Appendix: Diophantine Approximation on Hyperbolic Surfaces

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Abstract In this (independent) appendix, we study the Diophantine approximation properties for the particular case of the cusped hyperbolic surfaces, in the spirit of Sect. 2 (or [11]), and the many still open questions that arise for them. We refer to [9], [10] for fundamental results and further developments. We study in particular the distance to a cusp of closed geodesics on a hyperbolic surface.

Let S be a (smooth) connected orientable surface of genus g with n punctures, such that $\chi(S) = 2 - 2g - n < 0$ and $n \geq 1$.

Let $\mathcal{M}(S)$ be the moduli space of S, viewed as the space of puncture-preserving isometry classes of hyperbolic structures on S (i.e. complete, finite volume, constant curvature -1, Riemannian metrics on S).

For every end e of S, and every X in $\mathcal{M}(S)$, let $\beta_{X,e} : X \to \mathbb{R}$ be the *height function on X with respect to e*, defined as the Busemann function on X for the end e, normalized to converge to $+\infty$ towards e, and to vanish on the boundary of the maximal open Margulis neighbourhood of e in X (see for instance [11] or Sect. 2).

Define the *height function* β_X on X to be the maximum of the functions $\beta_{X,e}$ over the ends e of S.

If A is a subset of X, define $ht_e(A)$, the *height of A with respect to e*, as the supremum on A of the height function with respect to e, and $ht(A)$, the *height of A*, as the supremum on A of the height function.

Proposition 1 *The maps $\beta_{X,e}$ (resp. β_X) are proper, 1-Lipschitz, piecewise smooth on X. Their singular set is the cut locus of (X, e) (resp. X), defined as the set of points in X from which at least two minimizing geodesic rays converge to e (resp. to some end of X), which is a locally finite (resp. finite) graph.*

Proof. The fact that $\beta_{X,e}$ is proper and 1-Lipschitz is proven in [11]. Let $\Sigma_{X,e}$ be the cut locus of (X, e), as defined above. Its structure of a locally finite graph, more precisely a finite graph to which are glued finitely many (possibly none) half-lines, is described in [3, Lemma 2.2.1].

Let $\widetilde{X} \to X$ be a universal cover of X, with $\widetilde{\Sigma}_{X,e}$ the preimage of $\Sigma_{X,e}$, and let $\widetilde{\beta}_{X,e}$ be the lift of $\beta_{X,e}$. On each component U of $\widetilde{X} - \widetilde{\Sigma}_{X,e}$, the map $\widetilde{\beta}_{X,e}$ coincides with some Busemann function centered at the unique point at infinity of U corresponding to e, hence is smooth. Since a Busemann function centered at some point at infinity ξ has derivative $+1$ along each (unit speed) geodesic line converging to ξ, the map $\widetilde{\beta}_{X,e}$ is not smooth at any point of $\widetilde{\Sigma}_{X,e}$. □

Define $h_e : \mathcal{M}(S) \to \mathbb{R}$ and $h : \mathcal{M}(S) \to \mathbb{R}$, with $h_e(X)$ the lower bound of the heights with respect to e of the closed geodesics on X, and $h(X)$ the lower bound of the heights of the closed geodesics on X.

Define similarly h_e^{sim} and h^{sim} by taking lower bounds on the simple closed geodesics. Of course, $h_e \leq h_e^{sim}$, $h \leq h^{sim}$ and $h_e \leq h$.

Proposition 2 *For X in $\mathcal{M}(S)$, the value $h^{sim}(X)$ (resp. $h_e^{sim}(X)$) is equal to the minimum of the heights (resp. with respect to e) of the geodesic laminations on X.*

Proof. Recall (see for instance [5, Proposition 4.1.7, Theorem 4.2.14]) that the space $\mathcal{GL}(X)$ of (nonempty) geodesic laminations on X, endowed with the Chabauty topology, is compact, separable, and metrizable, and that the finite nonempty unions of simple closed geodesics are dense in it. Recall that $(L_i)_{i\in\mathbb{N}}$ converges to L_∞ in $\mathcal{GL}(X)$ if and only if

1. for every x_{i_k} in L_{i_k}, if $(x_{i_k})_{k\in\mathbb{N}}$ converges to x in X, then x belongs to L_∞,
2. for every x in L_∞, there exists a sequence $(x_i)_{i\in\mathbb{N}}$ in X, with x_i belonging to L_i, which converges to x.

