Farey neighbours, modular symbols and divergent geodesics

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Abstract

We give effective asymptotic counting results for pairs of Farey neighbours and for modular symbols in $\mathbb{Q}$, in imaginary quadratic number fields and in definite quaternion algebras over $\mathbb{Q}$, using the distribution of common perpendiculars between Margulis cusp neighbourhoods and divergent geodesics in hyperbolic manifolds. We describe the tangency properties of the canonical Margulis cusp neighbourhoods in Bianchi hyperbolic 3-orbifolds. [1]

1 Introduction

In this paper, we present effective asymptotic counting results for pairs of Farey neighbours in $\mathbb{Q}$, in imaginary quadratic number fields and in definite quaternion algebras over $\mathbb{Q}$, when the lower bound on the distances between the Farey neighbours shrinks to 0. These results appear to be new even in the classical rational case. They are contributions to the study of the distribution of pairs of the well known Farey fractions and their generalisations, see for instance [Hal, HaT, Hay, BZ, Mar1, Mar2, Mar3, Ath, Hee, BS, Lut, PP4, Say], and of modular symbols of Shimura, Eichler, Birch, Manin, see for instance [Man1, Cre, Man2, McM].

Let K be either $\mathbb{Q}$ or an imaginary quadratic number field, with ring of integers $\mathscr{O}_K$, discriminant D_K, class number h_K and Dedekind zeta function ζ_K. Recall that two elements $\alpha, \beta \in \mathbb{P}^1(K) = K \cup \{\infty\}$ are *Farey neighbours* if there exists $p, q, r, s \in \mathscr{O}_K$ with $\alpha = \frac{p}{q}$, $\beta = \frac{r}{s}$ and

$$|\, ps - qr \,| = 1 \tag{1}$$

or, equivalently, $ps - qr \in \mathscr{O}_K^\times$. The Diophantine equation $ps - qr = 1$ with integral unknowns p, q, r, s is called the *gcd equation*. See for instance [HN] for other distribution results of solutions to the gcd equation, and for instance [Sch, Duk, EMV, AES2, HK] for higher dimensional generalisations pioneered by Linnik and Maass.

When the class number h_K of K is greater than 1, there are infinitely many elements of $\mathbb{P}^1(K)$ that do not have Farey neighbours. In Section 3, we discuss a notion of generalized Farey neighbours due to Bestvina-Savin [BeS], that geometrically corresponds to the tangency in the real hyperbolic 3-space $\mathbb{H}^3_\mathbb{R}$ of Mendoza's canonical horoballs [Men] centered at two points of $\mathbb{P}^1(K)$. We prove in Theorem 12 the existence of generalized Farey neighbours of every element of $\mathbb{P}^1(K)$, extending [For, Theo. 2] when $K = \mathbb{Q}$.

[1] **Keywords:** Farey neighbours, divergent geodesics, common perpendiculars, hyperbolic spaces, counting, imaginary quadratic number field, modular symbols, Bianchi groups, quaternion algebra. **AMS codes:** 11B57, 20H10, 11N45, 53C22, 11R04, 22E40.

Let $\mathfrak{N}_K$ be the set of unordered pairs of Farey neighbours in K. The additive group $\mathscr{O}_K$ acts by simultaneous translations on the set $\mathfrak{N}_K$.

Theorem 1. *(1) As $\epsilon > 0$ tends to 0, we have*

$$\operatorname{Card}\big\{\{\alpha,\beta\} \in \mathfrak{N}_{\mathbb{Q}} : \alpha,\beta \in [0,1]\,,\ |\beta-\alpha| \geqslant \epsilon\big\} = -\frac{6}{\pi^2}\,\frac{\ln\epsilon}{\epsilon} + \operatorname{O}(\epsilon^{-1})\,.$$

(2) If K is an imaginary quadratic number field, then as $\epsilon > 0$ tends to 0, we have

$$\operatorname{Card}\big(\mathscr{O}_K\backslash\big\{\{\alpha,\beta\} \in \mathfrak{N}_K : |\beta-\alpha| \geqslant \epsilon\big\}\big) = \frac{4\,\pi}{|\mathscr{O}_K^\times|\ D_K\ \zeta_K(2)}\,\frac{\ln\epsilon}{\epsilon^2} + \operatorname{O}(\epsilon^{-2})\,.$$

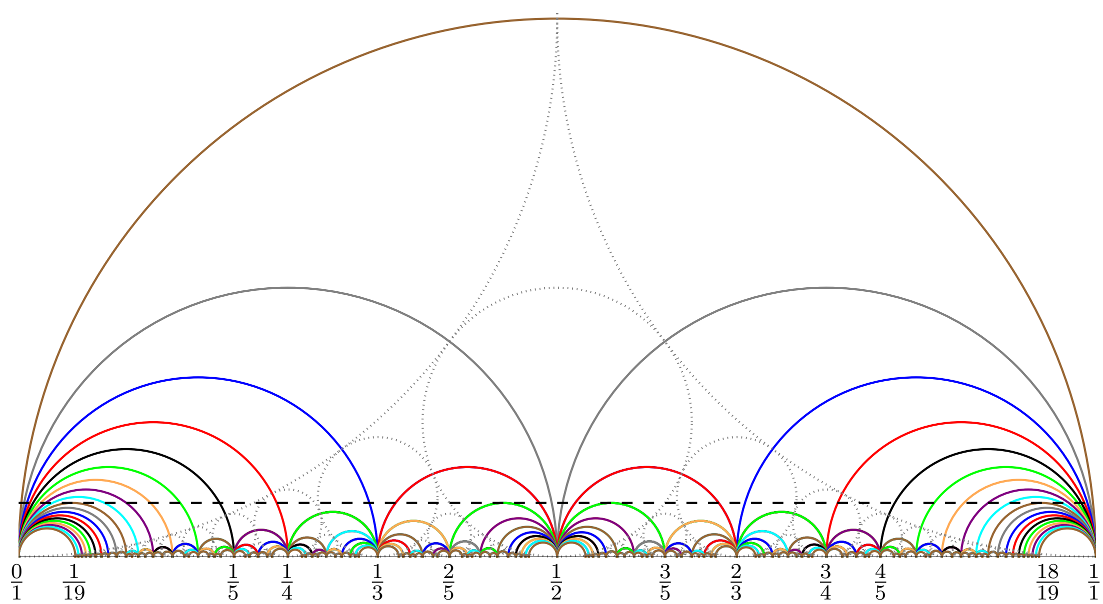
We refer to Section 4 (and in particular to Theorem 16) for analogous results on the effective asymptotic counting function of pairs of Farey neighbours in definite rational quaternion algebras, noting that their definition (see Equation (10)) needs to address the noncommutativity issue of the quaternion algebra. We have the following reformulation of the first claim of Theorem 1 in terms of the solutions of the gcd equation.

Corollary 2. *As $N \in \mathbb{N}\smallsetminus\{0\}$ tends to $+\infty$, we have*

$$\operatorname{Card}\Big\{(p,q,r,s) \in \mathbb{Z}^4 : \begin{array}{c} ps-qr=1\,,\ 0<qs\leqslant N \\ 0\leqslant p\leqslant q\,,\ 0\leqslant r\leqslant s \end{array}\Big\} = \frac{12}{\pi^2}\,N\log N + \operatorname{O}(N)\,.$$

The two above results follow (see Subsection 2.3) from the more general Theorem 4 when $K=\mathbb{Q}$ and Theorem 5 otherwise, that include versions with congruences and cover the more general case of modular symbols. Similarly, Theorem 16 in the quaternionic case follows from the more general Theorem 17.

For every $n \in \mathbb{N}$, let $\mathbb{H}^n_{\mathbb{R}}$ be the upper halfspace model of the real hyperbolic space. A *Farey arc* in $\mathbb{H}^2_{\mathbb{R}}$ is a hyperbolic geodesic line in $\mathbb{H}^2_{\mathbb{R}}$ whose pair of points at infinity is a pair of rational Farey neighbours. We prove Theorem 4 by relating its counting function to the counting function of common perpendicular geodesic arcs from the horoball $B_\infty = \{z \in \mathbb{H}^2_{\mathbb{R}} : \operatorname{Im} z \geqslant 1\}$ to the Farey arcs. This correspondence allows us to use the recent geometric counting results of [PP5] in the proof. The arguments for Theorems 5 and 16 are similar, using geodesic lines in $\mathbb{H}^3_{\mathbb{R}}$ and $\mathbb{H}^5_{\mathbb{R}}$ respectively.



The figure above shows the Farey arcs with endpoints $\frac{p}{q}$ and $\frac{r}{s}$ in $[0,1]$ that have denominators $1 \leqslant q, s \leqslant 19$ in reduced forms. The dotted circles show the $\mathrm{PSL}_2(\mathbb{Z})$-orbit of the horoball B_∞. The horizontal dashed line shows the points in $\mathbb{H}^2_\mathbb{R}$ at hyperbolic distance $\ln 20$ from B_∞. It meets 23 Farey arcs with endpoints in $[0,1]$.

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2 Counting common perpendiculars to divergent geodesics

For every $n \in \mathbb{N} \smallsetminus \{0,1\}$, let

$$\mathbb{H}^n_\mathbb{R} = \Big(\{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\},\ ds^2 = \frac{dx_1^2 + \dots + dx_n^2}{x_n^2}\Big)$$

be the upper halfspace model of the n-dimensional real hyperbolic space with constant sectional curvature -1. Let Γ be a discrete group of isometries of $\mathbb{H}^n_\mathbb{R}$ such that $M = \Gamma\backslash\mathbb{H}^n_\mathbb{R}$ is a finite volume complete noncompact real hyperbolic orbifold. Assume that ∞ is a parabolic fixed point of Γ, so that the image of the horoball $B_\infty = \{x \in \mathbb{H}^n_\mathbb{R} : x_n \geqslant 1\}$ in M is a properly immersed closed locally convex subset of M.

A locally geodesic line $\ell : \mathbb{R} \to M$ that is a proper mapping is a *divergent geodesic* in M. We denote by $m(\ell(\mathbb{R}))$ the cardinality of the orbifold pointwise stabiliser in Γ of the image $\ell(\mathbb{R})$ of ℓ. A locally geodesic line ℓ in M (or its image) is *weakly reciprocal* if it has a lift $\widetilde{\ell} : \mathbb{R} \to \mathbb{H}^n_\mathbb{R}$ such that an element of Γ interchanges the two endpoints at infinity of the geodesic line $\widetilde{\ell}$. We say[2] that ℓ (or its image) is *reciprocal* if there is such an element of order 2. Let $\iota_{\rm rec}(\ell(\mathbb{R})) = 1$ if ℓ is weakly reciprocal, and $\iota_{\rm rec}(\ell(\mathbb{R})) = 2$ otherwise.

Let D^- and D^+ be nonempty properly immersed closed locally convex subsets of M. For every $s > 0$, we denote by $\mathscr{N}_{D^-,\,D^+}(s)$ the cardinality of the set of common perpendiculars from D^- to D^+ with length at most s, considered with multiplicities (see [PP3] or [PP5] for precisions). The following result is the main tool in the proofs of this note.

Theorem 3. *Let D^- be a Margulis cusp neighbourhood in M and let D^+ be the image of a divergent geodesic in M. Then as $s \to +\infty$, we have*

$$\mathscr{N}_{D^-,D^+}(s) = \frac{\Gamma(\frac{n}{2})\,\iota_{\rm rec}(D^+)\ \mathrm{Vol}\,\partial D^-}{2\sqrt{\pi}\,\Gamma(\frac{n+1}{2})\,m(D^+)\ \mathrm{Vol}\,M}\ s\,e^{(n-1)\,s} + \mathrm{O}(e^{(n-1)s})\,.$$

Proof. Let $\|\sigma^+_{D^-}\|$ be the total mass of the outer skinning measure[3] $\sigma^+_{D^-}$ of D^-. By [PP3, Prop. 20 (2)], we have $\|\sigma^+_{D^-}\| = 2^{n-1}\,\mathrm{Vol}\,\partial D^-$. By [PP5, Thm. 6], we have

$$\mathscr{N}_{D^-,D^+}(s) = \frac{\Gamma(\frac{n}{2})\,\iota_{\rm rec}(D^+)\ \|\sigma^+_{D^-}\|}{2^n\sqrt{\pi}\,\Gamma(\frac{n+1}{2})\,m(D^+)\ \mathrm{Vol}\,M}\ s\,e^{(n-1)\,s} + \mathrm{O}(e^{(n-1)s})\,.$$

Theorem 3 follows. □

The boundary at infinity of $\mathbb{H}^n_\mathbb{R}$ is $\partial_\infty\mathbb{H}^n_\mathbb{R} = \mathbb{R}^{n-1} \cup \{\infty\}$. For all distinct $x, y \in \partial_\infty\mathbb{H}^n_\mathbb{R}$, let $\widetilde{\ell}_{x,y} : \mathbb{R} \to \mathbb{H}^n_\mathbb{R}$ be any geodesic line in $\mathbb{H}^n_\mathbb{R}$ with points at infinity $x = \widetilde{\ell}_{x,y}(-\infty)$ and $y = \widetilde{\ell}_{x,y}(+\infty)$, unique up to translation at the source. For every discrete group of isometries

[2] As in [PP5], see also [Sar].

[3] See Section 3 of [PP3] for the definition, that we won't need here.

