On the closedness of approximation spectra

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Abstract

Generalizing Cusick's theorem on the closedness of the classical Lagrange spectrum for the approximation of real numbers by rational ones, we prove that various approximation spectra are closed, using penetration properties of the geodesic flow in cusp neighbourhoods in negatively curved manifolds and a result of Maucourant [Mau]. [1]

The *approximation constant* of an irrational real number x by rational numbers is

$$c(x) = \liminf_{p,q\in\mathbb{Z},\, q\to+\infty} |q|^2 \left|x - \frac{p}{q}\right|$$

(though some references consider $c(x)^{-1}$ or even $(2c(x))^{-1}$). The *Lagrange spectrum* $\mathrm{Sp}_{\mathbb{Q}}$ is the subset of $\mathbb{R}$ consisting of the $c(x)$ for $x \in \mathbb{R}-\mathbb{Q}$. Many properties of $\mathrm{Sp}_{\mathbb{Q}}$ are known (see for instance [CF]), and have been known for a very long time, through the works of Korkine-Zolotareff, Hurwitz, Markoff, Hall, ... The fact that $\mathrm{Sp}_{\mathbb{Q}}$ is a closed subset of $\mathbb{R}$ was proved by Cusick only relatively recently, in 1975.

For many examples of a locally compact ring $\widehat{K}$ containing a dense countable subfield K, a linear algebraic group $\underline{G}$ defined over K and a left invariant distance d on the locally compact group $\underline{G}(\widehat{K})$, one can define a similar approximation spectrum of elements of $\underline{G}(\widehat{K})$ by elements of $\underline{G}(K)$. In this note, we prove that many such approximation spectra also are closed subsets of $\mathbb{R}$, in particular for

- the approximation of complex numbers by elements in imaginary quadratic number fields,
- the approximation of real Hamilton quaternions by rational ones, and
- the approximation of elements of a real Heisenberg group by rational points.

In each of the above cases, the approximating elements are restricted to certain subclasses of the irrational quadratic or rational elements, as explained below.

These arithmetic results will follow from a theorem in Riemannian geometry, that we will state and prove, after recalling some definitions.

Let M be a complete Riemannian manifold with dimension at least 2 and sectional curvature at most -1, which is geometrically finite (see for instance [Bow] for a general

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reference). Let e be a *cusp* of M, i.e. an asymptotic class of minimizing geodesic rays along which the injectivity radius goes to 0. In particular, when M has finite volume (which is going to be the case in all our arithmetic applications), it is geometrically finite, and moreover, the set of cusps of M is in natural bijection with the (finite) set of ends of M (see loc. cit.). Let $\rho_e : [0, +\infty[\to M$ be a minimizing geodesic ray in M in the class e and let $\beta_e : M \to \mathbb{R}$ be Busemann's *height function* relative to e (see for instance [BH, p. 268]) defined by

$$\beta_e(x) = \lim_{t \to +\infty} t - d(x, \rho_e(t)) \ .$$

Note that if another representative ρ'_e of e is considered, then the new height function β'_e only differs from β_e by an additive constant.

Recall that a (locally) geodesic line $\ell : \mathbb{R} \to M$ *starts from* (resp. *ends at*) e if the map from $]a, +\infty[$ to M, for some a big enough, defined by $t \mapsto \ell(-t)$ (resp. $t \mapsto \ell(t)$), is a minimizing geodesic ray in the class e. A geodesic line ℓ is *positively recurrent* if there exists a compact subset K of M and a sequence $(t_n)_{n \in \mathbb{N}}$ in $[0, +\infty[$ converging to $+\infty$ such that $\ell(t_n) \in K$ for every n. For every positively recurrent geodesic line ℓ starting from e, define the *asymptotic height* of ℓ (with respect to e) to be $\limsup_{t \to +\infty} \beta_e(\ell(t))$. Define (see for instance [HP1, HP2]) the *asymptotic height spectrum* of (M, e) as the set of asymptotic heights of positively recurrent geodesic lines starting from e. If C is a compact subset of M, define the *height* of C (with respect to e) as

