Diophantine Approximation in Negatively Curved Manifolds and in the Heisenberg Group

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Abstract This paper is a survey of the work of the authors [21], [2], [22], with a new application to Diophantine approximation in the Heisenberg group. The Heisenberg group, endowed with its Carnot–Carathéodory metric, can be seen as the space at infinity of the complex hyperbolic space (minus one point). The rational approximation on the Heisenberg group can be interpreted and developed using arithmetic subgroups of $SU(n,1)$. In the appendix, the case of hyperbolic surfaces is developed by Jouni Parkkonen and the second author.

1 Introduction

Let M be a geometrically finite pinched negatively curved Riemannian manifold with at least one cusp. Inspired by the theory of Diophantine approximation of a real (or complex) number by rational ones, we developed a theory of approximation of geodesic lines starting from a given cusp by ones returning to it. We proved [21] a Dirichlet-type theorem, expressing its Hurwitz-type constant in terms of the lengths of closed geodesics and their depths outside the cusp neighbourhood. Using the cut locus of the cusp, we defined [21] an explicit approximation sequence for geodesic lines starting from the cusp. We proved [22] a Khintchine–Sullivan type theorem on the Hausdorff measure of the geodesic lines starting from a cusp that are well approximated by cusp returning ones.

We state some of these results here, illustrating them by the well-known examples of the modular curve and the Bianchi orbifolds. We have tried to keep in section 2 the informal presentation of the talk at the March 2000 Cambridge conference. Connections between Diophantine approximation problems and hyperbolic geometry have already been studied a lot (see for instance [1], [9], [13], [14], [17], [18], [23], [31], [36], [34], [37], [39], [41], [26]) and we apologize for the omissions and a very partial presentation of the field. The main point of this contribution is to work in variable curvature, with surprises coming from the fact that the geometry of the cuspidal ends can be much more complicated. We get new applications even in the constant curvature case. The case when M is a complex hyperbolic manifold yields new important results about the Diophantine approximation in the Heisenberg group.

2 The Survey Part

Let M be a smooth complete Riemannian manifold, with sectional curvature $-\kappa^2 \leq K \leq -1$ where $1 \leq \kappa < +\infty$. We will assume in this paper that M has finite volume, and has exactly one cusp. See [21], [2] and [22] for the general case.

Two geodesic rays are *asymptotic* if they remain within bounded Hausdorff distance. A *cusp* e in M is an asymptotic class of minimizing geodesic rays in M along which the injectivity radius converges to 0. We say that a geodesic ray *converges* to e if some subray belongs to the class e.

Since M has finite volume and one cusp, it has exactly one end, whose neighbourhoods are exponentially thin near infinity. More precisely, by Gromov's theorem on almost flat manifolds (see for instance [6]), there exist an infranilmanifold N and an homeomorphism from a neighbourhood of the end to $N \times [0, +\infty[$, such that the diameter of the image in M of $N \times \{t\}$ is less than e^{-t}. (An *infranilmanifold* is just a quotient of a nilpotent Lie group, endowed with a left-invariant metric, by a discrete torsion-free group of isometries.)

Example 2.1. Let $\mathbb{H}^2_{\mathbb{R}}$ be the real hyperbolic plane, seen as the upper half-plane $\{(x,t) \in \mathbb{R}^2 \ : \ t > 0\}$ with the Riemannian metric $ds^2 = \frac{dx^2+dt^2}{t^2}$, which has constant curvature -1. We identify $\mathrm{PSL}_2(\mathbb{R})$ with the (orientation preserving) isometry group of $\mathbb{H}^2_{\mathbb{R}}$ by sending $\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to $z \mapsto \frac{az+b}{cz+d}$. Let $\mathrm{PSL}_2(\mathbb{Z})$ be the *modular subgroup* of $\mathrm{PSL}_2(\mathbb{R})$. Then $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2_{\mathbb{R}}$ is an orbifold, with two orbifold points of order 2 and 3, and one cusp.

Example 2.2. Let d be a square-free positive integer. Let $\mathcal{O}_{-d}$ be the ring of integers in the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$. Let $\mathbb{H}^3_{\mathbb{R}}$ be the real hyperbolic space of dimension 3, seen as the upper half-space $\{(x,y,t) \in \mathbb{R}^3 \ : \ t > 0\}$ with the Riemannian metric $ds^2 = \frac{dx^2+dy^2+dt^2}{t^2}$, which has constant curvature -1. We identify $\mathrm{PSL}_2(\mathbb{C})$ with the (orientation preserving) isometry group of $\mathbb{H}^3_{\mathbb{R}}$. Let $\mathrm{PSL}_2(\mathcal{O}_d)$ be the *Bianchi subgroup* of $\mathrm{PSL}_2(\mathbb{C})$. Then $\mathrm{PSL}_2(\mathcal{O}_d)\backslash\mathbb{H}^3_{\mathbb{R}}$ is an orbifold. Note that this orbifold has one and only one cusp if and only if $d = 1, 2, 3, 7, 11, 19, 43, 67, 163$ (see for instance [38]).

Both examples are orbifolds and not manifolds. But there are only minor modifications, left to the reader, in order for the following arguments to be adapted for them.

The end e of M has a canonical neighbourhood, called the *maximal Margulis neighbourhood*, constructed as follows (see for instance [6]).

We first recall some elementary facts about the universal covering of negatively curved manifolds, see for instance [5]. Let $\widetilde{M}$ be a fixed universal covering of M, with covering group Γ. The *boundary* $\partial\widetilde{M}$ of $\widetilde{M}$ is the set of asymptotic classes of geodesic rays in $\widetilde{M}$. The class of a geodesic ray is called its *point at infinity.* The set $\widetilde{M} \cup \partial\widetilde{M}$ is endowed with the *cone topology.* It is the unique metrizable compact topology such that $\rho(t)$, as t tends to $+\infty$, converges to the point at infinity of ρ, for every geodesic ray ρ in $\widetilde{M}$, and such that, for every point x in $\widetilde{M}$, the map $T^1\widetilde{M} \to \partial\widetilde{M}$, which sends a unit tangent vector to the point at infinity of the geodesic rays it defines, is a homeomorphism.

For ξ in $\partial\widetilde{M}$, the *Busemann function* $\beta_\xi : \widetilde{M} \times \widetilde{M} \to \mathbb{R}$ is defined by

$$\beta_\xi(x,y) = \lim_{t\to\infty} d(x,\xi_t) - d(y,\xi_t)$$

for any geodesic ray $t \mapsto \xi_t$ converging to ξ. The *horospheres* centered at ξ are the level sets of $x \mapsto \beta_\xi(x,y)$ (for any y in $\widetilde{M}$), and the (open) *horoballs* are the (strict) sublevel sets.

Let ξ_0 be a point on the boundary $\partial\widetilde{M}$ of $\widetilde{M}$, which is the endpoint of a lift of a geodesic ray converging to e. Let Γ_0 be the stabilizer of ξ_0 in Γ, which is nontrivial. Its nontrivial elements are parabolic isometries of $\widetilde{M}$, that is, their only fixed point in $\widetilde{M} \cup \partial\widetilde{M}$ is ξ_0, and they preserve each horosphere centered at ξ_0. By the Margulis lemma (see for instance [6]), there is a unique maximal open horoball HB_0 centered at ξ_0, such that $\Gamma_0\backslash HB_0$ embeds in M under the canonical map $\Gamma_0\backslash\widetilde{M} \to M$. This subset of M is called the *maximal Margulis neighbourhood* of the cusp e. For every $\gamma \in \Gamma$, note that γHB_0 meets HB_0 if and only if γ belongs to Γ_0. We denote by H_0 the horosphere centered at ξ_0 which is the boundary of HB_0.

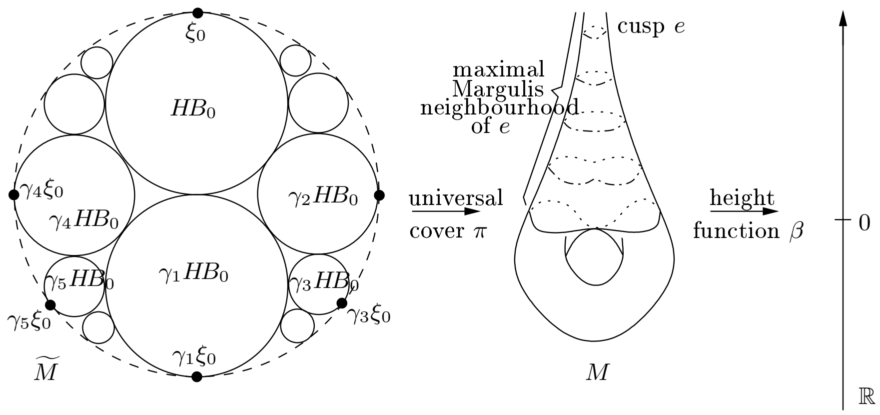

Fig. 1: Precisely invariant horoballs and height function

Example 2.3. The boundary of $\mathbb{H}^n_{\mathbb{R}}$ is the union of the horizontal coordinate hyperplane and the point at infinity ∞ of all vertical geodesic rays. The horospheres in $\mathbb{H}^n_{\mathbb{R}}$ are the horizontal hyperplanes and the Euclidean spheres tangent to the horizontal coordinate hyperplane. We will draw pictures using a projective transformation mapping the upper half space to the unit ball; the horospheres become the Euclidean spheres in the unit ball that are tangent to the unit sphere. For the modular group $\mathrm{PSL}_2(\mathbb{Z})$ or the Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_d)$, we will take $\xi_0 = \infty$. The parabolic subgroup Γ_0 is the subgroup of upper triangular matrices. The maximal open horoball as above for the modular group is $HB_0 = \{(x,t) \in \mathbb{R}^2 \ : \ t > 1\}$ and for the Bianchi groups is $HB_0 = \{(x,y,t) \in \mathbb{R}^3 \ : \ t > 1\}$.