It is then easy to show that the maps $ht_e(\cdot)$ and $ht(\cdot)$ from $\mathcal{GL}(X)$ to $\mathbb{R} \cup \{+\infty\}$ are continuous. The result follows. □

It is proved in [11] (see also Sect. 2) that the maps h_e and h are continuous, proper and, for $g = 1 and n = 1$, real analytic; that the infimum defining h_e is not always attained; that for $g = 1, n = 1$, this infimum is attained and $h = h^{sim}$. In contrast to the case of once-punctured tori, we have the following result.

Proposition 3 *The map h_e is not always real analytic. For $g \geq 2$ and $n \geq 1$, there exists X in $\mathcal{M}(S)$ such that $h(X) < h^{sim}(X)$.*

Proof. To prove the first claim, let us assume, for concreteness, that $g = 2$ and $n = 1$. Let γ_1 and γ_2 be simple closed curves on S such that $S - (\gamma_1 \cup \gamma_2)$ is the disjoint union of the interior of two tori with one boundary component

S_1, S_2, and a sphere with a puncture and two boundary components. Let $X \in \mathcal{M}(S)$ such that γ_1 and γ_2 are geodesic and $\ell(\gamma_1) = \ell(\gamma_2)$. By symmetry, we also have that $ht(\gamma_1) = ht(\gamma_2)$ (see Fig. 6 (a) below).

For $i = 1, 2$, let X_i be the surface S_i with the metric induced by X. Assume that the shortest closed geodesic σ_i of X_i is different from γ_i. Let δ_i be one of the two shortest geodesic arcs on X_i connecting σ_i to $\gamma_i = \partial X_i$. Rotate the gluing parameters at γ_1 and γ_2 in such a way that the endpoint of δ_i on γ_i coincides with the point realizing the height of γ_i. Assume from now on that the lengths of γ_1, γ_2, σ_1 and σ_2 are small enough. Then σ_1 and σ_2 are obvious candidates for lowest geodesics. In fact, it is easy to prove that

$$h(X) = h^{sim}(X) = \min(ht(\sigma_1), ht(\sigma_2)) .$$

If we further restrict to some (two-dimensional, real-analytic) suborbifold of the moduli space, it is easy to see the nondifferentiability of h. Consider the suborbifold of the moduli space defined by requiring $\ell(\gamma_1), \ell(\gamma_2)$ and the gluing parameters along γ_1, γ_2 to be fixed, and the two possible choices of δ_i to have a common starting point on σ_i. This slice can be parameterized real-analytically by the lengths $l(\sigma_1) > 0$, and $l(\sigma_2) > 0$, with $l(\sigma_1) \le l(\sigma_2)$ (by symmetry). The line $l(\sigma_1) = l(\sigma_2)$ is an orbifold line of order 2. If $\ell(\sigma_1) < \ell(\sigma_2)$, then the symmetry of the situation, along with the hyperbolic geometry of right-angled hexagons (see [2, Theorem 7.19.2]), implies that $h(X) = ht(\sigma_1)$, $\partial ht(\sigma_1)/\partial \ell(\sigma_2) = 0$ and $\partial ht(\sigma_1)/\partial \ell(\sigma_1) > 0$. Recall that a map f defined on an orbifold $\mathcal{O}$ is differentiable, if, locally, any lift of f to some manifold which orbifold-covers $\mathcal{O}$ is differentiable. By the above, h is nondifferentiable along the line $l(\sigma_1) = l(\sigma_2)$.

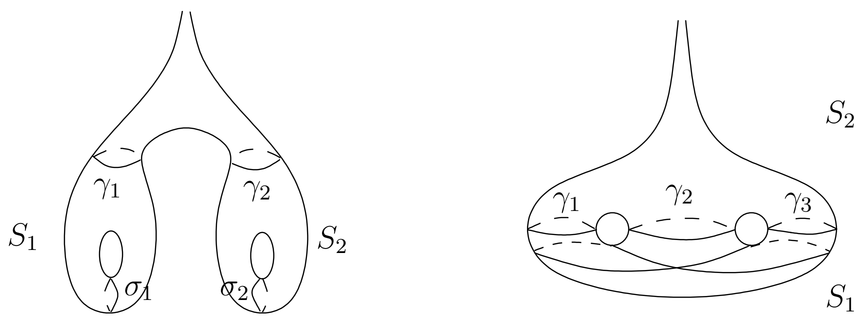

Fig. 6: Nondifferentiability (**a**) and nonsimplicity (**b**)