$$\mathrm{ht}_e(C) = \max\{\beta_e(x) \ : \ x \in C\} \ .$$

Note that the asymptotic height of a geodesic line, the height spectrum of (M, e) and the height of a closed geodesic depend on the choice of ρ_e only up to a uniform additive constant. There is a canonical normalization, by asking that $\rho_e(0)$ belongs to the boundary of the maximal Margulis neighbourhood of e, see [BK, HP1] for instance. In some cases, however, this is not an optimal choice in terms of computation lengths.

Theorem 2 answers a question raised during the work of the second author with S. Hersonsky, see for instance page 233 in [PP1]. In its proof, we will use the following result of F. Maucourant [Mau, Theo. 2 (2)], whose main tool is Anosov's closing lemma (and builds on a partial result of [HP1]). We denote the unit tangent bundle of a Riemannian manifold M by $\pi : T^1M \to M$. A unit tangent vector is *periodic* if it is tangent to a closed geodesic.

Theorem 1 *Let V be a complete Riemannian manifold with sectional curvature at most -1, let $(\phi^t)_{t \in \mathbb{R}}$ be its geodesic flow, and let J_0 be the subset of T^1V of periodic unit tangent vectors. If $f : T^1V \to \mathbb{R}$ is a proper continuous map, then*

$$\mathbb{R} \cap \{ \limsup_{t \to +\infty} f(\phi^t v) \ : \ v \in T^1V\} = \overline{\{ \max_{t \in \mathbb{R}} f(\phi^t v) \ : \ v \in J_0\}} \ . \quad \square$$

Here is the main geometric result of this note:

Theorem 2 *The asymptotic height spectrum of (M, e) is closed. It is equal to the closure of the heights of the closed geodesics in M.*

Notice that by [HP1, Theo. 3.2], the asymptotic height spectrum has a positive lower bound. But by [HP1, Prop. 4.1], its minimum is not always attained by the height of a closed geodesic. Hence, its minimum is not always isolated.

Proof. Busemann's height function β_e is continuous (in fact 1-Lipschitz). Let us prove that it is proper. Let $\widetilde{M} \to M$ be a universal Riemannian cover of M, with covering group Γ. Let $\partial_\infty \widetilde{M}$ be the sphere at infinity of $\widetilde{M}$, and endow $\widetilde{M} \cup \partial_\infty \widetilde{M}$ with the cone topology (see for instance [BH, p. 263]. Let $\Lambda\Gamma$ be the limit set of Γ and let $\Omega\Gamma = \partial_\infty \widetilde{M} - \Lambda\Gamma$ be its domain of discontinuity (see for instance [Bow] and notice that $\Omega\Gamma$ is empty if M has finite volume): The group Γ acts properly discontinuously on $\widetilde{M} \cup \Omega\Gamma$. The set of ends of the quotient manifold with boundary $M^* = \Gamma\backslash(\widetilde{M} \cup \Omega\Gamma)$ is in one-to-one bijection with the set of cusps of M, by the map which to a minimizing geodesic ray defining a cusp associates the end it converges to (see loc. cit.). By construction, $\beta_e(x)$ converges to $+\infty$ when x converges to the end of M^* corresponding to e, and tends to $-\infty$ when x tends to any other end or any boundary point of M^*. This implies that β_e is proper.

