From exponential counting to pair correlations

Jouni Parkkonen Frédéric Paulin

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Abstract

We prove an abstract result on the correlations of pairs of elements in an exponentially growing discrete subset $\mathscr{E}$ of $[0, +\infty[$ endowed with a weight function. Assume that there exist $\alpha \in \mathbb{R}$, $c, \delta > 0$ such that, as $t \to +\infty$, the weighted number $\widetilde{\omega}(t)$ of elements of $\mathscr{E}$ that are not greater than t is equivalent to $c\, t^{\alpha} e^{\delta t}$. We prove that the distribution function of the unscaled differences of elements of $\mathscr{E}$ is $t \mapsto \frac{\delta}{2}\, e^{-|t|}$, and that, under an error term assumption on $\widetilde{\omega}(t)$, the pair correlation with a scaling with polynomial growth exhibits a Poissonian behaviour. We apply this result to answer a question of Pollicott and Sharp on the pair correlations of closed geodesics and common perpendiculars in negatively curved manifolds and metric graphs. [1]

1 Introduction

When studying the asymptotic distribution of a sequence of finite subsets of $\mathbb{R}$, finer information is sometimes given by the statistics of the spacing between pairs or k-tuples of elements, seen at an appropriate scaling. This problematic is largely developped in quantum chaos, including energy level spacings or clusterings, and in statistical physics, including molecular repulsion or interstitial distribution. See for instance [Mon, Ber, RS, BZ, MaS, LS, HK]. In [PS1, PS2], Pollicott-Sharp study the pair correlations of lengths of closed geodesics in negatively curved manifolds as the word length of the elements of the fundamental group that represent them tends to $+\infty$. They mention that a result replacing the word length by the Riemannian length does not seem to be available. One aim of this note is to answer this problem, by a very general method.

For any set $\mathscr{E}$, a *weight function* (or multiplicity function when its values are positive integers) on $\mathscr{E}$ is simply a function $\omega : \mathscr{E} \to \,]\,0, +\infty\,[$. The *growth function* (or counting function when the weights are integers) of a locally finite subset $\mathscr{E}$ of $[\,0, +\infty\,[$ endowed with a weight function ω is the map $\mathscr{N}_{\mathscr{E},\omega} : [0, +\infty\,[\to\ [\,0, +\infty\,[$ defined by

$$\mathscr{N}_{\mathscr{E},\omega} : t \mapsto \sum_{x \in \mathscr{E} \cap [0,t]} \omega(x)\ .$$

Let $\mathscr{F} = (\,(F_N)_{N\in\mathbb{N}},\ \omega)$ be a nondecreasing sequence of finite subsets F_N of a finite dimensional Euclidean space E, endowed with a weight function $\omega : \bigcup_{N\in\mathbb{N}} F_N \to \,]\,0, +\infty\,[$. Let ψ be any function from $\mathbb{N}$ to $[1, +\infty\,[$, called a *scaling function*, and let $\psi' : \mathbb{N} \to]0, +\infty[$

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be an appropriately chosen function, called a *renormalising function.* The *pair correlation measure of $\mathscr{F}$ at time N with scaling $\psi(N)$* is the measure on E with finite support

$$\mathscr{R}_N^{\mathscr{F},\psi} = \sum_{x,y\in F_N} \omega(x)\,\omega(y)\,\Delta_{\psi(N)(y-x)}\ , \tag{1}$$

where Δ_z denotes the unit Dirac mass at z in any measurable space. When the sequence of measures $\mathscr{R}_N^{\mathscr{F},\psi}$, renormalised by $\psi'(N)$, weak-star converges to a measure $g_{\mathscr{F},\psi}\,\mathrm{Leb}_E$ absolutely continuous with respect to the Lebesgue measure Leb_E of E, the Radon-Nikodym derivative $g_{\mathscr{F},\psi}$ is called the asymptotic *pair correlation function* of $\mathscr{F}$ *for the scaling* ψ *and renormalisation* ψ'. When $g_{\mathscr{F},\psi}$ is a positive constant, we say that $\mathscr{F}$ has a *Poissonian behaviour for the scaling* ψ *and renormalisation* ψ'.

Theorem 1.1. *Let $\mathscr{E}$ be a locally finite subset of $[0,+\infty[$ endowed with a weight function ω. Assume that there exist $\alpha\in\mathbb{R}$, $c,\delta>0$ and $\kappa\geqslant 0$ such that, as $t\to+\infty$, we have*

$$\mathscr{N}_{\mathscr{E},\,\omega}(t)\sim c\,t^{\alpha}\,e^{\delta\,t}(1+o(e^{-\kappa\,t}))\ .$$

Let $\psi:\mathbb{N}\to[1+\infty[$ be an at most polynomially growing scaling function, with renormalising function $\psi':N\mapsto\frac{\mathscr{N}_{\mathscr{E},\,\omega}(N)^2}{\psi(N)}$. Then the family $\mathscr{F}=(\,(F_N=\{x\in\mathscr{E}:x\leqslant N\})_{N\in\mathbb{N}},\ \omega)$ has a pair correlation function $g_{\mathscr{F},1}:t\mapsto\frac{\delta}{2}\,e^{-\delta\,|t|}$ if $\psi=1$, and has Poissonian behaviour with $g_{\mathscr{F},\psi}=\frac{\delta}{2}$ if $\lim_{+\infty}\psi=\infty$ and $\kappa>0$.

We give some comments on the above statement at the beginning of Section 3. We refer to Theorem 3.1 for a more precise version, including error terms. The work on error terms constitutes the main technical parts of this paper.

Numerous settings in number theory, in geometry and in dynamical systems[2] give rise to counting functions that satisfy the assumption of Theorem 1.1. We will give some applications of the above result on geometry and dynamics in Section 4. Following the notation of [PS1], for all $a<b$ in $\mathbb{R}$ and $N\in\mathbb{N}$, let

$$\pi_{\mathscr{E}}(N,[a,b])=\mathscr{R}_N^{\mathscr{F},1}([a,b])=\sum_{x,y\in\mathscr{E}\,:\,x,y\leqslant N,\ a\leqslant y-x\leqslant b}\omega(x)\,\omega(y)$$

be the weighted number of differences of elements in $\mathscr{E}\cap[0,N]$ that lie in the interval $[a,b]$. Since the limit measure is atomless, under the assumptions of Theorem 3.1, we have the following corollary (see also Corollary 4.1).

Corollary 1.2. *For all $a<b$ in $\mathbb{R}$, as $N\to+\infty$, we have*

$$\pi_{\mathscr{E}}(N,[a,b])\sim\frac{\delta}{2}\,\mathscr{N}_{\mathscr{E},\,\omega}(N)^2\int_a^b e^{-\delta|t|}\,dt\ .\quad\Box$$

This answers the question of Pollicott-Sharp [PS1] when $\mathscr{E}$ is the set of lengths of closed geodesics in a closed negatively curved manifold, δ is the topological entropy of its geodesic flow and ω is the multiplicity function of these lengths (see Remark (3) in Section 3).

We suspect that when the scaling function has superexponential growth, the empirical measures $\frac{1}{\psi'(N)}\mathscr{R}_N^{\mathscr{F},\psi}$ have a total loss of mass as $N\to+\infty$ whatever the renormalising

[2]See, for instance [PPo], [EM].

function ψ' is, hence that the pair correlation function $g_{\mathscr{F},\psi}$ exists and is identically 0. The main open problem related to Theorem 1.1 is to study the pair correlations for scaling functions ψ which are at the threshold, that is, are just exponentially growing. For instance, the set $\mathscr{E} = \{\ln n : n \in \mathbb{N} - \{0\}\}$ endowed with the trivial multiplicity function $\omega : x \mapsto 1$ satisfies the assumption of the above theorem with $c = 1$, $\alpha = 0$ and $\delta = 1$. In [PP4], we study the pair correlations of this family for general scalings and some arithmetic weights functions, proving surprising level repulsion phenomena when $\psi(N) = e^N$.

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2 Preliminaries on the growth of positive sequences

In this section, we recall some standard terminology used in the paper, and we prove two technical results used in the proof of the main results in Section 3.

Recall that given a set of parameters P and a positive map h defined on a neighborhood of $+\infty$ in $\mathbb{N}$ or $\mathbb{R}$, we denote by $\mathrm{O}_P(h)$ (respectively o_P) any *Landau function* (as the variable goes to $+\infty$) from $\mathbb{R}$ to $\mathbb{R}$ such that there exists a constant $M > 0$ depending only on the parameters in P and $t_0 \geqslant 0$ (possibly depending on ambient data) such that for every $t \geqslant t_0$, we have $|\,\mathrm{O}_P(h)(t)| \leqslant M\, h(t)$ (respectively such that $\lim\limits_{+\infty} \frac{|\mathrm{o}_P(h)(t)|}{h(t)} = 0$).

A positive sequence $(x_n)_{n\in\mathbb{N}}$ is

- *subexponentially growing* if for every $\gamma > 0$, we have $\lim_{n\to+\infty} \frac{x_n}{e^{\gamma n}} = 0$,
- *at most polynomially growing* if there exists $\gamma > 0$ such that $\lim_{n\to+\infty} \frac{x_n}{n^\gamma} = 0$,
- *strictly sublinearly growing* if there exists $\gamma \in \,]0,1[$ such that $\lim_{n\to+\infty} \frac{x_n}{n^\gamma} = 0$.

The first result generalises a classical result on the geometric sums (when $b = 0$) to the generality needed for the proofs in Section 3.

