Divergent geodesics, ambiguous closed geodesics and the binary additive divisor problem

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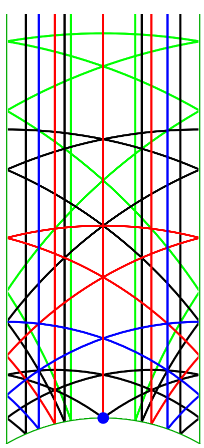
Abstract

We give an asymptotic formula as $t \to +\infty$ for the number of common perpendiculars of length at most t between two divergent geodesics or a divergent geodesic and a compact locally convex subset in negatively curved locally symmetric spaces with exponentially mixing geodesic flow, presenting a surprising non-purely exponential growth. We apply this result to count ambiguous geodesics in the modular orbifold recovering results of Sarnak, and to confirm and extend a conjecture of Motohashi on the binary additive divisor problem in imaginary quadratic number fields. [1]

1 Introduction

Let M be a noncompact finite volume complete connected negatively curved locally symmetric good orbifold.



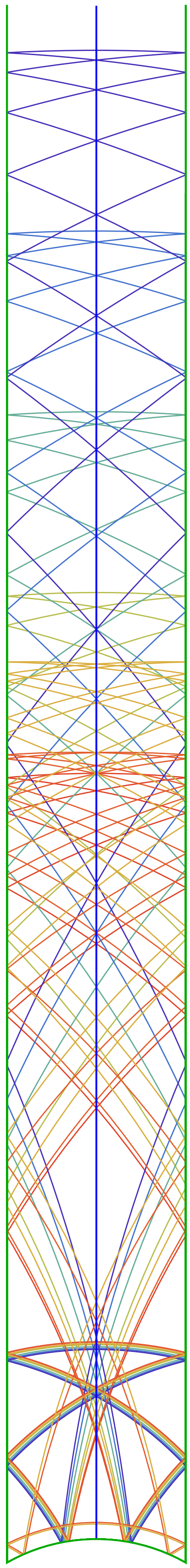
A locally geodesic line $\ell : \mathbb{R} \to M$ that is a proper mapping is a *divergent geodesic* in M. The distribution of divergent geodesics has been very actively studied in recent years. We refer for instance to [DaS1, PPaS] for equidistribution results of divergent orbits, in the space of lattices of $\mathbb{R}^2$ for the first one, in finite volume complete connected negatively curved good Riemannian orbifolds for the second one. See for instance [ShZ, DaS2, SoT, DaPS] for higher rank results.

With $\mathbb{H}^2_{\mathbb{R}}$ the upper halfspace model of the real hyperbolic plane, the picture on the left shows some divergent geodesics in the modular orbifold $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2_{\mathbb{R}}$ (lifted to the usual fundamental domain of $\mathrm{PSL}_2(\mathbb{Z})$ with its boundary identifications). See Section 4 for explanations.

Let D^- and D^+ be two properly immersed closed locally convex subsets of M. For instance, D^- and D^+ can be the images of two divergent geodesics in M. A *common perpendicular*[2] from D^- to D^+ is a locally geodesic path in M starting perpendicularly from D^- and arriving perpendicularly to D^+. In this paper, we prove an effective asymptotic counting result on the set of the common perpendiculars between the images of two divergent geodesics in M.

[1]**Keywords:** Common perpendiculars, divergent geodesics, negative curvature, symmetric spaces, counting, ambiguous classes, number of divisors, binary additive divisor problem, imaginary quadratic field. **AMS codes:** 53C22, 11N37, 37D40, 53C35, 32M15, 11N45, 11R04, 57K32.

[2]See [PP5, §2.3] for definitions when the boundary of D^- or D^+ is not smooth.



For instance, the image ℓ in the modular orbifold $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2_\mathbb{R}$ of the imaginary axis in $\mathbb{H}^2_\mathbb{R}$ is a divergent geodesic in $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2_\mathbb{R}$. The picture on the left shows several common perpendiculars (lifted to the usual fundamental domain of $\mathrm{PSL}_2(\mathbb{Z})$ with its boundary identifications) between ℓ and itself (with an extra symmetry, see Remark 13 for explanations).

For all $s > 0$, we denote by $\mathscr{N}_{D^-,\,D^+}(s)$ the cardinality of the set of common perpendiculars from D^- to D^+ with length at most s, considered with multiplicities (see Equations (7) and (8) for precisions). The counting function $\mathscr{N}_{D^-,\,D^+}$ has been studied for particular triples (M, D^-, D^+) at least since the 1940's for example in [Del, Hub, Her, Mar, EsMc, KO, OS1] and [Kim]. See [PP4] for a more detailed review. As we shall explain in Section 4, the general purely exponential asymptotic behaviour of $\mathscr{N}_{D^-,\,D^+}(s)$ as $s \to +\infty$ proven in [PP5, Thm. 1] does not apply when D^- or D^+ is a divergent geodesic. In this paper, we prove that when D^- or D^+ is a divergent geodesic, the number of common perpendiculars actually no longer has a purely exponential growth in terms of an upper bound on their lengths.

