Equidistribution of divergent diagonal orbits in positive characteristic

Nguyen-Thi Dang   Frédéric Paulin   Rafael Sayous

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Abstract

Given a local field $\widehat{K}$ with positive characteristic, we study the dynamics of the diagonal subgroup of the linear group $\mathrm{GL}_n(\widehat{K})$ on homogeneous spaces of discrete lattices in $\widehat{K}^n$. We first give a function field version of results by Margulis and Tomanov-Weiss, characterizing the divergent diagonal orbits. When $n = 2$, we relate the divergent diagonal orbits with the divergent orbits of the geodesic flow in the modular quotient of the Bruhat-Tits tree of $\mathrm{PGL}_2(\widehat{K})$. Using the (high) entropy method by Einsiedler-Lindentraus et al, we then give a function field version of a result of David-Shapira on the equidistribution of a natural family of these divergent diagonal orbits, with height given by a new notion of discriminant of the orbits. [1]

1 Introduction

Equidistribution problems of periodic orbits have been widely studied in many different settings. In hyperbolic dynamical systems, in particular for closed orbits of geodesic flows in negative curvature, see for instance [Mar, Bow] and many others, including [PauPS, §9.3] (see references therein). In homogeneous dynamics for diagonalisable group actions (sometimes in an arithmetic framework), see for instance [ELMV1] and many others, including [DaL] (see references therein). See also [Sha, SY, KePS] (this last one also over function fields) for possible chaotic behaviors of weak-star limits of homogeneous measures on periodic orbits, including surprising loss of mass phenomena.

Much less studied has been the problem of equidistribution of divergent orbits, as they require noncompact phase spaces and a specific study of equidistribution of (locally finite) infinite measures. See [PaPS] for geodesic flows in variable curvature, as well as [DS1, DS2] in homogeneous dynamics. Considering homogeneous dynamics over various local field is important and fruitful. The first purpose of this paper is to extend to local fields in positive characteristic works of Margulis, Tomanov-Weiss [TW] and Tomanov [Tom1] on the characterisation of divergent orbits. The second purpose is to extend David-Shapira [DS1, DS2] results on their equidistribution (with the challenges required for such an extension). The study of divergent orbits in homogeneous dynamics, through the Dani correspondence, has strong ties with Diophantine approximation problem, see for instance [KW, CG, KaKLM, DFSU, AK, BKL], these last two references also over function fields.

[1]**Keywords:** Equidistribution, divergent orbits, diagonal actions, positive characteristic, function fields. **AMS codes:** 22F30, 11N45, 20G30, 14G17, 28C10, 11J70, 11P21.

Throughout this paper, referring to [Gos, Ros2] and Subsection 2.1 for definitions and complements, we fix a function field K of genus $\mathfrak{g}$ over a finite field $\mathbb{F}_q$ of order q, a valuation v of K and a uniformiser π_v of v. We denote by K_v the completion of K for v, by $\mathscr{O}_v$ its valuation ring, by q_v the order of its residual field, by $|\cdot| = q_v^{-v(\cdot)}$ its (normalized) absolute value, by R_v the affine function ring associated with v, by ζ_v the Dedekind zeta function of R_v, and by φ_v the Euler function of R_v.

We fix $n \in \mathbb{N} \smallsetminus \{0, 1\}$. The unimodular group $\mathrm{GL}^1_n(K_v) = \{g \in \mathrm{GL}_n(K_v) : |\det g| = 1\}$ is endowed with the Haar measure giving mass 1 to the maximal compact subgroup $\mathrm{GL}_n(\mathscr{O}_v)$. We denote by $\mathscr{X}_1$ the $\mathrm{GL}^1_n(K_v)$-homogeneous space of R_v-lattices in ${K_v}^n$ with normalized covolume 1 (identified with $\mathrm{GL}^1_n(K_v)/\mathrm{GL}_n(R_v)$ when pointed at the standard R_v-lattice R_v^n). We endow $\mathscr{X}_1$ with the induced $\mathrm{GL}^1_n(K_v)$-invariant measure, that we denote by $\mathtt{m}_{\mathscr{X}_1}$ and which is finite.

We denote by A_1 the diagonal subgroup of $\mathrm{GL}^1_n(K_v)$, and we normalize its Haar measure to give mass 1 to its maximal compact subgroup $A_1 \cap \mathrm{GL}_n(\mathscr{O}_v)$. The diagonal orbit $A_1 x$ of an element $x \in \mathscr{X}_1$ is said to be *divergent* if the orbital map $a \mapsto ax$ from A_1 to $\mathscr{X}_1$ is proper. The *homogeneous measure* on $A_1 x$, that we denote by $\overline{\mu}_x$, is then the (locally finite) pushforward measure by this orbital map of the Haar measure of A_1.

The first main result of this paper (see Corollary 3.4 for a more general result and Theorem 3.1 for the analog result for the projective linear group $\mathrm{PGL}_n(K_v)$) is an algebraic characterisation of the divergent orbits, saying that they are the "rational" ones, that is, they come from a rational point (in $\mathrm{GL}^1_n(K)$) of $\mathrm{GL}^1_n(K_v)$, up to the action of an element of A_1.

Theorem 1.1 *Let $x \in \mathscr{X}_1$. The diagonal orbit $A_1 x$ is divergent if and only if there exists $g \in A_1 \,\mathrm{GL}^1_n(K)$ such that $x = g\, R_v^n$.*

This result has a long history. In the real field case with $n = 2$, an orbit of the diagonal subgroup of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathrm{PSL}_2(\mathbb{R})/\mathrm{PSL}_2(\mathbb{Z})$ is well known to be divergent if and only if it corresponds to a modular group orbit of a geodesic line in the upper halfspace model $\mathbb{H}^2_{\mathbb{R}}$ of the real hyperbolic plane both of whose endpoints are rational points (that is, are in $\mathbb{P}_1(\mathbb{Q})$) of the circle at infinity $\mathbb{P}_1(\mathbb{R})$ of $\mathbb{H}^2_{\mathbb{R}}$. In the function field case with $n = 2$, the corresponding result is also well known: The quotient of the Bruhat-Tits tree of $\mathrm{PGL}_2(K_v)$ by $\mathrm{PGL}_2(R_v)$ replaces the quotient of the real hyperbolic plane by $\mathrm{PSL}_2(\mathbb{Z})$, and the set of rational points at infinity is $\mathbb{P}_1(K)$ in $\mathbb{P}_1(K_v)$ (see for instance [Ser2] and Section 4.3). The real field case for any integer n is due to Margulis (see [TW, Appendix]). It has been extended to all reductive algebraic groups over $\mathbb{Q}$ in [TW, Theo. 1.1], and to the S-adic case over number fields in [Tom2]. The case of divergent orbits of proper subgroups of the full maximal $\mathbb{Q}$-torus subgroup has been studied by [Tam], with surprising differences.

For every $k \in \mathbb{N} \smallsetminus \{0\}$, we identify each element of ${K_v}^k$ with the column matrix of its coordinates in the canonical basis of ${K_v}^k$. For every element $\mathbf{t} \in {K_v}^{n-1}$, we define

$$\mathfrak{u}_{\mathbf{t}} = \begin{pmatrix} 1 & 0 \\ \mathbf{t} & I_{n-1} \end{pmatrix} \in \mathrm{SL}_n(K_v)\,. \tag{1}$$

Note that if $\mathbf{t} \in K^{n-1}$, then $\mathfrak{u}_{\mathbf{t}}\, R_v^n$ is an R_v-lattice with normalized covolume 1 whose diagonal orbit is divergent by Theorem 1.1.

The second main result of this paper (see Corollary 8.4 for a more general result) is the following equidistribution result in $\mathscr{X}_1$ for natural families of divergent diagonal orbits in $\mathscr{X}_1$. We emphazise the fact that the measures that equidistribute are infinite measures. But for the weak-star convergence, sequences of locally finite infinite measures may indeed converge to a finite measure. Such is not the case for the narrow convergence.

Theorem 1.2 *Let* $c_{K,n} = \frac{(n-1)!\,\prod_{i=1}^{n-1}\zeta_v(-i)}{q_v\,(q-1)\,\prod_{i=2}^{n-1}(q_v{}^i-1)}$. *For every nonzero* $s \in R_v$*, let us define* $\Lambda(s) = \{(\frac{r_2}{s},\dots,\frac{r_n}{s}) : r_2,\dots,r_n \in R_v,\ \forall j \in \{2,\dots,n\},\ r_jR_v + sR_v = R_v\} \mod R_v{}^{n-1}$. *Assume that* R_v *is principal. For the weak-star convergence of Radon measures on the locally compact space* $\mathscr{X}_1$*, we have*

$$\lim_{|s|\to+\infty,\ |s|\in q_v{}^{n\mathbb{Z}}} \frac{c_{K,n}}{(\varphi_v(s)\ \log_{q_v}|s|\,)^{n-1}} \sum_{\mathbf{t}\in\Lambda(s)} \overline{\mu}_{\mathfrak{u}_\mathbf{t} R_v{}^n} = \mathtt{m}_{\mathscr{X}_1}\,.$$

Let us discuss the scope of this result. We believe that the principal assumption on R_v may not be necessary, since we are only using it to prove the non-escape of mass property in Section 7, and an approach along the lines of [DKMS] could allow its removal. Over the real field, this result is due to [DS1] in dimension $n = 2$ and to [DS2] in general. Starting from Section 5, we will follow their scheme of proof. Our result has two new aspects, besides the fact that the algebraic properties of the ring R_v are much more involved than the ones of $\mathbb{Z}$. Firstly, we obtain an explicitely renormalized weak-star convergence, and not only a projective convergence of the measures. Secondly, we cover a larger set of types of divergent orbits, as we now explain. It follows from Theorem 1.1 that an A_1-orbit Θ is divergent if and only if it contains an element containing a sub-R_v-lattice Λ of $R_v{}^n$. We define the *type* of a sub-R_v-lattice Λ of $R_v{}^n$ as the isomorphism class of the torsion R_v-module $R_v{}^n/\Lambda$, and the *type* of Θ as the finite set of types of the sub-R_v-lattices of $R_v{}^n$ with minimal covolume contained in the elements of Θ. For instance, for every $\mathbf{t} \in \Lambda(s)$, the type of the A_1-orbit $A_1\mathfrak{u}_\mathbf{t} R_v{}^n$ is reduced to $\{(R_v/sR_v)^{n-1}\}$ (see Proposition 4.6 (4)). We choose this type in this Introduction for simplicity, but we refer to Corollary 8.4 for a generalisation.

The techniques of the second part of this paper, that we now present, rely in particular on the (high) entropy method in homogeneous dynamics (see for instance [EL]). Let

$$\mathfrak{a} = \begin{pmatrix} \pi_v{}^{n-1} & 0 \\ 0 & \pi_v{}^{-1} I_{n-1} \end{pmatrix} \in \mathrm{SL}_n(K_v)\,. \tag{2}$$

Let $U^- = \{\mathfrak{u}_\mathbf{t} : \mathbf{t} \in K_v{}^{n-1}\}$, which is the unipotent radical of the parabolic subgroup of $\mathrm{SL}_n(K_v)$ fixing the hyperplane $\{0\} \times K_v{}^{n-1}$ of $K_v{}^n$. Note that for all $k \in \mathbb{Z}$ and $\mathbf{t} \in K_v{}^{n-1}$, we have $\mathfrak{a}^k \mathfrak{u}_\mathbf{t} \mathfrak{a}^{-k} = \mathfrak{u}_{\pi_v{}^{-nk}\mathbf{t}}$, so that U^- is contained in (and actually equal to) the unstable horospherical group of the one-parameter diagonal group $(\mathfrak{a}^k)_{k\in\mathbb{Z}}$.

Using Mahler's criterion that a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathscr{X}_1$ goes out of every compact subset if and only if the systole of x_n goes to 0, the first step (see Section 4) is to describe a canonical "compact core" C_x of a divergent diagonal A_1-orbit x by trimming out the parts were the systole of the elements of x are small. When $n = 2$, this correspond to removing the first and last intersections with a cuspidal ray of the corresponding geodesic line in the quotient graph of groups by $\mathrm{PGL}_2(R_v)$ of the Bruhat-Tits tree of $\mathrm{PGL}_2(K_v)$ (see [Ser2, BPP] for background and Subsection 4.3). Let us denote by $\overline{\nu}_x$ the restriction of the homogeneous measure $\overline{\mu}_x$ to C_x normalized to be a probability measure, and by $\overline{\mu}_s = \sum_{\mathbf{t}\in\Lambda_s} \overline{\mu}_{\mathfrak{u}_\mathbf{t} R_v{}^n}$ and $\overline{\nu}_s = \frac{1}{\mathrm{Card}\Lambda_s} \sum_{\mathbf{t}\in\Lambda_s} \overline{\nu}_{\mathfrak{u}_\mathbf{t} R_v{}^n}$. It will follow from Subsection 4.2 (with the help of computations done in Subsection 4.4) that the measures $\frac{c_{K,n}}{(\varphi_v(s)\ \log_{q_v}|s|\,)^{n-1}}\ \overline{\mu}_s$ have as $|s| \to +\infty$ the same weak-star asymptotic properties as the probability measures $\overline{\nu}_s$. Furthermore, we prove that the measures $\overline{\nu}_s$ on $\mathscr{X}_1$ are averages over a compact subgroup C_n^1 of $\mathrm{GL}_n^1(K_v)$ of natural measures ν_s on $\mathrm{SL}_n(K_v)/\mathrm{SL}_n(R_v)$.

The second step (see Section 7) is to prove that the measures ν_s on $\mathrm{SL}_n(K_v)/\mathrm{SL}_n(R_v)$ as $|s| \to +\infty$ do not suffer any loss of mass, that is, any weak-star accumulation point ν of ν_s as $|s| \to +\infty$ is also a probability measure. We did not try to write our equidistribution result replacing the set Λ_s by a logarithmic full proportion of it, as it is done in [DS1, DS2]. This extension requires a non escape of mass assumption, that has been lifted in [DKMS] when $n = 2$ in the real case.

The third step (see Section 8) is to prove that the entropy $h_\nu(\mathfrak{a})$, which is well defined by the second step, of any weak-star accumulation point ν for the diagonal transformation $\mathfrak{a}$ on $\mathrm{SL}_n(K_v)/\mathrm{SL}_n(R_v)$ is equal to the maximal entropy of this transformation. This requires, as in [ELMV2], a construction of high entropy partitions for $\mathfrak{a}$, that are build using dynamical neighborhoods for the action of $\mathfrak{a}$ on its unstable horospherical group U^-.

The last step (see Section 5) is to apply Einsiedler-Lindenstraus [EL] uniqueness of the probability measure of maximal entropy on $\mathrm{SL}_n(K_v)/\mathrm{SL}_n(R_v)$ for $\mathfrak{a}$, which is the measure $\mathtt{m}_{\mathscr{X}_1}$ renormalized to be a probability measure, and to average back on the above-mentionned compact subgroup C_n^1 in order to prove Theorem 1.2.

Obtaining an error term in Theorem 1.2 would require an effective version of the uniqueness of measures of maximal entropy for diagonal actions in positive characteristic, and would constitute another project. We believe that our results could be extended to the S-adic case (working with a nonempty finite set of places S instead of just one v) or to the adelic setting (for the nonuniform lattice $\mathrm{PGL}_n(K)$ of $\mathrm{PGL}_n(\mathbb{A}_K)$, where $\mathbb{A}_K$ is the adèle ring of K).

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2 Background material

For all $r, r' \in \mathbb{Z}$ with $r \leqslant r'$, we denote $[\![r, r']\!] = [r, r'] \cap \mathbb{Z}$.

2.1 Function fields over finite fields

For the following notions and complements, we refer to [Gos, Ros2], as well as to [BPP, §14.2] whose notation we follow. Let $\mathbb{F}_q$ be a finite field of order a positive power q of a prime p. Let K be a (global) function field over $\mathbb{F}_q$ of genus $\mathfrak{g}$, that is, the function field of a geometrically connected smooth projective curve $\mathfrak{C}$ of genus $\mathfrak{g}$ defined over $\mathbb{F}_q$. We denote by h_K the number of divisor classes of degree 0 on $\mathfrak{C}$. Let v be a (normalised discrete) valuation of K, let K_v be the associated completion of K, let $\mathscr{O}_v = \{x \in K_v : v(x) \geqslant 0\}$ be its valuation ring, let $\pi_v \in K$ with $v(\pi_v) = 1$ be a uniformiser of v, let q_v be the order of the residual field $\mathbb{F}_{q_v} = \mathscr{O}_v/\pi_v\mathscr{O}_v$ (that we identify with its lift in $\mathscr{O}_v$), and let $|\cdot| = q_v^{-v(\cdot)}$ be the (normalized) absolute value associated with v. We denote by $\deg v \in \mathbb{N} \smallsetminus \{0\}$ the degree of the closed point of $\mathfrak{C}$ corresponding to v, so that $q_v = q^{\deg v}$. Let R_v be the affine function ring associated with v, that is, the affine algebra of the curve $\mathfrak{C}$ minus its closed point corresponding to v. Recall that R_v is a Dedekind ring whose field of fractions is K. The class number h_v of the Dedekind ring R_v is $h_v = (\deg v)\, h_K$ by [Gos, Coro. 4.1.3]. In particular R_v is principal if and only if $h_K = 1$ and $\deg v = 1$, which occurs in positive genus for exactly 4 isomorphisms classes of function fields K (one for each $(\mathfrak{g}, q) = (1,2), (1,3), (1,4), (2,2)$) by [MS, Theo. 1.1] and [MaQ, Theo. 2]. Note that $\mathscr{O}_v^\times = \{x \in K_v : |x| = 1\}$ and (see for instance [BPP, Eq. (14.2) and (14.3)]

$$R_v \cap \mathscr{O}_v = \mathbb{F}_q \quad \text{and} \quad R_v^\times = \mathbb{F}_q^\times \subset \mathscr{O}_v^\times \,. \tag{3}$$

The simplest example, used in Section 4.3, is given by the field $K = \mathbb{F}_q(Y)$ of rational fractions over $\mathbb{F}_q$ with one indeterminate Y, with genus $\mathfrak{g} = 0$, endowed with the valuation at infinity v with $\deg v = 1$ defined for all $P, Q \in \mathbb{F}_q[Y]$ with $Q \neq 0$ by $v(\frac{P}{Q}) \deg Q - \deg P$. Then $K_v = \mathbb{F}_q((Y^{-1}))$ is the field of formal Laurent series in Y^{-1} over $\mathbb{F}_q$, $\mathscr{O}_v = \mathbb{F}_q[[Y^{-1}]]$ is the local ring of formal power series in Y^{-1} over $\mathbb{F}_q$, $\pi_v = Y^{-1}$, $q_v = q$, and $R_v = \mathbb{F}_q[Y]$ is the ring of polynomials in Y over $\mathbb{F}_q$.

Let $\mathscr{I}_v^+$ be the semigroup of nonzero ideals of the ring R_v. As usual, $\mathfrak{p}$ ranges throughout the text over prime ideals in $\mathscr{I}_v^+$ and $\mathtt{N}(I) = [R_v : I] \in q^{\mathbb{N}}$ is the absolute norm of $I \in \mathscr{I}_v^+$. Let $\mathtt{N}(s) = \mathtt{N}(sR_v)$ for every $s \in R_v \smallsetminus \{0\}$, and note that $\mathtt{N}(s) = |s|$. For all $r, s \in R_v$, we write as usual $(r, s) = 1$ if r and s are coprime, that is, satisfy $r\,R_v + s\,R_v = R_v$.

We denote by $\mu_v : \mathscr{I}_v^+ \to \mathbb{Z}$ the Möbius function of R_v, so that $\mu_v(I) = 0$ if I has a squared prime factor, and otherwise $\mu_v(I) = (-1)^k$ where k is the number of prime factors of I.

We denote by $\varphi_v : \mathscr{I}_v^+ \to \mathbb{N}$ the Euler function of R_v, defined by

$$\varphi_v(I) = \operatorname{Card}\big(R_v/I\big)^\times = \mathtt{N}(I) \prod_{\mathfrak{p}\,|\,I} \big(1 - \frac{1}{\mathtt{N}(\mathfrak{p})}\big) = \mathtt{N}(I) \sum_{I' \in \mathscr{I}_v^+,\; I'\,|\,I} \frac{\mu_v(I')}{\mathtt{N}(I')}\,, \tag{4}$$

and $\varphi_v(s) = \varphi_v(sR_v)$ for every $s \in R_v$.

The Dedekind zeta function of the Dedeking ring R_v is (see, for instance, [Gos, §7.8]) the map $\zeta_v : \{z \in \mathbb{C} : \operatorname{Re} z > 1\} \to \mathbb{C}$ defined by

$$\zeta_v : z \mapsto \sum_{I \in \mathscr{I}_v^+} \frac{1}{\mathtt{N}(I)^z}\,.$$

By for instance [Gos, page 219, line 2] or [Ros2, page 244, Eq. (1)], it is related to the zeta function ζ_K of the field K (which is an Eulerian product over all closed points of $\mathfrak{C}$, including the one corresponding to v) by the formula

$$\zeta_K(z) = \frac{1}{1 - q_v^{-z}}\, \zeta_v(z)\,. \tag{5}$$

By for instance [Ros2, Theo. 5.9], ζ_K has an analytic continuation on $\mathbb{C} \smallsetminus \{0, 1\}$ with simple poles at $z = 0$ and $z = 1$ (it is actually a rational function of q^{-z}). Hence the value $\zeta_v(-k)$ for every $k \in \mathbb{N} \smallsetminus \{0\}$ is well defined. We recall the following counting result.

Lemma 2.1 *As $t \to +\infty$, we have*

$$\operatorname{Card}\{I \in \mathscr{I}_v^+ : \mathtt{N}(I) \leqslant t\} = \frac{h_K\, q^{2-\mathfrak{g}}\,(q_v - 1)}{(q-1)^2\, q_v}\, q^{[\log_q t]} + \operatorname{O}(1)\,.$$

Proof. We give a proof for completeness. Let $z \in \mathbb{C}$ with Re $z > 1$. For every $n \in \mathbb{N}$, let $c_n = \operatorname{Card}\{I \in \mathscr{I}_v^+ : \mathtt{N}(I) = q^n\}$. Since $\mathtt{N}(I) \in q^{\mathbb{N}}$ for every $I \in \mathscr{I}_v^+$, we have $\zeta_v(z) = \sum_{n=0}^{\infty} c_n\, q^{-nz}$. By for instance [Ros2, end of page 52], we have $\zeta_K(z) = \sum_{n=0}^{\infty} b_n\, q^{-nz}$ with $b_n = h_K\, \frac{q^{n-\mathfrak{g}+1}-1}{q-1}$ if $n > 2\,\mathfrak{g} - 2$. Hence by Equation (5), we have

$$\begin{aligned}\zeta_v(z) &= (1-q_v^{-z})\;\zeta_K(z) = (1-q^{-z\deg v})\sum_{n=0}^{\infty} b_n\, q^{-nz}\\ &= \sum_{n=0}^{\deg v\,-1} b_n\, q^{-nz} + \sum_{n=\deg v}^{\infty} (b_n - b_{n-\deg v})\, q^{-nz}\,.\end{aligned}$$

Hence by identification, if $n \geqslant 2\,\mathfrak{g} + \deg v$, we have

$$c_n = b_n - b_{n-\deg v} = h_K\, \frac{q^{n-\mathfrak{g}+1} - q^{n-\mathfrak{g}+1-\deg v}}{q-1} = \frac{h_K\; q^{1-\mathfrak{g}}\,(1-q_v^{-1})}{q-1}\; q^n\,.$$

Therefore, by a geometric series argument, for every $n \in \mathbb{N}$, we have

$$\begin{aligned}\operatorname{Card}\{I \in \mathscr{I}_v^+ : \mathtt{N}(I) \leqslant q^n\} &= \sum_{i=0}^{n} c_i = \sum_{i=2\mathfrak{g}+\deg v}^{n} c_i + \mathrm{O}(1)\\ = \frac{h_K\; q^{1-\mathfrak{g}}\,(1-q_v^{-1})}{(q-1)}\;\frac{q^{n+1}}{q-1} + \mathrm{O}(1) &= \frac{h_K\; q^{2-\mathfrak{g}}\,(1-q_v^{-1})}{(q-1)^2}\; q^n + \mathrm{O}(1)\,.\end{aligned}$$

Since $\mathtt{N}(I) \in q^{\mathbb{N}}$ for every $I \in \mathscr{I}_v^+$, this proves the result. □

The following lemma is an effective version of [Poo, Lem. 3]. Its proof follows the one of [HaW, Th. 328] given in Chap. XXII, §22.9 where $\mathbb{Z}$ is replaced by R_v. Again, we add a proof for completeness. We denote by γ the Euler constant.

Lemma 2.2 *If* $c' = \frac{q^{\mathfrak{g}-1}\,(q-1)\,\ln q}{(1-q_v^{-1})\,e^{\gamma}\,h_K}$, *then* $\liminf\limits_{\mathtt{N}(I)\to+\infty} \frac{\varphi_v(I)\ \ln\ln(\mathtt{N}(I))}{\mathtt{N}(I)} = c'$.

In particular, since $\mathtt{N}(s) = |s| = q_v^{-v(s)}$ for every $s \in R_v \smallsetminus \{0\}$, we have

$$\exists\; c_{\varphi_v} \in\,]0,1], \quad \forall s \in R_v \smallsetminus \{0\}, \qquad \varphi_v(s) \geqslant c_{\varphi_v} \frac{|s|}{\max\{1, \ln(-v(s))\}}\,. \tag{6}$$

Proof. Since we will only use the minoration (6), we only prove that the lower limit in the statement of Lemma 2.2 is at least c'. Let $F :]0,+\infty[\, \to \mathbb{R}$ be the map

$$t \mapsto F(t) = (\ln t)\Big(1 - \frac{1}{t}\Big)^{\frac{t}{\ln t}} \prod_{\mathtt{N}(\mathfrak{p})\leqslant t} \Big(1 - \frac{1}{\mathtt{N}(\mathfrak{p})}\Big)\,.$$

By [Ros1, Th. 3] (which gives an asymptotic expansion of the partial product over all closed points of $\mathfrak{C}$, including the one corresponding to v, which explains the factor $1 - q_v^{-1}$ in the constant c'), as $t \to +\infty$, we have $\prod_{\mathtt{N}(\mathfrak{p})\leqslant t}\big(1 - \frac{1}{\mathtt{N}(\mathfrak{p})}\big) \sim c' \frac{1}{\ln t}$. Hence as $t \to +\infty$ we have $F(t) \sim c'(1-\frac{1}{t})^{\frac{t}{\ln t}} \sim c'(1 - \frac{1}{\ln t})$, which tends to c'. For every $I \in \mathscr{I}_v^+$, let $A_I = \{\mathfrak{p} : \mathfrak{p} \mid I,\ \mathtt{N}(\mathfrak{p}) \leqslant \ln \mathtt{N}(I)\}$ and $B_I = \{\mathfrak{p} : \mathfrak{p} \mid I,\ \mathtt{N}(\mathfrak{p}) > \ln \mathtt{N}(I)\}$. Since $\mathtt{N}$ is completely multiplicative, we have $(\ln \mathtt{N}(I))^{\mathrm{Card}\, B_I} \leqslant \prod_{\mathfrak{p}\in B_I} \mathtt{N}(\mathfrak{p}) \leqslant \mathtt{N}(I)$, hence $\mathrm{Card}\, B_I \leqslant \frac{\ln \mathtt{N}(I)}{\ln\ln \mathtt{N}(I)}$. Thus as $\ln \mathtt{N}(I) \to +\infty$, we have

$$\begin{aligned}
&\frac{\varphi_v(I)\ \ln\ln \mathtt{N}(I)}{\mathtt{N}(I)}\\
&= \ln\ln \mathtt{N}(I) \prod_{\mathfrak{p}\,|\,I}\Big(1-\frac{1}{\mathtt{N}(\mathfrak{p})}\Big) \geqslant \ln\ln \mathtt{N}(I)\Big(1-\frac{1}{\ln \mathtt{N}(I)}\Big)^{\mathrm{Card}\, B_I} \prod_{\mathfrak{p}\in A_I}\Big(1-\frac{1}{\mathtt{N}(\mathfrak{p})}\Big)\\
&\geqslant \ln\ln \mathtt{N}(I)\Big(1-\frac{1}{\ln \mathtt{N}(I)}\Big)^{\frac{\ln \mathtt{N}(I)}{\ln\ln \mathtt{N}(I)}} \prod_{\mathtt{N}(\mathfrak{p})\leqslant \ln \mathtt{N}(I)}\Big(1-\frac{1}{\mathtt{N}(\mathfrak{p})}\Big) = F(\ln \mathtt{N}(I)) \sim c'\,. \square
\end{aligned}$$

Denote by $\varpi_v : \mathscr{I}_v^+ \to \mathbb{N}$ the omega function counting the prime factors of ideals:

$$\varpi_v : I \mapsto \mathrm{Card}\{\mathfrak{p} : \mathfrak{p} \mid I\}\,.$$

We define $\varpi_v(s) = \varpi_v(sR_v)$ for every $s \in R_v \smallsetminus \{0\}$. For every $I \in \mathscr{I}_v^+$, with the notation of the above proof, we have $\mathrm{Card}\, A_I = \mathrm{O}\big(\frac{\ln \mathtt{N}(I)}{\ln\ln \mathtt{N}(I)}\big)$ by the prime number theorem in K (see for instance [Ros2, Theo. 5.12]) and $\mathrm{Card}\, B_I \leqslant \frac{\ln \mathtt{N}(I)}{\ln\ln \mathtt{N}(I)}$. Hence as $N(I) \to +\infty$, we have $\varpi_v(I) = \mathrm{Card}\,(A_I \cup B_I) = \mathrm{O}\big(\frac{\ln \mathtt{N}(I)}{\ln\ln \mathtt{N}(I)}\big)$. In particular, since $\mathtt{N}(s) = |s| = q_v^{-v(s)}$ for every $s \in R_v \smallsetminus \{0\}$, and since $\varpi_v(s) = 0$ when $s \in R_v^\times$, we have

$$\exists\, c_{\varpi_v} > 0, \quad \forall s \in R_v \smallsetminus \{0\}, \qquad \varpi_v(s) \leqslant c_{\varpi_v} \frac{-v(s)}{\max\{1, \ln(-v(s))\}}\,. \tag{7}$$

We fix $n \in \mathbb{N} \smallsetminus \{0\}$ throughout this paper. We denote by $(e_1, \ldots, e_n)$ the canonical basis of the product K_v-vector space K_v^n. Let $\|\ \| : K_v^n \to [0, +\infty[$ be the standard norm $(x_1, \ldots, x_n) \mapsto \max_{1\leqslant i\leqslant n} |\,x_i\,|$. We denote by vol_v the normalized Haar measure on the locally compact additive group K_v such that $\mathrm{vol}_v(\mathscr{O}_v) = 1$. Let vol_v^n be the normalized Haar measure on K_v^n such that $\mathrm{vol}_v^n(\mathscr{O}_v^n) = 1$. Note that for every $g \in \mathrm{GL}_n(K_v)$, we have

$$d\,\mathrm{vol}_v^n(gx) = |\,\det(g)\,|\; d\,\mathrm{vol}_v^n(x)\,. \tag{8}$$

In particular, we have $\mathrm{vol}_v(\pi_v\mathscr{O}_v) = q_v^{-1}$ and

$$\mathrm{vol}_v(\mathscr{O}_v^\times) = \mathrm{vol}_v(\mathscr{O}_v \smallsetminus \pi_v\mathscr{O}_v) = 1 - q_v^{-1}\,. \tag{9}$$

If G is a discrete subgroup of the additive group K_v^n (for instance any nonzero, not necessarily principal, ideal of R_v when $n = 1$), we also denote by vol_v^n the unique Haar measure on the quotient abelian topological group K_v^n/G such that the covering map $K_v^n \to K_v^n/G$ locally preserves the measure.

2.2 Lattices

An R_v*-lattice* L in K_v^n is a free rank-n R_v-submodule in K_v^n that generates K_v^n as a K_v-vector space. It is a discrete cocompact additive subgroup of K_v^n. For instance, a nonzero ideal I of R_v is an R_v-lattice in K_v if and only if it is principal.

The *covolume* of L, denoted by $\mathrm{covol}(L)$, is defined as the measure of the (compact) quotient space K_v^n/L :

$$\mathrm{covol}(L) = \mathrm{vol}_v^n(K_v^n/L)\,.$$

For every $g \in \mathrm{GL}_n(K_v)$, by Equation (8), we have

$$\mathrm{covol}(gL) = |\det(g)| \ \mathrm{covol}(L) . \tag{10}$$

In particular, if $\lambda \in K_v^\times$, then $\mathrm{covol}(\lambda L) = |\lambda|^n \mathrm{covol}(L)$. Since the set of values of $|\cdot|$ is $\{0\} \cup q_v^{\mathbb{Z}}$, every R_v-lattice is hence homothetic under $K_v^\times$ to an R_v-lattice with covolume in $[1, q_v^n]$. For example, R_v^n is an R_v-lattice in K_v^n, and by for instance [BPP, Lem. 14.4], we have

$$\mathrm{covol}(R_v^n) = q^{(\mathfrak{g}-1)n} . \tag{11}$$

The *normalized covolume* of an R_v-lattice L is $\frac{\mathrm{covol}(L)}{\mathrm{covol}(R_v^n)}$, which belongs to $q_v^{\mathbb{Z}}$ since $\mathrm{GL}_n(K_v)$ acts transitively on the set of K_v-basis of K_v^n, hence on the set of R_v-lattices of K_v^n, and by using Equation (10).

An R_v-lattice L in K_v^n is said to be

- *unimodular* if $\mathrm{covol}(L) = \mathrm{covol}(R_v^n)$ (by Equation (11) for instance, as well as for other integrality purposes, it is not appropriate to define them by requiring $\mathrm{covol}(L) = 1$),
- *special unimodular* if L admits an R_v-basis $(b_1, \ldots, b_n)$ such that $b_1 \wedge \ldots \wedge b_n$ is equal to the canonical generator $e_1 \wedge \ldots \wedge e_n$ of the n-th exterior power $\wedge^n(K_v^n)$ (where, as already said, $(e_1, \ldots, e_n)$ is the canonical K_v-basis of K_v^n).
- *integral* if L is contained in R_v^n,
- *rational* if L is contained in K^n,
- *axial* if for every $i \in [\![1, n]\!]$, we have $(K_v e_i) \cap L \neq \{0\}$.

Any element of $\mathrm{GL}_n(K_v)$ mapping the canonical K_v-basis $(e_1, \ldots, e_n)$ of K_v^n to a K_v-basis $(b_1, \ldots, b_n)$ such that $b_1 \wedge \ldots \wedge b_n = e_1 \wedge \ldots \wedge e_n$ has determinant 1. Hence by Equation (10), special unimodular R_v-lattices are unimodular.

For instance, if $I_1, \ldots, I_n$ are nonzero principal ideals of R_v, then $\prod_{i=1}^n I_i$ is an integral R_v-lattice in K_v^n. Note that an integral R_v-lattice, being a finite index subgroup of R_v^n, is axial. If x is an axial R_v-lattice, since x has an R_v-basis $(b_1, \ldots, b_n)$ which is a K_v-basis of K_v^n, by Kramer's formula to solve a system of $n-1$ linearly independent linear equations in n variables in terms of one of these variables, for every $i \in [\![1, n]\!]$, the intersection $R_v e_i \cap x$ is a rank-1 R_v-submodule of $K_v e_i$. Hence there exists $\lambda_i \in K_v^\times$ such that $R_v e_i \cap x = R_v \lambda_i e_i$.

