UNCLOUDING THE SKY OF NEGATIVELY CURVED MANIFOLDS

JOUNI PARKKONEN AND FRÉDÉRIC PAULIN

ABSTRACT. Let M be a complete simply connected Riemannian manifold, with sectional curvature $K \le -1$. Under some assumptions on the geometry of ∂M, which are satisfied for instance if M is a symmetric space, or has dimension 2, we prove that given any family of horoballs in M, and any point x_0 outside these horoballs, it is possible to shrink uniformly, by a finite amount depending only on M, these horoballs so that some geodesic ray starting from x_0 avoids the shrunk horoballs. As an application, we give a uniform upper bound on the infimum of the heights of the closed geodesics in the finite volume quotients of M.

1. INTRODUCTION

Let M be a complete simply connected Riemannian manifold, with negative sectional curvature $K \le -1$, and ∂M be its space at infinity, i.e. the set of asymptotic classes of geodesic rays in M. A horoball HB in M can be defined as a proper non-empty subset of M which is the limit (for the Hausdorff distance on compact subsets) of a sequence of closed balls of radius r_n converging to ∞. For every $t \ge 0$, define the shrunk horoball $HB(t)$ to be the horoball which is the limit of the sequence of balls with the same center and radius $r_n - t$. Given a collection $(HB_n)_{n\in\mathbb{N}}$ of horoballs with pairwise disjoint interiors, and a point x_0 in M or ∂M, it may happen that every geodesic ray or line starting from x_0 meets at least one of the horoballs HB_n. We are interested in proving that there exists a finite and explicit lower bound on t such that at least one geodesic ray or line starting from x_0 avoids the uniformly shrunk horoballs $HB_n(t)$. The main point of this paper (besides the existence) is to get universal (and possibly as small as possible) such lower bounds. Thinking of ∂M as the sky of M, and of horoballs as clouds, we are interested in shrinking the clouds to be able to see the blue sky.

Theorem 1.1. *Let X be a proper $CAT(-1)$ metric space, and $t_{\min} > 0$. Assume that one of the following conditions holds:*

1. *X is the real hyperbolic n-space with $n \ge 2$, and $t_{\min} = -\log(4\sqrt{2} - 5)$;*
2. *X is a complete simply connected Riemannian manifold of dimension 2, with curvature $-a^2 \le K \le -1$, and $e^{-t_{\min}} = 2^{2/a}\left(\sqrt{1 + 2^{1-1/a}} - 1 - 2^{-1-1/a}\right)$, where $1 \le a \le 2$;*
3. *X is a locally finite tree (without vertices of degree 1 or 2, with edge lengths 1) and $t_{\min} = 1$;*
4. *X is a negatively curved symmetric space, and $t_{\min}$ is some constant depending only on X.*

Let $(HB_n)_{n\in\mathbb{N}}$ be a sequence of horoballs with pairwise disjoint interiors. If $t > t_{\min} + 2$, then for every x_0 in $X - \bigcup_n HB_n$, there exists at least one geodesic ray starting at x_0, which avoids $HB_n(t)$ for every n in $\mathbb{N}$.

Furthermore, if $t > t_{\min}$, then there exists at least one geodesic line starting at the point at infinity x_0 of HB_0 which avoids $HB_n(t)$ for every n in $\mathbb{N} - \{0\}$.

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When x_0 is at infinity, the first three cases are sharp results. In Sections 4 and 7, we give sufficient conditions on the geometry of the space at infinity of X for the results to be valid. The complete result is more general than the one above. In particular, the second result of Theorem 1.1 holds for X a homogeneous negatively curved Riemannian manifold of Carnot type (see Section 4), when x_0 is identified with the only point fixed by a simply transitive group of isometries of X. Note that the result is not true for every proper geodesic CAT(-1) metric space, as examples of trees with unbounded edge lengths show.

Consider now a finite volume complete negatively curved Riemannian manifold V, and e an end of V. Let β_e be the Busemann function on V with respect to e, normalized to be 0 on the boundary of the maximal Margulis neighbourhood of e (see section 7). Define the height of any compact subset A of V as the maximum value of β_e on A. As an application of Theorem 1.1, we give a uniform upper bound, which depends only on the universal cover of V, on the infimum $h_e(V)$ of the heights of closed geodesics on V.

Corollary 1.2. *Let $t_{\min} > 0$ and let V be a finite volume complete Riemannian manifold, with negative sectional curvature $K \leq -1$. If $t_{\min}$ and the universal cover of V satisfy the condition (1), (2) or (4) of the above theorem, then $h_e(V) \leq t_{\min}$.*

The results of this paper are related to the problem of finding geodesic rays avoiding obstacles in negatively curved manifolds (and constructing proper invariant subsets for the geodesic flow, see for instance the work of Burns and Pollicott [BP] and Buyalo, Schroeder and Walz [BSW]) or to the shrinking target problem of Hill and Velani [HV]. They are also related to the study of bounded geodesics in complete finite volume negatively curved manifolds, see for instance [Dani, Sch] and the references therein. In particular, V. Schroeder [Sch] proved the existence of a geodesic line through any given point having bounded height (though not with an effective bound, and in dimension at least 3, but without any other assumption besides having sectional curvature at most -1).

Let us describe briefly the contents of this paper. In Section 2, we collect geometric results on the shadows (in the sense of Sullivan) of balls and horoballs in CAT(-1) spaces in terms of the natural distance (due to Hamenstädt [Ham1], see [HP1, Appendix]) in the punctured boundary.

In Section 3, we prove an abstract uncovering result for collections of balls in metric spaces.

Proposition 1.3. *Let Y be a complete locally compact metric space such that through two points passes a geodesic line. Then for every family $(B(x_n, r_n))_{n\in\mathbb{N}}$ of balls in Y, with $0 < r_n \leq 1$ and $d(x_n, x_m)^2 \geq 4r_n r_m$ for $n \neq m$, the scaled family of balls $(B(x_n, sr_n))_{n\in\mathbb{N}}$ no longer covers Y if $s < \sqrt{5} - 2$.*

The fact that there is such a universal constant is amazing. In fact, the result is true, with different bounds on the scaling constant s, for a larger class of spaces which includes the Heisenberg group with its Carnot-Carathéodory metric. See Theorem 3.3 for a more precise statement. This result (or variations on its proof) will imply Theorem 1.1, as in Section 2, we prove that the disjointness of horoballs implies the above quadratic separation property for the packing of balls on the boundary at infinity obtained from the shadows of the horoballs.

In Section 4 we study the metric properties at infinity of negatively curved Riemannian manifolds, whose punctured boundaries are Carnot groups (see [FS, Gro2, HK] for instance). We show that these manifolds satisfy the sufficient conditions needed for Theorem 1.1 to be applicable.

The improvement of the constants, in order to get sharp results, is done for two-dimensional manifolds with pinched negative curvature in Section 5, and extended to the real hyperbolic space by a symmetry trick in Section 6.

Finally, in Section 7, we turn to the case of equivariant families of horoballs. This is the motivating problem, in particular for families of horoballs which arise from arithmetic constructions (see for instance Section 5). We prove Corollary 1.2 in this last section by showing the existence of a geodesic which does not enter deep in the lift of a maximal Margulis neighbourhood of the end e and applying results of [HP3]. The uniformity of the upper bound on $h_e(V)$ is surprising. The value $t_{\min} = -\log(4\sqrt{2}-5) \approx 0.4205$ of this upper bound in the constant curvature case is even not very far from the bound in particular cases coming from arithmetic constructions. For instance, if V is the orbifold $\mathrm{PSL}_2(\mathbb{R})\backslash\mathbb{H}^2_{\mathbb{R}}$, which has only one end e, then $h_e(V) = \log(\sqrt{5}/2) \approx 0.1115$. This has been known for a long time, as $1/\sqrt{5}$ is the Hurwitz constant of the classical diophantine approximation problem of real numbers by rational numbers, but see [HP2] and the references therein, and also [Ser, RWT]).

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2. The geometry of shadows in negatively curved manifolds

We refer to [Bou, BH, GH] for the definitions and the first properties of a CAT(-1) geodesic metric space X and its boundary ∂X, as well as the horospheres and horoballs in X. All balls and horoballs are assumed to be closed unless otherwise stated. Accordingly, in any metric space (Y,δ), we will denote by $B(x,r) = B_\delta(x,r)$ the closed ball of center x and radius r, and by $S(x,r) = S_\delta(x,r)$ the sphere of center x and radius r. If ξ,η are two points in ∂X, we will denote by $]\xi,\eta[$ the geodesic line in X between ξ and η.

Let (X,d) be a CAT(-1) geodesic metric space, let $\xi_\sharp$ be a point in ∂X, and let $H_\sharp$ be a horosphere in X centered at $\xi_\sharp$. The *Hamenstädt distance* $d_{\xi_\sharp,H_\sharp}$ on $\partial X - \{\xi_\sharp\}$ is defined by

$$d_{\xi_\sharp,H_\sharp}(a,b) = \lim_{t\to+\infty} e^{-\frac{1}{2}(2t-d(a_t,b_t))} \tag{1}$$

for $t\mapsto c_t$ the geodesic line from $\xi_\sharp$ to some point c in $\partial X - \{\xi_\sharp\}$, with c_0 belonging to $H_\sharp$. We refer to [HP1, Appendix][HP3] for more details.

For instance, if X is the upper halfplane model of the real hyperbolic n-space $\mathbb{H}^n_{\mathbb{R}}$, if $\xi_\sharp$ is the point at infinity ∞, and if $H_\sharp$ is the horosphere at Euclidean height 1, then the Hamenstädt distance $d_{\infty,H_\sharp}$ is the Euclidean distance on $\partial\mathbb{H}^n_{\mathbb{R}} - \{\infty\} = \mathbb{R}^n$.

For $H'_\sharp$ another horosphere centered at $\xi_\sharp$, denote by $d_{alg}(H'_\sharp,H_\sharp)$ the algebraic distance, on any geodesic line ℓ ending at $\xi_\sharp$ (and oriented towards $\xi_\sharp$), between the intersection points of ℓ with $H'_\sharp$ and $H_\sharp$. We then have

$$d_{\xi_\sharp,H'_\sharp} = e^{-d_{alg}(H'_\sharp,H_\sharp)}\, d_{\xi_\sharp,H_\sharp} \ . \tag{2}$$

Note that if γ is an isometry of X, then, for every a, b in $\partial M - \{\xi_\sharp\}$,

$$d_{\gamma\xi_\sharp,\gamma H_\sharp}(\gamma a, \gamma b) = d_{\xi_\sharp, H_\sharp}(a, b) \ . \tag{3}$$

Remark. The Hamenstädt distance has the following scaling property. For every ϵ in $(0, 1]$, the metric space $\epsilon X = (X, \epsilon d)$ is again a CAT(-1) geodesic metric space, with boundary $\partial(\epsilon X)$ naturally identified with ∂X, and $H_\sharp$ is again a horosphere in ϵX, centered at the point at infinity $\xi_\sharp$ of ϵX. It follows by definition that the Hamenstädt distance in $\partial(\epsilon X)$ defined by $\xi_\sharp$ and $H_\sharp$ is exactly ${d_{\xi_\sharp,H_\sharp}}^{\epsilon}$.

Remark 2.1. The Hamenstädt distances with respect to two points at infinity $\xi_\sharp, \xi'_\sharp$ are locally equivalent. More precisely, for every $\xi_\sharp, \xi'_\sharp, \eta_0$ in ∂X, for all horospheres $H_\sharp, H'_\sharp$ centered at $\xi_\sharp, \xi'_\sharp$ respectively, there exists a constant $\lambda > 0$ and a neighborhood U of η_0 such that for every η, η' in U, we have

$$\lambda\, e^{-8}\ d_{\xi_\sharp,H_\sharp}(\eta, \eta') \leq d_{\xi'_\sharp,H'_\sharp}(\eta, \eta') \leq \lambda\, e^{8}\ d_{\xi_\sharp,H_\sharp}(\eta, \eta') \ .$$

Proof. Assume that $\xi_\sharp, \xi'_\sharp, \eta_0, H_\sharp, H'_\sharp$ are as in the statement. Let x be a point on $]\xi_\sharp, \eta_0[$ at distance at most 1 from $]\xi'_\sharp, \eta_0[$ and $]\xi_\sharp, \xi'_\sharp[$, which exists by an easy comparison argument with the hyperbolic plane. Then for every small enough neighborhood U of η_0 and for every η in U, the geodesic lines between $\xi_\sharp$ and η, and between $\xi'_\sharp$ and η, are passing at distance at most 2 from the point x. Let $\eta, \bar\eta$ be in U. Let $p, \bar p$ (resp. $p', \bar p'$) be the closest points to x on $]\xi_\sharp, \eta[$ and $]\xi_\sharp, \bar\eta[$ (resp. on $]\xi'_\sharp, \eta[$ and $]\xi'_\sharp, \bar\eta[$). We chose U such that $p, \bar p, p', \bar p'$ belong to $B(x, 2)$. Up to changing the horoballs $H_\sharp, H'_\sharp$, which will only change the value of λ by Equation (2), we may assume that $H_\sharp$ (resp. $H'_\sharp$) contains the point p (resp. p'). The triangle inequality implies

$$\begin{aligned}&|(d(H_\sharp, \eta_t) + d(H_\sharp, \bar\eta_t) - d(\eta_t, \bar\eta_t)) - (d(x, \eta_t) + d(x, \bar\eta_t) - d(\eta_t, \bar\eta_t))|\\ &\leq d(H_\sharp, p) + d(H_\sharp, \bar p) + d(p, x) + d(\bar p, x) \leq d(\bar p, x) + d(x, p) + d(p, x) + d(\bar p, x) \leq 8 \ ,\end{aligned}$$

and similarly with $H'_\sharp$ replacing $H_\sharp$. Therefore

$$e^{-8}\ \ d_{\xi_\sharp,H_\sharp}(\eta, \bar\eta) \leq d_{\xi'_\sharp,H'_\sharp}(\eta, \bar\eta) \leq e^{8}\ d_{\xi_\sharp,H_\sharp}(\eta, \bar\eta) \ . \qquad \square$$

If A is a subset of X, define the *shadow* of A seen from $\xi_\sharp$ to be the subset $\mathcal{O}A = \mathcal{O}_{\xi_\sharp}(A)$ of $\partial X - \{\xi_\sharp\}$ consisting of the endpoints of the geodesic lines starting from $\xi_\sharp$ and meeting A. Note that if A is the (closed) horoball bounded by a horosphere H, then $\mathcal{O}H = \mathcal{O}A$, as H separates X.

It is geometrically clear that, in the upper halfspace model of the real hyperbolic n-space, the shadow seen from ∞ of a ball or horoball is a Euclidean ball. Our first result says that when X is assumed to be a pinched negatively curved Riemannian manifold, then the shadow of a ball or horoball is almost a ball for the Hamenstädt distance. Comparing shadows of balls with balls for some distance on the boundary has a long history, starting with the shadow lemma of [Sul], see also [Bou, Rob] and [HP2, Lemma 3.1].

Proposition 2.2. *Let M be a complete simply connected Riemannian manifold, with sectional curvature $-a^2 \leq K \leq -1$, let $\xi_\sharp$ be a point at infinity and $H_\sharp$ a horosphere centered at $\xi_\sharp$. For every ball B of center x and radius r, whose interior is disjoint from the open horoball bounded by $H_\sharp$, we have*

$$B_{d_{\xi_\sharp,H_\sharp}}(\xi, \frac{1}{2}e^{-d(H_\sharp,B)}(1 - e^{-2r})) \subset \mathcal{O}_{\xi_\sharp}(B) \subset B_{d_{\xi_\sharp,H_\sharp}}(\xi, 2^{-\frac{1}{a}}e^{-d(H_\sharp,B)}(1 - e^{-2ar})^{\frac{1}{a}}) \ ,$$

where ξ is the other endpoint of the geodesic line from $\xi_\sharp$ through x.