To prove the second claim, let $g \ge 2$ and $n = 1$, and let $\gamma_1, \gamma_2, \gamma_3$ be simple closed curves on S such that $S - (\gamma_1 \cup \gamma_2 \cup \gamma_3)$ is the disjoint union of the interior of a pair of pants S_1 (with boundary components $\gamma_1, \gamma_2, \gamma_3$), and the interior of another surface S_2 with three boundary components and a puncture. Let $X \in \mathcal{M}(S)$ such that $\gamma_1, \gamma_2, \gamma_3$ are closed geodesic and $ht(\gamma_1) =$

$ht(\gamma_2) = ht(\gamma_3)$. If $g = 2$, a surface like this can be constructed from a sphere with three holes of equal length and a puncture that has a symmetry of order three, fixing the puncture and permuting the boundary components, and a matching pair of pants (see Fig. 6 (b) above).

For $i = 1, 2$, let X_i be the surface S_i with the metric induced by X. Any simple closed geodesic in X is either contained in X_2 or intersects $\gamma_1 \cup \gamma_2 \cup \gamma_3$. Fix a pair of pants decomposition of S such that S_2 is one of the pairs of pants. Assume also that the lengths of the closed geodesics $\gamma_1, \gamma_2, \gamma_3$ are very short, while the twist parameters and the lengths of the other curves of the pants decomposition are fixed. Then,

$$h^{sim}(X) = ht(\gamma_1) = ht(\gamma_2) = ht(\gamma_3)\,.$$

On the other hand, there are nonsimple geodesics contained in $X_1 - \partial X_1$, and $ht(x) < ht(\gamma_i)$ for all $x \in X_1 - \partial X_1$. □

Theorem 4 *If $n = 1$, for every X in $\mathcal{M}(S)$, with $\pi : T^1X \to X$ the canonical projection from the unit tangent bundle of X to X, and $\phi^t : T^1X \to T^1X$ the geodesic flow, one has*

$$h_e(X) = \min_{v\in T^1X} \limsup_{t\to+\infty} \beta_{X,e}\,\pi\,\phi^t(v)\,.$$

Proof. Since $n = 1$, the map $\beta_{X,e}$ has a finite minimum. Define $f : T^1X \to \mathbb{R} \cup \{+\infty\}$ by $f(v) = \limsup_{t\to+\infty} \beta_{X,e}\,\pi\,\phi^t(v)$, and $h'_e(X) = \inf_{v\in T^1X} f(v)$. Note that by definition

$$h_e(X) = \inf_{\substack{v\in T^1X\\ v \text{ periodic}}} f(v)$$

where a unit tangent vector is said to be *periodic* if it is periodic under the geodesic flow. In particular, $h'_e(X) \le h_e(X)$. The opposite inequality follows from Anosov's closing lemma (see [1]).

Let us now prove that the infimum defining $h'_e(X)$ is a minimum. Take v_i in T^1X such that $f(v_i)$ tends to $h'_e(X)$. Since $f(v_i)$ is finite and $n = 1$, the geodesic defined by v_i accumulates in a compact subset of T^1X. Hence, up to extracting a subsequence, there exists $t_{i,k}$ tending to $+\infty$ as $k \to \infty$ such that $\phi^{t_{i,k}}(v_i)$ converges to v'_i as $k \to \infty$ and v'_i converges to v as $i \to \infty$.

Fix $\epsilon > 0$. Let $t_0 > 0$ such that $|f(v) - \beta_{X,e}\,\pi\,\phi^{t_0}(v)| \le \epsilon$. Choose i big enough so that $|\beta_{X,e}\,\pi\,\phi^{t_0}(v) - \beta_{X,e}\,\pi\,\phi^{t_0}(v'_i)| \le \epsilon$ and $f(v_i) \le h'_e(X) + \epsilon$. Choose now k big enough so that $|\beta_{X,e}\,\pi\,\phi^{t_0}\phi^{t_{i,k}}(v_i) - \beta_{X,e}\,\pi\,\phi^{t_0}(v'_i)| \le \epsilon$ and $\beta_{X,e}\,\pi\,\phi^{t_0+t_{i,k}}(v_i) \le f(v_i) + \epsilon$. Then