For every integral R_v-lattice L, by the structure theorem of finitely generated torsion modules over a Dedekind ring (see for instance [Nar, Theo. 1.41] without the uniqueness statement), there exist unique nonzero ideals $I_1, \ldots, I_n \in \mathscr{I}_v^+$ such that $I_1 \,|\, I_2 \,|\, \ldots \,|\, I_n$ and R_v^n/L is isomorphic to $\prod_{i=1}^n R_v/I_i$ as an R_v-module. The n-tuple $(I_1, \ldots, I_n)$, or the isomorphism class of the R_v-module R_v^n/L, is called the *type* of the integral lattice L. If $I_1 = s_1 R_v, \ldots, I_n = s_n R_v$ are principal ideals, we will also say that the type of L is $(s_1, \ldots, s_n)$ (which is well defined modulo $(R_v^\times)^n$). For instance, the type of R_v^n is $(1, \ldots, 1)$. The group $\mathrm{GL}_n(R_v)$ acts on the set of integral R_v-lattices of K_v^n and two integral R_v-lattices are in the same $\mathrm{GL}_n(R_v)$-orbit if and only if they have the same type.

2.3 Homogeneous spaces of lattices

We denote by I_n the $n \times n$ identity matrix. Let $PG = \mathrm{PGL}_n(K_v) = \mathrm{GL}_n(K_v)/(K_v^\times I_n)$, which is a totally disconnected metrisable locally compact topological group, and $P\Gamma = \mathrm{PGL}_n(R_v) = \mathrm{GL}_n(R_v)/(R_v^\times I_n)$, which is a nonuniform lattice in PG. Throughout the paper, for every element $g = (g_{ij})_{1 \leqslant i,j \leqslant n} \in \mathrm{GL}_n(K_v)$, we denote by $[g] = [g_{ij}]_{1 \leqslant i,j \leqslant n}$ its image in PG, when necessary. Otherwise, abusing notation, we omit the brackets.

Let $P\mathscr{X}$ be the homogeneous space $PG/P\Gamma$, that identifies PG-equivariantly with the space of the homothety classes $[L] = K_v^\times L$ under $K_v^\times$ of the R_v-lattices L in K_v^n by the orbital map $[g]P\Gamma \mapsto [gR_v^n]$. Contrarily to the case of the real field $\mathbb{R}$ and the ring of integers $\mathbb{Z}$ of the number field $\mathbb{Q}$, in positive characteristic, there is a difference between $\mathrm{SL}_n(K_v)/\mathrm{SL}_n(R_v)$ and $\mathrm{PGL}_n(K_v)/\mathrm{PGL}_n(R_v)$ and it is sometines preferable to work with the latter one, or with the following avatar.

Let

$$G_1 = \mathrm{GL}_n^1(K_v) = \{g \in \mathrm{GL}_n(K_v) : |\det g| = 1\},$$

which is a unimodular totally disconnected metrisable locally compact topological group with center $ZG_1 = \mathscr{O}_v^\times I_n$. We identify the image of G_1 in PG with G_1/ZG_1. Let us denote $\Gamma_1 = \mathrm{GL}_n(R_v)$. By Equation (3), we have $\Gamma_1 \subset G_1$ and $P\Gamma \subset G_1/ZG_1$. Besides, Γ_1 is a nonuniform lattice in G_1.

Finally, let $G = \mathrm{SL}_n(K_v)$, which is a unimodular closed normal subgroup of G_1 with a split exact sequence of topological groups

$$1 \longrightarrow G \longrightarrow G_1 \longrightarrow \mathscr{O}_v^\times \longrightarrow 1 \tag{12}$$

with section $\xi : \mathscr{O}_v^\times \to G_1$ defined by

$$\xi : \lambda \mapsto \begin{pmatrix} \lambda & 0 \\ 0 & I_{n-1} \end{pmatrix}.$$

Let $\Gamma = \mathrm{SL}_n(R_v)$, which is a nonuniform lattice in G, with an induced split exact sequence

$$1 \longrightarrow \Gamma \longrightarrow \Gamma_1 \longrightarrow R_v^\times \longrightarrow 1 \tag{13}$$

with section $\xi_{|R_v^\times}$.

We endow PG (respectively G_1 and G) with its right-invariant Haar measure $\mathtt{m}_{PG}$ (respectively $\mathtt{m}_{G_1}$ and $\mathtt{m}_G$) such that its maximal compact-open subgroup $PG(\mathscr{O}_v) = \mathrm{PGL}_n(\mathscr{O}_v)$ (respectively $G_1(\mathscr{O}_v) = \mathrm{GL}_n(\mathscr{O}_v)$ and $G(\mathscr{O}_v) = \mathrm{SL}_n(\mathscr{O}_v)$) has Haar measure 1. Equation (12) induces a split exact sequence $1 \longrightarrow G(\mathscr{O}_v) \longrightarrow G_1(\mathscr{O}_v) \longrightarrow \mathscr{O}_v^\times \longrightarrow 1$ of compact groups. Since $\mathrm{vol}_v(\mathscr{O}_v^\times) = 1 - q_v^{-1}$ by Equation (9), for all $\lambda \in \mathscr{O}_v^\times$ and $g \in G$, we hence have

$$d\,\mathtt{m}_{G_1}(\xi(\lambda)g) = \frac{q_v}{q_v - 1}\; d\,\mathrm{vol}_v(\lambda)\; d\,\mathtt{m}_G(g)\,. \tag{14}$$

Let $\mathscr{X}_1$ be the space of unimodular R_v-lattices in K_v^n, endowed with the Chabauty topology. As justified by Equation (10), we identify homeomorphically and G_1-equivariantly the homogeneous space G_1/Γ_1 with $\mathscr{X}_1$ by the orbital map $g\Gamma_1 \mapsto g\,R_v^n$.

Since Γ_1 is a discrete subgroup of the unimodular group G_1, we endow the homogeneous space G_1/Γ_1 with the unique G_1-invariant measure such that the orbital map $G_1 \to G_1/\Gamma_1$ defined by $g \mapsto g\,\Gamma_1$ locally preserves the measure, and we endow $\mathscr{X}_1$ with the corresponding measure $\mathtt{m}_{\mathscr{X}_1}$. By for instance [HoP, Eq. (41)] (building on [Ser1, §3]), we have

$$\|\mathtt{m}_{\mathscr{X}_1}\| = \frac{q_v - 1}{q_v(q-1)} \prod_{i=1}^{n-1} \frac{\zeta_v(-i)}{q_v^i - 1}\,. \tag{15}$$

Let $\mathscr{X}$ be the closed subspace of $\mathscr{X}_1$ consisting in the special unimodular R_v-lattices in K_v^n, which is equal to the orbit in $\mathscr{X}_1$ of the standard R_v-lattice R_v^n under the action of the subgroup G of G_1. The stabiliser of R_v^n in G is exactly Γ. The homogeneous space G/Γ

identifies homeomorphically and G-equivariantly with the space $\mathscr{X}$ by the map $g\Gamma \mapsto g\,R_v^{\,n}$. Since $\Gamma = \Gamma_1 \cap G$, the inclusion map $G \to G_1$ induces an injection $G/\Gamma \to G_1/\Gamma_1$ which is a homeomorphism onto its image, and the following diagram is commutative:

$$\begin{array}{ccc} G/\Gamma & \to & G_1/\Gamma_1 \\ {\scriptstyle\simeq}\downarrow & & \downarrow{\scriptstyle\simeq} \\ \mathscr{X} & \hookrightarrow & \mathscr{X}_1\,. \end{array}$$

The compact group $\mathscr{O}_v^{\,\times}/R_v^{\times}$ acts continuously and freely on the topological space $\mathscr{X}_1$ by $(\lambda R_v^{\times}, \Lambda) \mapsto \xi(\lambda)\Lambda$. By Equations (12) and (13), the inclusion map $\mathscr{X} \to \mathscr{X}_1$ induces a homeomorphism $\mathscr{X} \to (\mathscr{O}_v^{\,\times}/R_v^{\times})\backslash\mathscr{X}_1$.

Since Γ is a discrete subgroup of the unimodular group G, we endow the homogeneous space G/Γ with the unique G-invariant measure such that the orbital map $G \to G/\Gamma$ defined by $g \mapsto g\,\Gamma$ locally preserves the measure, and we endow $\mathscr{X}$ with the corresponding measure $\mathtt{m}_{\mathscr{X}}$. We denote by vol'_v the measure on $\mathscr{O}_v^{\,\times}/R_v^{\times}$ such that the map $\mathscr{O}_v^{\,\times} \to \mathscr{O}_v^{\,\times}/R_v^{\times}$ locally preserves the measure. Its total mass is $\operatorname{vol}'_v(\mathscr{O}_v^{\,\times}/R_v^{\times}) = \frac{\operatorname{vol}_v(\mathscr{O}_v^{\,\times})}{\operatorname{Card} R_v^{\times}} = \frac{1-q_v^{-1}}{q-1}$. By Equation (14), for all $\lambda\,R_v^{\times} \in \mathscr{O}_v^{\,\times}/R_v^{\times}$ and $x \in \mathscr{X}$, we have

$$d\,\mathtt{m}_{\mathscr{X}_1}(\xi(\lambda)x) = \frac{q_v}{q_v-1}\;d\operatorname{vol}'_v(\lambda\,R_v^{\times})\;d\,\mathtt{m}_{\mathscr{X}}(x) \quad\text{and}\quad \|\,\mathtt{m}_{\mathscr{X}}\| = (q-1)\,\|\,\mathtt{m}_{\mathscr{X}_1}\|\,. \tag{16}$$

2.4 Systoles of lattices

The *(normalized) systole* of an R_v-lattice L in $K_v^{\,n}$, which depends only on the homothety class $[L]$ of L modulo $K_v^{\,\times}$, is defined by

$$\operatorname{sys}([L]) = \operatorname{sys}(L) = \Big(\frac{\operatorname{covol}(R_v^{\,n})}{\operatorname{covol}(L)}\Big)^{\frac{1}{n}} \min_{w\in L\smallsetminus\{0\}} \|w\|\,. \tag{17}$$

If L is unimodular, we simply have $\operatorname{sys}(L) = \min_{w\in L\smallsetminus\{0\}} \|w\|$. Mahler's compactness criterion (see for instance [KlST, Theo. 1.1]) says that for every $\epsilon > 0$, the *ϵ-thick part* of $P\mathscr{X}$, defined by

$$P\mathscr{X}^{\geqslant\epsilon} = \big\{\, x \in P\mathscr{X} : \operatorname{sys}(x) \geqslant \epsilon \,\big\}\,,$$

is compact in $P\mathscr{X}$. For every compact subset K of $P\mathscr{X}$, there exists $\epsilon > 0$ such that $K \subset P\mathscr{X}^{\geqslant\epsilon}$. Similarly, the ϵ-thick parts

$$\mathscr{X}_1^{\geqslant\epsilon} = \big\{\, L \in \mathscr{X}_1 : \operatorname{sys}(L) \geqslant \epsilon \,\big\} \quad\text{and}\quad \mathscr{X}^{\geqslant\epsilon} = \big\{\, L \in \mathscr{X} : \operatorname{sys}(L) \geqslant \epsilon \,\big\}$$

of $\mathscr{X}_1$ and $\mathscr{X}$ respectively are compact. For every compact subset K of $\mathscr{X}_1$ or $\mathscr{X}$, there exists $\epsilon > 0$ such that K is contained in $\mathscr{X}_1^{\geqslant\epsilon}$ or $\mathscr{X}^{\geqslant\epsilon}$. We denote by $P\mathscr{X}^{<\epsilon} = P\mathscr{X} \smallsetminus P\mathscr{X}^{\geqslant\epsilon}$, $\mathscr{X}_1^{<\epsilon} = \mathscr{X}_1 \smallsetminus \mathscr{X}_1^{\geqslant\epsilon}$ and $\mathscr{X}^{<\epsilon} = \mathscr{X} \smallsetminus \mathscr{X}^{\geqslant\epsilon}$ the *ϵ-thin parts* of $P\mathscr{X}$, $\mathscr{X}_1$ and $\mathscr{X}$ respectively.

Since $\mathrm{GL}_n(K_v)$ acts transitively on the set of R_v-lattices in $K_v^{\,n}$ and by Equation (10), the set of values of the (continuous) *systole function* $\operatorname{sys} : P\mathscr{X} \to \mathbb{R}$ is contained in $q_v^{\frac{1}{n}\mathbb{Z}}$. More precisely, let us prove that we have

$$q_v^{-\frac{1}{n}\mathbb{N}} \subset \operatorname{sys}(P\mathscr{X}) \subset q_v^{\frac{1}{n}\mathbb{Z}} \cap [0, q_v q^{\mathtt{g}-1}]\,.$$

The left inclusion follows by considering, for every $k \in \mathbb{N}$, the lattice $L = g_k R_v^{\,n}$ with $g_k = \begin{pmatrix} \pi_v^{-k} & 0 \\ 0 & I_{n-1}\end{pmatrix}$ (giving $\operatorname{sys}(L) = q_v^{-\frac{k}{n}}$ since $|\pi_v^{-k}| = q_v^{k} > 1$). In order to prove the right inclusion, let $[L] \in P\mathscr{X}$. Up to rescaling, we may assume that $\min_{w \in L \smallsetminus \{0\}} \|w\| = 1$. Then the closed ball $B(0, q_v^{-1})$ in $K_v{}^n$ injects in $K_v{}^n/L$ by the ultrametric triangle inequality. Thus by Equation (11) and the line before Equation (9), we have as wanted

$$\operatorname{sys}(L) = \Big(\frac{\operatorname{covol}(R_v{}^n)}{\operatorname{covol}(L)}\Big)^{\frac{1}{n}} \leqslant \Big(\frac{q^{(\mathfrak{g}-1)n}}{\operatorname{vol}_v^n(B(0,q_v^{-1}))}\Big)^{\frac{1}{n}} = \frac{q^{\mathfrak{g}-1}}{\operatorname{vol}_v(\pi_v \mathscr{O}_v)} = q_v\, q^{\mathfrak{g}-1}\,.$$

When $\mathfrak{g} = 0$ and $\deg v = 1$, we have $\operatorname{sys}(L) \leqslant 1$, which is optimal since $\operatorname{sys}(R_v{}^n) = 1$.

Since the image $\operatorname{sys}(P\mathscr{X})$ is contained in $q_v^{\frac{1}{n}\mathbb{Z}}$, there is a partition $P\mathscr{X} = \bigcup_{k=1}^n P\mathscr{X}_k$ into nonempty closed and open subsets of $P\mathscr{X}$, defined by

$$\forall\, k \in [\![1, n]\!], \quad P\mathscr{X}_k = \{x \in P\mathscr{X} : n \log_{q_v}(\operatorname{sys}(x)) \equiv k - 1 \mod n\}\,.$$

The norm $\|\cdot\|$ having values on $K_v{}^n \smallsetminus \{0\}$ in $q_v^{\mathbb{Z}}$, for every R_v-lattice L and every $g \in \mathrm{GL}_n(K_v)$, by Equation (10), we have

$$\begin{aligned} n \log_{q_v}(\operatorname{sys}(gL)) &\equiv \log_{q_v}\Big(\frac{\operatorname{covol}(R_v{}^n)}{\operatorname{covol}(gL)}\Big) \equiv \log_{q_v}\Big(\frac{\operatorname{covol}(R_v{}^n)}{\operatorname{covol}(L)}\Big) - \log_{q_v}|\det g| \\ &\equiv n \log_{q_v}(\operatorname{sys} L) - \log_{q_v}|\det g| \mod n\,. \end{aligned}$$

Hence the image G_1/ZG_1 of G_1 in PG acts transitively on each one of the strata $P\mathscr{X}_k$ for $k \in [\![1, n]\!]$. Since $P\Gamma \subset G_1/ZG_1$, the stratum $P\mathscr{X}_1$ thus identifies (G_1/ZG_1)-equivariantly with the homogeneous space $(G_1/ZG_1)/P\Gamma$. Furthermore, for every $k' \in \mathbb{N}$, the element $g_{k'} = \begin{pmatrix} \pi_v^{k'} & 0 \\ 0 & I_{n-1} \end{pmatrix}$ maps $P\mathscr{X}_k$ to $P\mathscr{X}_{k''}$ where $k'' \in [\![1, n]\!]$ satisfies $k'' = k + k' \mod n$.

2.5 Diagonal subgroups

We denote by $\widetilde{A}$ the diagonal subgroup of $\mathrm{GL}_n(K_v)$, and by PA its image in PG, that we also call the *diagonal subgroup* of PG. Let

$$\widetilde{D} = \left\{ \begin{pmatrix} \pi_v^{-k_1} & & 0 \\ & \ddots & \\ 0 & & \pi_v^{-k_n} \end{pmatrix} : k_1, \dots, k_n \in \mathbb{Z} \right\} \subset \widetilde{A}\,.$$

We denote by PD the image of $\widetilde{D}$ in PG. Note that the diagonal subgroup PA is a closed noncompact subgroup of PG which is the direct product $PA = PA(\mathscr{O}_v)\, PD$ of its maximal compact subgroup $PA(\mathscr{O}_v) = PA \cap \mathrm{PGL}_n(\mathscr{O}_v)$ and its discrete subgroup PD. As seen at the end of the previous subsection 2.4, the group PD permutes transitively the strata $P\mathscr{X}_k$ for $k \in [\![1, n]\!]$.

Also note that if L is an axial R_v-lattice in $K_v{}^n$, then $a\,L$ is an axial R_v-lattice for every $a \in \widetilde{A}$, and in particular, every R_v-lattice homothetic to L is axial. Hence we may define an *axial PA-orbit* in $P\mathscr{X}$ to be a PA-orbit which contains the homothety class of an axial R_v-lattice, or equivalently a PA-orbit all of whose elements are homothety classes of axial R_v-lattices.

Let $\exp : \mathbb{Z}^n \to \mathrm{GL}_n(K_v)$ be the map $\mathbf{k} = (k_1, \dots, k_n) \mapsto \begin{pmatrix} \pi_v^{-k_1} & & 0 \\ & \ddots & \\ 0 & & \pi_v^{-k_n} \end{pmatrix}$, which is an injective group morphism with image $\widetilde{D}$. We have $\mathfrak{a} = \exp(1-n, 1, \dots, 1)$ by Equation

(2). We will also denote by exp its restriction to

$$\mathbb{Z}_0^n = \{\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n : k_1 + \ldots + k_n = 0\}.$$

Note that there exists no global exponential map of matrices in positive characteristic. The above map exp is a (very weak) ersatz for it.

We define $A_1 = \widetilde{A} \cap G_1$, $A = \widetilde{A} \cap G$ and $D = \widetilde{D} \cap G_1 = \widetilde{D} \cap G = \exp(\mathbb{Z}_0^n)$. Thus A_1 and A are the direct products $A_1 = A_1(\mathscr{O}_v)D$ and $A = A(\mathscr{O}_v)D$ of their maximal compact subgroups $A_1(\mathscr{O}_v) = A_1 \cap \mathrm{GL}_n(\mathscr{O}_v) = \widetilde{A} \cap \mathrm{GL}_n(\mathscr{O}_v)$ and $A(\mathscr{O}_v) = A \cap \mathrm{SL}_n(\mathscr{O}_v)$ respectively with their discrete subgroup D. The split exact sequence in Equation (12) gives a split exact sequence

$$1 \longrightarrow A \longrightarrow A_1 \longrightarrow \mathscr{O}_v^{\times} \longrightarrow 1 \tag{18}$$

with section ξ (which has values in A_1).

2.6 Homogeneous measures on diagonal orbits

Recalling that $A_1(\mathscr{O}_v)$ and $A(\mathscr{O}_v)$ are the maximal compact-open subgroups of the diagonal groups A_1 and A, we endow the abelian locally compact groups A_1 and A with their unique Haar measure $\mathtt{m}_{A_1}$ and $\mathtt{m}_A$ normalized so that

$$\mathtt{m}_{A_1}(A_1(\mathscr{O}_v)) = \mathtt{m}_A(A(\mathscr{O}_v)) = 1. \tag{19}$$

By Equation (18), for all $\lambda \in \mathscr{O}_v^{\times}$ and $a \in A$, we have

$$d\,\mathtt{m}_{A_1}(\xi(\lambda)a) = \frac{q_v}{q_v - 1}\, d\,\mathrm{vol}_v(\lambda)\; d\,\mathtt{m}_A(a). \tag{20}$$

We denote by $\mathtt{m}_{\mathbb{Z}_0^n}$ the counting measure on $\mathbb{Z}_0^n$, and, in order to simplify notation,

$$da = d\,\mathtt{m}_{A_1|A_1(\mathscr{O}_v)}(a), \quad da = d\,\mathtt{m}_{A|A(\mathscr{O}_v)}(a) \quad \text{and} \quad d\,\mathbf{k} = d\,\mathtt{m}_{\mathbb{Z}_0^n}(\mathbf{k}),$$

which are measures on $A_1(\mathscr{O}_v)$, $A(\mathscr{O}_v)$ and $\mathbb{Z}_0^n$ respectively.

The maps $(a, \mathbf{k}) \mapsto a\exp(\mathbf{k})$ from $A_1(\mathscr{O}_v) \times \mathbb{Z}_0^n$ to A_1 and from $A(\mathscr{O}_v) \times \mathbb{Z}_0^n$ to A are isomorphisms of topological groups, and we have

$$d\,\mathtt{m}_{A_1}(a\exp(\mathbf{k})) = da\; d\,\mathbf{k} \quad \text{and} \quad d\,\mathtt{m}_A(a\exp(\mathbf{k})) = da\; d\,\mathbf{k}.$$

For every $x \in \mathscr{X}_1$ (resp. $x \in \mathscr{X}$), with $\theta_x : a \mapsto ax$ the orbital map from A_1 to A_1x (resp. A to Ax), we define the *orbital measure* $\overline{\mu}_x = \overline{\mu}_{A_1x}$ (resp. $\mu_x = \mu_{Ax}$) on the orbit A_1x (resp. Ax) by

$$\overline{\mu}_x = \overline{\mu}_{A_1x} = (\theta_x)_*\mathtt{m}_{A_1} \quad (\text{resp. } \mu_x = \mu_{Ax} = (\theta_x)_*\mathtt{m}_A),$$

so that for $a \in A_1(\mathscr{O}_v)$ (resp. $a \in A(\mathscr{O}_v)$) and $\mathbf{k} \in \mathbb{Z}_0^n$, we have

$$d\,\overline{\mu}_x(a\exp(\mathbf{k})\,x) = da\; d\,\mathbf{k} \quad (\text{resp. } d\mu_x(a\exp(\mathbf{k})\,x) = da\; d\,\mathbf{k}).$$

If θ_x is a proper map (see Corollary 3.4 for characterisations), then $\overline{\mu}_x$ is an A_1-invariant infinite locally finite measure on $\mathscr{X}_1$ with support equal to the orbit A_1x (resp. μ_x is an A-invariant locally finite infinite measure on $\mathscr{X}$ with support equal to the orbit Ax).

By Equation (20) and by the definition of vol'_v at the end of Section 2.3, for every $x \in \mathscr{X}$, the measure $\overline{\mu}_x$ on $\mathscr{X}_1$ is an average of orbital measures on $\mathscr{X}$:

$$\overline{\mu}_x = \frac{q_v}{q_v - 1} \int_{\lambda \in \mathscr{O}_v^{\times}} \xi(\lambda)_*\mu_x \; d\,\mathrm{vol}_v(\lambda) = \frac{q_v(q+1)}{q_v - 1} \int_{\mathscr{O}_v^{\times}/R_v^{\times}} \xi(\lambda)_*\mu_x \; d\,\mathrm{vol}'_v(\lambda R_v^{\times}). \tag{21}$$

3 A classification of the divergent diagonal orbits

The following characterisation of the divergent diagonal orbits is due to Margulis in the case of the real field $\mathbb{R}$ and the ring of integers $\mathbb{Z}$, see [TW, Theo. 1.2]. See for instance [Wei1, Wei2, ST] for complementary studies in the real case. Our proof follows the same scheme of proof as in the real case.

Theorem 3.1 *For every $x = [L] \in P\mathscr{X}$, the following assertions are equivalent.*

(1) *The map $a \mapsto a\,x$ from PA to $P\mathscr{X}$ is a proper map.*
(2) *The map $d \mapsto d\,x$ from PD to $P\mathscr{X}$ is a proper map.*
(3) *There exists $g \in (PA)(\mathrm{PGL}_n(K))$ such that $[L] = g[R_v^{\,n}]$.*
(4) *The orbit $PA\,x$ contains the homothety class of an integral R_v-lattice.*
(5) *The orbit $PA\,x$ contains the homothety class of an axial R_v-lattice.*
(6) *Every element of the orbit $PA\,x$ is the homothety class of an axial R_v-lattice.*

If one of the above assertions is satisfied, we say that the orbit $PA\,x$ of x by the diagonal subgroup PA is *divergent*. Hence the divergent PA-orbits are the axial ones (as defined in Subsection 2.5).

Proof. Assertion (3) implies Assertion (4), since K is the field of fractions of R_v, hence for every $g' \in \mathrm{GL}_n(K)$, if $r \in R_v \smallsetminus \{0\}$ is the product of the denominators of the nonzero entries of g' written as fractions in R_v, then $rg'R_v^{\,n}$ is an integral R_v-lattice. It is immediate that Assertion (4) implies Assertion (5), since an integral R_v-lattice is an axial R_v-lattice. We have already seen in Subsection 2.5 that the Assertions (5) and (6) are equivalent.

Let us prove that Assertion (6) implies Assertion (1). Indeed, assume that L is axial and normalized in its homothety class to have covolume between 1 and $q_v^{\,n}$. Every element $a \in \widetilde{A}$ may be multiplied by a central element of $\mathrm{GL}_n(K_v)$ in order to have absolute value of its determinant between 1 and $q_v^{\,n}$. Then if a goes to infinity in $\widetilde{A}$, it has a diagonal entry that goes to 0. Hence the R_v-lattice aL has its covolume remaining between 1 and q_v^{2n}, and has a nonzero vector on the coordinate axis corresponding to that diagonal entry that goes to 0. Thus its image in $P\mathscr{X}$ leaves every compact subset of $P\mathscr{X}$ by Mahler's compactness criterion. Note that Assertion (1) and Assertion (2) are equivalent, since $PA(\mathscr{O}_v)$ is compact and $PA = PA(\mathscr{O}_v)\,PD$.

It remains to prove that Assertion (2) implies Assertion (3). We first give two lemmas.

Lemma 3.2 *There exist $c > 1$, a bounded open neighborhood W of 0 in $K_v^{\,n}$ and a finite subset F of $D = \widetilde{D} \cap \mathrm{SL}_n(K_v)$, such that for every $g'' \in \mathrm{GL}_n(K_v)$ with $\det(g'') \in [1, q_v^{\,n}]$, there exists $f \in F$ such that for every $w \in (g''R_v^{\,n}) \cap W$, we have*

$$\|fw\| \geqslant c\|w\|\,.$$

Proof. Each element in $\mathrm{GL}_n(K_v)$ with absolute value of its determinant in $[1, q_v^{\,n}]$ only multiplies the volume $\mathrm{vol}_v^{\,n}$ by a constant in $[1, q_v^{\,n}]$ by Equation (8). Hence there exists an open ball W centered at 0 in $K_v^{\,n}$ with small enough radius such that for every element $g'' \in \mathrm{GL}_n(K_v)$ with $\det(g'') \in [1, q_v^{\,n}]$, the K_v-linear subspace generated by $(g''R_v^{\,n}) \cap W$ is a proper K_v-linear subspace of $K_v^{\,n}$.

For every $d \in [\![1, n-1]\!]$, let $\mathrm{Gr}_d(K_v^{\,n})$ be the Grassmannian space of d-dimensional K_v-linear subspaces of $K_v^{\,n}$, endowed with the Chabauty topology. Let us prove that there exist $c_d > 1$ and a finite subset F_d of elements in D such that for every $V \in \mathrm{Gr}_d(K_v^{\,n})$, there exists $f \in F_d$ such that for every $w \in V$, we have $\|fw\| \geqslant c_d\|w\|$. This proves the result by taking $F = \bigcup_{1\leqslant d\leqslant n-1} F_d$ and $c = \min_{1\leqslant d\leqslant n-1} c_d > 1$. By the compactness of $\mathrm{Gr}_d({K_v}^n)$ and of the unit sphere of ${K_v}^n$, and by the homogeneity of the norm $\|\cdot\|$ (so that $\|{\pi_v}^{\log_{q_v}\|w\|}w\| = 1$ for every $w \in {K_v}^n \smallsetminus \{0\}$), we only have to prove that for every $V \in \mathrm{Gr}_d({K_v}^n)$, there exists $a \in D$ such that for every $w \in V$ with norm 1, we have $\|a\, w\| > 1$.

Since $d < n$, there exists $i_0 \in [\![1, n]\!]$ such that $K_v e_{i_0}$ is not contained in V. We claim that there exists $\epsilon_0 > 0$ such that for every $w = (w_1, \dots, w_n) \in V$ with norm 1, there exists $j_w \in [\![1, n]\!]$ different from i_0, such that $|w_{j_w}| \geqslant \epsilon_0$. Otherwise, for every $k \in \mathbb{N}$, there exists $w^{(k)} = (w_{1,k}, \dots, w_{n,k}) \in V$ with $\|w^{(k)}\| = 1$ and $|w_{i,k}| \leqslant \frac{1}{k+1}$ for every $i \in [\![1, n]\!] \smallsetminus \{i_0\}$. Up to extracting a subsequence by the compactness of the unit sphere of ${K_v}^n$ and since V is closed in ${K_v}^n$, the sequence $(w^{(k)})_{k\in\mathbb{N}}$ converges to a unit norm vector w_∞ in $V \cap (K_v e_{i_0})$, contradicting the fact that $K_v e_{i_0}$ is not contained in V.

Now, for every $k \in \mathbb{N}$, let a^k be the diagonal matrix with diagonal coefficients $a^k_{ii} = \pi_v^{-k}$ if $i \neq i_0$ and $a^k_{i_0 i_0} = \pi_v^{(n-1)k}$, which belongs to $D = \exp(\mathbb{Z}^n_0)$. Then, if k is large enough (for instance $k = \lceil -\log_{q_v} \epsilon_0 \rceil + 1$), for every element $w = (w_1, \dots, w_n) \in V$ with norm 1, we have

$$\|a^k\, w\| \geqslant |\pi_v^{-k} w_{j_w}| = q_v^k\, |w_{j_w}| \geqslant q_v^k\, \epsilon_0 > 1\ ,$$

which proves the result. □

Lemma 3.3 *For every element $g' \in \mathrm{GL}_n(K_v) \smallsetminus \widetilde{A}\ \mathrm{GL}_n(K)$, for every bounded open neighborhood W of 0 in ${K_v}^n$, for every finite subset J of $g'{R_v}^n \smallsetminus \{0\}$, for every finite subset C of D, there exists $a \in D \smallsetminus C$ such that*

$$(aJ) \cap W = \varnothing\,.$$

Proof. Let g', W, J, C be as in the statement. We first claim that there exists $i_0 \in [\![1, n]\!]$ such that

$$(K_v e_{i_0}) \cap (g'{R_v}^n) = \{0\}\,.$$

Assume for a contradiction that for every $i \in [\![1, n]\!]$ there exist $\underline{w}_i \in R_v^n$ and $a_i \in K_v \smallsetminus \{0\}$ such that $g'\underline{w}_i = a_i e_i$. Then $(\underline{w}_1, \dots, \underline{w}_n)$ is a K-basis of the K-linear space $\oplus_{1\leqslant i\leqslant n} K e_i$, and the transition matrix P from the canonical basis $(e_1, \dots, e_n)$ to this basis (so that $Pe_i = \underline{w}_i$ for every $i \in [\![1, n]\!]$) belongs to $\mathrm{GL}_n(K)$. Let $a' \in \widetilde{A}$ be the diagonal matrix with diagonal coefficients $a_1, \dots, a_n$ in this order. Then the linear map $(a')^{-1} g' P$ fixes the canonical basis, hence is the identity. Thus $g' = a' P^{-1} \in \widetilde{A}\ \mathrm{GL}_n(K)$, a contradiction to the assumption that $g' \notin \widetilde{A}\ \mathrm{GL}_n(K)$.

Now, for every $k \in \mathbb{N}$, let a^k be the diagonal matrix with diagonal coefficients $a^k_{ii} = \pi_v^{-k}$ if $i \neq i_0$ and $a^k_{i_0 i_0} = \pi_v^{(n-1)k}$, which belongs to D, and does not belong to C if k is large enough. For every $w = (w_1, \dots, w_n) \in g'{R_v}^n \smallsetminus \{0\}$, by the preliminary claim, there exists $i \neq i_0$ such that $w_i \neq 0$. Hence $\|a^k\, w\| \geqslant |\pi_v^{-k} w_i| = q_v^k\, |w_i|$ tends to $+\infty$ as $k \to +\infty$. Since J is finite and W is bounded, this implies that for k large enough, we have $(a^k J) \cap W = \varnothing$ and $a^k \in D \smallsetminus C$. □

Proof that Assertion (2) **implies Assertion** (3)**.** Let us fix $g \in PG \smallsetminus (PA)(\mathrm{PGL}_n(K))$, and let us prove that the map $d \mapsto dg[{R_v}^n]$ from PD to $P\mathscr{X}$ is not proper, which concludes the proof of Theorem 3.1. We fix a representative $g' \in \mathrm{GL}_n(K_v) \smallsetminus \widetilde{A}\ \mathrm{GL}_n(K)$ of g with $\det(g') \in [1, {q_v}^n]$. Let us prove that the map $d \mapsto [dg'{R_v}^n]$ from the discrete space D to $P\mathscr{X}$ is not proper, which implies the result.

Let c, W, F be as in Lemma 3.2. Without loss of generality, we may assume that $\mathrm{id} \in F$. Let

$$W_0 \subset \bigcap_{f \in F \cup F^{-1}} fW$$

be an open ball centered at 0 in $K_v{}^n$, contained in $\bigcap_{f \in F \cup F^{-1}} fW$ hence in W. Let

$$C' = \left\{ [L] \in P\mathscr{X} : \frac{\operatorname{covol}(L)}{\operatorname{covol}(R_v^n)} \in [1, q_v{}^n],\ \ L \cap W_0 = \{0\} \right\}, \tag{22}$$

which is a compact subset of $P\mathscr{X}$ by Mahler's compactness criterion (and by the definition (17) of the systole). For every finite subset C of D, let us prove that there exists an element $d_C \in D \smallsetminus C$ such that $[d_C g' R_v{}^n] \in C'$, which concludes the proof.

Let $J = (g' R_v{}^n \cap C^{-1} W) \smallsetminus \{0\}$, which is a finite subset of $g' R_v{}^n \smallsetminus \{0\}$ since $C^{-1}W$ is bounded. By Lemma 3.3, there exists $d_0 \in D \smallsetminus C$ such that

$$(d_0 J) \cap W = \varnothing\,. \tag{23}$$

Let us define by induction on $k \in \mathbb{N}$ an element $d_k \in D$ such that with the notation $\widetilde{d}_k = d_0 \dots d_k$, if $k \geqslant 1$, then $d_k \in F$ and

$$\forall\, w \in (\widetilde{d}_{k-1}\, g' R_v^n) \cap W, \quad \|d_k w\| \geqslant c\, \|w\|\,. \tag{24}$$

Let $k \in \mathbb{N}$, assume that $d_0, \dots, d_k$ have been constructed, and note that $\widetilde{d}_k = d_0 \dots d_k$ belongs to D. By Lemma 3.2, applied with $g'' = \widetilde{d}_k\, g'$ which has absolute value of its determinant in $[1, q_v{}^n]$, there exists $d_{k+1} \in F$ such that for every $w \in (\widetilde{d}_k\, g' R_v^n) \cap W$, we have $\|d_{k+1} w\| \geqslant c\, \|w\|$. This concludes the induction.

