On the critical exponent of a discrete group of hyperbolic isometries

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Abstract: Generalizing work of Bishop-Jones in the constant curvature case, we prove that if M is a complete Riemannian manifold with pinched negative curvature and with $\pi_1 M$ not virtually nilpotent, that the visual dimension of the set of geodesic rays , starting from a fixed base point, that are recurrent in some compact subset of M, is equal to the critical exponent of $\pi_1 M$. Similarly, the critical exponent of an infinite subgroup H of a given word hyperbolic group equals the visual dimension of the conical limit set of H.

Keywords: Word hyperbolic group, negative curvature, discrete groups, conical limit set, critical exponent, visual dimension.

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For a finitely generated group G, endowed with a word metric d, M. Gromov [Gro] defined a geometric notion, to be *word hyperbolic*. It is satisfied by a large class of groups, in particular by the fundamental groups of compact negatively curved manifolds, and captures from them their synthetic properties. The following result is a contribution to the study of the "dimension at infinity" of subgroups (possibly infinitely generated or with torsion) of G.

Theorem 0.1 *Let G be a word hyperbolic group and H be an infinite subgroup of G. The critical exponent of H acting on G equals the visual dimension of the conical limit set of H.*

If X is a hyperbolic space in Gromov's sense, pointed in x_0, and H is a discrete group of isometries of X, the *critical exponent* $\delta(H)$ of H is the infimum of the $s > 0$ such that the *Poincaré series* $\sum_{h\in H} e^{-sd(x_0,hx_0)}$ converges (this being independant of x_0).

If ∂X is the boundary (or space at infinity) of X, then the *conical limit set* $\Lambda_c H$ of H is the set of points a in ∂X such that there is a sequence of points in Hx_0 converging towards a while staying at bounded distance from a geodesic ray ending in a.

The *visual measure* on the boundary of X (see [Pau] Section 6.1) is a natural class of measures, generalizing the Hausdorff measure of the visual metric (à la Gromov) at infinity for a simply connected negatively curved manifold.

This theorem as been proved in the case of a discrete subgroup of the group of isometries of a (non compact) symmetric space of rank one by C. Bishop-P. Jones [BJ] (to which we refer for historical remarks). Our proof is a rather straihtforward generalization of the one in [BJ].

Note nevertheless that the following results are proved alike:

Theorem 0.2 *Let M be a complete Riemannian manifold, with sectional curvature $K \leq -a$ where $a > 0$, with $\pi_1 M$ having no global fixed point on the boundary of the universal* *cover $\tilde{M}$, and let $x \in M$. Then the visual dimension of the set of geodesic rays starting at x, that are recurrent in some compact subset, equals the critical exponent of $\pi_1 M$.*

Note that in pinched negative curvature, or if π_1 is finitely generated, the hypothesis on $\pi_1 M$ is equivalent to the requirement that $\pi_1 M$ is not virtually nilpotent, by [Bow].

Theorem 0.3 *Let T be a localy finite tree, having a discrete cocompact group of automorphisms. Let H be a discrete group of isometries of T, without a global fixed point on the boundary. Then the critical exponent of H equals the Hausdorff dimension of the conical limit set.*

This is in particular the case if H is a discrete subgroup of $\underline{G}(K)$, with G a simple algebraic group, of relative rank one, on a local field K with finite residual field, acting on its Bruhat-Tits building (see [Ser] page 124), as $\mathrm{SL}_2(\mathbb{Q}_p)$ acts on the regular tree of valence $p+1$.

1 Visual dimension

We use the notations of [GH], to which we refer for the background on hyperbolic spaces and groups. In what follows, X will denote a δ-hyperbolic space.