Lemma 2.1. *For every $b \in \mathbb{R}$, for every sequence $(a_M)_{M\in\mathbb{N}}$ in $]1,+\infty[$ such that the sequence $(\frac{1}{\ln a_M})_{M\in\mathbb{N}}$ is strictly sublinearly growing, as M tends to $+\infty$ in $\mathbb{N}$, we have*

$$\sum_{k=1}^{M} k^b (a_M)^k = \frac{a_M}{a_M - 1} M^b (a_M)^M \big(1 + \mathrm{O}_b(\frac{1}{\sqrt{M}})\big)\ .$$

Proof. Let $\gamma \in \,]0,1[$ be such that $\lim_{M\to+\infty} M^\gamma \ln a_M = +\infty$. As a preliminary remark, note that we have $n^b(a_M)^n = \mathrm{O}_b(M^b(a_M)^M)$ for every $n \in \{1,\dots,M\}$: This is immediate if $b \geqslant 0$, and follows when $b < 0$ by considering separately the case $n \geqslant \frac{M}{2}$ (in which case we have $\frac{n^b(a_M)^n}{M^b(a_M)^M} \leqslant 2^{|b|}$) and $n \leqslant \frac{M}{2}$ (in which case we have

$$\frac{n^b(a_M)^n}{M^b(a_M)^M} \leqslant M^{|b|}(a_M)^{-M/2} = e^{-\frac{M^{1-\gamma}}{2}\left(M^\gamma \ln a_M - 2|b| \frac{\ln M}{M^{1-\gamma}}\right)}\ ,$$

which converges to 0 as M tends to $+\infty$).

Recall that $(1 - \frac{1}{n+1})^b = 1 + \mathrm{O}_b(\frac{1}{n})$ for every $n \geqslant 1$. With $\Sigma_M = \sum_{n=1}^M n^b(a_M)^n$, for every $S \in [1, M]$, by the standard telescopic sum argument and by cutting the third sum below for $n \leqslant S$ and for $n > S$ (using in this second case the preliminary remark), we hence have

$$(a_M-1)\,\Sigma_M=\sum_{n=1}^{M}n^b(a_M)^{n+1}-\Sigma_M=\sum_{n=1}^{M}(n+1)^b(1-\frac{1}{n+1})^b(a_M)^{n+1}-\Sigma_M$$
$$=(M+1)^b(a_M)^{M+1}-a_M+\mathrm{O}_b\big(\sum_{n=1}^{M}\frac{1}{n}(n+1)^b(a_M)^{n+1}\big)$$
$$=(M+1)^b(a_M)^{M+1}-a_M+\mathrm{O}_b((S+1)^{b+1}(a_M)^{S+1})+\mathrm{O}_b\big(\frac{M-S}{S}(M+1)^b(a_M)^{M+1}\big)\;.$$

As $M\to+\infty$, by taking $S=\frac{4\,M^2}{(1+\sqrt{1+4M}\,)^2}\sim M$, so that $M-S\sim\sqrt{M}$ and $\frac{M-S}{S}\sim\frac{1}{\sqrt{M}}$, the sum of the $\mathrm{O}_b(\cdot)$ functions in the above centered line is an $\mathrm{O}_b\big(\frac{M^b(a_M)^M}{\sqrt{M}}\big)$ function. The result follows. □

In order to simplify the notation in the main body of this text, let

$$\widetilde{\omega}:t\mapsto\mathscr{N}_{\mathscr{E},\omega}(t)=\sum_{x\in\mathscr{E},\,x\leqslant t}\omega(x)\;. \tag{2}$$

This function is defined on $\mathbb{R}$ with the usual convention that a sum over an empty set of indices is 0. The local finiteness assumption of the subset $\mathscr{E}$ of $[0,+\infty[$ ensures the finiteness of the growth function $\widetilde{\omega}=\mathscr{N}_{\mathscr{E},\omega}$, and the local finiteness (hence regularity) of the pair correlation measures $\mathscr{R}_N^{\mathscr{F},\psi}$ on $\mathbb{R}$ for $\mathscr{F}=\big((F_N=\{x\in\mathscr{E}:x\leqslant N\})_{N\in\mathbb{N}},\ \omega\big)$, defined in Equation (1). We denote by

(PA) the assumption that $\widetilde{\omega}(t)\sim c\,t^\alpha\,e^{\delta t}$ as $t\to+\infty$ for some constants $c,\delta>0$ and $\alpha\in\mathbb{R}$, and by

(ET) the assumption that $\widetilde{\omega}(t)=c\,t^\alpha\,e^{\delta t}\big(1+\mathrm{O}(e^{-\kappa\,t})\big)$ as $t\to+\infty$ for some constants $c,\delta,\kappa>0$ and $\alpha\in\mathbb{R}$.

For all $t\in\mathbb{R}$ and $\eta>0$, we define the *(t,η)-slice of weights* as

$$\widetilde{\omega}(t,\eta)=\widetilde{\omega}(t)-\widetilde{\omega}(t-\eta)\;. \tag{3}$$

The next result describes the asymptotic behaviour for the thin, though not too thin, slices of weights under one of the two assumptions (PA) or (ET).

Lemma 2.2. *(1) Let $\eta>0$. Under Assumption (PA), as $t>0$ tends to $+\infty$, we have*

$$\widetilde{\omega}\,(t,\eta)=c\,t^\alpha\,e^{\delta\,t}(1-e^{-\delta\,\eta})\big(1+\mathrm{o}_{\alpha,\eta}(1)\big)\;.$$

(2) Let $\eta:t\mapsto\eta_t$ be a map from $[0,+\infty[$ to $]0,1]$. Under Assumption (ET), if $\lim_{t\to+\infty}\eta_t\,e^{\kappa\,t}=+\infty$, then, as t tends to $+\infty$, we have

$$\widetilde{\omega}\,(t,\eta_t)=c\,t^\alpha\,e^{\delta\,t}(1-e^{-\delta\,\eta_t})\Big(1+\frac{1}{\eta_t}\;\mathrm{O}_\delta(e^{-\kappa\,t})+\mathrm{O}_{\alpha,\delta}\,\big(\frac{1}{t}\big)\Big)\;.$$

In particular, as t tends to $+\infty$, under these assumptions, we have

$$\widetilde{\omega}\,(t,\eta_t)=\mathrm{O}_{\alpha,\delta}(t^\alpha\,e^{\delta\,t})\;.$$

If $\alpha=0$, then as t tends to $+\infty$, we have more precisely

$$\widetilde{\omega}\,(t,\eta_t)=c\,e^{\delta\,t}(1-e^{-\delta\,\eta_t})\big(1+\frac{1}{\eta_t}\;\mathrm{O}_\delta(e^{-\kappa\,t})\big)\;.$$

Proof. (1) For every $t\in\mathbb{R}$, let $r_t = c^{-1}t^{-\alpha}\, e^{-\delta\, t}\,\widetilde{\omega}\,(t)-1$, which converges to 0 as $t\to+\infty$ since Assumption (PA) holds. If $t>0$, we have

$$\begin{aligned}\widetilde{\omega}\,(t,\eta) &= \widetilde{\omega}(t)-\widetilde{\omega}(t-\eta) = c\, t^{\alpha}\, e^{\delta\, t}(1+r_t) - c\,(t-\eta)^{\alpha}\, e^{\delta\,(t-\eta)}(1+r_{t-\eta})\\ &= c\, t^{\alpha}\, e^{\delta\, t}\Big(1+r_t-\big(1-\frac{\eta}{t}\big)^{\alpha} e^{-\delta\,\eta}(1+r_{t-\eta}))\Big)\\ &= c\, t^{\alpha}\, e^{\delta\, t}\Big(1-e^{-\delta\,\eta}+r_t-e^{-\delta\,\eta}r_{t-\eta}-e^{-\delta\,\eta}\,\mathrm{O}_{\alpha}\,\big(\frac{\eta}{t}\big)\Big)\;.\end{aligned}$$

Since $\lim_{t\to+\infty}\frac{\max\{r_t,\, r_{t-\eta},\frac{1}{t}\}}{1-e^{-\delta\,\eta}}=0$, this concludes the proof of Assertion (1).

(2) Since $\eta_t\in\,]0,1]$, we have $\mathrm{O}(e^{\kappa\,\eta_t})=\mathrm{O}(1)$ as $t\to+\infty$. Recall that $(1+s)^{\alpha}=1+\mathrm{O}_{\alpha}(s)$ as $s\to 0$. Since Assumption (ET) holds, when $t>0$ tends to $+\infty$, we hence have

$$\begin{aligned}\widetilde{\omega}\,(t,\eta_t) &= \widetilde{\omega}(t)-\widetilde{\omega}(t-\eta_t) = c\, t^{\alpha}\, e^{\delta\, t}\Big(1+\mathrm{O}(e^{-\kappa\, t})-\big(1-\frac{\eta_t}{t}\big)^{\alpha} e^{-\delta\,\eta_t}(1+\mathrm{O}(e^{-\kappa\, t}))\Big)\\ &= c\, t^{\alpha}\, e^{\delta\, t}\Big(1-e^{-\delta\,\eta_t}+\mathrm{O}(e^{-\kappa\, t})+\mathrm{O}_{\alpha}(\frac{\eta_t}{t})\Big)\;.\end{aligned}$$

Since $\frac{1}{1-e^{-\delta\,\eta_t}}=\mathrm{O}_{\delta}(\frac{1}{\eta_t})$ as $t\to+\infty$, and since $\lim_{t\to+\infty}\frac{e^{-\kappa\, t}}{\eta_t}=0$, this proves the result for general α. The proof in the special case when $\alpha=0$ is even simplier. $\Box$

3 An extension of Theorem 1.1 with error terms

We will use in this section the notation $\mathscr{N}_{\mathscr{E},\,\omega}$ and $\mathscr{R}_N^{\mathscr{F},\psi}$ defined in the Introduction, as well as the notation $\widetilde{\omega}(\cdot)$ and $\widetilde{\omega}(\cdot,\cdot)$ of Equations (2) and (3). For every scaling function $\psi:\mathbb{N}\to[1,+\infty[$, we consider the renormalising function

$$\psi' : N\mapsto \frac{\mathscr{N}_{\mathscr{E},\,\omega}(N)^2}{\psi(N)} = \frac{\widetilde{\omega}(N)^2}{\psi(N)}\;. \tag{4}$$

We denote by $\|\mu\|$ the total mass of a measure μ.