For every $k \geqslant 1$, by the definition of W_0 which is contained in fW for every $f \in F$, we have

$$(\widetilde{d}_k\, g' R_v^n) \cap W_0 \subset (\widetilde{d}_k\, g' R_v^n) \cap (d_k W) = d_k\big((\widetilde{d}_{k-1}\, g' R_v^n) \cap W\big)\,. \tag{25}$$

Hence by Equation (24), the minimal norm $\mathtt{n}_k$ of a nonzero vector in $(\widetilde{d}_k\, g' R_v^n) \cap W_0$ is at least c times the minimal norm of a nonzero vector in $(\widetilde{d}_{k-1}\, g' R_v^n) \cap W$. Since W_0 is a ball centered at 0 contained in W, this implies that either $(\widetilde{d}_{k-1}\, g' R_v^n) \cap W_0 = \{0\}$ or that $\mathtt{n}_k \geqslant c\, \mathtt{n}_{k-1}$. Since W_0 is bounded and $c > 1$, this implies by a decreasing induction that there exists $k \in \mathbb{N}$ such that $(\widetilde{d}_k\, g' R_v^n) \cap W_0 = \{0\}$. Let $k_* \in \mathbb{N}$ be the smallest element $k \in \mathbb{N}$ for which this equality is true, so that

$$(\widetilde{d}_{k_*}\, g' R_v^n) \cap W_0 = \{0\} \quad \text{and} \quad \forall\, k \in [\![0, k_* - 1]\!], \quad (\widetilde{d}_k\, g' R_v^n) \cap W_0 \neq \{0\}\,. \tag{26}$$

By Equation (10), we have $\frac{\operatorname{covol}(\widetilde{d}_k\, g' R_v^n)}{\operatorname{covol}(R_v^n)} = |\det(\widetilde{d}_k\, g')| \in [1, q_v{}^n]$. In particular, by the definition of C' in Equation (22), we have $[\widetilde{d}_{k_*}\, g' R_v^n] \in C'$. Let us prove that $\widetilde{d}_{k_*} \notin C$, which gives the wanted result using $d_C = \widetilde{d}_{k_*}$.

If $k_* = 0$, this follows by the construction of $d_0 = \widetilde{d}_0$. Assume that $k_* \geqslant 1$ and for a contradiction that $\widetilde{d}_{k_*} \in C$. By the minimality of k_*, let w be a nonzero element of $(\widetilde{d}_{k_*-1}\, g' R_v^n) \cap W_0$. If $k_* \geqslant 2$, by Equation (25), there exists $w' \in (\widetilde{d}_{k_*-2}\, g' R_v^n) \cap W$ such that $w = d_{k_*-1}\, w'$. By Equation (26), we have $\|w\| \geqslant c\, \|w'\|$. Hence $\|w'\| \leqslant \|w\|$ since $c \geqslant 1$, and w' belongs as w to the ball W_0 centered at 0. Thus $w' \in (\widetilde{d}_{k_*-2}\, g' R_v^n) \cap W_0$ and $w = d_{k_*-1}\, w'$. By induction, we may therefore write $w = \widetilde{d}_{k_*-1} w_0$ for a nonzero element

w_0 in $(g'R_v^n) \cap W_0$ such that $\widetilde{d}_j w_0 \in W_0$ for all $j \in [\![0, k_* - 1]\!]$. By the definition of W_0 which contains w and is contained in $f^{-1}W$ for every $f \in F$ and since $k_* \geqslant 1$, we have $d_{k_*} w \in W$. Hence $\widetilde{d}_{k_*} w_0 = d_{k_*} \widetilde{d}_{k_*-1} w_0 = d_{k_*} w$ belongs to W. Since $\widetilde{d}_{k_*} \in C$, this implies that $w_0 \in ((C^{-1}W) \cap (g'R_v^n)) \smallsetminus \{0\} = J$. Since $d_0 w_0 = \widetilde{d}_0 w_0 \in W_0 \subset W$, this contradicts Equation (23). □

Corollary 3.4 *For every $L \in \mathscr{X}_1$, the following assertions are equivalent.*

(1) *The orbit map $a \mapsto aL$ from A_1 to $\mathscr{X}_1$ is a proper map.*
(2) *The orbit map $d \mapsto dL$ from D to $\mathscr{X}_1$ is a proper map.*
(3) *There exists $g \in A_1 \,\mathrm{GL}_n^1(K)$ such that $L = g\, R_v^n$.*
(4) *The orbit $\widetilde{A}\, L$ of L by the diagonal subgroup $\widetilde{A}$ of $\mathrm{GL}_n(K_v)$ contains an integral (possibly nonunimodular) R_v-lattice.*
(5) *The orbit $A_1 L$ contains an axial (unimodular) R_v-lattice.*
(6) *Every element of the orbit $A_1 L$ is an axial (unimodular) R_v-lattice.*

Proof. Using the notation of Subsection 2.4, we have a canonical onto map $\mathscr{X}_1 \to P\mathscr{X}_1$ which associates to a unimodular R_v-lattice its homothety class. This map is equivariant with respect to the canonical morphism $G_1 \to G_1/ZG_1$, hence with respect to the canonical morphisms $A_1 \to PA$ and $D \to PD$. The above map $\mathscr{X}_1 \to P\mathscr{X}_1$ is a proper map, since its fibers are the compact subsets $\mathscr{O}_v^\times L$ for $L \in \mathscr{X}_1$.

The image of A_1 in PA is a finite index subgroup, with index n (and representatives of the classes the elements $\begin{bmatrix} \pi_v^{-k} & 0 \\ & \\ 0 & I_{n-1} \end{bmatrix}$ for $k \in [\![0, n-1]\!]$). As seen in Subsection 2.5, the space $P\mathscr{X}$ is the finite union of the strata $P\mathscr{X}_k$ for $k \in [\![1, n]\!]$ that are transitively permuted by PA.

Therefore for every $L \in \mathscr{X}_1$, the orbit map $a \mapsto aL$ from A_1 to $\mathscr{X}_1$ (respectively $d \mapsto dL$ from D to $\mathscr{X}_1$) is a proper map if and only if the orbit map $[a] \mapsto [a]\,[L]$ from PA to $P\mathscr{X}$ (respectively $[d] \mapsto [d]\,[L]$ from PD to $P\mathscr{X}$) is a proper map.

The result then follows from Theorem 3.1. □

4 A description of the divergent diagonal orbits

4.1 Compact core and quasicenters of divergent diagonal orbits

Let $x \in \mathscr{X}_1$ be a unimodular R_v-lattice in K_v^n, which is axial, or equivalently by Corollary 3.4 such that its orbit in $\mathscr{X}_1$ under the diagonal subgroup A_1 is divergent. In this subsection, we define and study several invariants associated with x or with its A_1-orbit $A_1 x$.

For every $i \in [\![1, n]\!]$, we define

$$\mathrm{sys}_i(x) = \log_{q_v} \min\big\{\|w\| : w \in (x \cap K_v e_i) \smallsetminus \{0\}\big\} \in \mathbb{Z}\ ,$$

that we call the (logarithmic) *ith-directional systole*. As seen in Subsection 2.2, since x is axial, there exists $\lambda_i \in K_v^\times$ such that $x \cap K_v e_i = R_v \lambda_i e_i$, hence we have $\mathrm{sys}_i(x) = \log_{q_v} |\lambda_i|$. We define

$$\tau_x = \tau_{A_1 x} = \sum_{i=1}^n \mathrm{sys}_i(x) \in \mathbb{Z}\ , \tag{27}$$

that we call the (truncated) *covolume* of the divergent orbit A_1x, and that we will use as a complexity for divergent orbits. Assertion (2) of Proposition 4.1 below says that the covolume τ_x is indeed an invariant of the A_1-orbit of x. We will illustrate in Proposition 4.4 when $n = 2$ why we think of τ_x as the volume of a canonically truncated divergent orbit A_1x. Let

$$\Delta^x = \{\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}_0^n : \forall\, i \in [\![1, n]\!], \quad k_i \geqslant -\operatorname{sys}_i(x)\}\,, \tag{28}$$

which is a finite subset of $\mathbb{Z}_0^n$, and let $A_1^x = A_1(\mathscr{O}_v)\exp(\Delta^x)$. The subset

$$C_x = C_{A_1x} = A_1^x x = A_1(\mathscr{O}_v)\exp(\Delta^x)x \tag{29}$$

of the A_1-orbit of x is compact and open in A_1x since $A_1(\mathscr{O}_v)$ is a compact-open subgroup of A_1, and is called the *compact core* of the divergent orbit A_1x. Assertion (2) of Proposition 4.1 below says that the compact core C_x is indeed an invariant of the A_1-orbit of x.

The *coordinate sublattice* of x is

$$x^{\rm coo} = (x \cap K_v e_1) + \ldots + (x \cap K_v e_n)\,. \tag{30}$$

It is indeed an R_v-lattice contained in x, and $a(x^{\rm coo}) = (ax)^{\rm coo}$ for every $a \in A_1$. In particular, the covolume of $x^{\rm coo}$ is constant on the A_1-orbit of x.

The *quasicenter* of the A_1-orbit of x is the unique point $\widehat{x} \in A_1x$ modulo the left action of $A_1(\mathscr{O}_v)$ (see Assertion (4) of Proposition 4.1 below for its existence and uniqueness) such that if $\big(P = \lfloor \frac{\tau_x}{n} \rfloor, Q = \tau_x - n\lfloor \frac{\tau_x}{n} \rfloor\big) \in \mathbb{N}^2$ is the Euclidean division (with $0 \leqslant Q < n$) of $\tau_x = Pn + Q$ by n, then

$$\forall\, i \in [\![1, Q]\!], \quad \operatorname{sys}_i(\widehat{x}) = P + 1 \quad \text{and} \quad \forall\, i \in [\![Q+1, n]\!], \quad \operatorname{sys}_i(\widehat{x}) = P\,. \tag{31}$$

For instance (see Proposition 4.6 for other examples), if $x = {R_v}^n$, we have $\operatorname{sys}_i(x) = 0$ for every $i \in [\![1, n]\!]$, hence $x^{\rm coo} = \widehat{x} = x$ and

$$\tau_x = 0, \quad \Delta^x = \{0\} \quad \text{and} \quad C_x = A_1(\mathscr{O}_v)x\,.$$

Proposition 4.1 *Let $x \in \mathscr{X}_1$ be an axial unimodular R_v-lattive, and let $i \in [\![1, n]\!]$.*

(1) *We have $q_v^{\operatorname{sys}_i(x)} = q^{1-\mathfrak{g}} \operatorname{vol}_v\big((K_v e_i)/(x \cap K_v e_i)\big)$ and $\operatorname{covol}(x^{\rm coo}) = q^{n(\mathfrak{g}-1)} q_v^{\tau_x}$.*

(2) *For every $a = \operatorname{diag}(a_1, \ldots, a_n) \in A_1$, we have*

$$\operatorname{sys}_i(ax) = \operatorname{sys}_i(x) - v(a_i)\,, \quad \tau_{ax} = \tau_x \quad \text{and} \quad C_{ax} = C_x\,.$$

(3) *We have $\tau_x \in \mathbb{N}$. Furthermore $\tau_x = 0$ if and only if $x = x^{\rm coo}$.*

(4) *There exists a quasicenter $\widehat{x}$ of the A_1-orbit of x, unique modulo the action of $A_1(\mathscr{O}_v)$.*

(5) *With $c_n = \frac{1}{(n-1)!}$, as $\tau_x \to +\infty$, we have*

$$\operatorname{Card} \Delta^x = c_n \tau_x^{n-1} + \operatorname{O}(\tau_x^{n-2})\,.$$

Proof. Let x and i be as in the statement.

(1) As said above, there exists $\lambda_i \in K_v^\times$ such that $x \cap K_v e_i = R_v \lambda_i e_i$ and we have $\operatorname{sys}_i(x) = \log_{q_v} |\lambda_i|$. By Equations (10) and (11), we have

$$\operatorname{vol}_v\big((K_v e_i)/(x \cap K_v e_i)\big) = |\lambda_i| \operatorname{covol}(R_v) = q_v^{\operatorname{sys}_i(x)} q^{\mathfrak{g}-1}\,.$$

The first claim of Assertion (1) follows. The second one follows from the first one and the definition of τ_x.

(2) We have $\min_{w\in ax\cap K_v e_i}\|w\| = |a_i|\min_{w\in x\cap K_v e_i}\|w\|$. Hence the first claim of Assertion (2) follows since $|a_i| = q_v^{-v(a_i)}$. The second claim follows by summation since $|\det a| = 1$. By the definition (28) of Δ^x and the first claim, we have

$$\Delta^{ax} = \{\mathbf{k} = (k_1,\dots,k_n)\in\mathbb{Z}_0^n : \forall\, i\in[\![1,n]\!],\quad k_i \geqslant -\operatorname{sys}_i(x) + v(a_i)\}\,.$$

For every $i\in[\![1,n]\!]$, there exists $a'_i\in\mathscr{O}_v^\times$ such that $a_i = a'_i\,\pi_v^{-(-v(a_i))}$. We may hence write $a = a'\exp\mathbf{k}'$ with $a'\in A_1(\mathscr{O}_v)$ and $\mathbf{k}' = (-v(a_1),\dots,-v(a_n))\in\mathbb{Z}_0^n$. Therefore

$$\Delta^{ax} = \Delta^x - \mathbf{k}'\,, \tag{32}$$

so that since A_1 is Abelian, we have

$$C_{ax} = A_1(\mathscr{O}_v)\exp(\Delta^{ax})ax = A_1(\mathscr{O}_v)a'\exp(\Delta^{ax}+\mathbf{k}')x = A_1(\mathscr{O}_v)\exp(\Delta^x)x = C_x\,.$$

(3) Since the unimodular R_v-lattice x contains its coordinate sublattice x^{coo}, we have $\operatorname{covol}(x^{\mathrm{coo}}) \geqslant \operatorname{covol}(x) = \operatorname{covol}(R_v^{\,n})$, with equality if and only if $x = x^{\mathrm{coo}}$. Hence by Equation (11) and by Assertion (1), we have $q_v^{\tau_x} = \frac{\operatorname{covol}(x^{\mathrm{coo}})}{\operatorname{covol}(R_v^{\,n})} \geqslant 1$. Therefore $\tau_x\geqslant 0$, with equality if and only if $x = x^{\mathrm{coo}}$.

(4) Let $P = \lfloor\frac{\tau_x}{n}\rfloor$ and $Q = \tau_x - n\lfloor\frac{\tau_x}{n}\rfloor$. Let $a = \exp\mathbf{k}$ where

$$\mathbf{k} = (-\operatorname{sys}_1(x)+P+1,\dots,-\operatorname{sys}_Q(x)+P+1,-\operatorname{sys}_{Q+1}(x)+P,\dots,-\operatorname{sys}_n(x)+P)\,.$$

It is easy to check that $\mathbf{k}\in\mathbb{Z}_0^n$ by the definitions of $\tau_x = \sum_{i=1}^n\operatorname{sys}_i(x)$, and of P and Q so that $\tau_x = nP+Q$. By the first claim of Assertion (2) and by Equation (31), the element $\widehat{x} = ax$ is a quasicenter of A_1x. If $\widehat{\widehat{x}}$ is another quasicenter of A_1x, if $a\in A_1$ is such that $\widehat{\widehat{x}} = a\widehat{x}$, then for every $i\in[\![1,n]\!]$, we have $|a_i| = \frac{q_v^{\operatorname{sys}_i(\widehat{x})}}{q_v^{\operatorname{sys}_i(\widehat{\widehat{x}})}}$ by the first claim of Assertion (2). Hence $|a_i| = 1$ by the definition (31) of the quasicenter and since τ_x is constant on the A_1-orbit of x by Assertion (2). Therefore $a\in A_1(\mathscr{O}_v)$, thus proving the uniqueness claim.

(5) For every $m\in\mathbb{N}$, let $\widehat{\Delta}(m) = \{\mathbf{k} = (k_1,\dots,k_n)\in\mathbb{Z}_0^n : \forall\, i\in[\![1,n]\!],\quad k_i\geqslant -m\}$. By Equation (32) and the definition of the quasicenter, we have

$$\operatorname{Card}\big(\widehat{\Delta}\big(\lfloor\frac{\tau_x}{n}\rfloor\big)\big) \leqslant \operatorname{Card}(\Delta^x) = \operatorname{Card}(\Delta^{\widehat{x}}) \leqslant \operatorname{Card}\big(\widehat{\Delta}\big(\lfloor\frac{\tau_x}{n}\rfloor+1\big)\big)\,.$$

Let us prove that if $c'_n = \frac{n^{n-1}}{(n-1)!}$, then, as $m\to+\infty$, we have

$$\operatorname{Card}(\widehat{\Delta}(m)) = c'_n m^{n-1} + \mathrm{O}(m^{n-2})\,. \tag{33}$$

This implies Assertion (5) with $c_n = \frac{c'_n}{n^{n-1}} = \frac{1}{(n-1)!}$.

We start the proof by the following elementary integral computation. We consider the Euclidean subspace $\mathbb{R}_0^n = \{\mathbf{t} = (t_1,\dots,t_n)\in\mathbb{R}^n : \sum_{i=1}^n t_i = 0\}$ of the standard Euclidean space $\mathbb{R}^n$, endowed with its Lebesgue measure $\operatorname{Leb}_{\mathbb{R}_0^n}$. By invariance under translation, this measure is proportional to the measure $dt_1\dots dt_{n-1}$ on $\mathbb{R}_0^n$, and the proportionality constant is classically computed as follows. Let $u = \frac{1}{\sqrt{n}}(1,1,\dots,1)$ which is a unit normal vector to the hyperplane $\mathbb{R}^n_0$. Let P be the fundamental polytope of the $\mathbb{Z}$-lattice $\mathbb{Z}^n_0$ in $\mathbb{R}^n_0$ generated by the vectors

$$u_1 = (1, -1, 0, \dots, 0), \quad u_2 = (0, 1, -1, 0, \dots, 0), \ \dots \ , \quad u_{n-1} = (0, \dots, 0, 1, -1)\,.$$

Note that the first $n-1$ coordinates of a point $s_1 u_1 + \dots + s_{n-1} u_{n-1}$ of the polytope P, with $(s_1, \dots, s_{n-1}) \in [0,1]^{n-1}$, are $t_1 = s_1$, $t_2 = s_2 - s_1$, …, $t_{n-1} = s_{n-1} - s_{n-2}$, so that $dt_1 \dots dt_{n-1} = ds_1 \dots ds_{n-1}$ and $dt_1 \dots dt_{n-1}(P) = 1$. Therefore

$$\frac{d\,\mathrm{Leb}_{\mathbb{R}^n_0}}{dt_1 \dots dt_{n-1}} = \frac{\mathrm{Leb}_{\mathbb{R}^n_0}(P)}{dt_1 \dots dt_{n-1}(P)} = \mathrm{Leb}_{\mathbb{R}^n_0}(P) = |\det(u_1, \dots, u_{n-1}, u)| = \sqrt{n}\,.$$

Note for future use that

$$\mathrm{covol}(\mathbb{R}^n_0/\mathbb{Z}^n_0) = \mathrm{Leb}_{\mathbb{R}^n_0}(P) = \sqrt{n}\,. \tag{34}$$

For every $\alpha > 0$, let $\widehat{\Delta}'(\alpha) = \{\mathbf{t} = (t_1, \dots, t_n) \in \mathbb{R}^n_0 : \forall\, i \in [\![1, n]\!],\ t_i \geqslant -\alpha\}$.

Lemma 4.2 *For every $\alpha > 0$, we have* $\mathrm{Leb}_{\mathbb{R}^n_0}(\widehat{\Delta}'(\alpha)) = \sqrt{n}\,\frac{(n\alpha)^{n-1}}{(n-1)!}$.

Proof. Up to using an homothety of ratio α, we may assume by homogeneity that $\alpha = 1$. For every $k \in [\![1, n]\!]$, using the standard conventions that $*^0 = 1$ and $\sum_\varnothing * = 0$, we define a map $g_k : \mathbb{R}^{n-k} \to \mathbb{R}$ by

$$(t_1, \dots, t_{n-k}) \mapsto \frac{1}{(k-1)!}\Big(k - \sum_{i=1}^{n-k} t_i\Big)^{k-1}\,.$$

Note that $g_1 = 1$ and $g_n = \frac{n^{n-1}}{(n-1)!}$ are constant. For all $k \in [\![1, n-1]\!]$ and $t_1, \dots, t_{n-k-1} \in \mathbb{R}$, by a straightforward integration, we have

$$\begin{aligned}\int_{-1}^{k-\sum_{i=1}^{n-k-1} t_i} g_k(t_1, \dots, t_{n-k-1}, s)\, ds &= \frac{1}{(k-1)!}\Big[-\frac{1}{k}\Big(k - \sum_{i=1}^{n-k-1} t_i - s\Big)^k\Big]_{s=-1}^{s=k-\sum_{i=1}^{n-k-1} t_i} \\ &= g_{k+1}(t_1, \dots, t_{n-k-1})\,.\end{aligned}$$

We have $t_i \geqslant -1$ and $\sum_{i=1}^n t_i = 0$ for every $i \in [\![1, n]\!]$ if and only if $-1 \leqslant t_i \leqslant n - i - \sum_{j=1}^{i-1} t_j$ for every $i \in [\![1, n-1]\!]$. Hence by an easy induction, we have, for every $k \in [\![1, n-1]\!]$,

$$\begin{aligned}\mathrm{Leb}_{\mathbb{R}^n_0}(\widehat{\Delta}'(1)) &= \sqrt{n}\int_{-1}^{(n-1)}\int_{-1}^{(n-2)-t_1} \dots \int_{-1}^{1-\sum_{i=1}^{n-2} t_i} dt_{n-1} \dots dt_2\, dt_1 \\ &= \sqrt{n}\int_{-1}^{(n-1)}\int_{-1}^{(n-2)-t_1} \dots \int_{-1}^{k-\sum_{i=1}^{n-k-1} t_i} g_k(t_1, \dots, t_{n-k})\, dt_{n-k} \dots dt_2\, dt_1\,.\end{aligned}$$

When $k = n-1$, we get $\mathrm{Leb}_{\mathbb{R}^n_0}(\widehat{\Delta}'(1)) = \sqrt{n}\int_{-1}^{(n-1)} g_{n-1}(t_1)\, dt_1 = \sqrt{n}\, g_n$, as wanted. □

By the standard Gauss counting argument, by Lemma 4.2 and by Equation (34), Equation (33) follows since

$$\mathrm{Card}(\widehat{\Delta}(m)) \sim \frac{\mathrm{Leb}_{\mathbb{R}^n_0}(\widehat{\Delta}'(m))}{\mathrm{covol}(\mathbb{R}^n_0/\mathbb{Z}^n_0)} = \frac{(nm)^{n-1}}{(n-1)!}\,. \quad \square$$

4.2 The mass behavior of the compact cores of divergent diagonal orbits

In this subsection, we prove that for continuous functions with support in a fixed compact subset of $\mathscr{X}_1$, most of their mass for the orbital measure $\overline{\mu}_x = \overline{\mu}_{A_1 x}$ (defined in Subsection 2.6) on a divergent orbit $A_1 x$ is carried by the compact core of $A_1 x$ as the truncated covolume goes to infinity.

We keep denoting by x an axial unimodular R_v-lattice. We denote by

$$\overline{\nu}_x = \overline{\nu}_{A_1 x} = \frac{1}{\overline{\mu}_x(C_x)}\, \overline{\mu}_x|_{C_x} \tag{35}$$

the restriction of the orbital measure $\overline{\mu}_x$ to the compact core $C_x = C_{A_1 x}$ of the divergent orbit $A_1 x$, normalized to be a probability measure on $\mathscr{X}_1$. It is well defined since C_x is a nonempty compact open subset of $A_1 x$, hence $0 < \overline{\mu}_x(C_x) < +\infty$. It is independent of the choice x of an element in the orbit $A_1 x$, and its support is equal to C_x. By Equation (29), $C_x = A_1(\mathscr{O}_v)\exp(\Delta^x)\,x$ is the disjoint union of the clopen subsets $A_1(\mathscr{O}_v)(\exp\mathbf{k})\,x$ for $\mathbf{k}\in\Delta^x$. By the normalisation of the Haar measure of A_1 in Equation (19), we have

$$\overline{\mu}_x(C_x) = \mathtt{m}_{A_1}(A_1(\mathscr{O}_v)\exp(\Delta^x)) = \operatorname{Card}(\Delta^x)\,. \tag{36}$$

This formula, paired with Assertion (5) of Proposition 4.1, says that up to an error term, up to a multiplicative constant and up to a power constant depending only on n, the truncated covolume τ_x is the orbital measure of the compact core of the divergent orbit $A_1 x$, again justifying its name. With the simplified notation of Subsection 2.6, for $a\in A_1(\mathscr{O}_v)$ and $\mathbf{k}\in\Delta^x$, we have

$$d\overline{\nu}_x(a\exp(\mathbf{k})x) = \frac{1}{\operatorname{Card}(\Delta^x)}\, da\; d\mathbf{k}|_{\Delta^x}\ . \tag{37}$$

If $y\in\mathscr{X}$, then Ay is divergent in $\mathscr{X}$ if and only if $A_1 y$ is divergent in $\mathscr{X}_1$, and we then similarly denote by

$$\nu_y = \nu_{Ay} = \frac{1}{\mu_y(C_y\cap Ay)}\, \mu_y|_{C_y\cap Ay}$$

the restriction of the orbital measure μ_y to the compact core $C_y\cap Ay = A(\mathscr{O}_v)\exp(\Delta^y)y$ of the divergent orbit Ay, normalized to be a probability measure on $\mathscr{X}$. For $a\in A(\mathscr{O}_v)$ and $\mathbf{k}\in\Delta^y$, again with the simplified notation of Subsection (2.6) now for $A(\mathscr{O}_v)$, we have

$$d\nu_y(a\exp(\mathbf{k})y) = \frac{1}{\operatorname{Card}(\Delta^y)}\, da\; d\,\mathbf{k}|_{\Delta^y}\ . \tag{38}$$

By Equation (20), the measure $\overline{\nu}_y$ on $\mathscr{X}_1$ is an average of normalized restrictions of orbital measures on $\mathscr{X}$:

$$\overline{\nu}_y = \frac{q_v}{q_v-1}\int_{\lambda\in\mathscr{O}_v^\times} \xi(\lambda)_*\nu_y\; d\operatorname{vol}_v(\lambda)\,. \tag{39}$$

In the next lemma, we denote by $\|\ \|_\infty$ the uniform norm of continuous functions with compact support.

Lemma 4.3 *For every $m\in\mathbb{N}$, for every continuous function $f\in \mathrm{C}_c(\mathscr{X}_1)$ with compact support contained in $\mathscr{X}_1^{\geqslant q_v^{-m}}$, for every axial element $x\in\mathscr{X}_1$, as $\tau_x\to+\infty$, we have*

$$\frac{1}{c_n\tau_x^{\,n-1}}\,\overline{\mu}_x(f) = \overline{\nu}_x(f) + \mathrm{O}\Big(\frac{(m+1)\,\|f\|_\infty}{\tau_x}\Big)\,.$$

Similarly, for all $y \in \mathscr{X}$ axial and $f \in \mathrm{C_c}(\mathscr{X})$ with support in $\mathscr{X}^{\geqslant q_v^{-m}}$, as $\tau_y \to +\infty$, we have

$$\frac{1}{c_n \tau_y^{n-1}}\, \mu_y(f) = \nu_y(f) + \mathrm{O}\Big(\frac{(m+1)\,\|f\|_\infty}{\tau_y}\Big)\,.$$

Proof. Let m, f, x be as in the first statement (the second one is similar). We define

$$\Delta^{x,m} = \big\{\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}_0^n : \forall\, i \in [\![1,n]\!], \quad k_i \geqslant -\operatorname{sys}_i(x) - m\big\}\,,$$

$A_1^{x,m} = A_1(\mathscr{O}_v)\exp(\Delta^{x,m})$ and $C_{x,m} = A_1^{x,m}\, x$, which respectively contain Δ^x, A_1^x and C_x. By Equation (32) and the definition of the quasicenter, as in the proof of Proposition 4.1 (5), we have

$$\operatorname{Card}\big(\widehat{\Delta}\big(\lfloor\frac{\tau_x}{n}\rfloor + m\big)\big) \leqslant \operatorname{Card}(\Delta^{x,m}) = \operatorname{Card}(\Delta^{\widehat{x},m}) \leqslant \operatorname{Card}\big(\widehat{\Delta}\big(\lfloor\frac{\tau_x}{n}\rfloor + m + 1\big)\big)\,.$$

Hence by Equation (33), as $\tau_x \to +\infty$, we have

$$\operatorname{Card}(\Delta^{x,m}) = c_n(\tau_x + nm)^{n-1} + \mathrm{O}((\tau_x + nm)^{n-2})\,. \tag{40}$$

Note that if $a = \operatorname{diag}(a_1, \dots, a_n) \in A_1 \smallsetminus A_1^{x,m}$, then there exist $i \in [\![1,n]\!]$, $k_i \in \mathbb{Z}$ and $a_i' \in \mathscr{O}_v^\times$ such that $a_i = a_i'\pi_v^{-k_i}$ and $k_i = -v(a_i) < -\operatorname{sys}_i(x) - m$. Hence since x is unimodular, by the definition (17) of the systole in Subsection 2.4 and of the logarithmic directional systoles in this section, and by Proposition 4.1 (2), we have

$$\log_{q_v} \operatorname{sys}(ax) \leqslant \min_{1 \leqslant j \leqslant n} \operatorname{sys}_j(ax) = \min_{1 \leqslant j \leqslant n} (\operatorname{sys}_j(x) - v(a_j)) \leqslant \operatorname{sys}_i(x) - v(a_i) < -m\,.$$

Thus $ax \notin \mathscr{X}_1^{\geqslant q_v^{-m}}$ and $f(ax) = 0$. Therefore, using

- Equations (35) and (36) for the first and second equalities (and similarly for $C_{x,m}$),
- Equation (40) for the third equality,
- with a $\mathrm{O}(\ )$ which depends only on n for $\tau_x \geqslant nm$ for the last equality,

we have

$$\begin{aligned}
\Big|\frac{1}{c_n\tau_x^{n-1}}\,\overline{\mu}_x(f) - \overline{\nu}_x(f)\Big| &= \Big|\Big(\frac{1}{c_n\tau_x^{n-1}}\overline{\mu}_x|_{C_{x,m}} - \frac{1}{\operatorname{Card}(\Delta^x)}\overline{\mu}_x|_{C_x}\Big)(f)\Big| \\
\leqslant \Big(\frac{\overline{\mu}_x(C_{x,m})}{c_n\tau_x^{n-1}} - \frac{\overline{\mu}_x(C_x)}{\operatorname{Card}(\Delta^x)}\Big)\|f\|_\infty &= \Big(\frac{\operatorname{Card}(\Delta^{x,m})}{c_n\tau_x^{n-1}} - 1\Big)\|f\|_\infty \\
= \Big(\frac{c_n(\tau_x+nm)^{n-1} + \mathrm{O}((\tau_x+nm)^{n-2})}{c_n\tau_x^{n-1}} - 1\Big)\|f\|_\infty & \\
= \Big(\big(1 + \frac{nm}{\tau_x}\big)^{n-1} + \mathrm{O}\big(\big(1+\frac{nm}{\tau_x}\big)^{n-2}\frac{1}{\tau_x}\big) - 1\Big)\|f\|_\infty &= \mathrm{O}\Big(\frac{(m+1)\,\|f\|_\infty}{\tau_x}\Big),
\end{aligned}$$

as wanted. □

4.3 Zigzag length and continued fractions

We assume in this whole subsection that $n = 2$ (and we will then use n as a variable element of $\mathbb{N}$), that $K = \mathbb{F}_q(Y)$ and that $\deg v = 1$ so that $R_v = \mathbb{F}_q[Y]$ (see Subsection 2.1). We give in this particular case a geometric interpretation (using the geodesic flow on a Bruhat-Tits tree) and an arithmetic interpretation (using continued fraction expansions) of the quantities defined in Subsection 4.1. We refer for instance to [BPP, §15.1, 15.2] for the background information.

Let $\mathbb{T}_v$ be the *Bruhat-Tits tree* of (PGL_2, K_v), see for instance [Ser2]. It is a regular tree of degree $\mathrm{Card}\,\mathbb{P}_1(\mathscr{O}_v/\pi_v\mathscr{O}_v) = q+1$ (since $q_v = q$ here) and its set of vertices $V\mathbb{T}_v$ is the set of homothety classes (under $K_v^{\times}$) $[\Lambda]$ of $\mathscr{O}_v$-lattices Λ in $K_v \times K_v$. We denote by $*_v$ the homothety class of the $\mathscr{O}_v$-lattice $\mathscr{O}_v \times \mathscr{O}_v$ generated by the canonical basis of $K_v \times K_v$. The left linear action of $G_1 = \mathrm{GL}_2^1(K_v)$ on $K_v \times K_v$ induces a left action of G_1 on $\mathbb{T}_v$, which preserves and is transitive on the set $V_{\mathrm{even}}\mathbb{T}_v$ of vertices at even distance from $*_v$. Let $\mathfrak{a} = \begin{pmatrix} \pi_v & 0 \\ 0 & {\pi_v}^{-1} \end{pmatrix} \in D = \widetilde{D} \cap \mathrm{SL}_2(R_v)$, which generates $D \simeq \mathbb{Z}$. The lattice $\Gamma_1 = \mathrm{GL}_2(R_v)$ in G_1 (or its projective version $P\Gamma$) is called the *Nagao lattice*, see [Nag, Weil].

We identify the projective line $\mathbb{P}_1(K_v)$ with $K_v \cup \{\infty\}$ using the map $[x:y] \mapsto xy^{-1}$ as usual, and we endow $\mathbb{P}_1(K_v)$ with the projective action of G_1. The boundary at infinity $\partial_\infty \mathbb{T}_v$ of $\mathbb{T}_v$ identifies G_1-equivariantly with $\mathbb{P}_1(K_v)$. The Nagao lattice Γ_1 acts transitively on the subset $\mathbb{P}_1(K)$ of $\mathbb{P}_1(K_v)$.

We denote by $\mathscr{G}\mathbb{T}_v$ the space of *geodesic lines* in $\mathbb{T}_v$ (that is, the set of isometric maps $\ell : \mathbb{Z} \to V\mathbb{T}_v$ endowed with the compact-open topology), endowed with the action by post-composition of G_1 defined by $(g,\ell) \mapsto \{g\,\ell : k \mapsto g\,\ell(k)\}$. Let

$$\mathscr{G}_{\mathrm{even}}\mathbb{T}_v = \{\ell \in \mathscr{G}\mathbb{T}_v : \ell(0) \in V_{\mathrm{even}}\mathbb{T}_v\}\,,$$

which is invariant by G_1. Let $\ell_* \in \mathscr{G}\mathbb{T}_v$ be the unique geodesic line with $\ell_*(-\infty) = \infty \in \partial_\infty\mathbb{T}_v$, $\ell_*(+\infty) = 0 \in \partial_\infty\mathbb{T}_v$, and $\ell_*(0) = *_v$, so that $\ell_*(2n) = \mathfrak{a}^n *_v$ for every $n \in \mathbb{Z}$.

A geodesic line ℓ in $\mathscr{G}\mathbb{T}_v$, as well as its image in $\Gamma_1\backslash\mathscr{G}\mathbb{T}_v$, is called *birational* if its two points at infinity $\ell(\pm\infty)$ belong to $\mathbb{P}_1(K)$. For instance, ℓ_* belongs to $\mathscr{G}_{\mathrm{even}}\mathbb{T}_v$ and is birational.

