the convex hull of a finite set. A finite tree may be *degenerate*, i.e. consist of only one point.

Let K be a finite subtree of an $\mathbb{R}$-tree T equipped with an action of a finitely generated group G. Let $g_1, \dots, g_k$ be a finite generating system for G. Assuming that $K \cap g_j^{-1}K$ is nonempty, we consider the restriction $\varphi_j : K \cap g_j^{-1}K \to g_jK \cap K$ of the action of g_j.

More generally, a *system of isometries* is a pair $X = (K, \{\varphi_j\}_{j=1,\cdots,k})$, where K is a finite $\mathbb{R}$-tree and each $\varphi_j : A_j \to B_j$ (called a generator) is an isometry between nonempty finite subtrees of K.

If X is a system of isometries on a finite $\mathbb{R}$-tree K, we define (as in [GLP1], Part 1) a foliated 2-complex $(\Sigma(X), \mathcal{F})$ (or simply Σ) associated to X. Start with the disjoint union of K (foliated by points) and strips $A_j \times [0,1]$ (foliated by $\{*\} \times [0,1]$). We get Σ by glueing the strips $A_j \times [0,1]$ to K, identifying each $(t,0) \in A_j \times \{0\}$ with $t \in A_j \subset K$ and each $(t,1) \in A_j \times \{1\}$ with $\varphi_j(t) \in B_j \subset K$. Using Lemma 1.3, it is easy to subdivide each strip $A_j \times [0,1]$ in order to get the standard models of Figure 1.

We will identify K with its image in Σ. Fixing a base point in K, we identify $\pi_1\Sigma$ with the free group F_k on k generators $g_1, \dots, g_k$, the generator g_j corresponding to the j-th strip.

We will use an important general fact about foliated 2-complexes. Given a foliated 2-complex $(\Sigma, \mathcal{F})$, let $(\Sigma^*, \mathcal{F}^*)$ be the restriction to the complement of the set of vertices, and let E be the union of leaves of $\mathcal{F}^*$ that are closed in Σ^* but not compact. The set E contains no regular leaves, so that its complement in Σ^* is an open set with finitely many components.

Proposition 1.5 *Let U be a component of $\Sigma^* \setminus E$. Either every leaf contained in U is compact, or every leaf contained in U is dense in U.*

Proposition 1.5 relies upon work of Imanishi about foliations [Ima]. The case of foliated 2-complexes may be deduced from Part 3 of [GLP1]: we may associate to a foliated 2-complex a system of isometries X in the sense of [GLP1], on a multi-interval D consisting of the disjoint union of all closed edges of Σ (the generators of X are given by the holonomy within each 2-simplex).

Remark 1.6 We may be more precise in the statement of Proposition 1.5. In the first case, U is a (possibly twisted) family of compact leaves: U (or a 2-sheeted covering of U) is foliated as a product $\Gamma \times \{t\}$, with Γ a finite graph and t in an open subinterval of $\mathbb{R}$ (as a matter of fact, we may get rid of twisted families by subdividing Σ). In the second case U is called a *minimal component*. It may be shown that every leaf of $\mathcal{F}$ meeting the frontier of U meets U.

1.3 From foliated 2-complexes to $\mathbb{R}$-trees

Let Σ be a foliated 2-complex. We consider both the universal covering $\widetilde{\Sigma}$, and a normal covering $\theta : \overline{\Sigma} \to \Sigma$ associated to a normal subgroup $N \subset \pi_1\Sigma$. Let $T(\widetilde{\mathcal{F}})$ and $T(\overline{\mathcal{F}})$ be the spaces associated to the lifted foliations $\widetilde{\mathcal{F}}$ and $\overline{\mathcal{F}}$ as in the introduction.

We recall that $T(\widetilde{\mathcal{F}})$ is an $\mathbb{R}$-tree (see [GS]), and we note that $T(\overline{\mathcal{F}})$ is equal to $\widehat{T(\widetilde{\mathcal{F}})/N}$. Applying Theorem 1.1, we get:

Proposition 1.7 *Let $(\Sigma, \mathcal{F})$ be a foliated 2-complex. Let N be any normal subgroup of $\pi_1\Sigma$ generated by (free homotopy classes of) loops contained in leaves. If $\overline{\mathcal{F}}$ is the pullback of $\mathcal{F}$ to the covering $\overline{\Sigma}$ of Σ corresponding to N, then $T(\overline{\mathcal{F}})$ is an $\mathbb{R}$-tree (with an action of $\pi_1\Sigma/N$).*

Proof. The hypothesis on N guarantees that N is generated by elements that act elliptically on the $\mathbb{R}$-tree $T(\widetilde{\mathcal{F}})$. $\square$

Now suppose $(\Sigma, \mathcal{F})$ is associated to a system of isometries $(K, \{\varphi_j\}_{j=1,\cdots,k})$ as above. The group $F_k = \pi_1\Sigma$ acts on the $\mathbb{R}$-tree $T(\widetilde{\mathcal{F}})$. Let $i : K \to T(\widetilde{\mathcal{F}})$ be the map obtained by choosing a continuous lifting of K to $\widetilde{\Sigma}$ and applying the canonical map from $\widetilde{\Sigma}$ to $T(\widetilde{\mathcal{F}})$.