We start this section by some comments on the statement of Theorem 1.1.

Remarks. (1) When $\psi=1$, then $\psi'(N)=\|\mathscr{R}_N^{\mathscr{F},\psi}\|$, and the renormalisation in Theorem 1.1 (as well as in Theorem 3.1) is chosen in order to obtain probability measures $\frac{1}{\psi'(N)}\,\mathscr{R}_N^{\mathscr{F},\psi}$, which turns out to converge to a probability measure $g_{\mathscr{F},1}\;dt$ as $N\to+\infty$.

When $\lim\limits_{+\infty}\psi=+\infty$, as the proof below shows, the renormalisation is precisely chosen in order to obtain a locally finite nonzero measure, but there is an infinite loss of mass at infinity, in the sense that $\lim\limits_{N\to+\infty}\frac{1}{\psi'(N)}\,\|\,\mathscr{R}_N^{\mathscr{F},\psi}\,\|=+\infty$ even though $\lim\limits_{N\to+\infty}\frac{1}{\psi'(N)}\,\|\,(\mathscr{R}_N^{\mathscr{F},\psi})_{|K}\|$ is finite for every compact subset K of $[0,+\infty[$.

(2) The pair correlation measures are sometimes defined (see for instance [PP4]) by

$$\widetilde{\mathscr{R}}_N^{\mathscr{F},\psi} = \sum_{x,y\in F_N\,:\,x\neq y}\omega(x)\,\omega(y)\,\Delta_{\psi(N)(y-x)}\;,$$

that is by adding the assumption $x\neq y$ on the set of pair of indices $(x,y)\in {F_N}^2$ in the summation (compare with Equation (1)). Note that when the renormalised measures

$\frac{1}{\psi'(N)}\,\mathscr{R}_N^{\mathscr{F},\psi}$ weak-star converge to a measure μ on $\mathbb{R}$ which has no atom at 0 (for instance if the family $\mathscr{F}$ admits a pair correlation function for the scaling ψ and renormalisation ψ', as it is the case in Theorem 1.1), then we also have $\lim_{N\to+\infty} \frac{1}{\psi'(N)}\,\widetilde{\mathscr{R}}_N^{\mathscr{F},\psi} = \mu$, that is, the contribution of the diagonal set of indices in the sum defining $\frac{1}{\psi'(N)}\,\mathscr{R}_N^{\mathscr{F},\psi}$ is negligible.

(3) An archetypical example of a pair $(\mathscr{E},\omega)$ is given by a countable set $\widetilde{\mathscr{E}}$ endowed with a map $\ell : \widetilde{\mathscr{E}} \to \,]\,0,+\infty\,[$ with finite fibers whose image $\mathscr{E} = \ell(\widetilde{\mathscr{E}})$ is locally finite and endowed with the multiplicity function $\omega : x \mapsto \operatorname{Card}(\ell^{-1}(x))$. In this case, we have

$$\mathscr{R}_N^{\mathscr{F},\psi} = \sum_{x,y\,\in\,\widetilde{\mathscr{E}}\;:\;\ell(x),\,\ell(y)\leqslant N} \Delta_{\psi(N)(\ell(y)-\ell(x))}\;.$$

Theorem 1.1 when $\psi = 1$ then says that if there exist $\alpha\in\mathbb{R}$ and $c,\delta>0$ such that, as $t\to+\infty$, we have $\operatorname{Card}\{x\in\widetilde{\mathscr{E}} : \ell(x)\leqslant t\} \sim c\,t^\alpha\,e^{\delta\,t}$ then $\frac{1}{\|\mathscr{R}_N^{\mathscr{F},1}\|}\,\mathscr{R}_N^{\mathscr{F},1}$ weakstar converges to the measure $\frac{\delta}{2}\,e^{-\delta\,|t|}\,dt$ on the locally compact space $\mathbb{R}$ as $N\to+\infty$.

We now state an extended version with error terms of Theorem 1.1, from which it follows, using Assumption (ET). Let

$$g_\delta : t \mapsto \frac{\delta}{2}\,e^{-\delta\,|t|}\;.$$

Theorem 3.1. *Let $\mathscr{E}$ be a locally finite subset of $[0,+\infty[$ endowed with a weight function ω, and let $\alpha\in\mathbb{R}$ and $c,\delta>0$. Assume that there exists $\kappa>0$ such that as $t\to+\infty$, we have*

$$\mathscr{N}_{\mathscr{E},\,\omega}(t) = c\,t^\alpha\,e^{\delta\,t}(1+\mathrm{O}_{\mathscr{E},\omega}(e^{-\kappa\,t}))\;. \tag{5}$$

Let $\psi:\mathbb{N}\to[1+\infty[$ be a scaling function and let $A\geqslant 1$. For every function $f\in C^1(\mathbb{R})$ with compact support in $[-A,A]$, as $N\to+\infty$, with Landau functions $\mathrm{O}=\mathrm{O}_{\mathscr{E},\omega,c,\alpha,\delta,\kappa}$ and $\kappa'=\min\{\kappa,\delta\}$, we have

$$\frac{\mathscr{R}_N^{\mathscr{F},\psi}(f)}{\psi'(N)} = \begin{cases} \int_{\mathbb{R}} f(t)\,g_\delta(t)\,dt + \mathrm{O}(A\,e^{\delta A}e^{-\frac{\kappa'}{12}N}\,\|f\|_\infty) + \mathrm{O}(A\,e^{-\frac{\kappa'}{4}N}\,\|f'\|_\infty) \\ \text{if } \alpha=0 \text{ and } \psi=1, \\ \int_{\mathbb{R}} f(t)\,g_\delta(t)\,dt + \mathrm{O}\left(\frac{A^2}{N}\,(\|f\|_\infty+\|f'\|_\infty)\right) \\ \text{if } \alpha\neq 0 \text{ and } \psi=1, \\ \frac{\delta}{2}\int_{\mathbb{R}} f(t)\,dt + \mathrm{O}\left(\frac{A^2\|f\|_\infty}{\psi(N)}\right) + \mathrm{O}\left(A^2\,e^{-\frac{\kappa'}{4}N}\,\psi(N)\,\|f'\|_\infty\right) \\ \text{if } \alpha=0 \text{ and } \psi \text{ converges to } +\infty \text{ with subexponential growth} \\ \frac{\delta}{2}\int_{\mathbb{R}} f(t)\,dt + \mathrm{O}\left(\frac{A^2}{\min\{N,\psi(N)\}}(\|f\|_\infty+\|f'\|_\infty)\right) \\ \text{if } \alpha\neq 0 \text{ and } \psi \text{ converges to } +\infty \text{ with at most polynomial growth.} \end{cases}$$

Proof. Let $\mathscr{E}$, ω, c, α, δ, κ be the fixed data in the statement of Theorem 3.1. Though we won't indicate the dependency, the Landau functions $\mathrm{O}(\cdot)$ below will depend on these fixed data, in the sense defined at the beginning of Section 2. Up to replacing κ by $\min\{\kappa,\delta\}$ which does not change the conclusion of Theorem 3.1 and is implied by its hypothesis (5), we may assume that

$$\kappa\leqslant\delta\;. \tag{6}$$

Let ψ, f, A and N be the varying data in the statement of Theorem 3.1. The Landau functions $\operatorname{O}(\cdot)$ below will not depend on these varying data, in the sense defined at the beginning of Section 2.

Note that if $\iota : t \mapsto -t$, then $\iota_* \mathscr{R}_N^{\mathscr{F},\psi} = \mathscr{R}_N^{\mathscr{F},\psi}$ by using the change of variables $(x,y) \mapsto (y,x)$ in the summation of Equation (1), and that $g_\delta \circ \iota = g_\delta$. In order to prove Theorem 3.1, we may hence assume by additivity that the support of f is contained in $[0, A]$, and again by additivity that $f \geqslant 0$.

Note that $\lim_{+\infty} \psi' = +\infty$ by Equation (4) since $\widetilde{\omega}$ is exponentially growing and ψ is subexponentially growing in all the cases of Theorem 3.1. As $N \to +\infty$, by Equations (4) and (5), we have

$$\frac{1}{\psi'(N)} = \frac{\psi(N)}{c^2\, N^{2\alpha}\, e^{2\delta N}} \big(1 + \operatorname{O}(e^{-\kappa\, N})\big)\,. \tag{7}$$

We consider throughout this proof two small quantities $\epsilon, \tau' \in\,]0,1]$ which will depend on N and converge to 0. We define

$$\tau = \frac{\tau'}{\psi(N)}\,,$$

and we assume that $\tau \geqslant 2\epsilon$ when N is large. We will check this inequality after defining ϵ and τ' in Equations (22) and (27).