Theorem 1. *Let M be a noncompact finite volume complete connected real hyperbolic good orbifold of dimension $n \geqslant 2$. Let D^- and D^+ be the images of two divergent geodesics in M. Then there exists a constant $C_{D^-,\,D^+} > 0$ such that as $s \to +\infty$, we have*

$$\mathscr{N}_{D^-,D^+}(s) = \frac{C_{D^-,\,D^+}}{\mathrm{Vol}(M)}\, s^2\, e^{(n-1)\,s} + \mathrm{O}(s\, e^{(n-1)s})\,.$$

The constant $C_{D^-,\,D^+}$ is made explicit in Theorem 6. See Theorem 5 for a version of this theorem when D^- is instead assumed for instance to be compact, already providing a non-purely exponential growth. The size of the error term in Theorem 1 is optimal, as explained below. See Theorem 9 and its following comment for the version of Theorems 5 and 6 valid for the other locally symmetric spaces. In Sections 3 and 5, we give fine results on the lengths of common perpendiculars that are ending high in Margulis neighborhoods of the cusps of M. These results will be crucial for the proofs of our geometric main results, Theorems 5, 6 and 9, that are given in Sections 4 and 5, and that introduce a new counting disintegration process, that will explain the non-purely exponential behavior.

Earlier geometric counting results with growth that is not purely exponential in negatively curved spaces include the case of closed geodesics with an upper bound on their length starting with Bowen and Margulis, see for instance [PPo, EsMi] and [PPS, Coro. 9.15], as well as the results of [Vid] and [PTV] when the manifold M has infinite Bowen-Margulis measure or the covering group is of convergence type. See also [Sar].

As a first arithmetic application of our geometric counting results, we recover in Section 6 counting results of Sarnak [Sar] on ambiguous and reciprocal ambiguous conjugacy classes of primitive hyperbolic elements in $\mathrm{PSL}_2(\mathbb{Z})$, related to the ambiguous integral binary quadratic forms of Gauss.

In the very special case when $M = \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2_{\mathbb{R}}$ is the modular orbifold, we prove in Section 7 that Theorem 1 follows from the asymptotics on the binary additive divisor problem (see for instance [Ing, Est, HeB, Mot1]). These arithmetic results produce an error term of the form $b_1\, se^s + \mathrm{O}(e^s)$ with $b_1 \neq 0$, confirming that the general error term obtained in Theorem 1 is of the correct order.

Let K be an imaginary quadratic number field, with discriminant D_K, ring of integers $\mathscr{O}_K$ and Dedekind zeta function ζ_K. We denote by $\mathbf{d}_K : \mathscr{O}_K\smallsetminus\{0\} \to \mathbb{N}$ the (naive) *number of divisors function* of $\mathscr{O}_K$, with $\mathbf{d}_K(x) = \mathrm{Card}\{d \in \mathscr{O}_K\smallsetminus\{0\} : d \mid x\}$ for every $x \in \mathscr{O}_K\smallsetminus\{0\}$. In Section 8, we use Theorem 6 to prove the following new arithmetic application.

Theorem 2. *As $X \to +\infty$, we have*

$$\sum_{x\in\mathscr{O}_K\smallsetminus\{0,-1\}\,:\,|x|^2\leqslant X} \mathbf{d}_K(x)\,\mathbf{d}_K(x+1) = \frac{8\,\pi^3}{|D_K|^{3/2}\,\zeta_K(2)}\;X(\ln X)^2 + \mathrm{O}(X\ln X)\,.$$

In Remark 20 (2), we show that this result confirms a particular case of a conjecture of Motohashi [Mot2, p. 277] when $K = \mathbb{Q}(i)$, and gives a generalization to any imaginary quadratic number field. It might be possible to improve our error term given by Theorem 2 using arithmetic methods. See also [SaV] that solves the special case $K = \mathbb{Q}(i)$ of Theorem 2 with a less explicit constant.

The proof of Theorem 2 given in Section 8 uses arithmetic hyperbolic 3-manifolds. Let $\mathbb{H}^3_{\mathbb{R}}$ be the upper halfspace model of the 3-dimension real hyperbolic space, let M be the Bianchi orbifold $\mathrm{PSL}_2(\mathscr{O}_K)\backslash\mathbb{H}^3_{\mathbb{R}}$, and let ℓ be the image in M of the vertical axis of $\mathbb{H}^3_{\mathbb{R}}$. Then ℓ is a divergent geodesic, and the key idea is to link (this is not immediate) the counting of the binary additive divisor problem with the counting of common perpendiculars between ℓ and itsef. Then we apply the asymptotic of Theorem 1.

We will apply Theorems 5 and 9 (1) in [PP8] to count pairs of Farey neighbours in the rational numbers, in quadratic imaginary number fields and in the Heisenberg group.

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2 Geometric and measure-theoretic background

Let $\widetilde{M}$ be a negatively curved Riemannian symmetric space with dimension at least 2 and sectional curvature normalized to have maximum -1. Then $\widetilde{M}$ is isometric to the hyperbolic space $\mathbb{H}^n_{\mathbb{K}}$ with dimension n over $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ (with $n = 2$ in this last case), with the above normalization of its Riemannian metric. Let Γ be a discrete group of isometries of $\widetilde{M}$, and let $M = \Gamma\backslash\widetilde{M}$ be the quotient (complete, connected) locally symmetric good orbifold. We assume throughout this paper that M is noncompact and has finite volume. We refer for instance to [BPP, §2.1] for background on $\mathrm{CAT}(-1)$ spaces.