Let us recall the definition of the Gromov product

$$(x|y)_z = \frac{1}{2}(d(x,z) + d(y,z) - d(x,y)) \geq 0$$

for points x, y, z in X, and

$$(a|b)_z = \sup_{x_i \to a, y_j \to b} \liminf_{i,j \to +\infty} (x_i|y_j)_z$$

if $z \in X$ and $a, b \in X \cup \partial X$ (which does coincide with the first definition if $a, b \in X$). The usefullness of taking the least upper bound, on the sequences (x_i) tending to a and (y_j) tending to b, is to get an intrinsic definition (though the result depends on these sequences only up to a few δ's). In particular, if g is an isometry of X, then $(ga|gb)_{gz} = (a|b)_z$. Moreover

$$|(a|b)_z - (a|b)_{z'}| \leq d(z, z').$$

Let x be a base point in X, a be in ∂X and r be in $[0, +\infty[$. Define

$$\beta_x(a,r) = \{b \in \partial X \ / \ (a|b)_x \geq -\log r\}.$$

We call $\beta_x(a,r)$ ($\beta(a,r)$ by abuse) the *ball* in ∂X of *centre* a and *radius* r. See [Pau] Section 6.2 for the relationship with the visual metrics.

For every Borel subset E in ∂X and every $s > 0, \eta > 0$, define

$$\mathcal{H}^s_{\eta,x}(E) = \inf \sum_i (r_i)^s$$

where the infimum is taken on all coverings $\{\beta_x(a_i, r_i)\}_i$, finite or countable, of E by balls of radius $r_i \leq \eta$.

The *visual measure of dimension* s of E is then defined by

$$\mathcal{H}^s_x(E) = \lim_{\eta\to 0} \mathcal{H}^s_{\eta,x}(E).$$

As in the classic case (see [Fed]), this limit exists, and there exists a unique σ in $[0,+\infty]$ such that $\mathcal{H}^s_x(E) = +\infty$ if $0 \le s < \sigma$ and $\mathcal{H}^s_x(E) = 0$ if $\sigma < s \le +\infty$. We call $\sigma = \dim_{\text{vis}}(E)$ the *visual dimension* de E.

It is proved ([Pau] Section 6.2) that the visual dimension of E does not depend on x. Moreover (loc. cit.), if s is not zero and not infinity, then the (positive borelian) measures $\mathcal{H}^s_x$ and $\mathcal{H}^s_y$ on ∂X are absolutely continuous one with respect to the other, with

$$e^{-sd(x,y)} \le \frac{d\,\mathcal{H}^s_x}{d\,\mathcal{H}^s_y} \le e^{sd(x,y)}.$$

Here is a classic criterion (said to be Frostman's) to give a lower bound on the visual dimension.

Lemma 1.1 *Let E be a Borel subset of ∂X. Assume that there exist $K > 0$ and a positive non zero measure μ on E such that $\mu(\beta(a,r)\cap E) \le Kr^s$. Then the visual dimension of E is at least s.*

Proof. For any covering of E by balls $\beta(a_i, r_i)$, we have

$$\sum_i r_i^s \ge K^{-1}\sum_i \mu(\beta(a_i,r_i)) \ge K^{-1}\mu(E).$$

Passing to the limit, we get $\mathcal{H}^s_x(E) \ge K^{-1}\mu(E) > 0$, which proves the result. □

2 Critical exponent and visual dimension

We assume that X is geodesic, proper (see [GH]) and that there exists $\delta' > 0$ such that for every x, y dans X, there exists a geodesic ray starting from x passing at distance less than δ' from y. We call the point at infinity of this ray a *boundary projection* of y. This last assumption is for instance satisfied if X admits a discrete cocompact group of isometries (for instance if X is a rank one symmetric space, or a Cayley graph of some word hyperbolic group, or a regular tree T_p ou biregular tree $T_{p,q}$). Up to increasing δ, we will assume $\delta = \delta'$.

Theorem 2.1 *If G is a discrete group of isometries of X, with no global fixed point on the boundary, then*

$$\delta(G) = \dim_{\text{vis}}(\Lambda_c(G)).$$

The three theorems in the introduction follow easily from this one. The next proposition is the easy inequality. Its proof is essentially the same as Sullivan's one [Sul] page 192 for X the real hyperbolic n-space.

Proposition 2.2

$$\delta(G) \ge \dim_{\text{vis}}(\Lambda_c(G)).$$

Proof. Fix $R > 0$. For $x, y \in X$, define $\mathcal{O}_x(y, R)$ the *shadow* viewed from x of the ball of centre y and radius R, i.e. the set of points a in the boundary such that there exists a ray between x and a passing through this ball.