Theorem 1.8 *([GL], Part I)*

1. *The map $i : K \to T(\widetilde{\mathcal{F}})$ is an isometric embedding of K into $T(\widetilde{\mathcal{F}})$, and we identify K and $i(K)$. Every segment of $T(\widetilde{\mathcal{F}})$ is contained in the union of finitely many images of K (under the action of $F_k = \pi_1\Sigma$).*
2. *Let T be an $\mathbb{R}$-tree with an action of F_k. Assume K is isometrically embedded in T, with $g_j x = \varphi_j(x)$ for $j = 1, \ldots, k$ and $x \in A_j \subset K$. Then the embedding of K into T extends uniquely to a morphism from $T(\widetilde{\mathcal{F}})$ to T.*

2 Geometric actions and strong limits

An action of a countable group G on an $\mathbb{R}$-tree T is said to be *finitely supported* if there is a finite subtree K whose images under G cover T. Note that a minimal action of a finitely generated group is finitely supported (see [AB][Paul]), and that a tree with a finitely supported action is the union of an increasing sequence of finite trees.

Proposition 2.1 *A geometric action of a finitely generated group on an $\mathbb{R}$-tree is finitely supported.*

Proof. For every edge of Σ (as in Definition 0.1), fix a lift in $\overline{\Sigma}$. These lifts inject isometrically into $T = T(\overline{\mathcal{F}})$, so that there is a finite subtree K containing them all. □

The following result relies on a construction suggested by E. Rips.

Theorem 2.2 *Every finitely supported action of a finitely generated group G on an $\mathbb{R}$-tree T is a strong limit of geometric actions.*

Proof. Let $g_1, \cdots, g_k$ be a system of generators of G and $\rho : F_k \to G$ the corresponding epimorphism. We write T as an increasing union of finite subtrees K_n that meet every orbit, with $g_j^{-1}K_n \cap K_n$ nonempty for $j = 1, \cdots, k$. As in Section 1.2 we consider the system of isometries X_n of K_n defined by $\varphi_j^n : K_n \cap g_j^{-1}K_n \to g_jK_n \cap K_n$ and the associated foliated 2-complex $(\Sigma_n, \mathcal{F}_n)$. Taking basepoints in K_n, recall that we identify all groups $\pi_1\Sigma_n$ with the free group F_k on generators $g_1, \ldots, g_k$ (these identifications are compatible with the natural inclusions $\Sigma_n \to \Sigma_p$ for $n < p$).

Let N be the kernel of $\rho : F_k \to G$. Let $r = g_{i_1}^{\varepsilon_1} \ldots g_{i_r}^{\varepsilon_r}$ be any element of N. If we choose n such that K_n contains all points $g_{i_j}^{\varepsilon_j} \ldots g_{i_r}^{\varepsilon_r} x$ $(1 \leq j \leq r)$ for some $x \in T$, then there is a loop in Σ_n, contained in a leaf of $\mathcal{F}_n$, whose free homotopy class represents the conjugacy class of r.

Define N_n as the smallest normal subgroup of $\pi_1\Sigma_n$ containing all elements $r \in N$ represented by a loop contained in a leaf of $\mathcal{F}_n$. We let $G_n = \pi_1\Sigma_n/N_n$, and $\rho_n : \pi_1\Sigma_n \to G_n$ the quotient map. We see that N is the increasing union of the subgroups N_n, so that G is the direct limit of the sequence G_n.

Remark 2.3 If G is finitely presented, then for n large enough $N_n = N, G_n = G$, and the subgroup $N \subset \pi_1\Sigma_n$ is generated by free homotopy classes of loops contained in leaves of $\mathcal{F}_n$.

Let $\widetilde{\Sigma_n}$ and $\overline{\Sigma}_n$ be the universal covering and the covering corresponding to N_n respectively. Let $\widetilde{\mathcal{F}_n}$ and $\overline{\mathcal{F}}_n$ be the lifted foliations, and $T(\widetilde{\mathcal{F}_n})$, $T(\overline{\mathcal{F}}_n)$ the corresponding metric spaces. We know that $T(\widetilde{\mathcal{F}_n})$ is an $\mathbb{R}$-tree. By Proposition 1.7 the space $T(\overline{\mathcal{F}}_n) = \widehat{T(\widetilde{\mathcal{F}_n})/N_n}$ is an $\mathbb{R}$-tree T_n, equipped with an action of G_n.

We now proceed to show that the sequence of actions (G_n, T_n) converges strongly towards (G, T).

By Theorem 1.8 the tree $T(\widetilde{\mathcal{F}_n})$ contains K_n as an isometrically embedded subtree, and every segment in $T(\widetilde{\mathcal{F}_n})$ is contained in a finite union of images of K_n. Furthermore Theorem 1.8 yields a morphism from $T(\widetilde{\mathcal{F}_n})$ to T inducing the identity on K_n (we view T as a tree with an action of F_k, via the epimorphism $\rho : F_k \to G$).

This morphism from $T(\widetilde{\mathcal{F}_n})$ to T induces a map f_n from T_n to $T = \widehat{T/N_n}$. Since neither f_n nor the projection p_n from $T(\widetilde{\mathcal{F}_n})$ to T_n increases distances, we deduce first that the restriction of p_n to K_n is an isometric embedding, so that T_n contains K_n as a subtree. In particular the action of G_n on T_n satisfies the second condition of Definition 0.1: it is geometric.