Let $\partial_\infty\widetilde{M}$ be the boundary at infinity of $\widetilde{M}$, let $T^1\widetilde{M}$ be the unit tangent bundle of $\widetilde{M}$, and let T^1M be the unit tangent bundle of M, which identifies as an orbifold with $\Gamma\backslash T^1\widetilde{M}$. We denote the footpoint maps by $\widetilde{p}_\bullet : T^1\widetilde{M} \to \widetilde{M}$ and $p_\bullet : T^1M \to M$, so that the following diagram, whose vertical maps are the canonical projections modulo Γ, is commutative

$$\begin{array}{ccc} T^1\widetilde{M} & \xrightarrow{\tilde{p}_\bullet} & \widetilde{M} \\ {\scriptstyle\tilde{p}}\downarrow & & \downarrow{\scriptstyle p} \\ T^1M & \xrightarrow{p_\bullet} & M\,. \end{array} \qquad (1)$$

Let δ be the *critical exponent* of Γ, which equals the topological entropy of the geodesic flow on T^1M, see for instance [PPS, Theo. 6.1]. By for instance [Cor, Theo. 4.4 (i)], if $\widetilde{M} = \mathbb{H}^n_{\mathbb{K}}$ is the hyperbolic n-space over $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ (with $n = 2$ in this last case), then

$$\delta = (\dim_{\mathbb{R}} \mathbb{K})(n+1) - 2 \qquad (2)$$

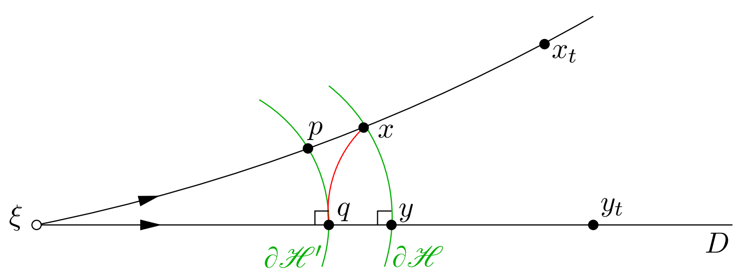
Let $\mathscr{H}$ be a (closed) horoball in $\widetilde{M}$, and let ξ be its point at infinity. For every $x \in \partial\mathscr{H}$, let $t \mapsto x_t$ be the geodesic line in $\widetilde{M}$ starting from the point at infinity ξ such that $x_0 = x$, and let $x_{+\infty} \in \partial_\infty\widetilde{M} \smallsetminus \{\xi\}$ be its terminal point at infinity. As defined for instance in [HP1, Appendix] on $\partial_\infty\widetilde{M} \smallsetminus \{\xi\}$ and using the homeomorphism $x \mapsto x_{+\infty}$ from $\partial\mathscr{H}$ to $\partial_\infty\widetilde{M} \smallsetminus \{\xi\}$, the *Hamenstädt distance* $d_{\mathscr{H}}$ on $\partial\mathscr{H}$ is defined by

$$\forall\, x, y \in \partial\mathscr{H}, \quad d_{\mathscr{H}}(x, y) = \lim_{t \to +\infty} e^{\frac{1}{2}\, d(x_t, y_t) - t}\,. \qquad (3)$$

As introduced in [HP2, §2.1], the *cuspidal distance* $d'_{\mathscr{H}}$ on $\partial\mathscr{H}$ is defined, for all $x, y \in \partial\mathscr{H}$ by setting $d'_{\mathscr{H}}(x, y)$ to be the greatest lower bound of all $r > 0$ such that the horosphere centered at $y_{+\infty}$, at signed distance $-\ln(2\ r)$ from $\partial\mathscr{H}$ along the geodesic line $t \mapsto y_t$, meets the geodesic line $t \mapsto x_t$. When $\widetilde{M} = \mathbb{H}^n_{\mathbb{R}}$, we have $d'_{\mathscr{H}} = d_{\mathscr{H}}$ by loc. cit.. The cuspidal distance is indeed a distance by loc. cit. since $\widetilde{M}$ is a symmetric space, and it is equivalent to the Hamenstädt distance by [HP2, Rem. 2.6].

For every isometry γ of $\widetilde{M}$, for all $x, y \in \partial\mathscr{H}$, we have $d_{\gamma\mathscr{H}}(\gamma x, \gamma y) = d_{\mathscr{H}}(x, y)$ and similarly $d'_{\gamma\mathscr{H}}(\gamma x, \gamma y) = d'_{\mathscr{H}}(x, y)$.