The following lemma describes the work on the set of indices of some of the following sums in order to be able to separate the variables x and y.

Lemma 3.2. *Let $x, y \in \mathscr{E}$ and $k, n \geqslant 1$. The system of inequalities*

$$0 < x \leqslant y \leqslant N, \quad (k-1)\epsilon < x \leqslant k\,\epsilon, \quad (n-1)\tau' < \psi(N)(y-x) \leqslant n\tau' \tag{8}$$

implies the system of inequalities

$$(k-1)\epsilon < x \leqslant k\,\epsilon,\;\; (n\tau + k\,\epsilon) - (\tau + \epsilon) < y \leqslant n\tau + k\,\epsilon,\;\; k \leqslant M_\epsilon^+ = \Big\lfloor \frac{N-(n-1)\tau}{\epsilon} \Big\rfloor + 1\,, \tag{9}$$

and is implied by the system of inequalities

$$(k-1)\epsilon < x \leqslant k\,\epsilon, \quad (n\tau + k\,\epsilon) - \tau < y \leqslant n\tau + k\,\epsilon - \epsilon, \quad k \leqslant M_\epsilon^- = \Big\lfloor \frac{N - n\tau}{\epsilon} \Big\rfloor + 1\,. \tag{10}$$

Proof. Since we have $\tau = \frac{\tau'}{\psi(N)}$, the last two inequalities of Equation (8) are equivalent to $(n-1)\tau + x < y \leqslant n\tau + x$. With the middle two inequalities of Equation (8), this implies that $(n\tau + k\,\epsilon) - (\tau + \epsilon) < y \leqslant n\tau + k\,\epsilon$ and is implied by $(n\tau + k\,\epsilon) - \tau < y \leqslant n\tau + k\,\epsilon - \epsilon$.

The inequalities $(n\tau + k\,\epsilon) - (\tau + \epsilon) < y$ and $y \leqslant N$ imply that $k\,\epsilon \leqslant N - (n-1)\tau + \epsilon$, hence that $k \leqslant M_\epsilon^+$.

The inequalities $y \leqslant n\tau + k\,\epsilon - \epsilon$ and $k \leqslant M_\epsilon^-$ imply that $y \leqslant n\tau + \epsilon\big(\frac{N-n\tau}{\epsilon} + 1\big) - \epsilon = N$. The inequalities $(k-1)\epsilon < x$ and $k \geqslant 1$ implies that $x > 0$. The inequalities $x \leqslant k\,\epsilon$, $(n\tau + k\,\epsilon) - \tau < y$ and $n \geqslant 1$ imply that $x \leqslant y$. The result follows. □

Note that by the definitions of $M_\epsilon^\pm$ in Equations (9) and (10), when $\tau \geqslant 2\epsilon$, we have

$$\epsilon\, M_\epsilon^\pm = \epsilon\Big(\Big\lfloor \frac{N - (n - \frac{1}{2}(1 \pm 1))\tau}{\epsilon} \Big\rfloor + 1\Big) = N - n\tau + \operatorname{O}(\tau)\,. \tag{11}$$

Let us define

$$\mu_N(f) = \sum_{x,y\in\mathscr{E},\ 0<x\leqslant y\leqslant N} \omega(x)\,\omega(y)\ f(\psi(N)(y-x))\ . \tag{12}$$

Since the support of f is contained in $[0,A]$ and $\psi \geqslant 1$, we have

$$\begin{aligned} 0 \leqslant \mathscr{R}_N^{\mathscr{F},\psi}(f) - \mu_N(f) &= \sum_{x,y\in\mathscr{E},\ 0=x\leqslant y\leqslant N} \omega(x)\,\omega(y)\ f(\psi(N)(y-x)) \\ &\leqslant \sum_{y\in\mathscr{E}\cap[0,A]} \omega(0)\,\omega(y)\ \|f\|_\infty\ , \end{aligned}$$

and the term on the right hand is independent of N. Note that $\frac{1}{\psi'(N)} = \mathrm{O}(e^{-\kappa N})$ by Equations (4), (5) and (6), and by the subexponential growth of ψ in all the cases of Theorem 3.1. Hence in order to prove Theorem 3.1, we therefore only have to prove that $\frac{\mu_N(f)}{\psi'(N)}$ converges, with the appropriate error terms, to $\int f\, g_\delta\ dt$ when $\psi = 1$ and to $\frac{\delta}{2}\int f\ dt$ when $\psi \to +\infty$.

We will use Riemann sums in order to approximate these integrals, with subdivision step given by τ'. Since the support of f is contained in $[0,A]$, we only need to subdivide this interval into $\lceil\frac{A}{\tau'}\rceil$ intervals of length τ'. For all $n \in \{1,\dots,\lceil\frac{A}{\tau'}\rceil\}$ and $t \in\ ](n-1)\tau', n\tau']$, we have

$$f(t) = f(n\tau') - \int_t^{n\tau'} f'(t)\ dt = f(n\tau') + \mathrm{O}\Big(\int_{(n-1)\tau'}^{n\tau'} |f'(t)|\ dt\Big) = f(n\tau') + \mathrm{O}(\tau'\,\|f'\|_\infty)\ . \tag{13}$$

Since $f \geqslant 0$, we may assume that $f(n\tau') + \mathrm{O}(\tau'\,\|f'\|_\infty) \geqslant 0$. Let us define

$$a_{n,N} = \sum_{k=1}^{+\infty} \sum_{\substack{x,y\in\mathscr{E}\,:\,0<x\leqslant y\leqslant N,\\ (k-1)\epsilon<x\leqslant k\,\epsilon,\\ (n-1)\tau'<\psi(N)(y-x)\leqslant n\tau'}} \omega(x)\,\omega(y) \tag{14}$$

which is a nonnegative finite sum and depends also on ϵ and on τ'. With the help of Equation (13), Equation (12) becomes, by subdividing the range of x into half-open intervals of length ϵ and the range of $\psi(N)(y-x)$ (with values at most A in order to be in the support of f) into half-open intervals of length τ',

$$\begin{aligned} \mu_N(f) &= \sum_{n=1}^{\lceil\frac{A}{\tau'}\rceil} \sum_{k=1}^{+\infty} \sum_{\substack{x,y\in\mathscr{E}\,:\,0<x\leqslant y\leqslant N,\\ (k-1)\epsilon<x\leqslant k\,\epsilon,\\ (n-1)\tau'<\psi(N)(y-x)\leqslant n\tau'}} \omega(x)\,\omega(y)\ f(\psi(N)(y-x)) \\ &= \sum_{n=1}^{\lceil\frac{A}{\tau'}\rceil} \big(f(n\tau') + \mathrm{O}(\tau'\,\|f'\|_\infty)\big)\ a_{n,N}\ . \end{aligned} \tag{15}$$

We now proceed by a majoration and minoration of $a_{n,N}$. Let us define

$$a^+_{n,N} = \sum_{k=1}^{M^+_\epsilon} \sum_{\substack{x,y\in\mathscr{E}\,:\,(k-1)\epsilon<x\leqslant k\,\epsilon\\ (n\tau+k\,\epsilon)-(\tau+\epsilon)<y\leqslant n\tau+k\,\epsilon}} \omega(x)\,\omega(y) \tag{16}$$

and

$$a^{-}_{n,N} = \sum_{k=1}^{M^{-}_{\epsilon}} \sum_{\substack{x,y\in\mathscr{E}\,:\,(k-1)\epsilon<x\leqslant k\,\epsilon\\ (n\tau+k\,\epsilon)-\tau<y\leqslant(n\tau+k\,\epsilon)+\epsilon}} \omega(x)\,\omega(y)\ . \tag{17}$$

By Lemma 3.2, we have

$$a^{-}_{n,N} \leqslant a_{n,N} \leqslant a^{+}_{n,N}\ .$$

Since the variables x, y are separated in the sums defining $a^{\pm}_{n,N}$ and by the definition (3) of the slices of weights, we have

$$a^{+}_{n,N} = \sum_{k=1}^{M^{+}_{\epsilon}} \widetilde{\omega}\,(k\epsilon,\epsilon)\,\widetilde{\omega}\,(n\tau + k\epsilon, \tau+\epsilon) \tag{18}$$

and

$$a^{-}_{n,N} = \sum_{k=1}^{M^{-}_{\epsilon}} \widetilde{\omega}\,(k\epsilon,\epsilon)\,\widetilde{\omega}\,(n\tau + k\epsilon+\epsilon, \tau-\epsilon)\ . \tag{19}$$

We study the quantities $a^{\pm}_{n,N}$ under the assumption (5) on the asymptotic behaviour of the weights.