Γ of $\mathbb{H}^n_\mathbb{R}$ with finite covolume such that x and y are parabolic fixed points of Γ, we denote by $\ell_{x,y} = \Gamma\,\widetilde{\ell}_{x,y}$ the divergent geodesic in $\Gamma\backslash\mathbb{H}^n_\mathbb{R}$ that is the image of $\widetilde{\ell}_{x,y}$ under the quotient mapping. Using the terminology of [McM], the image $\ell_{x,y}(\mathbb{R})$ of the divergent geodesic $\ell_{x,y}$, endowed with its "orientation" from x to y, pushforward by $\ell_{x,y}$ of the orientation of $\mathbb{R}$, is a *degree* 1 *modular symbol* in $\Gamma\backslash\mathbb{H}^n_\mathbb{R}$, that we denote by

$$\mathfrak{s}_\Gamma(x,y)\;.$$

Note that $\mathfrak{s}_\Gamma(x,y) = \mathfrak{s}_\Gamma(y,x)$ if and only if $\ell_{x,y}$ is reciprocal in $\Gamma\backslash\mathbb{H}^n_\mathbb{R}$, in which case $\mathfrak{s}_\Gamma(x,y)$ will be called a *reciprocal modular symbol*. We set

$$\iota_{\Gamma,\mathrm{rec}}(x,y) = \iota_{\mathrm{rec}}(\ell_{x,y}(\mathbb{R})) \quad \textit{and} \quad m_\Gamma(x,y) = m(\ell_{x,y}(\mathbb{R}))\,.$$

Note that $\iota_{\Gamma,\mathrm{rec}}(x,y)$ and $m_\Gamma(x,y)$ are constant on the Γ-orbits of pairs $\{x,y\}$ of distinct parabolic fixed points of Γ.

For $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$, we denote by $\left[\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right]\in\mathrm{PSL}_2(\mathbb{K})$ the image of $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{K})$.

2.1 Counting modular symbols: the rational case

We identify as usual $\mathbb{R}^2$ with $\mathbb{C}$. Note that $\mathbb{H}^2_\mathbb{R} = \big(\{z\in\mathbb{C} : \operatorname{Im} z>0\},\, ds^2 = \frac{|dz|^2}{(\operatorname{Im} z)^2}\big)$. The group $\mathrm{PSL}_2(\mathbb{R})$ acts on $\mathbb{H}^2_\mathbb{R}\cup\partial_\infty\mathbb{H}^2_\mathbb{R}$ by the homographies $g\cdot z = \frac{az+b}{cz+d}$ for all $z\in\mathbb{P}^1(\mathbb{C})$ and $g = \left[\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right]\in\mathrm{PSL}_2(\mathbb{R})$ with the usual convention when $z=\infty, -\frac{d}{c}$, and acts faithfully by isometries on $\mathbb{H}^2_\mathbb{R}$. The *modular group* $\Gamma_\mathbb{Q} = \mathrm{PSL}_2(\mathbb{Z})$ is an arithmetic lattice in $\mathrm{PSL}_2(\mathbb{R})$. It acts transitively on its set of parabolic fixed points $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q}\cup\{\infty\}$ in $\partial_\infty\mathbb{H}^2_\mathbb{R}$. The stabiliser $\Gamma_{\mathbb{Q},\infty}$ of ∞ in $\Gamma_\mathbb{Q}$ consists of the translations $z\mapsto z+k$ with $k\in\mathbb{Z}$.

Theorem 4. *Let Γ be a finite index subgroup of $\Gamma_\mathbb{Q} = \mathrm{PSL}_2(\mathbb{Z})$, and let Γ_∞ be the stabiliser of ∞ in Γ. For all distinct $x,y\in\mathbb{Q}\cup\{\infty\}$, as $\epsilon>0$ tends to 0, we have*

$$\operatorname{Card}\big(\Gamma_\infty\backslash\big\{\{\alpha,\beta\}\in\Gamma\cdot\{x,y\} :\ |\beta-\alpha|\geqslant\epsilon\big\}\big) = -\frac{6\,\iota_{\Gamma,\mathrm{rec}}(x,y)\,[\Gamma_{\mathbb{Q},\infty}:\Gamma_\infty]}{\pi^2\,[\Gamma_\mathbb{Q}:\Gamma]}\,\frac{\ln\epsilon}{\epsilon} + \mathrm{O}(\epsilon^{-1})\,.$$

For instance, for every $N\in\mathbb{N}\smallsetminus\{0,1\}$, with $\Gamma = \{\left[\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right]\in\mathrm{PSL}_2(\mathbb{Z}) : c\equiv 0 \bmod N\}$ the Hecke congruence sugbroup of level N of $\Gamma_\mathbb{Q} = \mathrm{PSL}_2(\mathbb{Z})$, we have $[\Gamma_{\mathbb{Q},\infty}:\Gamma_\infty] = 1$ and $[\Gamma_\mathbb{Q}:\Gamma] = N\prod_{p|N}\big(1+\frac{1}{p}\big)$ by [Shi, Prop. 1.43 (1)] (as usual, the index p ranges over primes).

Proof. Note that Γ and $\Gamma_\mathbb{Q}$ have the same sets of parabolic fixed points. We may hence apply Theorem 3 with $M = \Gamma\backslash\mathbb{H}^2_\mathbb{R}$, with $D^- = \Gamma_\infty\backslash B_\infty$ the image of B_∞ in M (which is a Margulis cusp neighbourhood in M), and with $D^+ = \ell_{x,y}(\mathbb{R}) = \Gamma\,\widetilde{\ell}_{x,y}(\mathbb{R})$ (which is the image of a divergent geodesic in M). We have

$$\operatorname{vol} M = [\Gamma_\mathbb{Q}:\Gamma]\operatorname{vol}(\Gamma_\mathbb{Q}\backslash\mathbb{H}^2_\mathbb{R}) = [\Gamma_\mathbb{Q}:\Gamma]\,\frac{\pi}{3}$$

and $\operatorname{Vol}\partial D^- = [\Gamma_{\mathbb{Q},\infty}:\Gamma_\infty]\operatorname{Vol}(\Gamma_{\mathbb{Q},\infty}\backslash\partial B_\infty) = [\Gamma_{\mathbb{Q},\infty}:\Gamma_\infty]$. Since the fixed point sets in $\mathbb{H}^2_\mathbb{R}$ of the elliptic elements of $\Gamma_\mathbb{Q}$ are singletons, the pointwise stabiliser in $\Gamma_\mathbb{Q}$ hence in Γ of any geodesic line in $\mathbb{H}^2_\mathbb{R}$ is trivial. Therefore $m(D^+)=1$.

If $\alpha,\beta\in\mathbb{R}$ satisfy $|\alpha-\beta|<2$, then the length of the common perpendicular from B_∞ to $\widetilde{\ell}_{\alpha,\beta}(\mathbb{R})$ is $\ln\big(\frac{2}{|\alpha-\beta|}\big)$. Since the stabilizer in $\Gamma_\mathbb{Q}$ hence in Γ of a nontrivial geodesic segment is trivial, the multiplicity of such a common perpendicular is 1 (see [PP3, §3.3]). Using these observations and Theorem 3 with $n = 2$, since $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$, as $\epsilon > 0$ tends to 0, we obtain

$$\begin{aligned}
&\operatorname{Card}\big(\Gamma_\infty\backslash\{\{\alpha,\beta\}\in\Gamma\cdot\{x,y\} :\ |\beta-\alpha|\geqslant\epsilon\}\big)\\
&=\operatorname{Card}\Big(\Gamma_\infty\backslash\Big\{\{\alpha,\beta\}\in\Gamma\cdot\{x,y\} : d\big(B_\infty,\widetilde{\ell}_{\alpha,\beta}(\mathbb{R})\big)\leqslant\ln\frac{2}{\epsilon}\Big\}\Big)\\
&=\mathscr{N}_{D^-,D^+}\Big(\ln\frac{2}{\epsilon}\Big)=\frac{1\cdot\iota_{\Gamma,\mathrm{rec}}(x,y)\cdot[\Gamma_{\mathbb{Q},\infty}:\Gamma_\infty]}{2\sqrt{\pi}\,\frac{\sqrt{\pi}}{2}\cdot1\cdot[\Gamma_{\mathbb{Q}}:\Gamma]\,\frac{\pi}{3}}\ln\Big(\frac{2}{\epsilon}\Big)\frac{2}{\epsilon}+\mathrm{O}\Big(\frac{2}{\epsilon}\Big).
\end{aligned}$$

Theorem 4 follows by simplification. □

2.2 Counting modular symbols: the imaginary quadratic case

The group $\mathrm{PSL}_2(\mathbb{C})$ acts on $\partial_\infty\mathbb{H}^3_\mathbb{R} = \mathbb{P}^1(\mathbb{C}) = \mathbb{C}\cup\{\infty\}$ by homographies (Möbius transformations) $g\cdot z = \frac{az+b}{cz+d}$ for all $g = \big[\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\big] \in \mathrm{PSL}_2(\mathbb{C})$ and $z\in\mathbb{P}^1(\mathbb{C})$ with the usual convention when $z = \infty, -\frac{d}{c}$. It acts faithfully isometrically on

$$\mathbb{H}^3_\mathbb{R} = \Big(\{(z,t)\in\mathbb{C}\times\mathbb{R} : t>0\},\ \ ds^2 = \frac{|dz|^2+dt^2}{t^2}\Big)$$

by Poincaré's extension.

In this Subsection 2.2, let K be an imaginary quadratic number field. Let $\mathscr{O}_K$, D_K, h_K, ζ_K be as in the introduction. Recall that the group of units $\mathscr{O}_K^\times$ of $\mathscr{O}_K$ is finite, and it is equal to $\{\pm1\}$ unless $D_K = -4, -3$. Let $\mathscr{I}_K$ be the group of ideal classes of $\mathscr{O}_K$, whose order is h_K. For every ideal $\mathfrak{a}$ of $\mathscr{O}_K$, recall that there exist $a, b \in \mathscr{O}_K$ such that $\mathfrak{a} = a\mathscr{O}_K + b\mathscr{O}_K$, and we denote by $[\mathfrak{a}]$ the class of $\mathfrak{a}$ in $\mathscr{I}_K$. The identity element of $\mathscr{I}_K$ is the principal class $[\mathscr{O}_K]$.

The *Bianchi group* $\Gamma_K = \mathrm{PSL}_2(\mathscr{O}_K)$ is an arithmetic lattice in $\mathrm{PSL}_2(\mathbb{C})$. The quotient space $M_K = \Gamma_K\backslash\mathbb{H}^3_\mathbb{R}$ is hence a finite volume noncompact complete real hyperbolic 3-orbifold. The Bianchi group Γ_K acts with h_K orbits on its set of parabolic fixed points $\mathbb{P}^1(K) = K\cup\{\infty\}$ in $\partial_\infty\mathbb{H}^3_\mathbb{R} = \mathbb{P}^1(\mathbb{C})$: the map

$$\Gamma_K\cdot[x_0 : x_1]\mapsto[x_0\mathscr{O}_K + x_1\mathscr{O}_K] \tag{2}$$

is a bijection[4] from $\Gamma_K\backslash\mathbb{P}^1(K)$ to $\mathscr{I}_K$. The stabiliser of ∞ in Γ_K is

$$\Gamma_{K,\infty} = \left\{\begin{bmatrix} a & b\\ 0 & a^{-1}\end{bmatrix}\in\Gamma_K : a\in\mathscr{O}_K^\times,\ b\in\mathscr{O}_K\right\}.$$

Note that $m_{\Gamma_K}(x,y)$ is not constant when x and y vary among the distinct elements of K. For example, we have $m_{\Gamma_K}(0,\infty) = \frac{|\mathscr{O}_K^\times|}{2}$, $m_{\Gamma_K}(1,-1) = 2$ and $m_{\Gamma_K}(\frac{1}{3},\infty) = 1$. As for $\iota_{\Gamma_K,\mathrm{rec}}(x,y)$ (see Subsection 2.4), an explicit arithmetic value of $m_{\Gamma_K}(x,y)$ as x and y vary in $\mathbb{P}^1(K)$ does not seem to be available. See Examples (1) to (3) in Section 3 for other examples of computation of $\iota_{\Gamma_K,\mathrm{rec}}(x,y)$ and $m_{\Gamma_K}(x,y)$ for some $x,y\in\mathbb{P}^1(K)$.