Let ρ be any minimizing geodesic ray in M starting from a point on the boundary of the maximal Margulis neighbourhood of e and converging to e. Define the *height function* on M (with respect to e) as

$$\beta(u) = \lim_{t \to +\infty} t - d(u, \rho(t)) \,.$$

Note that $\beta(u) = \inf_{\gamma \in \Gamma} \beta_{\gamma\xi_0}(\gamma x_0, \widetilde{u})$ for any lift $\widetilde{u}$ in $\widetilde{M}$ of a point u in M and any x_0 in H_0. This map is Lipschitz, piecewise smooth and proper on M, and positive precisely on the maximal Margulis neighbourhood of the cusp. We obviously define the height of a point of M as the value of β at that point. If A is a compact subset of M, we define its *height* as the maximum of the heights of its points. Here is our first invariant of M.

$\boxed{\text{Define } h_M \text{ as the lower bound of the heights of the closed geodesics in } M.}$

For instance, let $\mathcal{M}_{g,1}$ be the space of isometry classes of complete, finite volume hyperbolic metrics on $S_{g,1}$, the compact, connected, oriented surface with genus g and one point removed.

Proposition 2.1 ([21, Proposition 4.2]) *The map* $h : \begin{cases} \mathcal{M}_{g,1} \to \mathbb{R} \\ X \mapsto h_X \end{cases}$ *is continuous and proper.*

In fact, $h(X)$ converges to $-\infty$ as X leaves every compact subset of the moduli space $\mathcal{M}_{g,1}$. Indeed, by Mumford's lemma, if X leaves every compact set, then it develops a shorter and shorter closed geodesic. By Margulis' lemma, such a closed geodesic is simple, and has a tubular neighbourhood whose radius is bigger and bigger, and disjoint from the maximal Margulis neighbourhood of the cusp. Hence the closed geodesic is lower and lower for the height function. We refer to the appendix for speculations about the problem of finding a lowest closed curve, simply stating the following result here.

Theorem 2.2 ([17], [21, Theorem 4.7]) *The map* $h : \mathcal{M}_{1,1} \to \mathbb{R}$ *is* $\mathbb{R}$*-analytic, and reaches its maximum* $\log \frac{\sqrt{5}}{2}$ *exactly on the modular hyperbolic once-punctured torus* $PSL_2(\mathbb{Z})'\backslash\mathbb{H}^2_{\mathbb{R}}$.

The derived subgroup $G' = [G, G]$ of a group G is the smallest normal subgroup H of G such that G/H is abelian. Note that $\mathrm{PSL}_2(\mathbb{Z})'$ has index 6 in $\mathrm{PSL}_2(\mathbb{Z})$, and that $M_{\mathrm{mod}} = \mathrm{PSL}_2(\mathbb{Z})'\backslash\mathbb{H}^2_{\mathbb{R}}$ is the hyperbolic surface obtained by gluing isometrically opposite faces of a regular hyperbolic hexagon with three vertices at infinity and three vertices of angle $\frac{2\pi}{3}$. Note that M_{mod} is also the element of $\mathcal{M}_{1,1}$ which has a maximal order symmetry group 6, and the longest shortest curve (see for instance [35]). The dotted line represents the boundary of the maximal Margulis neighbourhood of the cusp, which is a circle with three self-tangent points. The three continuous lines represent the three closed geodesics whose heights are minimum on M_{mod}, hence are equal to $h_{M_{\mathrm{mod}}}$.

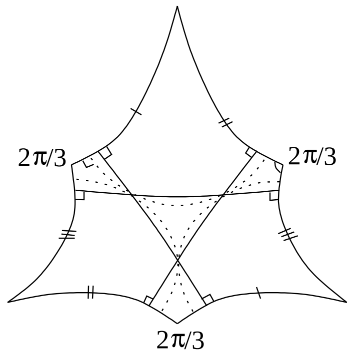

Fig. 2: The modular torus

We now consider the set $Lk(e)$ of oriented geodesic lines starting from the cusp e. (When we compactify M by adding a point at its end, we get a CW-complex, and $Lk(e)$ can be identified with the link of the added point.) By lifting to the universal cover, this set $Lk(e)$ identifies with the set of Γ_0-orbits of oriented geodesic lines starting from ξ_0. By taking the unique point of intersection with H_0 (resp. the point at infinity) of a geodesic line starting from ξ_0, the set $Lk(e)$ also identifies with $\Gamma_0\backslash H_0$ (resp. with $\Gamma_0\backslash\partial\widetilde{M} - \{\xi_0\}$).

Since M has finite volume and only one cusp, a geodesic line starting from e either converges to e, or accumulates inside M (see for instance [5]). We say that the geodesic line is *rational* in the first case, and *irrational* otherwise.

The reason for this terminology is the following. Let $\underline{G}$ be a connected semisimple algebraic group defined over $\mathbb{Q}$, with $\mathbb{R}$-rank and $\mathbb{Q}$-rank one. Let $\underline{P}$ be a minimal parabolic subgroup of $\underline{G}$, defined over $\mathbb{Q}$. Let $G = \underline{G}(\mathbb{R})_0$ be the identity component of the $\mathbb{R}$-points of $\underline{G}$, and K be a maximal compact subgroup of G. Let $P = \underline{P}(\mathbb{R}) \cap G$. The symmetric space $X = G/K$, when G is endowed with any left G-invariant and right K-invariant metric, has pinched negative curvature. The boundary of X identifies with the projective variety G/P, which is defined over $\mathbb{Q}$. Let ξ_0 be the trivial coset of G/P. Let $\Gamma = \underline{G}(\mathbb{Z}) \cap G$ be the lattice of integer points in G. The orbifold $\Gamma\backslash X$ has one and only one cusp if and only if there is one and only one Γ-orbit of $\mathbb{Q}$-points in G/P. See for instance [3], [4]. An oriented geodesic line starting from ξ_0 is rational in our sense if and only if its endpoint is a $\mathbb{Q}$-point in G/P.

For instance, for the orbifold $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2_{\mathbb{R}}$, a geodesic line starting from the cusp is rational (irrational) if and only if its lifts from $\xi_0 = \infty$ (in the upper half-plane model) end at rational (irrational) points on the real line.

In order to be able to effectively approximate irrational lines by rational ones, we need some complexity function on the rational lines and some uniform structure on the set $Lk(e)$ of geodesic lines starting from the cusp.

We define the *depth* $D(r)$ of a rational line r as the length of the subsegment of r between the first and last intersection point of r with the level set $\beta^{-1}(t)$, minus $2t$, for any $t > 0$. We proved in [21] that the set of depths of rational lines is a discrete subset of $\mathbb{R}$. In particular, $D(r)$ converges to $+\infty$ as r goes out of every finite set of rational lines.

Since $Lk(e)$ identifies with $\Gamma_0\backslash(\partial\widetilde{M} - \{\xi_0\})$, it is sufficient to define a natural uniform structure on $\partial\widetilde{M} - \{\xi_0\}$. Let $a, b \in \partial\widetilde{M}$. Their Gromov product with respect to a base point x in $\widetilde{M}$ is defined as the time when the geodesic rays starting from x and converging to a, b really start to diverge one from the other, i.e.

$$(a, b)_x = \lim_{t\to+\infty} \frac{1}{2}\big(d(x, a_t) + d(x, b_t) - d(a_t, b_t)\big)$$

independently of the geodesic rays $a_., b_. : [0, +\infty[\to \widetilde{M}$ converging to a, b. The *visual distance* d_x seen from x on $\partial\widetilde{M}$ is then defined by

$$d_x(a, b) = \begin{cases} 0 & \text{if } a = b \\ e^{-(a,b)_x} & \text{otherwise}\,. \end{cases}$$

These visual distances are natural, in the sense that every isometry γ of $\widetilde{M}$ extends to an homeomorphism of $\partial\widetilde{M}$ which is an isometry between d_x and $d_{\gamma x}$. The *Hamenstädt distance* d_{ξ_0} on $\partial\widetilde{M} - \{\xi_0\}$, which is invariant under Γ_0, is defined by scaling the visual distance seen from a point converging to ξ_0 (see [20, Appendix]):

$$d_{\xi_0}(a, b) = \lim_{t\to+\infty} e^{-t} d_{r(t)}(a, b),$$

with $a, b \in \partial\widetilde{M} - \{\xi_0\}$, and $r : [0, +\infty[\to \widetilde{M}$ a geodesic ray with origin on H_0 and converging to ξ_0. Note that (see [20, Appendix])

$$d_{\xi_0}(a, b) = \lim_{t\to+\infty} e^{-\frac{1}{2}(d(H_0, a_t) + d(H_0, b_t) - d(a_t, b_t))}$$

where $a_., b_. : \mathbb{R} \to \widetilde{M}$ are the geodesic lines starting from ξ_0, passing at time $t = 0$ through H_0 and converging to a, b. Taking the quotient by Γ_0, and identifying $\Gamma_0\backslash(\partial\widetilde{M} - \{\xi_0\})$ with $Lk(e)$, we get a distance d_e on $Lk(e)$, that we also call the *Hamenstädt distance* on $Lk(e)$.