For every g in G, choose a boundary projection a_g of gx. Since X is hyperbolic, it is easy to see that there exists a universal constant $C_1 > 0$ such that the shadow $\mathcal{O}_x(gx, R)$ is contained in the ball $\beta_x(a_g, r_g)$ of radius satisfying $-\log r_g \geq d(x, gx) - R - C_1\delta$.

Denote by E_R the set of boundary points that are in infinitely many shadows $\mathcal{O}_x(gx, R)$ for g's in G. Then if g lives outside a finite subset F of G, the balls $\beta_x(a_g, r_g)$ make a covering of E_R with small radii. If $s > \delta(G)$, the visual s-measure of E_R is finite, since

$$\sum_{g \in G-F} r_g^s \leq e^{s(R+C_1\delta)} \sum_{g \in G-F} e^{-sd(x,gx)} < +\infty.$$

As any point of the conical limite is in some E_R for some (integer) R, this proves the result. □

The difficult part lies in the reverse inequality.

Proposition 2.3 *If G has no global fixed point on ∂X, then*

$$\delta(G) \leq \dim_{\text{vis}}(\Lambda_c(G)).$$

Proof. Fix $\epsilon > 0$. Set $s = \delta(G) - 2\epsilon$. Fix a boundary projection a_x of every point x in the orbit Gx_0. Define $\mathcal{B}(a, r) = \{b \in X \cup \partial X \,/\, (a|b)_{x_0} \geq -\log r\}$, whose set at infinity is $\beta(a, r)$.

We will construct $C, D > 0$ and a subset $T = \coprod_{n \in \mathbb{N}} T_n$ of the orbit Gx_0 of G which is the set of vertices of a tree with root x_0, every element x in T_n having a unique parent in T_{n-1} and a finite set of children $T(x)$ in T_{n+1}, satisfying the following conditions:

1. if y is a child of x, then $y \in \mathcal{B}(a_x, Ce^{-d(x_0,x)})$
2. if y is a child of x, then $4C \leq d(x_0, y) - d(x_0, x) \leq D$
3. if y, z are distinct children of x, then $\mathcal{B}(a_y, 2Ce^{-d(x_0,y)})$ and $\mathcal{B}(a_z, 2Ce^{-d(x_0,z)})$ are disjoint
4. $\sum_{y \in T(x)} e^{-sd(x_0,y)} \geq e^{-sd(x_0,x)}$.

We will then prove that the set ∂T of accumulation points of T in the boundary is contained in the conical limit set, and has visual dimension at least s.

Since X is proper, ∂X is compact. The complementary subset of a neighborhood of the boundary in $X \cup \partial X$ contains only finitely many points of the orbit of x under G. Since $\mathcal{B}(a, r)$ for $r > 0$ is a neighborhood of a, and since $s + \epsilon > \delta(G)$, it follows by compactness that there exists a point a_1 in ∂X such that for every $r > 0$

$$\sum_{g \in G \,/\, gx_0 \in \mathcal{B}(a_1, r)} e^{-(s+\epsilon)d(x_0, gx_0)} = \infty. \qquad (*)$$

Since G does not fix globally a_1, choose $\gamma \in G$ such that $a_2 = \gamma a_1 \neq a_1$. Let $r > 0$ sufficiently small and $C > 0$ sufficiently large compared to $1, \delta$, and such that

$(x|y)_{x_0} \leq \frac{C}{2}$ for every x, y in respectively $B_1 = \mathcal{B}(a_1, r), B_2 = \mathcal{B}(a_2, r)$. Since B_2 contains $\gamma\mathcal{B}(a_1, re^{-d(x_0,\gamma x_0)})$, the equation (*) with a_2 remplacing a_1 is still valid.

Define A_n to be the annulus of the $x \in X$ such that $n \leq d(x_0, x) < n+1$. For $\ell = 1, 2$, let us prove that $\limsup_{n\to\infty} \sum_{g\,/\,gx_0 \in B_\ell \cap A_n} e^{-sd(x_0,gx_0)} = \infty$. Otherwise, there would exist an $N > 0$ such that for every n the above sum would have N as an upper bound. But then

$$\sum_{g\,/\,gx_0 \in B_\ell} e^{-(s+\epsilon)d(x_0,gx_0)} = \sum_{n\in\mathbb{N}} \sum_{g\,/\,gx_0 \in B_\ell \cap A_n} e^{-\epsilon d(x_0,gx_0)} e^{-sd(x_0,gx_0)} \leq N \sum_{n\in\mathbb{N}} e^{-\epsilon n} < \infty$$

which contradicts (*). Hence,

$$\limsup_{n\to\infty}\ e^{-ns} \text{ Card } \{g\,/\,gx_0 \in B_\ell \cap A_n\} = \infty.$$