[4] See for instance [EGM, §7, Th. 2.4]

Theorem 5. *Let Γ be a finite index subgroup of $\Gamma_K = \mathrm{PSL}_2(\mathscr{O}_K)$, and let Γ_∞ be the stabiliser of ∞ in Γ. For all distinct $x, y \in K \cup \{\infty\}$, as $\epsilon > 0$ tends to 0, we have*

$$\begin{aligned}&\operatorname{Card}\big(\Gamma_\infty\backslash\big\{\{\alpha,\beta\}\in\Gamma\cdot\{x,y\} : |\beta-\alpha|\geqslant\epsilon\big\}\big)\\ &= \frac{4\,\pi\,\iota_{\Gamma,\mathrm{rec}}(x,y)\,[\Gamma_{K,\infty}:\Gamma_\infty]}{|\mathscr{O}_K^\times|\,D_K\,\zeta_K(2)\,m_\Gamma(x,y)\,[\Gamma_K:\Gamma]}\;\frac{\ln\epsilon}{\epsilon^2}+\mathrm{O}(\epsilon^{-2}).\end{aligned}$$

Proof. Again Γ and Γ_K have the same sets of parabolic fixed points. As in the proof of Theorem 4, we apply Theorem 3 with $n = 3$, with $M = \Gamma\backslash\mathbb{H}^3_\mathbb{R}$, with $D^- = \Gamma_\infty\backslash B_\infty$, and with $D^+ = \ell_{x,y}(\mathbb{R})$. By Humbert's volume formula,[5] we have

$$\operatorname{Vol}(M) = [\Gamma_K:\Gamma]\,\operatorname{Vol}(\Gamma_K\backslash\mathbb{H}^3_\mathbb{R}) = [\Gamma_K:\Gamma]\,\frac{|D_K|^{3/2}\,\zeta_K(2)}{4\,\pi}\,.$$

The index $[\Gamma_{K,\infty}:\mathscr{O}_K]$ in $\Gamma_{K,\infty}$ of its unipotent subgroup consisting in the translations by elements of $\mathscr{O}_K$ is equal to $\frac{|\mathscr{O}_K^\times|}{2}$. By for instance the area computation in the proof of [PP1, Lemma 6], since $\mathscr{O}_K$ is generated as a $\mathbb{Z}$-module by 1 and $\frac{D_K+i\sqrt{|D_K|}}{2}$, we have

$$\begin{aligned}\operatorname{Vol}(\partial D^-) &= [\Gamma_{K,\infty}:\Gamma_\infty]\operatorname{Vol}(\Gamma_{K,\infty}\backslash\partial B_\infty) = [\Gamma_{K,\infty}:\Gamma_\infty]\frac{2}{|\mathscr{O}_K^\times|}\operatorname{Vol}(\mathscr{O}_K\backslash\mathbb{C})\\ &= [\Gamma_{K,\infty}:\Gamma_\infty]\;\frac{\sqrt{|D_K|}}{|\mathscr{O}_K^\times|}\,.\end{aligned}$$

Only finitely many $\Gamma_{K,\infty}$-orbits (hence Γ_∞- orbits) of geodesic lines $\widetilde{\ell}_{\infty,z}(\mathbb{R})$ for $z \in \mathbb{C}$ have nontrivial stabilizers in Γ_K (hence in Γ). A given geodesic line $\widetilde{\ell}_{\infty,z}(\mathbb{R})$ meets perpendicularly at most $\mathrm{O}(t)$ elements of $\Gamma\,\widetilde{\ell}_{x,y}(\mathbb{R})$ at distance at most t from B_∞. Hence only linearly many Γ_∞-orbits of common perpendiculars between B_∞ and elements of $\Gamma\,\widetilde{\ell}_{x,y}(\mathbb{R})$ have multiplicity different from 1. As in the proof of Theorem 4, when $\epsilon > 0$ tends to 0, we have

$$\begin{aligned}&\operatorname{Card}\big(\Gamma_\infty\backslash\big\{\{\alpha,\beta\}\in\Gamma\cdot\{x,y\} :\ |\beta-\alpha|\geqslant\epsilon\big\}\big) = \mathscr{N}_{D^-,D^+}\Big(\ln\frac{2}{\epsilon}\Big)+\mathrm{O}\Big(\ln\frac{2}{\epsilon}\Big)\\ &= \frac{\Gamma(\frac{3}{2})\,\iota_{\Gamma,\mathrm{rec}}(x,y)\,[\Gamma_{K,\infty}:\Gamma_\infty]\,\frac{\sqrt{|D_K|}}{|\mathscr{O}_K^\times|}}{2\sqrt{\pi}\,\Gamma(2)\,m_\Gamma(x,y)\,[\Gamma_K:\Gamma]\,\frac{|D_K|^{3/2}\,\zeta_K(2)}{4\,\pi}}\,\Big(\ln\frac{2}{\epsilon}\Big)\Big(\frac{2}{\epsilon}\Big)^2+\mathrm{O}(\epsilon^{-2})\,.\end{aligned}$$

Theorem 5 follows by simplification, using the values $\Gamma(2) = 1$ and $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$. □

2.3 Proofs of Theorem 1 and Corollary 2

In this Subsection, let K be as in the introduction, either $\mathbb{Q}$ or an imaginary quadratic number field. We are now ready to prove Theorem 1, by restricting Theorems 4 and 5 to the case when $x, y \in \mathbb{P}^1(K)$ are Farey neighbours. Since $\Gamma_K = \mathrm{PSL}_2(\mathscr{O}_K)$ has infinitely many orbits on the set of pairs of points of $\mathbb{P}^1(K)$, Lemma 6 below implies in particular that there are lots of unordered pairs of distinct elements $x, y \in K \cup \{\infty\}$ that are not Farey neighbours, though Theorems 4 and 5 still apply. See Examples (1) to (3) in Section 3

[5] See for instance [EGM, §8.8 and §9.6].

for explicit examples of degree 1 modular symbols, that are not pairs of Farey neighbours, in the imaginary quadratic case. The key translation between the arithmetics and the geometry is the following elementary lemma (that will be more involved in the quaternionic case).

Lemma 6. *Two distinct elements $\alpha, \beta \in \mathbb{P}^1(K) = K \cup \{\infty\}$ are Farey neighbours if and only if there exists $\gamma \in \Gamma_K = \mathrm{PSL}_2(\mathscr{O}_K)$ such that $\gamma \cdot \infty = \alpha$ and $\gamma \cdot 0 = \beta$.*

Proof. Let $\alpha, \beta \in \mathbb{P}^1(K)$ be distinct. If there exists $\gamma = \left[\begin{smallmatrix} p & r \\ q & s \end{smallmatrix}\right] \in \Gamma_K$ such that $\gamma \cdot \infty = \alpha$ and $\gamma \cdot 0 = \beta$, then $p, q, r, s \in \mathscr{O}_K$, $\alpha = \frac{p}{q}$, $\beta = \frac{r}{s}$ and $ps - qr = 1$, hence α, β are Farey neighbours.

Conversely, if α, β are Farey neighbours, let $p, q, r, s \in \mathscr{O}_K$ be such that $\alpha = \frac{p}{q}$, $\beta = \frac{r}{s}$ and $|\, ps - qr \,| = 1$. Then $u = ps - qr \in \mathscr{O}_K^\times$. Hence if $\gamma = \left[\begin{smallmatrix} pu^{-1} & r \\ qu^{-1} & s \end{smallmatrix}\right]$, then $\gamma \in \Gamma_K$ and we have $\gamma \cdot \infty = \alpha$ and $\gamma \cdot 0 = \beta$. □

Remark 7. (i) Lemma 6, and the fact that the ideal classes associated with $\infty = \frac{1}{0}$ and $0 = \frac{0}{1}$ by Equation (2) are the principal one $[\mathscr{O}_K]$, imply that if $x = [x_0 : x_1] \in \mathbb{P}^1(K)$ and $y = [y_0 : y_1] \in \mathbb{P}^1(K)$ are Farey neighbours, then their two associated ideal classes $[x_0\mathscr{O}_K + x_1\mathscr{O}_K]$ and $[y_0\mathscr{O}_K + y_1\mathscr{O}_K]$ are the principal one.

(ii) In this remark, assume that $\mathscr{O}_K$ is Euclidean, that is, that $D_K = 1$ ($K = \mathbb{Q}$) or $D_K = -3, -4, -7, -8, -11$. Let $n_K = [K : \mathbb{Q}]$, let Γ be a finite index subgroup of Γ_K and let $X_\Gamma = \Gamma\backslash(\mathbb{H}^{n_k}_\mathbb{R} \cup K \cup \{\infty\})$ be the cuspidal compactification of $\Gamma\backslash\mathbb{H}^{n_k}_\mathbb{R}$. By [Man1, §1.2] when $K = \mathbb{Q}$ and as extended in [Cre, §2.2] otherwise, the degree 1 modular symbols $\mathfrak{s}_\Gamma(x, y)$ for distinct $x, y \in K \cup \{\infty\}$, with the cusp points from which they start (respectively end) added at their beginning (respectively end), are integral 1-cycles when $\Gamma x = \Gamma y$, and define real 1-cycles when $\Gamma x \neq \Gamma y$, whose homology classes in $H_1(X_\Gamma, \mathbb{R})$ we denote by $[\mathfrak{s}_\Gamma(x, y)]$. By Lemma 6, precisely when x and y are Farey neighbours, the homology classes $[\mathfrak{s}_\Gamma(x, y)]$ are called *distinguished classes* in [Man1, §1.5] when $K = \mathbb{Q}$ and *special classes* in [Cre, §2.2] otherwise. Since $\mathscr{O}_K$ is Euclidean, these finitely many classes generate the real vector space $H_1(X_\Gamma, \mathbb{R})$ by [Man1, Prop. 1.6 a)] when $K = \mathbb{Q}$ and [Cre, page 287, lines - 9 to - 5] otherwise.

When $\mathscr{O}_K$ is not Euclidean, the same arguments prove that the homology classes of the degree 1 modular symbols corresponding to the 1-edges of the dual ideal tesselation of the Ford-Voronoi tesselation of $\mathbb{H}^3_\mathbb{R}$ (whose 2-skeleton is the Mendoza Γ_K-invariant spine of $\mathbb{H}^3_\mathbb{R}$, see [BeS, §4] and Section 3) generate the real vector space $H_1(X_\Gamma, \mathbb{R})$.

Proof of Theorem 1. In order to prove Claim (1), we apply Theorem 4 to $\Gamma = \Gamma_\mathbb{Q}$ and $(x, y) = (0, \infty)$, so that by Lemma 6, we have $\mathfrak{N}_\mathbb{Q} = \Gamma \cdot \{x, y\}$. The divergent geodesic $\ell_{0,\infty}$ in $\Gamma_\mathbb{Q}\backslash\mathbb{H}^2_\mathbb{R}$ is reciprocal since the elliptic element $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right] \in \Gamma_\mathbb{Q}$ of order 2 exchanges the two points at infinity 0 and ∞ of the geodesic line $\widetilde{\ell}_{0,\infty}(\mathbb{R})$. Hence $\iota_{\Gamma_\mathbb{Q},\mathrm{rec}}(0, \infty) = 1$. Two images of the geodesic line $\widetilde{\ell}_{0,\infty}(\mathbb{R})$ under $\Gamma_\mathbb{Q}$ either coincide or do not intersect in $\mathbb{H}^2_\mathbb{R}$. If the two endpoints of such an image are different from ∞, then their distance is at most one. Thus, except the images of $\widetilde{\ell}_{0,\infty}(\mathbb{R})$ under $\Gamma_{\mathbb{Q},\infty}$, every image of $\widetilde{\ell}_{0,\infty}(\mathbb{R})$ under an element of Γ has one and only one translate modulo $\Gamma_{\mathbb{Q},\infty} = \mathbb{Z}$ both of whose points at infinity lie in the unit interval $[0, 1]$. Thus Claim (1) of Theorem 1 follows from Theorem 4.

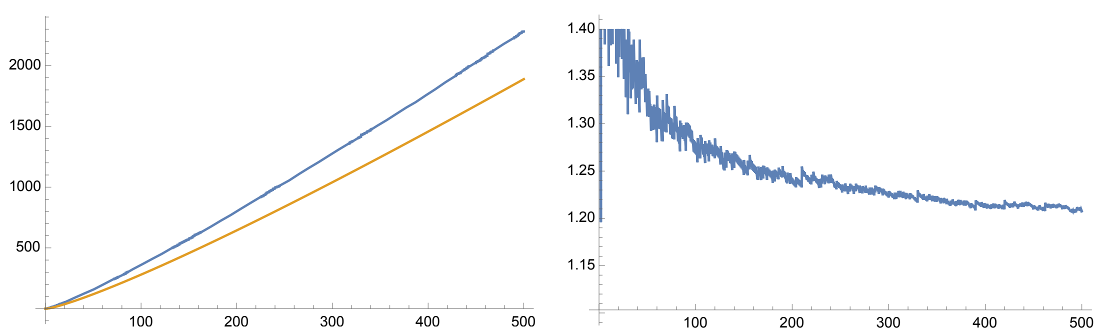
In order to prove Claim (2) of Theorem 1, we apply similarly Theorem 5 with $\Gamma = \Gamma_K$ and $(x, y) = (0, \infty)$, so that by Lemma 6, we have $\mathfrak{N}_K = \Gamma \cdot \{x, y\}$. Note that the pointwise stabiliser in Γ_K of the geodesic line $\widetilde{\ell}_{0,\infty}(\mathbb{R})$ has cardinality $\frac{|\mathscr{O}_K^\times|}{2}$, hence $m_{\Gamma_K}(0,\infty) = \frac{|\mathscr{O}_K^\times|}{2}$. Note that the divergent geodesic $\ell_{0,\infty}$ is reciprocal as in the rational case, hence $\iota_{\Gamma_K,\mathrm{rec}}(0,\infty) = 1$. The index in $\Gamma_{K,\infty}$ of its unipotent subgroup of translations by $\mathscr{O}_K$ is equal to $\frac{|\mathscr{O}_K^\times|}{2}$. Hence replacing the quotient modulo $\Gamma_{K,\infty}$ in the left-hand side of the formula in Theorem 5 by the quotient modulo $\mathscr{O}_K$ amounts to multiplying the right-hand side by $\frac{|\mathscr{O}_K^\times|}{2}$. □

Proof of Corollary 2. Note that Equation (1) is equivalent when $qs \neq 0$, to the equation $\left|\frac{p}{q} - \frac{r}{s}\right| = \frac{1}{|qs|}$. For every $N \in \mathbb{N} \smallsetminus \{0\}$, the map $(p,q,r,s) \mapsto \{\frac{p}{q}, \frac{r}{s}\}$ from the set

$$\big\{(p,q,r,s) \in \mathbb{Z}^4 : ps - qr = 1\,,\ 0 \leqslant p \leqslant q\,,\ 0 \leqslant r \leqslant s\,,\ 0 < qs \leqslant N\big\}$$

to the set $\big\{\{\alpha,\beta\} \in \mathfrak{N}_{\mathbb{Q}} : \alpha,\beta \in [0,1]\,,\ |\alpha - \beta| \geqslant \frac{1}{N}\big\}$ is easily checked to be 2-to-1. Hence Corollary 2 follows from Theorem 1 (1) with $\epsilon = \frac{1}{N}$. □

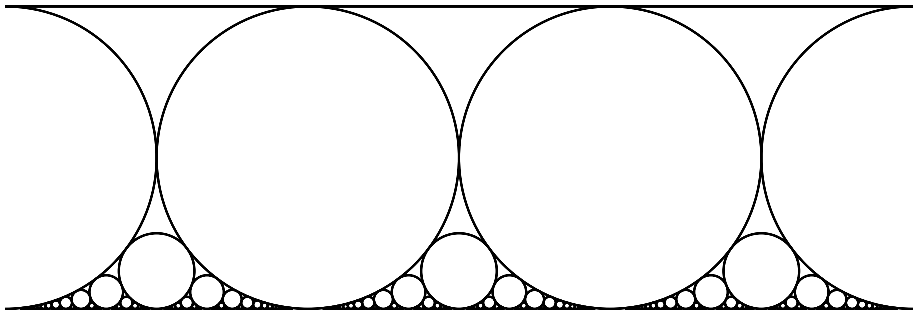
The next pictures illustrate Theorem 1 (1) and Corollary 2. The blue curve on the left represents the graph of the map $N \mapsto \operatorname{Card}\big\{\{\alpha,\beta\} \in \mathfrak{N}_{\mathbb{Q}} : \alpha,\beta \in [0,1]\,,\ |\beta - \alpha| \geqslant \frac{1}{N}\big\}$. The orange one represents the graph of the map $N \mapsto \frac{6}{\pi^2} N \ln N$. Note that the two graphs diverge slowly one from the other since there is only a logarithmic factor between the main term and the error term. The picture on the right represents the ratio map, slowly converging to 1.