In order to interpret constants correctly, we modify a bit this distance. For $a, b \in \partial\widetilde{M} - \{\xi_0\}$, let L_a, L_b be the oriented geodesic lines from ξ_0 to a, b. For $r > 0$, let H_r be the horosphere centered at a, meeting L_a at a point at signed distance $-\log 2r$ of $H_0 \cap L_a$ along L_a. Define $d'_{\xi_0}(a, b)$ to be the infimum of all $r > 0$ such that H_r meets L_b. Note that d'_{ξ_0} is invariant under Γ_0. Taking the quotient by Γ_0, and identifying $\Gamma_0 \backslash (\partial\widetilde{M} - \{\xi_0\})$ with $Lk(e)$, we get a map d'_e on $Lk(e) \times Lk(e)$, that we call the *cuspidal distance* on $Lk(e)$.

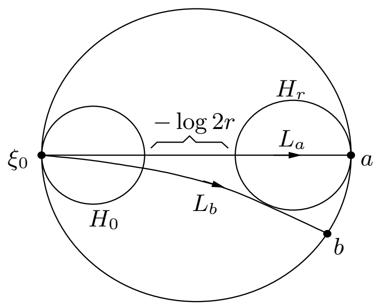

Fig. 3: The cuspidal distance

In constant curvature, both the Hamenstädt distance and the cuspidal distance coincide with the induced Riemannian metric on $\Gamma_0 \backslash H_0$, which is flat. In general, and contrarily to the Hamenstädt distance, the cuspidal distance might not be a distance. But since there exists a constant $c > 0$ such that $\frac{1}{c} d_e \leq d'_e \leq c d_e$ (see [21]), this doesn't make a big difference.

Example 2.4. We summarize in the following table the values of the corresponding notions that have just been introduced in our two test cases.

Finite volume one-cusped orbifold	$\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2_{\mathbb{R}}$	$\mathrm{PSL}_2(\mathcal{O}_d)\backslash\mathbb{H}^3_{\mathbb{R}}$ $d = 1, 2, 3, 7, 11, 19, 49, 67, 163$
Space $Lk(e)$ of geodesics starting from the cusp	$\mathbb{R}/\mathbb{Z}$	$\mathbb{C}/\mathcal{O}_d$
subset of rational lines	$\mathbb{Q}/\mathbb{Z}$	$\mathbb{Q}(\sqrt{-d})/\mathcal{O}_d$
depth of $\frac{p}{q}$	$\log \lvert q\rvert^2$	$\log \lvert q\rvert^2$
cuspidal distance	Euclidean distance	Euclidean distance
Hurwitz constant (see below)	$\frac{1}{\sqrt{5}}$	$\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[8]{8}}, \frac{2}{\sqrt{5}}, 1, ?, ?, ?$

Our first approximation result is the following. For $M = \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2_{\mathbb{R}}$, it is simply the Dirichlet theorem on Diophantine approximation of a real number by a rational one. See [40] and [41] for the extension to the case of constant curvature.

Theorem 2.3 ([21, Theorem 1.1]) *There exists a constant $c > 0$ such that for every irrational line α, there exist infinitely many rational lines r*

such that

$$d'_e(r,\alpha) \le c e^{-D(r)}\,.$$

This result allows us to define our second invariant for M.

Define the *Hurwitz constant* K_M of M as the lower bound of all such constants c.

The terminology again comes from the fact that the Hurwitz constant of the modular curve $M = \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2_{\mathbb{R}}$ is the classical Hurwitz constant for the Diophantine approximation for a real number by rational ones. Few values of the Hurwitz constant are known. For instance, it seems that the Hurwitz constant for the Bianchi groups for $d = 49, 67, 163$ are still unknown, as well as the Hurwitz constants at the cusp corresponding to ∞ for $\Gamma(N)\backslash\mathbb{H}^2_{\mathbb{R}}$ (which has more than one cusp if $N \neq 1$) where $\Gamma(N) = \mathrm{Ker}(\mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}/N\mathbb{Z}))$ is the N-th principal congruence subgroup. Note that the behaviour of the Hurwitz constant under covering is unknown.

There is an exact relationship between our two invariants. This formula is due to H. Cohn [8] for $M = \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2_{\mathbb{R}}$.

Theorem 2.4 ([21, Theorem 1.2]) $\boxed{\frac{1}{2K_M} = e^{h_M}.}$

Let us give an idea of the proof (see [21, Theorem 1.2] for the complete one). First, by definition of the cuspidal distance, an irrational line α which has the biggest approximation constant c (as in the statement of Theorem 2.3) is precisely one which asymptotically stays as far as possible from the cusp.

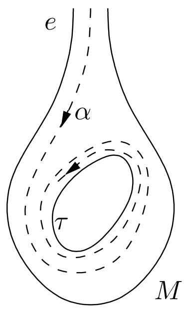

Fig. 4: Spiraling geodesic

Now, if τ is a closed geodesic whose height is almost minimal, let α be a geodesic line starting from the cusp e that spirals around τ. Then α is irrational and, after some time, is not much higher than τ.

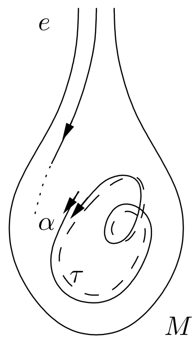

Fig. 5: Accumulating geodesic

Conversely, let α be an irrational geodesic line whose asymptotic height $\limsup_{t\to+\infty}\beta(\alpha(t))$ is almost the lowest possible. In particular, after some time, α goes not much higher than the lower bound. After that time, α accumulates both in space and direction, that is in T^1M. By Anosov's closing lemma, there will be a closed geodesic τ contained in the ϵ-neighbourhood of α. In particular the height of this closed geodesic is also not much higher than the lower bound.

Let us give a corollary of Theorem 2.4 in constant curvature, which appears to be new. Assume that M is a finite volume hyperbolic orbifold of dimension $n=2$ or 3. Identify M with $\Gamma\backslash\mathbb{H}^n_{\mathbb{R}}$, where Γ a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ if $n=2$ and $\mathrm{PSL}_2(\mathbb{C})$ if $n=3$. Write every element γ of Γ as $\gamma=\pm\begin{pmatrix} a(\gamma) & b(\gamma)\\ c(\gamma) & d(\gamma)\end{pmatrix}$. After normalization, assume that ∞ is a parabolic fixed point in Γ, and assume that the quotient by the stabilizer of ∞ in Γ of the open horoball centered at ∞ with equation $t>1$, is the maximal Margulis neighbourhood of the corresponding cusp.

Corollary 2.5 ([21, Theorem 1.4]) *With the above notations,*

$$\frac{1}{K_M}=\inf_{\gamma\in\Gamma\,:\,|\mathrm{tr}\,\gamma|>2}\quad\frac{|\sqrt{\mathrm{tr}^2\gamma-4}|}{\min_{\alpha\in\Gamma}|c(\alpha\gamma\alpha^{-1})|}\,.$$

Since the set of depths of rational lines in M is a discrete subset of $\mathbb{R}$, we may define the *depth counting function* $\mathcal{N}_e(t)$ as the number of rational lines whose depth is less than t. If r is a rational line in M, any lift of r to $\widetilde{M}$ starting from ξ_0 ends at $\gamma\xi_0$ for some γ in Γ. Note that by definition, $D(r)=d(H_0,\gamma H_0)$. The double class of γ in $\Gamma_0\backslash\Gamma/\Gamma_0$ is well defined, and gives a bijection from the set of rational lines in M with the set of nontrivial double cosets $\Gamma_0\backslash(\Gamma-\Gamma_0)/\Gamma_0$. Hence $\mathcal{N}_e(t)$ is the number of double classes $[\gamma]$ in $\Gamma_0\backslash(\Gamma-\Gamma_0)/\Gamma_0$ such that $d(H_0,\gamma H_0)\le t$.

Before giving the asymptotics of the depth counting function, we need some definitions. The *Poincaré series* of a group G of isometries of $\widetilde{M}$ is

$$P_G(x,s)=\sum_{g\,\in\,G}e^{-s\,d(x,\,gx)}$$

for x in $\widetilde{M}$ and s in $\mathbb{R}^+$. This series converges if $s>\delta_G$ and diverges if $s<\delta_G$ for some δ_G in $[0,+\infty]$, called the *critical exponent* of G, which is independent of x.

Let δ_Γ and δ_{Γ_0} be the critical exponents of Γ and Γ_0 respectively. Note that $0 < \delta_{\Gamma_0} \leq \delta_\Gamma < +\infty$ (see for instance [5]). For n in $\mathbb{N}$, let $f_{\Gamma_0}(n)$ be the number of elements γ in Γ_0 such that $d(x, \gamma x) \leq n$, where x is any base point in $\widetilde{M}$.

If f, g are maps from $\mathbb{N}$ to $\mathbb{R}^+$, write $f \asymp g$ if there is a constant $c > 0$ such that $\frac{1}{c} f(n) \leq g(n) \leq c f(n)$ for every n in $\mathbb{N}$.

If M is locally symmetric (for instance with constant curvature -1), then we have a strict inequality $\delta_{\Gamma_0} < \delta_\Gamma$ and $f_{\Gamma_0}(n) \asymp e^{\delta_{\Gamma_0} n}$. These properties may be false in the case of variable curvature (see [12]), due to the possibly much more complicated structure of the cusp neighbourhoods in variable curvature.