Lemma 3. *For all $x, y \in \partial\mathscr{H}$, if D is the image of the map $t \mapsto y_t$, then $d_{\mathscr{H}}(x, y) \leqslant e^{d(x,D)}$.*



Proof. Let ξ be the point at infinity of $\mathscr{H}$. Let q be the closest point to x on D and let $\mathscr{H}'$ be the horoball centered at ξ with $q \in \partial\mathscr{H}'$. Let p be the intersection point with the image of the geodesic line $t \mapsto x_t$ of the horosphere $\partial\mathscr{H}'$. Recall that two horospheres centered at the same point at infinity are equidistant. Since the points p and q are the closest points on $\partial\mathscr{H}'$ to x and y respectively, we have $d(y, q) = d(x, p) \leqslant d(x, q) = d(x, D)$. By the triangle inequality, for every $t \geqslant 0$, we have

$$d(x_t, y_t) \leqslant d(x_t, x) + d(x, q) + d(q, y) + d(y, y_t) \leqslant 2\, d(x, D) + 2\, t\,.$$

The result then follows by the definition (3) of the Hamenstädt distance. □

We denote by $\|\mathfrak{m}\|$ the total mass of any finite measure $\mathfrak{m}$. Since $\widetilde{M}$ is a negatively curved symmetric space and M has finite volume, there exists up to a positive scalar a unique (measurable) family $(\mu_x)_{x\in\widetilde{M}}$ of Patterson-Sullivan measures on $\partial_\infty\widetilde{M}$ for Γ, with full support, which is actually equivariant under the group of all isometries of $\widetilde{M}$ (see for instance [BPP, §4, §7] for definitions). When $\widetilde{M} = \mathbb{H}^n_\mathbb{R}$, we normalize these measures so that $\|\mu_x\| = \operatorname{Vol}(\mathbb{S}^{n-1})$ for every $x \in \widetilde{M}$. When $\widetilde{M} = \mathbb{H}^n_\mathbb{C}$ (respectively $\widetilde{M} = \mathbb{H}^n_\mathbb{H}$), we normalize these measures as in [PP6, §4] just before Lemma 12 (respectively as in [PP7, §7] just before Lemma 7·2). The details of these normalizations are not needed: We will directly use the computations of the above references that use them.

We denote by $m_{\rm BM}$ the *Bowen-Margulis measure* of M associated with this choice of Patterson-Sullivan measures. Under our assumption on M, it is finite nonzero and mixing, and it coincides, up to a positive scalar, with the Liouville measure on T^1M as well as, when normalized to be a probability measure, with the measure of maximal entropy for the geodesic flow on T^1M (see for instance [PPS, §6, §7]). Note that by the work of Li-Pan [LP] when $\widetilde{M} = \mathbb{H}^n_\mathbb{R}$ and by the Margulis arithmeticity result with the works of Kleinbock-Margulis and Clozel when $\widetilde{M} = \mathbb{H}^n_\mathbb{H}, \mathbb{H}^n_\mathbb{O}$ (see for instance [BPP, page 182], the only case when the geodesic flow of M is not yet known to be exponentially mixing is when $\widetilde{M} = \mathbb{H}^n_\mathbb{C}$.

By definition, a *properly immersed closed locally convex subset* D of M is the image by the orbifold covering map $\widetilde{M} \to M$ of a proper nonempty closed convex subset $\widetilde{D}$, thereafter called *a lift* of D in $\widetilde{M}$, with stabilizer $\Gamma_{\widetilde{D}}$ in Γ such that the family $(\gamma\widetilde{D})_{\gamma\in\Gamma/\Gamma_{\widetilde{D}}}$ of subsets of $\widetilde{M}$ is locally finite. We denote by $m(D)$ the order of the pointwise stabiliser of $\widetilde{D}$. We denote by ∂^1_+D and ∂^1_-D the *outer and inner unit normal bundles* of ∂D respectively. See [PP3], generalising [OS1, OS2], for definitions, in particular when ∂D is not smooth, or [BPP, §2.4].

Let us now recall (see [PP3, Eq. (11)]) the formula for the outer/inner skinning measures $\sigma^\pm_D$ of D associated with the above choice of Patterson-Sullivan measures $(\mu_x)_{x\in\widetilde{M}}$. Let $p_{\widetilde{D}}$ be the closest point projection from $(\widetilde{M} \cup \partial_\infty\widetilde{M})\smallsetminus\partial_\infty\widetilde{D}$ to $\widetilde{D}$. Let $\widetilde{\sigma}^\pm_{\widetilde{D}}$ be the measure on $T^1\widetilde{M}$ (with support contained in $\partial^1_\pm\widetilde{D}$) defined as follows: For every unit normal vector $w \in \partial^1_\pm\widetilde{D}$, with $w_\pm$ the point at $\pm\infty$ of the geodesic line it defines, we have

$$d\widetilde{\sigma}^\pm_{\widetilde{D}}(w) = d\mu_{p_{\widetilde{D}}(w_\pm)}(w_\pm)\ . \tag{4}$$

Then $\sigma^\pm_D$ is the measure induced by $\widetilde{\sigma}^\pm_D$ on T^1M, with support contained in $\partial^1_\pm D$, by the locally finite Γ-invariant measure $\sum_{\gamma\in\Gamma/\Gamma_{\widetilde{D}}} \gamma_*\widetilde{\sigma}^\pm_{\widetilde{D}}$ on $T^1\widetilde{M}$, using the orbifold covering $\widetilde{p} : T^1\widetilde{M} \to T^1M = \Gamma\backslash T^1\widetilde{M}$, see for instance [PPS, §2.6].