Hence, the map $f = \beta_e \circ \pi : T^1M \to \mathbb{R}$ is also continuous and proper. Note that if a geodesic line ℓ in M is not positively recurrent, then $\ell(t)$ converges, as t goes to $+\infty$, either to an end of M^* or to a boundary point of M^* (see loc. cit.). Hence, $\limsup_{t\to+\infty} \beta_e(\ell(t)) = \pm\infty$. Therefore, Theorem 2 follows from Maucourant's Theorem 1. □

For our arithmetic applications, we transform Theorem 2 into a form which is more applicable, using the framework of Diophantine approximation in negatively curved manifolds introduced in [HP1, HP2]. We recall the relevant definitions from these references:

Let ξ_e be the point at infinity of a lift $\widetilde{\rho}_e$ to $\widetilde{M}$ of the previously chosen minimizing geodesic ray ρ_e. Let $\widetilde{\beta}_e$ be Busemann's height function associated to $\widetilde{\rho}_e$. A *horoball* (resp. *horosphere*) centered at ξ_e is the preimage by $\widetilde{\beta}_e$ of $[s, +\infty[$ (resp. $\{s\}$) for some $s \in \mathbb{R}$. A horoball H centered at ξ_e is *precisely invariant* under the action of the stabilizer Γ_∞ of ξ_e in Γ if the interiors of H and γH do not meet for any $\gamma \in \Gamma - \Gamma_\infty$. Let H_e be a precisely invariant horoball centered at ξ_e, which exists by [Bow]. Assume without loss of generality that ρ_e starts in ∂H_e.

Let $\mathscr{R}_e$ be the set of geodesic lines in M starting from and ending at the cusp e. For every r in $\mathscr{R}_e$, define $D(r)$ as the length of the subsegment of r between its first and its last point whose height is 0. Let Lk_e be the set of positively recurrent geodesic lines starting from e.

For every distinct x, y in $\mathrm{Lk}_e \cup \mathscr{R}_e$, define the *cuspidal distance* $d'_e(x, y)$ between x and y as follows: Let $\widetilde{x}$ be a lift of x starting from ξ_e; for every $t > 0$, let $\mathscr{H}_t$ be the horosphere centered at $\widetilde{x}(+\infty)$, at signed distance $-\log 2t$ from ∂H_e along $\widetilde{x}$; then $d'_e(x, y)$ is the minimum, over all lifts $\widetilde{y}$ of y starting from ξ_e, of the greatest lower bound of $t > 0$ such that $\mathscr{H}_t$ meets $\widetilde{y}$ (see [HP1, Sect. 2.1]). The map d'_e is an actual distance in all our arithmetic applications, and it depends on the choice of ρ_e only up to a positive multiplicative constant.

For every x in Lk_e, define the *approximation constant* $c_{M,e}(x)$ of x by elements of $\mathscr{R}_e$ as

$$c_{M,e}(x) = \liminf_{r \in \mathscr{R}_e} \; d'_e(x, r)\, e^{D(r)} \;,$$

and the *Lagrange spectrum* of (M, e) as the subset of $\mathbb{R}$ consisting of the constants $c_{M,e}(x)$ for x in Lk_e.

Corollary 3 *The Lagrange spectrum of (M, e) is closed.*

By the remark following Theorem 2, the Lagrange spectrum is bounded but its maximum is not always isolated.

Proof. By [HP1, Theo. 3.4] (see also [PP1, p. 232]), the map $t \mapsto -\log(2t)$ is a homeomorphism from the Lagrange spectrum onto the asymptotic height spectrum. □

Let us now give some arithmetic applications of this corollary, using the notations introduced in [PP2, PP3].

Let m be a squarefree positive integer, and let $\mathscr{I}$ be a nonzero ideal of an order $\mathscr{O}$ in the ring of integers $\mathscr{O}_{-m}$ of the imaginary quadratic number field $K_{-m} = \mathbb{Q}(i\sqrt{m})$. For $p_1, \dots, p_k \in \mathscr{O}$, let $\langle p_1, \dots, p_k \rangle$ be the ideal of $\mathscr{O}$ generated by $p_1, \dots, p_k$. Let

$$\mathscr{E}_{\mathscr{I}} = \{(p,q) \in \mathscr{O} \times \mathscr{I} \,:\, \langle p,q \rangle = \mathscr{O}\} \,.$$