2.4 Ford circles, Farey neighbours and modular symbols

Assume in this Subsection that $K = \mathbb{Q}$. Being Farey neighbours in $\mathbb{P}^1(\mathbb{Q})$ has a well-known geometric characterisation, that we recall as a motivation for Section 3. For every $\alpha \in \mathbb{P}^1(\mathbb{Q})$, if $\alpha = \frac{p}{q}$ with $p,q \in \mathbb{Z}$ relatively prime and $q > 0$, let B_α be the intersection with $\mathbb{H}^2_{\mathbb{R}}$ of the closed Euclidean ball of center $x + \frac{i}{2q^2}$ and radius $\frac{1}{2q^2}$. The boundary of this disc is called the *Ford circle*[6] of $\frac{p}{q} \in \mathbb{Q}$, see the picture below. Let $B_\infty = \{z \in \mathbb{H}^2_{\mathbb{R}} : \operatorname{Im} z \geqslant 1\}$. The family $(B_\alpha)_{\alpha \in \mathbb{P}^1(\mathbb{Q})}$ is the unique $\mathrm{PSL}_2(\mathbb{Z})$-equivariant family of maximal horoballs with pairwise disjoint interiors. Two distinct $\alpha,\beta \in \mathbb{P}^1(\mathbb{Q})$ are Farey neighbours if and only if the horoballs B_α and B_β are tangent, or if and only if their Ford circles are tangent, see for instance [Zul, page 12] that was published before [For].

[6] Ford himself calls them *Speiser circles*.



Remark 8. (i) **Another counting of degree 1 modular symbols.** For all distinct $x, y \in \mathbb{P}^1(\mathbb{Q})$, the hyperbolic distance $d(B_x, B_y)$ is a natural complexity for the degree 1 modular symbol $\mathfrak{s}_{\Gamma_\mathbb{Q}}(x, y)$, see [PPS]. For every $T > 0$, let

$$\mathfrak{S}_{\Gamma_\mathbb{Q}}(T) = \{\mathfrak{s}_{\Gamma_\mathbb{Q}}(x, y) : x, y \in \mathbb{P}^1(\mathbb{Q}),\ x \neq y,\ d(B_x, B_y) \leqslant T\}$$

and let $\mathfrak{S}^{\rm rec}_{\Gamma_\mathbb{Q}}(T)$ be the subset of $\mathfrak{S}_{\Gamma_\mathbb{Q}}(T)$ that consists of its reciprocal modular symbols. Let $D^- = D^+$ be the Margulis neighbourhood of the cusp of $\Gamma_\mathbb{Q}\backslash\mathbb{H}^2_\mathbb{R}$ defined as the image of any B_α for $\alpha \in \mathbb{P}^1(\mathbb{Q})$ under the quotient mapping $\mathbb{H}^2_\mathbb{R} \to \Gamma_\mathbb{Q}\backslash\mathbb{H}^2_\mathbb{R}$. Its hyperbolic area is 1. By [PP3, Cor. 21], there exists $\kappa > 0$ such that as $T \to +\infty$, we have

$$\begin{aligned}\operatorname{Card} \mathfrak{S}_{\Gamma_\mathbb{Q}}(T) = \mathscr{N}_{D^-,\,D^+}(T) &= \frac{2^{2-1}(2-1)\operatorname{Vol}(D^-)\operatorname{Vol}(D^+)}{\operatorname{Vol}(\mathbb{S}^{2-1})\operatorname{Vol}(\Gamma_\mathbb{Q}\backslash\mathbb{H}^2_\mathbb{R})}\, e^T(1 + e^{-\kappa T}) \\ &= \frac{3}{\pi^2}\, e^T(1 + e^{-\kappa T})\,.\end{aligned}$$

For all distinct $x, y \in \mathbb{P}^1(\mathbb{Q})$, the degree 1 modular symbol $\mathfrak{s}_{\Gamma_\mathbb{Q}}(x, y)$ for $\Gamma_\mathbb{Q}$ is reciprocal if and only if the geodesic line $\widetilde{\ell}_{x,y}(\mathbb{R})$ in $\mathbb{H}^2_\mathbb{R}$ intersects the orbit $\Gamma_\mathbb{Q} \cdot i$. This (unique) point of intersection is the midpoint of the common perpendicular of B_x and B_y. Thus, for every $T > 0$, the number of reciprocal modular symbols of complexity at most T equals the number of common perpendiculars in $\Gamma_\mathbb{Q}\backslash\mathbb{H}^2_\mathbb{R}$ from D^- as above to $D'^+ = \Gamma_\mathbb{Q} \cdot i$ of length at most $\frac{T}{2}$. Since the stabilizer of i in $\Gamma_\mathbb{Q}$ has cardinality 2, by [PP3, Cor. 21], there exists $\kappa > 0$ such that as $T \to +\infty$, we have

$$\operatorname{Card} \mathfrak{S}^{\rm rec}_{\Gamma_\mathbb{Q}}(T) = \mathscr{N}_{D^-,\,D'^+}\Big(\frac{T}{2}\Big) = \frac{\operatorname{Vol}(D^-)}{2\operatorname{Vol}(\Gamma_\mathbb{Q}\backslash\mathbb{H}^2_\mathbb{R})}\, e^{\frac{T}{2}}(1 + e^{-\kappa\frac{T}{2}}) = \frac{3}{2\pi}\, e^{\frac{T}{2}}(1 + e^{-\kappa\frac{T}{2}})\,. \quad (3)$$

Thus, the proportion of reciprocal modular symbols in $\mathfrak{S}_{\Gamma_\mathbb{Q}}(T)$ is equivalent to $2\pi\, e^{-\frac{T}{2}}$ as $T \to +\infty$.

(ii) **Relationship between counting modular symbols and the primitive circle problem.** For every $n \in \mathbb{N}$, a *representation by primitive sums of two squares* of n is a pair $(p, q) \in \mathbb{Z}^2$ with p, q coprime such that $n = p^2 + q^2$. Let us denote by $r_{\rm prim}(n)$ their number. As $N \to +\infty$, we have

$$\sum_{n=1}^{N} r_{\rm prim}(n) = \frac{6}{\pi}\, N + \operatorname{O}\big(\sqrt{N}\,\big)\,, \quad (4)$$

see [Wu, Eq. (1.1)] (and Theorem 1 in loc. cit. for a better error term conditionally to the RH). Let us prove that Equation (3) follows from Equation (4).

For every $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_{\mathbb{Q}}$, the integers c and d are coprime, and we have

$$\operatorname{Re}(\gamma \cdot i) = \frac{ac+bd}{c^2+d^2} \quad \text{and} \quad \operatorname{Im}(\gamma \cdot i) = \frac{1}{c^2+d^2}\,. \tag{5}$$

In particular, the imaginary part of any element of $\Gamma_{\mathbb{Q}} \cdot i$ has the form $\frac{1}{n}$ for some $n \in \mathbb{N} \smallsetminus \{0\}$ which is a primitive sum of two squares. Fixing coprime integers $c, d \in \mathbb{Z}$ and a solution $(a, b) \in \mathbb{Z}^2$ of the gcd equation $ad - bc = 1$, all other solutions are $(a + kc, b + kd)$ with $k \in \mathbb{Z}$ by the uniqueness property of the Bézout identity. Since $\frac{(a+kc)c+(b+kd)d}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + k$, there exists a unique such solution $(a', b') \in \mathbb{Z}^2$ such that $R(c, d) = \frac{a'c+b'd}{c^2+d^2}$ belongs to $[0, 1[$. We define $\gamma_{c,d} = \left[\begin{smallmatrix} a' & b' \\ c & d \end{smallmatrix}\right]$. For every $n \in \mathbb{N}$, given a representation (c, d) of n by sums of two squares with $d \neq \pm c$, there are 8 representations of n obtained by changing the order and the signs of c and d. Among these 8 representations, the 4 pairs (c, d), $(-c, -d)$, $(d, -c)$, $(-d, c)$ do not change $R(c, d)$ (we have $\gamma_{-c,-d} = \gamma_{c,d}$ and $\gamma_{d,-c} = \gamma_{-d,c} = \gamma_{c,d} \circ \iota$ with $\iota = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ fixing i). The 4 remaining pairs obtained by exchanging c and d change $R(c, d)$ into $1 - R(c, d)$. The number of coprime pairs $(c, d) \in \mathbb{Z}^2$ such that $R(c, d) \in \{0, \frac{1}{2}, 1\}$ is finite, since a vertical geodesic line in $\mathbb{H}^2_{\mathbb{R}}$ meets at most one point of $\Gamma_{\mathbb{Q}} \cdot i$. The number of all representations of integers by primitive sums of squares of two equal or opposite integers is finite (equal to 4, since $c \in \mathbb{N}$ is coprime to $\pm c$ if and only if $c = \pm 1$). Thus as $T \to +\infty$, by the standard computation of the hyperbolic distance of a point of $\mathbb{H}^2_{\mathbb{R}}$ to the horizontal horosphere $\partial D^- = \{z \in \mathbb{H}^2_{\mathbb{R}} : \operatorname{Im} z = 1\}$ and by Equation (4), we have

$$\begin{aligned} \mathscr{N}_{D^-,\,D'^+}\Big(\frac{T}{2}\Big) &= \operatorname{Card}\big\{z \in \Gamma_{\mathbb{Q}} \cdot i : 0 \leqslant \operatorname{Re} z < 1,\ \operatorname{Im} z \in [e^{-\frac{T}{2}}, 1]\ \big\} \\ &= \frac{1}{4} \sum_{n=1}^{\lfloor e^{\frac{T}{2}} \rfloor} r_{\rm prim}(n) + \operatorname{O}\big(1\big) = \frac{3}{2\pi}\, e^{\frac{T}{2}} + \operatorname{O}\big(e^{\frac{T}{4}}\,\big)\,. \end{aligned}$$

This implies Equation (3), as wanted, with an explicit value $\kappa = \frac{1}{2}$.

(iii) **On the computation of the reciprocity indexes.** Given a finite index subgroup Γ of $\Gamma_{\mathbb{Q}}$, as x and y vary in $\mathbb{P}^1(\mathbb{Q})$, finding an explicit arithmetic value of the reciprocity index $\iota_{\Gamma,\rm rec}(x, y)$ is somewhat delicate, even when $\Gamma = \Gamma_{\mathbb{Q}}$. This also turns out to be related to problems of representations by primitive sums of two squares, as we now indicate.

By the diagonal $\Gamma_{\mathbb{Q}}$-invariance and by the transitivity of the action of $\Gamma_{\mathbb{Q}} = \mathrm{PSL}_2(\mathbb{Z})$ on $\mathbb{P}^1(\mathbb{Q})$, we only need to compute $\iota_{\Gamma_{\mathbb{Q}},\rm rec}(\infty, x)$ for $x \in K \cap [0, 1[$. We have $\iota_{\Gamma_{\mathbb{Q}},\rm rec}(\infty, x) = 1$ if and only if the geodesic line $\widetilde{\ell}_{\infty,x}(\mathbb{R})$ meets the $\Gamma_{\mathbb{Q}}$-orbit of i, that is, if and only if there exists $\gamma \in \Gamma_{\mathbb{Q}}$ such that $\operatorname{Re}(\gamma \cdot i) = x$. Let us write $x = \frac{p}{q}$ with $p, q \in \mathbb{Z}$ coprime and $q > 0$. Note that for all $a, b, c, d \in \mathbb{Z}$ such that $ad - bc = 1$, we have $c^2 + d^2 > 0$ and $(a^2 + b^2)(c^2 + d^2) - (ac + bd)^2 = (ad - bc)^2 = 1$ by the Diophantus identity. Hence $ac + bd$ and $c^2 + d^2$ are coprime by the Bézout identity, so that by the uniqueness property of reduced fractions, if $\frac{p}{q} = \frac{ac+bd}{c^2+d^2}$, then $q = c^2 + d^2$ and $p = ac + bd$. By Equation (5) and the discussion of Claim (ii), we hence have $\iota_{\Gamma_{\mathbb{Q}},\rm rec}(\infty, x) = 1$ if and only if $q = c^2 + d^2$ is a primitive sum of two squares and $p = qR(c, d)$.