Theorem 2.6 ([2, Theorem 1.1]) *If $\delta_{\Gamma_0} < \delta_\Gamma$, then* $\limsup_{n \to +\infty} \dfrac{\log \mathcal{N}_e(t)}{n} = \delta_\Gamma$.

Under the same hypothesis, this asymptotic of the depth counting function has been improved by [9] in constant curvature and very recently by [33] in general to $\mathcal{N}_e(t) \sim c e^{\delta_\Gamma n}$ for some constant $c > 0$ depending on M and the base point x.

We end with a result for the measure theoretical Diophantine approximation in negatively curved manifolds.

Let μ_e be the Hausdorff measure (in the Hausdorff dimension of $Lk(e)$, which is δ_Γ), defined by the Hamenstädt distance d_e on the set $Lk(e)$ of geodesic lines starting from the cusp e. For instance, in constant curvature, μ_e is just the Lebesgue measure on the flat manifold $Lk(e) = \Gamma_0 \backslash H_0$.

Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a slowly varying map, in the sense that there exists a constant $c > 0$ such that, if $|y - x| \leq 1$, then $\psi(y) \leq c\psi(x)$. For instance, one can take $\psi(t) = e^{-\alpha t}$ where $\alpha \geq 0$ is some constant.

Theorem 2.7 ([22, Theorem 1.3]) *Let E_ψ be the set of irrational lines α in $Lk(e)$ for which there exist infinitely many rational lines r with $d_e(\alpha, r) \leq \psi(D(r)) e^{-D(r)}$. If $\delta_{\Gamma_0} < \delta_\Gamma$ and $f_{\Gamma_0}(n) \asymp e^{\delta_{\Gamma_0} n}$, then $\mu_e(E_\psi)$ is zero if and only if the integral $\int_1^{+\infty} \psi(t)^{2(\delta - \delta_0)} dt$ converges.*

This theorem is exactly Khintchine's theorem [25] for the Diophantine approximation of a real number by rational ones, when one takes M to be the modular curve $\mathrm{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}^2_{\mathbb{R}}$. It is due to Sullivan [37] in constant curvature; see also [26].

In the next section, we will apply these theorems to get new results for the Diophantine approximation on the Heisenberg group.

3 Diophantine Approximation in the Heisenberg Group

Let $n \geq 1$ be an integer. Let ω be the standard symplectic form on the affine $2n$-space $\mathbb{A}^{2n}$, defined by

$$\omega(X, X') = \sum_{i=1}^{n} (x_i y_i' - y_i x_i')$$

if $X = (x_1, y_1, \cdots, x_n, y_n)$ and $X' = (x_1', y_1', \cdots, x_n', y_n')$. The Heisenberg group $\underline{H}_{2n+1}$ is the nilpotent connected algebraic group, defined over $\mathbb{Q}$, which is the set of points (X, t) in the affine $(2n+1)$ space $\mathbb{A}^{2n} \times \mathbb{A}^1$, endowed with the multiplication

$$(X, t)(X', t') = (X + X', t + t' + \omega(X, X'))\,.$$

The identity element is $(0, 0)$ and the inverse of (X, t) is $(-X, -t)$. The space $\underline{H}_{2n+1}(\mathbb{R})$ of $\mathbb{R}$-points of $\underline{H}_{2n+1}$ is a simply connected nilpotent Lie group, in which the set $\underline{H}_{2n+1}(\mathbb{Q})$ of $\mathbb{Q}$-points is a dense subgroup, and the set $\underline{H}_{2n+1}(\mathbb{Z})$ a discrete subgroup. Note that the algebraic group of unipotent upper triangular matrices is $\mathbb{Q}$-isomorphic (more generally over fields of characteristic different from 2) to $\underline{H}_3$ by the map $\begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto (x, y, 2t - xy)$.

If r belongs to $\underline{H}_{2n+1}(\mathbb{Q})$, define the *height* $h(r)$ of r as the absolute value of the lowest common multiple of the denominators of the components of r. Note that if $r = (\frac{p_i}{q_i})_{i=1\cdots 2n+1}$ with $(p_i, q_i) = 1$ and $|p_i| \leq |q_i|$, then $h(r) = \mathrm{lcm}(q_1, \cdots, q_{2n+1})$ is the (multiplicative) height of the $\mathbb{Q}$-point r in the projective variety $\underline{H}_{2n+1}$ over $\mathbb{Q}$ (see for instance [28, p. 52]).

The Lie group $\underline{H}_{2n+1}(\mathbb{R})$ has a natural distance, which does not come from a left-invariant Riemannian metric, but is a sub-Riemannian metric, called the *Carnot–Carathéodory distance* (see for instance [16]). This distance d_{CC} is constructed as follows. The tangent space $T_e\underline{H}_{2n+1}(\mathbb{R})$ at the identity contains an hyperplane $V_0 = \{(X, 0) : \ X \in \mathbb{R}^{2n}\}$. Endow V_0 with the standard Euclidean norm. The images of V_0 by the left translations define a (nonintegrable) distribution of Euclidean hyperplanes on $\underline{H}_{2n+1}(\mathbb{R})$. For any two points x, y in $\underline{H}_{2n+1}(\mathbb{R})$, there exists (see for instance [16]) a C^1 path from x to y which is tangent at each point to the hyperplane distribution. The Euclidean structure of the hyperplane distribution defines a length for each such path. The Carnot–Carathéodory distance between x and y is the lower bound of the lengths of C^1 paths from x to y tangent to the hyperplane distribution.

Our first result is analogous to Dirichlet's theorem for the Diophantine approximation of a real number by rational ones. After we wrote the first version of this paper, Stéphane Fischler gave us a short elementary proof of it, that we give in the next section.

Theorem 3.1 *There exists a constant $c = c_n > 0$ such that for every α in $\underline{H}_{2n+1}(\mathbb{R}) - \underline{H}_{2n+1}(\mathbb{Q})$, there exists infinitely many r in $\underline{H}_{2n+1}(\mathbb{Q})$ such that $d_{\mathrm{CC}}(\alpha, r) \leq c/h(r)$.*

Note that a ball of small radius ϵ centered at the origin for d_{CC} looks like a Euclidean ball of radius ϵ in the direction of V_0 and like a Euclidean ball of radius $\sqrt{\epsilon}$ in the direction of $\{0\} \times \mathbb{R}$ (see Sect. 3.3). This last fact explains the absence of a power 2 to $h(r)$ in the above result.

The Hausdorff dimension of the Carnot–Carathéodory distance is $2n+2$ (see for instance [16]), though its topological dimension is only $2n+1$. (See for instance [29] for the definition of the Hausdorff dimension and the Hausdorff measure of a metric space.) Let μ_{CC} be the Hausdorff measure of the Carnot–Carathéodory distance.

Recall that a map $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is slowly varying if there exists a constant $c > 0$ such that, if $|y - x| \leq 1$, then $\frac{1}{c}\psi(x) \leq \psi(y) \leq c\psi(x)$. Our last result is analogous to Khintchine's theorem for the Diophantine approximation of a real number by rational ones. It also follows from the work of Kleinbock–Margulis [26], but our proof is quite different.

Theorem 3.2 *Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a slowly varying map. Let E_ψ be the set of points α in $\underline{H}_{2n+1}(\mathbb{R}) - \underline{H}_{2n+1}(\mathbb{Q})$ such that there exist infinitely many r in $\underline{H}_{2n+1}(\mathbb{Q})$ with $d_{\mathrm{CC}}(\alpha, r) \leq \psi(h(r))/h(r)$. Then $\mu_{\mathrm{CC}}(E_\psi) = 0$ (resp. $\mu_{\mathrm{CC}}({}^c E_\psi) = 0$) if and only if the integral $\int_1^{+\infty} \psi(t)^{2n+2} \frac{dt}{t}$ converges (resp. diverges).*

3.3 The Heisenberg Group

In this subsection, we recall some properties of the Heisenberg group (see for instance [15, Sect. 2.6]), and we restate the above two theorems in a form which is more appropriate for our setting.

From now on, n is an integer at least 2. On $\mathbb{C}^{n-1}$, we use the standard hermitian product $\zeta \cdot \overline{\zeta}' = \sum z_i \overline{z}'_i$ if $\zeta = (z_1, \cdots, z_{n-1})$ and $\zeta' = (z'_1, \cdots, z'_{n-1})$.

Define the (real) *Heisenberg group* $\mathcal{H}_{2n-1}(\mathbb{R})$ as the manifold $\mathbb{C}^{n-1} \times \mathbb{R}$, with coordinates (ζ, v), endowed with the multiplication (with the conventions of Korányi–Reimann [27], Goldman [15], and [19])

$$(\zeta, v)(\zeta', v') = (\zeta + \zeta', v + v' + 2\ \mathrm{Im}\ \zeta \cdot \overline{\zeta}').$$

Note that $\mathrm{Im}\ \zeta \cdot \overline{\zeta}'$ is the standard symplectic form on $\mathbb{C}^{n-1}$. There is a factor 2 appearing here, but $\mathcal{H}_{2n-1}(\mathbb{R})$ is also the set of $\mathbb{R}$-points of a connected algebraic group $\mathcal{H}_{2n-1}$ defined over $\mathbb{Q}$, whose set of $\mathbb{Q}$-points is $\mathcal{H}_{2n-1}(\mathbb{Q}) = \mathbb{Q}[i]^{n-1} \times \mathbb{Q}$, and which is isomorphic over $\mathbb{Q}$ (more generally in characteristic different from 2) to $\underline{H}_{2n-1}$ by the map $(\zeta, v) \mapsto (\zeta, \frac{v}{2})$. This map is an isometry between $\mathcal{H}_{2n-1}(\mathbb{R})$ and $\underline{H}_{2n-1}(\mathbb{R})$ for the Carnot–Carathéodory metrics (defined on $\mathcal{H}_{2n-1}(\mathbb{R})$ in a similar way), and changes the heights (defined on $\mathcal{H}_{2n-1}(\mathbb{R})$ in a similar way) only up to a factor 2.