Remark. Note that this result is sharp, since the proof shows that if M has constant curvature -1, then the left inclusion is an equality, and if M has constant curvature $-a^2$, then the right inclusion is an equality.

Proof. To simplify notations, let $d_\infty = d_{\xi_\sharp, H_\sharp}$. For every $\kappa \geq 1$, let us denote by $\mathbb{H}^2_{-\kappa}$ the upper halfplane model of the real hyperbolic plane with constant curvature $-\kappa$, and $d_{-\kappa}$ its distance. Let $\overline{\xi}_\sharp$ be the point at infinity in this model, and let $\overline{H}_\sharp$ be the horosphere centered at $\overline{\xi}_\sharp$ at Euclidean height 1. Let $d_{-\kappa,\infty}$ be the Hamenstädt distance on $\partial\mathbb{H}^2_{-\kappa} - \{\overline{\xi}_\sharp\}$ with respect to the horoball $\overline{H}_\sharp$.

Let us start with a preliminary remark. Let $\overline{\xi}, \overline{\xi}'$ be two distinct points in $\partial\mathbb{H}^2_{-\kappa} - \{\overline{\xi}_\sharp\}$, and let $\overline{x}$ be a point in $]\overline{\xi}_\sharp, \overline{\xi}[$ below $\overline{H}_\sharp$. Define $\overline{p}'$ to be the orthogonal projection of $\overline{x}$ on $]\overline{\xi}_\sharp, \overline{\xi}'[$. Define $\overline{B}'$ to be the ball centered at $\overline{x}$ whose boundary contains $\overline{p}'$. Denote its radius by $\overline{r}$. Assume that the interior of $\overline{B}'$ is disjoint from the open horoball bounded by $\overline{H}_\sharp$ (see Figure 1 below).

First assume that $\kappa = 1$, hence in particular the Hamenstädt distance $d_{-1,\infty}$ is the Euclidean distance on $\partial\mathbb{H}^2_{-1} - \{\overline{\xi}_\sharp\} = \mathbb{R}$. Denote by u the Euclidean height of $\overline{x}$ and by R the Euclidean radius of the circle containing the geodesic segment between $\overline{x}$ and $\overline{p}'$. Then

$$u = e^{-(\overline{r} + d_{-1}(\overline{H}_\sharp, \overline{B}'))}$$

and

$$R^2 = u^2 + d_{-1,\infty}(\overline{\xi}, \overline{\xi}')^2$$

by Pythagoras' formula in the right angled Euclidean triangle $\overline{x}\,\overline{\xi}\,\overline{\xi}'$, and

$$R + d_{-1,\infty}(\overline{\xi}, \overline{\xi}') = e^{-d_{-1}(\overline{H}_\sharp, \overline{B}')}$$

as the Euclidean height of $\overline{p}'$ is R and the Euclidean center of the hyperbolic ball $\overline{B}'$ has the same Euclidean height as $\overline{p}'$. Eliminating u and R, we get

$$d_{-1,\infty}(\overline{\xi}, \overline{\xi}') = \frac{1}{2} e^{-d_{-1}(\overline{H}_\sharp, \overline{B}')}(1 - e^{-2\overline{r}}) \; .$$

Assuming now that $\kappa \geq 1$, by the scaling property of the Hamenstädt distance, and as $\mathbb{H}^2_{-\kappa} = \frac{1}{\sqrt{\kappa}} \mathbb{H}^2_{-1}$, we have

$$d_{-\kappa,\infty}(\overline{\xi}, \overline{\xi}') = \left(\frac{1}{2} e^{-\sqrt{\kappa}\, d_{-\kappa}(\overline{H}_\sharp, \overline{B}')}(1 - e^{-2\sqrt{\kappa}\,\overline{r}}) \right)^{\frac{1}{\sqrt{\kappa}}} = 2^{-\frac{1}{\sqrt{\kappa}}}\, e^{-d_{-\kappa}(\overline{H}_\sharp, \overline{B}')}(1 - e^{-2\sqrt{\kappa}\,\overline{r}})^{\frac{1}{\sqrt{\kappa}}} \; .$$

Let us first prove the inclusion on the right in the statement of Proposition 2.2. Take ξ' in $\mathcal{O}B$. Let us prove that $d_\infty(\xi, \xi') \leq 2^{-\frac{1}{a}} e^{-d(H_\sharp, B)}(1 - e^{-2ar})^{\frac{1}{a}}$.

We may assume that ξ and ξ' are different. Let p' be the orthogonal projection of x on $]\xi_\sharp, \xi'[$, that is, the closest point to x on the geodesic line $]\xi_\sharp, \xi'[$. Let B' be the ball in M centered at x whose boundary contains p', and hence is tangent to $]\xi_\sharp, \xi'[$ at p'. As $]\xi_\sharp, \xi'[$ meets B and p' is the closest point, the ball B' is contained in B.

There exists two distinct points $\overline{\xi}, \overline{\xi}'$ in $\partial\mathbb{H}^2_{-a^2} - \{\overline{\xi}_\sharp\}$, and a point $\overline{x}$ in $]\overline{\xi}_\sharp, \overline{\xi}[$ below $\overline{H}_\sharp$, such that

$$d_\infty(\xi, \xi') = d_{-a^2,\infty}(\overline{\xi}, \overline{\xi}') \;\text{ and }\; d(x, H_\sharp) = d_{-a^2}(\overline{x}, \overline{H}_\sharp) \; .$$

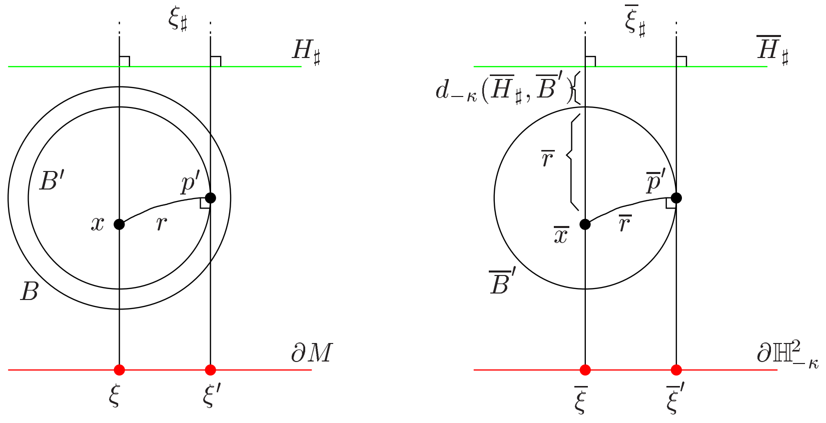

FIGURE 1

Define $\overline{p}', \overline{B}', \overline{r}$ as in the preliminary remark. By the comparison property, as the sectional curvature of M is at least $-a^2$, by taking limits of comparison triangles (see for instance [GH]), we have

$$\overline{r} = d_{-a^2}(\overline{x}, \overline{p}') \leq d(x, p') \leq r \ ,$$

which implies that

$$d_{-a^2}(\overline{H}_\sharp, \overline{B}') \geq d(H_\sharp, B') \geq d(H_\sharp, B) \ .$$

In particular, the interior of $\overline{B}'$ is disjoint from the open horoball bounded by $\overline{H}_\sharp$. Hence, by the preliminary remark,

$$d_\infty(\xi, \xi') = d_{-a^2,\infty}(\overline{\xi}, \overline{\xi}') = 2^{-\frac{1}{a}}\, e^{-d_{-a^2}(\overline{H}_\sharp, \overline{B}')}(1 - e^{-2a\overline{r}})^{\frac{1}{a}} \leq 2^{-\frac{1}{a}}\, e^{-d(H_\sharp, B)}(1 - e^{-2ar})^{\frac{1}{a}} \ .$$

This is what we wanted.

The proof of the other inclusion is similar (though easier, as we compare only with the constant curvature -1 hyperbolic plane), and is left to the reader. □

Corollary 2.3. *Under the assumption of Proposition 2.2, for every horosphere H centered at a point at infinity ξ, bounding an open horoball disjoint from the open horoball bounded by $H_\sharp$, we have*

$$B_{d_{\xi_\sharp, H_\sharp}}(\xi, \frac{1}{2} e^{-d(H_\sharp, H)}) \subset \mathcal{O}_{\xi_\sharp} H \subset B_{d_{\xi_\sharp, H_\sharp}}(\xi, 2^{-\frac{1}{a}}\, e^{-d(H_\sharp, H)}) \ .$$

As previously, this result is sharp.

Proof. Let p be the intersection point of H and $]\xi_\sharp, \xi[$. Let x_t be the point on $]\xi_\sharp, \xi[$, at distance $t > 0$ from p (and converging to ξ as t tends to $+\infty$). Let B_t be the ball of center x_t whose boundary contains p. Then B_t converges, for the topology of uniform convergence on compact subsets of M, to the horoball HB bounded by H. Hence $\mathcal{O}B_t$ converges, for the Hausdorff distance on closed subsets of $\partial M - \{\xi_\sharp\}$ to $\mathcal{O}H$. Note that the radius of B_t tends to $+\infty$. Hence the result follows from Proposition 2.2 by taking limits. □

Remark 2.4. It is well-known that some version of this sharp result holds for general CAT(-1) spaces, and here is one, where the constant $c_{\max}$ is probably not optimal. We claim that for $c_{\max} = e^2$, with the standing assumptions in this section on $(X, \xi_\sharp, H_\sharp)$, for every horosphere H in X centered at a point at infinity ξ, bounding an open horoball disjoint from the open horoball bounded by $H_\sharp$, we have

$$B_{d_{\xi_\sharp, H_\sharp}}(\xi, \frac{1}{2}e^{-d(H_\sharp, H)}) \subset \mathcal{O}_{\xi_\sharp} H \subset B_{d_{\xi_\sharp, H_\sharp}}(\xi, c_{\max}\, e^{-d(H_\sharp, H)}) \; .$$

Proof. The left inclusion is the one in the corollary above, since its proof only used the fact that M is CAT(-1). To prove the right inclusion, let η be the endpoint of a geodesic line starting from $\xi_\sharp$ and meeting H, that we may assume to be different from ξ. Let p be the orthogonal projection of η to $]\xi_\sharp, \xi[$, which is the closest point to η on the geodesic line between $\xi_\sharp$ and ξ in the sense of any Busemann function corresponding to η. Note that as in Proposition 2.2, the point p belongs to the horoball bounded by H. Let q (resp. $q_\sharp$) be the point, which is the closest to p, on the geodesic line between η and ξ (resp. between η and $\xi_\sharp$). By an easy computation in the upper-halfspace model, and by comparison using the CAT(-1) property of X, the distance between p and q, and the one between p and $q_\sharp$, are at most 1. Let $t \mapsto \xi_t$ (resp. $t \mapsto \eta_t$) be the geodesic line starting from $\xi_\sharp$, passing through $H_\sharp$ at time $t = 0$, and ending in ξ (resp. η). For every $\epsilon > 0$, for t big enough, as the geodesic segment between ξ_t and η_t converges to the geodesic line between ξ and η, there exists a point on $[\xi_t, \eta_t]$ which is at distance at most ϵ from q. Hence, by applying the triangular inequality several times, we have

$$\begin{aligned} & d(\xi_t, H_\sharp) + d(\eta_t, H_\sharp) - d(\xi_t, \eta_t) \\ & \geq d(\xi_t, p) + d(p, \xi_0) + d(\eta_t, q_\sharp) + d(q_\sharp, \eta_0) - d(\xi_t, q) - d(q, \eta_t) \\ & \geq -d(p, q) - d(q_\sharp, q) + d(\xi_0, p) + d(\eta_0, q_\sharp) \geq 2d(\xi_0, p) - 4 \; . \end{aligned}$$

Therefore

$$d_{\xi_\sharp, H_\sharp}(\eta, \xi) \leq e^{-d(p, H_\sharp)+2} \leq c_{\max}\, e^{-d(H, H_\sharp)} \; . \quad \square$$

If H is a horosphere in X, bounding an open horoball disjoint from $H_\sharp$, motivated by Proposition 2.2 and Remark 2.4, we call

$$r_i(H) = \frac{1}{2} e^{-d(H_\sharp, H)}$$

the *inner radius of the shadow* of H. When X is furthermore a complete Riemannian manifold with sectional curvature $-a^2 \leq K \leq -1$ (with $a \geq 1$), we call

$$r_e(H) = 2^{-\frac{1}{a}} e^{-d(H_\sharp, H)}$$

the *outer radius of the shadow* of H. These quantities depend on $H_\sharp$. Note that the outer radius is bounded above by $e^{-d(H_\sharp, H)}$.

We are now going to give an inequality relating the inner radii of two horospheres bounding disjoint open horoballs to the Hamenstädt distance between their points at infinity.

Given two balls $B = B(x, r)$ and $B' = B(x', r')$ in X, define the *algebraic distance* between B and B' to be

$$d_{\rm alg}(B, B') = d(x, x') - r - r' \; . \tag{4}$$

Note that $d_{\rm alg}(B, B') = d(B, B') \geq 0$ if B and B' have disjoint interior, as X is geodesic.

Given two points x, x' in X, that do not belong to the open horoball bounded by $H_\sharp$, define

$$d_{\xi_\sharp, H_\sharp}(x, x') = e^{-\frac{1}{2}(d(x, H_\sharp) + d(x, H_\sharp) - d(x, x'))} \; . \tag{5}$$

Note that, by the definition of the Hamenstädt distance, see Equation (1), the numbers $d_{\xi_\sharp,H_\sharp}(x_i,x'_i)$ converge to $d_{\xi_\sharp,H_\sharp}(\xi,\xi')$ as the points x_i,x'_i tend respectively to the points ξ,ξ' in $\partial X-\{\xi_\sharp\}$ while staying on a geodesic ray that converges to ξ,ξ'.

Lemma 2.5. *Let B,B' be two balls in X with center x,x', whose interior do not meet the open horoball bounded by $H_\sharp$. Then*

$$-2\log d_{\xi_\sharp,H_\sharp}(x,x')=d(H_\sharp,B)+d(H_\sharp,B')-d_{alg}(B,B')$$

Proof. If r is the radius of B then $d(H_\sharp,B)=d(H_\sharp,x)-r$. The result follows from the equations (4) and (5). □

If H,H' are two horospheres in X centered at the distinct points a,b in ∂X, denote by $d_{\rm alg}(H,H')$ the distance between the intersection points with H and H' of the geodesic line between a and b, with a positive sign if H and H' are disjoint, and a negative sign if the open horoballs bounded by H and H' meet.

Corollary 2.6. *Let H,H' be two horospheres in X centered at distinct points ξ,ξ' in ∂X respectively, with the open horoball bounded by $H_\sharp$ disjoint from the open horoballs bounded by H,H'. Then*

$$-2\log d_{\xi_\sharp,H_\sharp}(\xi,\xi')=d(H_\sharp,H)+d(H_\sharp,H')-d_{alg}(H,H')$$

Proof. This follows from Lemma 2.5 by taking limits, as in the proof of Corollary 2.3. □

Corollary 2.7. *With the notations of the previous corollary, assume furthermore that H and H' bound disjoint open horoballs. Then*

$$d_{\xi_\sharp,H_\sharp}(\xi,\xi')^2\geq 4r_i(H)r_i(H'). \tag{6}$$

This inequality is sharp, i.e. the equality holds if and only if H and H' meet in one (and only one) point (their common intersection point with the geodesic line between ξ and ξ').