For every $x \in \mathbb{C} - K_{-m}$, define the *approximation constant* of x by elements of $\mathscr{O}\mathscr{I}^{-1}$ as

$$c_{\mathscr{I}}(x) = \liminf_{(p,q) \in \mathscr{E}_{\mathscr{I}} \,,\, |q| \to \infty} |q|^2 \left| x - \frac{p}{q} \right|$$

(when $\mathscr{O}$ is principal, for instance if $\mathscr{O} = \mathscr{O}_{-m}$ for $m = 1, 2, 3, 7, 11, 19, 43, 67, 163$, the condition $\langle p,q \rangle = \mathscr{O}$ is not needed). Define the *Bianchi-Lagrange spectrum* for the approximation of complex numbers by elements of $\mathscr{O}\mathscr{I}^{-1} \subset K_{-m}$ as the subset $\mathrm{Sp}_{\mathscr{I}}$ of $\mathbb{R}$ consisting of the $c_{\mathscr{I}}(x)$ for $x \in \mathbb{C} - K_{-m}$.

Theorem 4 *The Bianchi-Lagrange spectrum* $\mathrm{Sp}_{\mathscr{I}}$ *is closed.*

When $\mathscr{I} = \mathscr{O} = \mathscr{O}_{-m}$, this result is due to Maucourant [Mau].

Proof. Let $X = \mathbb{H}^3_{\mathbb{R}}$ be the upper halfspace model of the real hyperbolic space of dimension 3 (and sectional curvature -1). The group $\mathrm{SL}_2(\mathbb{C})$ acts isometrically on X, so that its continuously extended action on $\partial_\infty X = \mathbb{C} \cup \{\infty\}$ is the action by homographies. Let Γ be the (discrete) image in $\mathrm{Isom}(X)$ of the preimage of the upper-triangular subgroup by the canonical morphism $\mathrm{SL}_2(\mathscr{O}) \to \mathrm{SL}_2(\mathscr{O}/\mathscr{I})$. Let $\mathscr{P}_\Gamma$ be the set of parabolic fixed points of elements of Γ. Let $M = \Gamma\backslash X$, and let e be its cusp corresponding to the parabolic fixed point ∞ of Γ. Note that M is not necessarily a manifold, as Γ may have torsion. However, Theorem 2 extends to this situation without any changes.

By standard results in arithmetic subgroups (see for instance [BHC, Bor], and the example (1) in [PP3, §6.3]), M has finite volume and we have

$$\mathscr{P}_\Gamma = K_{-m} \cup \{\infty\}$$

so that $\partial_\infty X - \mathscr{P}_\Gamma = \mathbb{C} - K_{-m}$. Let Γ_∞ be the stabilizer in Γ of the point ∞, which preserves the Euclidean distance in $\partial_\infty X - \{\infty\} = \mathbb{C}$.

By [HP1, Lem. 2.7], the map, which to $r \in \mathscr{R}_e$ associates the double class modulo Γ_∞ of an element $\gamma_r \in \Gamma - \Gamma_\infty$ such that $\gamma_r \infty$ is the other point at infinity of a lift of r to X starting from ∞, is a bijection

$$\mathscr{R}_e \to \Gamma_\infty \backslash (\Gamma - \Gamma_\infty) / \Gamma_\infty \,.$$

The map, which to $x \in \partial_\infty X - \{\infty\}$ associates the image ℓ_x in M of the geodesic line starting from ∞ and ending at x, induces a bijection

$$\Gamma_\infty \backslash (\partial_\infty X - \mathscr{P}_\Gamma) \to \mathrm{Lk}_e \,,$$

since ℓ_x is positively recurrent if and only if $x \notin \mathscr{P}_\Gamma$ (see for instance [Bow]). Furthermore (see for instance [EGM, page 314]), the map, which to $(p,q) \in \mathscr{E}_\mathscr{I}$ associates the image $r_{p/q}$ in M of the geodesic line starting from ∞ and ending at p/q, induces a bijection

$$\Gamma_\infty \backslash \left\{ \frac{p}{q} \ : \ (p,q) \in \mathscr{E}_\mathscr{I} \right\} \to \mathscr{R}_e \ .$$