Asymptotics on $a^{\pm}_{n,N}$. By Equation (18) and by two applications of Lemma 2.2 (2) with $(t,\eta_t) = (k\epsilon,\epsilon)$ and $(t,\eta_t) = (n\tau+k\epsilon,\tau+\epsilon)$ as $k\to+\infty$, and up to verifying when we will define ϵ and τ' that the assumption of this lemma (besides Assumption (ET) which holds by Equation (5)) is satisfied, for all N large enough and $n\in\{1,\dots,\lceil\frac{A}{\tau'}\rceil\}$, we have

$$\begin{aligned}
a^{+}_{n,N} &= \sum_{k=1}^{M^{+}_{\epsilon}} c\,(k\epsilon)^{\alpha}\,e^{\delta k\epsilon}(1-e^{-\delta\epsilon})\Big(1+\frac{1}{\epsilon}\ \mathrm{O}(e^{-\kappa k\epsilon}) + \mathrm{O}\,\big(\frac{1}{k\epsilon}\big)\Big)\\
&\quad\times c\,(n\tau+k\epsilon)^{\alpha}\,e^{\delta(n\tau+k\epsilon)}(1-e^{-\delta(\tau+\epsilon)})\Big(1+\frac{1}{\tau+\epsilon}\ \mathrm{O}(e^{-\kappa(n\tau+k\epsilon)}) + \mathrm{O}\,\big(\frac{1}{n\tau+k\epsilon}\big)\Big)\\
&= c^2(1-e^{-\delta\epsilon})(1-e^{-\delta(\tau+\epsilon)})\ e^{\delta n\tau}\sum_{k=1}^{M^{+}_{\epsilon}} z^{+}_{k}\ ,
\end{aligned} \tag{20}$$

where

$$z^{+}_{k} = (k\epsilon)^{2\alpha}\,e^{2\delta k\epsilon}\,\big(1+\frac{n\tau}{k\epsilon}\big)^{\alpha}\Big(1+\frac{1}{\epsilon}\ \mathrm{O}(e^{-\kappa k\epsilon})+\mathrm{O}\,\big(\frac{1}{k\epsilon}\big)\Big)\Big(1+\frac{1}{\tau+\epsilon}\ \mathrm{O}(e^{-\kappa(n\tau+k\epsilon)})+\mathrm{O}\,\big(\frac{1}{n\tau+k\epsilon}\big)\Big)\ .$$

Since $\tau\pm\epsilon\geqslant\epsilon$, $n\tau+k\epsilon\geqslant k\epsilon$ and $e^{-\kappa n\tau}\leqslant 1$, this simplifies as

$$z^{+}_{k} = (k\epsilon)^{2\alpha}\,e^{2\delta k\epsilon}\,\big(1+\frac{n\tau}{k\epsilon}\big)^{\alpha}\Big(1+\frac{1}{\epsilon}\ \mathrm{O}(e^{-\kappa k\epsilon})+\mathrm{O}\,\big(\frac{1}{k\epsilon}\big)\Big)^{2}\ . \tag{21}$$

Case 1: Let us first assume that $\alpha = 0$**.** We define in this case

$$\epsilon = e^{-\frac{\kappa}{3}N} \quad\text{and}\quad \tau' = e^{-\frac{\kappa}{4}N}\psi(N)\ . \tag{22}$$

In particular, we have $\lim_{N\to+\infty}\epsilon\,e^{\kappa N} = +\infty$, we have $\tau\geqslant 2\epsilon$ if N is large enough, and since ψ grows subexponentially under the two assumptions of Theorem 3.1 when $\alpha=0$, the quantity τ' tends to 0 as $N \to +\infty$. By the last claim of Lemma 2.2 (2), Equation (21) for z_k^+ simplifies as

$$z_k^+ = e^{2\delta k\epsilon}\big(1 + \frac{1}{\epsilon}\ \mathrm{O}(e^{-\kappa k\epsilon})\big)^2 = e^{2\delta k\epsilon} + \frac{1}{\epsilon^2}\ \mathrm{O}(e^{(2\delta-\kappa)k\epsilon})\ .$$

Hence by a geometric series summation, since $2\delta > \delta \geqslant \kappa$ by Equation (6), we have

$$\sum_{k=1}^{M_\epsilon^+} z_k^+ = \frac{e^{2\delta\epsilon(M_\epsilon^+ +1)} - 1}{e^{2\delta\epsilon} - 1} + \frac{1}{\epsilon^2}\,\mathrm{O}\,\big(\frac{e^{(2\delta-\kappa)\epsilon(M_\epsilon^+ +1)} - 1}{e^{(2\delta-\kappa)\epsilon} - 1}\big)\ .$$

Note that $\frac{e^{2\delta\epsilon}-1}{e^{(2\delta-\kappa)\epsilon}-1} = \mathrm{O}(1)$ as $\epsilon \to 0$, and recall that $n\tau = \frac{n\tau'}{\psi(N)} \leqslant \frac{A+1}{\psi(N)}$ for every $n \in \{1, \dots, \lceil \frac{A}{\tau'} \rceil\}$. By Equation (11), by the definition (22) of ϵ and since $2\delta \geqslant \delta \geqslant \kappa \geqslant \frac{\kappa}{3}$ by Equation (6), we hence have

$$\begin{aligned}
\sum_{k=1}^{M_\epsilon^+} z_k^+ &= \frac{e^{2\delta(N-n\tau)}}{e^{2\delta\epsilon} - 1}\big(e^{\mathrm{O}(\tau)} - e^{\delta(n\tau-N)} + \frac{1}{\epsilon^2}\,\mathrm{O}(e^{\kappa(n\tau-N)})\big)\\
&= \frac{e^{2\delta(N-n\tau)}}{e^{2\delta\epsilon} - 1}\big((1 + \mathrm{O}(e^{-\frac{\kappa}{4}N})) + \mathrm{O}(e^{\frac{\delta A}{\psi(N)}}e^{-\delta N}) + \mathrm{O}(e^{\frac{\kappa A}{\psi(N)}}e^{-\frac{\kappa}{3}N})\big)\\
&= \frac{e^{2\delta(N-n\tau)}}{e^{2\delta\epsilon} - 1}\big(1 + \mathrm{O}(e^{\frac{\delta A}{\psi(N)}}e^{-\frac{\kappa}{4}N})\big)\ .
\end{aligned}$$

By the definition (22) of ϵ and τ' (so that $\tau = e^{-\frac{\kappa}{4}N}$), we have

$$1 - e^{-\delta(\tau\pm\epsilon)} = \delta\tau(1 \pm \frac{\epsilon}{\tau})(1 + \mathrm{O}(\tau + \epsilon)) = \delta\tau(1 + \mathrm{O}(e^{-\frac{\kappa}{12}N}))\ ,$$

and

$$\frac{1 - e^{-\delta\epsilon}}{e^{2\delta\epsilon} - 1} = \frac{1}{2}(1 + \mathrm{O}(\epsilon)) = \frac{1}{2}(1 + \mathrm{O}(e^{-\frac{\kappa}{3}N}))\ .$$

Therefore Equation (20) becomes

$$\begin{aligned}
a_{n,N}^+ &= c^2\ e^{2\delta N}\frac{1 - e^{-\delta\epsilon}}{e^{2\delta\epsilon} - 1}(1 - e^{-\delta(\tau+\epsilon)})\,e^{-\delta n\tau}\,\big(1 + \mathrm{O}(e^{\frac{\delta A}{\psi(N)}}e^{-\frac{\kappa}{4}N})\big)\\
&= (c^2\ e^{2\delta N})(\frac{\delta}{2}\ \tau\ e^{-\delta n\tau})\big(1 + \mathrm{O}(e^{\frac{\delta A}{\psi(N)}}e^{-\frac{\kappa}{12}N})\big)\ .
\end{aligned}$$

Taking into account the small differences between $a_{n,N}^+$ and $a_{n,N}^-$ in Equations (16) and (17), a similar computation gives the same formula for $a_{n,N}^-$. Since $a_{n,N}^- \leqslant a_{n,N} \leqslant a_{n,N}^+$, we hence have

$$a_{n,N} = (c^2\ e^{2\delta N})(\frac{\delta}{2}\ \tau\ e^{-\delta n\tau})\big(1 + \mathrm{O}(e^{\frac{\delta A}{\psi(N)}}e^{-\frac{\kappa}{12}N})\big)\ . \tag{23}$$

End of the proof of Theorem 3.1 when $\alpha = 0$. Note that $\sum_{n=1}^{\lceil \frac{A}{\tau'} \rceil} e^{-\frac{\delta n\tau'}{\psi(N)}} = \mathrm{O}(\frac{A}{\tau'})$ as $N \to +\infty$. Since $\tau = \frac{\tau'}{\psi(N)}$, by Equations (15), (7) and (23), for N large enough, we have

$$\begin{aligned}
\frac{\mu_N(f)}{\psi'(N)} &= \sum_{n=1}^{\lceil \frac{A}{\tau'} \rceil}\big(f(n\tau') + \mathrm{O}(\tau'\,\|f'\|_\infty)\big)\,\frac{\delta}{2}\,\tau'\,e^{-\frac{\delta n\tau'}{\psi(N)}}\big(1 + \mathrm{O}(e^{\frac{\delta A}{\psi(N)}}e^{-\frac{\kappa}{12}N})\big)\big(1 + \mathrm{O}(e^{-\kappa N})\big)\\
&= \Big(\sum_{n=1}^{\lceil \frac{A}{\tau'} \rceil} f(n\tau')\,\frac{\delta}{2}\,\tau'\,e^{-\frac{\delta n\tau'}{\psi(N)}}\Big) + \mathrm{O}(A\,e^{\frac{\delta A}{\psi(N)}}e^{-\frac{\kappa}{12}N}\|f\|_\infty) + \mathrm{O}(A\,\tau'\,\|f'\|_\infty)\ .
\end{aligned} \tag{24}$$

Assume first that $\psi = 1$. Recall that $g_\delta : t \mapsto \frac{\delta}{2}\, e^{-\delta t}$ is bounded with bounded derivative on $[0, +\infty[$. By the standard Riemann sum approximation with error term of an integral, and since the support of f is contained in $[0, A]$ with $A \geqslant 1$, we have

$$\sum_{n=1}^{\lceil \frac{A}{\tau'} \rceil} f(n\tau')\, \frac{\delta}{2}\, \tau'\, e^{-\delta n \tau'} = \int_0^{+\infty} f(t)\, g_\delta(t)\; dt + \mathrm{O}\left(\tau'(\|f g_\delta\|_\infty + \mathrm{Var}(f g_\delta))\right)$$
$$= \int_0^{+\infty} f(t)\, g_\delta(t)\; dt + \mathrm{O}\left(A\, \tau'(\|f\|_\infty + \|f'\|_\infty)\right). \tag{25}$$

With Equation (24), this proves Theorem 3.1 when $\alpha = 0$ and $\psi = 1$.