$$\begin{aligned} h'_e(X) &\le f(v) \le \beta_{X,e}\,\pi\,\phi^{t_0}(v) + \epsilon \le \beta_{X,e}\,\pi\,\phi^{t_0}(v'_i) + 2\epsilon \\ &\le \beta_{X,e}\,\pi\,\phi^{t_0}\phi^{t_{i,k}}(v_i) + 3\epsilon \le f(v_i) + 4\epsilon \le h'_e(X) + 5\epsilon\,. \end{aligned}$$

As ϵ is arbitrary, this proves that $f(v) = h'_e(X)$. The result follows. □

Remark. This is true if X is replaced by any complete, finite volume Riemannian manifold with pinched negative curvature and one cusp. One also immediately obtains that

$$h(X) = \min_{v\in T^1X} \limsup_{t\to+\infty} \beta_X\,\pi\,\phi^t(v)\,.$$

Let X be in $\mathcal{M}(S)$ and let e be an end of S. Let $d = d_{X,e}$ be the Hamenstädt distance on the set of (oriented but not pointed) geodesic lines on X starting from the end e, and $D(r) = D_{X,e}(r)$ be the depth of a rational geodesic line starting from e (see Sect. 2 or [11]). For ξ an irrational geodesic line starting from e, let $K(\xi)$ be the infimum of the positive constants $c > 0$ such that there exist infinitely many rational geodesic lines r starting from e with

$$d(r,\xi) \leq ce^{-D(r)}\,.$$

If $\xi : \mathbb{R} \to X$ is a (unit speed, orientation preserving) geodesic parametrization of ξ, then the ω-limit set of ξ is

$$\omega\!-\lim \xi = \bigcap_{n\in\mathbb{N}} \overline{\xi([n,+\infty[)}\,.$$

Let us give some more definitions.

- The *Lagrange spectrum* Sp_{Lag} of (X,e) is the subset of $\mathbb{R}$ consisting of the $K(\xi)$ for ξ an irrational geodesic line starting from e.
- The *Cohn spectrum* Sp_{Coh} of (X,e) is the subset of Sp_{Lag} of the $K(\xi)$ such that the ω-limit set of ξ is a closed geodesic.
- The *Markov spectrum* Sp_{Mar} of (X,e) is the subset of Sp_{Coh} of the $K(\xi)$ such that the ω-limit set of ξ is simple.
- The *height spectrum* of (X,e) is the subset of $\mathbb{R}$ consisting of the elements of the form $\limsup_{t\to\infty} \beta_{X,e}(\xi(t))$ (if finite) for all irrational geodesic lines ξ in X starting from e.
- The *periodic height spectrum* of (X,e) is the subset of $\mathbb{R}$ consisting of the heights with respect to e of the closed geodesics in X.
- The *simple periodic height spectrum* of (X,e) is the subset of $\mathbb{R}$ consisting of the heights with respect to e of the simple closed geodesics in X.

The following results are known.

- ([12], [7], [8], [9], [10]) If X is the modular once-punctured hyperbolic torus (i.e. the quotient of the hyperbolic upper half-plane by the commutator subgroup of $\mathrm{PSL}_2(\mathbb{Z})$), then the Lagrange, Cohn, Markov spectra, as defined above, coincide with their usual definitions.
- ([9]) If X is an hyperbolic once-punctured torus, and $sys(X)$ is the *systole* of X, i.e. the length of the shortest simple closed geodesic in X, then

$$Sp_{\mathrm{Mar}} = Sp_{\mathrm{Lag}} \cap \left]\frac{1}{2\cosh\frac{sys(X)}{2}}, +\infty\right[$$

and,if $K(\xi) \in Sp_{\rm Mar}$, then

$$K(\xi) = \frac{1}{2\cosh\frac{sys(X)}{2}} \coth \frac{\ell(\omega - \lim \xi)}{2}\,.$$

In particular, $\max Sp_{\rm Lag} = \max Sp_{\rm Mar} = 1/(2\sinh\frac{sys(X)}{2})$. Since the systole of the modular once-punctured hyperbolic torus is $2\log\frac{3+\sqrt{5}}{2}$, one recovers the previous result.