Proof. By Corollary 2.6, since the open horoballs bounded by H and H' are disjoint, then

$$-2\log d_{\xi_\sharp,H_\sharp}(a,a')\leq d(H_\sharp,H)+d(H_\sharp,H'),$$

with equality if and only if $d_{alg}(H,H')=0$, that is when H and H' meet in one point. The corollary then follows by definition of the inner radius. □

Remark. Assume that H,H' are circles, bounding disjoint open discs, in the Euclidean upper halfplane, tangent to the horizontal line $\mathbb{R}$ at points x,x', with Euclidean radius $r,r'>0$, and let $u=d(x,x')$ be the Euclidean distance between x,x'. Then, by Pythagoras's equality on the Euclidean right-angled triangle with base the segment between the Euclidean centers of H,H', and right-angled vertex on the vertical line through the Euclidean center of H, we have

$$rr'\leq\frac{u^2}{4}\,.$$

Corollary 2.7 is a generalization of the above inequality. Indeed, one may assume that $r,r'\leq\frac{1}{2}$ by homogeneity. Let X be the upper halfplane model of the real hyperbolic plane $\mathbb{H}^2_{\mathbb{R}}$, with $\xi_\sharp$ the point at infinity, and $H_\sharp$ the horizontal line at Euclidean height 1, so that $\partial X=\mathbb{R}\cup\{\infty\}$. The above inequality follows from Corollary 2.7, as H,H' are horospheres in X, the Hamenstädt distance coincides with the Euclidean distance, and an easy computation gives $d(H_\sharp,H)=-\log(2r)$.

We end this section of preliminary results by the following lemma. Let $r : [0, +\infty) \to X$ be a geodesic ray in X converging to $\xi_\sharp$, with $r(0)$ belonging to $H_\sharp$. The *Busemann function* associated to $H_\sharp$ is (see for instance [Bou]) the convex map $\beta_\sharp : X \to \mathbb{R}$ defined by

$$\beta_\sharp(x) = \lim_{t\to\infty} t - d(x, r(t)) \ ,$$

so that $H_\sharp = \beta_\sharp^{-1}(\{0\})$ and $\beta_\sharp(r(t))$ tends to $+\infty$ as $t \to +\infty$. For any subset A of X define the *height* of A with respect to $H_\sharp$ as

$$ht(A) = \sup_{x\in A} \beta_\sharp(x) \ .$$

We will say that a subset A of X is *lower* than a subset B of X with respect to $\xi_\sharp$ if

$$\sup_{x\in A} \beta_\sharp(x) \leq \sup_{x\in B} \beta_\sharp(x) \ .$$

Note that this does not depend on the chosen horosphere $H_\sharp$ centered at $\xi_\sharp$.

Lemma 2.8. *Let H be a horosphere in X, bounding an open horoball which is disjoint from the open horoball bounded by $H_\sharp$. Let η, η' be two distinct points in $\mathcal{O}_{\xi_\sharp} H$. Then the geodesic $]\eta, \eta'[$ does not intersect $H_\sharp$.*

Proof. Recall that $t \mapsto \eta_t$ and $t \mapsto \eta'_t$ are the geodesic lines converging to η, η' respectively, with η_0 and η'_0 in $H_\sharp$. Let u, u' be points in the intersection of H with $]\xi_\sharp, \eta[,]\xi_\sharp, \eta'[$ respectively. By convexity of the horoballs, the geodesic segment $[u, u']$ between u and u' is contained in the horoball HB bounded by H, hence its height is negative. Let t be big enough, so that η_t lies between u and η on $]\eta_\sharp, \eta[$, and similarly for η'_t. Then by convexity of $\beta_\sharp$, the height of the geodesic segment between η_t and η'_t is lower than the height of $[u, u']$, and in particular is negative. The result follows. □

3. A metric uncovering theorem

In this section, we study the "scaling" properties in metric spaces of families of balls, which satisfy a "quadratic packing condition" on the radii and the mutual distances between the centers. The main result, Theorem 3.3, needs some elementary requirement on the metric, as we could get a counter-example by taking discrete spaces. We start the section by giving a few definitions about metric spaces.

If B is a ball of center x and radius r and $\epsilon \geq 0$, denote by ϵB the ball of center x and radius ϵr. If $s \geq 0$ and $\mathcal{B} = (B_\alpha)_{\alpha\in A}$ is a family of closed balls, define $s\mathcal{B}$ to be the family $(sB_\alpha)_{\alpha\in A}$ and $s \overset{\circ}{\mathcal{B}}$ to be the family of associated open balls.

A metric space is *proper* if every closed ball is compact, and is *geodesic* if between any two points there exists a *geodesic segment*, i.e. a path between these two points which is an isometry from some interval of $\mathbb{R}$ into the metric space. In any metric space (Y, δ), such that there exists a path of finite length between every two points, the *associated length distance* on Y is the distance δ_ℓ on Y defined to be the infimum of the lengths of paths between two points, see [Gro1, page 2]. Clearly, the length distance satisfies $\delta_\ell \geq \delta$.

A metric space *has extendable spheres* if for every $\epsilon > 0$, there exists δ in $]0, 1[$ such that for every $r > 0$ and x, y in Y, if $(1 - \delta)r \leq d(x, y) \leq r$, then there exists y' in Y such that $d(x, y') = r$ and $d(y, y') \leq \epsilon r$. We will call any map $\epsilon \mapsto \delta$ satisfying the above requirement a *modulus of sphere extendability*.

Examples 3.1. (i) If M is a nonpositively curved complete simply connected Riemannian manifold, with Riemannian distance d, then (M, d) has extendable spheres, with modulus $\delta(\epsilon) = \epsilon$.

More generally, any metric space such that through any two points passes at least one geodesic line (i.e. the image of an isometry from $\mathbb{R}$ to Y), has extendable spheres with modulus $\delta(\epsilon) = \epsilon$. This is the case for instance for the affine buildings with their natural metrics, and for the Moussong-Davis CAT(0) geometric realisation of any general building, see for instance [Dav].

But note that the Heisenberg group $\mathcal{H}$ (the three-dimensional simply connected non abelian nilpotent Lie group, see below), with its standard left-invariant Riemannian metric, does not have the property that through any two points passes at least one geodesic line. Indeed, by [Mar], the only (minimizing) geodesic rays starting from the origin are the ones orthogonal at the origin to the center $[\mathcal{H},\mathcal{H}]$ of $\mathcal{H}$. Similarly, almost no (in a sense we do not make precise here) germ of geodesic segment for the Carnot-Carathéodory distance on $\mathcal{H}$ starting from the origin can be extended to a (minimizing) geodesic ray, see for instance [BM].

(ii) The sphere $\mathbb{S}^n$ for $n \geq 1$ does not have extendable spheres, nor does any compact metric space having at least two points, as the definition implies that if (Y, d) has at least two points, then any sphere is non empty.

(iii) Recall that the Heisenberg group $\mathcal{H}$ is the Lie group whose underlying manifold is $\mathbb{C} \times \mathbb{R}$, with the group law

$$(\zeta, v)(\zeta', v') = (\zeta + \zeta', v + v' + 2\,\mathrm{Im}\,\zeta\overline{\zeta'}) \ .$$

We can endow $\mathcal{H}$ with the *Cygan distance*, which is the left-invariant distance d_{Cyg} on $\mathcal{H}$ such that

$$d_{\mathrm{Cyg}}((0,0),(\zeta,v)) = \sqrt[4]{|\zeta|^4 + v^2} \ ,$$

and with the *Carnot-Carathéodory distance*, which is (see [Gol, page 161]) the length distance d_{CC} associated to the distance d_{Cyg}. It is also left-invariant and satisfies (see [BM, Theo. 4.11])

$$d_{\mathrm{Cyg}} \leq d_{\mathrm{CC}} \leq \sqrt{\pi}\, d_{\mathrm{Cyg}} \ .$$

Proposition 3.2. *The metric space $(\mathcal{H}, d_{\mathrm{CC}})$ is a proper geodesic metric space having extendable spheres, with modulus $\delta(\epsilon) = 1 - (1 + \epsilon^2/\pi)^{-1/2}$.*

Proof. For every $\epsilon > 0$, define $\delta = 1 - (1 + \epsilon^2/\pi)^{-1/2}$, which belongs to $]0, 1[$. Take $x = (0, 0)$, $r > 0$ and $y = (\zeta, v)$ in $\mathcal{H}$ such that

$$(1 - \delta)r \leq d_{\mathrm{CC}}(x, y) \leq r \ .$$

For $t > 0$, let $h_t : \mathcal{H} \to \mathcal{H}$ be the group morphism defined by $(\zeta, v) \mapsto (t\zeta, t^2 v)$. Note that $d_{\mathrm{Cyg}}(h_t(u), h_t(u')) = t\, d_{\mathrm{Cyg}}(u, u')$ for every u, u' in $\mathcal{H}$, hence

$$d_{\mathrm{CC}}(h_t(u), h_t(u')) = t\, d_{\mathrm{CC}}(u, u') \ .$$

Let $\alpha = d_{\mathrm{CC}}(x, y)$, and define $y' = (\zeta', v') = (\frac{r}{\alpha}\zeta, \frac{r^2}{\alpha^2}v)$, so that $d_{\mathrm{CC}}(x, y') = r$. Now

$$d_{\mathrm{CC}}(y, y') \leq \sqrt{\pi}\, d_{\mathrm{Cyg}}(y, y') = \sqrt{\pi}\, \big(|\zeta' - \zeta|^4 + |v' - v - 2\,\mathrm{Im}\,\zeta'\overline{\zeta}\,|^2\big)^{\frac{1}{4}}$$

$$= \sqrt{\pi}\, \left(\big(\frac{r}{\alpha} - 1\big)^4 |\zeta|^4 + \big(\frac{r^2}{\alpha^2} - 1\big)^2 v^2 \right)^{\frac{1}{4}} \leq \sqrt{\pi} \left(\frac{r^2}{\alpha^2} - 1 \right)^{\frac{1}{2}} \sqrt[4]{|\zeta|^4 + v^2}$$

$$\leq \sqrt{\pi} \left(\frac{1}{(1 - \delta)^2} - 1 \right)^{\frac{1}{2}} d_{\mathrm{Cyg}}(x, y) \leq \epsilon\, d_{\mathrm{CC}}(x, y) \leq \epsilon\, r \ .$$

Hence the result follows, using the homogeneity to pass from $x = (0, 0)$ to any x. $\square$

(iv) More generally, let (Y,d) be a proper metric space. Assume that (Y,d) *has dilations*, i.e. that for every point x in Y, there exists a one-parameter group $(h_{x,t})_{t\in\mathbb{R}}$ of homeomorphisms of Y, such that $h_{x,t}\mid_{B(x,1)}$ converges uniformly to the identity map of $B(x,1)$ as t goes to 0, uniformly in x, such that $h_{x,t}(x)=x$, and such that

$$d(h_{x,t}(y),h_{x,t}(y'))=e^t\,d(y,y')$$

for every t in $\mathbb{R}$ and y,y' in Y. Then (Y,d) has extendable spheres.

Indeed, let $\delta>0$, r in $\mathbb{R}$ and x,y in Y such that $0<(1-\delta)e^r\le d(x,y)\le e^r$. With $\alpha=d(x,y)$, let $y'=h_{x,r-\log\alpha}(y)$. Then $d(x,y')=e^r$. Furthermore,

$$d(y,y')=e^r\,d\big(h_{x,-r}(y),h_{x,r-\log\alpha}\big(h_{x,-r}(y)\big)\big)\ .$$

Note that $h_{x,-r}(y)$ remains in $B(x,1)$, and that $r-\log\alpha$ converges to 0 as δ goes to 0, uniformly in r,x and y.

The main result of this section is the following one.

Theorem 3.3. *Let (Y,d) be a proper geodesic metric space with extendable spheres and $0<D\le\frac14$. Then there exists $s_0=s_0(D,\delta)>0$, depending only on D and on a modulus of sphere extendability δ of (Y,d), such that the following holds. Let $\mathcal{B}=\big(B_\alpha=B(\xi_\alpha,r_\alpha)\big)_{\alpha\in A}$ be a finite or countable family of balls, having uniformly bounded radii and satisfying the following condition:*

$$0<r_\alpha r_\beta\le D\,d(\xi_\alpha,\xi_\beta)^2 \tag{7}$$

for every distinct α,β in A. Then, the family $s\overset{\circ}{\mathcal{B}}$ does not cover Y if $0\le s<s_0$.

Remark 3.4. (i) The quadratic packing condition (7) is a generalisation of the inequality (6) which is satisfied by the inner radii of the shadows of disjoint horoballs.

(ii) Some restrictions like the quadratic packing condition (7) and the assumption on an upper bound on the radii are necessary:

Let (Y,d) be the real line with the usual Euclidean metric. Let $\mathcal{B}=\big(B(\alpha,1)\big)_{\alpha\in\mathbb{Q}}$. Clearly, the family $s\overset{\circ}{\mathcal{B}}$ covers the real line for all s. This shows that the boundedness of the radii is not sufficient.

Let HB_0 be the horoball in $\mathbb{H}^2_{\mathbb{R}}$ of Euclidean radius 1 tangent at 0 to the horizontal line. The transformation $Az=16z-8$ maps HB_0 to the horoball of Euclidean radius 16 tangent at -8 to the horizontal line. It is easy to check (using Pythagoras' theorem) that this horoball is tangent to HB_0. Thus, the sequence $\big(HB_n=A^n(HB_0)\big)_{n\in\mathbb{Z}}$ consists of horoballs with disjoint interiors. All the horoballs are tangent to the A-invariant Euclidean line that passes through the fixed point $8/15$ of A and is tangent to HB_0. The sequence $\mathcal{B}'=(\mathcal{O}_\infty HB_n)_{n\in\mathbb{Z}}$ of the shadows of these horoballs is a family of intervals in $\mathbb{R}$ satisfying the quadratic packing condition (7), because the horoballs are disjoint and by Corollary 2.7. However, it is clear that for $s>0$, the scaled collection $s\overset{\circ}{\mathcal{B}'}$ covers the real line.

For another more arithmetic example, let $\mathcal{HB}$ be the Apollonian packing of the upper halfspace: if HB_∞ denotes the closed halfspace above the horizontal line at Euclidean height 1, and Γ is the modular group $\mathrm{PSL}_2(\mathbb{Z})$, with Γ_∞ the stabiliser of the point at infinity, then $\mathcal{HB}=(\gamma HB_\infty)_{\gamma\in\Gamma/\Gamma_\infty}$. More explicitly, $\mathcal{HB}=(HB_r)_{r\in\mathbb{Q}\cup\{\infty\}}$ where $HB_{p/q}$, $p/q\in\mathbb{Q}$, is the horoball with Euclidean center $\big(p/q,1/(2q^2)\big)$ and Euclidean radius $1/(2q^2)$. Let ξ be a real number whose continued fraction expansion is unbounded. Let α be an element of $\mathrm{PSL}_2(\mathbb{R})$ sending ξ to ∞. Let $\mathcal{B}''$ be the family of shadows seen from ∞ of the horoballs in $\alpha\mathcal{HB}$, which satisfy the quadratic packing condition (7), because the horoballs are disjoint. By the properties of the continued fraction expansion (see for instance [Khi, Thm. 23]), any scaled collection $s\,\overset{\circ}{\mathcal{B}}{}''$ covers the real line.

(iii) If (Y,d) is a proper geodesic metric space, such that through any two points of Y passes a geodesic line, and if we define, for every D in $]0,\frac{1}{2}[$,

$$s_0(D) = \frac{\sqrt{1+16D}-1-4D}{4D} \;.$$

then the proof of Theorem 3.3 shows that the conclusion of Theorem 3.3 holds for every $s \leq s_0(D)$. This is a stronger result, because D can be taken bigger, and $s_0(D)$ depends only on D. We have decided not to emphasize this point, as we do not have any example (besides the real hyperbolic space) where the Hamenstädt distance on the boundary minus a point of a negatively curved complete simply connected Riemannian manifold satisfies the property that through any two points passes a geodesic line.

(iv) The constant s_0 is probably not optimal in general, even under the stronger assumption on (Y,d) of the previous remark, but see Section 5 and 6 for similar theorems with sharp constants in particular cases.