Furthermore f_n is a morphism, since it induces the identity on K_n and every segment in T_n is contained in a finite union of images of K_n. Similarly we get morphisms $f_{np} : T_n \to T_p$ inducing the inclusions $K_n \to K_p$ $(n < p)$. All these morphisms are surjective because K_n meets every orbit (of the action of G_n on T_n, and of the action of G on T).

The last thing to check is that the sequence T_n converges *strongly* towards T. Given $x, y \in T_n$, choose p so that K_p contains both $f_n(x)$ and $f_n(y)$. Then the distance between $f_{np}(x)$ and $f_{np}(y)$ in T_p equals the distance between $f_n(x)$ and $f_n(y)$ in T. $\square$

Remark 2.4 We note the following features of this proof:

1. If G is not finitely presented, let H_n be the kernel of the natural epimorphism $\tau_n : G_n \to G$. If the metric space $\widehat{T_n/H_n}$ is an $\mathbb{R}$-tree, then the action of G on $\widehat{T_n/H_n}$ is geometric (same proof as for the action of G_n on T_n).

2. We will show at the end of Section 3 that edge stabilizers of T_n are mapped injectively into G by τ_n. In particular (G_n, T_n) has trivial edge stabilizers if (G, T) does.

3. If G is not finitely presented, view it as the direct limit of a sequence of finitely presented groups G'_n obtained as quotients of F_k by the first n relations in a presentation of G. We claim that *every finitely supported action* (G, T) *is a strong limit of geometric actions* (G'_n, T'_n): in the construction above, we simply choose K_n so that the first n relations in the presentation of G are represented by loops contained in leaves of $\mathcal{F}_n$, and we let T'_n be the metric space associated to the lift of $\mathcal{F}_n$ to the G'_n-covering of Σ_n.

Theorem 2.5 *A geometric action of a finitely generated group G on an $\mathbb{R}$-tree is not a nontrivial strong limit of actions (in the sense of Section 1.1).*

Proof. Suppose T is an $\mathbb{R}$-tree with a geometric action of G. We thus have a foliated 2-complex $(\Sigma, \mathcal{F})$, with a covering $\theta : \overline{\Sigma} \to \Sigma$, as in Definition 0.1. The foliation defines a pseudodistance d on $\overline{\Sigma}$ and T is the associated metric space.

If e is an edge of $\overline{\Sigma}$ transverse to the foliation, the second condition in Definition 0.1 implies that the restriction of d to e is just the distance defined by the transverse measure on e. If e is contained in a leaf, the restriction of d to e is identically 0. For convenience, we shall say that a map f from a subset A of $\overline{\Sigma}$ to a metric space (X, δ) does not increase distances (resp. is an isometry) if $\delta(f(x), f(y)) \leq d(x, y)$ (resp. $\delta(f(x), f(y)) = d(x, y)$) for $x, y \in A$. In particular the canonical map $\pi : \overline{\Sigma} \to T$ is an isometry!

We first prove the theorem when G is finitely *presented*. We assume that a sequence of trees T_n, each with an action of G, converges strongly to T, and we want to show that the morphism $f_n : T_n \to T$ is an isometry for n large. Since f_n does not increase distances and $\pi : \overline{\Sigma} \to T$ is an isometry, it suffices to construct (for n large) a G-equivariant map $\phi_n : \overline{\Sigma} \to T_n$ which does not increase distances and is a lift of π (i.e. $f_n \circ \phi_n = \pi$).

$$\begin{array}{ccc} & & T_n \\ & {\scriptstyle \phi_n} \nearrow & \downarrow f_n \\ \overline{\Sigma} & \xrightarrow{\pi} & T \end{array}$$

First we construct ϕ_n over the 1-skeleton $\overline{\Sigma}^1$, as an equivariant lift of π such that the restriction of ϕ_n to every edge is an isometry. Note that $\phi_n' = f_{nn'} \circ \phi_n$ has the same properties for $n' > n$.

Fix a maximal tree L in the 1-skeleton of Σ. Let $\overline{L}$ be a component of the preimage $\theta^{-1}(L) \subset \overline{\Sigma}$. By the second condition in Definition 0.1, the image $\pi(\overline{L}) \subset T$ is a finite tree.

Since the convergence of T_n towards T is strong, we can construct an isometric section λ_p of f_p over $\pi(\overline{L})$ for p large, as follows. For each vertex v of $\pi(\overline{L})$, choose an arbitrary $v_1 \in f_1^{-1}(v)$. For p large, the distance between $f_{1p}(v_1)$ and $f_{1p}(v_1')$ in T_p equals the distance between v and v' in T for every couple of vertices v, v', and there is a section λ_p of f_p sending each v to $f_{1p}(v_1)$.

We define ϕ_p on $\overline{L}$ as $\lambda_p \circ \pi$, and we extend it to $\theta^{-1}(L)$ equivariantly.

Now let $\overline{e} = [a, b]$ be an edge of $\overline{\Sigma}$ above an edge $e \not\subset L$. For $n \geq p$ large enough, the distance between $f_{pn} \circ \phi_p(a)$ and $f_{pn} \circ \phi_p(b)$ in T_n equals $d(a, b)$. We may then extend the definition of $\phi_n = f_{pn} \circ \phi_p$ first to $\overline{e}$, and then to $\theta^{-1}(e)$ using equivariance. Repeating this operation finitely many times, we get the desired lift over the 1-skeleton.