Assume now that $\lim_{+\infty} \psi = +\infty$. Note that $e^{-\frac{\delta n \tau'}{\psi(N)}} = 1 + \mathrm{O}(\frac{A}{\psi(N)})$ since $n \leqslant \lceil \frac{A}{\tau'} \rceil$. Hence a similar Riemann sum argument gives

$$\sum_{n=1}^{\lceil \frac{A}{\tau'} \rceil} f(n\tau')\, \frac{\delta}{2}\, \tau'\, e^{-\delta n \frac{\tau'}{\psi(N)}} = \Big(\frac{\delta}{2} \int_0^{+\infty} f(t)\; dt + \mathrm{O}\left(A\, \tau'(\|f\|_\infty + \|f'\|_\infty)\right)\Big)(1 + \mathrm{O}(\frac{A}{\psi(N)}))$$
$$= \frac{\delta}{2} \int_0^{+\infty} f(t)\; dt + \mathrm{O}\left(\frac{A^2 \|f\|_\infty}{\psi(N)}\right) + \mathrm{O}\left(A^2\, \tau'(\|f\|_\infty + \|f'\|_\infty)\right). \tag{26}$$

Since ψ grows subexponentially, Equations (24) and (26) imply Theorem 3.1 when $\alpha = 0$ and $\lim_{+\infty} \psi = +\infty$.

Case 2: Let us now assume that $\alpha \neq 0$**.** The scheme of proof is the same one as in Case 1, though more technical. We will only be able to obtain the result under a bit stronger assumption on the scaling function ψ and with a much weaker error term, due to the polonomial term in the asymptotic growth of the counting function $\widetilde{\omega}$. Since ψ is assumed to have polynomial growth, we fix $\gamma' \geqslant 1$ such that $\lim_{N\to+\infty} \frac{\psi(N)}{N^{\gamma'-1}} = 0$. We define in this case

$$\epsilon = \frac{1}{N^{2\gamma'}} \quad \text{and} \quad \tau' = \frac{1}{N^{\gamma'}}\, \psi(N)\,. \tag{27}$$

In particular, we have $\lim_{N\to+\infty} \epsilon\, e^{\kappa N} = +\infty$, we have $\tau \geqslant 2\epsilon$ if N is large enough, and τ' tends to 0 as $N \to +\infty$.

For N is large enough, since $n\tau = \frac{n\tau'}{\psi(N)}$ is bounded by $A + 1 \leqslant 2A$ for $n \leqslant \lceil \frac{A}{\tau'} \rceil$ and since $A \geqslant 1$, as $k \to +\infty$, Equation (21) gives

$$z_k^+ = (k\epsilon)^{2\alpha}\, e^{2\delta k \epsilon}\left(1 + \mathrm{O}(\frac{n\tau}{k\epsilon})\right)\Big(1 + \frac{1}{\epsilon}\; \mathrm{O}(e^{-\kappa k \epsilon}) + \mathrm{O}\left(\frac{1}{k\epsilon}\right)\Big)^2$$
$$= (k\epsilon)^{2\alpha}\, e^{2\delta k \epsilon} + \mathrm{O}\left(A\, (k\epsilon)^{2\alpha - 1}\, e^{2\delta k\epsilon}\right) + \frac{1}{\epsilon^2}\; \mathrm{O}\left(A\, (k\epsilon)^{2\alpha}\, e^{(2\delta-\kappa)k\epsilon}\right).$$

We now apply Lemma 2.1 with $M = M_\epsilon^+$, with $b = 2\alpha$ or $b = 2\alpha - 1$, and with $a_M = e^{2\delta\epsilon}$ or $a_M = e^{(2\delta-\kappa)\epsilon}$. The hypothesis of this lemma is satisfied, since by the definition of ϵ in Equation (27) and of $M_\epsilon^\pm$ in Equations (9) and (10), and with $b' = 2\delta$ or $b' = 2\delta - \kappa$ for every $\gamma \in \,]\frac{2\gamma'}{2\gamma'+1}, 1[$, we have

$$(M_\epsilon^\pm)^\gamma \ln(e^{b'\epsilon}) \sim b'\, N^\gamma\, \epsilon^{1-\gamma} = b'\, N^{\gamma-(1-\gamma)(2\gamma')} = b'\, N^{\gamma(2\gamma'+1)-2\gamma'}\,,$$

which converges to $+\infty$ as $N \to +\infty$ by the assumption on γ. We hence have

$$\sum_{k=1}^{M_\epsilon^+} z_k^+ = \frac{e^{2\delta\epsilon}}{e^{2\delta\epsilon}-1}\,(\epsilon M_\epsilon^+)^{2\alpha}\, e^{2\delta\,\epsilon M_\epsilon^+}\big(1+\mathrm{O}(\frac{\sqrt{\epsilon}}{\sqrt{\epsilon M_\epsilon^+}})\big) + \mathrm{O}\big(A\frac{e^{2\delta\epsilon}}{e^{2\delta\epsilon}-1}\,(\epsilon M_\epsilon^+)^{2\alpha-1}e^{2\delta\epsilon M_\epsilon^+}\big)$$
$$+\frac{1}{\epsilon^2}\,\mathrm{O}\big(A\frac{e^{(2\delta-\kappa)\epsilon}}{e^{(2\delta-\kappa)\epsilon}-1}\,(\epsilon M_\epsilon^+)^{2\alpha}\, e^{(2\delta-\kappa)\epsilon M_\epsilon^+}\big)\ .$$

Since $\frac{e^{2\delta\epsilon}-1}{e^{(2\delta-\kappa)\epsilon}-1} = \mathrm{O}(1)$ as $\epsilon \to 0$, by Equation (11), since $n\tau \leqslant 2A$ as seen above, and since $\sqrt{\epsilon} = \frac{1}{N^{\gamma'}} \leqslant \frac{1}{\sqrt{N}}$, for N large enough, we have

$$\sum_{k=1}^{M_\epsilon^+} z_k^+ = \frac{N^{2\alpha}e^{2\delta N}}{e^{2\delta\epsilon}-1}\, e^{-2\delta n\tau}\Big(e^{\mathrm{O}(\epsilon)}(1-\frac{n\tau+\mathrm{O}(\tau)}{N})^{2\alpha}e^{\mathrm{O}(\tau)}\big(1+\mathrm{O}(\frac{\sqrt{\epsilon}}{\sqrt{N}})\big)$$
$$+\mathrm{O}(\frac{A}{N})+\frac{1}{\epsilon^2}\,\mathrm{O}(A\,e^{-\kappa N})\Big) = \frac{N^{2\alpha}e^{2\delta N}}{e^{2\delta\epsilon}-1}\, e^{-2\delta n\tau}\big(1+\mathrm{O}(\frac{A}{N})\big)\ .$$

By the definition (27) of ϵ and τ' (so that $\tau = \frac{1}{N^{\gamma'}}$), we have

$$1-e^{-\delta(\tau\pm\epsilon)} = \delta\tau(1\pm\frac{\epsilon}{\tau})(1+\mathrm{O}(\tau+\epsilon)) = \delta\tau\big(1+\mathrm{O}(\frac{1}{N^{\gamma'}})\big)\ ,$$

and

$$\frac{1-e^{-\delta\epsilon}}{e^{2\delta\epsilon}-1} = \frac{1}{2}(1+\mathrm{O}(\epsilon)) = \frac{1}{2}\big(1+\mathrm{O}(\frac{1}{N^{2\gamma'}})\big)\ .$$

Since $\gamma' \geqslant 1$, Equation (20) therefore becomes

$$a_{n,N}^+ = c^2\ N^{2\alpha}\ e^{2\delta N}\frac{1-e^{-\delta\epsilon}}{e^{2\delta\epsilon}-1}(1-e^{-\delta(\tau+\epsilon)})\,e^{-\delta n\tau}\ \big(1+\mathrm{O}(\frac{A}{N})\big)$$
$$= (c^2\ N^{2\alpha}\ e^{2\delta N})(\frac{\delta}{2}\ \tau\ e^{-\delta n\tau})\big(1+\mathrm{O}(\frac{A}{N})\big)\ .$$

As when $\alpha = 0$, computing similarly $a_{n,N}^-$, we have

$$a_{n,N} = (c^2\ N^{2\alpha}\ e^{2\delta N})(\frac{\delta}{2}\ \tau\ e^{-\delta n\tau})\big(1+\mathrm{O}(\frac{A}{N})\big)\ . \tag{28}$$

End of the proof of Theorem 3.1 when $\alpha \neq 0$. As when $\alpha = 0$, by Equations (15), (7) and (28), for N large enough, we have

$$\frac{\mu_N(f)}{\psi'(N)} = \sum_{n=1}^{\lceil\frac{A}{\tau'}\rceil}\big(f(n\tau')+\mathrm{O}(\tau'\,\|f'\|_\infty)\big)\,\frac{\delta}{2}\,\tau'\,e^{-\frac{\delta n\tau'}{\psi(N)}}\big(1+\mathrm{O}(\frac{A}{N})\big)\big(1+\mathrm{O}(e^{-\kappa N})\big)$$
$$= \Big(\sum_{n=1}^{\lceil\frac{A}{\tau'}\rceil} f(n\tau')\,\frac{\delta}{2}\,\tau'\,e^{-\frac{\delta n\tau'}{\psi(N)}}\Big)+\mathrm{O}(\frac{A^2}{N}\|f\|_\infty)+\mathrm{O}(A\,\tau'\,\|f'\|_\infty)\ . \tag{29}$$

When $\psi = 1$, since $\tau' = N^{-\gamma'}$ with $\gamma' \geqslant 1$ and $A \geqslant 1$, Equations (29) and (25) prove Theorem 3.1 when $\alpha \neq 0$ and $\psi = 1$.