Before proving Theorem 3.3, we start by the elementary observation that annuli in geodesic metric spaces contain balls of a definite size. This is used to prove Proposition 3.6 which provides the induction step in the proof of Theorem 3.3.

Lemma 3.5. *Let (Y,d) be a geodesic metric space. Let $\xi \in Y$ and $0 < r_1 < r_2$. Let $p \in S(\xi,r_2)$. Then there is a (closed) ball K of radius $(r_2-r_1)/2$, whose center is at distance $(r_2+r_1)/2$ from ξ, contained in the annulus*

$$\overline{B(\xi,r_2)-B(\xi,r_1)}\;,$$

containing p and meeting $B(\xi,r_1)$.

Proof. There exists a geodesic segment in $B(\xi,r_2)$ which connects p to ξ. Let η be the point at distance $(r_2-r_1)/2$ from p on this geodesic segment. A simple argument using the triangle inequality shows that the ball $K = B(\eta,(r_2-r_1)/2)$ satisfies the claim. $\square$

We call any ball K satisfying the conclusion of Lemma 3.5 a *canonical ball* in the annulus $\overline{B(\xi,r_2)-B(\xi,r_1)}$.

Proposition 3.6. *Let (Y,d) be a geodesic metric space with extendable spheres, and $0 < D \leq 1/4$. Then there exists $s_0 = s_0(D,\delta) > 0$, depending only on D and on a modulus of sphere extendability δ of (Y,d), such that the following holds. Let ξ,ξ' be points in Y and $r \geq r' > 0$ such that*

$$rr' \leq D\, d(\xi,\xi')^2. \tag{8}$$

If $0 \leq s < s_0$, then for every canonical ball K in $\overline{B(\xi,r)-B(\xi,sr)}$, either K does not meet $B(\xi',sr')$ or there exists a canonical ball K' in $\overline{B(\xi',r')-B(\xi',sr')}$ which is contained in K.

Proof. Let us begin by fixing the value of s_0. The significance of this particular value of s_0 will become apparent in the proofs of Cases 1 and 2 below. Consider the map $\phi :]0,+\infty[\, \to \mathbb{R}$ defined by

$$\phi(t) = \frac{\sqrt{4t+1}-t-1}{t} \;.$$

Note that this map is strictly decreasing on $]0,2]$ from $\phi(0^+)=1$ to $\phi(2)=0$, hence has values in $]0,1[$ on $]0,2[$.

Let $\delta(\cdot)$ be a modulus of sphere extendability of (Y,d), which has values in $]0,1[$. For every D in $]0,\frac{1}{4}]$, define

$$s_0=\sup_{0<\epsilon<1}\min\{\phi\big(4D(1+\epsilon)\big),\phi\big(4D(2-\delta(\epsilon))\big)\}\ . \tag{9}$$

Note that s_0 belongs to $]0,1]$. Also remark that if $\delta(\epsilon)=\epsilon$ (as in Example 3.1 (i)), then $s_0=\phi(6D)$.

Take s with $0\le s<s_0$. Choose an ϵ in $]0,1[$ such that $s\le\min\{\phi\big(4D(1+\epsilon)\big),\phi\big(4D(2-\delta(\epsilon))\big)\}$, and define $\delta=\delta(\epsilon)$. Note that in particular, we have $s<1$.

Let ξ,ξ',r,r',K be as in the statement. Assume that K meets $B(\xi',sr')$, and let us prove the existence of K' as in the statement. As $s<1$ the annuli $\overline{B(\xi,r)-B(\xi,sr)}$ and $\overline{B(\xi',r')-B(\xi',sr')}$ are well defined.

Let $K=B(\eta,R)$. By the properties of the canonical balls, we have

$$R=r\frac{1-s}{2}\ .$$

Since K meets $B(\xi',sr')$, say in a point z, and as K is contained in $B(\xi,r)$, we have

$$d(\xi,\xi')\le d(\xi,z)+d(z,\xi')\le r+sr'\le(1+s)r\ .$$

By Equation (8), we have $rr'\le D\,d(\xi,\xi')^2\le D(1+s)^2r^2$, so that

$$r'\le D(1+s)^2r=\frac{2D(1+s)^2}{1-s}R\ . \tag{10}$$

Case 1 : First assume that $d(\xi',\eta)\ge r'$.

Fix a geodesic segment between η and ξ'. Define x',y' to be the points at distance r',sr' respectively from ξ' on this segment. Let η' be the midpoint of x',y' on this segment, and $R'=r'\frac{1-s}{2}$. Note that the points η,η',y' and ξ' are in this order on this segment (see the picture below).

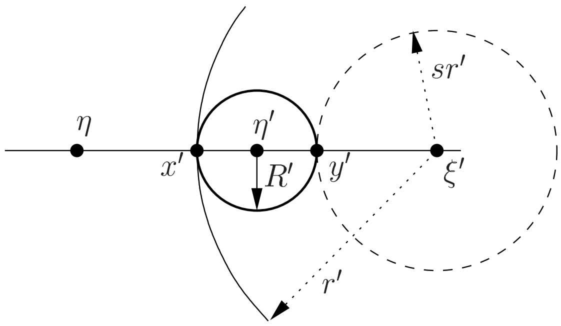

FIGURE 2

Define $K'=B(\eta',R')$, which is a canonical ball in $\overline{B(\xi',r')-B(\xi',sr')}$. Let us prove that K' is contained in K.

Let us first show that y' belongs to K. If this is not the case, then $d(y',\eta)>R$. Let z be a point in $K\cap B(\xi',sr')$. The triangle inequality gives

$$d(\eta,\xi')=d(\eta,y')+d(y',\xi')>R+sr'\ge d(\eta,z)+d(z,\xi')\ge d(\eta,\xi')\ ,$$

which is a contradiction.

Let w be a point in K', and let us prove that it belongs to K. We have

$$d(w,\eta) \le d(w,\eta') + d(\eta',\eta) \le R' + d(\eta',\eta) = d(y',\eta') + d(\eta',\eta) = d(y',\eta) \le R.$$

Thus, K' is contained in K.

Case 2 : Assume now that $r' \ge d(\xi',\eta) \ge (1-\delta)r'$.

Then since $\delta(\cdot)$ is a modulus of sphere extendability for Y, and $\delta = \delta(\epsilon)$, there exists a point ζ in Y such that $d(\zeta,\eta) \le \epsilon r'$ and $d(\zeta,\xi') = r'$. Let $K' = B(\eta', r'\frac{1-s}{2})$ be a canonical ball in $\overline{B(\xi',r') - B(\xi',sr')}$ containing ζ, so that ζ lies on the sphere of center η' and radius $r'\frac{1-s}{2}$. Let us prove that K' is contained in K.

For every y in K', we have

$$d(y,\eta) \le d(y,\eta') + d(\eta',\zeta) + d(\zeta,\eta) \le r'\frac{1-s}{2} + r'\frac{1-s}{2} + \epsilon\, r' = r'(1-s+\epsilon)\ .$$

So that by Equation (10) we get

$$d(y,\eta) \le \frac{2D(1-s+\epsilon)(1+s)^2}{1-s} R \le \frac{2D(1+\epsilon)(1+s)^2}{1-s} R\ .$$

Since $s \le \phi\big(4D(1+\epsilon)\big)$, an easy computation shows that the term on the right is at most R, which proves the result.

Case 3 : Finally assume that $d(\xi',\eta) \le (1-\delta)r'$.

Hence for every y in $B(\xi',r')$, we have, using again Equation (10),

$$d(\eta,y) \le d(y,\xi') + d(\xi',\eta) \le r' + (1-\delta)r' \le \frac{2D(2-\delta)(1+s)^2}{1-s} R\ .$$

Since $s \le \phi\big(4D(2-\delta)\big)$, an easy computation shows that $\frac{2D(2-\delta)(1+s)^2}{1-s} \le 1$, so that y belongs to K. Hence any canonical ball in $\overline{B(\xi',r') - B(\xi',sr')}$ is contained in K. Note that as

$$r'^{\,2} \le rr' \le D\, d(\xi,\xi')^2 \le d(\xi,\xi')^2\ ,$$

we have $d(\xi,\xi') \ge r'$. Since (Y,d) is a geodesic metric space, the sphere of center ξ' and radius r' is non empty, hence such a canonical ball does exists by Lemma 3.5. □

Remark. To prove Remark 3.4 (iii), we make an analogous proof as the one above, adding the assumption on Y that through any two points of Y passes a geodesic line, and requiring only D to be in $]0,1/2[$. Define now $s_0 = \phi(4D)$. Let ξ,ξ',r,r',s,K be as in the statement of Proposition 3.6. Assume that K meets $B(\xi',sr')$. The proof of Case 1 is unchanged. For the two other cases, that is if $d(\xi',\eta) < r'$, define ζ to be the point at distance r' from ξ' on the same side as η on some geodesic line ℓ passing through η and ξ'. Let $K' = B(\eta', r'(1-s)/2)$ be a canonical ball in $\overline{B(\xi',r') - B(\xi',sr')}$ containing ζ.

If η lies between η' and ξ' on ℓ, then for every y in K',

$$d(y,\eta) \le d(y,\eta') + d(\eta',\eta) \le r'\frac{1-s}{2} + r'\frac{1+s}{2} = r'\ .$$

If η lies between ζ and η' on ℓ, then for every y in K', as both η and y are in K',

$$d(y,\eta) \le 2r'\frac{1-s}{2} \le r'\ .$$

Using Equation (10), we have $d(y,\eta) \le 2D(1+s)^2R/(1-s)$. Hence, as $s < s_0 = \phi(4D)$, the points y belongs to K, which proves the result.

Taking $D = 1/4$ gives Proposition 1.3 in the introduction.

Proof of Theorem 3.3. Let Y, D be as in the statement. Take s with $0 < s < s_0$ where $s_0 = s_0(D, \delta) \leq 1$ is as in Proposition 3.6. Define

$$\epsilon = 1 - (1+s)\sqrt{D} \;,$$

which is easily seen to be positive, as $D \leq \frac{1}{4}$ and $s_0 < 1$.

Let $\mathcal{B}$ be as in the statement. We may assume that A is non empty. As the balls in $\mathcal{B}$ have uniformly bounded radii, we can assume that $\sup_{\alpha \in A} r_\alpha = 1$, up to replacing Y by $\frac{1}{\sup_{\alpha\in A} r_\alpha} Y$ (which satisfies the same assumptions as Y) and $\mathcal{B}$ by $\frac{1}{\sup_{\alpha\in A} r_\alpha}\mathcal{B}$, since the quadratic packing condition (7) is scaling invariant.

Let us prove the stronger statement that given α_0 in A such that $r_{\alpha_0} \geq 1-\epsilon$, then there exists at least one point in B_{α_0} that does not belong to $\bigcup_{\alpha\in A} s\overset{\circ}{B}_\alpha$.

For every α in $A - \{\alpha_0\}$ such that $r_\alpha > r_{\alpha_0}$, by the quadratic packing condition, we have

$$d(\xi_\alpha, \xi_{\alpha_0}) > \frac{1}{\sqrt{D}}(1-\epsilon) \;.$$

If the balls B_{α_0} and sB_α were meeting, then

$$d(\xi_\alpha, \xi_{\alpha_0}) \leq r_{\alpha_0} + s r_\alpha \leq 1 + s \;,$$

which would contradict the definition of ϵ. Hence, up to removing such α's, we may assume that $r_\alpha \leq r_{\alpha_0} = 1$ for every α in A.

For every α in $A - \{\alpha_0\}$ such that $d(\xi_\alpha, \xi_{\alpha_0}) > 3$, the balls B_{α_0} and sB_α are disjoint, by the triangular inequality and as r_{α_0}, sr_α are at most 1. Hence, up to removing such α's, we may assume that $d(\xi_\alpha, \xi_{\alpha_0}) \leq 3$ for every α in A.

For every $\epsilon' > 0$, by compactness of the ball $B(\xi_{\alpha_0}, 3)$ and the quadratic packing condition, there are only finitely many elements α in A such that $r_\alpha \geq \epsilon'$. Hence we can order the elements of $A - \{\alpha_0\}$ by $\alpha_1, \alpha_2, \dots$ such that the radii of the balls B_{α_i} are non-increasing. Define $r_i = r_{\alpha_i}$ and $\xi_i = \xi_{\alpha_i}$ for i in $\mathbb{N}$.

Let us construct by induction on i in some initial subsegment of $\mathbb{N}$, an element n_i in $\mathbb{N}$ and a canonical ball K_i in $\overline{B(\xi_{n_i}, r_{n_i}) - B(\xi_{n_i}, sr_{n_i})}$, contained in K_{i-1} for $i > 0$, and which does not meet the interior of $B(\xi_j, sr_j)$ for any $j \leq n_i$.

Let $n_0 = 0$ and K_0 be any canonical ball in $\overline{B(\xi_0, r_0) - B(\xi_0, sr_0)}$ (which exists by Lemma 3.5). They satisfy the requirement for $i = 0$. Assume the construction done for i in $\mathbb{N}$.

If K_i does not meet any $B(\xi_n, sr_n)$ for $n > n_i$, then the sequence (n_i, K_i) stops. Otherwise, let n_{i+1} be the first integer $n \geq n_i + 1$ for which K_i meets $B(\xi_n, sr_n)$. Proposition 3.6 implies that there exists a canonical ball K_{i+1} in $\overline{B(\xi_{n_{i+1}}, r_{n_{i+1}}) - B(\xi_{n_{i+1}}, sr_{n_{i+1}})}$ which is contained in K_i. The couple (n_{i+1}, K_{i+1}) satisfies the requirement.

Clearly, any point in the nonempty intersection $\bigcap_i K_i$ of the nested nonempty compact subsets K_i is in the complement of $\bigcup_{\alpha\in A} s\overset{\circ}{B}_\alpha$. □

Remark 3.7. At the beginning of the induction in the proof of Theorem 3.3, we fixed a canonical ball K_0. We obtained a point in the complement of $s\overset{\circ}{\mathcal{B}}$ which is contained in K_0. Assume that in (Y, d) the *spheres have antipodal points*, i.e. for every y in Y and $r > 0$, there exists p, q in the sphere $S(y, r)$ such that $d(p, q) = 2r$. This is the case if through any point in Y passes a geodesic line, in particular if Y is a Carnot group endowed with its Carnot-Carathéodory metric (see Remark 4.2). Then take antipodal points p, p' on the sphere of center ξ_{α_0} and radius r_{α_0}. If K_0, K'_0 are canonical balls in

$\overline{B(\xi_{\alpha_0}, r_{\alpha_0}) - B(\xi_{\alpha_0}, sr_{\alpha_0})}$ containing p, p' respectively (which exist by Lemma 3.5), then K_0 and K'_0 are disjoint (they are even at distance at least $sr_{\alpha_0} > 0$, by the triangular inequality). Thus, in the conclusion of Theorem 3.3, we can obtain that if $s < s_0$, then there exist not only one, but at least two points outside $s\overset{\circ}{\mathcal{B}}$. We will use this fact in the next Section (see Remark 4.6).

4. Unclouding the sky of negatively curved manifolds

In this section, we consider a proper CAT(-1) space X, a point $\xi_\sharp$ in ∂X, and a horoball $H_\sharp$ centered at $\xi_\sharp$. We denote by $d_\infty = d_{\xi_\sharp, H_\sharp}$ the Hamenstädt distance on $\partial X - \{\xi_\sharp\}$ and by d_ℓ its associated length distance. We assume that the following two conditions on $(X, \xi_\sharp)$ hold:

(i) the metrics d_∞ and d_ℓ are equivalent, i.e. there exists $C \geq 1$ such that

$$d_\infty \leq d_\ell \leq C d_\infty. \tag{11}$$

(ii) the metric space $(\partial X - \{\xi_\sharp\}, d_\ell)$ has extendable spheres, with modulus $\delta(\cdot)$.