The *Cygan distance* on $\mathcal{H}_{2n-1}(\mathbb{R})$ is defined as follows (see [10], [15]):

$$d_{\mathrm{Cyg}}((\zeta, v), (\zeta', v')) = \left(|\zeta - \zeta'|^4 + |v - v'|^2\right)^{\frac{1}{4}} .$$

Note that the Cygan distance on $\mathcal{H}_{2n-1}(\mathbb{R})$ is equivalent to the Carnot–Caratheodory distance. Moreover, d_{CC} is the length metric induced by d_{Cyg} (see for instance [15, p. 161]). Let μ_{Cyg} be the Hausdorff measure of the Cygan distance. To prove Theorems 3.1 and 3.2, we only have to prove the following results.

Theorem 3.4 *There exists a constant $c = c_n > 0$ such that for every α in $\mathcal{H}_{2n-1}(\mathbb{R}) - \mathcal{H}_{2n-1}(\mathbb{Q})$, there exist infinitely many r in $\mathcal{H}_{2n-1}(\mathbb{Q})$ such that $d_{\mathrm{Cyg}}(\alpha, r) \leq c/h(r)$.*

Theorem 3.5 *Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a slowly varying map. Let E_ψ be the set of points α in $\mathcal{H}_{2n-1}(\mathbb{R}) - \mathcal{H}_{2n-1}(\mathbb{Q})$ such that there exist infinitely many r in $\mathcal{H}_{2n-1}(\mathbb{Q})$ with $d_{\mathrm{Cyg}}(\alpha, r) \leq \psi(h(r))/h(r)$. Then $\mu_{\mathrm{Cyg}}(E_\psi) = 0$ (resp. $\mu_{\mathrm{Cyg}}({}^cE_\psi) = 0$) if and only if the integral $\int_1^{+\infty} \psi(t)^{2n} \frac{dt}{t}$ converges (resp. diverges).*

The next result gives a (probably not sharp) lower bound on the Hurwitz constant for the Diophantine approximation in the Heisenberg group.

Theorem 3.6 *The lower bound of the constants c as in Theorem 3.4 is at least $\frac{1}{\sqrt[4]{5}}$.*

The next result is a counting result for $\mathbb{Q}$-points modulo $\mathbb{Z}$-points with bounded height.

Theorem 3.7 *Let $\mathcal{N}(t)$ be the number of points r in $\mathcal{H}_{2n-1}(\mathbb{Q})/\mathcal{H}_{2n-1}(\mathbb{Z})$ with $h(r) \leq t$. Then there exists a constant $c > 0$ such that $\mathcal{N}(t) \sim ct^{2n}$ as t tends to $+\infty$.*

All these results will be proven in Sect. 3.19, as an application of the results of Sect. 2. As said in the previous subsection, here is a simple proof of Theorem 3.4, due to Stéphane Fischler.

Let $\alpha = (\zeta_1, \zeta_2, ..., \zeta_{2n-2}, v)$ be an irrational point in $\mathcal{H}_{2n-1}(\mathbb{R})$. First assume that v is irrational. By the classical Dirichlet theorem, there exist infinitely many rational numbers p/q such that $|v - p/q| \leq 1/q^2$. For $i = 1, \ldots, 2n-2$, let p_i be the integer part of $q\zeta_i$. Then the Cygan distance between α and the rational point $(p_1/q, p_2/q, \ldots, p_{2n-2}/q, p/q)$ is at most $(2n-1)/q$. Assume now that $v = \frac{p'}{q'}$ with $(p', q') = 1$ and $q' \geq 1$. By the simultaneous approximation theorem in $\mathbb{R}^{2n-2}$, there exists infinitely many $q \in \mathbb{N} - \{0\}$ for which there exists $p_i \in \mathbb{Z}$, $i = 1, \ldots, 2n-1$, such that $\max_i |\zeta_i - \frac{p_i}{q}| \leq 1/q^{1+\frac{1}{2n-2}}$. Note that $\mathrm{lcm}(q, q') \leq q'q \leq q^{1+\frac{1}{2n-2}}$ if q is big enough. Hence the Cygan distance between α and the rational point $(p_1/q, p_2/q, \ldots, p_{2n-2}/q, p'/q')$ is at most $1/\mathrm{lcm}(q, q')$. This proves the result.

3.8 The complex Hyperbolic Space

In this section, we recall some properties of the complex hyperbolic n-space $\mathbb{H}^n_{\mathbb{C}}$, and in particular the fact that its boundary $\partial\mathbb{H}^n_{\mathbb{C}}$ is the one-point compactification of the Heisenberg group. The main reference is [15], though as in [19], we will use a different Hermitian form, better suited for our purpose.

Let $q = -(z_0\overline{z}_1 + z_1\overline{z}_0) + z \cdot \overline{z}$ be our chosen hermitian form of signature $(n, 1)$, defined on $\mathbb{C}^{n+1} = \mathbb{C} \times \mathbb{C} \times \mathbb{C}^{n-1}$, with coordinates (z_0, z_1, z), where $z \cdot \overline{z}$ is the standard Hermitian form on $\mathbb{C}^{n-1}$.

We will use two models for $\mathbb{H}^n_{\mathbb{C}}$. The first one is the *Siegel domain*. It is better suited for the understanding of the boundary with a chosen point at infinity and is analogous to the upper half-space model for $\mathbb{H}^n_{\mathbb{R}}$. This is the domain

$$\mathbb{H}^n_{\mathbb{C}} = \{(w_1, w) \in \mathbb{C} \times \mathbb{C}^{n-1} \ : \ 2 \text{ Re } w_1 - |w|^2 > 0\}$$

endowed with the Riemannian metric

$$ds^2 = \frac{4}{(2 \text{ Re } w_1 - |w|^2)^2} \big((dw_1 - dw \cdot \overline{w})(d\overline{w}_1 - w \cdot \overline{d}w) +$$

$$(2 \text{ Re } w_1 - |w|^2) dw \cdot \overline{d}w\big)$$

which has constant holomorphic sectional curvature -1, hence sectional curvatures between -1 and $-\frac{1}{4}$.

The second model is the *projective model*, obtained by mapping $\mathbb{H}^n_{\mathbb{C}}$ into the complex projective space $\mathbb{P}^n(\mathbb{C})$ (with its standard homogeneous coordinates) by the map $(w_1, w) \mapsto [1, w_1, w]$. The image of this embedding is the open cone defined by $q < 0$. In particular, $PU(q)$ acts naturally on $\mathbb{H}^n_{\mathbb{C}}$. It is well known (see for instance [15]) that $PU(q)$ is the group of orientation preserving isometries of $\mathbb{H}^n_{\mathbb{C}}$.

The subspace of $\mathbb{H}^n_{\mathbb{C}}$ defined by the equation $w = 0$ is totally geodesic, and isometric to the real hyperbolic plane with sectional curvature -1. In particular, the map $c : \mathbb{R} \to \mathbb{H}^n_{\mathbb{C}}$ defined by $t \mapsto (e^{-t}, 0)$ is a unit speed geodesic line. Denote by ∞ the point of $\partial\mathbb{H}^n_{\mathbb{C}}$ corresponding to the limit of $c(t)$ as t tends to $-\infty$. In the projective model, ∞ corresponds to the point $[0, 1, 0]$.

The horospheres centered at ∞ are

$$H_t = \{(w_1, w) \in \mathbb{C} \times \mathbb{C}^{n-1} \ : \ 2 \text{ Re } w_1 - |w|^2 = t\}$$

for $t > 0$, which bound the open horoballs

$$HB_t = \{(w_1, w) \in \mathbb{C} \times \mathbb{C}^{n-1} \ : \ 2 \text{ Re } w_1 - |w|^2 > t\}\,.$$

The Heisenberg group acts on $\mathbb{H}^n_{\mathbb{C}}$ by

$$(\zeta, v)(w_1, w) = (w_1 + w \cdot \overline{\zeta} + \frac{1}{2}|\zeta|^2 - \frac{i}{2}v, w + \zeta)\,.$$

This action is isometric, and preserves ∞ and the horospheres centered at ∞. Furthermore, $\mathcal{H}_{2n-1}(\mathbb{R})$ acts simply transitively on each horosphere H_t for $t > 0$. In particular, the unit speed geodesic lines starting from ∞ are the paths

$$t \mapsto (e^{t_0 - t} + \frac{1}{2}|\zeta|^2 - \frac{i}{2}v, \zeta)$$

for t_0 in $\mathbb{R}$ and (ζ, v) in $\mathbb{C}^{n-1} \times \mathbb{R}$. Hence the boundary of $\mathbb{H}^n_{\mathbb{C}}$ is:

$$\partial\mathbb{H}^n_{\mathbb{C}} = \{(w_1, w) \in \mathbb{C} \times \mathbb{C}^{n-1} \;:\; 2 \operatorname{Re} w_1 - |w|^2 = 0\} \cup \{\infty\} .$$

Note that the continuous extension to $\partial\mathbb{H}^n_{\mathbb{C}}$ of the isometric action on $\mathbb{H}^n_{\mathbb{C}}$ of $\mathcal{H}_{2n-1}(\mathbb{R})$ is also simply transitive on $\partial\mathbb{H}^n_{\mathbb{C}} - \{\infty\}$. In particular every nontrivial element of $\mathcal{H}_{2n-1}(\mathbb{R})$ is a parabolic isometry of $\mathbb{H}^n_{\mathbb{C}}$.