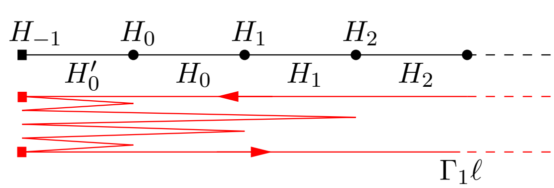
We denote by $(\phi^n)_{n\in\mathbb{Z}}$ the (discrete-time) *geodesic flow* on the space $\mathscr{G}\mathbb{T}_v$, defined by $\phi^n\ell : k \mapsto \ell(k+n)$ for all $n \in \mathbb{Z}$ and $\ell \in \mathscr{G}\mathbb{T}_v$, which commutes with the action of G_1, as well as its quotient flow on $\Gamma_1\backslash\mathscr{G}\mathbb{T}_v$. The stabilizer of ℓ_* for the transitive action of G_1 on $\mathscr{G}_{\mathrm{even}}\mathbb{T}_v$ is exactly $A_1(\mathscr{O}_v)$. Hence the map $A_1(\mathscr{O}_v)g \mapsto g^{-1}\,\ell_*$ is a homeomorphism $\widetilde{\Xi}$ from $A_1(\mathscr{O}_v)\backslash G_1$ to $\mathscr{G}_{\mathrm{even}}\mathbb{T}_v$, which is (anti-)equivariant with respect to the actions of Γ_1 on the right on $A_1(\mathscr{O}_v)\backslash G_1$ and on the left on $\mathscr{G}_{\mathrm{even}}\mathbb{T}_v$:

$$\forall\ g \in G_1,\ \ \forall\ \gamma \in \Gamma_1, \quad \widetilde{\Xi}(A_1(\mathscr{O}_v)g\gamma) = \gamma^{-1}\widetilde{\Xi}(A_1(\mathscr{O}_v)g)\,.$$

We denote by $\Xi : A_1(\mathscr{O}_v)\backslash G_1/\Gamma_1 \to \Gamma_1\backslash\mathscr{G}_{\mathrm{even}}\mathbb{T}_v$ the homeomorphism induced by $\widetilde{\Xi}$. We have the following crucial property relating the right action of A_1 (or the commuting right actions of D and $A_1(\mathscr{O}_v)$) on $\mathscr{X}_1 = G_1/\Gamma_1$ and the even-time geodesic flow on $\Gamma_1\backslash\mathscr{G}_{\mathrm{even}}\mathbb{T}_v$: for every $n \in \mathbb{Z}$, we have $\widetilde{\Xi} \circ \mathfrak{a}^{-n} = \phi^{2n} \circ \widetilde{\Xi}$, or equivalently

$$\forall\ g \in G_1, \quad \phi^{2n}(g\,\ell_*) = g\,\mathfrak{a}^n\,\ell_*\,. \tag{41}$$

Refering to [Ser2] for background, the quotient graph of groups $\Gamma_1\backslash\!\backslash\mathbb{T}_v$ is the following *modular ray*



where $H_{-1} = \Gamma_1 \cap \mathrm{GL}_2(\mathbb{F}_q)$, $H'_0 = H_0 \cap H_{-1}$ and, for every $n \in \mathbb{N}$,

$$H_n = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \Gamma_1 \;:\; a, d \in \mathbb{F}_q^\times, b \in \mathbb{F}_q[Y], \deg b \leqslant n+1 \right\}.$$

A birational geodesic line in $\Gamma_1\backslash\mathscr{G}\mathbb{T}_v$ starts from the point at infinity of the ray $\Gamma_1\backslash\mathbb{T}_v$, goes down to the origin $\Gamma_1 *_v$, then makes some back-and-forth to the origin for some (even, possibly zero) finite time, then goes up all the way to the point at infinity of the ray (see [Ser2, page 116], [Pau, §6.1] and the above picture).

Half the (even) length of the birational geodesic line $\Gamma_1\ell$ in $\Gamma_1\backslash\mathscr{G}_{\mathrm{even}}\mathbb{T}_v$ between the first and last time of passage through the origin $\Gamma_1 *_v$ is called the *zigzag length* of $\Gamma_1\ell$, and denoted by $\mathrm{zz}(\Gamma_1\ell) \in \mathbb{N}$. It is invariant under the action of the geodesic flow. For instance, $\mathrm{zz}(\Gamma_1\ell_*) = 0$.

Any element $f \in K_v = \mathbb{F}_q((Y^{-1}))$ may be uniquely written as a sum $f = [f] + \{f\}$ with $[f] \in R_v = \mathbb{F}_q[Y]$ (called the *integral part* of f) and $\{f\} \in \pi_v\mathscr{O}_v$ (called the *fractional part* of f). The *Artin map* $\Psi : \pi_v\mathscr{O}_v \smallsetminus \{0\} \to \pi_v\mathscr{O}_v$ is defined by $f \mapsto \left\{\frac{1}{f}\right\}$. Any $f \in K = \mathbb{F}_q(Y)$ has a unique finite *continued fraction expansion*

$$f = [a_0; a_1, \dots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\dots + \cfrac{1}{a_n}}}} = \frac{P_n}{Q_n}\,,$$

with $a_0 = a_0(f) = [f] \in R_v$ and if $f \neq a_0$ then $n = n(f) \in \mathbb{N} \smallsetminus \{0\}$ is such that we have $\Psi^n(f - a_0) = 0$ and $a_i = a_i(f) = \left[\frac{1}{\Psi^{i-1}(f-a_0)}\right]$ is a nonconstant polynomial for $1 \leqslant i \leqslant n$. The elements $a_0, a_1, \dots, a_n \in R_v$ are called the *coefficients* of the continued fraction expansion of f. The fraction $\frac{P_i}{Q_i} = [a_0; a_1, \dots, a_i] \in K$ is called the *i-th convergent* of f and is uniquely defined by induction by

$$\begin{array}{lll} P_{-1} = 1 & P_0 = a_0, & P_i = P_{i-1}a_i + P_{i-2} \\ Q_{-1} = 0 & Q_0 = 1, & Q_i = Q_{i-1}a_i + Q_{i-2} \end{array} \tag{42}$$

for $1 \leqslant i \leqslant n$. We refer for instance to [Las, Sch, Pau] for background on the above notions.

The stabilizer of $\infty \in \mathbb{P}_1(K_v)$ for the projective action of Γ_1 is its upper triangular subgroup $H_\infty = \bigcup_{n\in\mathbb{N}} H_n$. For every $f \in K_v = \mathbb{P}_1(K_v) \smallsetminus \{\infty\}$, there exists $g \in H_\infty$ such that $gf \in \pi_v\mathscr{O}_v$ (for instance $g = \left(\begin{smallmatrix} 1 & -[f] \\ 0 & 1 \end{smallmatrix}\right)$) and if $g' \in H_\infty$ also satisfies that $g'f \in \pi_v\mathscr{O}_v$, then there exists $u \in \mathbb{F}_q^\times$ such that $g'f = u(gf)$. Hence every birational geodesic line ℓ in $\mathscr{G}\mathbb{T}_v$ has a representative $\widetilde{\ell}$ in its class $\phi^{\mathbb{Z}}\Gamma_1\ell$ modulo the (commuting) actions of the geodesic flow and of Γ_1 which starts at time $-\infty$ from $\infty \in \partial_\infty\mathbb{T}_v$, passes at time $t = 0$ through $* \in V\mathbb{T}_v$, and ends at a point in $\pi_v\mathscr{O}_v \subset \partial_\infty\mathbb{T}_v$, unique up to multiplication by an element of $\mathbb{F}_q^\times$. Note that $t = 0$ is the time when $\widetilde{\ell}$ starts its zigzag part (see Proposition 4.4 for the computation of the time $\widetilde{\ell}$ ends its zigzag part). We define the *continued fraction total length* $\mathrm{cf}(\Gamma_1\ell)$ of $\Gamma_1\ell$ as the sum of the degrees of the coefficients of the continued fraction expansion $[0; a_1, a_2 \dots, a_n]$ of $\widetilde{\ell}(+\infty)$:

$$\mathrm{cf}(\Gamma_1\ell) = \sum_{i=1}^{n} \deg(a_i)\,.$$

This does not depend on the choice of $\widetilde{\ell}$, since for all $u \in \mathbb{F}_q^\times$ and $a_1, \dots, a_n \in R_v \smallsetminus \mathbb{F}_q$, we have $u[0; a_1, a_2 \dots, a_n] = [0; u^{-1}a_1, ua_2, \dots, u^{(-1)^n} a_n]$. We define the *height* $\mathrm{ht}(\Gamma_1 \ell)$ of $\Gamma_1 \ell$ as the degree of the denominator of the last convergent $\frac{P_n}{Q_n}$ of $\widetilde{\ell}(+\infty)$:

$$\mathrm{ht}(\Gamma_1 \ell) = \deg(Q_n)\,.$$

The following result says that the truncated covolume of a divergent orbit in G_1/Γ_1 coincides with the zigzag length, with the continued fraction total length and with the height of the corresponding orbit of the even-time geodesic flow in $\Gamma_1 \backslash \mathscr{G}_{\rm even}\mathbb{T}_v$.

Proposition 4.4 *For every $g \in G_1$, the A_1-orbit $A_1 g R_v^2$ of the R_v-lattice $g R_v^2$ is divergent in $\mathscr{X}_1$ if and only if the geodesic line $\Xi(A_1(\mathscr{O}_v) g \Gamma_1) = \Gamma_1 g^{-1} \ell_* \in \Gamma_1 \backslash \mathscr{G}_{\rm even}\mathbb{T}_v$ is birational, and we then have*

$$\tau_{A_1 g R_v^2} = \mathrm{ht}(\Gamma_1 g^{-1} \ell_*) = \mathrm{cf}(\Gamma_1 g^{-1} \ell_*) = \mathrm{zz}(\Gamma_1 g^{-1} \ell_*)\,. \qquad (43)$$

Proof. Let $g \in G_1$. By Corollary 3.4, we know that $A_1 g R_v^2$ is divergent if and only if $g \in A_1 \,\mathrm{GL}_2^1(K)$.

The group $\mathrm{GL}_2^1(K)$ acts transitively on the set of ordered pairs of distinct points of $\mathbb{P}_1(K)$, since for all $x, y \in K$, the element $\left(\begin{smallmatrix} 0 & 1 \\ 1 & -x \end{smallmatrix}\right) \in \mathrm{GL}_2^1(K)$ maps x to ∞ and the element $\left(\begin{smallmatrix} 1 & -y \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2^1(K)$ maps y to 0 while fixing ∞. Hence if the geodesic line $g^{-1}\ell_* \in \mathscr{G}_{\rm even}\mathbb{T}_v$ is birational, then there exist $h \in \mathrm{GL}_2^1(K)$ and $n \in \mathbb{N}$ such that $h g^{-1} \ell_* = \phi^{2n} \ell_* = \mathfrak{a}^n \ell_*$, using Equation (41) for the last equality. Hence $\mathfrak{a}^{-n} h g^{-1}$ fixes ℓ_*, whose stabilizer is $A_1(\mathscr{O}_v)$. Therefore $g \in A_1 \,\mathrm{GL}_2^1(K)$.

Conversely, assume that $g \in A_1 \,\mathrm{GL}_2^1(K)$. Since $A_1 = D A_1(\mathscr{O}_v)$, there exist $n \in \mathbb{Z}$, $h' \in A_1(\mathscr{O}_v)$ and $h \in \mathrm{GL}_2^1(K)$ such that $g = \mathfrak{a}^n h' h$. The points at infinity of $g^{-1}\ell_*$ are hence equal to the points at infinity of $h^{-1}\ell_*$, which are both in $\mathbb{P}_1(K)$, hence $g^{-1}\ell_*$ is birational. This proves the first claim.

Let us prove the first equality of Equation (43). If $A_1 g R_v^2$ is divergent, we may assume that $g \in \mathrm{GL}_2^1(K)$ by Corollary 3.4. By the transitivity properties of Γ_1, up to multiplying g on the left by an element of $A_1 \cap \mathrm{GL}_2^1(K)$ and on the right by an element of Γ_1, we may assume that $g^{-1} *_v = *_v$ and that the projective action of g^{-1} fixes ∞ and sends 0 to the last convergent $\frac{P_n}{Q_n}$ of $g^{-1}\ell_*(+\infty)$. Thus g has the form $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ with $a, b, d \in K$ with $|ad| = 1$. In particular, we have $g^{-1}\ell_* = \widetilde{g^{-1}\ell_*}$ with the above notation. Since multiplying g on the left by an element of $A_1 \cap \Gamma_1$ does not change $g^{-1}\ell_*(+\infty)$, we may assume that $a = d = 1$. Then $b = -\frac{P_n}{Q_n}$. Now, we have $g R_v^2 = R_v e_1 + (b e_1 + e_2) R_v$. Hence $g R_v^2 \cap K_v e_1 = R_v e_1$ and $g R_v^2 \cap K_v e_2 = R_v Q_n e_2$. Thus by Equation (3) and the definition of the directional systoles, we have

$$\mathrm{sys}_1(g R_v^2) = \log_q 1 = 0 \quad \text{and} \quad \mathrm{sys}_2(g R_v^2) = \log_q |Q_n| = \deg Q_n\ ,$$

so that $\tau_{A_1 g R_v^2} = \deg Q_n = \mathrm{ht}(\Gamma_1 g^{-1} \ell_*)$, as wanted. For use in the following remark, note that if $m = \left\lceil \frac{\deg Q_n}{2} \right\rceil$, then by Proposition 4.1 (2), we have

$$\mathrm{sys}_1(\mathfrak{a}^{-m} g R_v^2) = -v(\pi_v^{-m}) = m \quad \text{and} \quad \mathrm{sys}_2(\mathfrak{a}^{-m} g R_v^2) = \deg Q_n - v(\pi_v^{-m}) = \left\lfloor \frac{\deg Q_n}{2} \right\rfloor. \qquad (44)$$

The middle equality of Equation (43) follows by induction from Equation (42), noting that $\deg a_i \geqslant 1$ if $i \geqslant 1$. The last equality follows from [Pau, §6.3, Remarque 2.]. □

Remark. The above proof also gives that the compact core of a divergent A_1-orbit $A_1 g R_v^2$ corresponds to the part of the geodesic flow orbit of a birational geodesic line $\Gamma_1 \ell$ in the space $\Gamma_1 \backslash \mathscr{G}_{\rm even} \mathbb{T}_v$ where the base point $\Gamma_1 \ell(0)$ lies exactly in the zigzag part: more precisely

$$\Xi(A_1(\mathscr{O}_v) C_{A_1 g \Gamma_1}) = \big\{\Gamma_1 \phi^{2k}\big(\widetilde{g^{-1}\ell_*}\big) : 0 \leqslant k \leqslant {\rm zz}(\Gamma_1 g^{-1} \ell_*)\big\}\,.$$

Note that a quasicenter $\widehat{x}$ of a divergent A_1-orbit x is well defined up to the action of $A_1(\mathscr{O}_v)$, and Ξ is a homeomorphism from $A_1(\mathscr{O}_v)\backslash \mathscr{X}_1$ to $\Gamma_1 \backslash \mathscr{G}_{\rm even} \mathbb{T}_v$, hence looking at the image $\Xi(A_1(\mathscr{O}_v)\widehat{x})$ of the set of quasicenters is well defined. Equation (44) in the above proof gives besides that the quasicenter of a divergent A_1-orbit $A_1 g R_v^2$ (defined by Equation (31)) corresponds to the geodesic flow orbit point of a birational geodesic line in the space $\Gamma_1 \backslash \mathscr{G}_{\rm even} \mathbb{T}_v$ where the base point is almost at the midpoint of the zigzag part: more precisely

$$\Xi(A_1(\mathscr{O}_v)\widehat{A_1 g R_v^2}) = \Gamma_1 \phi^{2m}\big(\widetilde{g^{-1}\ell_*}\big) \quad \text{where} \quad m = \Big\lceil \frac{{\rm zz}(\Gamma_1 g^{-1} \ell_*)}{2} \Big\rceil\,.$$

4.4 Type and discriminant of the divergent diagonal orbits

In this section, we introduce two new invariants of the divergent diagonal orbits in $\mathscr{X}_1$, we gather the technical notation that will be used in the following sections, and we give a precise description of the divergent orbits whose equidistribution we will study.

Let $x \in P\mathscr{X}$ be the homothety class of an R_v-lattice whose PA-orbit is divergent in $P\mathscr{X}$. By Theorem 3.1 (4), we know that the orbit $PA\,x$ contains at least one homothety class $[L]$ of an integral R_v-lattice L. Since the normalized covolume $\frac{{\rm covol}(L)}{{\rm covol}(R_v^n)} \in q_v^{\mathbb{Z}}$ of L is at least 1 as $L \subset R_v^n$, and since any nonempty subset of $\mathbb{N}$ has a lower bound, there exists at least one integral R_v-lattice L_x whose homothety class belongs to $PA\,x$ and whose covolume is minimal. We define

- the *discriminant* of the divergent PA-orbit $PA\,x$ as ${\rm disc}(PA\,x) = \log_{q_v} \frac{{\rm covol}(L_x)}{{\rm covol}(R_v^n)} \in \mathbb{N}$ and
- the *type* of the divergent PA-orbit $PA\,x$ as the set of types (see Subsection 2.2) of the finitely many integral lattices L_x minimizing the covolume among the integral R_v-lattices whose homothety class belongs to $PA\,x$.

We refer to the proof of Proposition 4.6 (4) for examples of divergent PA-orbits having nonunique homothety classes of covolume-minimizing integral R_v-lattices.

We endow the infinite sets $(R_v \smallsetminus \{0\})^{n-1}$ and $R_v \smallsetminus \{0\}$ with the Fréchet filter of the complementary sets of their finite subsets, and we will denote as usual by $\lim\limits_{+\infty}$ the limits along this filter.

Let us introduce the notation that will be used in the remainder of this paper. In this section, we fix an element $\mathbf{s} = (s_2, \dots, s_n) \in (R_v \smallsetminus \{0\})^{n-1}$ such that

$$\exists\, \sigma \in {\rm Bij}([\![2, n]\!]), \qquad s_{\sigma^{-1}(2)} \mid s_{\sigma^{-1}(3)} \mid \dots \mid s_{\sigma^{-1}(n)} \quad \text{and } s_{\sigma^{-1}(n)} \in n\,\mathbb{Z}\,. \tag{45}$$

We then define $\mathbf{s}_* = s_{\sigma^{-1}(n)} \in n\,\mathbb{Z}$. Since the valuation v is nonpositive on $R_v \smallsetminus \{0\}$ by Equation (3), we have $v(\mathbf{s}_*) \leqslant 0$. Note that $\mathbf{s} \to +\infty$ if and only if $\mathbf{s}_* \to +\infty$, hence if and only if $-v(s) \to +\infty$. We define (independently of the choice of such a permutation σ)

$$\Lambda_{\mathbf{s}} = \Big\{ \big(\frac{r_2}{s_2}, \frac{r_3}{s_3}, \dots, \frac{r_n}{s_n}\big) : \forall\, i \in [\![2,n]\!],\ r_i \in R_v \text{ and } r_i R_v + s_i R_v = R_v \Big\} \mod R_v^{\,n-1}\,,$$

$$\Delta_{\mathbf{s}} = \big\{ \mathbf{k} = (k_1,\dots,k_n) \in \mathbb{Z}_0^n : k_1 \geqslant v(\mathbf{s}_*) \text{ and } \forall\, i \in [\![2,n]\!],\ k_i \geqslant 0 \big\}\,,$$

$$\lozenge_{\mathbf{s}} = \big\{ \mathbf{k} = (k_1,\dots,k_n) \in \mathbb{Z}_0^n : \forall\, i \in [\![2,n]\!],\ 0 \leqslant k_i \leqslant k_1 - v(\mathbf{s}_*) \big\}\,,$$

$$\mathbf{k_s} = \Big(\frac{n-1}{n}\, v(\mathbf{s}_*), -\frac{1}{n}\, v(\mathbf{s}_*), \dots, -\frac{1}{n}\, v(\mathbf{s}_*) \Big) \in \mathbb{Z}_0^n\,,$$

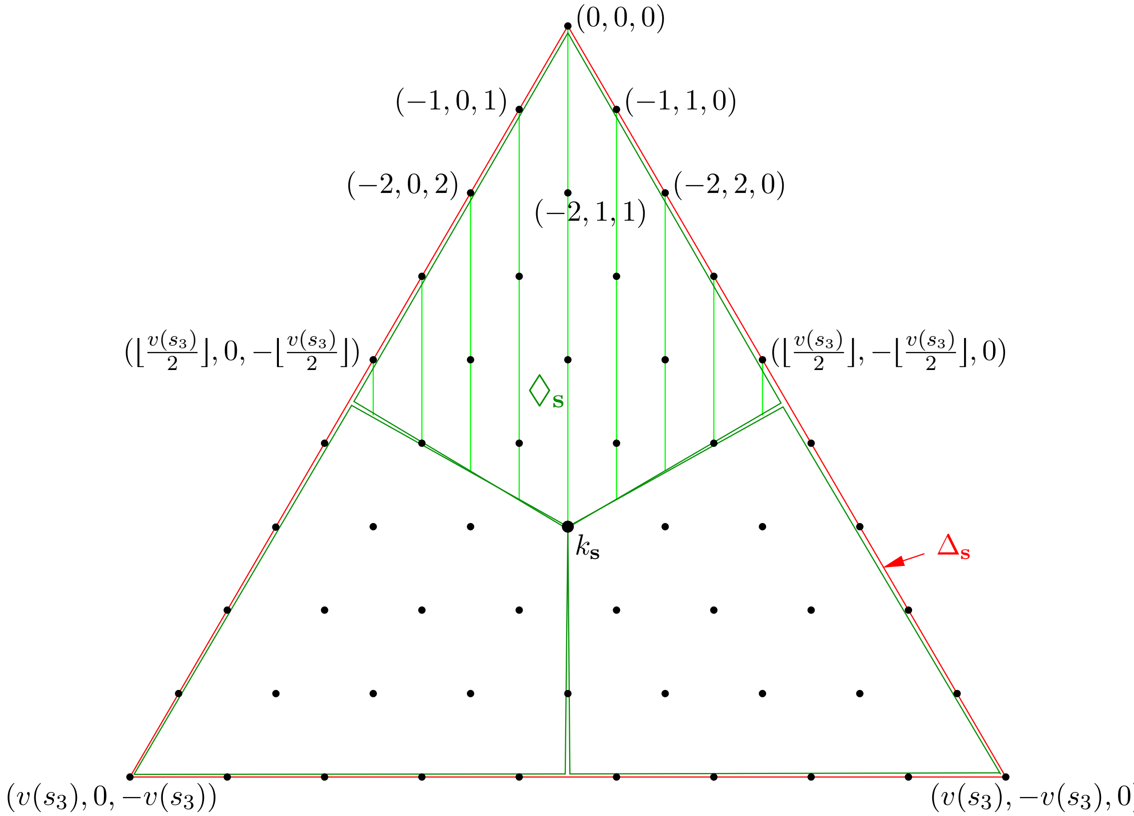
$$\widetilde{\Delta}_{\mathbf{s}} = \Delta_{\mathbf{s}} - \mathbf{k_s} = \Big\{ \mathbf{k} = (k_1,\dots,k_n) \in \mathbb{Z}_0^n : \forall\, i \in [\![1,n]\!],\ k_i \geqslant \frac{v(\mathbf{s}_*)}{n} \Big\}\,,$$

$$\widetilde{\lozenge}_{\mathbf{s}} = \lozenge_{\mathbf{s}} - \mathbf{k_s} = \big\{ \mathbf{k} = (k_1,\dots,k_n) \in \mathbb{Z}_0^n : \forall\, i \in [\![2,n]\!],\ \frac{v(\mathbf{s}_*)}{n} \leqslant k_i \leqslant k_1 \big\}\,.$$

Let us make some comments on this notation. We have $\mathbf{k_s} \in \lozenge_{\mathbf{s}} \subset \Delta_{\mathbf{s}}$ (see the following picture when $n = 3$). We have

$$\operatorname{Card}(\Lambda_{\mathbf{s}}) = \prod_{i=2}^{n} \varphi_v(s_i)\,, \quad \text{hence} \quad \operatorname{Card}(\Lambda_{(s,\dots,s)}) = (\varphi_v(s))^{n-1}\,, \tag{46}$$

where φ_v is the Euler function of R_v defined in Equation (4).



We denote by δ_y the unit Dirac mass at y. With $\mathfrak{u}_{\mathbf{t}}$ for $\mathbf{t} \in K_v^{\,n-1}$ defined in Equation (1), we define two probability measures on $\mathscr{X}$ by

$$\nu_{\mathbf{s}}^{\Delta} = \frac{1}{\operatorname{Card} \Lambda_{\mathbf{s}} \operatorname{Card} \Delta_{\mathbf{s}}} \sum_{\mathbf{t} \in \Lambda_{\mathbf{s}},\, \mathbf{k} \in \Delta_{\mathbf{s}}} \int_{a \in A(\mathscr{O}_v)} \delta_{a \exp(\mathbf{k})\, \mathfrak{u}_{\mathbf{t}} R_v^{\,n}}\; da\,, \tag{47}$$

and similarly, replacing $\Delta_{\mathbf{s}}$ by $\lozenge_{\mathbf{s}}$,

$$\nu_{\mathbf{s}}^{\lozenge} = \frac{1}{\operatorname{Card}\Lambda_{\mathbf{s}}\ \operatorname{Card}\lozenge_{\mathbf{s}}} \sum_{\mathbf{t}\in\Lambda_{\mathbf{s}},\ \mathbf{k}\in\lozenge_{\mathbf{s}}} \int_{a\in A(\mathscr{O}_v)} \delta_{a\exp(\mathbf{k})\,\mathfrak{u}_{\mathbf{t}}R_v^{\,n}}\ da\,. \tag{48}$$

We define similarly two probability measures $\overline{\nu}_{\mathbf{s}}^{\Delta}$ and $\overline{\nu}_{\mathbf{s}}^{\lozenge}$ on $\mathscr{X}_1$ by replacing A by A_1 in Equations (47) and (48), so that by Equation (20) we have

$$\overline{\nu}_{\mathbf{s}}^{\Delta} = \frac{q_v}{q_v-1}\int_{\lambda\in\mathscr{O}_v^{\,\times}} \xi(\lambda)_*\nu_{\mathbf{s}}^{\Delta}\ d\operatorname{vol}_v(\lambda)\,. \tag{49}$$

We denote by $\mathscr{S}_n$ the permutation group of $[\![1,n]\!]$, and by $\mathscr{S}_{n-1}$ the stabilizer of $\{1\}$ in $\mathscr{S}_n$. The group $\mathscr{S}_{n-1}$ acts K-linearly on K^{n-1} (preserving the subset $(R_v\smallsetminus\{0\})^{n-1}$) by $\sigma\cdot\mathbf{t} = (t_{\sigma^{-1}(2)},\dots,t_{\sigma^{-1}(n)})$ for all $\sigma\in\mathscr{S}_{n-1}$ and $\mathbf{t}=(t_2,\dots,t_n)\in K^{n-1}$. By construction, for all $\sigma\in\mathscr{S}_{n-1}$ and $\mathbf{s}\in(R_v\smallsetminus\{0\})^{n-1}$, the element $\mathbf{s}$ satisfies Equation (45) if and only if $\sigma\cdot\mathbf{s}$ does, and we have

$$\Lambda_{\sigma\cdot\mathbf{s}} = \sigma\cdot\Lambda_{\mathbf{s}},\quad v((\sigma\cdot\mathbf{s})_*) = v(\mathbf{s}_*),\quad \Delta_{\sigma\cdot\mathbf{s}} = \Delta_{\mathbf{s}},\quad \lozenge_{\sigma\cdot\mathbf{s}} = \lozenge_{\mathbf{s}},\quad\text{and}\quad \mathbf{k}_{\sigma\cdot\mathbf{s}} = \mathbf{k}_{\mathbf{s}}\,. \tag{50}$$

The group $\mathscr{S}_n$ acts $\mathbb{Z}$-linearly on $\mathbb{Z}_0^n$ by $\sigma\cdot\mathbf{k} = (k_{\sigma^{-1}(1)},\dots,k_{\sigma^{-1}(n)})$ for all $\sigma\in\mathscr{S}_n$ and $\mathbf{k}=(k_1,\dots,k_n)\in\mathbb{Z}_0^n$. Note that the subset $\widetilde{\Delta}_{\mathbf{s}}$ of $\mathbb{Z}_0^n$ is invariant under the action of $\mathscr{S}_n$. As said above, the subsets $\Delta_{\mathbf{s}}$ and $\lozenge_{\mathbf{s}}$ of $\mathbb{Z}_0^n$, as well as their point $\mathbf{k}_{\mathbf{s}}$, are invariant under the action of $\mathscr{S}_{n-1} = \operatorname{Stab}_{\mathscr{S}_n}\{1\}$. Let $\sigma_n = (1\dots n)$ be the standard n-cycle in $\mathscr{S}_n$. The cyclic group $\sigma_n^{\mathbb{Z}}$ of order n generated by σ_n acts freely on $\widetilde{\Delta}_{\mathbf{s}}\smallsetminus\{\mathbf{0}\}$ where $\mathbf{0}=(0,\dots,0)$. The subset $\widetilde{\lozenge}_{\mathbf{s}}$ is a (weak) fundamental domain for the action of $\sigma_n^{\mathbb{Z}}$ on $\widetilde{\Delta}_{\mathbf{s}}$: we have (with nondisjoint union)

$$\widetilde{\Delta}_{\mathbf{s}} = \bigcup_{j=0}^{n-1}\ \sigma_n^{\,j}\cdot\widetilde{\lozenge}_{\mathbf{s}}\,.$$

For every $\sigma\in\mathscr{S}_n$, we denote by $P_\sigma\in\operatorname{GL}_n(K_v) = \operatorname{GL}(K_v^{\,n})$ the corresponding permutation matrix of the canonical basis $(e_1,\dots,e_n)$ of $K_v^{\,n}$, so that $P_\sigma(e_i) = e_{\sigma(i)}$ for every $i\in[\![1,n]\!]$. The map $\sigma\mapsto P_\sigma$ is a group morphism from $\mathscr{S}_n$ to $\operatorname{GL}_n(K_v)$. This K_v-linear action of $\mathscr{S}_n$ on $K_v^{\,n}$ gives an action of $\mathscr{S}_n$ on the set of R_v-lattices x of $K_v^{\,n}$ by $x\mapsto\sigma(x) = \{P_\sigma(w) : w\in x\}$, that preserves $\mathscr{X}_1$ since the determinant of P_σ is the signature $\varepsilon(\sigma)\in\{\pm1\}$ of σ for every $\sigma\in\mathscr{S}_n$.

Lemma 4.5 *For every $\mathbf{s}\in(R_v\smallsetminus\{0\})^{n-1}$ satisfying Equation (45) and for every $\sigma\in\mathscr{S}_{n-1}$, we have $\nu_{\sigma\cdot\mathbf{s}}^{\Delta} = \sigma_*(\nu_{\mathbf{s}}^{\Delta})$, $\nu_{\sigma\cdot\mathbf{s}}^{\lozenge} = \sigma_*(\nu_{\mathbf{s}}^{\lozenge})$, $\overline{\nu}_{\sigma\cdot\mathbf{s}}^{\Delta} = \sigma_*(\overline{\nu}_{\mathbf{s}}^{\Delta})$ and $\overline{\nu}_{\sigma\cdot\mathbf{s}}^{\lozenge} = \sigma_*(\overline{\nu}_{\mathbf{s}}^{\lozenge})$.*

Proof. We only prove the first equality. The proofs of the other ones are similar. Let $\sigma,\mathbf{s}$ be as in the statement. Let $a\in A(\mathscr{O}_v)$, $\mathbf{k}\in\mathbb{Z}_0^n$ and $\mathbf{t}\in K^{n-1}$. Since $\mathfrak{u}_{\sigma\cdot\mathbf{t}} = P_\sigma\mathfrak{u}_{\mathbf{t}}P_\sigma^{-1}$ and $R_v^{\,n}$ is invariant by P_σ^{-1}, we have

$$\begin{aligned}\sigma(a\,\exp(\mathbf{k})\,\mathfrak{u}_{\mathbf{t}}\,R_v^{\,n}) &= P_\sigma\,a\,\exp(\mathbf{k})\,\mathfrak{u}_{\mathbf{t}}\,R_v^{\,n} = P_\sigma\,a\,P_\sigma^{-1}P_\sigma\,\exp(\mathbf{k})\,P_\sigma^{-1}P_\sigma\mathfrak{u}_{\mathbf{t}}\,P_\sigma^{-1}\,R_v^{\,n}\\ &= (P_\sigma\,a\,P_\sigma^{-1})\exp(\sigma\cdot\mathbf{k})\,\mathfrak{u}_{\sigma\cdot\mathbf{t}}\,R_v^{\,n}\,.\end{aligned}$$

For every $\sigma \in \mathscr{S}_n$, the conjugation by P_σ in $\mathrm{GL}_n(K_v)$ preserves $A(\mathscr{O}_v)$ and its Haar measure. Hence, by Equation (50) and by changes of variables in the sums and integral, we have

$$\begin{aligned}
\nu^{\Delta}_{\sigma\cdot\mathbf{s}} &= \frac{1}{\mathrm{Card}\,\Lambda_{\sigma\cdot\mathbf{s}}\ \mathrm{Card}\,\Delta_{\sigma\cdot\mathbf{s}}} \sum_{\mathbf{t}\in\Lambda_{\sigma\cdot\mathbf{s}},\ \mathbf{k}\in\Delta_{\sigma\cdot\mathbf{s}}} \int_{a\in A(\mathscr{O}_v)} \delta_{a\exp(\mathbf{k})\,\mathfrak{u}_{\mathbf{t}} R_v^n}\, da \\
&= \frac{1}{\mathrm{Card}\,\Lambda_{\mathbf{s}}\ \mathrm{Card}\,\Delta_{\mathbf{s}}} \sum_{\mathbf{t}\in\Lambda_{\mathbf{s}},\ \mathbf{k}\in\Delta_{\mathbf{s}}} \int_{a\in A(\mathscr{O}_v)} \delta_{(P_\sigma a P_\sigma^{-1})\exp(\sigma\cdot\mathbf{k})\,\mathfrak{u}_{\sigma\cdot\mathbf{t}} R_v^n}\, da = \sigma_*\big(\nu^{\Delta}_{\mathbf{s}}\big)\,. \quad \Box
\end{aligned}$$

Note for future use that for all $\sigma \in \mathscr{S}_n$, $a \in A_1(\mathscr{O}_v)$, $\mathbf{k} \in \mathbb{Z}_0^n$ and $\mathbf{t} \in K^{n-1}$, we have

$$\sigma\big(a\,\exp(\mathbf{k})\,\mathfrak{u}_{\mathbf{t}}\,R_v^n\big) = \exp(\sigma\cdot\mathbf{k})\,\sigma(a\,\mathfrak{u}_{\mathbf{t}}\,R_v^n)\,. \tag{51}$$

For every $\mathbf{t} = \big(t_2, \dots, t_n\big) \in K_v^{n-1}$, let

$$x_{\mathbf{t}} = \mathfrak{u}_{\mathbf{t}} R_v^n = R_v\Big(e_1 + \sum_{i=2}^{n} t_i\, e_i\Big) + R_v\, e_2 + \dots + R_v\, e_n\,. \tag{52}$$

Note that $x_{\mathbf{t}} \in \mathscr{X}$ since $\mathfrak{u}_{\mathbf{t}} \in G = \mathrm{SL}_n(K_v)$. We have $x_{\mathbf{0}} = R_v^n$, and if $\mathbf{t}' \in \mathbf{t} + R_v^{n-1}$, then $x_{\mathbf{t}'} = x_{\mathbf{t}}$. Note that $x_{\mathbf{t}}$ is a unimodular R_v-lattice, which is rational if and only if $\mathbf{t} \in K^{n-1}$ and integral if and only if $\mathbf{t} \in R_v^{n-1}$. For every $i \in [\![2, n]\!]$, we have $x_{\mathbf{t}} \cap (K_v e_i) = R_v\, e_i$. Hence by the definition of the directional systoles and by Equation (3), we have

$$\forall\, i \in [\![2, n]\!], \qquad \mathrm{sys}_i(x_{\mathbf{t}}) = \log_{q_v} \min_{s\in R_v\smallsetminus\{0\}} |s| = 0\,. \tag{53}$$

Since

$$x_{\mathbf{t}} \cap (K_v e_1) = \{\lambda_1 e_1 : \lambda_1 \in R_v \ \text{ and } \ \forall\, i \in [\![2, n]\!],\ \ \lambda_1 t_i \in R_v\}\,, \tag{54}$$

the R_v-lattice $x_{\mathbf{t}}$ is axial if and only if it is rational, hence if and only if $\mathbf{t} \in K^{n-1}$.