The inequalities (11) in particular imply that d_ℓ is finite, so that $(\partial X - \{\xi_\sharp\}, d_\ell)$ is a proper geodesic metric space. Note that these two properties do not depend on $H_\sharp$. By Remark 2.1, if $H'_\sharp$ is another horosphere in X, centered at the point at infinity $\xi'_\sharp$, then the Hamenstädt distance $d'_\infty = d_{\xi'_\sharp, H'_\sharp}$ and its associated length distance d'_ℓ also satisfy the first property (i) locally, and locally

$$\lambda e^{-8} d_\ell \leq d'_\ell \leq \lambda e^{8} d_\ell$$

for some $\lambda > 0$, so that $(\partial X - \{\xi'_\sharp\}, d'_\ell)$ also has extendable spheres restricted (in a sense we won't make precise) to a compact subset of $\partial X - \{\xi_\sharp, \xi'_\sharp\}$.

The class of spaces for which (i) and (ii) hold includes (besides the real hyperbolic n-space $\mathbb{H}^n_\mathbb{R}$, with the constant $C = 1$, as we have already seen) the complex hyperbolic n-space $\mathbb{H}^n_\mathbb{C}$, with the constant $C = \sqrt{\pi}$. To prove this, we use the upper halfspace model of $\mathbb{H}^n_\mathbb{C}$ with $\xi_\sharp$ the point at infinity, where the space $\partial\mathbb{H}^n_\mathbb{C} - \{\infty\}$ is identified with the Heisenberg group $\mathcal{H}$. Then the Hamenstädt distance on $\partial\mathbb{H}^n_\mathbb{C} - \{\infty\}$ is (up to a constant factor) the Cygan metric d_{Cyg} on $\mathcal{H}$ (see [HP4, page 219]). We have already said that the Carnot-Carathéodory metric d_{CC} on $\mathcal{H}$ is the length distance associated to the Cygan distance on $\mathcal{H}$, and that the Cygan and Carnot-Carathéodory distances satisfy $d_{\mathrm{Cyg}} \leq d_{CC} \leq \sqrt{\pi} d_{\mathrm{Cyg}}$. We proved in Proposition 3.2 that the Carnot-Carathéodory distance on $\mathcal{H}$ has extendable spheres with modulus $\delta(\epsilon) = 1 - (1 + \epsilon^2/\pi)^{-1/2}$. Hence the two properties (i), (ii) above are satisfied for $(\mathbb{H}^n_\mathbb{C}, \infty)$ (and any point at infinity $\xi_\sharp$ instead of ∞ by homogeneity).

More generally, the two conditions (i) and (ii) are true if X is any negatively curved symmetric space, see [Ham2, Lemmas 2.1, 2.2] (where the distance defined in this paper is not exactly the Hamenstädt distance, but is equivalent to it, see [HP2]), that is, besides the above cases, for the quaternionic hyperbolic spaces and the octonionic hyperbolic plane.

There are infinitely many more examples. Let us first recall the basic structural properties of a connected homogeneous negatively curved Riemannian manifold M, which are essentially due to [Kob, Wolf, Hei] (see also [Ale, AW1, AW2]).

Such a manifold M is simply connected and there exists a simply transitive solvable Lie group S of isometries of M [Hei, Prop. 1]. We identify M and S from now on, so that S carries a left-invariant negatively curved Riemannian metric. Let $N = [S, S]$, and let $\mathfrak{S}, \mathfrak{N}$ be the Lie algebras of S and N. If X is a unit length vector in $\mathfrak{S}$ which is orthogonal to $\mathfrak{N}$, then $\mathfrak{S} = \mathbb{R}X \oplus \mathfrak{N}$ and (up to changing X to $-X$) the eigenvalues of the derivation $A = \mathrm{ad}\ X\ |_{\mathfrak{N}}$ of $\mathfrak{N}$ have positive real parts (see [Hei, Prop. 2] and [EH, Sect. 1.3]). Note that S and A are not uniquely determined by M, but $\mathfrak{N}$ is (up to isomorphism) (see [AW1, AW2] who give the precise relation between two such S's, and see also [EH, Sect. 1.8]). Conversely, if $\mathfrak{N}$ is a nilpotent Lie algebra endowed with a derivation A whose eigenvalues have positive real parts, then the simply connected Lie group, whose Lie algebra is the one-dimensional extension of $\mathfrak{N}$ constructed using A, carries a left-invariant negatively curved Riemannian metric (see [Hei, Theo. 3]). This metric is not unique in general (see for instance [EH] for precise examples).

Now, let $\mathfrak{N}_\lambda$ be the eigenspace of A associated to a complex number λ (which is trivial if λ is not an eigenvalue). Note that $[\mathfrak{N}_\lambda, \mathfrak{N}_\mu]$ is contained in $\mathfrak{N}_{\lambda+\mu}$, as A is a derivation.

We say that M is a homogeneous negatively curved Riemannian manifold *of Carnot type* if $\mathfrak{N}_1$ generates $\mathfrak{N}$ as a Lie algebra. In particular, N is a *Carnot group* (see for instance [FS, Pan2, Gro3]), which means that $\mathfrak{N}$ is graduated by $\mathfrak{N} = \bigoplus_{i=1}^k \mathfrak{N}_i$ (for some k) as a Lie algebra, with $\mathfrak{N}_1$ generating $\mathfrak{N}$ as a Lie algebra. We will make some comments on this definition after the proof of the following proposition.

Proposition 4.1. *Any homogeneous negatively curved Riemannian manifold of Carnot type satisfies the above two properties (i) and (ii), for some point at infinity.*

Proof. Let $\exp : \mathfrak{S} \to S$ be the exponential map. Note that by [Hei] (see also [EH]), the curve $t \mapsto \exp tX$ is a (unit speed) geodesic line passing through the identity e of S at time $t = 0$. Furthermore, N is the horosphere containing e centered at the point at infinity $\xi_\sharp$ of $t \mapsto \exp tX$. Identify the punctured boundary at infinity $\partial S - \{\xi_\sharp\}$ with N by the map sending the point g in N to the endpoint of the geodesic line starting from $\xi_\sharp$ and passing through g at time $t = 0$. Note that this geodesic line is $t \mapsto g\exp(-tX)$. Hence the isometry of S which is the left translation by $\exp(tX)$, fixes the point $\xi_\sharp$ and extends to the punctured boundary at infinity, identified to N as above, by the map $\psi^t : g \mapsto \exp(tX)g\exp(-tX)$.

As $\exp(tX)N$ is the horosphere centered at $\xi_\sharp$ containing $\exp tX$, its algebraic distance to N is exactly t. By the formulas (3) and (2) in Section 2, for every a, b in $\partial S - \{\xi_\sharp\}$, we have

$$d_{\xi_\sharp, N}(\psi^t(a), \psi^t(b)) = d_{\xi_\sharp, \exp(-tX)N}(a, b) = e^t\ d_{\xi_\sharp, N}(a, b)\ .$$

Hence $(\psi^t)_{t\in\mathbb{R}}$ is a one-parameter group of dilations for the Hamenstädt distance $d_{\xi_\infty, N}$.

Recall that the Carnot-Carathéodory distance d_{CC} on N is defined as follows (see for instance [Gro3]). Endow $\mathfrak{N}_1$ with a Euclidean norm (for instance the one induced by the Riemannian metric on S). Consider the left-invariant Euclidean vector sub-bundle Δ of TN defined by the Euclidean subspace $\mathfrak{N}_1$ of T_eN. Then $d_{CC}(x, y)$ is the lower bound of the Euclidean lengths of the smooth paths between x and y in N that are tangent to Δ at every point.

We make an independent remark, which will be used in Section 7.

Remark 4.2. *In a Carnot group endowed with its Carnot-Carathéodory distance, the spheres have antipodal points (in the sense of Remark 3.7).*

Proof. Indeed, the map $t \mapsto exp(tX)$ for X in $\mathfrak{N}_1$ is a geodesic line (parametrized proportionally to arc length) through e, and then one uses the homogeneity of N. □

Recall that

$$\exp tX \exp Y \exp -tX = \exp\left(e^{t\ \mathrm{ad}\ X}(Y)\right)\ ,$$

and that $\exp : \mathfrak{N} \to N$ is a diffeomorphism. As ψ^t is a group morphism of N and as $e^{t \text{ ad } X}$ is the homothety of ratio e^t on $\mathfrak{N}_1$, the map ψ^t acts on the Euclidean sub-bundle Δ by a dilation of ratio e^t from Δ_g to $\Delta_{\psi^t(g)}$. Hence $(\psi^t)_{t\in\mathbb{R}}$ is also a one-parameter group of dilations for the Carnot-Carathéodory distance d_{CC}.

Since the Carnot-Carathéodory and Hamenstädt distances induce the same topology, they are equivalent on the spheres of center e and radius 1 in N. As $(\psi^t)_{t\in\mathbb{R}}$ is a one-parameter group of dilations for both of them, they are also equivalent on every sphere of center e. As they are both invariant by left-translation under N, these two distances are equivalent.

As $d_{\xi_\sharp,N}$ is equivalent to d_{CC} which is a length distance, it follows easily that the length distance d_ℓ associated to $d_{\xi_\sharp,N}$ is finite, and equivalent to d_{CC}, hence to $d_{\xi_\sharp,N}$.

By Example 3.1 (iv), since $(\partial S - \{\xi_\sharp\}, d_{\xi_\sharp,N})$ has dilations, it has extendable spheres. This proves the result. □

Remark 4.3. (i) Many authors have considered homogeneous negatively curved Riemannian manifolds M of Carnot type (see [Gro3]). For instance, it is proved in [HK] that the local and global definitions of quasi-conformal maps on the boundary of M coincide. The negatively curved 3-steps Carnot solvmanifolds of [EH] belong to this class (they are precisely the ones such that $\mathfrak{N} = \mathfrak{N}_1 \oplus \mathfrak{N}_2$ and $\mathfrak{N}_2$ is the full center of $\mathfrak{N}$).

(ii) If M is a negatively curved symmetric space different from the real hyperbolic space, then M is a negatively curved 3-step Carnot solvmanifold, such that furthermore the center of $\mathfrak{N}$ has dimension $1, 3$ or 7. There exist infinitely many pairwise non isomorphic Carnot groups with order of nilpotency at least 3 (if $\mathfrak{G}$ is any finite dimensional nilpotent Lie algebra, with $\mathfrak{G}_0 = \mathfrak{G}$ and $\mathfrak{G}_{i+1} = [\mathfrak{G}, \mathfrak{G}_i]$, then the graded Lie algebra associated to $\mathfrak{G}$, which is $\bigoplus_i \mathfrak{G}_i/\mathfrak{G}_{i+1}$ with the obvious Lie bracket, is the Lie algebra of a Carnot group). There exist infinitely many Carnot groups with order of nilpotency 2 whose center has dimension at least 8 (see for instance [Kap]). Each Lie algebra $\mathfrak{N}$ of a Carnot group carries a derivation A whose eigenvalues have positive real parts, defined, if $\mathfrak{N} = \oplus_{i=1}^k \mathfrak{N}_i$ with $[\mathfrak{N}_i, \mathfrak{N}_j] \subset \mathfrak{N}_{i+j}$, by $A\mid_{\mathfrak{N}_j} = j \ \mathrm{id}_{\mathfrak{N}_j}$. As seen above, we get infinitely many (pairwise non homothetic) non symmetric homogeneous negatively curved Riemannian manifolds.

(iii) Let M be a homogeneous negatively curved Riemannian manifold. If we assume that the curvature of M is normalized so that its maximum is -1, and that A is diagnalisable over $\mathbb{R}$, then the assumption (in the definition of being of Carnot type) that $\mathfrak{N}_1$ generates $\mathfrak{N}$ as a Lie algebra is in fact necessary.

To prove this, we start with the following lemma, whose proof is contained in the proof of Lemma 1, page 486 of [EH].

Lemma 4.4. *(Eberlein-Heber) If the curvature of M is at at most -1, then the real part of any eigenvalue λ of A is at least* 1.

Proof. As A is a linear map of the Euclidean space $\mathfrak{N}$, we can consider the symmetric and antisymmetric part $D_0 = \frac{1}{2}(A + A^*)$, $S_0 = \frac{1}{2}(A - A^*)$ of A. Recall that D_0 is symmetric and positive definite [Hei, Prop2]. Define $N_0 = D_0^2 + D_0 S_0 - S_0 D_0$. Recall that (see for instance [Hei]) if Y is a unit vector in $\mathfrak{N}$, then the curvature in S of the tangent 2-plane generated by X (where X is as above) and Y is

$$K(X,Y) = -\langle N_0 Y, Y\rangle \ .$$

As N_0 is symmetric, hence diagonalisable, and since $K \leq -1$, the eigenvalues of N_0 are at least 1. Let $(Y_i)_{i\in I}$ be an orthonormal basis of eigenvectors of D_0, with (positive) associated eigenvalues $(\lambda_i)_{i\in I}$. Note that

$$\lambda_i^2 = \langle D_0^2 Y_i, Y_i\rangle = \langle N_0 Y_i, Y_i\rangle = -K(X, Y_i) \geq 1 \ .$$

Hence $\lambda_i \geq 1$ for every i. In particular, $\langle D_0\zeta, \zeta\rangle \geq ||\zeta||^2$ for every ζ in $\mathfrak{N}$. Now let $\xi = \xi' + i\xi''$ be a complex eigenvector for the complex eigenvalue $\lambda = \lambda' + i\lambda''$ of A. We have, by an easy computation,

$$\lambda'(||\xi'||^2 + ||\xi''||^2) = \langle A\xi', \xi'\rangle + \langle A\xi'', \xi''\rangle = \langle D_0\xi', \xi'\rangle + \langle D_0\xi'', \xi''\rangle \geq ||\xi'||^2 + ||\xi''||^2 \ .$$

Therefore $\lambda' \geq 1$. □

Now assume that the curvature of M has maximum -1, and that A is diagonalisable over $\mathbb{R}$. Identify N with $\mathfrak{N}$ by the exponential map. Let Λ be the set of eigenvalues of A. Consider the map d' from $N \times N$ to $[0, +\infty)$, which is left-invariant by the diagonal action of N on $N \times N$, such that for $Y = \sum_{\lambda\in\Lambda} Y_\lambda$ in $\mathfrak{N} = \bigoplus_{\lambda\in\Lambda} \mathfrak{N}_\lambda$, we have

$$d'(0, Y) = \sum_{\lambda\in\Lambda} ||Y_\lambda||^{\frac{1}{\lambda}} \ .$$

Note that the map d' (which is a priori not a distance) is continuous and does not vanish outside the diagonal of $N \times N$, and that the map ψ^t is a dilation of ratio e^t for the map d'. Hence there is a constant $c > 0$ such that, for every x, y in N,

$$\frac{1}{c} d'(x, y) \leq d_{\xi_\sharp, N}(x, y) \leq c d'(x, y) \ .$$

Therefore, to check if a path has finite length or not, we may replace the Hamenstädt distance by d'. As the eigenvalues of A are at least 1 and $||\ ||$ is an Euclidean norm on N, the only smooth paths that have finite length are the ones that are tangent at every point to the left-invariant vector sub-bundle Δ of TN defined by the subspace $\mathfrak{N}_1$ of T_eN. In order for every pair of points to be joined by a finite length curve, it is hence necessary that $\mathfrak{N}_1$ generates $\mathfrak{N}$ as a Lie algebra.

For every horosphere H in a CAT(-1) metric space X, and every $t \geq 0$, define $H(t)$ to be the horosphere at distance t from H, contained in the horoball bounded by H.