Define as usual $M^* = \overline{M}^t$ for any $m \times m'$ complex matrix M. Let I be the identity matrix in any dimension. We denote by

$$X = \begin{pmatrix} a & b & \gamma^* \\ c & d & \delta^* \\ \alpha & \beta & A \end{pmatrix}$$

a generic matrix in $U(q)$. If Q is the matrix representing q in the canonical basis of $\mathbb{C}^{n+1}$, one has

$$X^{-1} = Q^{-1} X^* Q = \begin{pmatrix} \overline{d} & \overline{b} & -\beta^* \\ \overline{c} & \overline{a} & -\alpha^* \\ -\delta & -\gamma & A^* \end{pmatrix} .$$

The isomorphism between the Heisenberg group $\mathcal{H}_{2n-1}(\mathbb{R})$ and a subgroup of $PU(q)$ (which is the full orientation preserving isometry group of $\mathbb{H}^n_{\mathbb{C}}$) is given by

$$(\zeta, v) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2}|\zeta|^2 - \frac{i}{2}v & 1 & \zeta^* \\ \zeta & 0 & I \end{pmatrix} .$$

An element of $PU(q)$ preserving each horosphere centered at ∞ is of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2}|\zeta|^2 - \frac{i}{2}v & 1 & \zeta^* \\ A\zeta & 0 & A \end{pmatrix} .$$

with $v \in \mathbb{R}$, $\zeta \in \mathbb{C}^{n-1}$ and $A \in U(n-1)$. Denote by $PU(q)_\infty$ the subgroup of such elements in $PU(q)$.

The following lemma is in [15, p. 77], with a different Hermitian form of signature $(n, 1)$, but the proof is the same.

Lemma 3.9 *If x, y are points in $\mathbb{H}^n_{\mathbb{C}}$, denote by V, W points in $\mathbb{C}^{n+1}$ whose images in $\mathbb{P}^n(\mathbb{C})$ correspond to x, y. Then*

$$\cosh^2 \frac{d(x,y)}{2} = \frac{|\langle V, W\rangle|^2}{q(V)q(W)}$$

where $\langle \cdot, \cdot \rangle$ is the hermitian bilinear form on $\mathbb{C}^{n+1}$ associated to q.

3.10 The Rational Lines are $\mathbb{Q}$-Points

Consider the group $\Gamma = PU(q)(\mathbb{Z}[i])$ of elements in $PU(q)$ with coefficients in $\mathbb{Z}[i]$. Then Γ is a discrete subgroup of $PU(q)$. It is well known that $\Gamma \backslash \mathbb{H}^n_{\mathbb{C}}$ has finite volume (see for instance [32, p. 214]) and, if $n = 2$, it has one and only one cusp (see for instance [42]). We do not know the number of cusps if $n > 2$.

The point ∞ in $\partial \mathbb{H}^n_{\mathbb{C}}$ is a parabolic point of Γ, since for instance the element of $PU(q)$ corresponding to the element $(0, 2)$ of the Heisenberg group is in Γ. Let e_∞ be the cusp of $\Gamma \backslash \mathbb{H}^n_{\mathbb{C}}$ corresponding to ∞ (see Sect. 2 for definitions).

Recall from Sect. 2 that the rational lines in $\Gamma \backslash \mathbb{H}^n_{\mathbb{C}}$ are the (orbifold) geodesic lines starting from the cusp e_∞ and converging to it. A geodesic line in $\mathbb{H}^n_{\mathbb{C}}$ starting from ∞ is the lift of a rational line if and only if its endpoint belongs to the orbit of ∞ by Γ.

Let X be an element in $U(q)$, with the notations as above. Let $\Omega = (0, 1, 0)$ in $\mathbb{C}^{n+1}$. Since $X\Omega = (b, d, \beta)$, the isometry γ of $\mathbb{H}^n_{\mathbb{C}}$ defined by X fixes ∞ if and only if $b = 0$. If $b \neq 0$, then γ sends the point ∞ to the point $(\frac{d}{b}, \frac{\beta}{b})$ in $\partial \mathbb{H}^n_{\mathbb{C}}$. In particular, the orbit of ∞ by Γ is contained (except ∞) in the set of $\mathbb{Q}$-points of the real affine algebraic set $\partial \mathbb{H}^n_{\mathbb{C}} - \{\infty\}$. (In fact, when $\Gamma \backslash \mathbb{H}^n_{\mathbb{C}}$ has only one cusp, then $\Gamma\infty$ is equal to the set of $\mathbb{Q}$-points – see for instance [4]; if $n = 2$, this also follows from an elementary but tedious argument.)

Let O be the point $(0, 0)$ in $\partial \mathbb{H}^n_{\mathbb{C}} - \{\infty\}$. The map $\phi : (\zeta, v) \mapsto (\zeta, v)O = (\frac{1}{2}|\zeta|^2 - \frac{i}{2}v, \zeta)$ is a diffeomorphism from $\mathcal{H}_{2n-1}(\mathbb{R})$ to $\partial \mathbb{H}^n_{\mathbb{C}} - \{\infty\}$ such that $\phi^{-1}(\Gamma\infty - \{\infty\})$ is contained in $\mathcal{H}_{2n-1}(\mathbb{Q})$.

3.11 The Hamenstädt Distance is a Multiple of the Cygan Distance

In this section, we determine the maximal Margulis neighbourhood of the cusp e_∞ of $\Gamma \backslash \mathbb{H}^n_{\mathbb{C}}$, and we compute the Hamenstädt distance on $\partial \mathbb{H}^n_{\mathbb{C}} - \{\infty\}$. (Note that the Hamenstädt distance coincides with the cuspidal distance in $\mathbb{H}^n_{\mathbb{C}}$, see [21].)

Let $(0, v)$ be a nontrivial element in $\mathcal{H}_{2n-1}(\mathbb{R})$ with smallest $|v|$ such that the corresponding element in $PU(q)$ belongs to Γ. Such an element exists since the commutator subgroup of $\mathcal{H}_{2n-1}(\mathbb{R})$ is equal to $\{0\} \times \mathbb{R}$, and since Γ is discrete. Let $\Gamma_\infty = \Gamma \cap PU(q)_\infty$ be the stabilizer of ∞ in Γ.

Using Kamiya's discreteness criterion [24, Theorem 3.2] (or [30, Proposition 5.2]), we proved in [19, Proposition 5.7] that the horoball $HB_{|v|}$ is precisely invariant under Γ. This means that if γ belongs to Γ, then $\gamma HB_{|v|} \cap HB_{|v|}$ is nonempty if and only if γ is in Γ_∞.

In our situation, $(0, v)$ corresponds to an element of Γ if and only if $\begin{pmatrix} 1 & 0 & 0 \\ -\frac{i}{2}v & 1 & 0 \\ 0 & 0 & I \end{pmatrix}$ has coefficients in $\mathbb{Z}[i]$, that is, if and only if v is in $2\mathbb{Z}$. Hence the horoball HB_2 is precisely invariant. Let $\gamma_0 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & I \end{pmatrix}$, which is an element in $U(q)$ with coefficients in $\mathbb{Z}[i]$. Then γ_0 preserves the totally geodesic plane P defined by the equation $w = 0$. Note that $P \cap HB_2$ is the horoball in the real hyperbolic plane P defined by the equations $w = 0$ and $2 \text{ Re } w_1 > 2$, that is $\text{Re } w_1 > 1$. Hence $\gamma_0 \overline{HB_2} \cap \overline{HB_2}$ contains the point $(1, 0)$. Therefore no horoball strictly containing HB_2 is precisely invariant under Γ. This implies that $\Gamma \backslash HB_2$ is the maximal Margulis neighbourhood of the cusp e_∞ of $\Gamma \backslash \mathbb{H}^n_{\mathbb{C}}$.

Recall that $\mathbb{H}^n_{\mathbb{C}}$ has sectional curvatures between -1 and $-\frac{1}{4}$. Hence, if d is the Riemannian distance on $\mathbb{H}^n_{\mathbb{C}}$, then $d' = \frac{1}{2}d$ is the Riemannian distance on $\mathbb{H}^n_{\mathbb{C}}$ normalized to have curvature at most -1.

Recall that (see Sect. 2) the Hamenstädt distance d_∞ on $\partial\mathbb{H}^n_{\mathbb{C}} - \{\infty\}$ is defined as follows. For every ξ, ξ' in $\partial\mathbb{H}^n_{\mathbb{C}} - \{\infty\}$,

$$d_\infty(\xi, \xi') = \lim_{t \to +\infty} e^{-\frac{1}{2}(d'(H_2, \xi_t) + d'(H_2, \xi'_t) - d'(\xi_t, \xi'_t))}$$

where $t \mapsto \xi_t, t \mapsto \xi'_t$ are the geodesic lines with unit speed for the distance d', starting from ∞, passing at time $t = 0$ through the horosphere H_2, and with endpoints ξ, ξ'.

The following result tells us that the Hamenstädt distance coincides with the Cygan distance (up to a multiplicative constant).