There exists a universal constant $C_2 > 0$ such that for every n big enough, if x, y belongs to A_n with $d(x, y) \geq M = 2\log(3C) + C_2(\delta+1)$, then $\mathcal{B}(a_x, 3Ce^{-d(x_0,x)})$, $\mathcal{B}(a_y, 3Ce^{-d(x_0,y)})$ are disjoint.

Since G is discrete, the orbit Gx_0 is the union of finitely many subsets such that two distinct points in each subset are at distance at least M.

For $\ell = 1, 2$, there exists therefore, for n_ℓ that might be choosen arbitrarily big, a (finite) subset V_ℓ of the orbit, contained in $\mathcal{B}_\ell \cap A_{n_\ell}$ such that

i) $\mathcal{B}(a_x, 3Ce^{-d(x_0,x)})$ and $\mathcal{B}(a_y, 3Ce^{-d(x_0,y)})$ are disjoint for any two distinct points x, y in V_ℓ,

ii) $\sum_{g\,/\,gx_0 \in V_\ell} e^{-sd(x_0,gx_0)} \geq e^{-sn_\ell}$ Card $\{g\,/\,gx_0 \in V_\ell\} \geq e^{sC}|G_{x_0}|$, where $|G_{x_0}|$ is the cardinal of the (finite) stabilizer of x_0 in G.

We will assume that $n_\ell \geq 6C$, and we set $D = \sup\{n_1, n_2\} + 1$.

Let us now define $T = \coprod_{n\in\mathbb{N}} T_n$ by induction. Set $T_0 = \{x_0\}$ and (for example) $T_1 = V_1$. The conditions (1)-(4) are clearly satisfied at level 1.

Assume that T_n is constructed for $n \geq 1$ and consider $x = gx_0 \in T_n$. Recall that by construction, two geodesic segments starting at x_0, ending respectively in B_1, B_2, follow each other at a distance less than or equal to 8δ for at most $\frac{C}{2} + C_3\delta$ with C_3 a universal constant. By condition (2) applied at level n, $d(x, x_0) \geq 2C$. Hence if C is big enough with respect to δ, then two geodesic segments starting at x, the first one going to x_0, the other to a point in one of gB_1, gB_2, say gB_1, are following each other at distance less than 4δ for at most $\frac{C}{2} + C_4\delta$ (with $C_4 > 0$ a universal constant). Then set $T(x) = gV_1$ and $T_{n+1} = \bigcup_{x\in T_n} T(x)$.

Let us check the above conditions (1)-(4). By construction, a child y of x belongs to $\mathcal{B}(a_x, Ce^{-d(x_0,x)}))$. Since G acts by isometries, $d(x_0, y) - d(x_0, x) \leq d(y, x) \leq n_1 + 1 \leq D$. Since two geodesic segments between respectively x, x_0 and x, y follow each other closer than 4δ, for at most a distance $\frac{C}{2} + C_4\delta$, we get $d(x_0, y) - d(x_0, x) \geq n_1 - 2C \geq 4C$.

Recall that gV_1 is far (with respect to δ) from every geodesic segment between x and x_0, and that C is big compared to δ. Hence $\mathcal{B}(a_y, 2Ce^{-d(x_0,y)})$ and $\mathcal{B}(a_z, 2Ce^{-d(x_0,z)})$, for y, z distinct children of x, have their images by g^{-1} at distance bounded by $C_5\delta$ (with $C_5 > 0$ a universal constant), from respectively $\mathcal{B}(a_{g^{-1}y}, 2Ce^{-d(x_0,g^{-1}y)})$, $\mathcal{B}(a_{g^{-1}z}, 2Ce^{-d(x_0,g^{-1}z)})$. Hence $\mathcal{B}(a_y, 2Ce^{-d(x_0,y)})$, $\mathcal{B}(a_z, 2Ce^{-d(x_0,z)})$ are distinct by i).