Theorem 4.5. *Let X be a proper CAT(-1) space, $\xi_\sharp$ a point in ∂X, and $H_\sharp$ a horoball centered at $\xi_\sharp$. Assume that $(X, \xi_\sharp)$ satisfies the conditions (i) and (ii) as in the beginning of Section 4. There exists $t_0 > 0$ such that for every sequence $(H_n)_{n\in\mathbb{N}}$ of horospheres bounding open horoballs that are pairwise disjoint and disjoint from the open horoball bounded by $H_\sharp$, if $t > t_0$, then there exists at least one geodesic line starting at $\xi_\sharp$ which avoids $H_n(t)$ for every n in $\mathbb{N}$.*

Proof. We use the notations of the beginning of the section. Define

$$t_0 = t_0(C, \delta) = -\log \frac{s_0(\frac{1}{4}, \delta)}{2c_{\max} C} \ ,$$

where $s_0(\epsilon, \delta)$ is as in Theorem 3.3, and $c_{\max}$ as in Remark 2.4 (and $c_{\max} = 2^{-\frac{1}{a}}$ if X is a Riemannian manifold with sectional curvature $-a^2 \leq K \leq -1$, with $a \geq 1$).

Let ξ_n be the point at infinity of H_n and

$$r_n = r_i(H_n) = \frac{1}{2} e^{-d(H_\sharp, H_n)} > 0$$

be the inner radius of the shadow of H_n. Consider the family of balls $(B_n = B_{d_\ell}(\xi_n, r_n))_{n\in\mathbb{N}}$ in $(\partial X - \{\xi_\sharp\}, d_\ell)$. By Corollary 2.7, with $D = \frac{1}{4}$, we have, if $n \neq m$,

$$0 < r_n r_m \leq D d_\infty(\xi_n, \xi_m)^2 \leq D d_\ell(\xi_n, \xi_m)^2 \ .$$

Note that the radii $(r_n)_{n\in\mathbb{N}}$ are uniformly bounded (by $\frac{1}{2}$), and recall that the proper geodesic metric space $(\partial X - \{\xi_\sharp\}, d_\ell)$ has extendable spheres with modulus δ. Hence, by Theorem 3.3, for every $s < s_0(\frac{1}{4},\delta)$, there exists a point ξ_s in $\partial X - \{\xi_\sharp\}$ which does not belong to $\bigcup_{n\in\mathbb{N}} sB_n$. By Remark 2.4 (and Corollary 2.3 if X is a Riemannian manifold with sectional curvature $-a^2 \leq K \leq -1$, with $a \geq 1$), as $d(H_\sharp, H_n(t)) = t + d(H_\sharp, H_n)$, we have

$$\mathcal{O}_{\xi_\sharp} H_n(t) \subset B_{d_\infty}(\xi_n, 2c_{\max}\, e^{-t} r_n) \subset B_{d_\ell}(\xi_n, 2c_{\max}\, e^{-t} C r_n)\ .$$

Let $s(t) = 2c_{\max}\, e^{-t} C$, so that if $t > t_0$, then $s(t) < s_0(\frac{1}{4},\delta)$. Therefore the point $\xi_{s(t)}$ in $\partial X - \{\xi_\sharp\}$ does not belong to $\bigcup_{n\in\mathbb{N}} \mathcal{O}_{\xi_\sharp} H_n(t)$, which proves the result. $\square$

Note that if X is a complete simply connected Riemannian manifold with sectional curvature $-a^2 \leq K \leq -1$ (with $a \geq 1$), if the Hamenstädt distance d_∞ is geodesic and if through any two points of $\partial X - \{\xi_\sharp\}$ passes a geodesic line for d_∞, then, by Remark 3.4 (iii), the minimal value t_0 is

$$t_0 = -\log \frac{\sqrt{5}-2}{2^{1-\frac{1}{a}}}\ .$$

In the case X is the real hyperbolic n-space $\mathbb{H}^n_\mathbb{R}$, we get $t_0 = -\log(\sqrt{5}-2) \approx 1.44$. In Section 6, we will show how to improve this, by using the fact that the boundary of $\mathbb{H}^n_\mathbb{R}$ minus a point is the $(n-1)$-dimensional Euclidean space to prove a stronger version of Theorem 3.3. If $X = \mathbb{H}^2_\mathbb{C}$, a straightforward calculation using Proposition 3.2 yields $t_0 \approx 4.9157$.

Remark 4.6. Let $(X, \xi_\sharp, H_\sharp)$ be as in Theorem 4.5. Assume that furthermore the spheres in $(\partial X - \{\xi_\sharp\}, d_\ell)$ have antipodal points (in the sense of Remark 3.7). Then it follows from Remark 3.7 that there exists $t_0 > 0$ such that for every sequence $(H_n)_{n\in\mathbb{N}}$ of horospheres bounding open horoballs that are pairwise disjoint and disjoint from the open horoball bounded by $H_\sharp$, if $t > t_0$, then there exist at least two geodesic lines starting at $\xi_\sharp$ which avoid $H_n(t)$ for every n in $\mathbb{N}$. We will use this fact in Section 7.

Remark 4.7. (1) Let T be a locally finite metric tree, with degrees at least 3, and denote by $\ell_{\max}$ the upper bound on the length of the edges. Recall that T is a proper CAT(-1) space (see for instance [BH]). We claim that for every sequence $(H_n)_{n\in\mathbb{N}}$ of horospheres in T, bounding open horoballs that are pairwise disjoint, for every point x_0 not contained in an open horoball bounded by some H_n, if $t > \ell_{\max}$, then there exist at least two geodesic rays in T starting at x_0 which avoid $H_n(t)$ for every n in $\mathbb{N}$.

Proof. Fix a geodesic ray starting from x_0. Follow this ray until it reaches its first intersection point y_1 with $\bigcup_n H_n$ or otherwise till infinity. In the first case, let H_{i_1} be a horosphere containing y_1. Follow a segment starting at y_1 and entering in H_{i_1}, and when arriving at the first vertex v_1 of T, follow a segment starting at v_1 that does not point towards the point at infinity of H_{i_1} or back towards x_0. Let x_1 be the intersection point with H_{i_1} after v_1. Fix a geodesic ray starting at x_1 that does not enter H_{i_1}. Iterating this construction, we get a geodesic ray starting from x_0 which avoids $H_n(t)$ for every n in $\mathbb{N}$ and $t > \ell_{\max}$. This gives our first geodesic ray. Using the fact that the degree at y_1 is at least 3, we easily get a second geodesic ray. $\square$

(2) The above result is sharp, as shown by the 3-regular tree T_3 whose edge lengths are equal to some constant N, with a family $(H_n)_{n\in\mathbb{N}}$ of horospheres bounding disjoint open horoballs, such that the union of the (closed) horoballs bounded by the H_n's covers T_3. The existence of such a family is easy, but see also [BL, Pau] for algebraic constructions.

(3) The above sharpness result (where N can be taken arbitrarily large) shows that it is not possible to generalize Theorem 4.5 to the full collection of CAT(-1) metric spaces, even if we assume that ∂X has no isolated point to avoid trivial examples (as for $X = \mathbb{R}$). To be more precise, there exists no constant $c > 0$ such that for every proper geodesic CAT(-1) metric space X, for every horosphere $H_\sharp$ centered at $\xi_\sharp$, for every every sequence $(H_n)_{n\in\mathbb{N}}$ of horospheres in X, bounding open horoballs that are pairwise disjoint and disjoint from the open horoball bounded by $H_\sharp$, if $t > c$, then there exists at least one geodesic ray starting at $\xi_\sharp$ which avoids $H_n(t)$ for every n in $\mathbb{N}$. Some conditions on X, as for instance (i) and (ii), (which in particular imply that ∂X is path-connected, which excludes the case of trees for instance) are hence necessary.

5. Unclouding the sky of two-dimensional spaces

In this section, we prove a stronger version of Theorem 4.5 in the two-dimensional manifold case. Assume in the whole section that M is any two-dimensional complete simply connected Riemannian manifold with curvature $-a^2 \leq K \leq -1$ (for some $a \geq 1$). In this situation, the boundary of M is homeomorphic to a circle, which allows us to use techniques not available in higher-dimensional situations. Note that in most two-dimensional cases there are no curves of finite length between two distinct points in the boundary. Thus, Theorem 4.5 does not apply in general.

Define $t_1 = t_1(a)$ by

$$e^{-t_1} = \begin{cases} 2^{2/a}\left(\sqrt{1+2^{1-1/a}} - 1 - 2^{-1-1/a}\right) & \text{if } a \leq 2 \\ \min\left\{1 - 2^{-2/a}\ ,\ 2^{2/a}\left(\sqrt{1+2^{1-1/a}} - 1 - 2^{-1-1/a}\right)\right\} & \text{if } a \geq 2\ . \end{cases}$$

Note that this is well defined, as the right terms are positive, and that $t_1(a)$ converges to $+\infty$ as a converges to $+\infty$.

Theorem 5.1. *With M as above, let $\xi_\sharp$ be a point in ∂M, and $H_\sharp$ be a horosphere in M centered at $\xi_\sharp$. For every sequence $(H_n)_{n\in\mathbb{N}}$ of horospheres bounding open horoballs that are pairwise disjoint and disjoint from the open horoball bounded by $H_\sharp$, if $t > t_1$, then there exist at least two geodesic lines starting at $\xi_\sharp$ which avoid $H_n(t)$ for every n in $\mathbb{N}$.*

Before giving the proof, we first fix some notations. The shadows will be taken with respect to the point at infinity $\xi_\sharp$. We denote by d_∞ the Hamenstädt distance $d_{\xi_\sharp, H_\sharp}$, and the balls will be taken with respect to this distance. For η, η' in $\partial M - \{\xi_\sharp\}$, we denote by $[\eta, \eta']_\infty$ the closure of the connected component of $\partial M - \{\eta, \eta'\}$ which does not contain $\xi_\sharp$, and similarly for half-open intervals.

If H is a horosphere centered at a point at infinity ξ different from $\xi_\sharp$, then by convexity, there exist two and only two geodesics lines starting from $\xi_\sharp$ and tangent to H (at time $t = 0$). If ξ_+, ξ_- are the points at infinity of these two geodesic lines, then by connectedness, we have $\mathcal{O}\,H = [\xi_+, \xi_-]_\infty$. In particular, for any $t > 0$, the closure $\overline{\mathcal{O}\,H - \mathcal{O}\,H(t)}$ is the union of exactly two (topological) compact intervals. Note that $\overline{B(\xi, r) - B(\xi, r')}$ for $r' < r$ may a priori have more than two components. We claim that

$$r_i(H) \leq d_\infty(\xi, \xi_\pm) \leq r_e(H)\ . \tag{12}$$

The upper bound follows from the right inclusion in the statement of Corollary 2.3. The lower bound follows from its proof (take $B' = B$ in the paragraph around Figure 1, compare with $\mathbb{H}^2_{-1}$ instead of $\mathbb{H}^2_{-a^2}$, which reverses the comparison inequality, and take limits to pass from balls to horoballs).

The main step of the proof of Theorem 5.1 is the following proposition.

Proposition 5.2. *Let M, $\xi_\sharp$ and $H_\sharp$ be as in Theorem 5.1. Let H and H' be horospheres bounding open horoballs that are disjoint and disjoint from the open horoball bounded by $H_\sharp$, with $r_i(H') \leq r_i(H)$. Let $t \geq t_1(a)$. For every connected component K of $\overline{\mathcal{O}\,H - \mathcal{O}\,H(t)}$, the following dichotomy holds :*

- *either K does not meet $\mathcal{O}\,H'(t)$*
- *or there exists a connected component K' in $\overline{\mathcal{O}\,H' - \mathcal{O}\,H'(t)}$, which is contained in K.*

Proof. Let H, H', t, K be as in the statement. Up to replacing $H_\sharp$ by the horosphere centered at $\xi_\sharp$ and tangent to H, we may assume that H and $H_\sharp$ are tangent, which is equivalent to assuming that $r_i(H) = 1/2$. Let ξ, ξ' be the points at infinity of H, H' respectively. Orient the topological line $\partial M - \{\xi_\sharp\}$ so that K is on the right side of ξ, and let $K = [k_1, k_2]_\infty$ with ξ, k_1, k_2 in this order on $\partial M - \{\xi_\sharp\}$.

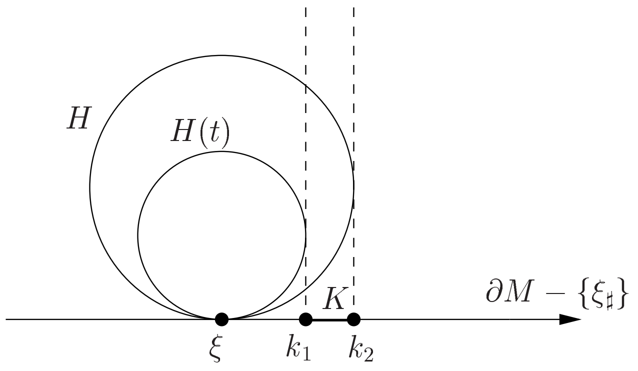

FIGURE 3

Case 1: Assume that ξ' belongs to K. We will show the stronger conclusion that at least one of the two components of $\mathcal{O}\,H' - \{\xi'\}$ is contained in K. This follows if

$$r_e(H') < \max\{d_\infty(\xi', k_1), d_\infty(\xi', k_2)\}.$$

Indeed, recall that $\mathcal{O}\,H'$ is contained in $B(\xi', r_e(H'))$ by Corollary 2.3. Hence, if for $i = 1$ (resp. $i = 2$) we have $r_e(H') < d_\infty(\xi', k_i)$, then the left (resp. right) connected component of $\mathcal{O}\,H' - \{\xi'\}$ is contained in $[\xi', k_i[_\infty$ hence in K. Note that in constant curvature, and more generally if the Hamenstädt distance to a given point is monotonous on the line $\partial M - \{\xi_\sharp\}$, then we may replace the strict inequality in the displayed formula above by a large one.

Define $u = d_\infty(\xi, \xi')$. Corollary 2.7 implies that

$$r_e(H') = 2^{1-\frac{1}{a}} r_i(H') = 2^{-\frac{1}{a}} 4 r_i(H') r_i(H) \leq 2^{-\frac{1}{a}} u^2.$$

On the other hand, the triangle inequality and Equation (12) implies that

$$d_\infty(\xi', k_2) \geq d_\infty(k_2, \xi) - d_\infty(\xi', \xi) \geq r_i(H) - u = \frac{1}{2} - u\ .$$

Thus we have $r_e(H') < d_\infty(\xi', k_2)$ if

$$2^{-\frac{1}{a}} u^2 < 1/2 - u. \tag{13}$$

Similarly, if

$$2^{-\frac{1}{a}} u^2 < u - 2^{-\frac{1}{a}} e^{-t} \tag{14}$$

then the inequality $r_e(H') < d_\infty(\xi', k_1)$ is satisfied.

The first condition (13) holds if (and only if)

$$2^{\frac{1}{a}-1}\left(-1-\sqrt{1+2^{1-\frac{1}{a}}}\right) < u < 2^{\frac{1}{a}-1}\left(-1+\sqrt{1+2^{1-\frac{1}{a}}}\right) .$$

The lower bound is nonpositive, so this reduces to

$$u < u_1(a) = 2^{\frac{1}{a}-1}\left(\sqrt{1+2^{1-\frac{1}{a}}}-1\right) . \tag{15}$$

Let us define $s = 2^{-\frac{2}{a}}e^{-t}$. As $2^{2-\frac{2}{a}}e^{-t_1(a)} < 1$ (by an easy computation), we have $4s < 1$. Hence the second condition (14) holds if (and only if)

$$2^{\frac{1}{a}-1}\left(1-\sqrt{1-4s}\right) < u < 2^{\frac{1}{a}-1}\left(1+\sqrt{1-4s}\right) . \tag{16}$$

Note that ξ' belongs to $\mathcal{O}\,H$, hence by Corollary 2.3, we have $u \leq r_e(H) = 2^{-\frac{1}{a}}$. Now, by an easy computation, the upper bound in (16) is strictly bigger than $2^{-\frac{1}{a}}$ if $a < 2$, or if $a \geq 2$ and $e^{-t} < 1 - 2^{-\frac{2}{a}}$. Hence, as $t > t_1(a)$, Equation (16) reduces to

$$u > u_2(a) = 2^{\frac{1}{a}-1}\left(1-\sqrt{1-4s}\right) . \tag{17}$$

At least one of the conditions (15) and (17) holds if $u_1(a) > u_2(a)$, that is if

$$\sqrt{1+2^{1-\frac{1}{a}}} - 1 > 1 - \sqrt{1-4s} .$$

This last equation is equivalent to $e^{-t} < 2^{2/a}\left(\sqrt{1+2^{1-1/a}} - 1 - 2^{-1-1/a}\right)$, which proves the result in the first case, as $t > t_1(a)$.