Proposition 3.12 *For every ξ, ξ' in $\partial\mathbb{H}^n_{\mathbb{C}} - \{\infty\}$, we have*

$$d_\infty(\xi, \xi') = \frac{1}{\sqrt{2}} d_{\text{Cyg}}(\xi, \xi') \,.$$

Proof. To compute d_∞, we use the invariance under $\mathcal{H}_{2n-1}(\mathbb{R})$ of both distances and the transitivity of the action of $\mathcal{H}_{2n-1}(\mathbb{R})$ on $\partial\mathbb{H}^n_{\mathbb{C}} - \{\infty\}$. Hence we only have to prove the result for $\xi = O = (0, 0)$ and $\xi' = (\zeta, v)O$ for every (ζ, v) nontrivial in $\mathcal{H}_{2n-1}(\mathbb{R})$. Recall that (see Sect. 3.8) the geodesic line with unit speed for the distance d', starting from ∞, passing at time $t = 0$ through the horosphere H_2, and with endpoint $(\zeta, v)O$ is the path $t \mapsto \xi'_t = (e^{-2t} + \frac{1}{2}|\zeta|^2 - \frac{i}{2}v, \zeta)$.

By Lemma 3.9, as $d(\xi_t, \xi'_t)$ tends to $+\infty$, we have, when $t \to +\infty$,

$$e^{\,d(\xi_t, \xi'_t)} \sim 4 \cosh^2 \frac{d(\xi_t, \xi'_t)}{2} = 4 \, \frac{|\,\langle\, (1, e^{-2t}, 0), (1, e^{-2t} + \frac{1}{2}|\zeta|^2 - \frac{i}{2}v, \zeta) \,\rangle\,|^2}{q(1, e^{-2t}, 0)\; q(1, e^{-2t} + \frac{1}{2}|\zeta|^2 - \frac{i}{2}v, \zeta)}$$

$$= \frac{4\left|2e^{-2t} + \frac{1}{2}|\zeta|^2 + \frac{i}{2}v\right|^2}{4e^{-4t}} \sim \frac{e^{4t}}{4}(|\zeta|^4 + v^2)\,.$$

Hence

$$d_\infty(O, (\zeta, v)O) = \lim_{t\to+\infty} e^{-t+\frac{1}{4}d(\xi_t,\xi'_t)}$$
$$= \frac{1}{\sqrt{2}}(|\zeta|^4 + v^2)^{\frac{1}{4}} = \frac{1}{\sqrt{2}} d_{\mathrm{Cyg}}(O, (\zeta, v)O)\,.$$

The result follows. □

3.13 The Depth is the Logarithmic Height

In this section, we compute the depth $D(r)$ of a rational line r in $\Gamma\backslash\mathbb{H}^n_{\mathbb{C}}$.

Proposition 3.14 *If $\Gamma_\infty\gamma\Gamma_\infty$ is the nontrivial double coset corresponding to r, where γ is the image of $X = \begin{pmatrix} a & b & \gamma^* \\ c & d & \delta^* \\ \alpha & \beta & A \end{pmatrix}$ in $PU(q)$, then $D(r) = \log|b|$.*

Proof. Since $XX^{-1} = I$, we have the following set of identities

$$a\overline{d} + b\overline{c} - \gamma^*\delta = 1 \tag{1a}$$
$$a\overline{b} + b\overline{a} - \gamma^*\gamma = 0 \tag{1b}$$
$$c\overline{d} + d\overline{c} - \delta^*\delta = 0 \tag{1c}$$
$$\overline{a}\beta + \overline{b}\alpha - A\gamma = 0 \tag{1d}$$
$$\overline{c}\beta + \overline{d}\alpha - A\delta = 0 \tag{1e}$$
$$\alpha\beta^* + \beta\alpha^* - AA^* = I \tag{1f}$$

Note that by definition of γ, there exists a lift of r to $\mathbb{H}^n_{\mathbb{C}}$ starting from ∞ and ending at $\gamma\infty$. In particular, $\gamma\infty \neq \infty$, which implies that $b \neq 0$.

Recall that (see Sect. 2)

$$D(r) = d'(H_2, \gamma H_2) = \frac{1}{2}d(H_2, \gamma H_2)\,.$$

The point at infinity of the horosphere γH_2 is $\gamma\infty$. We may assume, by multiplying γ on the left by an element of $\mathcal{H}_{2n-1}(\mathbb{R})$, that $\gamma\infty = O$. Indeed, the Heisenberg group acts transitively on $\partial\mathbb{H}^n_{\mathbb{C}} - \{\infty\}$ and preserves the horosphere H_2. Furthermore, the left multiplication of X by an element of $\mathcal{H}_{2n-1}(\mathbb{R})$ leaves b unchanged.

Since $\gamma\infty = (\frac{d}{b}, \frac{\beta}{b})$, we have $d = 0, \beta = 0$. Equation (1f) implies that A is in $U(n-1)$ and (1d) implies that $\delta = 0$.

By multiplying X on the left by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & A^{-1} \end{pmatrix}$, which belongs to $U(q)$, preserves H_2 and does not change the value of b, we may assume that $A = I$.

Note that

$$\begin{pmatrix} a & b & \gamma^* \\ c & 0 & 0 \\ \alpha & 0 & I \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2}|\zeta|^2 - \frac{i}{2}v & 1 & \zeta^* \\ \zeta & 0 & I \end{pmatrix} = \begin{pmatrix} a + b(\frac{1}{2}|\zeta|^2 - \frac{i}{2}v) + \gamma^*\zeta & b & b\zeta^* + \gamma^* \\ c & 0 & 0 \\ \alpha + \zeta & 0 & I \end{pmatrix}.$$

It follows from (1b) that $c = \frac{1}{b}$, from (1e) that $\gamma = \overline{b}\alpha$, and from (1c) that $2 \text{ Re } (a\overline{b}) = |\gamma|^2$, which implies that $2 \text{ Re } (\frac{a}{b}) = |\frac{\gamma}{b}|^2$.

The right multiplication of X by an element of $\mathcal{H}_{2n-1}(\mathbb{R})$ does not change $d(H_2, \gamma H_2)$ nor b. Hence, multiplying X on the right by $\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2}|\zeta|^2 - \frac{i}{2}v & 1 & \zeta^* \\ \zeta & 0 & I \end{pmatrix}$ with $\zeta = -\alpha$ and $v = 2 \text{ Im } \frac{a}{b}$, we may assume that $\alpha = \gamma = 0$ and $a = 0$.

Summarizing the above computations, we may assume that γ is the image in $PU(q)$ of $\begin{pmatrix} 0 & b & 0 \\ \frac{1}{b} & 0 & 0 \\ 0 & 0 & I \end{pmatrix}$. This element preserves the nonoriented geodesic line ρ between ∞ and O, as well as the totally geodesic plane P in $\mathbb{H}^n_{\mathbb{C}}$ defined by the equation $w = 0$. If $x = (1, 0)$ and y are the intersection points of ρ with H_2 and γH_2 respectively, then $d(H_2, \gamma H_2) = d(x, y)$. A simple computation in the constant curvature -1 plane P shows that $d(x, y) = \log |b|^2$. This proves the result. □

Note that $\gamma\infty$ is the $\mathbb{Q}$-point $(\frac{d}{b}, \frac{\beta}{b})$ in the real affine algebraic set $\partial\mathbb{H}^n_{\mathbb{C}} - \{\infty\}$. The coefficients of β and b, d are relatively prime, since γ belongs to $U(q)(\mathbb{Z}[i])$. By acting on the left by an element of Γ_∞, we may assume that d and the coefficients of β have absolute value at most $|b|$. In particular, $D(r)$ is then the logarithmic height of the $\mathbb{Q}$-point corresponding to $(\frac{d}{b}, \frac{\beta}{b})$ in $\mathbb{P}^n(\mathbb{C})$ (see for instance [28, p. 52]).

3.15 Computations of the Critical Exponents

In this section, we compute the critical exponent δ of Γ and the critical exponent δ_0 of Γ_∞ (see also [7]).

Recall that (see Sect. 2) the Poincaré series of a discrete group of isometries G of $\mathbb{H}^n_{\mathbb{C}}$ is

$$P_G(s) = \sum_{g \in G} e^{-sd'(x_0, gx_0)}$$

where x_0 is any base point in $\mathbb{H}^n_{\mathbb{C}}$. This series diverges for $s < \sigma$ and converges for $s > \sigma$ for some σ in $[0, +\infty]$, which does not depend on x_0, and is called the critical exponent of G. Take $x_0 = (\frac{1}{4}, 0)$ in what follows.

Since $\Gamma\backslash\mathbb{H}^n_{\mathbb{C}}$ has finite volume, its critical exponent is equal to the Hausdorff dimension of $\partial\mathbb{H}^n_{\mathbb{C}}$ endowed with the Carnot–Carathéodory distance (see for instance [7, Theorem 6.1]). Hence $\delta = 2n$.

Note that Γ_∞ is commensurable with $\mathcal{H}_{2n-1}(\mathbb{Z})$. Indeed, $\Gamma_\infty \cap \mathcal{H}_{2n-1}(\mathbb{R})$ has finite index in Γ_∞, and consists of the (ζ, v) in $\mathcal{H}_{2n-1}(\mathbb{Z})$ such that v and $|\zeta|^2$ belongs to $2\mathbb{Z}$. Hence $\Gamma_\infty \cap \mathcal{H}_{2n-1}(\mathbb{R})$ has finite index in $\mathcal{H}_{2n-1}(\mathbb{Z})$.