Proposition 4.6 *Let $\mathbf{s} \in (R_v\smallsetminus\{0\})^{n-1}$ satisfying Equation (45). Let $\mathbf{t} = \big(\frac{r_2}{s_2}, \dots, \frac{r_n}{s_n}\big) \in \Lambda_{\mathbf{s}}$.*

(1) *The first directional systole of the R_v-lattice $x_{\mathbf{t}}$ is $\mathrm{sys}_1(x_{\mathbf{t}}) = -v(\mathbf{s}_*)$. The truncated covolume of the A_1-orbit of $x_{\mathbf{t}}$ is $\tau_{x_{\mathbf{t}}} = -v(\mathbf{s}_*)$. The coordinate sublattice of $x_{\mathbf{t}}$ is $(x_{\mathbf{t}})^{\mathrm{coo}} = R_v\,\mathbf{s}_*\,e_1 + R_v\,e_2 + \cdots + R_v\,e_n$. The compact core of the A_1-orbit of $x_{\mathbf{t}}$ is $C_{x_{\mathbf{t}}} = A_1(\mathscr{O}_v)\exp(\Delta_{\mathbf{s}})x_{\mathbf{t}}$. A quasicenter of the A_1-orbit of $x_{\mathbf{t}}$ is $\widehat{x_{\mathbf{t}}} = \exp(\mathbf{k}_{\mathbf{s}})x_{\mathbf{t}}$.*

(2) *As $\mathbf{s} \to +\infty$, we have*

$$\mathrm{Card}\,\Diamond_{\mathbf{s}} = \frac{1}{n!}(-v(\mathbf{s}_*))^{n-1} + \mathrm{O}((-v(\mathbf{s}_*))^{n-2}) = \frac{1}{n}\,\mathrm{Card}\,\Delta_{\mathbf{s}} + \mathrm{O}((-v(\mathbf{s}_*))^{n-2})\,. \tag{55}$$

(3) *We have $x_{\mathbf{t}} \cap R_v^n = R_v\,\mathbf{s}_*\,e_1 + R_v\,e_2 + \cdots + R_v\,e_n = (x_{\mathbf{t}})^{\mathrm{coo}}$. Hence the type of the integral lattice $x_{\mathbf{t}} \cap R_v^n$ is $(1, \dots, 1, \mathbf{s}_*)$.*

(4) *The PA-orbit of the homothety class of $x_{\mathbf{t}}$ has discriminant $\prod_{i=2}^n |s_i|$ and has type $\{(1, s_{\sigma^{-1}(2)}, \dots, s_{\sigma^{-1}(n)})\}$ where $\sigma \in \mathscr{S}_{n-1}$ is such that $s_{\sigma^{-1}(2)} \mid \dots \mid s_{\sigma^{-1}(n)}$.*

(5) *With ν_y for $y \in \mathscr{X}$ the probability measure given by Equation (38), we have*

$$\nu^{\Delta}_{\mathbf{s}} = \frac{1}{\mathrm{Card}\,\Lambda_{\mathbf{s}}} \sum_{\mathbf{t}\in\Lambda_{\mathbf{s}}} \nu_{x_{\mathbf{t}}}\,.$$

As $s \to +\infty$ in $R_v \smallsetminus \{0\}$ with $v(s) \in n\mathbb{Z}$, defining $\mathfrak{s} = (s, \dots, s) \in (R_v \smallsetminus \{0\})^{n-1}$, for every $f \in \mathrm{C}_c(\mathscr{X})$, we have

$$\nu_{\mathfrak{s}}^{\Delta}(f) = \frac{1}{n} \sum_{j=0}^{n-1} (\sigma_n^{\,j})_* \nu_{\mathfrak{s}}^{\Diamond}(f) + \mathrm{O}\Big(\frac{\|f\|_\infty}{-v(s)}\Big) .$$

Proof. If $r_2, \dots, r_n, r'_2, \dots, r'_n \in R_v$ satisfy $r'_2 \equiv r_2 \mod s_2, \dots, r'_n \equiv r_n \mod s_n$, and if $\mathbf{t} = \big(\frac{r_2}{s_2}, \dots, \frac{r_n}{s_n}\big)$, $\mathbf{t}' = \big(\frac{r'_2}{s_2}, \dots, \frac{r'_n}{s_n}\big)$, then we have $\mathbf{t}' - \mathbf{t} \in R_v^{\,n-1}$ and $x_{\mathbf{t}} = x_{\mathbf{t}'}$. In particular, the R_v-lattice $x_{\mathbf{t}}$, as well as the measures $\nu_{x_{\mathbf{t}}}$, $\nu_{\mathbf{s}}^{\Delta}$ and $\nu_{\mathbf{s}}^{\Diamond}$, do not depend on the choice of representatives of the elements $\mathbf{t}$ in the index set $\Lambda_{\mathbf{s}} \subset K^{n-1}/R_v^{n-1}$.

(1) By Equation (54), the set $x_{\mathbf{t}} \cap (K_v e_1)$ consists in the elements $\lambda_1 e_1$ where $\lambda_1 \in R_v$ is such that, for every $i \in [\![2, n]\!]$, we have $\lambda_1 \frac{r_i}{s_i} \in R_v$. Since r_i is invertible modulo s_i and by Equation (45), this occurs if and only if $\lambda_1 \in \bigcap_{i=2}^n s_i R_v = \mathbf{s}_* R_v$. Hence we have $x_{\mathbf{t}} \cap (K_v e_1) = \mathbf{s}_* R_v$. By the definition of the first directional systole, this proves the first claim. By the definition (27) of the truncated covolume and by Equation (53), we hence have $\tau_{x_{\mathbf{t}}} = \sum_{i=1}^n \mathrm{sys}_i(x_{\mathbf{t}}) = -v(\mathbf{s}_*)$. By the definition (30) of the coordinate sublattice, we have

$$(x_{\mathbf{t}})^{\mathrm{coo}} = (x_{\mathbf{t}} \cap K_v e_1) + \dots + (x_{\mathbf{t}} \cap K_v e_n) = R_v\, \mathbf{s}_*\, e_1 + R_v\, e_2 + \dots + R_v\, e_n .$$

By Equation (28), by the above computation of the directional systoles of $x_{\mathbf{t}}$ and by the definition of $\Delta_{\mathbf{s}}$, we have $\Delta^{x_{\mathbf{t}}} = \Delta_{\mathbf{s}}$. This gives the description of the compact core of the A_1-orbit of $x_{\mathbf{t}}$ by Equation (29).

Recall that $-v(\mathbf{s}_*) \in n\mathbb{N}$ and that $\mathbf{k}_{\mathbf{s}} = \big(k_1 = \frac{n-1}{n} v(\mathbf{s}_*), k_2 = -\frac{v(\mathbf{s}_*)}{n}, \dots, k_n = -\frac{v(\mathbf{s}_*)}{n}\big)$. By the first claim of Proposition 4.1 (2), we have that

$$\mathrm{sys}_i(\exp(\mathbf{k}_{\mathbf{s}}) x_{\mathbf{t}}) = \mathrm{sys}_i(x_{\mathbf{t}}) - v(\pi_v^{-k_i}) = \mathrm{sys}_i(x_{\mathbf{t}}) + k_i$$

is equal to $-\frac{v(\mathbf{s}_*)}{n}$ if $i \in [\![2, n]\!]$ by Equation (53) and to $-v(\mathbf{s}_*) + \frac{n-1}{n} v(\mathbf{s}_*) = -\frac{v(\mathbf{s}_*)}{n}$ if $i = 1$. The description of the quasicenters of the A_1-orbit of $x_{\mathbf{t}}$ (well-defined up to left translation by an element of $A_1(\mathscr{O}_v)$) then follows from their definition that requires, by Equation (31) and since $\tau_{x_{\mathbf{t}}} = -v(\mathbf{s}_*) \in \mathbb{Z}$, that we have $\mathrm{sys}_i(\widehat{x_{\mathbf{t}}}) = \frac{\tau_{x_{\mathbf{t}}}}{n} = -\frac{v(\mathbf{s}_*)}{n}$ for every $i \in [\![1, n]\!]$.

(2) Using the facts that $\Delta^{x_{\mathbf{t}}} = \Delta_{\mathbf{s}}$ and $\tau_{x_{\mathbf{t}}} = -v(\mathbf{s}_*)$ seen in Assertion (1), the first equality in Equation (55) follows by the same proof as the one of Proposition 4.1 (5), by comparing with the Euclidean volume of the polytop

$$\widehat{\Diamond}'(\alpha) = \big\{\mathbf{t} = (t_1, \dots, t_n) \in \mathbb{R}_0^n : \forall\, i \in [\![2, n]\!],\ 0 \leqslant t_i \leqslant t_1 + \alpha\big\}$$

for $\alpha = -v(\mathbf{s}_*)$ whose images under the powers $\sigma_n^{\,0}, \sigma_n^{\,1}, \dots, \sigma_n^{\,n-1}$ of σ_n have pairwise disjoint interior and cover $\Delta'(\alpha)$. The second equality in Equation (55) then follows from Proposition 4.1 (5).

(3) Let $\lambda_1, \dots, \lambda_n \in R_v$. We have $\lambda_1\big(e_1 + \sum_{i=2}^n \frac{r_i}{s_i} e_i\big) + \sum_{i=2}^n \lambda_i\, e_i \in R_v^{\,n}$ if and only if for every $i \in [\![2, n]\!]$ we have $\lambda_1 \frac{r_i}{s_i} \in R_v$, hence if and only if $\lambda_1 \in \mathbf{s}_* R_v$ as seen in Assertion (1). The result follows by Equation (52).

(4) Let $a \in \widetilde{A}$ be a diagonal matrix with diagonal coefficients $a_1, \dots, a_n \in K_v^\times$. The R_v-lattice $a\, x_{\mathbf{t}} = R_v\big(a_1\, e_1 + \sum_{i=2}^n \;\frac{r_i}{s_i}\; a_i\, e_i\big) + R_v\, a_2\, e_2 + \ldots + R_v\, a_n\, e_n$ is integral, that is contained in $R_v{}^n$, if and only if $a_i \in R_v$ for every $i \in [\![2, n]\!]$, $a_1 \in R_v$ and $\frac{r_i}{s_i}\; a_i \in R_v$ for every $i \in [\![2, n]\!]$, hence if and only if $a_1 \in R_v$ and $a_i \in s_i\, R_v$ for every $i \in [\![2, n]\!]$ since r_i is invertible modulo s_i,

Recall that by Equation (3), for every $z \in R_v \smallsetminus \{0\}$, we have $|z| \geqslant 1$, with equality if and only if $z \in R_v^\times$. Recall that $x_{\mathbf{t}}$ is unimodular. Hence by Equation (10), we have

$$\frac{\operatorname{covol}(a\, x_{\mathbf{t}})}{\operatorname{covol}(R_v^n)} = |\det a|\;\frac{\operatorname{covol}(x_{\mathbf{t}})}{\operatorname{covol}(R_v^n)} = \prod_{i=1}^n |a_i| \geqslant \prod_{i=2}^n |s_i|$$

with equality if and only if $a_1 \in R_v^\times$ and $a_i \in s_i\, R_v^\times$ for every $i \in [\![2, n]\!]$. Therefore the integral R_v-lattices contained in elements of the orbit $\widetilde{A}\, x_{\mathbf{t}}$ with minimal covolume are exactly the R_v-lattices $L = R_v\big(e_1 + \sum_{i=2}^n \; r_i\; a_i'\, e_i\big) + R_v\, s_2\, e_2 + \ldots + R_v\, s_n\, e_n$ where $a_2', \dots, a_n' \in R_v^\times$. Note that there is no uniqueness of such an R_v-lattice L in general.

Since $R_v^n = R_v\big(e_1 + \sum_{i=2}^n \; r_i\; a_i'\, e_i\big) + R_v\, e_2 + \ldots + R_v\, e_n$ by an immediate change of R_v-basis, the R_v-module $R_v{}^n/L$ is isomorphic to $\prod_{i=2}^n R_v/(s_i R_v)$ for every such L. All such integral lattices L hence have the same type, and this proves the result by Equation (45).

(5) Since $\Delta^{x_{\mathbf{t}}} = \Delta_{\mathbf{s}}$ for every $\mathbf{t} \in \Lambda_{\mathbf{s}}$ by Assertion (1), the first claim of Assertion (5) follows from the definitions of the measure $\nu_{\mathbf{s}}^{\Delta}$ in Equation (47) and of the measure ν_y for any R_v-lattice $y \in \mathscr{X}$ in Equation (38).

Let us prove the second claim of Assertion (5), which uses the symmetry of the $(n-1)$-uples $(s, \dots, s)$ for $s \in R_v$. We start with a computational lemma.

We fix $s \in R_v$ with $v(s) \in n\mathbb{Z}$ and we consider the $(n-1)$-uple $\mathfrak{s} = (s, \dots, s)$. Let $a \in A(\mathscr{O}_v)$ with diagonal coefficients $a_1, \dots, a_n$ and let $\mathbf{t} = \big(\frac{r_2}{s}, \dots, \frac{r_n}{s}\big) \bmod R_v{}^{n-1} \in \Lambda_{\mathfrak{s}}$. Let $s' \in \mathscr{O}_v^\times$ be such that $s = \pi_v{}^{v(s)}\, s'$, and let $\overline{r_n} \in R_v$ be an inverse of r_n modulo s. Let a' be the element of $A(\mathscr{O}_v)$ with diagonal coefficients

$$a_1' = (s')^{-1} a_n, a_2' = s' a_1, a_3' = a_2, \dots, a_n' = a_{n-1}\,.$$

Note that the map $a \mapsto a'$ is a homeomorphism of $A(\mathscr{O}_v)$ preserving its Haar measure. Let $\mathbf{t}' = \big(\frac{\overline{r_n}}{s}, \frac{\overline{r_n}\, r_2}{s}, \dots, \frac{\overline{r_n}\, r_{n-1}}{s}\big) \bmod R_v{}^{n-1}$, and note that the map $\mathbf{t} \mapsto \mathbf{t}'$ is a bijection of $\Lambda_{\mathfrak{s}}$ with inverse $\big(\frac{r_2'}{s}, \dots, \frac{r_n'}{s}\big) \bmod R_v{}^{n-1} \mapsto \big(\frac{\overline{r_2'}\, r_3'}{s}, \dots, \frac{\overline{r_2'}\, r_n'}{s}, \frac{\overline{r_2'}}{s}\big) \bmod R_v{}^{n-1}$.

Lemma 4.7 *Denoting by $y_{a,\mathbf{t}}$ the R_v-lattice $a\, \exp(\mathbf{k}_{\mathfrak{s}})\, x_{\mathbf{t}} \in \mathscr{X}$, we have $\sigma_n(y_{a,\mathbf{t}}) = y_{a',\mathbf{t}'}$.*

Proof. Let $w = s^{-1}\big(a_2'\; e_1 + \sum_{j=2}^n \; r_j\; a_j\; e_j\big) \in K_v{}^n$. Since $\pi_v{}^{-\frac{n-1}{n} v(s)} a_1 = \pi_v{}^{\frac{v(s)}{n}}\, s^{-1} a_2'$, by Equation (52) and by the diagonal action, we have

$$y_{a,\mathbf{t}} = a\, \exp(\mathbf{k}_{\mathfrak{s}})\, x_{\mathbf{t}} = \pi_v{}^{\frac{v(s)}{n}} \Big(R_v\, w + \sum_{j=2}^n R_v\, a_j\; e_j\Big)\,.$$

In particular, $\pi_v{}^{\frac{v(s)}{n}}\, w, \pi_v{}^{\frac{v(s)}{n}}\, a_2 e_2, \dots, \pi_v{}^{\frac{v(s)}{n}}\, a_n e_n$ belong to $y_{a,\mathbf{t}}$. Hence $\pi_v{}^{\frac{v(s)}{n}}\, a_2'\; e_1$ belongs to $y_{a,\mathbf{t}}$ since $a_2'\; e_1 = s\; w - \sum_{j=2}^n r_j\; a_j\; e_j$ and $s, r_2, \dots, r_n \in R_v$. Let $w' = \overline{r_n}\; w$, so that $\pi_v{}^{\frac{v(s)}{n}}\, w' = \overline{r_n}\; (\pi_v{}^{\frac{v(s)}{n}}\, w)$ belongs to the R_v-lattice $y_{a,\mathbf{t}}$, and

$$y_{a,\mathbf{t}} = \pi_v{}^{\frac{v(s)}{n}} \Big(R_v\, w + R_v\, w' + R_v\, a_2'\; e_1 + \sum_{j=2}^n R_v\, a_j\; e_j\Big)\,.$$

Let $s_0 \in R_v$ be such that $1 = \overline{r_n}\, r_n + s\, s_0$, so that $w = r_n w' + s_0\big(a'_2\, e_1 + \sum_{j=2}^n\ r_j\ a_j\ e_j\big)$ and we can remove $R_v\, w$ in the above expression of $y_{a,\mathbf{t}}$. Therefore, plugging in the expressions of $w' = \overline{r_n}\ s^{-1}\big(a'_2\ e_1 + \sum_{j=2}^n\ r_j\ a_j\ e_j\big)$ and $a'_2 = s' a_1$, we have

$$y_{a,\mathbf{t}} = \pi_v^{\frac{v(s)}{n}} \Big(R_v\, s^{-1}\big(\overline{r_n}\,(s'\,a_1)\, e_1 + (\overline{r_n}\, r_2)\, a_2\, e_2 + \ldots + (\overline{r_n}\, r_{n-1})\, a_{n-1} e_{n-1} + s'(s'^{-1}\, a_n) e_n\big) \\ + R_v\,(s'\,a_1)\, e_1 + R_v\, a_2\, e_2 + \ldots + R_v\, a_{n-1}\, e_{n-1} + R_v\, s'(s'^{-1}\, a_n)\, e_n\Big)\,.$$

The result follows by the action $P_{\sigma_n} : e_i \mapsto e_{i+1}$ (for $i \in [\![1, n]\!]$ modulo n) of σ_n on the canonical basis, and by the definition of a' and $\mathbf{t}'$. □

Recall that $\mathfrak{s} = (s, \ldots, s)$ till the end of the proof of the second claim of Assertion (5). Since we have $a \exp(\mathbf{k})\, \mathfrak{u}_{\mathbf{t}} R_v^n = \exp(\mathbf{k} - \mathbf{k}_{\mathfrak{s}})\, y_{a,\mathbf{t}}$ and by using the change of variable $\mathbf{k} \in \Diamond_{\mathfrak{s}} \mapsto \mathbf{k} - \mathbf{k}_{\mathfrak{s}} \in \widetilde{\Diamond}_{\mathfrak{s}}$ in Equation (48), we have

$$\nu_{\mathfrak{s}}^{\Diamond} = \frac{1}{\operatorname{Card} \Lambda_{\mathfrak{s}} \operatorname{Card} \Diamond_{\mathfrak{s}}} \sum_{\mathbf{t} \in \Lambda_{\mathfrak{s}},\ \mathbf{k} \in \widetilde{\Diamond}_{\mathfrak{s}}} \int_{a \in A(\mathscr{O}_v)} \delta_{\exp(\mathbf{k})\, y_{a,\mathbf{t}}}\, da\,. \tag{56}$$

Similarly, by Equation (47), we have

$$\nu_{\mathfrak{s}}^{\Delta} = \frac{1}{\operatorname{Card} \Lambda_{\mathfrak{s}} \operatorname{Card} \Delta_{\mathfrak{s}}} \sum_{\mathbf{t} \in \Lambda_{\mathfrak{s}},\ \mathbf{k} \in \widetilde{\Delta}_{\mathfrak{s}}} \int_{a \in A(\mathscr{O}_v)} \delta_{\exp(\mathbf{k})\, y_{a,\mathbf{t}}}\, da\,. \tag{57}$$

Let $\widetilde{\Diamond}_{\mathfrak{s}}^{\sharp}$ be a strict fundamental domain for the (free) action of $\sigma_n^{\mathbb{Z}}$ on $\widetilde{\Delta}_{\mathfrak{s}} \smallsetminus \{\mathbf{0}\}$, so that

$$\Big\{\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}_0^n : \forall\, j \in [\![2, n]\!],\ \frac{v(s)}{n} \leqslant k_j < k_1\Big\} \subset \widetilde{\Diamond}_{\mathfrak{s}}^{\sharp} \subset \widetilde{\Diamond}_{\mathfrak{s}}$$

and $\widetilde{\Delta}_{\mathfrak{s}} \smallsetminus \{\mathbf{0}\} = \bigsqcup_{j=0}^{n-1}\ \sigma_n^{\ j} \cdot \widetilde{\Diamond}_{\mathfrak{s}}^{\sharp}$ (see the picture at the beginning of this Section for an illustration when $n = 3$ of this partition, after translating by $-\mathbf{k}_{\mathfrak{s}}$). By the standard Gauss counting argument and by Assertion (2) applied with $\mathbf{s} = \mathfrak{s} = (s, \ldots, s)$, we have

$$\frac{\operatorname{Card}\big(\widetilde{\Diamond}_{\mathfrak{s}} \smallsetminus \widetilde{\Diamond}_{\mathfrak{s}}^{\sharp}\big)}{\operatorname{Card} \Diamond_{\mathfrak{s}}} = \operatorname{O}\Big(\frac{1}{-v(s)}\Big) \quad \text{and} \quad \frac{\operatorname{Card} \Delta_{\mathfrak{s}}}{\operatorname{Card} \Diamond_{\mathfrak{s}}} = n + \operatorname{O}\Big(\frac{1}{-v(s)}\Big)\,. \tag{58}$$

Let

$$\nu_{\mathfrak{s}}^{\sharp} = \frac{1}{\operatorname{Card} \Lambda_{\mathfrak{s}} \operatorname{Card} \Diamond_{\mathfrak{s}}} \sum_{\mathbf{t} \in \Lambda_{\mathfrak{s}},\ \mathbf{k} \in \widetilde{\Diamond}_{\mathfrak{s}}^{\sharp}} \int_{a \in A(\mathscr{O}_v)} \delta_{\exp(\mathbf{k})\ y_{a,\mathbf{t}}}\, da\,,$$

so that by Equation (56) and by the left hand part of Equation (58), for every $f \in \operatorname{C}_c(\mathscr{X})$, we have

$$\nu_{\mathfrak{s}}^{\Diamond}(f) = \nu_{\mathfrak{s}}^{\sharp}(f) + \operatorname{O}\Big(\frac{\|f\|_\infty}{-v(s)}\Big)\,. \tag{59}$$

By Equation (51) and by Lemma 4.7, we have

$$(\sigma_n)_* \delta_{\exp(\mathbf{k})\, y_{a,\mathbf{t}}} = \delta_{\sigma_n(\exp(\mathbf{k})\, y_{a,\mathbf{t}})} = \delta_{\exp(\sigma_n \cdot \mathbf{k})\ \sigma_n(y_{a,\mathbf{t}})} = \delta_{\exp(\sigma_n \cdot \mathbf{k})\ y_{a',\mathbf{t}'}}\,.$$

By the changes of variable $a \mapsto a'$ in the integral, and $\mathbf{k} \mapsto \sigma_n \cdot \mathbf{k}$ as well as $\mathbf{t} \mapsto \mathbf{t}'$ in the sum, we hence have

$$(\sigma_n)_* \nu_{\mathfrak{s}}^{\sharp} = \frac{1}{\operatorname{Card} \Lambda_{\mathfrak{s}} \operatorname{Card} \Diamond_{\mathfrak{s}}} \sum_{\mathbf{t} \in \Lambda_{\mathfrak{s}},\ \mathbf{k} \in \sigma_n \cdot \widetilde{\Diamond}_{\mathfrak{s}}^{\sharp}} \int_{a \in A(\mathscr{O}_v)} \delta_{\exp(\mathbf{k})\ y_{a,\mathbf{t}}}\, da\,.$$

By iteration, we therefore have

$$\frac{1}{n}\sum_{j=0}^{n-1}(\sigma_n^{\,j})_*\nu^\sharp_{\mathfrak s}=\frac{1}{n\ \mathrm{Card}\ \Lambda_{\mathfrak s}\ \mathrm{Card}\ \Diamond_{\mathfrak s}}\sum_{\mathbf t\in\Lambda_{\mathfrak s},\ \mathbf k\in\bigsqcup_{j=0}^{n-1}\sigma_n^{\,j}\cdot\widetilde{\Diamond}^\sharp_{\mathfrak s}}\int_{a\in A(\mathscr O_v)}\delta_{\exp(\mathbf k)\ y_{a,\mathbf t}}\ da\,.$$

By Assertion (2), for every $f\in \mathrm C_c(\mathscr X)$, we have

$$\frac{1}{n\ \mathrm{Card}\ \Lambda_{\mathfrak s}\ \mathrm{Card}\ \Diamond_{\mathfrak s}}\sum_{\mathbf t\in\Lambda_{\mathfrak s}}\int_{a\in A(\mathscr O_v)}f(y_{a,\mathbf t})\ da=\mathrm O\Big(\frac{\|f\|_\infty}{(-v(s))^{n-1}}\Big)=\mathrm O\Big(\frac{\|f\|_\infty}{-v(s)}\Big)\,.$$

Since $\widetilde\Delta_{\mathfrak s}=\{\mathbf 0\}\cup\bigsqcup_{j=0}^{n-1}\sigma_n^{\,j}\cdot\widetilde\Diamond^\sharp_{\mathfrak s}$, by Equations (59) and (57) and by the right hand part of Equation (58), the result follows. □

The aim of the following sections will be to prove that the probability measures $\nu^\Diamond_{\mathbf s}$ weak-star converge as $\mathbf s\to+\infty$ (with appropriate conditions the components of $\mathbf s$ that are satisfied if they are all equal and have absolute values equal to a multiple of n) to the G-homogeneous measure $\mathtt m_{\mathscr X}$ on $\mathscr X$, renormalized to be a probability measure, see Theorem 8.1. In the particular case when $\mathbf s=(s,s,\dots,s)$, this will imply Theorem 1.2 by the following lemma. We denote by $c_{K,n}$ the constant defined in the statement of Theorem 1.2.

Lemma 4.8 *Assume that* $\lim_{s\to\infty,\ v(s)\in n\mathbb Z}\ \nu^\Diamond_{(s,\dots,s)}=\frac{\mathtt m_{\mathscr X}}{\|\mathtt m_{\mathscr X}\|}$. *Then*

$$\lim_{s\to\infty,\ v(s)\in n\mathbb Z}\ \frac{c_{K,n}}{(\varphi_v(s)\ \log_{q_v}|s|\,)^{n-1}}\sum_{\mathbf t\in\Lambda_{(s,\dots,s)}}\overline\mu_{\mathfrak u_{\mathbf t}R_v^n}\ =\ \mathtt m_{\mathscr X_1}\,.$$

Proof. For every $s\in R_v$ such that $v(s)\in n\mathbb Z$, let $\mathfrak s=(s,s,\dots,s)\in(R_v\smallsetminus\{0\})^{\,n-1}$. By the assumption of the lemma and by a finite average using the last claim of Proposition 4.6 (5) (the only point where we use the symmetry of $\mathfrak s$), we have $\lim_{s\to\infty,\ v(s)\in n\mathbb Z}\ \nu^\Delta_{\mathfrak s}=\frac{\mathtt m_{\mathscr X}}{\|\mathtt m_{\mathscr X}\|}$. By the first claim of Proposition 4.6 (5) and by Equation (46) on the right, we have $\nu^\Delta_{\mathfrak s}=\frac{1}{(\varphi_v(s))^{n-1}}\sum_{\mathbf t\in\Lambda_{\mathfrak s}}\nu_{x_{\mathbf t}}$. By Proposition 4.6 (1), we have $\tau_{x_{\mathbf t}}=-v(s)=\log_{q_v}|s|$ for all $\mathbf t\in\Lambda_{\mathfrak s}$. Hence by the last claim of Lemma 4.3 (and the definition of the weak-star convergence), we have

$$\lim_{s\to\infty,\ v(s)\in n\mathbb Z}\ \frac{1}{c_n\,(\varphi_v(s)\ \log_{q_v}|s|\,)^{n-1}}\sum_{\mathbf t\in\Lambda_{\mathfrak s}}\mu_{x_{\mathbf t}}=\frac{\mathtt m_{\mathscr X}}{\|\mathtt m_{\mathscr X}\|}\,.\tag{60}$$

Again by averaging, this time over the compact probability space $(\mathscr O_v^{\times}/R_v^{\times},\frac{q_v(q-1)}{q_v-1}\ \mathrm{vol}'_v)$ defined at the end of Section 2.3, and using Equations (21) and (16), we have

$$\lim_{s\to\infty,\ v(s)\in n\mathbb Z}\ \frac{1}{c_n\,(\varphi_v(s)\ \log_{q_v}|s|\,)^{n-1}}\sum_{\mathbf t\in\Lambda_{\mathfrak s}}\overline\mu_{x_{\mathbf t}}=\frac{\mathtt m_{\mathscr X_1}}{\|\mathtt m_{\mathscr X_1}\|}\,.$$

By Equation (15) giving the value of $\|\mathtt m_{\mathscr X_1}\|$ and by Proposition 4.1 (5) giving the value of c_n, we have

$$\frac{\|\mathtt m_{\mathscr X_1}\|}{c_n}=\frac{(n-1)!\,(q_v-1)}{q_v\,(q-1)}\prod_{i=1}^{n-1}\frac{\zeta_v(-i)}{q_v^{\,i}-1}\,.$$

This is exactly the value of the constant $c_{K,n}$ defined in the statement of Theorem 1.2. The result follows. $\square$

Remark. Equation (60) says that the distribution statement in $\mathscr{X}$ analogous to the one of Theorem 1.2 in $\mathscr{X}_1$ will also follow from Theorem 8.1.

5 The high entropy method for equidistribution problem

In this section, we explain how we are going to use entropy technics in homogeneous dynamics of diagonal actions (that are currently more and more used, see for instance [EL, LSS, ELW, KiLP]) in order to prove the equidistribution of our families of measures. We start by a brief reminder of entropy theory.

Let (X', μ') be a Borel probability space, $\phi : X' \to X'$ a measurable map, and $\mathscr{P}, \mathscr{P}'$ finite measurable partitions of X'.

We denote by $\phi^{-1}\mathscr{P} = \{\phi^{-1}(B) : B \in \mathscr{P},\ \phi^{-1}(B) \neq \varnothing\}$ the pull-back partition and by $\mathscr{P} \vee \mathscr{P}'' = \{B \cap B' : B \in \mathscr{P},\ B' \in \mathscr{P}',\ B \cap B' \neq \varnothing\}$ the joint partition. Using the convention $0 \log_{q_v} 0 = 0$, the *entropy* of the partition $\mathscr{P}$ with respect to μ' is

$$H_{\mu'}(\mathscr{P}) = -\sum_{P \in \mathscr{P}} \mu'(P) \log_{q_v} \mu'(P) \ \in [0, \infty[\ .$$

The usual definition of the entropy of a partition uses the Neperian logarithm ln instead of $\log_{q_v}$, but the above convention will be technically easier in this paper. We have $H_{\phi_*\mu'}(\mathscr{P}) = H_{\mu'}(\phi^{-1}\mathscr{P})$. We have the following concavity properties of the entropy of a partition as a function of the measure.

Lemma 5.1 *(David-Shapira [DS2, Lem 3.4]) Let $(X', \mu'), \phi, \mathscr{P}$ be as above.*

(1) *For all $M \leqslant N$ in $\mathbb{N} \smallsetminus \{0\}$, we have*

$$\frac{1}{M} H_{\frac{1}{N}\sum_{i=0}^{N-1}(\phi^i)_*\mu'}\Big(\bigvee_{i=0}^{M-1} \phi^{-i}\mathscr{P}\Big) \geqslant \frac{1}{N} H_{\mu'}\Big(\bigvee_{i=0}^{N-1} \phi^{-i}\mathscr{P}\Big) - \frac{M}{N} \log_{q_v} \operatorname{Card} \mathscr{P}\,.$$

(2) *Let (Ω, ω) be a probability space and let $x \mapsto \mu'_x$ be a measurable map from Ω to the space of probability measures on X' such that $\mu' = \int_{x\in\Omega} \mu'_x \, d\omega(x)$. Then we have $H_{\mu'}(\mathscr{P}) \geqslant \int_{x\in\Omega} H_{\mu'_x}(\mathscr{P}) \, d\omega(x)$.* $\square$

If ϕ preserves the measure μ', the *(dynamical) entropy* of ϕ with respect to μ' is defined by $h_{\mu'}(\phi) = \sup_{\mathscr{P}} \ h_{\mu'}(\phi, \mathscr{P})$ where the least upper bound is taken over all finite measurable partitions $\mathscr{P}$ of X' and

$$h_{\mu'}(\phi, \mathscr{P}) = \lim_{M\to+\infty} \frac{1}{M} H_{\mu'}\Big(\bigvee_{i=0}^{M-1} \phi^{-i}\mathscr{P}\Big)\,. \tag{61}$$

The following result says that the homogeneous measure $m_{\mathscr{X}}$ is after renormalisation the unique probability measure of maximal entropy on the space $\mathscr{X}$ for the transformation $\mathfrak{a} = \begin{pmatrix} \pi_v^{\,n-1} & 0 \\ 0 & \pi_v^{\,-1} I_{n-1} \end{pmatrix}$ given by Equation (2).

Theorem 5.2 (Einsiedler-Lindenstrauss) *Let ν be an $\mathfrak{a}$-invariant probability measure on $\mathscr{X}$. Then $h_\nu(\mathfrak{a}) \leqslant h_{\frac{m_{\mathscr{X}}}{\|m_{\mathscr{X}}\|}}(\mathfrak{a}) = n(n-1)$ with equality if and only if $\nu = \frac{m_{\mathscr{X}}}{\|m_{\mathscr{X}}\|}$.*

We will apply this theorem in Section 8 to every weak-star accumulation point ν of the measures $\nu_{\mathbf{s}}^{\Diamond}$ as $\mathbf{s}$ tends appropriately to $+\infty$. Since the space $\mathscr{X}$ is not compact, we will first need to prove that ν is a probability measure (see the arguments in Section 7), and then that its entropy $h_\nu(\mathfrak{a})$ is equal to $h_{\frac{m_{\mathscr{X}}}{\|m_{\mathscr{X}}\|}}(\mathfrak{a})$ (see Section 8).

Proof. Let $\mathbb{G}$ be the algebraic group SL_n over the local field K_v and let $G = \mathrm{SL}_n(K_v)$ be its locally compact group of K_v-points, so that $\Gamma = \mathrm{SL}_n(R_v)$ is a lattice in G. Let

$$G^- = \{g \in G : \lim_{i\to-\infty} \mathfrak{a}^i g \mathfrak{a}^{-i} = I_n\} = \left\{\left(\begin{smallmatrix} 1 & 0 \\ b & I_{n-1}\end{smallmatrix}\right) : b \in K_v^{\,n-1}\right\}$$

and

$$G^+ = {}^tG^- = \{g \in G : \lim_{i\to+\infty} \mathfrak{a}^i g \mathfrak{a}^{-i} = I_n\} = \left\{\left(\begin{smallmatrix} 1 & {}^tb \\ 0 & I_{n-1}\end{smallmatrix}\right) : b \in K_v^{\,n-1}\right\}$$

be respectively the unstable and stable horospherical groups of $\mathfrak{a}$ in G. By [BoT, Prop 4.11], the groups G^- and G^+ generate a normal subgroup H of G. It is well known that $H = G$ when $n = 2$. Hence H contains the copies of $\mathrm{SL}_2(K_v)$ with upper and lower unipotent subgroups contained in G^- and G^+ respectively. Therefore H contains the diagonal subgroup A of G, thus contains properly the center of G, hence is equal to G since $\mathrm{PSL}_n(K_v)$ is simple. By [EL, Th. 7.10], the normalized Haar measure $\frac{m_{\mathscr{X}}}{\|m_{\mathscr{X}}\|}$ of the homogeneous space G/Γ is hence the unique measure of maximal entropy on G/Γ for the left action of $\mathfrak{a}$.