Case 2: Assume that ξ' lies on the left of k_1. Then, we assume that K meets $\mathcal{O}\,H'(t)$, and we prove that every point η in the right component of $\overline{\mathcal{O}\,H' - \mathcal{O}\,H'(t)}$ belongs to K. By connectedness, we have that η lies on the right of k_1, and it is sufficient to prove that η lies in $\mathcal{O}\,H$. By Corollary 2.3, we only have to prove that $d_\infty(\xi,\eta) \leq r_i(H) = \frac{1}{2}$.

As H and H' bound disjoint open horoballs, as $r_i(H) \leq r_i(H')$ and as K meets $\mathcal{O}\,H'(t)$, the point ξ' belongs to $[\xi, k_1]_\infty$. Hence by connectedness, the point ξ' belongs to $\mathcal{O}\,H(t)$. By Corollary 2.3, we have

$$d_\infty(\xi,\xi') \leq r_e(H(t)) = 2^{-\frac{1}{a}}e^{-t} ,$$

and

$$d_\infty(\xi',\eta) \leq r_e(H') .$$

By Corollary 2.7, we have $r_i(H') = 2r_i(H')r_i(H) \leq \frac{1}{2}d_\infty(\xi,\xi')^2$, so that

$$d_\infty(\xi',\eta) \leq r_e(H') = 2^{1-\frac{1}{a}}r_i(H') \leq 2^{-\frac{1}{a}}d_\infty(\xi,\xi')^2 \leq 2^{-\frac{3}{a}}e^{-2t} .$$

By the triangular inequality, we have

$$d_\infty(\xi,\eta) \leq d_\infty(\xi,\xi') + d_\infty(\xi',\eta) \leq 2^{-\frac{1}{a}}e^{-t} + 2^{-\frac{3}{a}}e^{-2t} .$$

As $t > t_1(a)$, we have $2^{-\frac{1}{a}}e^{-t} + 2^{-\frac{3}{a}}e^{-2t} \leq \frac{1}{2}$, by an easy computation. Hence $d_\infty(\xi,\eta) \leq \frac{1}{2}$, which proves the result in the second case.

Case 3: Finally, assume that ξ' lies to the right of k_2. Then, we assume that K meets $\mathcal{O}\,H'(t)$, and we prove that the left component of $\overline{\mathcal{O}\,H' - \mathcal{O}\,H'(t)}$ belongs to K.

We start by proving two lemmas, which are obvious in the constant curvature case, but not immediate otherwise.

Lemma 5.3. *If H, H' are horospheres in M centered at points ξ, ξ' different from $\xi_\sharp$, bounding disjoint open horoballs, and such that there exists a geodesic line ℓ starting from $\xi_\sharp$ that first meets H' and then meets H, then*

$$r_i(H) \leq 2^{1-\frac{1}{a}} r_i(H') \ .$$

Note that the same result with the constant $2^{1-\frac{1}{a}}$ replaced by 1 is not true, as shows an easy example built by deforming a constant curvature -2 situation with two tangent horospheres of the same height.

Proof. Up to pushing ℓ, we may assume that ℓ is tangent to H, and that the point at infinity of ℓ different from $\xi_\sharp$ belongs to $[\xi, \xi']_\infty$. Up to replacing H' by a horosphere also centered at ξ' and contained in the open horoball bounded by H' (which does not increase $r_i(H')$), we may assume that ℓ is also tangent to H'. Orient $\partial M - \{\xi_\sharp\}$ such that H is on the left of ℓ. Note that H' is on the right of ℓ. If the point x on ℓ moves towards $\xi_\sharp$ on ℓ, then, with H_x the horosphere tangent at x to ℓ and on the left of ℓ, the quantity $d(H_x, H_\sharp)$ is non increasing (we assume, as we may, that $H_\sharp$ is close to $\xi_\sharp$). Hence, up to replacing H by the horosphere on the left of ℓ tangent to ℓ at the same point than H', we may assume that H and H' are tangent. Let x be the (only) tangency point. Consider the quantity

$$\Delta_H = d(x, H_\sharp) - d(H, H_\sharp) \ .$$

If the curvature of M was constant, equal to $-b^2$ (with $1 \leq b \leq a$), then, as an easy computation in the upper halfspace model proves, we would have $\Delta_H = \log 2^{\frac{1}{b}}$. Since the curvature of M lies between $-a^2$ and -1, as an easy comparison argument shows, we have

$$\log 2^{\frac{1}{a}} \leq \Delta_H \leq \log 2 \ .$$

By symmetry, this is also true for H'. Hence $r_i(H)/r_i(H') = e^{\Delta_{H'} - \Delta_H} \leq 2^{1-\frac{1}{a}}$, which proves the result. □

Lemma 5.4. *Let H, H', H'' be horospheres in M centered at points at infinity respectively ξ, ξ', ξ'' distinct from $\xi_\sharp$ and with ξ'' between ξ and ξ' on $\partial M - \{\xi_\sharp\}$, such that H is tangent to H'' and the open horoball bounded by H is disjoint from both open horoballs bounded by H' and H''. Assume that there exists no geodesic line starting from $\xi_\sharp$ that first meets H' and then meets H. Then*

$$\mathcal{O}\, H' \cap [\xi, \xi'']_\infty \subset \mathcal{O}\, H'' \ .$$

Proof. Assume by absurd that there exists a point in $(\mathcal{O}\, H' - \mathcal{O}\, H'') \cap [\xi, \xi'']_\infty$. Let ℓ be the geodesic line starting from $\xi_\sharp$, which is tangent to H'', and whose point at infinity lies in $[\xi, \xi'']_\infty$, which exists as $H \cup H''$ separates M. By convexity of horoballs, the complement in ℓ of the open horoball bounded by H is the union of two geodesic rays ℓ_1, ℓ_2 with ℓ_2 ending in $[\xi, \xi'']_\infty$. Since ℓ cannot first meet H' and then meet H, as H, H' are bounding disjoint open horoballs, and by convexity, the horosphere H' meets ℓ_2 in two distinct points x, y.

Assume that x is closer to $\xi_\sharp$ on ℓ, and orient the topological line H' from y to x. Let p be the first intersection point of H' with H'' after x, and ρ'', ρ' the geodesic rays starting from p and ending at ξ'', ξ'. As ξ' is on the right of ξ'' on $\partial M - \{\xi_\sharp\}$, and as H' enters the horoball bounded by H'' at p, the tangent vector v' to ρ' at the origin is contained in the angular sector between the tangent vector of H' and the tangent vector v'' to ρ'' at the origin. This contradicts the fact that an horosphere is perpendicular to the geodesic lines ending at the point at infinity of a horosphere. □

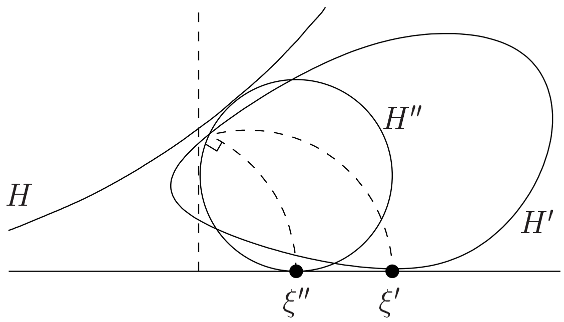

FIGURE 4

Let us end now the proof of Case 3. Consider the (unique by convexity) horosphere H'' centered at k_2 and tangent to H. Let ℓ_{k_2} be the geodesic line starting from $\xi_\sharp$ tangent to H at a point x and ending at k_2. As K meets $\mathcal{O}\,H'(t)$, the horoball bounded by $H'(t)$ has to meet ℓ_{k_2} in a segment. If ℓ_{k_2} was first meeting $H'(t)$ and then H, then by Lemma 5.3, we would have

$$r_i(H) \leq 2^{1-\frac{1}{a}} r_i(H'(t)) \ .$$

As $0 < r_i(H') \leq r_i(H)$, we would have $e^{-t} \geq 2^{\frac{1}{a}-1}$. This is impossible, as $t > t_1(a)$.

Hence no geodesic line starting from $\xi_\sharp$ first meets H' and then meets H. By Lemma 5.4, we have $\mathcal{O}\,H' \cap [\xi, k_2]_\infty \subset \mathcal{O}\,H'' \cap [\xi, k_2]_\infty$. Hence the result follows from Case 1 (applied by replacing H' by H''.) $\square$

Proof of Theorem 5.1. Theorem 5.1 follows from Proposition 5.2 in the same way as Theorem 4.5 follows from Proposition 3.6. Note that the assumption about the space being geodesic and having extendable spheres was only used in Proposition 3.6, and not in the deduction of Theorem 4.5 from Proposition 3.6. Note that in the proof, we fix a connected component K_0 of $\overline{\mathcal{O}\,H_0 - \mathcal{O}\,H_0(t)}$ and we find a geodesic line starting from $\xi_\sharp$ as wanted, which furthermore has its endpoint in K_0. But there exists another such connected components K_0' (which is different from K_0) and working with K_0' instead of K_0, we get a second geodesic line as wanted. $\square$

Remark 5.5. (a) Theorem 5.1 gives the value $t_1(1) = -\log(4\sqrt{2}-5) \approx 0.42$ for the case of constant curvature -1, improving the value $t_0(1) = -\log(\sqrt{5}-2) \approx 1.44$ given by Theorem 4.5.

(b) If $M = \mathbb{H}^2_{\mathbb{R}}$, we have seen in the proof that we may replace the condition $t > t_1$ by $t \geq t_1$, and the triangle inequalities used in the proof of Proposition 5.2 are replaced by equalities. In fact Theorem 5.1 is sharp, in the following sense: there exists a family of horoballs $(H_n)_{n\in\mathbb{N}}$ in $\mathbb{H}^2_{\mathbb{R}}$ (with the upper halfspace model) such that there exists a geodesic line starting at ∞ and meeting the open horoball bounded by H_0 which avoids the open horoballs bounded by $H_n(t)$ for every n in $\mathbb{N}$ if and only if $t \geq t_1(1) = -\log(4\sqrt{2}-5)$. To construct such a family, we start with a horosphere H_0 (not centered at ∞). We define H_1, H_2 to be the boundaries of the two maximal horoballs whose interiors are disjoint from the open horoball bounded by H_0, and which are contained in the shadow of H_0 seen from ∞. An easy computation shows that $\mathcal{O}\,H_1$ and $\mathcal{O}\,H_2$ are exactly the two components of $\overline{\mathcal{O}\,H - \mathcal{O}\,H(t_1(1))}$. And then one iterates the construction (see picture below).

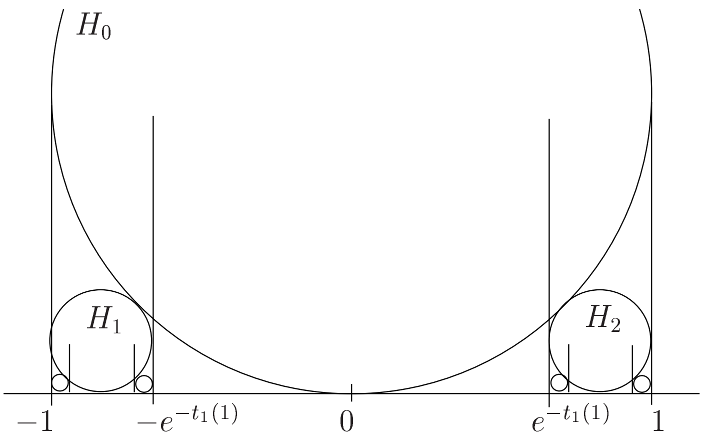

FIGURE 5. An extremal packing for Theorem 5.1.

(c) Let $(HB_r)_{r\in\mathbb{Q}\cup\{\infty\}}$ be the Apollonian packing in the upper half plane model of $\mathbb{H}^2_\mathbb{R}$ as in Remark 3.4(ii). Let $\xi \in \mathbb{R}$. If $[\xi, \infty]$ meets $HB_{p/q}(t)$, then

$$\left|\xi - \frac{p}{q}\right| < \frac{e^{-t}}{2q^2}. \tag{18}$$

It is well known (see [Ford, HW, Khi]) that there are irrational numbers ξ such that Equation (18) has only finitely many solutions $\frac{p}{q}$ if $t > -\log\frac{2}{\sqrt{5}} \approx 0.11$. Furthermore, if $\xi = \frac{1+\sqrt{5}}{2}$ is the golden ratio, then by [RWT], Equation (18) has no solution at all if $t > -\log(3-\sqrt{5}) \approx 0.269$. Theorem 5.1 implies that there exists at least one irrational number ξ such that (18) has no solution for $t \geq t_1(1) = -\log(4\sqrt{2}-5) \approx 0.42$. This is not much more than what is actually gotten for the golden ratio.

(d) Clearly, the two possible choices of the maximal interval, in the analog of the first step of the proof of Theorem 3.3 within the proof of Theorem 5.1, give two different geodesics which avoid the scaled horospheres such that the endpoints of these geodesics are both in the shadow of the first horosphere used in the construction.

6. Unclouding the sky of real hyperbolic spaces

Let $\mathbb{H}^n_\mathbb{R}$ be the real hyperbolic n-space (with constant curvature -1), for $n \geq 2$. In this section, we will use the symmetries of $\partial\mathbb{H}^n_\mathbb{R}$ to show that the condition $t \geq t_1(1) = -\log(4\sqrt{2}-5)$, which is our best estimate for the real hyperbolic plane (see Theorem 5.1 and Remark 5.5 (a)), also works in the higher-dimensional situation.

Theorem 6.1. *Let $\xi_\sharp$ be a point in $\partial\mathbb{H}^n_\mathbb{R}$, and $H_\sharp$ be a horosphere in $\mathbb{H}^n_\mathbb{R}$ centered at $\xi_\sharp$. For every sequence $(H_n)_{n\in\mathbb{N}}$ of horospheres in $\mathbb{H}^n_\mathbb{R}$ bounding open horoballs that are pairwise disjoint and disjoint from the open horoball bounded by $H_\sharp$, if $t \geq t_1(1)$, then there exists at least two geodesic lines in $\mathbb{H}^n_\mathbb{R}$, starting at $\xi_\sharp$, which avoid $H_n(t)$ for every n in $\mathbb{N}$.*

Proof. By homogeneity, we may use the upper halfspace model for $\mathbb{H}^n_\mathbb{R}$, with $\xi_\sharp$ the point at infinity ∞ and $H_\sharp$ the horizontal Euclidean hyperplane at Euclidean height 1. In particular, as we have already seen, the space $\partial\mathbb{H}^n_\mathbb{R} - \{\infty\}$ is the horizontal coordinate hyperplane $\mathbb{R}^{n-1}$, the shadows of horoballs (seen from ∞) are Euclidean balls in $\mathbb{R}^{n-1}$, and the Hamenstädt distance is the Euclidean distance.