3 Bianchi cusps are very maximal

Let K, $\mathscr{O}_K$, $\mathscr{I}_K$, h_K, D_K, $\Gamma_K = \mathrm{PSL}_2(\mathscr{O}_K)$ and $M_K = \Gamma_K\backslash\mathbb{H}^3_{\mathbb{R}}$ be as in Subsection 2.2. Let f_K be a square-free negative integer such that $K = \mathbb{Q}(f_K)$, with $D_K = 4f_K$ if $f_K \equiv 2, 3 \bmod 4$ and $D_K = f_K$ otherwise. When $h_K \neq 1$, there are elements of $\mathbb{P}^1(K)$ that have no Farey neighbour as defined in Equation (1), by Remark 7 (i). The aim of this Section is to advertise a more general notion of Farey neighbours and to prove that it solves this issue. We refer for instance to [Men], [EGM, Chap. 7], [BeS, Sect. 4] for background material on this Section.

Two distinct $x, y \in \mathbb{P}^1(K)$ are said to be *K-Farey neighbours* if for any $a, b, c, d \in \mathscr{O}_K$ such that $x = \frac{a}{b}$ and $y = \frac{c}{d}$, we have

$$(a\mathscr{O}_K + b\mathscr{O}_K)(c\mathscr{O}_K + d\mathscr{O}_K) = (ad - bc)\mathscr{O}_K\,. \tag{6}$$

This does not depend on the choices of a, b, c, d. We refer to Examples 14 (1) to (3) below for examples of K-Farey neighbours. After some remarks, we will recall the geometric interpretation of this property, due to [BeS], that proves without computation that being K-Farey neighbours is a property invariant by the diagonal action of Γ_K on the set of unordered pairs $\{x, y\}$ of distinct elements of $\mathbb{P}^1(K)$.

Remark 9. (1) If $x, y \in \mathbb{P}^1(K)$ are Farey neighbours, then they are K-Farey neighbours. Indeed, $(a, b, c, d) = (1, 0, 0, 1)$ is a solution of Equation (6), hence $\infty = \frac{1}{0}, 0 = \frac{0}{1}$ are K-Farey neighbours. Being K-Farey neighbours is invariant by Γ_K, hence this Claim (1) follows from Lemma 6. For example, Equation (6) implies that the K-Farey neighbours of $x = \infty = \frac{1}{0}$ are the points $\frac{c}{d}$ with $c\mathscr{O}_K + d\mathscr{O}_K = d\mathscr{O}_K$ or equivalently $d \mid c$, hence are the points in $\mathscr{O}_K$, that is, are its Farey neighbours. By Γ_K-invariance, the K-Farey neighbours of an element $x \in \mathbb{P}^1(K)$ whose associated ideal class by Equation (2) is principal are its Farey neighbours. In particular if there exist pairs of K-Farey neighbours that are not pairs of Farey neighbours, then $h_K \geqslant 2$.[7]

(2) If $x, y \in \mathbb{P}^1(K)$ are K-Farey neighbours, then their associated ideal classes are inverse one of the other in the group $\mathscr{I}_K$: by Equation (6), if $a, b, c, d \in \mathscr{O}_K$ are such that $x = \frac{a}{b}$ and $y = \frac{c}{d}$ are K-Farey neighbours, then $[a\mathscr{O}_K + b\mathscr{O}_K]^{-1} = [c\mathscr{O}_K + d\mathscr{O}_K]$. Hence if furthermore the divergent geodesic $\ell_{x,y}$ is reciprocal, then x and y in particular are in the same Γ_K-orbit, thus have same associated ideal class, which is either trivial or has order 2 in the group $\mathscr{I}_K$. We refer to Examples (1) to (3) below for examples of such order 2 ideal classes.

(3) Let $\mathtt{N}(\mathfrak{a}) = [\mathscr{O}_K : \mathfrak{a}]$ be the norm of a nonzero ideal $\mathfrak{a}$ of $\mathscr{O}_K$, extended by multiplicativity to the norm of fractional ideals. For every $a \in K$, let $\mathtt{N}(a) = \mathtt{N}(a\mathscr{O}_K)$. Equation (6) implies that $\mathtt{N}(\mathscr{O}_K + x\mathscr{O}_K)\,\mathtt{N}(\mathscr{O}_K + y\mathscr{O}_K) = \mathtt{N}(x - y)$, which is the equality case in the inequalities with $c_1 = c_2 = 1$ pages 10 and 11 of [Men].

Let us turn to a geometric characterisation of being K-Farey neighbours in the complex case, that is analogous to the tangency property of Ford circles discussed in Subsection 2.4. For every $\alpha \in \mathbb{P}^1(K) \smallsetminus \{\infty\}$, writing $\alpha = \frac{a}{b}$ with any $a, b \in \mathscr{O}_K$, the *canonical (closed) horoball* B_α in $\mathbb{H}^3_{\mathbb{R}}$ centered at α is the intersection with $\mathbb{H}^3_{\mathbb{R}}$ of the Euclidean closed ball with center $\big(\alpha, \frac{\mathtt{N}(a\mathscr{O}_K + b\mathscr{O}_K)}{2\,\mathtt{N}(b)}\big) \in \mathbb{H}^3_{\mathbb{R}}$ and radius $\frac{\mathtt{N}(a\mathscr{O}_K + b\mathscr{O}_K)}{2\,\mathtt{N}(b)}$. This does not depend on the

[7] See the comment after Theorem 12 for the converse.

choices of a and b. Furthermore, the *canonical horoball* in $\mathbb{H}^3_\mathbb{R}$ centered at ∞ is the already defined horoball $B_\infty = \{(z,t) \in \mathbb{H}^3_\mathbb{R} : t \geqslant 1\}$.

This family $(B_x)_{x\in\mathbb{P}^1(K)}$, constructed and studied in [Men], is a Γ_K-equivariant family of horoballs with pairwise disjoint interiors. In particular, for every $\gamma \in \Gamma_K$, two canonical horoballs B_x and B_y with distinct $x, y \in \mathbb{P}^1(K)$ touch at one point (or equivalently are tangent) if and only if $B_{\gamma\cdot x}$ and $B_{\gamma\cdot y}$ are tangent. The image of $\bigcup_{x\in\mathbb{P}^1(K)} B_x$ in the quotient orbifold $M_K = \Gamma_K\backslash\mathbb{H}^3_\mathbb{R}$ is the union of closed Margulis cusp neighbourhoods with pairwise disjoint interiors of the h_K ends of M_K.

The geometric characterisation alluded to above, proving the Γ_K-invariance of being K-Farey neighbours, is the following one, see [BeS, Prop. 4.1] for the proof.

Proposition 10. *Two distinct elements $x, y \in \mathbb{P}^1(K)$ are K-Farey neighbours if and only if the canonical horoballs B_x and B_y are tangent.* □

Remark 11. (1) It follows from Proposition 10 and Remark 9 (2) that for all distinct $x, y \in \mathbb{P}^1(K)$, if the canonical horoballs B_x and B_y are tangent, then the ideal classes associated with x and y are inverse one of the other.

(2) Since $\mathtt{N}(a\mathscr{O}_K + b\mathscr{O}_K) = \mathtt{N}(b\mathscr{O}_K)$ implies that $a\mathscr{O}_K + b\mathscr{O}_K = b\mathscr{O}_K$ hence that $b \mid a$ for all $a, b \in \mathscr{O}_K$, it follows from their construction that the canonical horoballs that are tangent (and distinct) to the canonical horoball B_∞ are the ones centered at $c = \frac{c}{1} \in \mathscr{O}_K$, confirming the example claim of Remark 9 (1), by Proposition 10.

The main result of this Section 3, proving the maximality of $(B_x)_{x\in\mathbb{P}^1(K)}$ at all cusps, is the following one.

Theorem 12. *Every element of $\mathbb{P}^1(K)$ has infinitely many K-Farey neighbours.*

By Proposition 10, we have an equivalent, more geometric formulation of Theorem 12.

Theorem 13. *For every $x \in \mathbb{P}^1(K)$, the canonical horoball B_x is tangent to infinitely many elements of Mendoza's canonical family $(B_x)_{x\in\mathbb{P}^1(K)}$ of horoballs.* □

If $h_K \geqslant 2$, then any element of $\mathbb{P}^1(K)$ whose associated ideal class is not principal admits K-Farey neighbours by Theorem 12, and they are not Farey neighbours since this ideal is not principal. Therefore by Remark 9 (1), there exist pairs of K-Farey neighbours that are not pairs of Farey neighbours if and only if $h_K \geqslant 2$.

Proof of Theorem 12. Since Γ_K preserves the set of pairs of K-Farey neighbours and since the stabilizer of any element of $\mathbb{P}^1(K)$ is an infinite parabolic subgroup, we only have to prove that every element $x' \in \mathbb{P}^1(K)$ admits an element x in its Γ_K-orbit that has a K-Farey neighbour y.

Let $a', b' \in \mathscr{O}_K$ be such that $x' = \frac{a'}{b'}$, and let $\mathfrak{a}' = a'\mathscr{O}_K + b'\mathscr{O}_K$. For all $a, b \in \mathscr{O}_K$, if $\mathfrak{a} = a\mathscr{O}_K + b\mathscr{O}_K$ belongs to the same ideal class as $\mathfrak{a}'$, then $x = \frac{a}{b}$ belongs to the same Γ_K-orbit as x' by the bijection (2). If $\mathfrak{a}$ is principal, then x is the same Γ_K-orbit as ∞, hence has K-Farey neighbours by Remark 9 (1), and so does x'. The norm of a nonprincipal prime ideal is a prime integer. By Weber's theorem in [Coh, Sect. X.12], there are infinitely many prime ideals in each ideal class. By for instance [Lem, Thm. 6.14], we may hence assume that $\mathfrak{a}$ is a nonprincipal prime ideal such that $[\mathfrak{a}] = [\mathfrak{a}']$ and $\mathtt{N}(\mathfrak{a}) = p_0$ is an odd prime such that one of the following two claims hold.[8]

[8] The third case of [Lem, Thm. 6.14] does not occur, since otherwise $p_0\mathscr{O}_K$ would be a prime ideal in that case, and $\mathtt{N}(\mathfrak{a}) = p_0$ implies that $\mathfrak{a} \mid p_0\mathscr{O}_K$, so that $\mathfrak{a} = p_0\mathscr{O}_K$ and $\mathtt{N}(\mathfrak{a}) = p_0^2$, a contradiction.

Case i). The prime p_0 ramifies in K, that is $p_0 \mid D_K$. With $\mathfrak{a} = \sqrt{f_K}\mathscr{O}_K + p_0\mathscr{O}_K$, we have $p_0\mathscr{O}_K = \mathfrak{a}^2$ by loc. cit.. We define $a_0 = 0$ in Case i). Note that we have $p_0 \mid -f_K = \mathtt{N}(a_0 + \sqrt{f_K})$ since p_0 is odd (and D_K and f_K have the same odd prime factors), and $p_0^2 \nmid -f_K = \mathtt{N}(a_0 + \sqrt{f_K})$ since f_K is square-free.

Case ii). The prime p_0 splits in K, that is the discriminant D_K is a quadratic residue modulo p_0. Since p_0 is odd, there exists $a_0 \in \mathbb{Z} \smallsetminus p_0\mathbb{Z}$ such that $a_0^2 = f_K \bmod p_0$. Let us define $\mathfrak{a} = (a_0 + \sqrt{f_K}\,)\mathscr{O}_K + p_0\mathscr{O}_K$. We then have $p_0\mathscr{O}_K = \mathfrak{a}\,\overline{\mathfrak{a}}$ by loc. cit.. We have $p_0 \mid a_0^2 - f_K = \mathtt{N}(a_0 + \sqrt{f_K}\,)$. If $p_0^2 \mid \mathtt{N}(a_0 + \sqrt{f_K}\,)$, then p_0^2 does not divide $\mathtt{N}(a_0 + p_0 + \sqrt{f_K}\,) = \mathtt{N}(a_0 + \sqrt{f_K}\,) + p_0^2 + 2a_0p_0$ since p_0 is odd and $a_0 \neq 0 \bmod p_0$. Hence up to replacing a_0 by $a_0 + p_0$, which does not change $\mathfrak{a}$ nor the fact that $p_0 \mid \mathtt{N}(a_0 + \sqrt{f_K}\,)$, we have $p_0^2 \nmid \mathtt{N}(a_0 + \sqrt{f_K})$.

In both cases, $\frac{\mathtt{N}(a_0+\sqrt{f_K}\,)}{p_0}$ and p_0 are relatively prime integers. By Bézout's identity for $\mathbb{Z}$, there exist $t, u \in \mathbb{Z}$ such that

$$\frac{\mathtt{N}(a_0 + \sqrt{f_K}\,)}{p_0}\,u - p_0\,t = 1\,. \tag{7}$$

Thus, setting $a = a_0 + \sqrt{f_K}$, $b = p_0$, $c = t(a_0 + \sqrt{f_K}\,)$ and $d = u\,\frac{\mathtt{N}(a_0+\sqrt{f_K}\,)}{p_0}$ that all belong to $\mathscr{O}_K$, we have $\mathfrak{a} = a\,\mathscr{O}_K + b\,\mathscr{O}_K$ and (a, b, c, d) satisfies Equation (6) : Using Equation (7) for the last two equalities, we have

$$\begin{aligned}
&(a\mathscr{O}_K + b\mathscr{O}_K)(c\mathscr{O}_K + d\mathscr{O}_K)\\
&= \big((a_0 + \sqrt{f_K}\,)\mathscr{O}_K + p_0\,\mathscr{O}_K\big)\big(t\,(a_0 + \sqrt{f_K}\,)\mathscr{O}_K + u\,\frac{\mathtt{N}(a_0 + \sqrt{f_K}\,)}{p_0}\mathscr{O}_K\big)\\
&= (a_0 + \sqrt{f_K}\,)\Big(t\,(a_0 + \sqrt{f_K}\,)\mathscr{O}_K + u\,\frac{\mathtt{N}(a_0 + \sqrt{f_K}\,)}{p_0}\mathscr{O}_K + p_0\,t\,\mathscr{O}_K + u\,(a_0 - \sqrt{f_K}\,)\mathscr{O}_K\Big)\\
&= (a_0 + \sqrt{f_K}\,)\mathscr{O}_K = (ad - bc)\mathscr{O}_K\,.
\end{aligned} \tag{8}$$

Thus $x = \frac{a}{b}$ is in the same Γ_K-orbit as x' since they have the same associated ideal class $[\mathfrak{a}] = [\mathfrak{a}']$, and $y = \frac{c}{d}$ is a K-Farey neighbour of x as wanted. □

Note that the computation (8) in the proof of Theorem 12 is valid as long as the integers $\frac{\mathtt{N}(a_0+\sqrt{f_K}\,)}{p_0}$ and p_0 in Equation (7) are relatively prime. Thus, in order to produce examples of K-Farey neighbours, we can use the tables at the end of [Som] (and their reproduction at the end of [Coh]) where representatives are listed for all ideal classes of imaginary quadratic number fields with $-97 \leqslant f_K \leqslant -1$. Note that the representatives in these tables are not always prime ideals, though.