The horoball $\mathscr{H}_\infty$ of points with Euclidean height at least 1 in X is precisely invariant, by Shimizu's Lemma (see also [HP1, §5]). Let ρ_e be the image by the canonical projection $X \to M$ of a geodesic ray from a point of $\partial\mathscr{H}_\infty$ to ∞. We use this minimizing geodesic ray to define Busemann's height function β_e and the cuspidal distance d'_e. Hence, by definition, for every r in $\mathscr{R}_e$, we have

$$D(r) = d_X(\mathscr{H}_\infty, \gamma_r \mathscr{H}_\infty) \ .$$

If q is the lower-left entry of an element γ in $\Gamma - \Gamma_\infty$, then we have

$$d_X(\mathscr{H}_\infty, \gamma\mathscr{H}_\infty) = 2\log|q|$$

by [HP1, Lem. 2.10]. Hence, for every $(p,q) \in \mathscr{E}_\mathscr{I}$, we have $D(r_{p/q}) = 2\log|q|$.

It has been proved in [HP1, §2.1] (for the real hyperbolic space X of any dimension) that, for every x, y in Lk_e, the cuspidal distance $d'_e(x,y)$ is equal to the minimum of the Euclidean distances between the other points at infinity of two lifts to X of x, y starting from ∞.

From the above, it follows that, for every $x \in \partial_\infty X - \mathscr{P}_\Gamma = \mathbb{C} - K_{-m}$,

$$c_{(M,e)}(\ell_x) = \liminf_{(p,q)\in\mathscr{E}_\mathscr{I}, \, |q|\to+\infty} e^{D(r_{p/q})} \, d'_e(\ell_x, r_{p/q}) = c_\mathscr{I}(x) \ .$$

Hence, Theorem 4 follows from Corollary 3. □

Let $\mathscr{I}'$ be a nonzero two-sided ideal in an order $\mathscr{O}'$ of a quaternion algebra $A(\mathbb{Q})$ over $\mathbb{Q}$ ramifying over $\mathbb{R}$, for instance the Hurwitz ring $\mathscr{O}' = \mathbb{Z}[\frac{1}{2}(1+i+j+k), i, j, k]$ in Hamilton's quaternion algebra over $\mathbb{Q}$ with basis $(1, i, j, k)$, and let N be the reduced norm on $A(\mathbb{R}) = A(\mathbb{Q}) \otimes_\mathbb{Q} \mathbb{R}$ (see for instance [Vig]). Consider the set

$$\mathscr{E}_{\mathscr{I}'} = \{(p,q) \ \in \ \mathscr{O}' \times \mathscr{I}' \ : \ \exists \, r, s \ \in \ \mathscr{O}', \ N(qr - qpq^{-1}s) = 1\}.$$

For every $x \in A(\mathbb{R}) - A(\mathbb{Q})$, define the *approximation constant* of x by elements of $\mathscr{O}'\mathscr{I}'^{-1} \subset A(\mathbb{Q})$ as

$$c_{\mathscr{I}'}(x) = \liminf_{(p,q) \, \in \, \mathscr{E}_{\mathscr{I}'} \, , \, N(q)\to\infty} \ N(q) N(x - pq^{-1})^{\frac{1}{2}} \ ,$$

and the *Hamilton-Lagrange spectrum* for the approximation of elements of $A(\mathbb{R})$ by elements of $\mathscr{O}'\mathscr{I}'^{-1} \subset A(\mathbb{Q})$ as the subset $\mathrm{Sp}_{\mathscr{I}'}$ of $\mathbb{R}$ consisting of the $c_{\mathscr{I}'}(x)$ for $x \in A(\mathbb{R}) - A(\mathbb{Q})$.