Proposition 6.2. *Let H and H' be horospheres in $\mathbb{H}^n_{\mathbb{R}}$ bounding open horoballs that are disjoint and disjoint from the open horoball bounded by $H_\sharp$, with H' lower than H (i.e. $r_i(H') \leq r_i(H)$). Let $t \geq t_1(1)$. For every maximal Euclidean ball K of $\overline{\mathcal{O}\,H - \mathcal{O}\,H(t)}$, the following dichotomy holds :*

- *either K does not meet $\mathcal{O}\,H'(t)$*
- *or there exists a maximal Euclidean ball K' in $\overline{\mathcal{O}\,H' - \mathcal{O}\,H'(t)}$, which is contained in K.*

Proof. Assume that K meets $\mathcal{O}\,H'(t)$. Let x be the common center of the Euclidean balls $\mathcal{O}\,H(t)$, let x' be the common center of the Euclidean balls $\mathcal{O}\,H'(t)$, let y be the center of K (note that $y \neq x$), let D be the Euclidean line passing through x and y, and let D' be the Euclidean line through x' and y (take $D' = D$ if $x' = y$).

Case 1: Assume first that $D = D'$ (see the picture below).

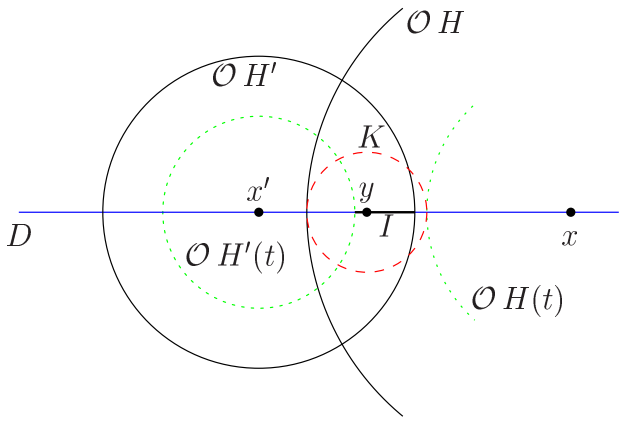

FIGURE 6

Let A be the hyperbolic plane in $\mathbb{H}^n_{\mathbb{R}}$ with boundary $D \cup \{\infty\}$. Note that $H \cap A$ and $H' \cap A$ are horoballs in H, and that $\mathcal{O}\,H \cap D$ is the shadow of $H \cap A$ seen from ∞ in A. Note that, by maximality, $K \cap D$ is a connected component of $\left(\overline{\mathcal{O}\,H - \mathcal{O}\,H(t)}\right) \cap D$. Also note that $K \cap D$ meets $\mathcal{O}\,H'(t) \cap D$. Hence, by Proposition 5.2 applied to $M = A$, there exists a connected component I of $\left(\overline{\mathcal{O}\,H' - \mathcal{O}\,H'(t)}\right) \cap D$ which is contained in $K \cap D$. Consider the ball K' in $\partial \mathbb{H}^n_{\mathbb{R}} - \{\infty\}$ having I as one diameter. Then K' is contained in $\overline{\mathcal{O}\,H' - \mathcal{O}\,H'(t)}$ and is maximal there, since D goes through the center of the balls $\mathcal{O}\,H'(s)$. Furthermore K' is contained in K, since the ball with diameter a segment contained in K of a line passing through the center of K is contained in K.

Case 2: Now, assume that D' and D are different (see the picture below).

Let P be the orthogonal subspace of $\mathbb{R}^{n-1}$ through y to the plane containing D and D'. Let f be the isometry of X fixing ∞, preserving the horosphere $H_\sharp$, and inducing on $\partial \mathbb{H}^n_{\mathbb{R}} - \{\infty\} = \mathbb{R}^{n-1}$ the Euclidean rotation fixing P and sending D' to D such that y is between x and x'. Then the map f preserves K, and sends H' to a horoball H'' which is still disjoint from (and lower than) H. By the case $D = D'$, let K'' be a maximal Euclidean ball in $\overline{\mathcal{O}\,H'' - \mathcal{O}\,H''(t)}$, which is also contained in K. Then, since f is an isometry of X, the Euclidean ball $f^{-1}(K'')$ is a maximal Euclidean ball in $\overline{\mathcal{O}\,H' - \mathcal{O}\,H'(t)}$, which is also contained in $f^{-1}(K) = K$. $\square$

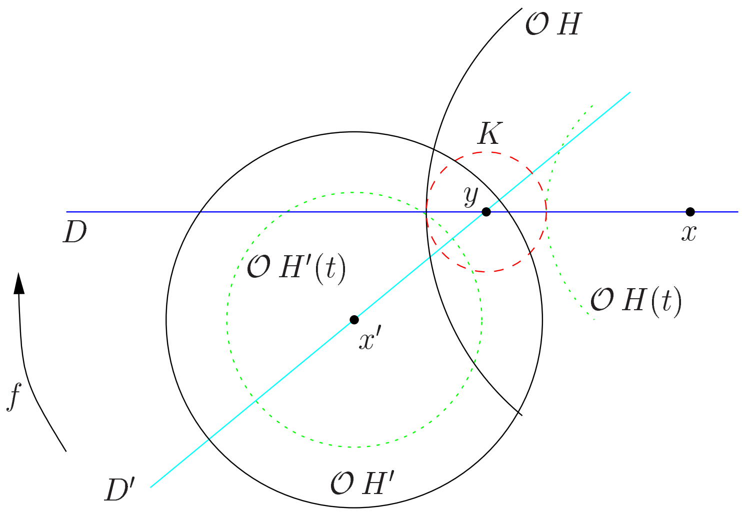

FIGURE 7

Now, Theorem 6.1 follows from the above Proposition 6.2 exactly in the same way as Theorem 5.1 followed from Proposition 5.2. □

7. Bi-infinite geodesics, geodesic rays, and closed geodesics in finite volume cusped manifolds

In this concluding section, we give some applications of the results obtained in the previous sections. We start by studying the problem of finding geodesic lines which avoid the scaled family of horoballs in both directions, as well as geodesic rays starting from given points.

Theorem 7.1. *Let X be a proper geodesic CAT(-1) space, and $t_{\min} > 0$. Assume that one of the following conditions holds:*

1. *X is the real hyperbolic n-space with $n \geq 2$, and $t_{\min} = t_1(1) = -\log(4\sqrt{2} - 5)$;*
2. *X is a complete simply connected Riemannian manifold of dimension 2, with pinched curvature $-a^2 \leq K \leq -1$, and $t_{\min} = t_1(a)$;*
3. *X is a locally finite metric tree, without vertices of degree 1 or 2, with edge lengths at most $\ell_{\max}$, and $t_{\min} = \ell_{\max}$;*
4. *X is a symmetric space, and $t_{\min} = t_0$ (see Theorem 4.5).*

Let $(H_n)_{n\in\mathbb{N}}$ be a sequence of horospheres in X bounding pairwise disjoint open horoballs $(HB_n)_{n\in\mathbb{N}}$. If $t > t_{\min}$ in case (3) and $t > \log(2 + \sqrt{5}) + t_{\min}$ otherwise, for every x in $X - \bigcup_n \overset{\circ}{HB}_n$, then there exists at least one geodesic ray starting at x which avoids $H_n(t)$ for every n in $\mathbb{N}$.

Furthermore, if $t > t_{\min}$ in Case (3) and $t > t_{\min} + \log(1 + \sqrt{2})$ otherwise, then there exists at least one geodesic line which completely avoids $H_n(t)$ for every n in $\mathbb{N}$.

The constants in this theorem are probably not optimal (except for the case (3), where they are optimal). The first assertion of this theorem implies the first assertion in Theorem 1.1 in the introduction. The second assertion of Theorem 1.1 follows from Theorem 6.1, Theorem 5.1, Remark 4.7 and Theorem 4.5.

Proof. Let $(H_n)_{n\in\mathbb{N}}$ be as in the statement, and define ξ_n to be the point at infinity of H_n.

Let us prove the first assertion. Let x, t be as in the statement. Let H_{n_0} be one of the closest horoballs to x (which exists since the family of horoballs is locally finite). Let H_{n_1} be the first horosphere met after x by the geodesic line starting from ξ_0 and passing through x, with y the intersection point. If H_{n_1} does not exists, then we are done. We may assume that $n_0 = 0$ and $n_1 = 1$.

Lemma 7.2. *For every point at infinity $\xi_\sharp$ and every horosphere H not centered at $\xi_\sharp$ in a CAT(-1) space, define $\mathcal{C}^-_{\xi_\sharp} H$ to be the union of the geodesic rays meeting H exactly in their starting point, and converging to $\xi_\sharp$. For every such ray ρ, the subset $\mathcal{C}^-_{\xi_\sharp} H$ is contained in the* $\log(2+\sqrt{5})$*-neighborhood of ρ.*

Proof. By comparison, this follows from an easy computation in the upper halfspace model of the real hyperbolic plane. Note that in a CAT$(-a^2)$ space (resp. a tree), we may replace the constant $\log(2+\sqrt{5})$ by $\frac{1}{a}\log(2+\sqrt{5})$ (resp. 0). $\square$

Define $\alpha = 0$ in the case (3) and $\alpha = \log(2+\sqrt{5})$ otherwise. By construction, no horosphere H_n besides H_0 meets the geodesic ray from y to ξ_0. Hence by the lemma, no horosphere $H_n(s)$ with $n \geq 1$ and $s > \alpha$ meets $\mathcal{C}^-_{\xi_0} H_1$. As $t - \alpha > t_{\min}$, we can apply Theorem 6.1, Theorem 5.1, Remark 4.7 and Theorem 4.5 respectively, while taking H_1 as the starting horosphere in our inductive constructions. Hence there exists at least one geodesic line ℓ starting from ξ_0 which avoids $H_n(t-\alpha)$ for $n \geq 1$, and whose other endpoint belongs to $\mathcal{O}_{\xi_0} H_1$. Note that in Case (4), we may indeed take $\xi_\sharp = \xi_0$ to apply Theorem 4.5, as the isometry group of X then acts transitively on ∂X.

By the above lemma, the point x lies at distance at most α from the geodesic line ℓ. Hence by convexity, the geodesic ray ρ_0 starting from x and converging to the point at infinity of ℓ different from ξ_0 is contained in the α-neighborhood of ℓ. Hence ρ_0 avoids $H_n(t)$ for $n \geq 1$. The result follows, because, by construction, ρ_0 also avoids H_0.

Now, let us prove the second assertion. Let t be as in the statement. By Theorem 6.1, Theorem 5.1, Remark 4.7 and Remark 4.6 respectively, there exist at least two geodesic lines starting from ξ_0 which, for $n \geq 1$, avoid $H_n(t)$ in Case (3) and $H_n(t - \log(1+\sqrt{2}))$ otherwise, and are contained in $\mathcal{O}_{\xi_0} H_{n_0}$ for some n_0 by construction. Note that in Case (4), we may indeed take $\xi_\sharp = \xi_0$ to apply Remark 4.6, as the isometry group of X then acts transitively on ∂X, and as the spheres in $\partial X - \{\xi_0\}$ have antipodal points (see Remark 4.2).

Let η, η' be the endpoints of these two geodesic lines. By Lemma 2.8, the geodesic line between η and η' does not intersect the open horoball bounded by H_0, and in particular avoids $H_0(t)$.

In case (3), the geodesic line between η and η' is contained in the union of the geodesic lines between ξ_0 and η and between ξ_0 and η', which proves the result.

For the other cases, recall that in a CAT(-1) space, given three points at infinity, and three geodesic lines between the pairs of them, each geodesic line is contained in the $\log(1+\sqrt{2})$-neighborhood of the union of the two others. This follows, by comparison with an ideal triangle in the hyperbolic plane, from an easy computation in the upper halfplane model (see also [GH, Proposition 2.21], where the worse constant $\log 3$ is given). Hence the geodesic line between η and η' also avoids $H_n(t)$ for $n \geq 1$. $\square$

Remark 7.3. Let X be a proper geodesic CAT(-1) space, $\xi_\sharp$ a point at infinity, $H_\sharp$ a horosphere in X centered at $\xi_\sharp$, and $t_{\min} > 0$. With d_ℓ the length distance associated to the Hamenstädt distance $d_{\xi_\sharp, H_\sharp}$, assume that there exists $C \geq 1$ such that $d_{\xi_\sharp, H_\sharp} \leq d_\ell \leq C d_{\xi_\sharp, H_\sharp}$, the metric space $(\partial X - \{\xi_\sharp\}, d_\ell)$ has extendable spheres with modulus $\delta(\cdot)$, and the spheres in $(\partial X - \{\xi_\sharp\}, d_\ell)$ have antipodal points. Then for every sequence $(H_n)_{n\in\mathbb{N}}$ of horospheres bounding open horoballs that are pairwise disjoint and disjoint from the open horosphere bounded by $H_\sharp$, if $t > t_0(C,\delta) + \log(1+\sqrt{2})$, then there exists at least one geodesic line which completely avoids $H_\sharp$ and $H_n(t)$ for every n in $\mathbb{N}$.

Proof. The proof is the same as the previous one, we just added the hypothesis necessary to handle the more general case. □

Let us now give an application of our results to equivariant families of horospheres. Let V be a complete noncompact nonelementary geometrically finite Riemannian manifold with sectional curvature $K \leq -1$. Let β_e be the Busemann function on V with respect to a cusp e of V, normalised to be 0 on the boundary of the maximal open Margulis neighbourhood N_e of e in the convex core of V. Recall that (see for instance [HP2]) if $\rho : [0,+\infty) \to V$ is a geodesic ray contained in the closure of N_e with $\rho(0)$ not in N_e, then $\beta_e(x) = \lim_{t\to\infty}(t - d(\rho(t),x))$. The height with respect to e of any compact subset A in V can be defined as

$$\mathrm{ht}_e(A) = \max_{x\in A} \beta_e(x).$$

We denote by $h_e(V)$ the infimum of the heights of the closed geodesics on V with respect to e. The results of the previous sections give an upper bound on $h_e(V)$.

Theorem 7.4. *Let V be a complete noncompact nonelementary geometrically finite Riemannian manifold (for instance a manifold with finite volume), e be a cusp of V, and $t_{\min} > 0$. Assume that one of the following conditions holds:*

1. *V has constant sectional curvature -1, and $t_{\min} = t_1(1) = -\log(4\sqrt{2}-5)$;*
2. *V is two-dimensional with pinched curvature $-a^2 \leq K \leq -1$, and $t_{\min} = t_1(a)$;*
3. *the universal cover of V is a CAT(-1) space and satisfies the conditions (i) and (ii) of Section 4 with $\xi_\sharp$ a lift of the cusp e, and $t_{\min} = t_0$ (see Theorem 4.5).*

Then $h_e(V) \leq t_{\min}$.

This theorem implies Corollary 1.2 in the introduction. Note that if a negatively curved homogeneous Riemannian manifold has a finite volume quotient, then it is symmetric [Hei2]. Hence, in Corollary 1.2, we only gave the result for locally symmetric spaces. For related results on bounded geodesics in Riemannian manifolds, see [Sch] and the references therein.

Proof. The lift, to the universal cover $\widetilde{V}$ of V, of the union of the maximal open Margulis neighbourhoods of ends of V is a disjoint collection of open horoballs in $\widetilde{V}$. Thus, the theorems 6.1, 5.1, 4.5 respectively can be applied to the corresponding family of horospheres. By [HP3, Theorem 3.4], $h_e(V)$ is equal to the lower bound of all h in $\mathbb{R}$ such that there exists a geodesic line starting from e, which does not converge into a cusp of V, and eventually avoids $\beta_e^{-1}([h,+\infty))$. This implies the result. □

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Department of Mathematics and Statistics
P.O. Box 35
40014 University of Jyväskylä, FINLAND.
e-mail: parkkone@maths.jyu.fi

Département de Mathématique et Applications, UMR 8553 CNRS
Ecole Normale Supérieure
45 rue d'Ulm
75230 PARIS Cedex 05, FRANCE
e-mail: Frederic.Paulin@ens.fr