- ([11]) The map $x \mapsto -\log 2x$ sends the Lagrange spectrum to the height spectrum, the Cohn spectrum to the periodic height spectrum and the Markov spectrum to the simple periodic height spectrum. In particular, if X is an hyperbolic once-punctured torus, then

$$h(X) = \log\sinh\frac{sys(X)}{2}\,.$$

Though the following result might be well known, we give a simple geometric proof.

Proposition 5 *If X is the (unique) hyperbolic thrice punctured sphere, then*

$$h(X) = \log\sqrt{5}\,.$$

Proof. The thrice-punctured sphere has a unique configuration of maximal pairwise tangent horocycles c_∞, c_0 and c_1. The complement of the union of the corresponding cusp neighbourhoods U_∞, U_0 and U_1 does not contain any closed geodesics. Let γ be the closed geodesic that passes twice through each of the points of tangency $c_i \cap c_j$. The height of γ is $\log\sqrt{5}$ (see Fig. 7 below).

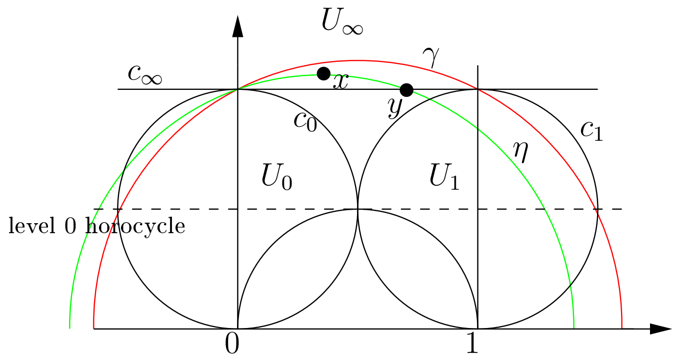

Fig. 7: The averaged Hurwitz constant of the thrice punctured sphere

Let η be a closed geodesic, such that $ht(\eta) < ht(\gamma)$. Assume that the height of η is attained at a point $x \in \eta \cap U_\infty$. One of the endpoints y of

$\eta \cap U_\infty$ must lie strictly between the points of tangency of U_∞ with U_0 and U_1, and the angle θ at y between η and c_∞ must be smaller than $\arcsin(1/2)$, which is the angle between γ and c_∞. We can assume η is as in the Fig. 7 above. Clearly, η enters U_1 at an angle that is greater than θ. Thus, η enters deeper in U_1 than in U_∞, which is a contradiction.

The same argument gives the uniqueness of γ. $\square$

Now come the questions:

1. Is h_e (resp. h) piecewise real analytic? What is its singular locus? (Compare with [3].)
2. On what subset of the moduli space is the infimum defining h_e, h, h^{sim} or h_e^{sim} attained?
3. More generally, when is the height spectrum closed?
4. How do the maps h_e, h change under finite covers? In particular for the congruence subgroups of $\mathrm{PSL}_2(\mathbb{Z})$?
5. Is it possible to compute $h(X)$ for X arithmetic?

Concerning question 2, here is a partial answer.

Proposition 6 *If $X \in \mathcal{M}(S)$ has a pants decomposition by closed geodesics with lengths at most $2\operatorname{arcsinh}\frac{1}{2}$, then the lower bounds defining $h_e^{sim}(X)$ and $h^{sim}(X)$ are attained on one of these closed geodesics.*

Note that the value $2\operatorname{arcsinh}\frac{1}{2} \simeq 0.962423$ is not sharp, it can be replaced by 1.363223 for instance, but we do not know what is the sharp value (nor if the lower bounds are always attained).

Proof. We start with some hyperbolic estimates. Let γ be any closed geodesic on X with length $\ell = \ell(\gamma)$. By [4, p. 99 and p. 97] (taking limits to handle the case with cusps), if $\ell \leq 4\operatorname{arcsinh} 1$, then γ is simple, and if $w = \operatorname{arcsinh}(1/\sinh\frac{\ell}{2})$, then the w-neighbourhood of γ is an embedded cylinder C.