When $\lim\limits_{+\infty}\psi = +\infty$, since $\lim_{N\to+\infty}\frac{\psi(N)}{N^{\gamma'-1}} = 0$ so that

$$\tau' = \frac{\psi(N)}{N^{\gamma'}} = \frac{1}{N}\frac{\psi(N)}{N^{\gamma'-1}} = \mathrm{O}\left(\frac{1}{\min\{N,\psi(N)\}}\right),$$

Equations (29) and (26) imply Theorem 3.1 when $\alpha \neq 0$ and $\lim\limits_{+\infty}\psi = +\infty$. □

Let us now prove the analogous result under Assumption (PA) when $\psi = 1$, following closely the scheme of proof of Theorem 3.1, and using the same notation. This is useful in order to deal with counting asymptotics that sometimes do not come with a known error term.

Theorem 3.3. *Let $\mathscr{E}$ be a locally finite subset of $[0,+\infty[$ endowed with a weight function ω, and let $\alpha\in\mathbb{R}$ and $c,\delta>0$. Assume that as $t\to+\infty$, we have*

$$\mathscr{N}_{\mathscr{E},\omega}(t) \sim c\, t^{\alpha}\, e^{\delta\, t}\,.$$

Then the family $\mathscr{F} = \big((F_N = \{x\in\mathscr{E} : x \leqslant N\})_{N\in\mathbb{N}},\ \omega\big)$ admits a pair correlation function for the scaling function $\psi = 1$ and renormalizing function $\psi' : N \mapsto \mathscr{N}_{\mathscr{E},\omega}(N)^2$, which is equal to $g_\delta : t \mapsto \frac{\delta}{2}\, e^{-\delta|t|}$.

Proof. In order to prove, as requested for the weak-star convergence, that $\frac{1}{\psi'(N)}\,\mathscr{R}_N^{\mathscr{F},\psi}(f)$ converges to $\int_{\mathbb{R}} f\, g_\delta\, dt$ for every continuous fonction f with compact support on $\mathbb{R}$, we may assume by density that f is of class C^1.

Since $\psi = 1$, analogously with the beginning of the proof of Theorem 3.1, as $N\to+\infty$, we have

$$\frac{1}{\psi'(N)} = \frac{1}{c^2\ N^{2\alpha}\, e^{2\delta N}}(1+\mathrm{o}(1))\,. \tag{30}$$

The first part of the proof is identical with the proof of Theorem 3.1 until Equation (19), and we will not repeat it here. In the same way, we are lead to study the quantities $a^{\pm}_{n,N}$ given by the equations (18) and (19). Here, we do not separate the treatment of the cases $\alpha = 0$ and $\alpha \neq 0$.

Asymptotics on $a^{\pm}_{n,N}$. Let $\sigma_{\pm} = \frac{1-\pm 1}{2}$. Note that $(k\epsilon)^{2\alpha}e^{2\delta k\epsilon}$ tends to $+\infty$ as $k\to+\infty$, in order to control the beginning of the following summations over k. Furthermore, since $\epsilon \leqslant 1 \leqslant A$ and $n\tau \leqslant n\tau' \leqslant A+1$, as $k\to+\infty$, we have $(1+\frac{n\tau+\sigma_{\pm}\epsilon}{k\epsilon})^{\alpha} = 1+\mathrm{O}(\frac{A}{k\epsilon})$. In the following estimates, except for $\mathrm{O}(\tau)$ taken as $\tau\to 0$ uniformly on everything else unless indicated by a subscript, the Landau functions O and o are taken as $N\to+\infty$, uniformly in $n\in\{1,\dots,\lceil\frac{A}{\tau'}\rceil\}$ up to Equation (32). By Equations (18) and (19), we hence have

$$\begin{aligned} a^{\pm}_{n,N} = \Big(\sum_{k=1}^{M^{\pm}_{\epsilon}} & c\,(k\epsilon)^{\alpha}\, e^{\delta k\epsilon}(1-e^{-\delta\epsilon}) \\ & \times c\,(n\tau+k\epsilon+\sigma_{\pm}\epsilon))^{\alpha}\, e^{\delta(n\tau+k\epsilon+\sigma_{\pm}\epsilon)}(1-e^{-\delta(\tau\pm\epsilon)})\Big)(1+\mathrm{o}_{\epsilon,\tau}(1)) \\ = c^2(1-e^{-\delta\epsilon})&(1-e^{-\delta(\tau\pm\epsilon)})\, e^{\delta n\tau}\Big(\sum_{k=1}^{M^{+}_{\epsilon}}(k\epsilon)^{2\alpha}e^{2\delta k\epsilon}\Big)(1+\mathrm{o}_{\epsilon,\tau,A}(1))\quad. \end{aligned} \tag{31}$$

By Lemma 2.1 with $M = M_\epsilon^\pm$ which goes to $+\infty$ as $N \to +\infty$ when ϵ is fixed, and $a_M = e^{2\delta\epsilon}$ which is constant when ϵ is fixed, by Equation (11), since $\epsilon \leqslant \frac{\tau}{2}$ and since $(1 - \frac{n\tau + \mathrm{O}(\tau)}{N})^\alpha = 1 + \mathrm{O}(\frac{A}{N})$, as $N \to +\infty$, we have

$$\begin{aligned}\sum_{k=1}^{M_\epsilon^+} (k\epsilon)^{2\alpha} e^{2\delta k\epsilon} &= \frac{e^{2\delta\epsilon}}{e^{2\delta\epsilon}-1}\,(\epsilon M_\epsilon^+)^{2\alpha}\, e^{2\delta\,\epsilon M_\epsilon^+}(1+\mathrm{o}_{\epsilon,\tau}(1))\\ &= \frac{e^{\mathrm{O}(\tau)}}{e^{2\delta\epsilon}-1}\,N^{2\alpha}\,e^{2\delta(N-n\tau)}(1+\mathrm{o}_{\epsilon,\tau,A}(1))\ .\end{aligned}$$

Therefore Equation (31) becomes

$$a_{n,N}^\pm = c^2\,N^{2\alpha}\,e^{2\delta N}\,e^{\mathrm{O}(\tau)}\,\frac{1-e^{-\delta\epsilon}}{e^{2\delta\epsilon}-1}(1-e^{-\delta(\tau\pm\epsilon)})\,e^{-\delta n\tau}(1+\mathrm{o}_{\epsilon,\tau,A}(1))\ . \qquad (32)$$

End of the proof. By Equations (15) and (30), since $a_{n,N}^- \leqslant a_{n,N} \leqslant a_{n,N}^+$ and $\tau' = \tau$, we have

$$\limsup_{N\to+\infty}\frac{\mu_N(f)}{\psi'(N)} \leqslant \sum_{n=1}^{\lceil \frac{A}{\tau'}\rceil} \frac{1-e^{-\delta\epsilon}}{e^{2\delta\epsilon}-1} e^{\mathrm{O}(\tau')}\,\big(f(n\tau')+\mathrm{O}_f(\tau')\big)\,(1-e^{-\delta(\tau'+\epsilon)})\,e^{-\delta n\tau'}\ .$$

By taking the limit as $\epsilon \to 0$, we then have

$$\limsup_{N\to+\infty}\frac{\mu_N(f)}{\psi'(N)} \leqslant \sum_{n=1}^{\lceil \frac{A}{\tau'}\rceil} e^{\mathrm{O}(\tau')}\,\big(f(n\tau')+\mathrm{O}_f(\tau')\big)\,\frac{1-e^{-\delta\tau'}}{2}\,e^{-\delta n\tau'}\ .$$

Since $1-e^{-\delta\tau'} \sim \delta\tau'$ as $\tau' \to 0$, by taking the limit as $\tau' \to 0$ and by a Riemann sum argument, we have

$$\limsup_{N\to+\infty}\frac{\mu_N(f)}{\psi'(N)} \leqslant \int_0^A f(t)\,g_\delta(t)\,dt\ .$$

A similar computation gives

$$\liminf_{N\to+\infty}\frac{\mu_N(f)}{\psi'(N)} \geqslant \int_0^A f(t)\,g_\delta(t)\,dt\ ,$$

which proves Theorem 3.3. □

4 Geometric applications

In this section, we apply the Theorems 3.3 and 3.1 to the sets (with multiplicities) of the lengths of closed geodesics and common perpendiculars in negatively curved spaces, and to other discrete sets with similar growth properties that arise in geometry and dynamics. We assume familiarity with geometry and ergodic theory in negative curvature, and we refer, for instance, to [BPP] for more background and for definitions of the various objects below.