Finally, let Δ be a 2-simplex of $\overline{\Sigma}$. The map ϕ_n is already defined on the boundary of Δ, and it is an isometry on each edge. Since T_n is an $\mathbb{R}$-tree, the image of the edges is determined by that of the vertices. It follows that two points of the boundary of Δ that lie on the same leaf of $\overline{\mathcal{F}} \mid_\Delta$ have the same image in T_n. This gives a canonical way to extend ϕ_n to Δ.

Doing this for every 2-simplex, we obtain an equivariant lift ϕ_n of π over the whole of $\overline{\Sigma}$. It has the property that, for any path γ contained in a 2-simplex, the total mass placed on γ by the transverse measure equals the length of $\phi_n(\gamma)$ in T_n.

Since any path may be approximated by a finite union of paths, each contained in a simplex, this implies that ϕ_n does not increase distances, completing the proof when G is finitely presented.

The proof is basically the same in the general case, but now T_n is equipped with an action of a group G_n. Recall that H_n denotes the kernel of the epimorphism $\tau_n : G_n \to G$. The metric space $\widehat{T_n/H_n}$ has an action of G, but we do not know that it is an $\mathbb{R}$-tree. We want to show that it is G-equivariantly isometric to T, and we do so by constructing an equivariant, distance-nonincreasing, lift ϕ_n of π to $\widehat{T_n/H_n}$.

$$\begin{tikzcd}
 & & (T_n, G_n) \arrow[dl] \arrow[dd, "f_n"] \\
 & (\widehat{T_n/H_n}, G) \arrow[dr] & \\
\overline{\Sigma} \arrow[ur, "\phi_n"] \arrow[rr, "\pi"] & & (T, G)
\end{tikzcd}$$

The construction of ϕ_n on the 1-skeleton is the same as before: we did not use the fact that T_n is a tree, but only the existence of sections of f_n over finite subtrees of T. Such sections exist a fortiori for the map from $\widehat{T_n/H_n}$ to T: simply compose a section of f_n with the projection from T_n to $\widehat{T_n/H_n}$.

Let $\Delta \subset \overline{\Sigma}$ be a 2-simplex as above. We have a map δ_n from the boundary of Δ to $\widehat{T_n/H_n}$, and we wish to extend it to Δ. This is done by projecting δ_n to T, lifting it to T_n (increasing n if needed), extending it to a map from Δ to T_n, and finally projecting it onto $\widehat{T_n/H_n}$.

This gives a way to define ϕ_n on every 2-simplex in the same G-orbit as Δ. Since there are finitely many such orbits, we get the required distance-nonincreasing map from $\overline{\Sigma}$ to $\widehat{T_n/H_n}$. $\square$

Let us now combine Theorems 2.2 and 2.5. A minimal action of G is finitely supported. It is a strong limit of geometric actions (of G if G is finitely presented) by Theorem 2.2. If the action is nongeometric, the strong limit is nontrivial (in the sense of Section 1.1): if it were, then $T = \widehat{T_n/H_n}$ would be geometric by Remark 2.4. We have proved:

Theorem 2.6 *Let T be an $\mathbb{R}$-tree with a minimal action of a finitely generated group G.*

1. *If the action is not geometric, then it is a nontrivial strong limit of geometric actions (of G if G is finitely presented).*

2. *If the action is geometric, then it may only be a strong limit in a trivial way.*

Corollary 2.7 *An action is nongeometric if and only if it is a nontrivial strong limit (in the sense of Section 1.1).*

3 Standard forms for geometric actions, and finiteness results

Let T be an $\mathbb{R}$-tree with an action of a group G. Recall that a *branch point* of T is a point $x \in T$ such that $T \setminus \{x\}$ has at least 3 components. If x and x' are branch points, the orbit closures $\overline{Gx}$, $\overline{Gx'}$ are disjoint or equal. As in [Lev2], we say that the action is a *J-action* if there are only finitely many distinct orbit closures of branch points. A minimal J-action of a finitely generated group has the properties that the quotient space, made Hausdorff, is homeomorphic to a finite graph, and the closure of an orbit cannot meet a segment in a Cantor set (see [Lev2]).

Our first finiteness result is :

Theorem 3.1 *A geometric action of a finitely generated group is a J-action.*

Proof. Let $(\Sigma, \mathcal{F})$ be as in Definition 0.1. Let a be a point of $\overline{\Sigma}$ whose image in Σ belongs to a compact regular leaf. By Remark 1.6, this point a has a neighborhood foliated as a product $L \times \{*\}$. This implies that its image in T cannot be a branch point.

Now there exist finitely many leaves $\ell_1, \ldots, \ell_p$ of $\mathcal{F}$ such that every leaf either is compact and regular, or is contained in the closure of some ℓ_i: we simply take every singular leaf and one leaf in each minimal component. Lifting to $\overline{\Sigma}$ and projecting to T, we find finitely many orbits Gx_i ($i = 1, \ldots, p$) such that each branch point of T belongs to some $\overline{Gx_i}$. This shows that the action is a J-action. $\square$

Using a general foliated 2-complex in Definition 0.1 might cause difficulties for applications. But Theorems 2.2 and 2.5 imply that a geometric action may be represented by a very special foliated 2-complex.