Counting results for pairs of K-Farey neighbours in an orbit of a given pair $\{x, y\}$ by a finite index subgroup Γ of Γ_K follow immediately from Theorem 5. The results become more explicit in the cases where the values of the reciprocity index $\iota_\Gamma(x, y)$ and the multiplicity $m_\Gamma(x, y)$ are known. If the K-Farey neighbours $x, y \in K$ are in two different Γ_K-orbits, then $\iota_{\Gamma_K}(x, y) = 2$. The following examples provide in particular infinite collections of K-Farey neighbours $x, y \in K$ with $\iota_{\Gamma_K}(x, y) = 1$.

Examples 14. (1) Assume that $-f_K$ is at least 6 and is not a prime. Let p_0 be a prime factor of $-f_K$. Then p_0 is ramified in K. As in Case i) in the proof of Theorem 12, the ideal $\mathfrak{a} = \sqrt{f_K}\mathscr{O}_K + p_0\mathscr{O}_K$ satisfies $\mathfrak{a}^2 = p_0\mathscr{O}_K$. In particular $\mathtt{N}(\mathfrak{a}) = p_0$, hence $\mathfrak{a}$ is prime. Furthermore $\mathfrak{a}$ is not principal by the following result.

Lemma 15. *There are no elements of norm p_0 in $\mathscr{O}_K$.*

Proof. Let $\omega_K = \sqrt{f_K}$ if $f_K \equiv 2,3 \mod 4$ and $\omega_K = \frac{1+\sqrt{f_K}}{2}$ if $f_K \equiv 1 \mod 4$, so that $\mathscr{O}_K = \mathbb{Z} + \mathbb{Z}\omega_K$. The norm of an element of $\mathscr{O}_K \cap \mathbb{Z}$ is not a prime. Let $u, v \in \mathbb{Z}$ with $v \neq 0$, and assume for a contradiction that $\mathtt{N}(u + v\,\omega_K) = p_0$. If $f_K \not\equiv 1 \mod 4$, then $\mathtt{N}(u + v\,\omega_K) \geqslant -v^2 f_K \geqslant -f_K > p_0$ since $p_0 \mid -f_K$ and f_K is not a prime, a contradiction. If $|v| \geqslant 2$ or if $\frac{-f_K}{p_0} > 4$, then $\mathtt{N}(u + v\,\omega_K) \geqslant v^2 (\operatorname{Im}\, \omega_K)^2 \geqslant -v^2 \frac{f_K}{4} > p_0$, a contradiction. Thus $f_K \equiv 1 \mod 4$ and in particular f_K is odd with $-f_K = 3\,p_0$, and $v = \pm 1$. Therefore $\mathtt{N}(u + v\,\omega_K) = (u \pm \frac{1}{2})^2 + \frac{3\,p_0}{4}$. Since the solutions of the equations $(u \pm \frac{1}{2})^2 + \frac{3\,p_0}{4} = p_0$ with unknown u, that are $u = \frac{\pm 1 \pm \sqrt{p_0}}{2}$, are irrational, this contradicts the assumption. $\square$

Equation (7) becomes

$$-\frac{f_K}{p_0}\, u - p_0\, t = 1\,. \tag{9}$$

Let $(t, u) \in \mathbb{Z}^2$ be an integral solution of Equation (9). By the end of the proof of Theorem 12, the element $\beta = \frac{t\sqrt{f_K}}{-u\frac{f_K}{p_0}} = -\frac{t\,p_0}{u\sqrt{f_K}} \in K$ is a K-Farey neighbour of $\alpha = \frac{\sqrt{f_K}}{p_0}$ (but note that they are not Farey neighbours since $\mathfrak{a}$ is not principal). By Equation (9), we have $C = \begin{bmatrix} \sqrt{f_K} & t \\ p_0 & -\frac{u\sqrt{f_K}}{p_0} \end{bmatrix} \in \mathrm{PSL}_2(K)$. We have $C \cdot \infty = \alpha$ and $C \cdot 0 = \beta$. Note that $\iota_{p_0} = \begin{bmatrix} 0 & \frac{1}{p_0} \\ -p_0 & 0 \end{bmatrix} \in \mathrm{PSL}_2(K)$ is an involution exchanging the points at infinity ∞ and 0 of $\partial_\infty \mathbb{H}^3_{\mathbb{R}}$. The conjugate involution

$$E = C\, \iota_{p_0}\, C^{-1} = \begin{bmatrix} (tu-1)\sqrt{f_K} & t^2\, p_0 + \frac{f_K}{p_0} \\ -\frac{u^2\, f_K}{p_0} - p_0 & (1-tu)\sqrt{f_K} \end{bmatrix}$$

belongs to $\mathrm{PSL}_2(\mathscr{O}_K)$ since $p_0 \mid -f_K$ and satisfies $E \cdot \alpha = (C\, \iota_{p_0}) \cdot \infty = C \cdot 0 = \beta$ and similarly $E \cdot \beta = \alpha$. Thus, with the notation of Section 2, the divergent geodesic $\ell_{\alpha,\beta}$ in M_K is reciprocal and $\iota_{\Gamma_K,\mathrm{rec}}(\alpha, \beta) = 1$.

The pointwise stabilizer $\Gamma_{\alpha,\beta}$ of $\widetilde{\ell}_{\alpha,\beta}(\mathbb{R}) = C \cdot \widetilde{\ell}_{\infty,0}(\mathbb{R})$, which is the conjugate by C of the pointwise stabilizer $\Big\{M(\theta) = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} : \theta \in \mathbb{R}\Big\}$ of $\widetilde{\ell}_{\infty,0}(\mathbb{R})$, can also be determined. Note that by Equation (9), we have

$$C\, M(\theta)\, C^{-1} = \begin{bmatrix} \cos\theta - i\,(1 + \frac{2 f_K u}{p_0}) \sin\theta & -2i\, t\,\sqrt{f_K}\sin\theta \\ -2i\, u\,\sqrt{f_K}\sin\theta & \cos\theta + i\,(1 + \frac{2 f_K u}{p_0}) \sin\theta \end{bmatrix}.$$

Let $\theta \in \mathbb{R}$ be such that $C\, M(\theta)\, C^{-1}$ belongs to $\mathrm{PSL}_2(\mathscr{O}_K)$. Then the trace of this matrix, which is $\pm 2\cos\theta$, belongs to $\mathscr{O}_K \cap \mathbb{R} = \mathbb{Z}$. Therefore $\cos\theta = 0, \pm 1, \pm\frac{1}{2}$ and correspondingly $\sin\theta = \pm 1, 0, \pm\frac{\sqrt{3}}{2}$. If, for a contradiction, $\sin\theta \neq 0$, then the 2-1 entry of the above matrix, that is equal to $\pm\, 2\, u\,\sqrt{-f_K}$ or $\pm\, u\,\sqrt{-3\, f_K}$, also belongs to $\mathscr{O}_K \cap \mathbb{R} = \mathbb{Z}$. But since $-f_K$ is squarefree and at least $6 > 3$, these entries are irrational, a contradiction. Thus, the stabilizer $\Gamma_{\alpha,\beta}$ is trivial and $m_{\Gamma_K}(\alpha, \beta) = 1$.

(2) Assume in this family of examples that $K = \mathbb{Q}(\sqrt{f_K}\,)$ with $f_K \equiv 3 \mod 4$ and $-f_K \geqslant 5$. Then $\mathscr{O}_K = \mathbb{Z} + \sqrt{f_K}\,\mathbb{Z}$, $D_K = 4\, f_K$ and $2 \mid D_K$, so that 2 ramifies in K. Let $\mathfrak{a} = (1 + \sqrt{f_K}\,)\mathscr{O}_K + 2\mathscr{O}_K$. Since $\overline{1 + \sqrt{f_K}} = 2 - (1 + \sqrt{f_K}\,)$ and by for instance [Art,

Lem. 13.8.4], we have $\mathfrak{a}^2 = \mathfrak{a}\,\overline{\mathfrak{a}} = 2\mathscr{O}_K$, so that the class of $\mathfrak{a}$ has order 2 in the ideal class group $\mathscr{I}_K$. Even if the case $p_0 = 2$ does not appear in the proof of Theorem 12, the analogous computations work in the present infinite collection of examples.

Now, $\mathtt{N}(1+\sqrt{f_K}\,) = 1 - f_K \equiv 2 \bmod 4$. Hence $\frac{1-f_K}{2}$ and 2 are coprime integers, and Equation (7) becomes $\frac{1-f_K}{2}\,u - 2\,t = 1$, satisfied for instance by $t = -\frac{1+f_K}{4}$ and $u = 1$. Let $a = 1+\sqrt{f_K}$, $b = 2$, $c = -\frac{1+f_K}{2}$ and $d = 1-\sqrt{f_K}$, that belong to $\mathscr{O}_K$ and satisfy $ad-bc = 2$. By a computation similar to the one in Equation (8), the quadruple (a,b,c,d) is a solution of Equation (6). Hence $\alpha = \frac{a}{b} = \frac{1+\sqrt{f_K}}{2}$ and $\beta = \frac{c}{d} = -\frac{1+f_K}{2(1-\sqrt{f_K}\,)}$ are K-Farey neighbours (that are not Farey neighbours since $\mathfrak{a}$ is not principal).

The element $C = \begin{bmatrix} 1+\sqrt{f_K} & -\frac{1+f_K}{4} \\ 2 & \frac{1-\sqrt{f_K}}{2} \end{bmatrix} \in \mathrm{PSL}_2(K)$ maps ∞ and 0 to α and β respectively. Note that $\iota_2 = \left[\begin{smallmatrix} 0 & \frac{1}{2} \\ -2 & 0 \end{smallmatrix}\right] \in \mathrm{PSL}_2(K)$ is an involution exchanging ∞ and 0. The involution

$$E = C\,\iota_2\,C^{-1} = \begin{bmatrix} \frac{-3+f_K-(5+f_K)\sqrt{f_K}}{4} & \frac{5+6f_K+f_K^2}{8} + \sqrt{f_K} \\ -\frac{5+f_K}{2} + \sqrt{f_K} & \frac{3-f_K+(5+f_K)\sqrt{f_K}}{4} \end{bmatrix}$$

belongs to $\mathrm{PSL}_2(\mathscr{O}_K)$ since $f_K \equiv 3 \bmod 4$, and satisfies $E\cdot\alpha = \beta$ and $E\cdot\beta = \alpha$. Thus, the divergent geodesic $\ell_{\alpha,\beta}$ in M_K is reciprocal and $\iota_{\Gamma_K,\mathrm{rec}}(\alpha,\beta) = 1$.

The pointwise stabilizer $\Gamma_{\alpha,\beta}$ of the geodesic line $\widetilde{\ell}_{\alpha,\beta}(\mathbb{R}) = C\cdot\widetilde{\ell}_{\infty,0}(\mathbb{R})$ can be determined as in the previous examples (1). Let $\theta\in\mathbb{R}$ be such that the entries of the elliptic element

$$C\;M(\theta)\;C^{-1} = \begin{bmatrix} \cos\theta - i f_K\sin\theta & i(1+\sqrt{f_K})\frac{1+f_K}{2}\sin\theta \\ -2i(\sqrt{f_K}-1)\sin\theta & \cos\theta + i f_K\sin\theta \end{bmatrix}$$

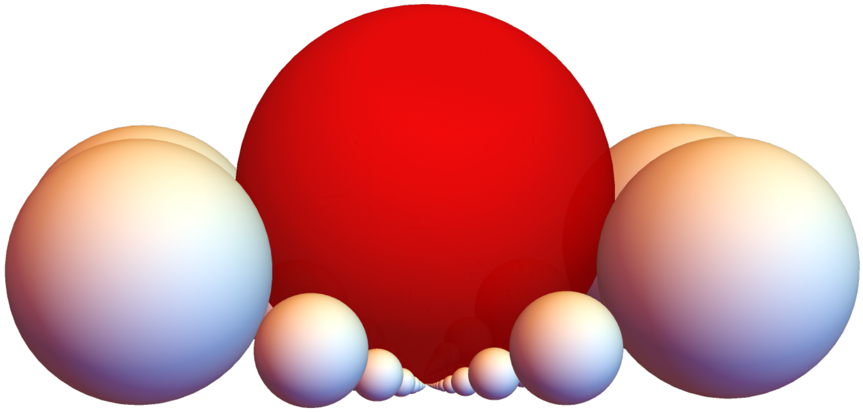
are in $\mathscr{O}_K$. Then by taking the sum and the difference of the diagonal entries, we have $2\cos\theta\in\mathscr{O}_K\cap\mathbb{R} = \mathbb{Z}$ and $2if_K\sin\theta\in\mathscr{O}_K\cap(i\mathbb{R}) = \sqrt{f_K}\,\mathbb{Z}$. Hence $\cos\theta = 0, \pm1, \pm\frac{1}{2}$ and $2\sqrt{-f_K}\sin\theta\in\mathbb{Z}$. Since $-f_K \geqslant 5 > 3$, this implies as in the previous examples (1) that the stabilizer $\Gamma_{\alpha,\beta}$ is trivial, so that $m_{\Gamma_K}(\alpha,\beta) = 1$. Note that if $K = \mathbb{Q}(\sqrt{-1})$, then $\Gamma_{\alpha,\beta}$ consists of id and $\begin{bmatrix} i & 0 \\ 2(1+i) & -i \end{bmatrix}$, so that $m_{\Gamma_K}(\alpha,\beta) = 2$.