Theorem 5 *The Hamilton-Lagrange spectrum* $\mathrm{Sp}_{\mathscr{I}'}$ *is closed.*

Proof. The proof is the same as the previous one, with the following changes.

- Let $X = \mathbb{H}^5_{\mathbb{R}}$ be the upper halfspace model of the real hyperbolic space of dimension 5 (and sectional curvature -1). With $\mathbb{H}$ the field of quaternions of Hamilton identified with $\mathbb{R}^4$ by its standard basis $1, i, j, k$, we have $\partial_\infty X = \mathbb{H} \cup \{\infty\}$. The group $\mathrm{SL}_2(\mathbb{H})$, of 2×2 matrices with coefficients in $\mathbb{H}$ and Dieudonné determinant 1, acts isometrically on X, so that its continuously extended action on $\partial_\infty X = \mathbb{C} \cup \{\infty\}$ is, with the obvious particular cases, $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \right) \mapsto (az+b)(cz+d)^{-1}$, see for instance [Kel].

- Let Γ be the image in $\mathrm{Isom}(X)$ of the preimage of the upper-triangular subgroup by the canonical morphism $\mathrm{SL}_2(\mathscr{O}') \to \mathrm{SL}_2(\mathscr{O}'/\mathscr{I}')$.

- Fix an identification of the quaternion algebras $A(\mathbb{R})$ and $\mathbb{H}$. We have $\mathscr{P}_\Gamma = A(\mathbb{Q}) \cup \{\infty\}$ by the example (3) in [PP3, §6.3], so that $\partial_\infty X - \mathscr{P}_\Gamma = \mathbb{H} - A(\mathbb{Q})$.

- By definition of $\mathscr{E}_{\mathscr{I}'}$ and of the Dieudonné déterminant, the map, which to $(p,q) \in \mathscr{E}_{\mathscr{I}'}$ associates the image $r_{pq^{-1}}$ in $M = \Gamma\backslash X$ of the geodesic line starting from ∞ and ending at pq^{-1}, induces a bijection $\Gamma_\infty \backslash \{pq^{-1} \ : \ (p,q) \in \mathscr{E}_{\mathscr{I}'}\} \to \mathscr{R}_e$.

- The fact that the horoball $\mathscr{H}_\infty$ of points with Euclidean height at least 1 in X is precisely invariant is proved in [Kel, page 1091].

- If q is the lower-left entry of an element γ in $\Gamma - \Gamma_\infty$, then we have $d_X(\mathscr{H}_\infty, \gamma\mathscr{H}_\infty) = \log N(q)$ by [PP3, Lem. 6.7], so that $D(r_{pq^{-1}}) = \log N(q)$.

- Recall that the reduced norm N on $\mathbb{H}$ is the square of the Euclidean distance on $\mathbb{H}$ making the basis $(1, i, j, k)$ orthonormal. □

Our last result concerns Diophantine approximation in Heisenberg groups. For every integer $n \geq 2$, consider the Lie group

$$\mathrm{Heis}_{2n-1}(\mathbb{R}) = \{(z, w) \in \mathbb{C} \times \mathbb{C}^{n-1} \ : \ 2\,\mathrm{Re}\, z - |w|^2 = 0\} \ ,$$

where $w' \cdot \overline{w} = \sum_{i=1}^{n-1} w'_i \overline{w}_i$ is the standard Hermitian scalar product on $\mathbb{C}^{n-1}$ and $|w|^2 = w \cdot \overline{w}$, with law

$$(z, w)(z', w') = (z + z' + w' \cdot \overline{w}, w + w') \ .$$

Consider the *modified Cygan distance* d'_{Cyg} on $\mathrm{Heis}_{2n-1}(\mathbb{R})$, which is (uniquely) defined as the distance which is invariant under left translations and satisfies

$$d'_{\mathrm{Cyg}}((z, w), (0, 0)) = \sqrt{|w|^2 + 2\,|z|} \ ,$$

see [PP3, §6.1]. Notice that its induced length distance is equivalent to the Cygan distance and to the Carnot-Carathéodory distance (see [Gol]).