Consider the sequence of trees T_n constructed in the proof of Theorem 2.2. If T is geometric, then $T = \widehat{T_n/H_n}$ for n large: in other words, T is the space of leaves (made Hausdorff) of the foliation induced on the covering $\overline{\Sigma}_n$ of Σ_n associated to the epimorphism $\rho : \pi_1\Sigma_n \to G$. Furthermore transverse edges of $\overline{\Sigma}_n$ isometrically embed into T. We get:

Proposition 3.2 *Let G be a finitely generated group acting on an $\mathbb{R}$-tree T. Let $g_1, \ldots, g_k$ be a finite generating system for G. If the action is geometric, then in Definition 0.1 we may take $\Sigma = \Sigma(X)$, where $X = (K, \{\varphi_j\}_{j=1,\cdots,k})$ is the system of isometries associated to a large enough finite subtree $K \subset T$ as in Section 1.2.*

We shall often represent a geometric action in this way. If G is finitely presented, we may also assume that $N = \mathrm{Ker}\rho$ is normally generated by free homotopy classes of loops contained in leaves of Σ (Remark 2.3). We will call such a description of a geometric action a *standard form*.

Remark 3.3 When G is finitely presented, our definition of a geometric action is closely related to that in [BF2]. Indeed, suppose that the action is geometric in standard form. The subgroup N is normally generated by a finite number of loops contained in leaves of Σ. One may then replace Σ by another foliated 2-complex Σ', by glueing discs along these loops. Such a disc is understood to be part of a leaf. The tree T is then the leaf space made Hausdorff for the lift of the foliation to the *universal* covering of Σ'.

Our next goal will be to prove Theorem 0.8.

Let G be a finitely generated group, and T an $\mathbb{R}$-tree with a geometric action of G in standard form. In other words, we assume that $T = T(\overline{\mathcal{F}})$, where $(\Sigma, \mathcal{F})$ is the foliated 2-complex associated to a system of isometries $X = (K, \{\varphi_j : A_j \to B_j\}_{j=1,\cdots,k})$ and $\overline{\mathcal{F}}$ is the lift of $\mathcal{F}$ to a G-covering $\theta : \overline{\Sigma} \to \Sigma$. When needed, we will view K as a subtree of T.

If $A \subset \overline{\Sigma}$, let $\mathrm{sat}(A)$ be the union of all leaves meeting A. Let $\overline{C}_1, \ldots, \overline{C}_m, \ldots$ be the components of $\theta^{-1}(K)$. Since $\overline{\Sigma}$ is connected, we may order them in such a way that $I_m = \overline{C}_m \cap \mathrm{sat}(\overline{C}_1 \cup \ldots \cup \overline{C}_{m-1})$ is nonempty for all m.

Lemma 3.4 *Assume that $\pi_1\overline{\Sigma}$ is generated by free homotopy classes of loops contained in leaves of $\overline{\mathcal{F}}$. Then the sets I_m are connected, and the natural map from the set of leaves of $\overline{\mathcal{F}}$ to T is one-to-one outside of a countable set.*

If the sets I_m are closed, then T is obtained from $\overline{C}_1$ by successively glueing $\overline{C}_m$ isometrically along I_m.

Note that the assumption on $\pi_1\overline{\Sigma}$ involves no loss of generality if G is finitely presented (see Remark 2.3). It is satisfied by the actions (G_n, T_n) constructed in the proof of Theorem 2.2.

Proof. We argue as in Part 3 of [GLP2]. Say that a path $\gamma : [0,1] \to \overline{\Sigma}$ (or by abuse its image) is *taut* if $\gamma^{-1}(L)$ is connected for every leaf L of $\overline{\mathcal{F}}$. Any path contained in some $\overline{C}_m$ is taut. Since $\pi_1\overline{\Sigma}$ is generated by free homotopy classes of loops contained in leaves, the proof of Lemma 3.3 of [GLP2] applies: *if a path γ between $x, y \in \overline{\Sigma}$ is taut, then $|\gamma|_{\overline{\mathcal{F}}} = d(x,y)$; if γ' is another path between x and y, then γ is contained in sat(γ').* The same argument as in [GLP2] (proof of Lemma 3.5) then shows that I_m is connected.

First assume that the sets I_m are closed. Arguing again as in Part 3 of [GLP2], we see that any two points of Σ may be joined by a taut path. The canonical map from $\overline{\Sigma}$ to T maps each $\overline{C}_m$ isometrically, and T may be obtained from $\overline{C}_1$ by successively glueing $\overline{C}_m$ isometrically along I_m.

The key fact in this is that T is precisely the space of leaves of $\overline{\mathcal{F}}$: the natural mapping taking a leaf of $\overline{\mathcal{F}}$ to its image in T is one-to-one.

Without the assumption that the sets I_m are closed, it is conceivable that a leaf containing an endpoint of some I_m be identified with another leaf. But this may affect only countably many leaves. In other words, we have proved that the map taking a leaf of $\overline{\mathcal{F}}$ to its image in T is one-to-one outside of a countable set. $\square$

Lemma 3.5 *If the action (G,T) has trivial edge stabilizers, then I_m is closed.*

Proof. Assume it is not. As in [GLP2] (proof of Lemma 3.5), we find $\eta' > 0$ and isometric maps $\overline{p} : [0,\eta'] \to \overline{C}_m$ and $\overline{q} : [0,\eta'] \to \overline{C}_i$ (for some $i < m$) such that $\overline{p}(t)$ is in the same leaf as $\overline{q}(t)$ if and only if $t > 0$.