The entropy of $\mathfrak{a}$ with respect to the homogeneous measure of $\frac{m_{\mathscr{X}}}{\|m_{\mathscr{X}}\|}$ is well-known (see for instance [EL, §7.8]) to be the logarithm (in basis q_v for entropy computations in nonarchimedian local fields with residual fields of order q_v) of the unstable Jacobian of $\mathfrak{a}$. That is, with $\underline{\mathfrak{u}}^-$ the strict lower triangular linear subspace of the Lie algebra $\mathfrak{sl}_n(K_v)$ of $\mathrm{SL}_n(K_v)$, with basis the family of elementary matrices $(E_{i,j})_{1\leqslant j<i\leqslant n}$, since we have $\mathrm{Ad}\,\mathfrak{a}\,(E_{i,j}) = \pi_v^{\,-n}E_{i,j}$ if $j = 1$ and $\mathrm{Ad}\,\mathfrak{a}\,(E_{i,j}) = E_{i,j}$ otherwise, we have

$$h_{\frac{m_{\mathscr{X}}}{\|m_{\mathscr{X}}\|}}(\mathfrak{a}) = \log_{q_v} |\det(\mathrm{Ad}\,\mathfrak{a})_{|\,\underline{\mathfrak{u}}^-}| = \log_{q_v} \Big|\prod_{j=2}^{n} \pi_v^{\,-n}\Big| = n(n-1)\,. \quad \square$$

6 Constructing high entropy partitions from dynamical neighborhoods

In this section, using the contraction and dilation properties of the action of the diagonal element $\mathfrak{a} = \left(\begin{smallmatrix} \pi_v^{\,n-1} & 0 \\ 0 & \pi_v^{\,-1}I_{n-1}\end{smallmatrix}\right)$ on its unstable and stable horospherical subgroups G^- and G^+, we give a construction of good measurable partitions in the homogeneous space $\mathscr{X}$, that will turn out in Section 8 to be well adapted in order to obtain entropy lower bounds of $\mathfrak{a}$-invariant measures. This construction is essentially due to [ELMV2, Lem. 4.5] in dimension 2 (see [DS1, Lem. 2.9] correcting a small inaccuracy in [ELMV2]) and to [DS2, Lem. 3.7] for any dimension, see also [LSS, §2.3], [KiKL, §3.3], all these references in the real case, and [KiLP, §6.1] in the function fields case.

6.1 Dynamical neighbourhoods in $\mathrm{SL}_n(K_v)$

We first define the dynamical neighbourhoods of the identity element I_n in $\mathrm{SL}_n(K_v)$ that we will consider.

We denote by $\| \ \| : \mathscr{M}_n(K_v) \to [0, +\infty[$ the ultrametric norm on $\mathscr{M}_n(K_v)$ defined by $(x_{ij})_{1 \leqslant i,j \leqslant n} \mapsto \max_{1 \leqslant i,j \leqslant n} |\, x_{ij} \,|$, which is, since the absolute value of K_v is ultrametric, a submultiplicative norm on the K_v-algebra $\mathscr{M}_n(K_v)$.

For all $\ell, N \in \mathbb{Z}$, let

$$\begin{aligned} W_{\ell,N} &= \{ w = (w_{i,j})_{1 \leqslant i,j \leqslant n} \in \mathscr{M}_n(K_v) : \|w\| \leqslant {q_v}^{-\ell} \text{ and } \forall\, i \in [\![2, n]\!],\ |w_{i,1}| \leqslant {q_v}^{-(\ell+nN)} \}, \\ &= \{ w \in \mathscr{M}_n({\pi_v}^{\ell} \mathscr{O}_v) : \forall\, i \in [\![2, n]\!],\ w_{i,1} \in {\pi_v}^{\ell+nN} \mathscr{O}_v \}, \\ B_{\ell,N} &= (I_n + W_{\ell,N}) \cap \mathrm{SL}_n(K_v)\,. \end{aligned} \tag{62}$$

We also define $W_\ell = W_{\ell,0} = \mathscr{M}_n(\pi_v^\ell \mathscr{O}_v)$ and $B_\ell = B_{\ell,0}$. For all $\ell, \ell', N \in \mathbb{Z}$, by the ultrametric inequalities, we have

$$W_{\ell,N} + W_{\ell',N} \subset W_{\min(\ell,\ell'),N} \quad \text{and} \quad W_{\ell,N} W_{\ell',N} \subset W_{\ell+\ell',N}\,. \tag{63}$$

We also have the following decreasing properties $W_{\ell,N+1} \subset W_{\ell,N}$, $B_{\ell,N+1} \subset B_{\ell,N}$ in the parameter N and, in the parameter ℓ,

$$W_{\ell+1,N} \subset W_{\ell,N}, \quad B_{\ell+1,N} \subset B_{\ell,N}, \quad \bigcap_{\ell \in \mathbb{N}} W_{\ell,N} = \{0\}, \quad \bigcap_{\ell \in \mathbb{N}} B_{\ell,N} = \{I_n\}\,. \tag{64}$$

Since the multiplication by an element of $\mathscr{O}_v^\times$ preserves the absolute value on K_v, for every $a \in A(\mathscr{O}_v)$, we have

$$a\, W_{\ell,N}\, a^{-1} = W_{\ell,N} \qquad \text{and} \qquad a\, B_{\ell,N}\, a^{-1} = B_{\ell,N} \tag{65}$$

The action by conjugation of the transformation $\mathfrak{a}$ on these dynamical balls $W_{\ell,N}$ and $B_{\ell,N}$ satisfy the following contraction/dilation properties: For all $\ell, \ell', N \in \mathbb{Z}$, we have

$$\begin{aligned} \mathfrak{a}^{\ell'} W_{\ell,N}\, \mathfrak{a}^{-\ell'} &= \Big\{ w \in \mathscr{M}_n(K_v) : \begin{array}{l} \forall\, i,j \in [\![2,n]\!],\ w_{1,1}, w_{i,j} \in {\pi_v}^{\ell} \mathscr{O}_v, \\ w_{i,1} \in {\pi_v}^{\ell+(N-\ell')n} \mathscr{O}_v,\ w_{1,j} \in {\pi_v}^{\ell+n\ell'} \mathscr{O}_v \end{array} \Big\} \\ &\subset W_{\min\{\ell,\, \ell+n\ell'\},\ N-\ell'} \\ \text{hence} \quad \mathfrak{a}^{\ell'} B_{\ell,N}\, \mathfrak{a}^{-\ell'} &\subset B_{\min\{\ell,\, \ell+n\ell'\},\ N-\ell'}\,. \end{aligned} \tag{66}$$

The dynamical neighbourhoods $W_{\ell,N}$ and $B_{\ell,N}$ satisfy the following three elementary lemmas. The first one says that the balls $B_{\ell,N}$ are invariant upon taking inverses.

Lemma 6.1 *Let $N \in \mathbb{N}$, $\ell \in \mathbb{N} \smallsetminus \{0\}$ and $w \in W_{\ell,N}$. Then $(I_n + w)^{-1} \in I_n - w + W_{2\ell,N}$. In particular, we have $(B_{\ell,N})^{-1} = B_{\ell,N}$.*

Proof. By Equations (63) and (64), we have $w^i \in W_{i\ell,N}$ and $\lim_{i \to +\infty} w^i = 0$ since $\ell > 0$. Hence $I_n + w$ is invertible with $(I_n + w)^{-1} = I_n - w + \sum_{i=2}^{\infty} (-1)^i w^i$. By Equation (63) and since $W_{2\ell,N}$ is closed, we have $\sum_{i=2}^{\infty} (-1)^i w^i \in W_{2\ell,N}$. In particular we have the inclusion $(B_{\ell,N})^{-1} \subset B_{\ell,N}$, and equality holds by taking the inverses. □

The next lemma is a version, for dynamical balls in $\mathscr{X}$ of the form $B_{\ell,N}\, x$ centred at any point $x \in \mathscr{X}$, of the intersection property of ultrametric balls.

Lemma 6.2 *For all $N \in \mathbb{N}$, $\ell \in \mathbb{N} \smallsetminus \{0\}$ and $x, y \in \mathscr{X}$ with $B_{\ell,N}\, x \cap B_{\ell,N}\, y \neq \varnothing$, we have $B_{\ell,N}\, x = B_{\ell,N}\, y$.*

Proof. First notice the inclusion

$$\begin{aligned} B_{\ell,N}B_{\ell,N} &\subset ((I_n + W_{\ell,N})(I_n + W_{\ell,N})) \cap \mathrm{SL}_n(K_v) \\ &\subset (I_n + W_{\ell,N} + W_{2\ell,N}) \cap \mathrm{SL}_n(K_v) \subset B_{\ell,N}\,. \end{aligned} \tag{67}$$

Let $g, h \in B_{\ell,N}$ be such that $gx = hy$. Using Lemma 6.1 and the latter inclusion, we have

$$B_{\ell,N}\, x = B_{\ell,N}\, g^{-1}hy \subset B_{\ell,N}(B_{\ell,N})^{-1}B_{\ell,N}\, y = B_{\ell,N}B_{\ell,N}B_{\ell,N}\, y \subset B_{\ell,N}\, y\,.$$

By symmetry, the result follows. □

The final lemma gives a quantitative covering property for any dynamical ball $B_{\ell,N}$ in $\mathrm{SL}_n(K_v)$ by smaller dynamical balls $B_{\ell+\ell',N}\,g_i$.

Lemma 6.3 *Let $N \in \mathbb{N}$ and $\ell, \ell' \in \mathbb{N}\smallsetminus\{0\}$ with $\ell' \leqslant \ell$. Let $S \subset B_{\ell,N}$. Then, there exist an integer $C \leqslant (q_v^{\ell'})^{n^2}$ and matrices $g_1, \dots, g_C \in S$ such that*

$$S \subset \bigsqcup_{i=1}^{C} B_{\ell+\ell',N}\, g_i\,.$$

Proof. We may assume that S is nonempty, otherwise $C = 0$ works. As a preliminary remark, let us prove that for all integers $\ell, \ell' > 0$, there exist an integer $C \leqslant (q_v^{\ell'})^{n^2}$ and points $w_1, \dots, w_C \in W_{\ell,N}$ such that

$$W_{\ell,N} = \bigsqcup_{i=1}^{C}(w_i + W_{\ell+\ell',N})\,. \tag{68}$$

Indeed, recall that Equation (8) when $n = 1$ gives $\mathrm{vol}_v(\pi_v^{\ell+\ell'}\mathscr{O}_v) = q_v^{-\ell'}\,\mathrm{vol}_v(\pi_v^{\ell}\mathscr{O}_v)$. Let $\{x_i : i \in I\}$ be a set of representatives of the classes in $\pi_v^{\ell}\mathscr{O}_v/\pi_v^{\ell+\ell'}\mathscr{O}_v$, so that we have a partition $\pi_v^{\ell}\mathscr{O}_v = \bigsqcup_{i\in I}(x_i + \pi_v^{\ell+\ell'}\mathscr{O}_v)$. Furthermore, by the invariance of vol_v under translations, we have $\mathrm{Card}(I) \leqslant \mathrm{vol}_v(\pi_v^{\ell}\mathscr{O}_v)/\,\mathrm{vol}_v(\pi_v^{\ell+\ell'}\mathscr{O}_v) = q_v^{\ell'}$. The same argument replacing ℓ by $\ell+nN$ proves that $\pi_v^{\ell+nN}\mathscr{O}_v$ can be covered by at most $q_v^{\ell'}$ pairwise disjoint translates of the ball $\pi_v^{\ell+\ell'+nN}\mathscr{O}_v$. Equation (68) follows by applying this construction for each matrix entries.

Now, take $C \leqslant (q_v^{\ell'})^{n^2}$ and $w_1, \dots, w_C \in W_{\ell,N}$ as in Equation (68). We obtain a partition

$$S = \bigsqcup_{i=1}^{C}(I_n + w_i + W_{\ell+\ell',N}) \cap \mathrm{SL}_n(K_v) \cap S\,.$$

Up to decreasing C, we may assume that the set $S_i = (I_n + w_i + W_{\ell+\ell',N}) \cap \mathrm{SL}_n(K_v) \cap S$ is nonempty for every $i \in [\![1, C]\!]$. Let us fix an element $g_i \in S_i$ and let us prove that we have $(I_n + w_i + W_{\ell+\ell',N})\, g_i^{-1} \cap \mathrm{SL}_n(K_v) \subset B_{\ell+\ell',N}$. By Equations (63) and (64), since $\ell' \leqslant \ell$ and $I_n \in W_{0,N}$, we have

$$w_i^2 \in W_{\ell,N}W_{\ell,N} \subset W_{2\ell,N} \subset W_{\ell+\ell',N} \quad \text{and} \quad W_{\ell+\ell',N}W_{2\ell,N} \subset W_{3\ell+\ell',N} \subset W_{\ell+\ell',N}\,,$$

$$(I_n + w_i)W_{2\ell,n} \subset (W_{0,N} + W_{\ell,N})W_{2\ell,N} \subset W_{0,N}W_{2\ell,N} \subset W_{2\ell,N} \subset W_{\ell+\ell',N}\,.$$

By Lemma 6.1, we know that $g_i^{-1} \in I_n - w_i + W_{2\ell,N}$. We hence have

$$\begin{aligned}(I_n + w_i + W_{\ell+\ell',N})\, g_i^{-1} &\subset (I_n + w_i + W_{\ell+\ell',N})(I_n - w_i + W_{2\ell,N})\\ &\subset I_n - w_i^2 + (I_n + w_i)W_{2\ell,N} + W_{\ell+\ell',N}(I_n - w_i) + W_{\ell+\ell',N}W_{2\ell,N}\\ &\subset I_n + W_{\ell+\ell',N}\,.\end{aligned}$$

We obtain the inclusions $S_i \subset B_{\ell+\ell',N}\, g_i$ by taking the intersection with $\mathrm{SL}_n(K_v)$, so that $S \subset \bigcup_{i=1}^{C} B_{\ell+\ell',N}\, g_i$. Up to decreasing C, we may assume that this intersection is disjoint by using Lemma 6.2. This concludes the proof. □

6.2 Dynamical partitions in $\mathscr{X}$

We now construct measurable partitions of $\mathscr{X}$, that will be useful for entropy lower bounds computations. We start with a systole minoration result for R_v-lattices that are close enough to the standard "cubic" R_v-lattice. Recall that the map exp is defined just before Subsection 2.6 and the systole function in Subsection 2.4.

Lemma 6.4 *Let $\ell, d \in \mathbb{N} \smallsetminus \{0\}$, $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{Z}^d$ and $g \in \mathscr{M}_d(K_v)$ be such that $\|g\| \leqslant q_v^{-\ell}$. Then if $L = (I_d + \exp(\mathbf{k})\, g\, \exp(-\mathbf{k}))R_v^{\,d}$, we have $\mathrm{sys}(L) \geqslant 1 - q_v^{-\ell}$.*

Proof. By the equality case of the ultrametric triangular inequality and since $\|g\| < 1$, note that $|\det(I_d + g)| = \max_{\sigma\in\mathscr{S}_d} \prod_{1\leqslant i\leqslant d} |(I_d + g)_{i\sigma(i)}| = 1$, hence $I_d + g$ belongs to G_1 and so does $\exp(\mathbf{k})(I_d + g)\exp(-\mathbf{k})$. Therefore L is indeed an R_v-lattice in $K_v^{\,d}$ and $\mathrm{covol}(L) = \mathrm{covol}(R_v^{\,d})$ by Equation (10).

By Equation (17), assume for a contradiction that $\mathrm{sys}(L) = \min_{w\in L\smallsetminus\{0\}} \|w\| < 1 - q_v^{-\ell}$. Let $\mathbf{x} = (x_1, \dots, x_d) \in R_v^{\,d} \smallsetminus \{0\}$ be such that $\|(I_d + \exp(\mathbf{k})\, g\, \exp(-\mathbf{k}))\mathbf{x}\| < 1 - q_v^{-\ell}$. For every $i_0 \in [\![1, d]\!]$ such that $x_{i_0} \neq 0$, by computing the i_0-th coordinate of the vector $(I_d + \exp(\mathbf{k})\, g\, \exp(-\mathbf{k}))\mathbf{x}$, by the ultrametric triangular inequality and since $\|g\| \leqslant q_v^{-\ell}$, we then have

$$\begin{aligned}1 - q_v^{-\ell} > \Big|x_{i_0} + \sum_{j=1}^{d} x_j\, \pi_v^{k_j - k_{i_0}} g_{i_0 j}\Big| &\geqslant |x_{i_0}| - \max_{1\leqslant j\leqslant d} |x_j|\, |g_{i_0 j}|\, q_v^{-k_j + k_{i_0}}\\ &\geqslant |x_{i_0}| - q_v^{-\ell} \max_{1\leqslant j\leqslant d} |x_j|\, q_v^{-k_j + k_{i_0}}\,.\end{aligned}$$

Noting that $|x_{i_0}| \geqslant 1$ since $x_{i_0} \in R_v \smallsetminus \{0\}$, and since $q_v^{\ell} - 1 \geqslant 0$, we have $|x_{i_0}|(q_v^{\ell} - 1) \geqslant q_v^{\ell} - 1$, thus $|x_{i_0}| \leqslant q_v^{\ell}\big(|x_{i_0}| - 1 + q_v^{-\ell}\big)$. Therefore

$$|x_{i_0}|\, q_v^{-k_{i_0}} \leqslant q_v^{\ell}\big(|x_{i_0}| - 1 + q_v^{-\ell}\big)\, q_v^{-k_{i_0}} < \max_{1\leqslant j\leqslant d} |x_j|\, q_v^{-k_j}\,.$$

Thus, there exists $i_1 \neq i_0$ such that $|x_{i_0}|\, q_v^{-k_{i_0}} < |x_{i_1}|\, q_v^{-k_{i_1}}$. By iteration, and since there is no strictly increasing sequence in the finite subset $\{|x_i|\, q_v^{-k_i} : i \in [\![1, d]\!], x_i \neq 0\}$ of $[0, +\infty[$, we obtain a contradiction. □

Let us fix $\mathbf{s} = (s_2, \dots, s_n) \in (R_v \smallsetminus \{0\})^{n-1}$ satisfying Equation (45): There exists a permutation σ of $[\![2, n]\!]$ such that $s_{\sigma^{-1}(2)} \mid s_{\sigma^{-1}(3)} \mid \dots \mid \mathbf{s}_* = s_{\sigma^{-1}(n)}$ and

$$v(\mathbf{s}_*) = \min_{2\leqslant i\leqslant n} v(s_i) \in n\,\mathbb{Z}\,. \tag{69}$$

The next lemma gives a cardinality estimate for the number of R_v-lattices whose equidistribution we want to study, that belong to a small dynamical ball of $\mathscr{X}$.

Lemma 6.5 *For every $x \in \mathscr{X} = G/\Gamma$, every integer $\ell > \max\{0, -\log_{q_v}(\mathrm{sys}(x))\}$, every $\mathbf{k} = (k_1, \dots, k_n) \in \Delta_{\mathbf{s}}$ and every $N \in [\![0, \frac{-v(\mathbf{s}_*)-\ell+k_1-\max_{2\leqslant i\leqslant n} k_i}{n}]\!]$, we have*

$$\mathrm{Card}\big(\{\mathbf{t} + R_v^{\,n-1} \in \Lambda_{\mathbf{s}} : \exp(\mathbf{k})\,\mathfrak{u}_{\mathbf{t}}\,\Gamma \in B_{\ell,N}\, x\}\big) \leqslant 2^{n-1}\, q_v^{-\ell(n-1)-v(\mathbf{s}_*)(n-1)-n(n-1)N}\,.$$

Proof. Fix x, ℓ, $\mathbf{k}$ and N as in the lemma. Let $g \in G$ be such that $x = g\Gamma$. For simplicity, we fix a lift $\widetilde{\Lambda_{\mathbf{s}}}$ of $\Lambda_{\mathbf{s}}$ in K^{n-1}, hence having the same cardinality $\prod_{i=2}^n \varphi_v(s_i)$ as $\Lambda_{\mathbf{s}}$.

We want to evaluate the number of points $\mathbf{t} \in \widetilde{\Lambda_{\mathbf{s}}}$ such that $\exp(\mathbf{k})\mathfrak{u}_{\mathbf{t}}\Gamma \in B_{\ell,N}\, g\Gamma$, in other words such that there exist $\gamma \in \Gamma$ and $h \in B_{\ell,N}$ verifying $h^{-1}\exp(\mathbf{k}) = g\gamma\mathfrak{u}_{-\mathbf{t}}$. Since $B_{\ell,N}^{-1} = B_{\ell,N}$ by Lemma 6.1, we have to bound from above the nonnegative quantity

$$\mathrm{Card}\{\mathbf{t} \in \widetilde{\Lambda_{\mathbf{s}}} : \exists \gamma \in \Gamma,\ g\gamma\mathfrak{u}_{-\mathbf{t}} \in B_{\ell,N}\exp(\mathbf{k})\}\,.$$

We may assume that this quantity is nonzero. Let us take $\mathbf{t} = (\frac{r_2}{s_2}, \dots, \frac{r_n}{s_n}) \in \widetilde{\Lambda_{\mathbf{s}}}$ and $\gamma \in \Gamma$ whose column matrices are denoted by $\gamma_1, \dots, \gamma_n$. Recall the notation $(e_1, \dots, e_n)$ for the canonical K_v-basis of $K_v^{\,n}$. Let us consider the constraints on the columns in the condition $g\gamma\mathfrak{u}_{-\mathbf{t}} \in (I_n + W_{\ell,N})\exp(\mathbf{k})$, which is equivalent to the condition $g\gamma\mathfrak{u}_{-\mathbf{t}} \in B_{\ell,N}\exp(\mathbf{k})$ since $\det(g\gamma\mathfrak{u}_{-\mathbf{t}}\exp(-\mathbf{k})) = 1$. We obtain the equivalent system of conditions

$$g\gamma_1 - \sum_{i=2}^n \frac{r_i}{s_i} g\gamma_i \ \in\ \pi_v^{-k_1}e_1 + (\pi_v^{\ell-k_1}\mathscr{O}_v) \times (\pi_v^{\ell+nN-k_1}\mathscr{O}_v)^{n-1}\,, \tag{70}$$

$$\forall\, i \in [\![2, n]\!], \quad g\gamma_i \ \in\ \pi_v^{-k_i}e_i + (\pi_v^{\ell-k_i}\mathscr{O}_v)^n\,. \tag{71}$$

Since $\ell > -\log_{q_v}(\mathrm{sys}(x))$, the lattice $x = gR_v^{\,n}$ contains at most one point in each translate of $(\pi_v^{\,\ell}\mathscr{O}_v)^n$. As seen in the proof of Equation (68), for every $i \in [\![2, n]\!]$, since $k_i \geqslant 0$ as $\mathbf{k} \in \Delta_{\mathbf{s}}$, we can cover (any translate of) $(\pi_v^{\,\ell-k_i}\mathscr{O}_v)^n$ by at most $q_v^{\,k_i n}$ pairwise disjoint translates of $(\pi_v^{\ell}\mathscr{O}_v)^n$. Hence for each fixed $i \in [\![2, n]\!]$, the condition on $\gamma_i \in R_v^{\,n}$ given in Equation (71) has at most $q_v^{\,k_i n}$ solutions, and this set of solutions is independent of $\mathbf{t}$.

Let us fix a solution $(\gamma_2, \dots, \gamma_n)$ of the system of equations (71). Given an element $\mathbf{t} = (\frac{r_2}{s_2}, \dots, \frac{r_n}{s_n}) \in \widetilde{\Lambda_{\mathbf{s}}}$, let us fix a solution $\gamma = (\gamma_1\ \gamma_2\ \dots\ \gamma_n) \in \Gamma$ of Equation (70) with prescribed last $n-1$ columns $\gamma_2, \dots, \gamma_n$. Note that since $\gamma \in \Gamma = \mathrm{SL}_n(R_v)$, the vectors of $K_v^{\,n}$ with column matrices $\gamma_1, \dots, \gamma_n$ form an R_v-basis of $R_v^{\,n}$. Hence any other solution $\gamma' = (\gamma_1'\ \gamma_2\ \dots\ \gamma_n) \in \Gamma$ of this equation with these last $n-1$ columns has a first column γ_1' such that there exists $\lambda_1, \dots, \lambda_n \in R_v$ with $\gamma_1' = \lambda_1\gamma_1 + \lambda_2\gamma_2 + \dots + \lambda_n\gamma_n$. Since the determinant of n-tuples of elements of $K_v^{\,n}$ is multilinear and alternating, and since $\det\gamma = \det\gamma' = 1$, we have

$$\lambda_1 = \det(\lambda_1\gamma_1, \gamma_2, \dots, \gamma_n) = \det(\gamma_1', \gamma_2, \dots, \gamma_n) = 1\,.$$

Hence Equation (70) for the matrix γ' becomes

$$g\gamma_1 + \sum_{i=2}^n \big(\lambda_i - \frac{r_i}{s_i}\big)g\gamma_i \ \in\ \pi_v^{-k_1}e_1 + (\pi_v^{\ell-k_1}\mathscr{O}_v) \times (\pi_v^{\ell+nN-k_1}\mathscr{O}_v)^{n-1}\,. \tag{72}$$

We denote by $\mathrm{pr} : K_v^{\,n} \to K_v^{\,n-1}$ the projection onto the last $n-1$ coordinates. Let $\mathfrak{d} = \mathrm{diag}(\pi_v^{\,k_2}, \dots, \pi_v^{\,k_n})$. Let us multiply Equation (72) by the scalar $\mathbf{s}_*$ (defined above Equation (69), so that $\frac{r_i\,\mathbf{s}_*}{s_i} \in R_v$ for every $i \in [\![2, n]\!]$). Let then project it to $K_v^{\,n-1}$ as well as Equation (71). Let us then multiply them by the matrix $\mathfrak{d} = \operatorname{diag}(\pi_v^{k_2}, \dots, \pi_v^{k_n})$ on the left. The condition $g\gamma\mathfrak{u}_{-\mathbf{t}} \in B_{\ell,N}\exp(\mathbf{k})$ thus provides the system of conditions

$$\sum_{i=2}^{n}\Big(\lambda_i\,\mathbf{s}_* - \frac{r_i\,\mathbf{s}_*}{s_i}\Big)\mathfrak{d}\operatorname{pr}(g\gamma_i) \;\in\; \mathbf{s}_*\mathfrak{d}\operatorname{pr}(g\gamma_1) + \mathbf{s}_*\prod_{i=2}^{n}(\pi_v^{\ell+nN+k_i-k_1}\mathscr{O}_v)\,, \tag{73}$$

$$\forall\, i \in [\![2,n]\!],\quad \mathfrak{d}\operatorname{pr}(g\gamma_i) \;\in\; e_i + \mathfrak{d}(\pi_v^{\ell-k_i}\mathscr{O}_v)^{n-1}\,. \tag{74}$$

We want to give an upper bound on the number of elements $\mathbf{t} \in \widetilde{\Lambda_{\mathbf{s}}}$ such that there exists $\lambda_2, \dots, \lambda_n \in R_v$ satisfying this system. By Equation (74), since $\ell > 0$, there exists a matrix $\widetilde{g} \in \mathscr{M}_{n-1}(K_v)$ with $\|\widetilde{g}\| \leqslant q_v^{-\ell}$ such that the R_v-lattice $L = \oplus_{2\leqslant i\leqslant n} R_v\mathfrak{d}\operatorname{pr}(g\gamma_i)$ in K_v^{n-1} is equal to $(I_{n-1} + \mathfrak{d}\,\widetilde{g}\,\mathfrak{d}^{-1})R_v^{n-1}$. By Lemma 6.4 applied with $d = n-1$, we have $\operatorname{sys}(L) \geqslant 1 - q_v^{-\ell} \geqslant \frac{1}{2}$. Note that for every $i \in [\![2,n]\!]$, the assumption on N of Lemma 6.5 gives the inequalities $-v(\mathbf{s}_*) - \ell - nN - k_i + k_1 \geqslant 0$. Note that $\mathbf{s}_* \in \pi^{v(\mathbf{s}_*)}\mathscr{O}_v^{\times}$. Since each solution $(\lambda_2\,\mathbf{s}_* + \frac{r_2\,\mathbf{s}_*}{s_2}, \dots, \lambda_n\,\mathbf{s}_* - \frac{r_n\,\mathbf{s}_*}{s_n}) \in R_v^{n-1}$ of Equation (73) corresponds to one point of the lattice L, the associated number of solutions is bounded from above by

$$\begin{aligned}
&\Big\lceil \frac{1}{\operatorname{sys}(L)^{n-1}}\operatorname{vol}_v^{n-1}\Big(\mathbf{s}_*\prod_{i=2}^{n}\pi_v^{\ell+nN+k_i-k_1}\mathscr{O}_v\Big)\Big\rceil\\
&\leqslant \Big\lceil 2^{n-1}\prod_{2\leqslant i\leqslant n}\operatorname{vol}_v\big(\pi_v^{v(\mathbf{s}_*)+\ell+nN+k_i-k_1}\mathscr{O}_v\big)\Big\rceil = 2^{n-1}\prod_{2\leqslant i\leqslant n} q_v^{-v(\mathbf{s}_*)-\ell-nN-k_i+k_1}\\
&\leqslant 2^{n-1}\,q_v^{-\ell(n-1)-v(\mathbf{s}_*)(n-1)-n(n-1)N+(n-1)k_1-\sum_{i=2}^{n}k_i}\,.
\end{aligned}$$

Combining this with the previous counting results for the columns $\gamma_2, \dots, \gamma_n$, recalling that $\sum_{i=1}^{n} k_i = 0$ (since $\mathbf{k} \in \Delta_{\mathbf{s}} \subset \mathbb{Z}_0^n$), we finally obtain, as wanted,

$$\begin{aligned}
&\operatorname{Card}\big(\{\exp(\mathbf{k})\mathfrak{u}_{\mathbf{t}}\Gamma : \mathbf{t} \in \widetilde{\Lambda}_{\mathbf{s}}\} \cap (B_{\ell,N}\,x)\big)\\
&\leqslant 2^{n-1}\,q_v^{-\ell(n-1)-v(\mathbf{s}_*)(n-1)-n(n-1)N+(n-1)k_1-\sum_{i=2}^{n}k_i}\prod_{i=2}^{n} q_v^{nk_i}\\
&\leqslant 2^{n-1}\,q_v^{-\ell(n-1)-v(\mathbf{s}_*)(n-1)-n(n-1)N}\,. \quad \square
\end{aligned}$$

Before stating the main result of Subsection 6.2, let us give some definitions. For all $\ell, m \in \mathbb{N}$, an (m,ℓ)-*partition* of $\mathscr{X}$ is a finite measurable partition $\mathscr{P} = \{P_1, P_2, \dots, P_{|\mathscr{P}|}\}$ of $\mathscr{X}$ such that P_1 is equal to the q_v^{-m}-thin part $\mathscr{X}^{<q_v^{-m}} = \{x \in \mathscr{X} : \operatorname{sys}(x) < q_v^{-m}\}$ of $\mathscr{X}$ (see Subsection 2.4) and such that for every $i \in [\![2, |\mathscr{P}|]\!]$, there exists $x_i \in \mathscr{X}$ with $P_i \subset B_\ell\,x_i$ with $B_\ell = B_{\ell,0}$ defined in Equation (62). Note that for every (m,ℓ)-partition $\mathscr{P}$ of $\mathscr{X}$ and every $a \in A(\mathscr{O}_v)$, since $\operatorname{sys}(ax) = \operatorname{sys}(x)$ for every $x \in \mathscr{X}$ and by Equation (65), the partition $a^{-1}\mathscr{P}$ is also an (m,ℓ)-partition of $\mathscr{X}$.

For every $N \in \mathbb{N} \smallsetminus \{0\}$, the N-*th dynamical partition* for $\mathfrak{a}$ associated with a finite measurable partition $\mathscr{P}$ of $\mathscr{X}$ is the finite measurable partition

$$\mathscr{P}^N = \bigvee_{i=0}^{N-1}\mathfrak{a}^{-i}\mathscr{P}\,. \tag{75}$$

Note that for every $a \in A(\mathscr{O}_v)$, since a commutes with $\mathfrak{a}$, we have $a^{-1}(\mathscr{P}^N) = (a^{-1}\mathscr{P})^N$. The *N-th Birkhoff average* for $\mathfrak{a}$ of a Borel probability measure μ on $\mathscr{X}$ is

$$S_N\mu = \frac{1}{N}\sum_{i=0}^{N-1} \mathfrak{a}^i_*\mu\,. \tag{76}$$

The next lemma says that the thick part of the space $\mathscr{X}$ of special unimodular R_v-lattices may be almost entirely covered by dynamical balls $B_{\ell,N}\,x_j$ that are essentially finer than the partition $\mathscr{P}^N$, with a good control of the cardinality of this cover.

Lemma 6.6 *For every $m \in \mathbb{N}$, there exists $\ell_m \in \mathbb{N}\smallsetminus\{0\}$ such that for every integer $\ell \geqslant \ell_m$, for every (m,ℓ)-partition $\mathscr{P}$ of $\mathscr{X}$, for every $\kappa \in \,]0,1[$, and for every $N \in \mathbb{N}\smallsetminus\{0\}$, the q_v^{-m}-thick part $\mathscr{X}^{\geqslant q_v^{-m}}$ of $\mathscr{X}$ contains a measurable subset $\mathscr{X}' = \mathscr{X}'_{\mathscr{P},\kappa,N}$ satisfying the two following conditions.*

1. *There exists a subset $\mathscr{P}'$ of $\mathscr{P}^N$ such that $\mathscr{X}' = \bigcup \mathscr{P}'$ and such that, for every $P \in \mathscr{P}'$, there exists a finite subset F_P of P with cardinality at most $q_v^{\,n^3\kappa N}$ such that $P \subset \bigcup_{x\in F_P} B_{\ell,N-1}\,x$.*
2. *For every Borel probability measure μ on $\mathscr{X}$, we have $\mu(\mathscr{X}') \geqslant 1 - \frac{1}{\kappa}\,S_N\mu(\mathscr{X}^{<q_v^{-m}})$.*

Proof. Let m, κ, N and μ be as in the statement. Since the action by left translations of G on G/Γ is locally free and since $\mathscr{X}^{\geqslant q_v^{-m}}$ is compact, there exists $\ell_m \in \mathbb{N}\smallsetminus\{0\}$ such that for every $x \in \mathscr{X}^{\geqslant q_v^{-m}}$, the map $g \mapsto gx$ is injective on the dynamical ball $B_{\ell_m - n}$. We may assume that $\ell_m \geqslant n$ for future use. Let $\ell \geqslant \ell_m$ and let $\mathscr{P} = \{P_1, \dots, P_{|\mathscr{P}|}\}$ be an (m,ℓ)-partition of $\mathscr{X}$ so that $P_1 = \mathscr{X}^{<q_v^{-m}}$ and for every $k \in [\![2, |\mathscr{P}|]\!]$, there exists $x_k \in \mathscr{X}$ such that $P_k \subset B_\ell\,x_k$. We define a function $f_N : \mathscr{X} \to [0,+\infty[$ counting in average the excursions before time N of the diagonal orbits under $\mathfrak{a}$ into the q_v^{-m}-thin part of $\mathscr{X}$ by

$$f_N : x \mapsto \frac{1}{N}\sum_{j=0}^{N-1} \mathbb{1}_{\mathscr{X}^{<q_v^{-m}}}(\mathfrak{a}^j x)\,.$$

We define $\mathscr{X}' = \{x \in \mathscr{X} : f_N(x) \leqslant \kappa\}$. By Markov's inequality applied to the nonnegative random variable f_N, we have

$$\begin{aligned} 1 - \mu(\mathscr{X}') = \mu\big(\{x \in \mathscr{X} : f_N(x) > \kappa\}\big) &\leqslant \frac{1}{\kappa}\int_{\mathscr{X}} f_N\, d\mu = \frac{1}{\kappa N}\sum_{j=0}^{N-1}\int_{\mathscr{X}} \mathbb{1}_{\mathscr{X}^{<q_v^{-m}}}(\mathfrak{a}^j x)\, d\mu(x)\\ &= \frac{1}{\kappa N}\sum_{j=0}^{N-1}\int_{\mathscr{X}} \mathbb{1}_{\mathscr{X}^{<q_v^{-m}}}\, d(\mathfrak{a}^j{}_*\mu) = \frac{1}{\kappa} S_N\mu(\mathbb{1}_{\mathscr{X}^{<q_v^{-m}}})\,. \end{aligned}$$

Hence the set $\mathscr{X}'$ satisfies Assertion (2).