Let a be a highest point of γ with respect to the cusp e. Let ρ be a minimizing geodesic ray from a to e, cutting one of the two boundary components $\partial_1 C$ of C at some point b. Let δ be any closed geodesic cutting γ transversally in at least one point. Since δ cannot stay in C, it cuts $\partial_1 C$ transversally in at least one point c. By the triangular inequality, one has $\beta_{X,e}(c) \geq \beta_{X,e}(b) - d(c,b)$. Note that $ht_e(\delta) \geq \beta_{X,e}(c)$ and $ht_e(\gamma) = \beta_{X,e}(b) - w$. Hence $ht_e(\delta) \geq ht_e(\gamma) + w - d(c,b)$. Since δ is closed, by the equality case in the triangle inequality and the fact that δ is not tangential to $\partial_1 C$ at c, one even has $ht_e(\delta) > ht_e(\gamma) + w - d(c,b)$.

By the hyperbolic formulae of the trirectangle (see for instance [4, p. 38]),

$$\sinh\frac{d(c,b)}{2} \leq \sinh\frac{\ell}{4}\cosh w\,.$$

Lemma 7 *Let $\lambda \leq 2$ be the only root bigger than* 1 *of the polynomial $P(u) = -2u^5 + 4u^4 - 2u^3 + 4u^2 - 2u + 1$. If $\ell \leq 2\operatorname{arcsinh}\frac{1}{\lambda} \simeq 1.363223$, then $\sinh\frac{\ell}{4}\cosh w \leq \sinh\frac{w}{2}$.*

Proof. Let $u = \sinh w = (\sinh\frac{\ell}{2})^{-1}$. Then $\sinh^2(w/2) = (\sqrt{1+u^2} - 1)/2$, $\cosh^2 w = 1 + u^2$ and $\sinh^2(\ell/4) = (\sqrt{1+u^2} - u)/(2u)$. Hence

$$\begin{aligned}\sinh\frac{\ell}{4}\cosh w \leq \sinh\frac{w}{2} \quad &\Leftrightarrow \quad \frac{\sqrt{1+u^2}-u}{u}(1+u^2) \leq \sqrt{1+u^2} - 1 \\ &\Leftrightarrow \quad P(u) \leq 0\,.\end{aligned}$$

The result follows (note that $P(2) < 0$). $\square$

End of the Proof of Proposition 6. This lemma, and the formula above it, imply that if $\ell \leq 2\operatorname{arcsinh}\frac{1}{2}$, then $w \geq d(c, b)$. Hence $ht_e(\delta) > ht_e(\gamma)$.

Let us now complete the proof of the result. Assume that X has a pants decomposition by closed geodesics $\gamma_1, ..., \gamma_k$ with lengths at most $2\operatorname{arcsinh}\frac{1}{2}$. By the above, every closed geodesic meeting transversally γ_i has height strictly bigger than the height of γ_i. Every simple closed geodesic in an hyperbolic pair of pants (with totally geodesic boundary, possibly degenerate) is a boundary one. This proves the result. $\square$

Concerning question 4, here are partial answers.

Proposition 8 *Let $X \in \mathcal{M}(S)$ and $\pi : \widehat{X} \to X$ be a finite Riemannian covering. Let e be a cusp of X and U_e be its maximal Margulis neighbourhood in X. For every $\widehat{e}$ in $\pi^{-1}(e)$, let $U_{\widehat{e}}$ be the maximal Margulis neighbourhood of $\widehat{e}$ in $\widehat{X}$, and $V_{\widehat{e}}$ be the connected component of $\pi^{-1}(U_e)$ contained in $U_{\widehat{e}}$. Assume that the distance $\tau_{e,\widehat{e}}$ in $\widehat{X}$ between $\partial V_{\widehat{e}}$ and $\partial U_{\widehat{e}}$ does not depend on $\widehat{e}$, and denote it by τ_e. Then*

$$\max_{\widehat{e}\in\pi^{-1}(e)} h_{\widehat{e}}(\widehat{X}) = h_e(X) + \tau_e\,.$$

As the proof below will show, this result is valid more generally for any X which is a finite volume pinched negatively curved Riemannian orbifold.