For every horoball $\mathscr{H}$ in $\widetilde{M}$, let $\widetilde{\sigma}^-_{\mathscr{H}}$ be the inner skinning measure of $\mathscr{H}$ (associated with the above choice of Patterson-Sullivan measures). Since $\widetilde{M}$ is a negatively curved symmetric space, the group of isometries of $\widetilde{M}$ acts transitively on the set of horoballs of $\widetilde{M}$. Furthermore, the group of isometries of $\widetilde{M}$ preserving $\mathscr{H}$ acts transitively on $\partial\mathscr{H}$ and leaves $\widetilde{\sigma}^-_{\mathscr{H}}$ invariant since M has finite volume. Let $B_{d'_{\mathscr{H}}}(x, r)$ be the ball of radius $r > 0$ and center $x \in \partial\mathscr{H}$ for the cuspidal distance $d'_{\mathscr{H}}$ on $\partial\mathscr{H}$. Let us define

$$\Xi_{\widetilde{M}} = \widetilde{\sigma}^-_{\mathscr{H}}\big(\widetilde{p}_\bullet^{\ -1}(B_{d'_{\mathscr{H}}}(x, 1))\big)\ , \tag{5}$$

which depends neither on the horoball $\mathscr{H}$ in $\widetilde{M}$ nor on the point $x \in \partial \mathscr{H}$. A computation of this constant will be given in Equation (14) and in Lemma 10.

3 A lemma in real hyperbolic geometry

Let

$$\mathbb{H}^n_{\mathbb{R}} = \Big(\{(x,y) \in \mathbb{R}^{n-1} \times \mathbb{R} : y > 0\}, \quad ds^2_{\mathbb{H}^n_{\mathbb{R}}} = \frac{1}{y^2}(dx^2 + dy^2) \Big)$$

be the upper halfspace model of the real hyperbolic space of dimension n (with constant sectional curvature -1). Recall that $\partial_\infty \mathbb{H}^n_{\mathbb{R}} = (\mathbb{R}^{n-1} \times \{0\}) \cup \{\infty\}$, that

$$\mathscr{H}_\infty = \{(x,y) \in \mathbb{H}^n_{\mathbb{R}} : y \geqslant 1\} \tag{6}$$

is a horoball in $\mathbb{H}^n_{\mathbb{R}}$ centered at ∞, and that the geodesic line in $\mathbb{H}^n_{\mathbb{R}}$ from $(x,0) \in \partial_\infty \mathbb{H}^n_{\mathbb{R}}$ to ∞, through the horosphere $\partial \mathscr{H}_\infty = \{(x,y) \in \mathbb{H}^n_{\mathbb{R}} : y = 1\}$ at time $t = 0$, is the map $t \mapsto (x, e^t)$. Furthermore, the map $x \mapsto (x,1)$, from $\mathbb{R}^{n-1}$ endowed with the standard Euclidean distance to $\partial \mathscr{H}_\infty$ endowed with the Hamenstädt distance $d_{\mathscr{H}_\infty}$, is an isometry.

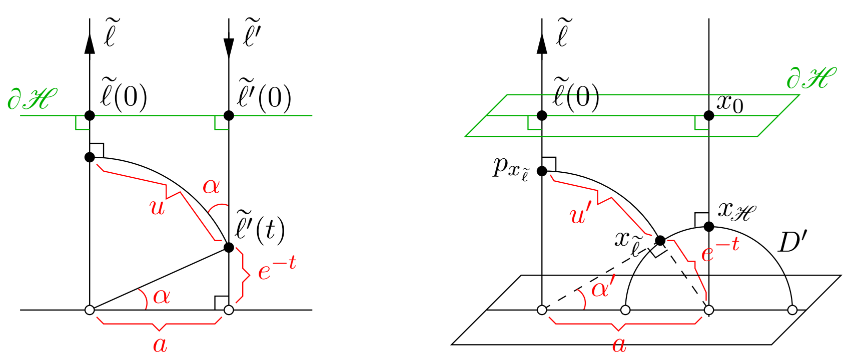
Lemma 4. *Let $\mathscr{H}$ be a horoball in $\mathbb{H}^n_{\mathbb{R}}$ and let $\widetilde{\ell}$ be a geodesic line in $\mathbb{H}^n_{\mathbb{R}}$ that enters $\mathscr{H}$ perpendicularly at $\widetilde{\ell}(0) \in \partial \mathscr{H}$.*
(i) Let $\widetilde{\ell}'$ be a geodesic line in $\mathbb{H}^n_{\mathbb{R}}$ that exits $\mathscr{H}$ perpendicularly at $\widetilde{\ell}'(0) \in \partial \mathscr{H}$. For every $t \geqslant 0$, we have

$$d(\widetilde{\ell}'(t), \widetilde{\ell}\,) = t + \ln d_{\mathscr{H}}(\widetilde{\ell}'(0), \widetilde{\ell}(0)) + \ln 2 + \mathrm{O}\left(d_{\mathscr{H}}(\widetilde{\ell}'(0), \widetilde{\ell}(0))^{-2} e^{-2t}\right).$$

(ii) Let D be a closed convex subset of $\mathbb{H}^n_{\mathbb{R}}$ disjoint from $\mathscr{H}$ and let $x_0 \in \partial \mathscr{H}$ be the closest point to D in $\mathscr{H}$. If $d_{\mathscr{H}}(x_0, \widetilde{\ell}(0)) \geqslant 1$, then

$$d(D, \widetilde{\ell}\,) = d(D, \mathscr{H}) + \ln d_{\mathscr{H}}(x_0, \widetilde{\ell}(0)) + \ln 2 + \mathrm{O}\left(d_{\mathscr{H}}(x_0, \widetilde{\ell}(0))^{-2} e^{-2\,d(D,\mathscr{H})}\right).$$

Furthermore, if $d_{\mathscr{H}}(x_0, \widetilde{\ell}(0)) \geqslant 2$, then the closest point to D on $\widetilde{\ell}$ belongs to $\mathscr{H}$.