In order to prove Assertion (1), first notice that f_N can be described on the partition $\mathscr{P}^N$ as follows. For all $P = \bigcap_{j=0}^{N-1} \mathfrak{a}^{-j} P_{k_j} \in \mathscr{P}^N$, where $k_0, \dots, k_{N-1} \in [\![1, |\mathscr{P}|]\!]$, since $P_1 = \mathscr{X}^{<q_v^{-m}}$, we have, for all $x \in P$,

$$f_N(x) = \frac{1}{N}\,\mathrm{Card}\{j \in [\![0, N-1]\!] : k_j = 1\}\,. \tag{77}$$

In particular, f_N is constant on every $P \in \mathscr{P}^N$. The definition of $\mathscr{X}'$ then implies that there exists a subset $\mathscr{P}'$ of the partition $\mathscr{P}^N$ such that $\mathscr{X}' = \bigcup \mathscr{P}'$.

Now let $P \in \mathscr{P}'$, so that we have $f_{N|P} \leqslant \kappa < 1$. By the definition of the partition $\mathscr{P}^N$ and since $\mathscr{P}$ is an (m,ℓ)-partition of $\mathscr{X}$, for every $j \in [\![0, N-1]\!]$, we have either $\mathfrak{a}^j P \subset \mathscr{X}^{\geqslant q_v^{-m}}$ or $\mathfrak{a}^j P \subset \mathscr{X}^{< q_v^{-m}}$. The set $\{j \in [\![0, N-1]\!] : \mathfrak{a}^j P \subset \mathscr{X}^{\geqslant q_v^{-m}}\}$ is nonempty, since otherwise we would have $\mathfrak{a}^j P \subset \mathscr{X}^{< q_v^{-m}}$ for every $j \in [\![0, N-1]\!]$, hence $f_{N|P} = 1$, a contradiction. Let $j_0 \in [\![0, N-1]\!]$ be the minimum of this nonempty subset of $\mathbb{N}$. By the definition of the partition $\mathscr{P}^N$ and since $\mathscr{P}$ is an (m,ℓ)-partition of $\mathscr{X}$, there exists $k_0 \in [\![2, |\mathscr{P}|]\!]$ such that $\mathfrak{a}^{j_0} P \subset P_{k_0} \subset B_\ell\, x_{k_0}$. This inclusion $\mathfrak{a}^{j_0} P \subset B_\ell\, x_{k_0}$ is the starting point in order to prove Assertion (1) by using iterations of Lemma 6.3 and of Equation (66). We define, for every $j \in [\![0, N-1]\!]$,

$$V_j = \{i \in [\![0, j]\!] : \mathfrak{a}^i P \subset \mathscr{X}^{< q_v^{-m}}\}\,,$$

and we denote by $|V_j|$ its cardinality.

Let $C = q_v^{n^2}$ be the constant satisfying Lemma 6.3 for $\ell' = 1$ (allowing multiplicities), so that $C^n = q_v^{n^3}$ is the constant satisfying Lemma 6.3 for $\ell' = n$.

Claim. For every $j \in [\![0, N-1]\!]$ such that $|V_j| \neq j+1$, there exist R_v-lattices $y_{1,j}, \dots, y_{C^{n|V_j|},\, j} \in P$ such that

$$P \subset \bigcup_{i=1}^{C^{n|V_j|}} B_{\ell,\, j}\, y_{i,\, j}\,. \tag{78}$$

We have $|V_{N-1}| \leqslant \kappa N < N$ (hence $C^{n|V_{N-1}|} \leqslant q_v^{n^3 \kappa N}$) since $f_{N|P} = \frac{|V_{N-1}|}{N}$ by Equation (77) and since $P \subset \mathscr{X}'$ so that $f_{N|P} \leqslant \kappa$. Therefore the case $j = N-1$ of this claim implies Assertion (1).

Proof of the claim. We proceed by induction on $j \in [\![0, N-1]\!]$. By definition, we have $j_0 = \min\{j \in [\![0, N-1]\!] : |V_j| \neq j+1\}$, hence we begin the induction at the step j_0, the previous cases being empty. If $j_0 = 0$, we have $P \subset B_\ell\, x_{k_0}$ and by Lemma 6.2 applied with $N = 0$, we can assume that $x_{k_0} \in P$, which proves the Claim at the j_0-th step (since then $|V_0| = 0$ or equivalently $C^{n|V_0|} = 1$). If $j_0 \geqslant 1$, then we apply nj_0 times Lemma 6.3 with $N = 0$, $\ell' = 1$ and S successively equal to $B_\ell, B_{\ell+1}, B_{\ell+2}, \dots, B_{\ell+nj_0-1}$. This gives the existence of $g_1, \dots, g_{C^{nj_0}} \in G$ such that $B_\ell \subset \bigcup_{i=1}^{C^{nj_0}} B_{\ell+nj_0}\, g_i$. Hence by the inclusion $\mathfrak{a}^{j_0} P \subset B_\ell\, x_{k_0}$, we have $\mathfrak{a}^{j_0} P \subset \bigcup_{i=1}^{C^{nj_0}} B_{\ell+nj_0}\, g_i x_{k_0}$. Up to allowing multiplicities, we may assume that for every $i \in [\![1, C^{nj_0}]\!]$, the intersection $(\mathfrak{a}^{j_0} P) \cap (B_{\ell+nj_0}\, g_i x_{k_0})$ is nonempty, hence contains an element $\mathfrak{a}^{j_0} y_{i,\, j_0}$ with $y_{i,\, j_0} \in P$. By Lemma 6.2 and since $I_n \in B_{\ell+nj_0}$, we have $B_{\ell+nj_0}\, g_i x_{k_0} = B_{\ell+nj_0}\, \mathfrak{a}^{j_0} y_{i,\, j_0}$. Therefore $\mathfrak{a}^{j_0} P \subset \bigcup_{i=1}^{C^{nj_0}} B_{\ell+nj_0}\, \mathfrak{a}^{j_0} y_{i,\, j_0}$. Hence by Equation (66) applied with $N = 0$, $\ell' = -j_0$ and ℓ replaced by $\ell + nj_0$, we have

$$P \subset \bigcup_{i=1}^{C^{nj_0}} \mathfrak{a}^{-j_0} B_{\ell+nj_0}\, \mathfrak{a}^{j_0} y_{i,\, j_0} \subset \bigcup_{i=1}^{C^{nj_0}} B_{\ell,\, j_0}\, y_{i,\, j_0}\,.$$

Since $|V_{j_0}| = j_0$, this proves the j_0-th step of the Claim. If $j_0 = N-1$, there is nothing more to be proved, hence we assume that $j_0 \leqslant N-2$.

Now let $j \in [\![j_0, N-2]\!]$ and assume that the j-th step of the Claim is satisfied, so that $P \subset \bigcup_{i=1}^{C^{n|V_j|}} B_{\ell,\, j}\, y_{i,\, j}$ where $y_{1,j}, \dots, y_{C^{n|V_j|},\, j} \in P$.

- First assume that $\mathfrak{a}^{j+1}P \subset \mathscr{X}^{\geqslant q_v^{-m}}$ or equivalently that $|V_{j+1}| = |V_j|$. Let us fix an element $i \in [\![1, C^{n|V_j|}]\!]$ and let us prove that $B_{\ell, j+1}\, y_{i,j} \cap P = B_{\ell, j}\, y_{i,j} \cap P$. This will imply the $(j+1)$-th step of the Claim by setting $y_{i,j+1} = y_{i,j}$. The inclusion $B_{\ell, j+1}\, y_{i,j} \cap P \subset B_{\ell, j}\, y_{i,j} \cap P$ is clear by the inclusion just above Equation (64). For the converse one, let $g \in B_{\ell,j}$ be such that $gy_{k,j} \in P$. Since we have $\mathfrak{a}^{j+1}P \subset \mathscr{X}^{\geqslant q_v^{-m}}$ and since $\mathscr{P}$ is an (m,ℓ)-partition of $\mathscr{X}$, there exists $k \in [\![2, |\mathscr{P}|]\!]$ such that $\mathfrak{a}^{j+1}P \subset P_k \subset B_\ell\, x_k$. Let us define $\widetilde{x} = \mathfrak{a}^{j+1} y_{i,j}$ and $\widetilde{g} = \mathfrak{a}^{j+1} g\, \mathfrak{a}^{-(j+1)}$. Since $y_{i,j} \in P$, we have

$$\widetilde{x} = \mathfrak{a}^{j+1} y_{i,j} \in \mathfrak{a}^{j+1} P \subset \mathscr{X}^{\geqslant q_v^{-m}} \cap (B_\ell\, x_k)\,.$$

  Similarly, since $gy_{i,j} \in P$, we have $\widetilde{g}\,\widetilde{x} = \mathfrak{a}^{j+1}(gy_{k,j}) \in B_\ell\, x_k$. Therefore we have $\widetilde{g}\,\widetilde{x} \in B_\ell (B_\ell)^{-1} \widetilde{x} \subset B_\ell\, \widetilde{x}$ by Lemma 6.1 and Equation (67) (both with $N = 0$ therein). By Equations (66) and (64), we have $\widetilde{g} = \mathfrak{a}^{j+1} g\, \mathfrak{a}^{-(j+1)} \in B_{\ell,-1} \subset B_{\ell - n} \subset B_{\ell_m - n}$. We have $B_\ell \subset B_{\ell_m} \subset B_{\ell_m - n}$ since $\ell \geqslant \ell_m \geqslant \ell_m - n$, again by Equation (64). Since $\widetilde{g}\,\widetilde{x} \in B_\ell\, \widetilde{x}$, since $\widetilde{x} \in \mathscr{X}^{\geqslant q_v^{-m}}$ and by the definition of ℓ_m, we have $\widetilde{g} \in B_\ell$. Therefore, by Equation (66) again, we finally obtain

$$g = \mathfrak{a}^{-(j+1)} \widetilde{g}\, \mathfrak{a}^{j+1} \in \mathfrak{a}^{-(j+1)} B_\ell\, \mathfrak{a}^{j+1} \cap B_{\ell, j} \subset B_{\ell - n(j+1), j+1} \cap B_{\ell, j} \subset B_{\ell, j+1}\,,$$

  so that $gy_{k,j} \in B_{\ell, j+1}\, y_{k,j} \cap P$, thus proving the wanted converse inclusion.
- Now assume that $\mathfrak{a}^{j+1}P \subset \mathscr{X}^{< q_v^{-m}}$ or equivalently that $|V_{j+1}| = |V_j| + 1$. The proof of the $(j+1)$-th step of the Claim is then straightforward by applying Lemma 6.3 with $\ell' = n$ (so that $\ell \geqslant \ell_m \geqslant n \geqslant \ell'$) and $N = j$ to $S = B_{\ell,j}$ in order to cover each $B_{\ell, j}\, y_{i,j} \cap P$ for $i \in [\![1, C^{n|V_j|}]\!]$ by C^n subsets of $\mathscr{X}$ of the form

$$B_{\ell+n, j}\, y^{(i)}_{i', j+1} \cap P \subset B_{\ell, j+1}\, y^{(i)}_{i', j+1} \cap P\,,$$

  where $i' \in [\![1, C^n]\!]$ and $y^{(i)}_{i', j+1} \in \mathscr{X}$, thus covering P by $C^{n|V_j|} C^n = C^{n|V_{j+1}|}$ subsets $B_{\ell, j+1}\, y^{(i)}_{i', j+1}$. As in the j_0-th step, by Lemma 6.2, we may assume that $y^{(i)}_{i', j+1} \in P$. □

7 Non-escape of mass in the thin part

In this section, according to the first step of the program announced after the statement of Theorem 5.2, we provide the material that will be used in Section 8 in order to prove that every weak-star accumulation point μ of the measures $\nu^{\lozenge}_{\mathbf{s}}$ as $\mathbf{s}$ tends appropriately to $+\infty$ is a probability measure on $\mathscr{X}$.

For every fixed $J \in \mathscr{I}_v^+$, we first estimate the number of nonzero ideals that are coprime to J and whose norm is comparatively small with respect to the one of J. Recall (see Subsection 2.1) that $\varpi_v(J)$ is the number of prime factors of J. For every $\epsilon > 0$, let

$$E_{J,\,\mathrm{prim}}(\epsilon) = \big\{ I \in \mathscr{I}_v^+ : (I, J) = 1,\ \mathtt{N}(I) \leqslant \epsilon\, \mathtt{N}(J) \big\}\,.$$

Lemma 7.1 *There exists $c_1 \geqslant 0$ such that for all $J \in \mathscr{I}_v^+$ and $\epsilon \in q^{\mathbb{Z}} \cap \,]\,0, 1\,[\,$, we have*

$$\Big|\ \mathrm{Card}(E_{J,\,\mathrm{prim}}(\epsilon)) - \epsilon\, \frac{h_K\, q^{2-\mathfrak{g}}\, (q_v - 1)}{(q-1)^2\, q_v}\, \varphi_v(J)\ \Big| \leqslant c_1\, 2^{\,\varpi_v(J)}\,.$$

Proof. Let $c_K = \frac{h_K\, q^{2-\mathfrak{g}}\,(q_v-1)}{(q-1)^2\, q_v} > 0$. Let $E_J(\epsilon) = \big\{I \in \mathscr{I}_v^+ : \mathtt{N}(I) \leqslant \epsilon\, \mathtt{N}(J)\big\}$. By a standard sieving argument, with μ_v the Möbius function defined in Subsection 2.1, by Lemma 2.1 since $\epsilon \in q^{\mathbb{Z}}$ and $N(I') \in q^{\mathbb{N}}$ for every $I' \in \mathscr{I}_v^+$, and by Equation (4), we have

$$\begin{aligned}
\operatorname{Card}(E_{J,\,\mathrm{prim}}(\epsilon)) &= \sum_{I\in\mathscr{I}_v^+,\; I\,|\,J} \mu_v(I)\, \operatorname{Card}(E_{JI^{-1}}(\epsilon))\\
&= \sum_{I\in\mathscr{I}_v^+,\; I\,|\,J} \mu_v(I)\big(c_K\, \epsilon\, N(JI^{-1}) + \mathrm{O}(1)\big)\\
&= \epsilon\, c_K\, N(J) \sum_{I\in\mathscr{I}_v^+,\; I\,|\,J} \frac{\mu_v(I)}{N(I)} + \mathrm{O}\Big(\sum_{I\in\mathscr{I}_v^+,\; I\,|\,J} |\mu_v(I)|\Big)\\
&= \epsilon\, c_K\, \varphi_v(J) + \mathrm{O}(2^{\varpi_v(J)})\,.
\end{aligned}$$

This proves the result. $\square$

Lemma 7.2 *Assume that R_v is principal. There exists a constant $c_2 \geqslant 1$ such that for all $\mathbf{s} = (s_2, \dots, s_n) \in (R_v \smallsetminus \{0\})^{n-1}$ and $\mathbf{k} = (k_1, \dots, k_n) \in \Delta_{\mathbf{s}}$ with*

$$\forall\, i \in [\![2, n]\!], \quad 2^{\varpi_v(s_i)} q_v^{\,k_i - k_1} \leqslant \frac{|s_i|}{\max\{1, \ln\ln|s_i|\}}\,, \tag{79}$$

and for every $\epsilon \in q_v^{\mathbb{Z}} \cap\,]0, 1[\,$, we have

$$\Big(\frac{1}{\operatorname{Card}\Lambda_{\mathbf{s}}} \sum_{\mathbf{t}\in\Lambda_{\mathbf{s}}} \delta_{\exp(\mathbf{k})x_{\mathbf{t}}}\Big)(\mathscr{X}^{\leqslant\epsilon}) \leqslant c_2\, \epsilon^n\,.$$

Proof. Let $\mathbf{s}$, $\mathbf{k}$, ϵ be fixed as in the statement. Let $\mathbf{t} = (\frac{r_2}{s_2}, \dots, \frac{r_n}{s_n}) \mod R_v^{n-1}$ that varies in $\Lambda_{\mathbf{s}}$. Recall that $x_{\mathbf{t}} = \mathfrak{u}_{\mathbf{t}} R_v^n$. By the definition in Subsection 2.4 of the ϵ-thin part $\mathscr{X}^{\leqslant\epsilon}$ of $\mathscr{X}$, we have $\exp(\mathbf{k})\, x_{\mathbf{t}} \in \mathscr{X}^{\leqslant\epsilon}$ if and only if there exists a nonzero element $\lambda \in R_v^n$ such that $\|\exp(\mathbf{k})\, \mathfrak{u}_{\mathbf{t}}\, \lambda\| \leqslant \epsilon$, or equivalently by an easy computation if and only if the following joint system of inequalities with unknown $(\lambda_1, \lambda_2, \dots, \lambda_n)$ in R_v^n has a nonzero solution

$$|\lambda_1| \;\leqslant\; \epsilon\, q_v^{-k_1} \tag{80}$$

$$\forall\, i \in [\![2, n]\!], \quad \Big|\lambda_1 \frac{r_i}{s_i} + \lambda_i\Big| \;\leqslant\; \epsilon\, q_v^{-k_i}\,. \tag{81}$$

Note that if $(\lambda_1, \dots, \lambda_n) \in R_v^n$ is a nonzero solution to the joint system (80) and (81), then $\lambda_1 \neq 0$. Indeed, by Equation (3), the only element of R_v contained in the closed ball $B(0, q_v^{-1})$ of center 0 and radius q_v^{-1} is 0. Hence if $\lambda_1 = 0$, then for every $i \in [\![2, n]\!]$, since $\epsilon < 1$ and $k_i \geqslant 0$ as $\mathbf{k} \in \Delta_{\mathbf{s}}$, we have $\lambda_i \in R_v \cap B(0, q_v^{-k_i}\epsilon) = \{0\}$, which contradicts the fact that $(\lambda_1, \dots, \lambda_n) \neq 0$.

By the ultrametric triangle inequality, if λ, λ' are distinct elements of R_v, then the closed balls $B(\lambda, q_v^{-1})$ and $B(\lambda', q_v^{-1})$ are disjoint. Again by the ultrametric triangle inequality, for every $\rho \geqslant q_v^{-1}$, the closed ball $B(0, \rho)$ contains $\bigcup_{\lambda\in R_v\cap B(0,\rho)} B(\lambda, q_v^{-1})$, and this union is a disjoint union. Recall that $\mathrm{vol}_v(B(0, q_v^{-1})) = q_v^{-1}$ by the normalisation of the Haar measure vol_v of K_v. Separating the cases when $\epsilon\, q_v^{-k_1} < q_v^{-1}$ or the contrary, and since $\epsilon \in q_v^{\mathbb{Z}}$, the number of nonzero solutions λ_1 of Equation (80) hence satisfies

$$\operatorname{Card}((R_v \smallsetminus \{0\}) \cap B(0, \epsilon\, q_v^{-k_1})) \leqslant \frac{\mathrm{vol}_v(B(0, \epsilon\, q_v^{-k_1}))}{\mathrm{vol}_v(B(0, q_v^{-1}))} = \epsilon\, q_v^{-k_1+1}\,. \tag{82}$$

Let us now fix $\lambda_1 \in (R_v \smallsetminus \{0\}) \cap B(0, q_v^{-k_1}\epsilon)$ and $i \in [\![2, n]\!]$. In this proof, the O( ) functions do not depend on λ_1, $\mathbf{s}$, $\mathbf{k}$, ϵ. Let

$$\mathscr{N}_i(\lambda_1) = \operatorname{Card}\Big\{\frac{r_i}{s_i} \mod R_v : (r_i, s_i) = 1,\ \exists\ \lambda_i \in R_v,\ \big|\lambda_1 \frac{r_i}{s_i} + \lambda_i\big| \leqslant \epsilon\, q_v^{-k_i}\Big\}\,.$$

Claim 1 : We have $\mathscr{N}_i(\lambda_1) = \mathrm{O}\big(\epsilon\, q_v^{-k_i}\, \varphi_v(s_i)\big)$.

By the discussion above Equations (80), (81), by Equation (82) and this claim, since $k_1 = -\sum_{i=2}^n k_i$ as $\mathbf{k} \in \Delta_{\mathbf{s}} \subset \mathbb{Z}_0^n$, and by Equation (46) on the left, this will imply the inequality

$$\begin{aligned}\operatorname{Card}\big\{\mathbf{t} \in \Lambda_{\mathbf{s}} : \exp(\mathbf{k})x_{\mathbf{t}} \in \mathscr{X}^{\leqslant \epsilon}\big\} &\leqslant \sum_{\lambda_1 \in (R_v \smallsetminus \{0\}) \cap B(0,\, q_v^{-k_1}\epsilon)} \ \prod_{i=2}^n \mathscr{N}_i(\lambda_1)\\ &= \mathrm{O}\Big(\epsilon\, q_v^{-k_1+1} \prod_{i=2}^n \epsilon\, q_v^{-k_i}\, \varphi_v(s_i)\Big)\\ &= \mathrm{O}\big(\epsilon^n \operatorname{Card}\Lambda_{\mathbf{s}}\big)\,.\end{aligned}$$

This estimate will prove Lemma 7.2.

Proof of Claim 1. Let $J_i = s_i R_v \in \mathscr{I}_v^+$. Since R_v is assumed to be principal, let $d_i \in R_v$ be such that $\lambda_1 R_v + J_i = d_i R_v$. Let $\widetilde{\lambda}_1 = \frac{\lambda_1}{d_i}$, $\widetilde{s}_i = \frac{s_i}{d_i}$ and $\widetilde{J}_i = \widetilde{s}_i R_v \in \mathscr{I}_v^+$. By dividing by d_i and since the fibers of the canonical morphism $(R_v/J_i)^\times \to (R_v/\widetilde{J}_i)^\times$ have order $\frac{\varphi_v(J_i)}{\varphi_v(\widetilde{J}_i)}$, we have

$$\begin{aligned}\mathscr{N}_i(\lambda_1) &= \operatorname{Card}\big\{r_i + J_i \in (R_v/J_i)^\times : \exists\ \lambda_i \in R_v,\ \big|\lambda_1\, r_i + \lambda_i\ s_i\big| \leqslant \epsilon\, q_v^{-k_i}\,|s_i|\big\}\\ &= \frac{\varphi_v(J_i)}{\varphi_v(\widetilde{J}_i)} \operatorname{Card}\big\{r_i + \widetilde{J}_i \in (R_v/\widetilde{J}_i)^\times : \exists\ \lambda_i \in R_v,\ \big|\widetilde{\lambda}_1\, r_i + \lambda_i\ \widetilde{s}_i\big| \leqslant \epsilon\, q_v^{-k_i}\,|\widetilde{s}_i|\big\}\,.\end{aligned}$$

Since $\widetilde{\lambda}_1$ and $\widetilde{s}_i$ are coprime, let $\widetilde{\lambda}_1^- \in R_v$ be such that $\lambda_1\,\widetilde{\lambda}_1^- - 1 \in \widetilde{J}_i$. The multiplication by $\widetilde{\lambda}_1^-$ is a bijective map from $(R_v/\widetilde{J}_i)^\times$ to itself. Hence by Lemma 7.1, we have

$$\begin{aligned}\mathscr{N}_i(\lambda_1) &= \frac{\varphi_v(J_i)}{\varphi_v(\widetilde{J}_i)} \operatorname{Card}\big\{r_i + \widetilde{J}_i \in (R_v/\widetilde{J}_i)^\times : \exists\ \lambda_i \in R_v,\ \big|r_i + \lambda_i\ \widetilde{s}_i\big| \leqslant \epsilon\, q_v^{-k_i}\,|\widetilde{s}_i|\big\}\,.\\ &\leqslant \frac{\varphi_v(J_i)}{\varphi_v(\widetilde{J}_i)} \operatorname{Card}\big\{I \in \mathscr{I}_v^+ : (I, \widetilde{J}_i) = 1,\ \mathtt{N}(I) \leqslant \epsilon\, q_v^{-k_i}\ \mathtt{N}(\widetilde{J}_i)\big\}\\ &= \frac{\varphi_v(J_i)}{\varphi_v(\widetilde{J}_i)}\ \mathrm{O}\big(\epsilon\, q_v^{-k_i}\, \varphi_v(\widetilde{J}_i) + 2^{\,\varpi_v(\widetilde{J}_i)}\big) = \mathrm{O}\Big(\epsilon\, q_v^{-k_i}\, \varphi_v(J_i) + \frac{2^{\,\varpi_v(\widetilde{J}_i)}}{\varphi_v(\widetilde{J}_i)}\, \varphi_v(J_i)\Big)\,.\end{aligned}$$

Claim 2 : We have $\frac{2^{\,\varpi_v(\widetilde{J}_i)}}{\varphi_v(\widetilde{J}_i)} = \mathrm{O}(\epsilon\, q_v^{-k_i})$.

With the previous formula, this implies Claim 1, hence concludes Lemma 7.2

Proof of Claim 2. By Lemma 2.2, since $\varpi_v(\widetilde{J}_i) \leqslant \varpi_v(J_i)$ as $\widetilde{J}_i$ divides J_i, since $\mathtt{N}(\widetilde{J}_i) \leqslant \mathtt{N}(J_i) = \mathtt{N}(\widetilde{J}_i)\ \mathtt{N}(\lambda_1 R_v + \widetilde{J}_i)$, and since $\mathtt{N}(\lambda_1 R_v + \widetilde{J}_i) \leqslant \mathtt{N}(\lambda_1 R_v) = |\lambda_1| \leqslant \epsilon\, q_v^{-k_1}$ by Equation (80), we have

$$\begin{aligned}\frac{2^{\,\varpi_v(\widetilde{J}_i)}}{\varphi_v(\widetilde{J}_i)} &= \mathrm{O}\Big(2^{\,\varpi_v(\widetilde{J}_i)}\frac{\ln\ln \mathtt{N}(\widetilde{J}_i)}{\mathtt{N}(\widetilde{J}_i)}\Big) = \mathrm{O}\Big(2^{\,\varpi_v(J_i)}\frac{\ln\ln \mathtt{N}(J_i)}{\mathtt{N}(J_i)}\, \mathtt{N}(\lambda_1 R_v + \widetilde{J}_i)\Big)\\ &= \mathrm{O}\Big(\epsilon\, q_v^{-k_1}\, 2^{\,\varpi_v(J_i)}\frac{\ln\ln \mathtt{N}(J_i)}{\mathtt{N}(J_i)}\Big)\,.\end{aligned}$$

Claim 2 hence follows by the technical Assumption (79) of Lemma 7.2. □□□

Let $\mathbf{s} = (s_2, \ldots, s_n) \in (R_v \smallsetminus \{0\})^{n-1}$ satisfying Equation (45), so that there exists a permutation σ of $[\![2, n]\!]$ with $s_{\sigma^{-1}(2)} \mid s_{\sigma^{-1}(3)} \ldots \mid \mathbf{s}_* = s_{\sigma^{-1}(n)}$ and $v(\mathbf{s}_*) \in n\,\mathbb{Z}$. Let

$$\mathbf{w} = (1-n, 1, \ldots, 1) \in \mathbb{Z}_0^n, \quad \text{so that} \quad \mathbf{k_s} = -\frac{v(\mathbf{s}_*)}{n}\,\mathbf{w} \quad \text{and} \quad \mathfrak{a} = \exp(\mathbf{w})\,.$$

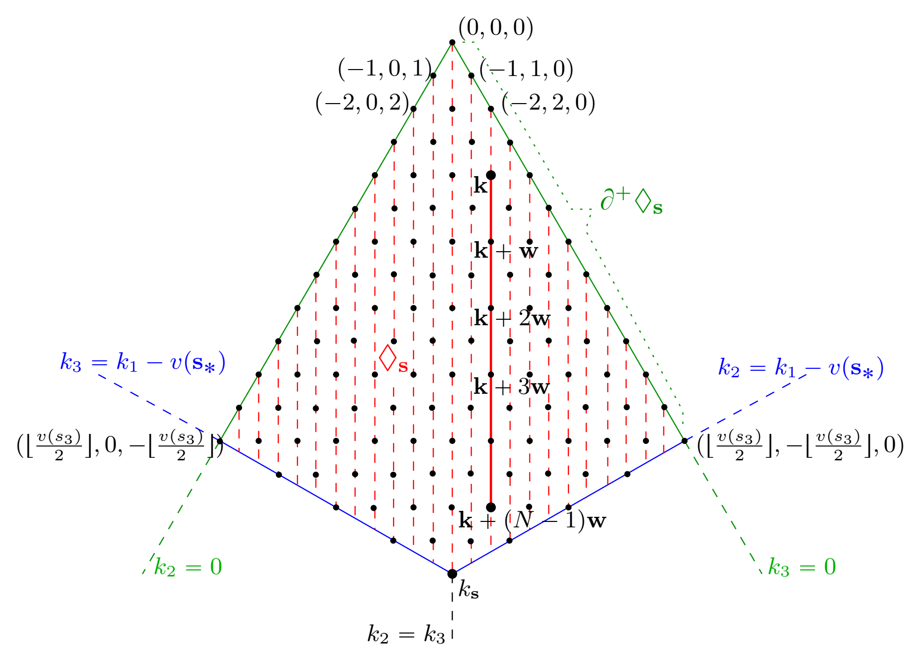
For all $\mathbf{k} \in \lozenge_{\mathbf{s}}$ and $N \in \mathbb{N}\smallsetminus\{0\}$, we denote by $[\mathbf{k}, N]$ (see picture below) the discrete interval in $\mathbb{Z}_0^n$ defined by

$$[\mathbf{k}, N] = \{\mathbf{k} + \ell'\mathbf{w} : \ell' \in [\![0, N-1]\!]\}\,.$$

By the convexity of $\lozenge_{\mathbf{s}}$, we have $[\mathbf{k}, N] \subset \lozenge_{\mathbf{s}}$ if and only if $\mathbf{k} + (N-1)\mathbf{w} \in \lozenge_{\mathbf{s}}$. Let

$$\nu_{\mathbf{s},[\mathbf{k},N]} = \frac{1}{N\ \mathrm{Card}\ \Lambda_{\mathbf{s}}} \sum_{\mathbf{t}\in\Lambda_{\mathbf{s}},\ \ell'\in[\![0,\,N-1]\!]} \delta_{\exp(\mathbf{k}+\ell'\mathbf{w})x_{\mathbf{t}}}\,, \tag{83}$$

which is a probability measure on $\mathscr{X}$.



The next corollary proves the non-escape of mass at infinity property for averages of measures parametrized by the discrete interval $[\mathbf{k}, N]$. Let us recall the positive constant c_{ϖ_v} introduced in Equation (7). For every $\mathbf{s} = (s_2, \ldots, s_n) \in (R_v \smallsetminus \{0\})^{n-1}$ satisfying Equation (45), let

$$\kappa'(\mathbf{s}) = \frac{1}{n}\Big(-v(\mathbf{s}_*) + \max_{i\in[\![2,n]\!]} \log_{q_v} \frac{2^{\varpi_v(s_i)}\ \max\{1, \ln\ln|s_i|\}}{|s_i|} \Big)\,. \tag{84}$$

Remark 7.3 *If there exists* $c_0 \geqslant 0$ *such that* $\max_{i\in[\![2,n]\!]} v(s_i) - v(\mathbf{s}_*) \leqslant c_0 \frac{-v(\mathbf{s}_*)}{\max\{1, \ln(-v(\mathbf{s}_*))\}}$, *then* $\kappa'(\mathbf{s}) \leqslant \frac{1}{n}(c_{\varpi_v} + c_0 + 1)\frac{-v(\mathbf{s}_*)}{\max\{1, \ln(-v(\mathbf{s}_*))\}}$. *In particular,* $\kappa'(\mathbf{s})$ *is negligible with respect to* $-v(\mathbf{s}_*)$ *as* $-v(\mathbf{s}_*) \to +\infty$.

Proof. Since $|s_i| = q_v^{-v(s_i)}$, since $\frac{1}{2} \leqslant \ln 2 \leqslant \ln q_v$ and by Equation (7), since the maps $f_1 : t \mapsto \frac{t}{\max\{1, \ln t\}}$ and $f_2 : t \mapsto 2\ln(\max\{1, \ln t\})$ on $[0, +\infty[$ are nondecreasing with $f_2 \leqslant f_1$, and by the assumption of the remark, we have

$$\begin{aligned} n\,\kappa'(\mathbf{s}) &= \max_{i\in[\![2,n]\!]} \Big(\log_{q_v} 2^{\varpi_v(s_i)} + \log_{q_v}(\max\{1, \ln(-v(s_i))\}) + v(s_i) - v(\mathbf{s}_*)\Big) \\ &\leqslant \max_{i\in[\![2,n]\!]} \Big(c_{\varpi_v} \frac{-v(s_i)}{\max\{1, \ln(-v(s_i))\}} + 2\ln(\max\{1, \ln(-v(s_i))\}) + v(s_i) - v(\mathbf{s}_*)\Big) \\ &\leqslant (c_{\varpi_v} + 1) \frac{-v(\mathbf{s}_*)}{\max\{1, \ln(-v(\mathbf{s}_*))\}} + c_0 \frac{-v(\mathbf{s}_*)}{\max\{1,\, \ln(-v(\mathbf{s}_*))\}}\,, \end{aligned}$$

which proves the result. □

Corollary 7.4 *Assume that R_v is principal. For all $\mathbf{s} \in (R_v \smallsetminus \{0\})^{n-1}$ satisfying Equation (45), $\mathbf{k} \in \Diamond_{\mathbf{s}}$, $N \in \mathbb{N} \smallsetminus \{0\}$ and $\epsilon \in q_v^{\mathbb{Z}} \cap \,]0, 1[$ such that*

$$\mathbf{k} + (N-1)\mathbf{w} \in \Diamond_{\mathbf{s}} \quad \textit{and} \quad N \geqslant \frac{\kappa'(\mathbf{s})}{c_2\, \epsilon^n}\,, \tag{85}$$

we have

$$\nu_{\mathbf{s},[\mathbf{k},N]}(\mathscr{X}^{\leqslant\epsilon}) \leqslant 2\, c_2\, \epsilon^n\,.$$

Proof. Let $\mathbf{s} = (s_2, \dots, s_n)$, $\mathbf{k} = (k_1, \dots, k_n)$, N, ϵ be fixed as in the statement. Let ℓ' be an integer that will vary in $[\![0, N-1]\!]$. First assume that $\ell' \leqslant N - 1 - \kappa'(\mathbf{s})$. Note that $\kappa'(\mathbf{s}) \geqslant \frac{1}{n} \log_{q_v}(2^{\varpi_v(\mathbf{s}_*)} \max\{1, \ln\ln|\mathbf{s}_*|\}) \geqslant 0$.