Let $\mathscr{I}''$ be a nonzero ideal of an order $\mathscr{O}''$ in the ring of integers $\mathscr{O}_{-m}$ of the imaginary quadratic number field $K_{-m} = \mathbb{Q}(i\sqrt{m})$, and let ω be the element of $\mathscr{O}_{-m}$ with $\mathrm{Im}\,\omega > 0$ such that $\mathscr{O}'' = \mathbb{Z} + \omega\mathbb{Z}$. Notice that $\mathrm{Heis}_{2n-1}(\mathbb{R})$ is the set of real points of a $\mathbb{Q}$-form Heis_{2n-1} (depending on m) of the $(2n-1)$-dimensional Heisenberg group, whose set of $\mathbb{Q}$-points is $\mathrm{Heis}_{2n-1}(\mathbb{R}) \cap ({K_{-m}} \times {K_{-m}}^{n-1})$.

If $n = 2$ and $\mathscr{O}'' = \mathscr{O}_{-m}$, then let $\mathscr{E}_{\mathscr{I}''}$ be the set of $(a, \alpha, c) \in \mathscr{O}_{-m} \times \mathscr{I}'' \times \mathscr{I}''$ such that $2\ \mathrm{Re}\ a\overline{c} = |\alpha|^2$ and $\langle p, \alpha, q \rangle = \mathscr{O}_{-m}$. Otherwise, see the fifth point below for the

definition of $\mathscr{E}_{\mathscr{J}''}$. For every $x \in \mathrm{Heis}_{2n-1}(\mathbb{R}) - \mathrm{Heis}_{2n-1}(\mathbb{Q})$, define the *approximation constant* $c_{\mathscr{J}''}(x)$ of x by

$$c_{\mathscr{J}''}(x) = \liminf_{(a,\alpha,c)\in \mathscr{E}_{\mathscr{J}''},\ |c|\to\infty} |c|\ d'_{\mathrm{Cyg}}(x,(a/c,\alpha/c))\ ,$$

and the *Heisenberg-Lagrange spectrum* for the approximation of elements of $\mathrm{Heis}_{2n-1}(\mathbb{R})$ by elements of $\mathrm{Heis}_{2n-1}(\mathbb{Q})$ as the subset $\mathrm{Sp}_{\mathscr{J}''}$ of $\mathbb{R}$ consisting of the $c_{\mathscr{J}''}(x)$ for $x \in \mathrm{Heis}_{2n-1}(\mathbb{R}) - \mathrm{Heis}_{2n-1}(\mathbb{Q})$.

Theorem 6 *The Heisenberg-Lagrange spectrum* $\mathrm{Sp}_{\mathscr{J}''}$ *is closed.*

Proof. The proof is the same as the one of Theorem 4, with the following changes.

- Let $X = \mathbb{H}^n_{\mathbb{C}}$ be the Siegel domain model of the complex hyperbolic n-space, which is the manifold $\{(w_0,w) \in \mathbb{C}\times\mathbb{C}^{n-1}\ :\ 2\,\mathrm{Re}\,w_0 - |w|^2 > 0\}$ with Riemannian metric

  $$ds^2 = \frac{1}{(2\,\mathrm{Re}\,w_0 - |w|^2)^2}\big((dw_0 - dw\cdot\overline{w})(\overline{dw_0} - w\cdot\overline{dw}) + (2\,\mathrm{Re}\,w_0 - |w|^2)\ dw\cdot\overline{dw}\big)$$

  (we normalized the metric so that the maximal sectional curvature is -1). Its boundary at infinity is $\partial_\infty X = \mathrm{Heis}_{2n-1}(\mathbb{R}) \cup \{\infty\}$.