Proof. Let $\widehat{\beta}_{X,e}$ be the lift to $\widehat{X}$ of the height function $\beta_{X,e}$. For any subset A in X, define

$$\widehat{ht}_e(A) = \sup_{x\in A}\widehat{\beta}_{X,e}(x)\,.$$

Fix universal coverings $\widetilde{X} \to X$ and $\widetilde{X} \to \widehat{X}$ such that the second covering group is contained in the first one. By finiteness, any element of the first covering group has a power which is contained in the second one. Hence

$$\inf_{\widehat{\gamma}\text{ closed geodesic in }\widehat{X}} \widehat{ht}_e(\widehat{\gamma}) = \inf_{\gamma\text{ closed geodesic in }X} ht_e(\gamma)\,.$$

Let Σ_e be the cut locus of e in X. On any component of $\widehat{X} - \pi^{-1}(\Sigma_e)$, containing a cusp $\widehat{e}$ in $\pi^{-1}(e)$, the maps $\beta_{\widehat{X},\widehat{e}}$ and $\widehat{\beta}_{X,e}$ differ by the constant $\tau_{e,\widehat{e}}$. By assumption, this last term is independent of $\widehat{e}$ in $\pi^{-1}(e)$. Hence

$$\inf_{\widehat{\gamma}\text{ closed geodesic in }\widehat{X}} \ \max_{\widehat{e}\in\pi^{-1}(e)} ht_{\widehat{e}}(\widehat{\gamma}) = \inf_{\gamma\text{ closed geodesic in }X} ht_e(\gamma) + \tau_e\,.$$

The result follows. □

Corollary 9 *If $n = 1$, if $X \in \mathcal{M}(S)$ and $\widehat{X} \to X$ is a finite normal Riemannian covering, then*

$$h(\widehat{X}) = h(X) + \tau$$

with τ the distance between the maximal Margulis neighbourhood of a cusp in $\widehat{X}$ and the component contained in it of the preimage of the maximal Margulis neighbourhood of the cusp in X.

Proof. This follows immediately from Proposition 8. □

Corollary 10 *If $\Gamma(N) = \mathrm{Ker}(\mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}/N\mathbb{Z}))$ is the N-th principal congruence subgroup of the modular group $\mathrm{PSL}_2(\mathbb{Z})$, and $X_N = \Gamma(N)\backslash\mathbb{H}^2_{\mathbb{R}}$, then*

$$h(X_N) = \log\frac{N\sqrt{5}}{2}\,.$$

Proof. The covering $X_N \to X_1$ is finite and normal, and X_1 has only one cusp. By Sect. 2 or [11], namely $h(X_1) = \log\frac{\sqrt{5}}{2}$. It is easy to compute the constant τ in the previous corollary, one has $\tau = \log N$. The result follows (and gives another proof of Proposition 5, since the hyperbolic thrice punctured sphere is X_2). □

Proposition 11 *With the notations of Corollary 10, if e_∞ is the cusp of X_N corresponding to the parabolic fixed point ∞ of $\Gamma(N)$, then there exists a constant $c \geq 0$ such that*

$$-3\log N - c \leq h_{e_\infty}(X_N) \leq -3\log N + c\,.$$

Proof. By [6], if

$$k(N) = \sup_{\xi\in\mathbb{R}-\mathbb{Q}} \ \liminf_{p\in\mathbb{N},\, q\in\mathbb{N}-\{0\},\, (p,q)=1,\, p\equiv 1\,[\mathrm{mod}\,N],\, q\equiv 0\,[\mathrm{mod}\,N]} |q(q\xi - p)|\,,$$

then there exists a constant $C \geq 1$ such that $\frac{1}{C}N^2 \leq k(N) \leq CN^2$. The horosphere centered at the point at infinity ∞ of $\mathbb{H}^2_{\mathbb{R}}$, mapping onto the boundary of the maximal Margulis neighbourhood of e_∞ in $X(N)$, is the horizontal line of Euclidean height $\frac{1}{N}$. Hence, by [11], if x, y are geodesic lines starting from the cusp e_∞ in X_N, the Hamenstädt distance between x, y is $N|\xi_x - \xi_y|$ where ξ_x, ξ_y are the closest endpoints in $\mathbb{R}$ of the lifts starting from ∞ of x, y respectively. If $K_{e_\infty}(N)$ is the Hurwitz constant for (X_N, e_∞) (see Sect. 2 or [11]), then $K_{e_\infty}(X_N) = Nk(N)$. Recall (see Sect. 2 or [11]) that $\frac{1}{2K_{e_\infty}(X_N)} = e^{h_{e_\infty}(X_N)}$. Hence the result follows. □

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