By the definition of $\Diamond_{\mathbf{s}}$, since $\mathbf{k} + (N-1)\mathbf{w} \in \Diamond_{\mathbf{s}}$ by the left hand side of Assumption (85), we have $\max_{2\leqslant i\leqslant n}(\mathbf{k} + (N-1)\mathbf{w})_i \leqslant (\mathbf{k} + (N-1)\mathbf{w})_1 - v(\mathbf{s}_*)$. Hence for every $i \in [\![2, n]\!]$, since $\ell' \leqslant N - 1 - \kappa'(\mathbf{s})$ and by the definition of $\kappa'(\mathbf{s})$, we have

$$\begin{aligned} (\mathbf{k} + \ell'\mathbf{w})_i - (\mathbf{k} + \ell'\mathbf{w})_1 &= (k_i + \ell') - (k_1 + (1-n)\ell') \\ &= (\mathbf{k} + (N-1)\mathbf{w})_i - (\mathbf{k} + (N-1)\mathbf{w})_1 + n(\ell' - N + 1) \\ &\leqslant -\,v(\mathbf{s}_*) - n\,\kappa'(\mathbf{s}) \leqslant \log_{q_v} \frac{|s_i|}{2^{\varpi_v(s_i)}\, \max\{1, \ln\ln|s_i|\}}\,. \end{aligned}$$

Therefore the element $\mathbf{k} + \ell'\mathbf{w}$ of $\Diamond_{\mathbf{s}}$ satisfies the technical Assumption (79) of Lemma 7.2, and we have

$$\Big(\frac{1}{\operatorname{Card} \Lambda_{\mathbf{s}}} \sum_{\mathbf{t}\in\Lambda_{\mathbf{s}}} \delta_{\exp(\mathbf{k}+\ell'\mathbf{w})x_{\mathbf{t}}}\Big)(\mathscr{X}^{\leqslant\epsilon}) \leqslant c_2\, \epsilon^n\,.$$

There are at most N (resp. $\kappa'(\mathbf{s})$) integral elements in the real interval $[0, N-1-\kappa'(\mathbf{s})]$ (resp. $]N-1-\kappa'(\mathbf{s}), N-1]$). Therefore separating, in Equation (83) that defines the measure $\nu_{\mathbf{s},[\mathbf{k},N]}$, the summation over $\ell' \in [\![0, N-1]\!]$ in firstly $\ell' \in [0, N-1-\kappa'(\mathbf{s})]$ and secondly $\ell' \in \,]N-1-\kappa'(\mathbf{s}), N-1]$, we have

$$\nu_{\mathbf{s},[\mathbf{k},N]}(\mathscr{X}^{\leqslant\epsilon}) \leqslant \frac{1}{N}(N\, c_2\, \epsilon^n) + \frac{\kappa'(\mathbf{s})}{N}\,.$$

By the right hand side of Assumption (85), this proves Corollary 7.4. □

8 Optimal entropy lower bound

In this final section, we prove the main equidistribution result of this paper, in the space $\mathscr{X} = \operatorname{SL}_n(K_v)/\operatorname{SL}_n(R_v)$ of special unimodular R_v-lattices in $K_v^{\,n}$ towards its homogeneous measure $\mathtt{m}_{\mathscr{X}}$, of the measures supported on large subsets of divergent orbits of type $(1, s_2, \ldots, s_n)$ (up to permutation) as $\mathbf{s} = (s_2, \ldots, s_n) \in (R_v \smallsetminus \{0\})^{n-1}$ tends to infinity (for the Fréchet filter or equivalently when $\min_{i \in [\![2,n]\!]} v(s_i)$ tends to $-\infty$). We will actually require some uniform convergence to $-\infty$ of the valuations of the components $s_2, \ldots, s_n$ of $\mathbf{s}$, and precisely

$$\exists\, c_0 \geqslant 0, \qquad \max_{i,j \in [\![2,n]\!]} |\, v(s_i) - v(s_j)\,| = \max_{i \in [\![2,n]\!]} v(s_i) - v(\mathbf{s}_*) \leqslant c_0 \frac{-v(\mathbf{s}_*)}{\max\{1,\, \ln(-v(\mathbf{s}_*))\}}\,. \quad (86)$$

Note that this is for instance satisfied if $s_2 = \ldots = s_n$ as in Theorem 1.2 in the Introduction, and that this assumption is optimal by Remark 7.3.

Theorem 8.1 *Assume that R_v is principal. As $\mathbf{s} \in (R_v \smallsetminus \{0\})^{n-1}$ satisfying Equations* (45) *and* (86) *tends to infinity, the measures $\nu_{\mathbf{s}}^{\Diamond}$ weak-star converge to $\frac{\mathtt{m}_{\mathscr{X}}}{\|\mathtt{m}_{\mathscr{X}}\|}$ on $\mathscr{X}$.*

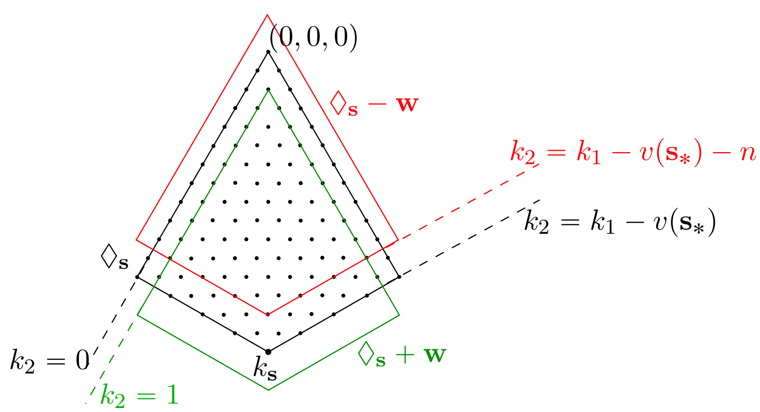
Proof. Let us fix a weak-star accumulation point ν of the measures $\nu_{\mathbf{s}}^{\Diamond}$ as $\mathbf{s} \in (R_v \smallsetminus \{0\})^{n-1}$ satisfying Equations (45) and (86) tends to infinity. We will prove that ν is a probability measure using the work of Section 7 and that $\nu = \frac{\mathtt{m}_{\mathscr{X}}}{\|\mathtt{m}_{\mathscr{X}}\|}$ using the entropy method described in Section 5, which will conclude using the Banach-Alaoglu theorem.

Lemma 8.2 *The measure ν is $\mathfrak{a}$-invariant.*

Proof. Recall that $\mathbf{w} = (1-n, 1, \ldots, 1) \in \mathbb{Z}_0^n$. Using the definitions (48) and (2), since $\mathfrak{a} = \exp(\mathbf{w})$ commutes with $A(\mathscr{O}_v)$, we have

$$\mathfrak{a}_* \nu_{\mathbf{s}}^{\Diamond} = \frac{1}{\operatorname{Card} \Lambda_{\mathbf{s}} \operatorname{Card} \Diamond_{\mathbf{s}}} \sum_{\mathbf{t} \in \Lambda_{\mathbf{s}},\, \mathbf{k} \in \Diamond_{\mathbf{s}}} \int_{a \in A(\mathscr{O}_v)} \delta_{a \exp(\mathbf{k}+\mathbf{w})\, x_{\mathbf{t}}}\, da\,.$$

In order to compare $\nu_{\mathbf{s}}^{\Diamond}$ and $\mathfrak{a}_* \nu_{\mathbf{s}}^{\Diamond}$, let us give an upper estimate on the cardinality of the symmetric difference between $\Diamond_{\mathbf{s}}$ and $\Diamond_{\mathbf{s}} + \mathbf{w}$.



By construction, the boundary of $\Diamond_{\mathbf{s}}$ is contained in the hyperplanes with equations $k_i = 0$ and $k_j = k_1 - v(\mathbf{s}_*)$ for $i, j \in [\![2,n]\!]$. Hence $\Diamond_{\mathbf{s}} \smallsetminus (\Diamond_{\mathbf{s}} + \mathbf{w}) = \bigcup_{i \in [\![2,n]\!]} \{\mathbf{k} \in \Diamond_{\mathbf{s}} : 0 \leqslant k_i < 1\}$

(see the above picture). By Proposition 4.6 (2) in dimension $n-1$, for every $i \in [\![2, n]\!]$, we have $\operatorname{Card}\{\mathbf{k} \in \Diamond_{\mathbf{s}} : k_i = 0\} = \operatorname{O}((-v(\mathbf{s}_*))^{n-2})$. Therefore

$$\operatorname{Card}\big(\Diamond_{\mathbf{s}} \smallsetminus (\Diamond_{\mathbf{s}} + \mathbf{w})\big) = \operatorname{O}((-v(\mathbf{s}_*))^{n-2})\,.$$

Similarly, we have $\Diamond_{\mathbf{s}} \smallsetminus (\Diamond_{\mathbf{s}} - \mathbf{w}) = \bigcup_{i \in [\![2,n]\!]} \{\mathbf{k} \in \Diamond_{\mathbf{s}} : k_1 - v(\mathbf{s}_*) - n < k_i \leqslant k_1 - v(\mathbf{s}_*)\}$ (see the above picture) and $\operatorname{Card}\big((\Diamond_{\mathbf{s}} + \mathbf{w}) \smallsetminus \Diamond_{\mathbf{s}}\big) = \operatorname{Card}(\Diamond_{\mathbf{s}} \smallsetminus (\Diamond_{\mathbf{s}} - \mathbf{w})) = \operatorname{O}((-v(\mathbf{s}_*))^{n-2})$. Therefore by Proposition 4.6 (2), the cardinality of the symmetric difference between $\Diamond_{\mathbf{s}}$ and $\Diamond_{\mathbf{s}} + \mathbf{w}$ is negligible with respect to the cardinality of $\Diamond_{\mathbf{s}}$.

This implies the weak-star convergence $\nu_{\mathbf{s}} - \mathfrak{a}_* \nu_{\mathbf{s}} \overset{*}{\rightharpoonup} 0$ as $\mathbf{s} \to +\infty$. Finally, since the transformation $\mathfrak{a} : \mathscr{X} \to \mathscr{X}$ is a homeomorphism (in particular, it is continuous and proper), we have $\mathfrak{a}_* \nu = \nu$. □

Let us recall the notation $\ell_m \in \mathbb{N} \smallsetminus \{0\}$ introduced in Lemma 6.6 for every $m \in \mathbb{N}$ and $\kappa'(\mathbf{s})$ introduced in Equation (84) for every $\mathbf{s} \in (R_v \smallsetminus \{0\})^{n-1}$ satisfying Equation (45). Let us recall the notation $\mathscr{P}^N = \bigvee_{i=0}^{N-1} \mathfrak{a}^{-i} \mathscr{P}$ introduced in Equation (75) for every $N \in \mathbb{N} \smallsetminus \{0\}$ and every finite measurable partition $\mathscr{P}$ of $\mathscr{X}$.

Lemma 8.3 *Assume that R_v is principal. For every $\eta \in \,]0,1[\,$, there exists $m = m(\eta) \in \mathbb{N}$ such that with $\ell = \max\{\ell_m, m+1\}$, for every (m,ℓ)-partition $\mathscr{P}$ of $\mathscr{X}$ and for every $M \in \mathbb{N} \smallsetminus \{0\}$, there exists $N_0 = N_0(\eta, \mathscr{P}, M) \in \mathbb{N} \smallsetminus \{0\}$ such that for every $\mathbf{s} \in (R_v \smallsetminus \{0\})^{n-1}$ satisfying Equations (45) and (86), for every $\mathbf{k} = (k_1, \dots, k_n) \in \Diamond_{\mathbf{s}}$ and for every $N \in \mathbb{N}$ satisfying the three assumptions*

$$\mathbf{k} + (N-1)\mathbf{w} \in \Diamond_{\mathbf{s}}\,,$$

$$\max\Big\{N_0,\ \frac{4(1-\eta)}{\eta}(n+1)(c_0+1)\frac{-v(\mathbf{s}_*)}{\max\{1, \ln(-v(\mathbf{s}_*))\}},\ \frac{\kappa'(\mathbf{s})}{c_2\, q_v^{-mn}}\Big\} \leqslant N$$

$$\text{and} \quad N \leqslant \frac{-v(\mathbf{s}_*) - \ell + k_1 - \max_{2 \leqslant i \leqslant n} k_i}{n}\,,$$

we have

$$\frac{1}{M}\, H_{\nu_{\mathbf{s},[\mathbf{k},N]}}(\mathscr{P}^M) \geqslant (1-\eta)^2\, n(n-1) - \eta\,.$$

Proof. For every $\eta \in \,]0,1[$, let $\kappa = \frac{n(n-1)\,\eta}{n^3} \in \,]0,1[$, let $m = \big\lceil -\frac{1}{n} \log_{q_v} \frac{\eta^2 n(n-1)}{2c_2 n^3} \big\rceil$ which belongs to $\mathbb{N}$ since $\eta^2 < 1 \leqslant 2c_2$, and let $\epsilon = q_v^{-m} \in q_v^{\mathbb{Z}} \cap \,]0,1[\,$. Let $\ell, \mathscr{P}, M$ be as in the statement. With c_{φ_v} the constant introduced in Equation (6), let

$$N_0 = \max\Big\{M, \frac{2M}{\eta} \log_{q_v} \operatorname{Card} \mathscr{P}, \frac{4(1-\eta)}{\eta}(n-1)(n+1-\log_{q_v} c_{\varphi_v})\Big\} \in \mathbb{N} \smallsetminus \{0\}$$

and let $N \in \mathbb{N}$ with $N \geqslant N_0$. Let $\mathbf{s}, \mathbf{k}$ be as in the statement. Let

$$\nu_{\mathbf{s},\mathbf{k}} = \frac{1}{\operatorname{Card} \Lambda_{\mathbf{s}}} \sum_{\mathbf{t} \in \Lambda_{\mathbf{s}}} \delta_{\exp(\mathbf{k}) x_{\mathbf{t}}}\,.$$

Note that by the definition of $\nu_{\mathbf{s},[\mathbf{k},N]}$ in Equation (83), since $\mathfrak{a} = \exp(\mathbf{w})$ and by the definition in Equation (76) of the N-th Birkhoff average of measures for $\mathfrak{a}$, we have

$$\nu_{\mathbf{s},[\mathbf{k},N]} = \frac{1}{N} \sum_{i=0}^{N-1} (\mathfrak{a}^i)_* \nu_{\mathbf{s},\mathbf{k}} = S_N \nu_{\mathbf{s},\mathbf{k}}\,.$$

By Lemma 5.1 (1) applied with $\mu' = \nu_{\mathbf{s},\mathbf{k}}$ and $\phi = \mathfrak{a}$ since $N \geqslant N_0 \geqslant M \geqslant 1$, we have

$$\frac{1}{M} H_{\nu_{\mathbf{s},[\mathbf{k},N]}}(\mathscr{P}^M) \geqslant \frac{1}{N} H_{\nu_{\mathbf{s},\mathbf{k}}}(\mathscr{P}^N) - \frac{M}{N} \log_{q_v} \operatorname{Card} \mathscr{P} \,. \tag{87}$$

Since $N \geqslant N_0$, we have $\frac{M}{N} \log_{q_v} \operatorname{Card} \mathscr{P} \leqslant \frac{\eta}{2}$.

As in Lemma 6.6 (1), let $\mathscr{X}' = \mathscr{X}'_{\mathscr{P},\,\kappa,\,N}$ be a measurable subset of $\mathscr{X}^{\geqslant q_v^{-m}}$, let $\mathscr{P}'$ be a subset of the partition $\mathscr{P}^N$ and for every $P \in \mathscr{P}'$, let F_P be a finite subset of P with cardinality at most $q_v^{n^3\kappa N} = q_v^{n(n-1)\eta N}$ such that $\mathscr{X}' = \bigcup \mathscr{P}'$ and $P \subset \bigcup_{x \in F_P} B_{\ell,N-1}\, x$. Since $F_P \subset P \subset \mathscr{X}' \subset \mathscr{X}^{\geqslant q_v^{-m}}$, for every $x \in F_P$, we have $\operatorname{sys}(x) \geqslant q_v^{-m}$. Hence $\max\{0, -\log_{q_v}(\operatorname{sys}(x))\} \leqslant m < \ell$ by the definition of ℓ. Therefore by the assumptions on $\mathbf{s}$ and $\mathbf{k}$, by Lemma 6.5 and by Equation (46) on the left, for every $x \in F_P$, we have

$$\begin{aligned}\nu_{\mathbf{s},\mathbf{k}}(B_{\ell,N-1}\, x) &= \frac{1}{\operatorname{Card}(\Lambda_{\mathbf{s}})} \operatorname{Card}\big(\{\mathbf{t} + R_v^{\,n-1} \in \Lambda_{\mathbf{s}} : \exp(\mathbf{k})\, x_{\mathbf{t}} \in B_{\ell,N-1}\, x\}\big)\\ &\leqslant \frac{2^{n-1}}{\prod_{i=2}^n \varphi_v(s_i)}\; q_v^{-\ell(n-1)-v(\mathbf{s}_*)(n-1)-n(n-1)(N-1)}\,.\end{aligned}$$

Thus, since $\mathscr{P}' \subset \mathscr{P}^N$, since we have $P \subset \bigcup_{x \in F_P} B_{\ell,N-1}\, x$ and $\operatorname{Card} F_P \leqslant q_v^{\,n(n-1)\,\eta N}$ for every $P \in \mathscr{P}'$, and since $\mathscr{X}' = \bigsqcup_{P \in \mathscr{P}'} P$, we have

$$\begin{aligned} H_{\nu_{\mathbf{s},\mathbf{k}}}(\mathscr{P}^N) &= -\sum_{P \in \mathscr{P}^N} \nu_{\mathbf{s},\mathbf{k}}(P) \log_{q_v} \nu_{\mathbf{s},\mathbf{k}}(P) \geqslant -\sum_{P \in \mathscr{P}'} \nu_{\mathbf{s},\mathbf{k}}(P) \log_{q_v} \nu_{\mathbf{s},\mathbf{k}}(P)\\ &\geqslant -\sum_{P \in \mathscr{P}'} \nu_{\mathbf{s},\mathbf{k}}(P) \log_{q_v} \Big(q_v^{\,n(n-1)\,\eta N} \;\frac{2^{n-1}}{\prod_{i=2}^n \varphi_v(s_i)}\; q_v^{(n-\ell)\ell(n-1)-v(\mathbf{s}_*)(n-1)-n(n-1)N}\Big)\\ &= \nu_{\mathbf{s},\mathbf{k}}(\mathscr{X}')\Big((1-\eta)n(n-1)N - \frac{\ln 2}{\ln q_v}(n-1) - (n-\ell)(n-1) + \sum_{i=2}^n \log_{q_v} \frac{\varphi_v(s_i)}{|\mathbf{s}_*|}\Big)\,. \end{aligned} \tag{88}$$

By Equation (6), by computations similar to the ones done in the proof of Remark 7.3 and by the assumptions on $\mathbf{s}$ in the statement of Lemma 8.3, we have

$$\begin{aligned} &\sum_{i=2}^n \log_{q_v} \frac{\varphi_v(s_i)}{|\mathbf{s}_*|} \geqslant \sum_{i=2}^n \log_{q_v} \frac{c_{\varphi_v}\, |s_i|}{\max\{1, \ln(-v(s_i))\}\, |\mathbf{s}_*|}\\ &= \sum_{i=2}^n - \log_{q_v}\big(\max\{1, \ln(-v(s_i))\}\big) + \log_{q_v} c_{\varphi_v} - v(s_i) + v(\mathbf{s}_*)\\ &\geqslant -(n-1)\Big(\log_{q_v}(\max\{1, \ln(-v(\mathbf{s}_*))\}) - \log_{q_v} c_{\varphi_v} + c_0 \frac{-v(\mathbf{s}_*)}{\max\{1,\, \ln(-v(\mathbf{s}_*))\}}\Big)\\ &\geqslant -(n-1)\Big((c_0+1)\frac{-v(\mathbf{s}_*)}{\max\{1,\, \ln(-v(\mathbf{s}_*))\}} - \log_{q_v} c_{\varphi_v}\Big)\\ &\geqslant -\frac{\eta}{4(1-\eta)} N + (n-1)\log_{q_v} c_{\varphi_v}\,. \end{aligned}$$

Since $N \geqslant N_0$, we have

$$\begin{aligned} &\frac{\ln 2}{\ln q_v}(n-1) + (n-\ell)(n-1) - (n-1)\log_{q_v} c_{\varphi_v}\\ &\leqslant (n-1)(n+1-\log_{q_v} c_{\varphi_v}) \leqslant \frac{\eta}{4(1-\eta)}\, N\,. \end{aligned}$$

Therefore Equation (88) becomes

$$\frac{1}{N}\,H_{\nu_{\mathbf{s},\mathbf{k}}}(\mathscr{P}^N) \geqslant \nu_{\mathbf{s},\mathbf{k}}(\mathscr{X}')\Big((1-\eta)n(n-1) - \frac{\eta}{2(1-\eta)}\Big)\,.$$

We have $N \geqslant \frac{\kappa'(\mathbf{s})}{c_2\,q_v^{-mn}} = \frac{\kappa'(\mathbf{s})}{c_2\,\epsilon^n}$ by the assumptions on $\mathbf{s}$ and the definition of ϵ at the beginning of this proof. By Lemma 6.6 (2) applied with $\mu = \nu_{\mathbf{s},\mathbf{k}}$, by Corollary 7.4 proving that there is no escape of mass (the only place in this proof that requires the principal assumption on R_v) whose assumptions (85) are satisfied, and by the definitions of κ and $m = m(\eta)$ at the beginning of this proof, we have

$$\nu_{\mathbf{s},\mathbf{k}}(\mathscr{X}') \geqslant 1 - \frac{1}{\kappa}\nu_{\mathbf{s},[\mathbf{k},N]}(\mathscr{X}^{<q_v^{-m}}) \geqslant 1 - \frac{2\,c_2\,\epsilon^n}{\kappa} = 1 - \frac{2\,c_2\,q_v^{-mn}n^3}{n(n-1)\,\eta} \;\geqslant 1-\eta\,.$$

Hence Lemma 8.3 follows using Equation (87). □

End of the proof of Theorem 8.1. Let us fix a sequence $(\mathbf{s}^{(j)})_{j\in\mathbb{N}}$ of elements of $(R_v\smallsetminus\{0\})^{n-1}$ satisfying Equations (45) and (86) and tending to infinity such that we have $\nu = \lim\limits_{j\to+\infty} \nu_{\mathbf{s}^{(j)}}$.

Let us fix $\eta > 0$, that will tend to 0 at the very end of the proof. Let $m = m(\eta)$, $\ell = \max\{\ell_m, m+1\}$, $\mathscr{P}$ a (m,ℓ)-partition of $\mathscr{X}$, $M \in \mathbb{N}\smallsetminus\{0\}$ and $N_0 = N_0(\eta,\mathscr{P},M)$ be as in Lemma 8.3. Since $h_\nu(\mathfrak{a})$ is the upper bound of $h_\nu(\mathfrak{a},\mathscr{P}')$ where $\mathscr{P}'$ varies over all finite measurable partitions of $\mathscr{X}$, and by Equation (61) applied with $\mu' = \nu$ and $\phi = \mathfrak{a}$, if M is large enough, we have

$$h_\nu(\mathfrak{a}) \geqslant h_\nu(\mathfrak{a},\mathscr{P}) \geqslant \frac{1}{M}\,H_\nu(\mathscr{P}^M) - \eta\,.$$

For all $\mathbf{s} \in (R_v\smallsetminus\{0\})^{n-1}$ satisfying Equations (45) and (86), and $\mathbf{k} = (k_1,\dots,k_n) \in \Diamond_{\mathbf{s}}$, let $N_{\mathbf{k}} = \max\{\ell' \in \mathbb{N}\smallsetminus\{0\} : \mathbf{k} + (\ell'-1)\mathbf{w} \in \Diamond_{\mathbf{s}}\}$. For every $\ell'' \in \mathbb{N}$, the point $\mathbf{k}+\ell''\mathbf{w}$ belongs to $\Diamond_{\mathbf{s}}$ if and only if, for every $i \in [\![2,n]\!]$, we have $0 \leqslant k_i + \ell'' \leqslant k_1 + (1-n)\ell'' - v(\mathbf{s}_*)$. Hence

$$N_{\mathbf{k}} - 1 = \Big\lfloor \frac{1}{n}\big(-v(\mathbf{s}_*) + k_1 - \max_{i\in[\![2,n]\!]} k_i\big)\Big\rfloor\,.$$

Let

$$\partial^+\Diamond_{\mathbf{s}} = \{\mathbf{k} = (k_1,\dots,k_n) \in \Diamond_{\mathbf{s}} : \min_{i\in[\![2,n]\!]} k_i = 0\}$$

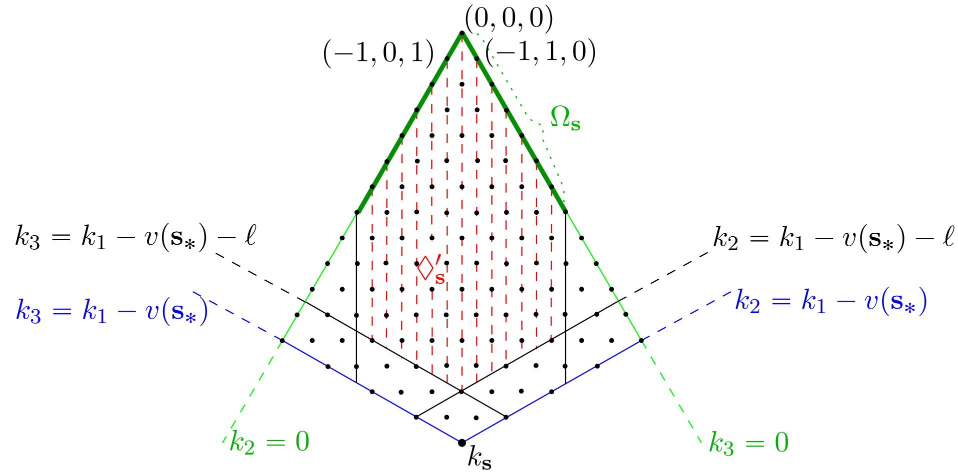
be the upper part of the boundary of $\Diamond_{\mathbf{s}}$ (see the picture above Remark 7.3). Since $\Diamond_{\mathbf{s}}$ is the disjoint union of the maximal vertical (directed by $\mathbf{w}$) segments contained in it, we have $\Diamond_{\mathbf{s}} = \bigsqcup_{\mathbf{k}\in\partial^+\Diamond_{\mathbf{s}}}\{\mathbf{k}+\ell'\mathbf{w} : \ell' \in [\![0,N_{\mathbf{k}}-1]\!]\}$. Let

$$N'_{\mathbf{k}} = \Big\lfloor \frac{1}{n}\big(-v(\mathbf{s}_*) - \ell + k_1 - \max_{i\in[\![2,n]\!]} k_i\big)\Big\rfloor + 1\,.$$

Note that $0 \leqslant N_{\mathbf{k}} - N'_{\mathbf{k}} \leqslant \lceil\frac{\ell}{n}\rceil$ is uniformly bounded for η fixed and that $N'_{\mathbf{k}}$ satisfies the upper bound assumption on N in Lemma 8.3. With c_{ϖ_v} the constant introduced in Equation (7), let $c_\eta = \max\big\{\frac{4(1-\eta)}{\eta}(n+1)(c_0+1),\ \frac{c_{\varpi_v}+c_0+1}{n\,c_2\,q_v^{-m(\eta)\,n}}\big\}$ and

$$\Omega_{\mathbf{s}} = \Big\{\mathbf{k} \in \partial^+\Diamond_{\mathbf{s}} : N'_{\mathbf{k}} \geqslant \max\big\{N_0, c_\eta\,\frac{-v(\mathbf{s}_*)}{\max\{1,\,\ln(-v(\mathbf{s}_*))\}}\big\}\Big\}\,.$$

Note that for every $\mathbf{k} \in \Omega_{\mathbf{s}}$, by Remark 7.3 whose assumption holds true since $\mathbf{s}$ verifies Equation (86), the number $N'_{\mathbf{k}}$ satisfies the lower bound assumption on N in Lemma 8.3.



Let $\lozenge'_{\mathbf{s}} = \bigsqcup_{\mathbf{k}\in\Omega_{\mathbf{s}}} \{\mathbf{k}+\ell'\mathbf{w} : \ell' \in [\![0, N'_{\mathbf{k}}-1]\!]$, which is obtained from $\lozenge_{\mathbf{s}}$ by removing a bounded size neighborhood of the lower part of the boundary of $\lozenge_{\mathbf{s}}$ and a comparatively small part of the vertical side of $\lozenge_{\mathbf{s}}$ (see the above picture). More precisely, $\lozenge'_{\mathbf{s}} = \lozenge_{\mathbf{s}} \smallsetminus (\lozenge''_{\mathbf{s}} \cup \lozenge'''_{\mathbf{s}})$ where $\lozenge''_{\mathbf{s}} = \bigsqcup_{\mathbf{k}\in\Omega_{\mathbf{s}}} \{\mathbf{k} + \ell'\mathbf{w} : \ell' \in [\![N'_{\mathbf{k}}, N_{\mathbf{k}} - 1]\!]$, whose cardinality is $\mathrm{O}((-v(\mathbf{s}_*))^{n-2})$ as seen in the proof of Lemma 8.2, and $\lozenge'''_{\mathbf{s}} = \bigsqcup_{\mathbf{k}\in\partial^+\lozenge_{\mathbf{s}}\smallsetminus\Omega_{\mathbf{s}}} \{\mathbf{k} + \ell'\mathbf{w} : \ell' \in [\![0, N_{\mathbf{k}} - 1]\!]$. Since we have $N_{\mathbf{k}} = \mathrm{O}\big(\frac{-v(\mathbf{s}_*)}{\max\{1, \ln(-v(\mathbf{s}_*))\}}\big)$ when $\mathbf{k} \in \partial^+\lozenge_{\mathbf{s}} \smallsetminus \Omega_{\mathbf{s}}$, and since $\mathrm{Card}\,(\partial^+\lozenge_{\mathbf{s}}) = \mathrm{O}((-v(\mathbf{s}_*))^{n-2})$ as seen in the proof of Lemma 8.2, the cardinality of $\lozenge'''_{\mathbf{s}}$ is $\mathrm{O}\big(\frac{(-v(\mathbf{s}_*))^{n-1}}{\max\{1, \ln(-v(\mathbf{s}_*))\}}\big)$. Modifying Equation (48), let

$$\nu_{\mathbf{s}}^{\lozenge'} = \frac{1}{\mathrm{Card}\,\Lambda_{\mathbf{s}}\ \mathrm{Card}\,\lozenge'_{\mathbf{s}}} \sum_{\mathbf{t}\in\Lambda_{\mathbf{s}},\ \mathbf{k}\in\lozenge'_{\mathbf{s}}} \int_{a\in A(\mathscr{O}_v)} \delta_{a\exp(\mathbf{k})\,x_{\mathbf{t}}}\, da\,.$$

Since the cardinalities of $\lozenge''_{\mathbf{s}}$ and $\lozenge'''_{\mathbf{s}}$ are negligible compared to the one of $\lozenge_{\mathbf{s}}$ (given by Proposition 4.6 (2)) we have $\lim_{j\to+\infty} \nu_{\mathbf{s}^{(j)}}^{\lozenge'} = \lim_{j\to+\infty} \nu_{\mathbf{s}^{(j)}}^{\lozenge} = \nu$. In particular, for every $j \in \mathbb{N}$ large enough, we have

$$h_\nu(\mathfrak{a}) \geqslant \frac{1}{M} H_{\nu_{\mathbf{s}^{(j)}}^{\lozenge'}}(\mathscr{P}^M) - 2\eta\,.$$

Let $\omega_{\mathbf{s}}$ be the probability measure on the finite (discrete) set $\Omega_{\mathbf{s}}$ defined by $\omega_{\mathbf{s}}(\mathbf{k}) = \frac{N'_{\mathbf{k}}}{\mathrm{Card}\,\lozenge'_{\mathbf{s}}}$ for every $\mathbf{k} \in \Omega_{\mathbf{s}}$. Then by Equation (83), we have

$$\nu_{\mathbf{s}}^{\lozenge'} = \int_{a\in A(\mathscr{O}_v)} \int_{\mathbf{k}\in\Omega_{\mathbf{s}}} a_*\nu_{\mathbf{s},[\mathbf{k},N'_{\mathbf{k}}]}\, d\omega_{\mathbf{s}}(\mathbf{k})\, da\,.$$

By Lemma 5.1 (2) applied with $(\Omega, \omega) = (A(\mathscr{O}_v) \times \Omega_{\mathbf{s}}, da \otimes \omega_{\mathbf{s}})$, since $a^{-1}\mathscr{P}$ is also an (m, ℓ)-partition for every $a \in A(\mathscr{O}_v)$, and by Lemma 8.3 applied with $N = N'_{\mathbf{k}}$ and integrated over $(a, \mathbf{k}) \in A(\mathscr{O}_v) \times \Omega_{\mathbf{s}}$ for the probability measure $da \otimes \omega_{\mathbf{s}}$, we have

$$\frac{1}{M} H_{\nu_{\mathbf{s}}^{\lozenge'}}(\mathscr{P}^M) \geqslant \int_{a\in A(\mathscr{O}_v)} \int_{\mathbf{k}\in\Omega_{\mathbf{s}}} \frac{1}{M} H_{a_*\nu_{\mathbf{s},[\mathbf{k},N'_{\mathbf{k}}]}}(\mathscr{P}^M)\, d\omega_{\mathbf{s}}(\mathbf{k})\, da \geqslant (1-\eta)^2\ n(n-1) - \eta\,.$$

Thus $h_\nu(\mathfrak{a}) \geqslant (1-\eta)^2\ n(n-1) - 3\,\eta$. By letting $\eta \to 0$, we have $h_\nu(\mathfrak{a}) \geqslant n(n-1)$. By the Einsiedler-Lindenstrauss Theorem 5.2, we hence have $h_\nu(\mathfrak{a}) = n(n-1)$ and then $\nu = \frac{m_{\mathscr{X}}}{\|m_{\mathscr{X}}\|}$, as wanted at the beginning of the proof of Theorem 8.1. □

The following result follows by averaging Theorem 8.1 over the permutations of $[\![2,n]\!]$ and over the compact probability space $(\mathscr{O}_v^{\times}/R_v^{\times}, \frac{q_v(q+1)}{q_v-1}\operatorname{vol}'_v)$ as in the proof of Lemma 4.8.

Corollary 8.4 *Assume that R_v is principal. For every $\mathbf{s}$ in the set S_n (endowed with the Fréchet filter) of elements $(s_2, \dots s_n) \in (R_v \smallsetminus \{0\})^{n-1}$ with $s_2 \mid s_3 \mid \dots \mid s_n$, $v(s_n) \in n\mathbb{Z}$ and $v(s_2) - v(s_n) \leqslant \frac{-v(s_n)}{\max\{1, \ln(-v(s_n))\}}$, let us define*

$$\Lambda'_{\mathbf{s}} = \left\{ \Big(\frac{r_2}{s'_2}, \dots, \frac{r_n}{s'_n}\Big) \mod R_v^{n-1} : \begin{array}{l} r_2, \dots, r_n, s'_2, \dots, s'_n \in R_v, \\ \forall\, j \in [\![2,n]\!],\ r_j R_v + s'_j R_v = R_v, \\ \{s'_2, \dots, s'_n\} = \{s_2, \dots, s_n\}\,. \end{array} \right\}$$

For the weak-star convergence of Radon measures on the locally compact space $\mathscr{X}_1$, we have

$$\lim_{\mathbf{s}\in S_n,\ |s_n|\to+\infty} \ \frac{1}{\operatorname{Card}\Lambda'_{\mathbf{s}}} \sum_{\mathbf{t}\,\in\,\Lambda'_{\mathbf{s}}} \overline{\mu}_{\mathtt{u}_{\mathbf{t}} R_v^n} \ = \ \frac{\mathtt{m}_{\mathscr{X}_1}}{\|\mathtt{m}_{\mathscr{X}_1}\|}\,. \quad \square$$

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N-T. D, F. P. & R. S. : Laboratoire de mathématique d'Orsay, UMR 8628 CNRS,
Université Paris-Saclay, 91405 ORSAY Cedex, FRANCE
nguyen-thi.dang@universite-paris-saclay.fr,
frederic.paulin@universite-paris-saclay.fr,
rafael.sayous@universite-paris-saclay.fr

R. S. : Department of Mathematics and Statistics, P.O. Box 35
40014 University of Jyväskylä, FINLAND.
sayousr@jyu.fi