The critical exponent of Γ_∞ is then the same as the critical exponent of the series

$$Q(s) = \sum_{g \in \mathcal{H}_{2n-1}(\mathbb{Z})} \left(\cosh \frac{d(x_0, g x_0)}{2}\right)^{-s} .$$

By Lemma 3.9, we have

$$\begin{aligned} Q(s) &= \sum_{(\zeta,v) \in \mathcal{H}_{2n-1}(\mathbb{Z})} \left(\frac{\left|\frac{1}{2} + \frac{1}{2}|\zeta|^2 + \frac{i}{2}v\right|^2}{(\frac{1}{2})^2}\right)^{-\frac{s}{2}} \\ &= \sum_{(\zeta,v) \in \mathcal{H}_{2n-1}(\mathbb{Z})} \left((1 + |\zeta|^2)^2 + |v|^2\right)^{-\frac{s}{2}} . \end{aligned}$$

By comparison, this sum converges if and only if the following integral converges:

$$\begin{aligned} \int_{\mathbb{C}^{n-1}} \int_{\mathbb{R}} \frac{d\zeta \; dv}{(1 + |\zeta|^4 + |v|^2)^{\frac{s}{2}}} &= 2v_{2n-3} \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \frac{x^{2n-3} dx dv}{(1 + x^4 + v^2)^{\frac{s}{2}}} \\ &= 2v_{2n-3} \int_{\mathbb{R}^+} \frac{x^{2n-3} dx}{(1 + x^4)^{\frac{s}{2} - \frac{1}{2}}} \int_{\mathbb{R}^+} \frac{dy}{(1 + y^2)^{\frac{s}{2}}} . \end{aligned}$$

where v_k is the area of the unit sphere in $\mathbb{R}^{k+1}$ and by setting $v = y\sqrt{1 + x^4}$. The last line converges if and only if $s > n$. Hence $\delta_0 = n$.

3.16 The Height of Closed Geodesics

In this section, we compute the "distance to the cusps" of the closed geodesics in $\Gamma\backslash\mathbb{H}^n_{\mathbb{C}}$. More generally, let G be a discrete nonelementary group of isometries of $\mathbb{H}^n_{\mathbb{C}}$. Assume that ∞ is a parabolic fixed point, with G_∞ its stabilizer in G. Assume that $G\backslash HB_2$ is the maximal Margulis neighbourhood of the cusp corresponding to ∞.

Let τ be a geodesic line in $\mathbb{H}^n_{\mathbb{C}}$, whose points at infinity differ from ∞. Define the *height* $ht(\tau)$ of τ as the maximum of $\frac{1}{2}\log\frac{t}{2}$ for $t > 0$ such that τ meets HB_t. The first factor $\frac{1}{2}$ comes from the fact that we need to normalize so that the sectional curvature is at most -1; the second part of the formula comes from the fact that the signed distance between H_2 and H_t is $\log\frac{t}{2}$.

An element g in $PU(q)$ is *loxodromic* if it has a unique pair of fixed points in $\mathbb{H}^n_{\mathbb{C}} \cup \partial\mathbb{H}^n_{\mathbb{C}}$, that are in $\partial\mathbb{H}^n_{\mathbb{C}}$. The geodesic line between these points is called the *translation axis* of g, and will be denoted by A_g. Note that g is loxodromic if and only if its *translation length* $\ell(g) = \inf_{x\in\mathbb{H}^n_{\mathbb{C}}} d(x, gx)$ is nonzero. The isometry g preserves A_g, and acts by a translation of length $\ell(g)$ on it. The translation length is a conjugation invariant.

Lemma 3.17 *For every loxodromic element g in $PU(q)$, whose fixed points are different from ∞, one has*

$$ht(A_g) = \frac{1}{2}\log\frac{\sinh\frac{\ell(g)}{2}}{|b(g)|}\,,$$

where $b(g)$ is the coefficient 1-2 *of any matrix X in $U(q)$ mapping to g in $PU(q)$.*

Proof. By conjugating g by an element of $PU(q)_\infty$, which does not change $\ell(g)$ nor $|b(g)|$, we may assume that the fixed points of g are $O = (0, 0)$ and $(u, 0)$ with $\operatorname{Re} u = 0$ and $\operatorname{Im} u > 0$.

Set $X = \begin{pmatrix} a & b & \gamma^* \\ c & d & \delta^* \\ \alpha & \beta & A \end{pmatrix}$. By the assumptions on g, together with (1b)–(1f) in the proof of Proposition 3.14, we obtain

$$b \neq 0,\ c = 0,\ \alpha = 0,\ A \in U(n-1),\ \delta = 0,\ \beta = 0,\ \gamma = 0,$$
$$\operatorname{Re}\overline{a}b = 0,\ a\overline{d} = 1,\ \text{and}\ u = \frac{a-d}{b}\,.$$

If $a = |a|e^{i\theta}$, then $d = \frac{1}{|a|}e^{i\theta}$, $u = \frac{|a|-|d|}{be^{-i\theta}}$ and $\operatorname{Re} be^{-i\theta} = 0$. The element γ' of $PU(q)$, which is the projection of $X' = \begin{pmatrix} |a| & be^{-i\theta} & 0 \\ 0 & |d| & 0 \\ 0 & 0 & I \end{pmatrix}$, fixes $(0, 0)$ and $(u, 0)$. Hence γ' is a loxodromic element with the same translation axis as γ. Since $\gamma'\gamma^{-1}$ fixes the point $(\frac{|u|}{2}, 0)$, the isometries γ, γ' also have the same translation length. Since γ' sends ∞ to $(\frac{|d|}{be^{-i\theta}}, 0)$ whose first component is purely imaginary, it preserves the hyperbolic plane P defined by the equation $w = 0$. In particular, $\ell(\gamma') = |\log|a|^2|$. By replacing γ by its inverse, we may assume that $|a| \geq 1$. Then $|a| = e^{\frac{\ell(\gamma)}{2}}$. Hence

$$\frac{|u|}{2} = \frac{\sinh\frac{\ell(\gamma)}{2}}{|b|}\,.$$

An easy computation in the real hyperbolic plane P shows that $ht(\gamma) = \frac{1}{2}\log\frac{|u|}{2}$. Hence the result follows. □

Recall from Sect. 2 that the invariant h_M of $M = G\backslash\mathbb{H}^n_{\mathbb{C}}$ is defined as

$$h_M = \inf_{g\in G\,:\,\ell(g)>0}\ \max_{h\in G}\ ht(A_{hgh^{-1}})\,.$$

Corollary 3.18 *The Hurwitz constant K_M of $M = G\backslash\mathbb{H}^n_{\mathbb{C}}$ satisfies*

$$K_M = \frac{1}{2}\sup_{g\in G\,:\,\ell(g)>0}\left(\frac{\sinh\frac{\ell(g)}{2}}{\min\limits_{h\in G}\ |b(hgh^{-1})|}\right)^{-\frac{1}{2}}.$$

Proof. This follows from the previous lemma and from Theorem 2.4. □

3.19 Proofs of the Results

Theorem 3.4 is a corollary of Theorem 2.3 applied to $M = \Gamma\backslash\mathbb{H}^n_{\mathbb{C}}$, with the help of the Sects. 3.10, 3.11, and 3.13.

The assertion $\mu_{\mathrm{Cyg}}(E_\psi) = 0$ if and only if $\int_1^{+\infty}\psi(t)^{2n}\frac{dt}{t}$ converges in Theorem 3.5 is a corollary of Theorem 2.7 applied to $M = \Gamma\backslash\mathbb{H}^n_{\mathbb{C}}$, with the help of the Sects. 3.10, 3.11, 3.13 and 3.15. Note that $\mu_{\mathrm{Cyg}}(E_\psi) > 0$ implies that $\mu_{\mathrm{Cyg}}({}^cE_\psi) = 0$, by the same argument as in [37, p. 223].

Theorem 3.6 follows from Corollary 3.18 applied to $G = \Gamma$, from the fact that $d_{\mathrm{Cyg}} = \sqrt{2}d_\infty$, and the fact that the matrix $\begin{pmatrix} 2 & i & 0\\ -i & 1 & 0\\ 0 & 0 & I\end{pmatrix}$ is in Γ, has coefficient 1-2 with absolute value 1 (which is the least possible since the coefficients are in $\mathbb{Z}[i]$), and has translation length $2\log\frac{3+\sqrt{5}}{2}$.

Theorem 3.7 follows from the paragraph after 2.6 with the computations of Sect. 3.13.

We have concentrated on one particular finite volume discrete subgroup of $PU(q)$, but the results of Sect. 2 apply more generally to other geometrically finite discrete subgroups of isometries of $\mathbb{H}^n_{\mathbb{C}}$ with cusps, for instance the groups of integer points of any split $\mathbb{Q}$-forms of $PU(q)$.

Note that by a theorem of Feustel and Zink [42], if $n = 2$, if d is a squarefree positive integer, then the number of cusps of $PU(q)(\mathcal{O}_{-d})$ is the class number of the imaginary quadratic fields $\mathbb{Q}(\sqrt{-q})$, and in particular is 1 if and only if $d = 1, 2, 3, 7, 11, 19, 43, 67, 163$.

Note that similar results apply for the quaternionic hyperbolic space $\mathbb{H}^n_{\mathbb{H}}$ and the octonionic hyperbolic plane $\mathbb{H}^2_{\mathbb{O}}$, the computations of the Hamenstädt distance, of the depth and of the critical exponents being similar to what we have done.

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