Proof. (i) Since two geodesic lines meeting perpendicularly a horosphere have one common point at infinity, the geodesic lines $\widetilde{\ell}$ and $\widetilde{\ell}'$ are coplanar. By symmetry, we may assume that $n = 2$, that $\mathscr{H} = \mathscr{H}_\infty$ and that there exists $a > 0$ such that for every $t \in \mathbb{R}$, we have $\widetilde{\ell}(t) = (0, e^t)$ and $\widetilde{\ell}'(t) = (a, e^{-t})$. Note that then $a = d_{\mathscr{H}}(\widetilde{\ell}(0), \widetilde{\ell}'(0))$. Recall that in a right-angled hyperbolic triangle with one vertex at infinity, with finite opposite side length

u and acute angle α, the *angle of parallelism formula* gives $\cosh u = \frac{1}{\sin\alpha}$.[3] Hence (see the above picture on the left), we have as wanted

$$\begin{aligned} d(\widetilde{\ell}'(t),\widetilde{\ell}\,) &= \operatorname{arcosh}\frac{\sqrt{a^2+e^{-2t}}}{e^{-t}} = \ln(\sqrt{a^2e^{2t}+1}+a\,e^t) \\ &= \ln(a\,e^t(1+\sqrt{1+a^{-2}e^{-2t}}\,)) = t+\ln a+\ln 2+\mathrm{O}(a^{-2}e^{-2t})\,. \end{aligned}$$

(ii) Let $x_{\widetilde{\ell}}\in D$ and $p_{x_{\widetilde{\ell}}}$ be the endpoints of the common perpendicular between D and $\widetilde{\ell}$, and let $x_{\mathscr{H}}\in D$ be the closest point in D to $\mathscr{H}$. Let $\widetilde{\ell}'$ be the geodesic line through $x_{\mathscr{H}}$ exiting $\mathscr{H}$ at time 0 perpendicularly at x_0. Let $t=d(D,\mathscr{H})$ and $a=d_{\mathscr{H}}(x_0,\widetilde{\ell}(0))$. Since

$$d(x_{\widetilde{\ell}},\widetilde{\ell}\,)=d(D,\widetilde{\ell}\,)\leqslant d(x_{\mathscr{H}},\widetilde{\ell}\,)=t+\ln a+\ln 2+\mathrm{O}(a^{-2}e^{-2t})$$

by Assertion (i) applied with the above $\widetilde{\ell}'$, we only prove a similar lower bound on $d(x_{\widetilde{\ell}},\widetilde{\ell})$.

The union of the geodesic lines perpendicular to $\widetilde{\ell}'$ at $x_{\mathscr{H}}$ is a (totally geodesic) hypersurface S that separates D and $\mathscr{H}$ (and is a supporting hyperplane of D). Replacing $x_{\widetilde{\ell}}$ by the intersection point of the geodesic segment $[p_{x_{\widetilde{\ell}}},x_{\widetilde{\ell}}]$ with S does not increase $d(x_{\widetilde{\ell}},\widetilde{\ell})$. Let D' be the geodesic line in S through $x_{\mathscr{H}}$ and $x_{\widetilde{\ell}}$. Up to rotating D' around $\widetilde{\ell}'$ until it lies in the copy of the hyperbolic plane containing $\widetilde{\ell}$ and $\widetilde{\ell}'$, which does not increase $d(x_{\widetilde{\ell}},\widetilde{\ell}\,)$, we may assume that $\widetilde{\ell}$, $\widetilde{\ell}'$ and D' are coplanar. We may assume that $n=2$, $\mathscr{H}=\mathscr{H}_\infty$, $\widetilde{\ell}(t)=(0,e^t)$, $\widetilde{\ell}'(t)=(0,e^{-t})$ as in Assertion (i), so that D' is the geodesic line with points at infinity $(a-e^{-t},0)$ and $(a+e^{-t},0)$. Since by assumption $a\geqslant 1>e^{-t}$, the point $x_{\widetilde{\ell}}$ is the vertex at its right angle of the right-angled Euclidean triangle with other vertices $(0,0)$ and $(a,0)$. When furthermore $a\geqslant 2$, then $a-e^{-t}\geqslant 1$, hence $p_{x_{\widetilde{\ell}}}$ belongs to $\mathscr{H}_\infty$. The Euclidean length of the side between $x_{\widetilde{\ell}}$ and $(a,0)$ is equal to e^{-t}, since D is the open Euclidean halfcircle centered at $(a,0)$ with Euclidean radius e^{-t}. Again applying the angle of parallelism formula (see the above picture on the right), we have as wanted

$$\begin{aligned} d(x_{\widetilde{\ell}},\widetilde{\ell}\,) &= \operatorname{arcosh}\big(\frac{a}{e^{-t}}\big) = \ln(a\,e^t+\sqrt{a^2e^{2t}-1}) \\ &= \ln(a\,e^t(1+\sqrt{1-a^{-2}e^{-2t}}\,)) = t+\ln a+\ln 2+\mathrm{O}(a^{-2}e^{-2t})\,. \quad \square \end{aligned}$$

4 Common perpendiculars of divergent geodesics

Before giving precisely the counting function whose asymptotic we will study in this paper, let us recall the structural properties of the noncompact finite volume complete connected negatively curved good orbifold $M=\Gamma\backslash\widetilde{M}$. Let Par_Γ be the subset of $\partial_\infty\widetilde{M}$ consisting of the fixed points of the parabolic elements of Γ. The *set of cusp* of M is the finite set $\Gamma\backslash\mathrm{Par}_\Gamma$ of Γ-orbits of parabolic fixed points of Γ, whose elements are denoted by $e_1,\dots,e_m$.