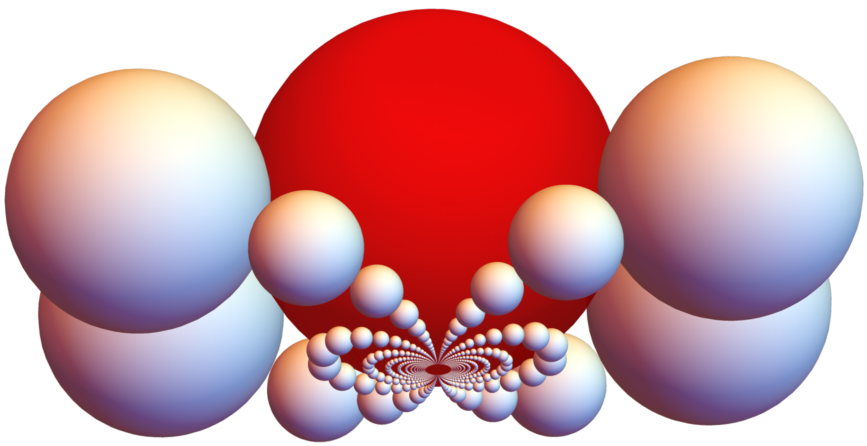
$(2)^{\mathrm{bis}}$ Let us consider the particular case $f_K = -5$ of the previous family of examples (2). Recall that the class number of $K = \mathbb{Q}(\sqrt{-5})$ is $h_K = 2$. The ideal $\mathfrak{a} = (1+i\sqrt{5})\mathscr{O}_K + 2\mathscr{O}_K$ (that satisfies $\mathfrak{a} = \overline{\mathfrak{a}}$ and $\mathtt{N}(\mathfrak{a}) = 2$) is a prime representative of the unique nonprincipal ideal class. By the general computation above, the elements $\alpha = \frac{1+i\sqrt{5}}{2}$ and $\beta = -\frac{2}{1-i\sqrt{5}}$ in $\mathbb{Q}(\sqrt{-5})$ are K-Farey neighbours, that are not Farey neighbours.

The orbit of β under the stabilizer Γ_α of α in Γ_K gives an infinite collection of K-Farey neighbours of α. For example by the proof of [PP1, Lemma 6], we have

$$\Gamma_\alpha = \left\{ \begin{bmatrix} 1+(1+i\sqrt{5})x & (2-i\sqrt{5})x \\ 2x & 1-(1+i\sqrt{5})x \end{bmatrix} : x\in\mathscr{O}_K \right\}.$$

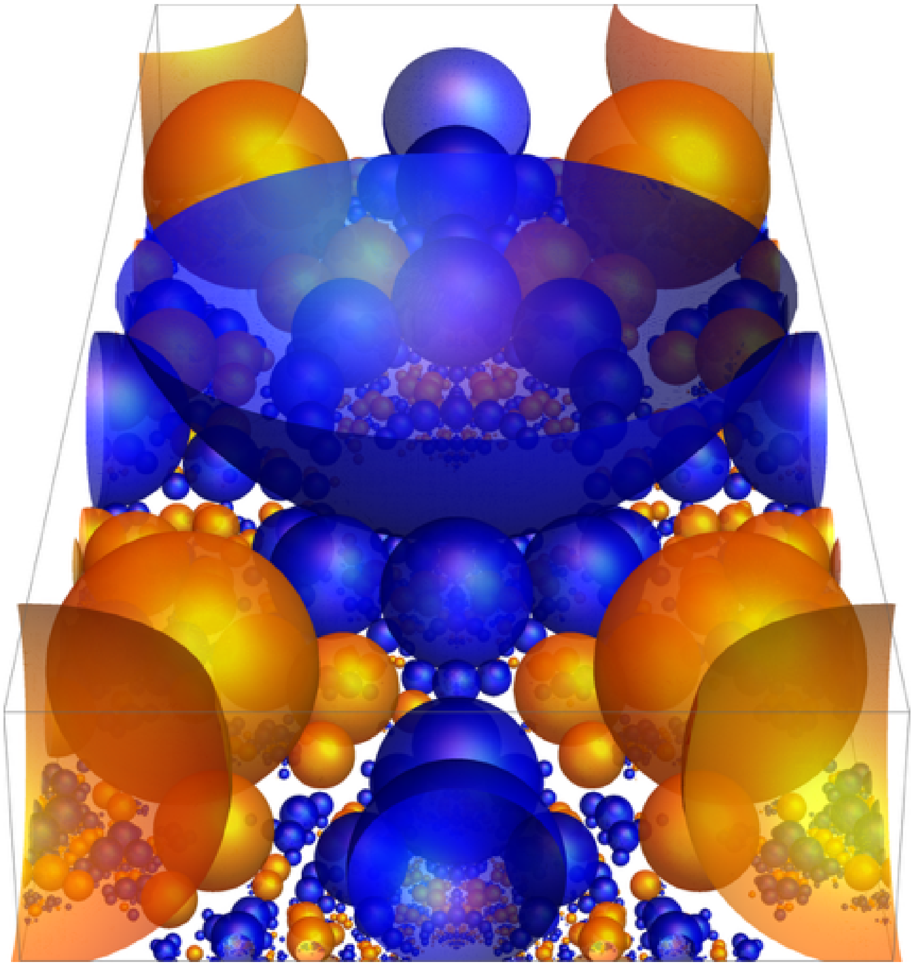
The figures below show the canonical horoball B_α (drawn in red) and the canonical horoballs (drawn in beige) of some of the K-Farey neighbours of α (images of B_β by elements of Γ_α), that are hence tangent.





As seen above, the elliptic element $\begin{bmatrix} -2 & i\sqrt{5} \\ i\sqrt{5} & 2 \end{bmatrix} \in \mathrm{PSL}_2(\mathscr{O}_K)$ of order 2 exchanges α and β. Hence for all K-Farey neighbours α' and β' in K, the divergent geodesic $\ell_{\alpha',\beta'}$ in M_K is reciprocal.

The figure below shows parts of the two families of horospheres that correspond to the classes of principal (blue) and non-principal (orange) ideals. The horospheres are somewhat translucent, and the horospheres can be seen even if they are behind other canonical horoballs as seen from the viewpoint. The horospheres are restricted to the symmetric closed fundamental domain $\{(z,t) \in \mathbb{H}^3_{\mathbb{R}} : |\operatorname{Re} z| \leqslant \frac{1}{2},\ |\operatorname{Im} z| \leqslant \frac{\sqrt{5}}{2}\}$ of the stabilizer of ∞ in $\mathrm{PSL}_2(\mathscr{O}_K)$, and the picture is cut at height $\frac{1}{3}$.



(3) Assume in this other family of examples that $K = \mathbb{Q}(\sqrt{f_K}\,)$ with $f_K \equiv 2 \bmod 4$ and $-f_K \geqslant 6$. Similarly as for the examples (2), let $a = \sqrt{f_K}$, $b = 2$, $c = \frac{f_K+2}{2}$ and $d = \sqrt{f_K}$, that belong to $\mathscr{O}_K$. The nonprincipal ideal class of $\mathfrak{a} = a\mathscr{O}_K + b\mathscr{O}_K$ has order 2 in $\mathscr{I}_K$ again by for instance [Art, Lem. 13.8.4] since 2 ramifies in K. Then we have $ad - bc = -2$ and (a, b, c, d) satisfies the condition (6). Thus the elements $\alpha = \frac{a}{b} = \frac{\sqrt{f_K}}{2}$ and $\beta = \frac{2+f_K}{2\sqrt{f_K}}$ of K are K-Farey neighbours that are not Farey neighbours (since $[\mathfrak{a}]$ has order 2 hence is not the principal class). The element $C = \begin{bmatrix} \sqrt{f_K} & -\frac{2+f_K}{4} \\ 2 & -\frac{\sqrt{f_K}}{2} \end{bmatrix} \in \mathrm{PSL}_2(K)$ maps ∞ and 0 to α and β respectively. The involution

$$E = C\,\iota_2\,C^{-1} = \begin{bmatrix} -\frac{6+f_K}{4}\sqrt{f_K} & \frac{4+8f_K+f_K^2}{8} \\ -\frac{4+f_K}{2} & \frac{6+f_K}{4}\sqrt{f_K} \end{bmatrix}$$

belongs to $\mathrm{PSL}_2(\mathscr{O}_K)$ and satisfies $E \cdot \alpha = \beta$ and $E \cdot \beta = \alpha$. Thus, the divergent geodesic $\ell_{\alpha,\beta}$ in M_K is reciprocal and $\iota_{\Gamma_K,\mathrm{rec}}(\alpha, \beta) = 1$.

The pointwise stabilizer $\Gamma_{\alpha,\beta}$ of the geodesic line $\widetilde{\ell}_{\alpha,\beta}(\mathbb{R})$ can be determined in the same way. For $\theta \in \mathbb{R}$, the entries of the elliptic element

$$C\ M(\theta)\ C^{-1} = \begin{bmatrix} \cos\theta - i(1+f_K)\sin\theta & i\sqrt{f_K}\,\frac{2+f_K}{2}\sin\theta \\ -2i\sqrt{f_K}\sin\theta & \cos\theta + i(1+f_K)\sin\theta \end{bmatrix}$$

are in $\mathscr{O}_K$ if and only if $\sin\theta = 0$ (as for the examples (2), otherwise the 2-1 entry would be an irrational real number), since $-f_K \geqslant 6 > 3$. This implies that $\Gamma_{\alpha,\beta}$ is trivial, thus $m_{\Gamma_K}(\alpha,\beta) = 1$.

(4) The last congruence property on f_K is when $f_K \equiv 1 \bmod 4$. If furthermore $-f_K$ is a prime integer, then there are no elements of order 2 in the class group $\mathscr{I}_K$ by, for instance, [Cox, Prop. 3.11]. By Remark 9 (2), all reciprocal K-Farey neighbours in K are then Farey neighbours.

4 Farey neighbours in rational definite quaternion algebras

In this Section, we study similar asymptotic countings of quaternionic Farey neighbours. Let $\mathbb{H}$ be the standard Hamilton quaternion algebra over $\mathbb{R}$, with canonical $\mathbb{R}$-basis $(1, i, j, k)$ and with conjugation $x \mapsto \overline{x}$, reduced norm $\mathtt{n}$ and reduced trace $\mathtt{tr}$. We denote by $\mathbb{P}^1_r(\mathbb{H})$ the right projective line of $\mathbb{H}$, identified as usual with the Alexandrov compactification $\mathbb{H} \cup \{\infty\}$ where $[x : y] = xy^{-1}$ if $y \neq 0$ and $[1 : 0] = 1\,0^{-1} = \infty$.

Let $\mathscr{O}$ be a maximal order in a quaternion algebra A over $\mathbb{Q}$, which is definite (that is, $A \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{H}$), with class number h_A and discriminant D_A. Its group $\mathscr{O}^\times$ of invertible elements is finite, of order 2, 4, 6, 12 (when $D_A = 3$) or 24 (when $D_A = 2$). An example is given by the *Hurwitz order* $\mathscr{O} = \mathbb{Z} + \mathbb{Z}\,i + \mathbb{Z}\,j + \mathbb{Z}\,\frac{1+i+j+k}{2}$ in $A = \mathbb{Q} + \mathbb{Q}\,i + \mathbb{Q}\,j + \mathbb{Q}\,k$, in which case $h_A = 1$ and $D_A = 2$. We refer for these informations and more to [Vig].

We will say that two elements α and β in $\mathbb{P}^1_r(A) = A \cup \{\infty\}$ are *Farey neighbours with respect to* $\mathscr{O}$ if there exist $p, q, r, s \in \mathscr{O}$ with $\alpha = pq^{-1}$, $\beta = rs^{-1}$, and either we have $q = 0$ and $p, s \in \mathscr{O}^\times$ or we have $q \neq 0$ and

$$\mathtt{n}(qpq^{-1}s - qr) = 1\,. \tag{10}$$

This condition is the appropriate noncommutative analog of Equation (1). Let $\mathfrak{N}_{\mathscr{O}}$ be the set of unordered pairs of Farey neighbours in $\mathbb{P}^1_r(A)$ with respect to $\mathscr{O}$. It is easy to check that the additive group $\mathscr{O}$ acts by simultaneous translations on the set $\mathfrak{N}_{\mathscr{O}}$. The following theorem gives an effective asymptotic counting result for pairs of quaternionic Farey neighbours with respect to $\mathscr{O}$ when the lower bound on their distances shrinks to 0.