Moreover, according to ii),

$$\sum_{y\in T(x)} e^{-sd(x_0,y)} \geq (\text{ Card } V_1)e^{-s(n_1+1)}e^{-sd(x_0,x)} \geq e^{-sd(x_0,x)}.$$

Let us now use the tree T to end the proof. Every point of ∂T is the limit of a sequence of points $(x_n)_{n\in\mathbb{N}}$ with x_{n+1} a child of x_n. But it follows from conditions (1), (2) that this sequence is a quasi-geodesic (by the criterion page 92 of [GH]), hence stays at bounded distance from a geodesic ray ([GH] page 101). Therefore ∂T is contained in the conical limit set.

Let us prove that the visual dimension of ∂T is at least s. Let us construct a measure μ as in Lemma 1.1 for $E = \partial T$.

Set $r_x = Ce^{-d(x_0,x)}$ and $\beta(x) = \beta(a_x, r_x)$. For x in T_n, the balls $\beta(a_x, 2r_x)$ are disjoint, by (3) and since by (1),(2), if y is a child of x, then $\beta(y, 2r_y) \subset \beta(x, 2r_x)$. If $E_n = \coprod_{x\in T_n} \beta(x)$, then $\partial T = \bigcap_n E_n$ by (1).

Let us define a probability measure μ with support in E by setting $\mu(E_0) = 1$ and if y is a child of x, then

$$\mu(\beta(y)) = \frac{e^{-sd(x_0,y)}}{\sum_{z\in T(x)} e^{-sd(x_0,z)}}\mu(\beta(x)).$$

It is clear by induction and by (4) that

$$\mu(\beta(x)) \leq e^{-sd(x_0,x)}. \qquad (**)$$

Let us prove that a similar inequality is still valid for every ball $\beta = \beta(a,t)$. Let $\beta(y)$ be a lowest generation ball meeting β such that $\beta(a_y, 2r_y)$ does not contain β. (Such a ball exists if β meets ∂T.) Let x be the parent of y. By minimality, we get $\beta \subset \beta(a_x, 2r_x)$. Since $\beta(a_x, 2r_x)$ is disjoint from the balls $\beta(z)$ of the same generation as x, we get

$$\mu(\beta) \leq \mu(\beta(a_x, 2r_x)) = \mu(\beta(x)).$$

Hence by (2) and (**),

$$\mu(\beta) \leq e^{sD}e^{-sd(x_0,y)} = C^{-s}e^{sD}r_y^s.$$

Recall that $\beta = \beta(a,t)$ meets, say in b, the ball $\beta(y)$, as well as, say in c, the complementary subset of $\beta(a_y, 2r_y)$. Assume that $\log 2$ is big enough compared to δ (which may be assumed up to multiplying the metric of X by a very small constant, which multiplies the critical exponent and the visual dimension by the same factor). Therefore $t \geq Kr_y$ with $K > 0$ a constant depending only on δ. Indeed,

$$(b|c)_{x_0} \geq \inf\{(a|b)_{x_0}, (a|c)_{x_0}\} - \delta \geq -\log t\ - \delta.$$

And if δ is small compared to $\log 2$, then $(b|c)_{x_0}$ is less than $-\log r_y - \log 2 + C_6\delta$ (with $C_6 > 0$ a universal constant).

Hence

$$\mu(\beta) \leq (CK)^{-s}e^{sD}t^s,$$

which is what had to be proved. □

If the group $\text{Isom}(X)$ of isometries of X is endowed with the compact-open topology, if Γ is a finitely generated group (with the discrete topology), the space of morphisms from

Γ to $\mathrm{Isom}(X)$ is endowed with the compact-open topology (called the *algebraic topology*). Since as in [BJ], one can deduce from 2.3 (the conditions i), ii) requiring the control of only finitely many elements) that $\delta(G)$ is lower semi-continuous for the algebraic topology.

Theorem 2.4 *If $\phi_n : \Gamma \to \mathrm{Isom}(X)$ converges algebraically to ϕ with $\phi_n(\Gamma), \phi(\Gamma)$ discrete and $\phi(\Gamma)$ having no global fixed point at infinity, then*

$$\delta(\phi(\Gamma)) \leq \liminf_{n\to\infty} \delta(\phi_n(\Gamma)).$$

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