Let $(H_\xi)_{\xi\in\mathrm{Par}_\Gamma}$ be a Γ-equivariant family of (closed) horoballs with pairwise disjoint interiors, with H_ξ centered at ξ for every ξ in Par_Γ, which is *precisely invariant*: If the intersection $\gamma\,\overset{\circ}{\mathscr{H}}_\xi\cap\overset{\circ}{\mathscr{H}}_{\xi'}$ is nonempty, then $\gamma\xi=\xi'$. For every $i\in[\![1,m]\!]$, the closure $\mathscr{V}_{e_i}$ of $\Gamma\backslash\big(\bigcup_{\xi\in e_i}\overset{\circ}{H}_\xi\,\big)$ is called the *Margulis neighbourhood* of the cusp e_i. The closure of $M\smallsetminus\bigcup_{1\leqslant i\leqslant m}\mathscr{V}_{e_i}=\Gamma\backslash\big(\widetilde{M}\smallsetminus\bigcup_{\xi\in\mathrm{Par}_\Gamma}H_\xi\big)$ is a compact subset of $M=\Gamma\backslash\widetilde{M}$.

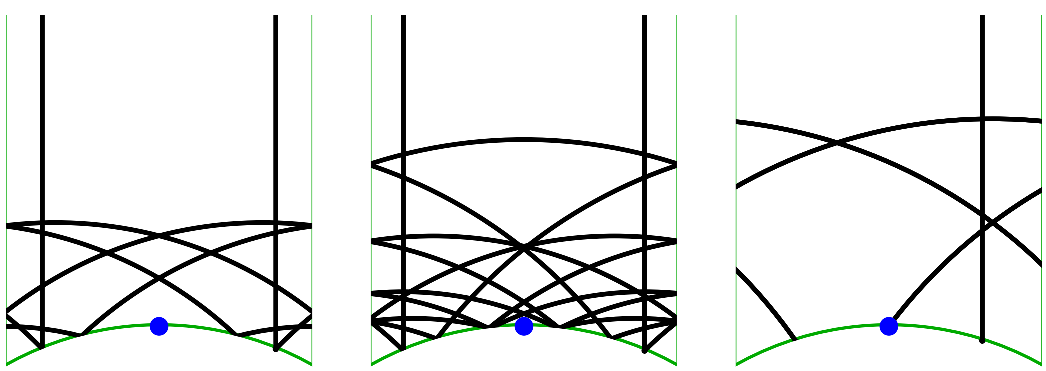
[3]See for instance [Bea1, Thm. 7.9.1 (ii)].

We understand the locally geodesic lines ℓ in M in the orbifold sense. They are possibly not injective maps from $\mathbb{R}$ to M with multiplicities. Since the fixed point set of an isometry of $\widetilde{M}$ is totally geodesic, the orbifold stabilizer of a positive length subsegment of ℓ is equal to the orbifold stabilizer of the whole ℓ. Specific examples when ℓ is not injective are the following ones.

We say that a locally geodesic line ℓ in M (or its image) is *weakly reciprocal* if it has a lift $\widetilde{\ell}:\mathbb{R}\to\widetilde{M}$ such that an element of Γ interchanges the two endpoints at infinity of the geodesic line $\widetilde{\ell}$. Let $\iota_{\rm rec}(\ell(\mathbb{R}))=1$ if ℓ is weakly reciprocal, and $\iota_{\rm rec}(\ell(\mathbb{R}))=2$ otherwise. We say that ℓ (or its image) is *reciprocal* if there is such an element of order 2. Note that when $\widetilde{M}=\mathbb{H}^2_{\mathbb{R}}$ and $\Gamma=\mathrm{PSL}_2(\mathbb{Z})$, a locally geodesic line in $M=\Gamma\backslash\widetilde{M}$ is weakly reciprocal if and only it is reciprocal if and only it it passes through $\Gamma\cdot i$. See [Sar] and [ErPP] for counting and equidistribution results of reciprocal closed geodesics in negatively curved spaces, and Remark 13 for an example.