Let X be either a proper $\mathbb{R}$-tree without terminal points or a complete simply connected Riemannian manifold with pinched negative curvature at most -1. Let Γ be a nonelementary discrete group of isometries of X. Assume that the critical exponent δ_Γ of Γ is finite. Let $\mathscr{D}^\pm = (D_k^\pm)_{k\in I^\pm}$ be locally finite Γ-equivariant families of nonempty proper closed convex subsets of X. Assume that the outer and inner skinning measures $\sigma^\pm_{\mathscr{D}^\mp}$ of the families $\mathscr{D}^\mp$ are finite and nonzero, and that the Bowen-Margulis measure $m_{\rm BM}$ is finite and mixing for the geodesic flow on the space $\Gamma\backslash\mathscr{G}X$ of geodesic lines of X modulo Γ.

A *common perpendicular* from $\pi(D_k^-)$ to $\pi(D_j^+)$ is a locally geodesic path γ in $\Gamma\backslash X$ starting perpendicularly from $\pi(D_k^-)$ and arriving perpendicularly to $\pi(D_j^+)$. For every $t > 0$, we denote by $\operatorname{Perp}(\mathscr{D}^-, \mathscr{D}^+, t)$ the (locally finite) set of lengths $\ell(\gamma)$ at most t of common perpendiculars γ from elements of $\pi(D^-)$ to elements of $\pi(D^+)$ (considered with multiplicities). Let $\operatorname{Perp} = (\operatorname{Perp}(\mathscr{D}^-, \mathscr{D}^+, N))_{N\in\mathbb{N}}$.

For every $t > 0$, we denote by $\operatorname{Geod}(t)$ the (locally finite) set of lengths at most t of primitive closed geodesics in $\Gamma\backslash X$ (considered with multiplicities). If Γ is furthermore assumed to be geometrically finite, let $\operatorname{Geod} = (\operatorname{Geod}(N))_{N\in\mathbb{N}}$.

We refer to Remark (3) at the beginning of Section 3 for the use of sets with multiplicities in order to compute pair correlations.

Corollary 4.1. *Let X, Γ and $\mathscr{D}^\pm$ be as above. Then the families* Perp *and* Geod *admit pair correlation functions $g_{\rm Perp}$ and $g_{\rm Geod}$ for the scaling function $\psi = 1$ (and renormalisation to probability measures) with*

$$g_{\rm Perp} = g_{\rm Geod} : t \mapsto \frac{\delta_\Gamma}{2}\, e^{-\delta_\Gamma\,|t|}\ .$$

Proof. By [BPP, Thm. 1.5], the number of common perpendiculars with length at most t (counted with multiplicities) is asymptotic with $\frac{\|\sigma^+_{\mathscr{D}^-}\|\,\|\sigma^-_{\mathscr{D}^+}\|}{\delta_\Gamma\,\|m_{\rm BM}\|}\, e^{\delta_\Gamma\, t}$. If Γ is furthermore assumed to be geometrically finite, by [PPS, Cor. 1.7] and [BPP, Cor. 13.5(1)], the number of primitive closed geodesics with length at most t (counted with multiplicities) is asymptotic with $\frac{e^{\delta_\Gamma t}}{\delta_\Gamma\, t}$ as $t \to +\infty$. The claim follows from Theorem 3.3 with constants respectively $(c = \frac{\|\sigma^+_{\mathscr{D}^-}\|\,\|\sigma^-_{\mathscr{D}^+}\|}{\delta_\Gamma\,\|m_{\rm BM}\|}, \alpha = 0, \delta = \delta_\Gamma)$ and $(c = \frac{1}{\delta_\Gamma}, \alpha = -1, \delta = \delta_\Gamma)$. □

Remarks. (1) Refering to [PPS] and [BPP] for the terminology, when $\widetilde{F}$ is a bounded Γ-invariant potential on $\Gamma\backslash T^1X$ which is Hölder-continuous if X is a manifold, assuming the pressure of $\widetilde{F}$ to be positive and finite, the Gibbs measure on $\Gamma\backslash\mathscr{G}X$ for $\widetilde{F}$ to be finite and mixing for the geodesic flow, and the outer and inner skinning measures of the families $\mathscr{D}^\mp$ for the potential $\widetilde{F}$ to be finite and nonzero, then the same statement as Corollary 4.1 is satisfied when Perp and Geod are endowed with weights defined by the potential as in [BPP, §1.2].

(2) The assumptions of Corollary 4.1 are satisfied, as a very special case, when X is a real, complex or quaternionic hyperbolic symmetric space with finite covolume under Γ, and the images of the elements of $\mathscr{D}^\pm$ in $\Gamma\backslash X$ are points, finite volume totally geodesic submanifolds or Margulis cusp neighbourhoods, see [PP1, Cor. 21], [PP2, Theo. 3], [PP3, Thm. 8·1]. For instance, if $x \in X$ and $\mathscr{D}^- = \mathscr{D}^+ = \Gamma x = \{\gamma x : \gamma \in \Gamma\}$, then $\operatorname{Perp}(\mathscr{D}^-, \mathscr{D}^+, t) = \{d(x, \gamma x) : \gamma \in \Gamma\} \cap [0, t]$, and the number of common perpendiculars of length at most N (counted with multiplicities) is given by the growth function of the orbit Γx.

(3) In [PTV], Peigné, Tapie and Vidotto construct for all $1 < \alpha < 2$ examples of complete simply connected Riemannian manifolds X with pinched negative sectional curvature

and geometrically finite convergent groups Γ of isometries of X such that the growth function of the orbit of any point $x \in X$ is asymptotic with $t \mapsto C\, t^{\alpha} e^{\delta_\Gamma t}$ for some $C > 0$. Theorem 3.3 implies that, also in this case, the family Perp for $\mathscr{D}^- = \mathscr{D}^+ = \Gamma x$ admits a pair correlation function $g_{\rm Perp}$ for the scaling function $\psi = 1$ (and renormalisation to probability measures), given by $g_{\rm Perp} : t \mapsto \frac{\delta_\Gamma}{2}\, e^{-\delta_\Gamma |t|}$ as in Corollary 4.1.

(4) Discrete sets with growth functions for which Theorem 3.3 can be applied to prove analogs of Corollary 4.1 arise in many important dynamical systems. To name some notable ones, Parry and Pollicott [PPo] proved that the number of lengths at most t of closed orbits of Axiom A flows on compact manifolds (counted with multiplicities) is asymptotic with $t \mapsto \frac{e^{ht}}{ht}$ with h the topological entropy of the flow, and Eskin and Mirzakhani [EM] proved the analogous behaviour for the lengths of closed Teichmüller geodesics in the moduli space of closed Riemann surfaces of genus g. Athreya, Bufetov, Eskin and Mirzakhani [ABEM] proved the exponential growth of orbits of the mapping class group in the Teichmüller space of closed Riemann surfaces of genus g.

Under additional assumptions, the asymptotic behaviour of counting functions used in the proof of Corollary 4.1 comes with an error term required for an application of Theorem 3.1.

Corollary 4.2. *Let X, Γ and $\mathscr{D}^\pm$ be as in the beginning of Section 4. Assume that $\Gamma\backslash X$ is a compact Riemannian manifold and $m_{\rm BM}$ is exponentially mixing under the geodesic flow for the Hölder regularity, or that $\Gamma\backslash X$ is a locally symmetric space, the boundary of $D_k^\pm$ is smooth, $m_{\rm BM}$ is finite, smooth, and exponentially mixing under the geodesic flow for the Sobolev regularity. Assume that the strong stable/unstable ball masses by the conditionals of $m_{\rm BM}$ are Hölder-continuous in their radius.*

Let $\psi : \mathbb{N} \to [1+\infty[$ be an at most polynomially growing scaling function, and let $\psi' : N \mapsto \frac{{\rm Card}({\rm Perp}(\mathscr{D}^-,\mathscr{D}^+,N))^2}{\psi(N)}$ be the associated renormalizing function. Then the family Perp *has a pair correlation function $g_{{\rm Perp},1} : t \mapsto \frac{\delta_\Gamma}{2}\, e^{-\,\delta_\Gamma |t|}$ if $\psi = 1$, and has Poissonian behaviour with $g_{{\rm Perp},\psi} = \frac{\delta_\Gamma}{2}$ if $\lim\limits_{+\infty} \psi = \infty$, with error terms as in Theorem 3.1.*

Proof. By [BPP, Thm. 1.8 (2)], the family Perp of common perpendiculars has exponential growth $C\, e^{\delta_\Gamma t}(1 + {\rm O}(e^{\kappa t}))$ for some $\kappa > 0$. Thus, Theorem 3.1 implies the claim. □

The geodesic flow is known to have exponential decay of Hölder correlations for compact manifolds $M = \Gamma\backslash\widetilde{M}$ when M is two-dimensional by [Dol], M is 1/9-pinched [GLP, Coro. 2.7], and when M is locally symmetric by [Sto]. When X is a symmetric space and Γ is an arithmetic lattice, the geodesic flow has exponential decay of Sobolev correlations by for some $\ell \in \mathbb{N}$ by [KM1, Theorem 2.4.5], with the help of [Clo, Theorem 3.1] to check its spectral gap property, and of [KM2, Lemma 3.1] to deal with finite cover problems. See also [MO, LT].

Corollary 4.2 also has generalisations when the lengths are weighted by potentials. See, for instance, the introduction of [BPP] for counting results in this generality.

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Department of Mathematics and Statistics, P.O. Box 35
40014 University of Jyväskylä, FINLAND.
e-mail: jouni.t.parkkonen@jyu.fi

Laboratoire de mathématique d'Orsay, UMR 8628 CNRS,
Bâtiment 307, Université Paris-Saclay,
91405 ORSAY Cedex, FRANCE
e-mail: frederic.paulin@universite-paris-saclay.fr