- Using matrices by blocks in the decomposition $\mathbb{C}^{n+1} = \mathbb{C}\times\mathbb{C}^{n-1}\times\mathbb{C}$ with coordinates (z_0,z,z_n), let Q be the matrix of the Hermitian form $-z_0\overline{z_n} - z_n\overline{z_0} + |z|^2$ of signature $(1,n)$, and let SU_Q be the group of complex matrices of determinant 1 preserving this Hermitian form. We identify $X \cup \partial_\infty X$ with its image in the complex projective n-space $\mathbb{P}_n(\mathbb{C})$ by the map (using homogeneous coordinates) $(w_0,w) \mapsto [w_0,w,1]$. The group SU_Q, acting projectively on $\mathbb{P}_n(\mathbb{C})$, then preserves X and acts isometrically on it.

- Let Γ be the image in $\mathrm{Isom}(X)$ of the preimage, by the canonical morphism from $\mathrm{SU}_Q \cap \mathrm{SL}_{n+1}(\mathscr{O}'')$ to $\mathrm{SL}_{n+1}(\mathscr{O}''/\mathscr{I}'')$, of the subgroup of matrices all of whose coefficients in the first column vanish except the first one.

- We have $\mathscr{P}_\Gamma = \mathrm{Heis}_{2n-1}(\mathbb{Q})\cup\{\infty\}$ by the example (2) in [PP3, §6.3], so that $\partial_\infty X - \mathscr{P}_\Gamma = \mathrm{Heis}_{2n-1}(\mathbb{R}) - \mathrm{Heis}_{2n-1}(\mathbb{Q})$.

- Let $\mathscr{E}_{\mathscr{J}''}$ be the set of $(a,\alpha,c) \in \mathscr{O}''\times\mathscr{I}''^{n-1}\times\mathscr{I}''$ such that there exists a matrix of the form $\begin{pmatrix} a & \gamma & b \\ \alpha & A & \beta \\ c & \delta & d \end{pmatrix}$ that belongs to Γ. If $n=2$ and $\mathscr{O}'' = \mathscr{O}_{-m}$, we recover the previous notation, by [PP3, §6.1]. By definition, the map, which to $(a,\alpha,c) \in \mathscr{E}_{\mathscr{J}''}$ associates the image in M of the geodesic line starting from ∞ and ending at $(ac^{-1},\alpha c^{-1})$, induces a bijection $\Gamma_\infty\backslash\big\{(ac^{-1},\alpha c^{-1})\ :\ (a,\alpha,c)\in\mathscr{E}_{\mathscr{J}''}\big\} \to \mathscr{R}_e$.

- Let $\mathscr{H}_\infty$ be the horoball $\{(w_0,w)\in\mathbb{C}\times\mathbb{C}^{n-1}\ :\ 2\,\mathrm{Re}\,w_0 - |w|^2 \geq 4\,\mathrm{Im}\,\omega\}$. The fact that $\mathscr{H}_\infty$ is precisely invariant is proved in [PP3, Lem. 6.4].

- If c is the lower-left entry of an element γ in $\Gamma - \Gamma_\infty$, then we have $d_X(\mathscr{H}_\infty,\gamma\mathscr{H}_\infty) = \log|c| + \log(2\,\mathrm{Im}\,\omega)$ by [PP3, Lem. 6.3].

- By [PP3, Prop. 6.2], the cuspidal distance is equal to a multiple of the modified Cygan distance. Hence, there exists a constant $\kappa > 0$ such that $c_{\mathscr{I}''}(x) = \kappa\, c_{(M,e)}(\ell_x)$ for every $x \in \partial_\infty X - \mathscr{P}_\Gamma$. □

Other applications could be obtained by varying the nonuniform arithmetic lattices in $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{K}})$ with $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ (where $n = 2$ in this last octonion case).

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