Let $p = \theta \circ \overline{p}$ and $q = \theta \circ \overline{q}$. By the "segment closed" property ([GLP2], Theorem 2.3), there exists a word $\varphi_{j_1}^{\varepsilon_1} \dots \varphi_{j_r}^{\varepsilon_r}$ defined on some nondegenerate segment $I = p([0,\eta''])$ and sending it to $J = q([0,\eta''])$ (Theorem 2.3 of [GLP2] is stated for a system of isometries on a multi-interval; it also applies to a system of isometries on a finite tree K, as may be seen by splitting K into a multi-interval as in [GLP1], Part 2).

Geometrically, we get a "band of leaves" joining I to J via $\varphi_{j_r}^{\varepsilon_r}(I)$, $\varphi_{j_{r-1}}^{\varepsilon_{r-1}}\varphi_{j_r}^{\varepsilon_r}(I)$ etc. Lift this band to $\overline{\Sigma}$ with origin $\overline{p}([0,\eta''])$. It ends at some component $\overline{C}_j = g\overline{C}_i$. Observe that $g \in G$ is nontrivial, since otherwise $\overline{p}(0)$ and $\overline{q}(0)$ would be on the same leaf.

Now choose a point $\overline{q}(t_0)$ belonging to a regular leaf L of $\overline{\mathcal{F}}$ with $0 < t_0 < \eta''$, and a path $\gamma_{t_0} \subset L$ from $\overline{q}(t_0)$ to $\overline{p}(t_0)$. Push this path onto nearby leaves, obtaining a continuous family of paths γ_t contained in leaves L_t, joining $\overline{q}(t)$ to $\overline{p}(t)$ for $|t-t_0| < \varepsilon$. We see that g fixes the image of $\overline{q}([t_0-\varepsilon, t_0+\varepsilon])$ in T, contradicting triviality of edge stabilizers. $\square$

Theorem 3.6 *Let T be an $\mathbb{R}$-tree with a geometric action of a finitely presented group G. If the action has trivial edge stabilizers, there are finitely many orbits of branch points and the action has finite $\mathbb{Z}$-rank.*

Proof. By lemmas 3.5 and 3.4, the tree T may be obtained from $\overline{C}_1$ (which we identify with K) by successively glueing $\overline{C}_m$ isometrically along I_m.

A branch point of T thus belongs to the same orbit as a vertex of K or an endpoint of some I_m. Let $B \subset T$ be the union of orbits containing a vertex of K, or a vertex of some domain A_j ($1 \le j \le k$). Geometrically, B is the image in T of all singular leaves of $\overline{\mathcal{F}}$. The image in T of an endpoint of I_m belongs to B: otherwise one could extend I_m past the point. It follows that every branch point belongs to B, the union of finitely many orbits.

To see that the action has finite $\mathbb{Z}$-rank, we simply note that T is a Λ-tree, where Λ is the subgroup of $\mathbb{R}$ generated by all distances between vertices of K, $A_1, \dots, A_k$, $B_1, \dots, B_k$. $\square$

We now prove:

Theorem 3.7 *If a finitely generated group G acts on an $\mathbb{R}$-tree T, then G is the direct limit of finitely generated groups G_n acting geometrically on $\mathbb{R}$-trees T_n such that the edge stabilizers for T_n are subgroups of edge stabilizers for T. If G is finitely presented, then we may take $G = G_n$.*

Proof. Let us consider the geometric actions (G_n, T_n) constructed in the proof of Theorem 2.2. Recall that G is the direct limit of the sequence G_n. If G is finitely presented, then $G_n = G$ for n big enough (Remark 2.3), and the result is clear since there is a G-equivariant morphism from T_n to T.

If G is not finitely presented, we apply Lemma 3.4 to the action (G_n, T_n) (it does satisfy the assumption that $\pi_1\overline{\Sigma}_n$ is generated by free homotopy classes of loops contained in leaves of $\overline{\mathcal{F}}_n$).

Suppose that $h \in G_n$ fixes a nondegenerate segment in T_n. Then it fixes some $x \in T_n$ which is the image of only one leaf of $\overline{\mathcal{F}}_n$. In geometric terms, this means that in Σ_n there is a loop contained in a leaf of $\mathcal{F}_n$, whose homotopy class is mapped to (a conjugate of) h by the epimorphism $\rho_n : \pi_1\Sigma_n \to G_n$ (in fact there is a whole band of leaves representing h, but we don't need this).

Thus h is the image of a word $r \in F_k$ that is represented by a loop in a leaf of $\mathcal{F}_n$. If h is mapped to the identity in G by the canonical map $\tau_n : G_n \to G$, then r belongs to N, hence to N_n, and h is the identity in G_n. We have proved that $\tau_n : G_n \to G$ is injective on each edge stabilizer. The result follows. □

4 Minimal subtrees of geometric actions

We start with a very general fact.

Proposition 4.1 *Let T be an $\mathbb{R}$-tree with an action of a group G. Let $K \subset T$ be a subtree such that every closed segment may be covered by finitely many images h_iK. The mapping $T_0 \mapsto A = T_0 \cap K$ defines a bijection between the set of nonempty invariant subtrees $T_0 \subset T$, and the set of nonempty subtrees $A \subset K$ such that:*

1. If $a \in A$ and $ga \in K$, then $ga \in A$.

2. The set $\{h \in G \mid hA \cap A \neq \emptyset\}$ generates G.

The inverse mapping is given by $T_0 = \cup_{g\in G} gA$. If K is closed, then T_0 is closed if and only if A is closed.