Theorem 16. *As $\epsilon > 0$ tends to 0, we have*

$$\begin{aligned}&\operatorname{Card}\big(\mathscr{O}\backslash\big\{\{\alpha,\beta\} \in \mathfrak{N}_{\mathscr{O}} : \mathtt{n}(\beta-\alpha) \geqslant \epsilon\big\}\big)\\ &= -\frac{2160\ D_A}{\zeta(3)\ |\mathscr{O}^\times|^2\ \prod_{p\,|D_A}(p^3-1)(p-1)}\ \frac{\ln\epsilon}{\epsilon^2} + \mathrm{O}(\epsilon^{-2})\,.\end{aligned}$$

As usual, the above index p ranges over primes. We will actually prove a much stronger result, that requires some information on the Hamilton-Bianchi groups $\mathrm{PSL}_2(\mathscr{O})$. See for instance [Kel] for background; we will follow the presentation of [PP2, §3].

The *Dieudonné determinant* is the group morphism $\mathrm{Det} : \mathrm{GL}_2(\mathbb{H}) \to]0, +\infty[$ defined by

$$\mathrm{Det} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \big(\mathtt{n}(a\,d) + \mathtt{n}(b\,c) - \mathtt{tr}(a\,\overline{c}\,d\,\overline{b})\big)^{\frac{1}{2}} . \tag{11}$$

The Lie group $\mathrm{SL}_2(\mathbb{H})$ is the kernel of Det. We denote by $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{PSL}_2(\mathbb{H}) = \mathrm{SL}_2(\mathbb{H})/\{\pm\,\mathrm{id}\}$ the image of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{H})$. The group $\mathrm{PSL}_2(\mathbb{H})$ acts faithfully by homographies on the right projective plane $\mathbb{P}^1_r(\mathbb{H}) = \mathbb{H} \cup \{\infty\}$, by $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \cdot z = (az+b)(cz+d)^{-1}$ for all $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{PSL}_2(\mathbb{H})$ and $z \in \mathbb{P}^1_r(\mathbb{H})$, with the usual conventions when $z = \infty$ or $z = -c^{-1}d$. With $ds^2_{\mathbb{H}}$ the usual translation-invariant flat Riemannian metric on $\mathbb{H}$ (making the canonical $\mathbb{R}$-basis $(1, i, j, k)$ of $\mathbb{H}$ orthonormal at each point), we identify $\mathbb{H}^5_{\mathbb{R}}$ with

$$\Big(\{(z,t) \in \mathbb{H} \times \mathbb{R} : t > 0\},\ \frac{ds^2_{\mathbb{H}} + dt^2}{t^2}\Big) .$$

The group $\mathrm{PSL}_2(\mathbb{H})$ acts faithfully on $\mathbb{H}^5_{\mathbb{R}}$ by the Poincaré extension procedure,[9] and $\mathrm{PSL}_2(\mathbb{H})$ thus identifies with the orientation preserving isometry group of $\mathbb{H}^5_{\mathbb{R}}$.

Let ${}_{\mathscr{O}}\mathscr{I}$ be the set of left ideal classes of $\mathscr{O}$, whose cardinality is the class number h_A. The subgroup $\Gamma_{\mathscr{O}} = \mathrm{PSL}_2(\mathscr{O})$ is an arithmetic lattice in $\mathrm{PSL}_2(\mathbb{H})$. By for instance [KO, Satz 2.1, 2.2], it acts with $(h_A)^2$ orbits on its set of parabolic fixed points $\mathbb{P}^1_r(A) = A \cup \{\infty\}$ in $\partial_\infty \mathbb{H}^5_{\mathbb{R}}$. The stabiliser of ∞ in $\Gamma_{\mathscr{O}}$ is

$$\Gamma_{\mathscr{O},\infty} = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in \Gamma_{\mathscr{O}} : a, d \in \mathscr{O}^\times\ ,\ b \in \mathscr{O} \right\} .$$

Theorem 17. *Let Γ be a finite index subgroup of $\Gamma_{\mathscr{O}} = \mathrm{PSL}_2(\mathscr{O})$, and let Γ_∞ be the stabiliser of ∞ in Γ. For all distinct $x, y \in A \cup \{\infty\}$, as $\epsilon > 0$ tends to 0, we have*

$$\begin{aligned} &\mathrm{Card}\,\big(\Gamma_\infty \backslash \big\{\{\alpha,\beta\} \in \Gamma \cdot \{x,y\} : \mathtt{n}(\beta-\alpha) \geqslant \epsilon\big\}\big) \\ &= -\,\frac{2160\ D_A\ \iota_{\Gamma,\mathrm{rec}}(x,y)\ [\Gamma_{\mathscr{O},\infty} : \Gamma_\infty]}{\zeta(3)\ |\mathscr{O}^\times|^2\ \big(\prod_{p\,|D_A}(p^3-1)(p-1)\big)\ m_\Gamma(x,y)\ [\Gamma_{\mathscr{O}} : \Gamma]}\ \frac{\ln\epsilon}{\epsilon^2} + \mathrm{O}(\epsilon^{-2}). \end{aligned}$$

Proof. As in the proof of Theorem 4, we apply Theorem 3 with $n = 5$, with $M = \Gamma \backslash \mathbb{H}^5_{\mathbb{R}}$, with $D^- = \Gamma_\infty \backslash B_\infty$, and with $D^+ = \ell_{x,y}(\mathbb{R})$. By Emery's volume formula [PP2, Theo. 8 and Appendix], we have

$$\mathrm{Vol}(M) = [\Gamma_{\mathscr{O}} : \Gamma]\ \mathrm{Vol}(\Gamma_{\mathscr{O}} \backslash \mathbb{H}^5_{\mathbb{R}}) = [\Gamma_{\mathscr{O}} : \Gamma]\ \frac{\zeta(3)\ \prod_{p\,|D_A}(p^3-1)(p-1)}{11520} .$$

The index $[\Gamma_{\mathscr{O},\infty} : \mathscr{O}]$ in $\Gamma_{\mathscr{O},\infty}$ of its unipotent subgroup consisting in the translations by elements of $\mathscr{O}$ is equal to $\frac{|\mathscr{O}^\times|^2}{2}$. By the Remark just above [PP2, Lemma 15], we have

$$\mathrm{Vol}(\partial D^-) = [\Gamma_{\mathscr{O},\infty} : \Gamma_\infty]\,\mathrm{Vol}(\Gamma_{\mathscr{O},\infty} \backslash \partial B_\infty) = [\Gamma_{\mathscr{O},\infty} : \Gamma_\infty]\ \frac{D_A}{8\ |\mathscr{O}^\times|^2} .$$

The Euclidean distance in $\mathbb{H}$ between two elements $\alpha, \beta \in \mathbb{H}$ is $\mathtt{n}(\beta-\alpha)^{\frac{1}{2}}$, so that the length of the common perpendicular from B_∞ to $\widetilde{\ell}_{\alpha,\beta}(\mathbb{R})$ when it exists is $\ln\Big(\frac{2}{\mathtt{n}(\beta-\alpha)^{\frac{1}{2}}}\Big)$.

[9] See for instance [PP2, Eq. (14)]

Hence as in the proof of Theorem 4, since $\Gamma\big(\frac{5}{2}\big) = \frac{3\sqrt{\pi}}{4}$ and $\Gamma(3) = 2! = 2$, as $\epsilon > 0$ tends to 0, we have

$$\begin{aligned}
&\mathrm{Card}\,\big(\Gamma_\infty\backslash\{\{\alpha,\beta\}\in\Gamma\cdot\{x,y\}:\ \mathtt{n}(\beta-\alpha)\geqslant\epsilon\}\big) = \mathscr{N}_{D^-,D^+}\Big(\ln\frac{2}{\sqrt{\epsilon}}\Big) + \mathrm{O}\Big(\ln\frac{2}{\sqrt{\epsilon}}\Big)\\
&= \frac{\Gamma(\frac{5}{2})\ \iota_{\Gamma,\mathrm{rec}}(x,y)\ [\Gamma_{\mathscr{O},\infty}:\Gamma_\infty]\ \frac{D_A}{8\,|\mathscr{O}^\times|^2}}{2\sqrt{\pi}\ \Gamma(3)\ m_\Gamma(x,y)\ [\Gamma_{\mathscr{O}}:\Gamma]\ \frac{\zeta(3)\ \prod_{p\,|D_A}(p^3-1)(p-1)}{11520}}\ \Big(\ln\frac{2}{\sqrt{\epsilon}}\Big)\Big(\frac{2}{\sqrt{\epsilon}}\Big)^4 + \mathrm{O}(\epsilon^{-2})\\
&= -\ \frac{2160\ D_A\ \iota_{\Gamma,\mathrm{rec}}(x,y)\ [\Gamma_{\mathscr{O},\infty}:\Gamma_\infty]}{\zeta(3)\ |\mathscr{O}^\times|^2\ \big(\prod_{p\,|D_A}(p^3-1)(p-1)\big)\ m_\Gamma(x,y)\ [\Gamma_{\mathscr{O}}:\Gamma]}\ \frac{\ln\epsilon}{\epsilon^2} + \mathrm{O}(\epsilon^{-2})\,. \qquad \Box
\end{aligned}$$

In order to prove Theorem 16, the key translation between the arithmetics and the geometry is the following lemma.

Lemma 18. *Two distinct elements $\alpha,\beta\in\mathbb{P}^1_r(A) = A\cup\{\infty\}$ are Farey neighbours with respect to $\mathscr{O}$ if and only if there exists $\gamma\in\Gamma_{\mathscr{O}} = \mathrm{PSL}_2(\mathscr{O})$ such that $\gamma\cdot\infty = \alpha$ and $\gamma\cdot 0 = \beta$.*

Proof. For every $\big(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\big)\in\mathrm{GL}_2(\mathbb{H})$ such that $c\neq 0$, by for instance [PP2, Eq. (12)], we have $\mathrm{Det}\big(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\big) = \mathtt{n}(cac^{-1}d - cb)^{\frac{1}{2}}$.

Let $\alpha,\beta\in\mathbb{P}^1_r(A)$ be distinct elements. Assume that there exists $\gamma = \big[\begin{smallmatrix} p & r\\ q & s\end{smallmatrix}\big]\in\Gamma_{\mathscr{O}}$ such that $\gamma\cdot\infty = \alpha$ and $\gamma\cdot 0 = \beta$. If $q = 0$, then $\alpha = \infty$ and by Equation (11), we have $\mathtt{n}(ps) = (\mathrm{Det}(\gamma))^2 = 1$. Since $p,s\in\mathscr{O}$, we have $\mathtt{n}(p) = \mathtt{n}(s) = 1$ and $p,s\in\mathscr{O}^\times$, hence α,β are Farey neighbours with respect to $\mathscr{O}$. If $q\neq 0$, then $p,q,r,s\in\mathscr{O}$, $\alpha = pq^{-1}$, $\beta = rs^{-1}$ and $\mathtt{n}(qpq^{-1}s - qr) = (\mathrm{Det}(\gamma))^2 = 1$, hence α,β are Farey neighbours with respect to $\mathscr{O}$.

Conversely, assume that α,β are Farey neighbours with respect to $\mathscr{O}$. First assume that there exists $p,q,r,s\in\mathscr{O}$ such that $\alpha = pq^{-1}$, $\beta = rs^{-1}$, $q = 0$ and $p,s\in\mathscr{O}^\times$. Then $\gamma = \big[\begin{smallmatrix} p & r\\ 0 & s\end{smallmatrix}\big]$ belongs to $\Gamma_{\mathscr{O}}$, and $\alpha = \infty = \gamma\cdot\infty$ and $\beta = rs^{-1} = \gamma\cdot 0$, as wanted. Otherwise, there exists $p,q,r,s\in\mathscr{O}$ such that $\alpha = pq^{-1}$, $\beta = rs^{-1}$, $q\neq 0$ and $\mathtt{n}(qpq^{-1}s - qr) = 1$. Then $\big(\begin{smallmatrix} p & r\\ q & s\end{smallmatrix}\big)$ belongs to $\mathrm{SL}_2(\mathscr{O})$ by the preliminary comment. Hence $\gamma = \big[\begin{smallmatrix} p & r\\ q & s\end{smallmatrix}\big]$ belongs to $\mathrm{PSL}_2(\mathscr{O})$ and maps ∞ and 0 to α and β respectively, as wanted. $\Box$

Proof of Theorem 16. We apply Theorem 17 to $\Gamma = \Gamma_{\mathscr{O}}$ and $(x,y) = (0,\infty)$, so that by Lemma 18, we have $\mathfrak{N}_{\mathscr{O}} = \Gamma\cdot\{x,y\}$. The locally geodesic line $\ell_{0,\infty}$ in $\Gamma_{\mathscr{O}}\backslash\mathbb{H}^5_{\mathbb{R}}$ is reciprocal as in the rational case (the order two element $\big[\begin{smallmatrix} 0 & -1\\ 1 & 0\end{smallmatrix}\big]\in\Gamma_{\mathscr{O}}$ exchanges the two points at infinity of $\widetilde{\ell}_{0,\infty}(\mathbb{R})$), hence $\iota_{\Gamma_{\mathscr{O}},\mathrm{rec}}(0,\infty) = 1$. The pointwise stabiliser in $\Gamma_{\mathscr{O}}$ of the geodesic line $\widetilde{\ell}_{0,\infty}(\mathbb{R})$ has cardinality $\frac{|\mathscr{O}^\times|^2}{2}$, hence $m_{\Gamma_{\mathscr{O}}}(0,\infty) = \frac{|\mathscr{O}^\times|^2}{2}$. The index in $\Gamma_{\mathscr{O},\infty}$ of its unipotent subgroup of translations by $\mathscr{O}$ is equal to $\frac{|\mathscr{O}^\times|^2}{2}$. Hence replacing the quotient modulo $\Gamma_{\mathscr{O},\infty}$ in the left-hand side of the formula in Theorem 17 by the quotient modulo $\mathscr{O}$ amounts to multiplying the right-hand side by $\frac{|\mathscr{O}^\times|^2}{2}$. Therefore Theorem 16 does follow from Theorem 17. $\Box$

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