Proof. Let T_0 be nonempty invariant. Since K meets every orbit, the set $A = T_0 \cap K$ satisfies $T_0 = \cup_{g\in G} gA$. It clearly satisfies condition 1. To show that A satisfies 2, fix $a \in A$ and $g \in G$. Cover the segment $[a, ga]$ by finitely many images g_iK, with $g_1 = 1_G$, $g_p = g$, and $g_iK \cap g_{i+1}K \cap [a, ga] \neq \emptyset$. Setting $h_i = g_i^{-1}g_{i+1}$ we have $g = h_1 \dots h_{p-1}$ with $A \cap h_iA \neq \emptyset$.

Conversely, take $A \subset K$ satisfying 1 and 2. Let $T_0 = \cup_{g\in G} gA$. This is a subtree because of condition 2, and $T_0 \cap K = A$ because of condition 1.

This shows that we have two inverse bijections. There remains to show that T_0 is closed if K and A are. If T_0 is not closed, there exists a segment $[a, b]$ with $[a, b] \cap T_0 = (a, b]$. Changing b if necessary, we may assume that $[a, b]$ is contained in some gK. This means that A is not closed. □

Theorem 4.2 *Let T be an $\mathbb{R}$-tree with a geometric action of a finitely generated group G. If there is no global fixed point, then the minimal subtree $T_{\min}$ is closed in T, and the restriction of the action to $T_{\min}$ is geometric.*

Proof. As mentioned in Proposition 3.2, we may assume that the geometric action is in standard form, that is $T = T(\overline{\mathcal{F}})$, where $(\Sigma, \mathcal{F})$ is the foliated 2-complex associated to a system of isometries $X = (K, \{\varphi_j : A_j \to B_j\}_{j=1,\cdots,k})$ and $\overline{\mathcal{F}}$ is the lift of $\mathcal{F}$ to a G-covering $\theta : \overline{\Sigma} \to \Sigma$. We view K both as a subcomplex of Σ and a closed subtree of T. We may further assume $K \cap g_j K \cap T_{\min} \neq \emptyset$ $(j = 1, \ldots, k)$.

Let $A_{\min} = T_{\min} \cap K$. We want to show that it is closed. A subset of Σ is *$\mathcal{F}$-saturated* if it contains the leaf through any of its points.

Since $A_{\min}$ satisfies condition 1 of Proposition 4.1, it is the intersection with K of an $\mathcal{F}$-saturated subset $\Sigma_{\min} \subset \Sigma$. Let U be a minimal component of $\mathcal{F}^*$ (as defined in Section 1.2). By Remark 1.6 every leaf of $\mathcal{F}$ meeting $\overline{U}$ meets U. It follows that either $\Sigma_{\min}$ is disjoint from $\overline{U}$, or $\Sigma_{\min}$ contains $\overline{U}$.

If $a \in \overline{A_{\min}} \setminus A_{\min}$, there is then a segment $(a, b] \subset A_{\min}$ meeting only compact leaves of $\mathcal{F}$. The image of $(a, b]$ in T is a segment $(x, y]$ containing no branch point, with $[x, y] \cap T_{\min} = (x, y]$. This contradicts the minimality of $T_{\min}$: the union of all orbits contained in $T_{\min}$ and disjoint from $(x, y]$ is a proper invariant subtree. We have thus proved that $T_{\min}$ is closed.

Now consider $\Sigma_{\min} \subset \Sigma$. It is compact and $\mathcal{F}$-saturated. The hypothesis $K \cap g_j K \cap T_{\min} \neq \emptyset$ implies that the inclusion $\Sigma_{\min} \to \Sigma$ is a homotopy equivalence. (Recall indeed that Σ collapses to a bouquet of circles, K collapsing to the vertex and each strip $A_j \times [0, 1]$ to one of the circles. Since $K \cap g_j K \cap T_{\min}$ is connected and nonempty, so does $\Sigma_{\min}$.) In fact there is a natural strong deformation retraction from Σ to $\Sigma_{\min}$. This retraction does not increase distances, in the following sense: if a path $\gamma \subset \Sigma$ joins two points of $\Sigma_{\min}$, it is homotopic to a path $\gamma' \subset \Sigma_{\min}$ such that $|\gamma'|_{\mathcal{F}} \leq |\gamma|_{\mathcal{F}}$. This implies that $T_{\min}$ is geometric: it is isometric to $T(\overline{\mathcal{F}_{\min}})$, where $\overline{\mathcal{F}_{\min}}$ is the pullback by θ of the restriction of $\mathcal{F}$ to $\Sigma_{\min}$. $\square$

5 Geometric simplicial actions

Let us first give an alternative construction of the Bass-Serre tree (see [Ser, SW]) of a graph of groups.

Suppose G is the fundamental group of a graph of groups $\mathcal{G}$, with underlying graph Γ, edge groups G_e, and vertex groups G_v (we consider edges as unoriented). For every edge group G_e, let S_e be a set of generators and F_e be the free group on S_e. For every vertex v, choose a set S_v of generators of G_v which contains the disjoint union of the (images by the monomorphisms of $\mathcal{G}$ of the) sets S_e, for e adjacent to v (it should contain two copies of S_e if e is a loop). Let F_v be the free group on S_v. If e is adjacent to v, the monomorphism $G_e \to G_v$ fits in a natural commutative diagram

$$\begin{array}{ccc} F_e & \to & F_v \\ \downarrow & & \downarrow \\ G_e & \to & G_v \end{array}$$

For every vertex v (resp. edge e), let B_v (resp. B_e) be a bouquet of circles indexed by S_v (resp. S_e). Let $(\Sigma = \Sigma_{\mathcal{G}}, \mathcal{F})$ be the foliated 2-complex obtained as follows (after a suitable triangulation, see Lemma 1.3). For every edge e, take $B_e \times [0,1]$ foliated by $B_e \times \{*\}$ If v and w are the endpoints of e, glue $B_e \times \{0\} \subset B_e \times [0,1]$ to the sub-bouquet of circles of B_v corresponding to the generators of S_e, and glue similarly $B_e \times \{1\}$ to B_w. The leaves of $\mathcal{F}$ are the bouquets of circles B_v, and the sets $B_e \times \{t\}$ for $t \in (0,1)$.

The group $\pi_1\Sigma$ is the fundamental group of the graph of groups with underlying graph Γ, edge groups F_e, and vertex groups F_v. The epimorphisms $\rho_e : F_e \to G_e$ and $\rho_v : F_v \to G_v$ induce an epimorphism $\rho : \pi_1\Sigma \to G$.

Let $\overline{\Sigma}$ be the covering of Σ associated to ρ, and $\overline{\mathcal{F}}$ the lifted foliation. The space $T(\overline{\mathcal{F}})$ is endowed with the natural action of G.

Proposition 5.1 *$T(\overline{\mathcal{F}})$ is the Bass-Serre tree of $\mathcal{G}$.*

Proof. The kernel of ρ is normally generated by elements belonging to the kernels of the epimorphisms ρ_v. These elements are represented geometrically by loops in the leaves B_v of $\mathcal{F}$. According to Proposition 1.7, the space $T(\overline{\mathcal{F}})$ is an $\mathbb{R}$-tree.

Because of the product structure of $\mathcal{F}$ in each piece $B_e \times (0,1)$, this tree is simplicial. The quotient of $T(\overline{\mathcal{F}})$ by the action of G is Γ, the leaf space of $\mathcal{F}$.

The product structure also implies that the pseudodistance $d(x,y)$ between $x, y \in \overline{\Sigma}$ is 0 if and only if x, y belong to the same leaf. This means that, up to conjugacy, stabilizers of edges (resp. vertices) of $T(\overline{\mathcal{F}})$ are images in G of the fundamental groups of the leaves B_e (resp. B_v) of $\mathcal{F}$, that is G_e (resp. G_v).

The result now follows by the characteristic property of the Bass-Serre tree. □

In what follows, we consider a finitely generated group G acting on a tree T. We assume that the action is simplicial (T is a simplicial tree, with every edge of length one) and minimal (finitely supported would suffice). Let $\mathcal{G}$ be the associated graph of groups.

Theorem 5.2 *The minimal simplicial action (T, G) is geometric if and only if every edge group is finitely generated.*

Remark 5.3 (1) Since $G = \pi_1\mathcal{G}$ is finitely generated, finite generation of edge groups implies finite generation of vertex groups (see [Coh] p. 218).

(2) A finitely generated group may be the fundamental group of a graph of groups with non finitely generated edge and vertex groups. For instance, the free group of rank 2 on generators a_0, t has the presentation $< t, \{a_i\}_{i\in\mathbb{N}} \,;\, a_{i+1} = t^{-1}a_i t >$ corresponding to an HNN extension of the free group with countable basis $< \{a_i\}_{i\in\mathbb{N}} >$.

Proof. First assume that the action is geometric, associated to $(\Sigma, \rho, \mathcal{F})$ as in Definition 0.1. Since orbits in T are discrete and transverse edges of $\overline{\Sigma}$ isometrically inject into T, all leaves of $\mathcal{F}$ are compact.

Using Remark 1.6, we see that the pseudodistance $d(x, y)$ between any two points $x, y \in \overline{\Sigma}$ is realized by a path. In particular $d(x, y) = 0$ if and only if x, y are in the same leaf of $\overline{\mathcal{F}}$. Thus the stabilizer of any point in T is finitely generated: it is obtained by mapping the fundamental group of some leaf of $\mathcal{F}$ into $\pi_1\Sigma$, and then into G by ρ. Since the stabilizer of an edge is the stabilizer of any of its points (except maybe its midpoint), edge groups also are finitely generated.

Conversely, assume finite generation of edge groups (hence also of vertex groups by Remark 5.3). Then the 2-complex $\Sigma_{\mathcal{G}}$ constructed at the beginning of the section may be taken to be a finite complex. This implies that the corresponding action of G is geometric.

We sketch another proof, based on Theorem 2.6. For simplicity we assume that G is finitely presented. We suppose that T is a strong limit of $\mathbb{R}$-trees T_n, each equipped with an action of G. We show that this limit is trivial by constructing an equivariant section $\varphi : T \to T_n$ for n large. Arguments are similar to those used in the proof of Theorem 2.5. Finite generation of vertex groups allows us to construct φ over the 0-skeleton, and finite generation of edge groups allows us to extend φ to the whole of T. □

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