Recall that a locally geodesic line $\ell:\mathbb{R}\to M$ that is a proper mapping is a *divergent geodesic* in M. By the above description of M, a locally geodesic line $\ell:\mathbb{R}\to M$ is a divergent geodesic in M if and only if there are times $t_-,t_+\in\mathbb{R}$ with $t_-\leqslant t_+$ at which ℓ meets at a right angle the boundary of two Margulis neighbourhoods $\mathscr{V}_-$ and $\mathscr{V}_+$ of cusps of M, that we refer to as the *initial* and *terminal Margulis neighbourhoods* of ℓ. They are possibly equal, as when M has only one cusp or when ℓ is weakly reciprocal. The images of the subrays $\ell|_{]-\infty,t_-]}$ and $\ell|_{[t_+,+\infty[}$ of ℓ are contained in $\mathscr{V}_-$ and $\mathscr{V}_+$. The image D_ℓ of ℓ is a properly immersed closed locally convex subset in M, and we have $\partial^1_+D_\ell=\partial^1_-D_\ell$ and $\iota_*\sigma^-_{D_\ell}=\sigma^+_{D_\ell}$, where $\iota:v\mapsto -v$ is the time reversal map of T^1M.

Examples: Since the set of parabolic fixed points of $\mathrm{PSL}_2(\mathbb{Z})$ is $\mathbb{Q}\cup\{\infty\}$, the divergent geodesics in the modular orbifold $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2_{\mathbb{R}}$ are the images of the vertical geodesic lines starting from ∞ and ending at a rational point $\frac{p}{q}$ with p,q coprime. The first picture in the introduction gives all the divergent geodesics ending at a rational point with positive denominators at most 6. Here are three further pictures, with divergent geodesics defined by the rational numbers 3/8, 31/80 and 3/10 from the left to the right. The last one, passing through i, is reciprocal. Following the path of each geodesic in the quotient orbifold requires to use the boundary identifications $z\mapsto z+1$ and $z\mapsto -\frac{1}{z}$ of the usual fundamental domain $\{z\in\mathbb{C}:-\frac{1}{2}\leqslant\operatorname{Re}z\leqslant\frac{1}{2},|z|\geqslant 1\}$ of $\mathrm{PSL}_2(\mathbb{Z})$.



Let D^- and D^+ be two properly immersed closed locally convex subsets of M, with lifts $\widetilde{D}^-$ and $\widetilde{D}^+$. For all γ,γ' in Γ, the convex sets $\gamma\widetilde{D}^-$ and $\gamma'\widetilde{D}^+$ have a common perpendicular (as defined in the introduction) if and only if their closures $\overline{\gamma\widetilde{D}^-}$ and $\overline{\gamma'\widetilde{D}^+}$ in $\widetilde{M}\cup\partial_\infty\widetilde{M}$

do not intersect. This common perpendicular $\alpha_{\gamma\widetilde{D}^-,\,\gamma'\widetilde{D}^+}$ starts from $\gamma\widetilde{D}^-$ at time $t=0$ with unit tangent vector $\dot\alpha_{\gamma\widetilde{D}^-,\,\gamma'\widetilde{D}^+}(0)$ and ends in $\gamma'\widetilde{D}^+$ at time $t=d(\gamma\widetilde{D}^-,\gamma'\widetilde{D}^+)$ with unit tangent vector $\dot\alpha_{\gamma\widetilde{D}^-,\,\gamma'\widetilde{D}^+}(d(\gamma\widetilde{D}^-,\gamma'\widetilde{D}^+))$. Its *multiplicity*[4] is

$$m_{\gamma\widetilde{D}^-,\gamma'\widetilde{D}^+}=\frac{1}{\operatorname{Card}(\gamma\Gamma_{\widetilde{D}^-}\gamma^{-1}\cap\gamma'\Gamma_{\widetilde{D}^+}\gamma'^{-1})}\,, \tag{7}$$

which equals 1 when Γ acts freely on $T^1\widetilde{M}$ (for instance when Γ is torsion-free). Note that for every $\gamma''\in\Gamma$, we have

$$\gamma''\alpha_{\gamma\widetilde{D}^-,\,\gamma'\widetilde{D}^+}=\alpha_{\gamma''\gamma\widetilde{D}^-,\,\gamma''\gamma'\widetilde{D}^+}\quad\text{and}\quad m_{\gamma''\gamma\widetilde{D}^-,\gamma''\gamma'\widetilde{D}^+}=m_{\gamma\widetilde{D}^-,\gamma'\widetilde{D}^+}\,.$$

Recall that $\widetilde{p}:T^1\widetilde{M}\to T^1M=\Gamma\backslash T^1\widetilde{M}$ is the canonical projection. Let Ω^- and Ω^+ be measurable subsets of $\partial^1_+D^-$ and $\partial^1_-D^+$, and let $\widetilde{\Omega}^-=\partial^1_+\widetilde{D}^-\cap\widetilde{p}^{-1}(\Omega^-)$ and $\widetilde{\Omega}^+=\partial^1_-\widetilde{D}^+\cap\widetilde{p}^{-1}(\Omega^+)$ be the subsets of all elements of $\partial^1_+\widetilde{D}^-$ and $\partial^1_-\widetilde{D}^+$ mapping to Ω^- and Ω^+ by $\widetilde{p}$. Note that $\widetilde{\Omega}^-$ and $\widetilde{\Omega}^+$ are invariant under $\Gamma_{\widetilde{D}^-}$ and $\Gamma_{\widetilde{D}^+}$ respectively. The counting function $\mathscr{N}_{\Omega^-,\,\Omega^+}$ is defined[5] by

$$\mathscr{N}_{\Omega^-,\,\Omega^+}:t\mapsto\sum_{\substack{\Gamma(\gamma\Gamma_{\widetilde{D}^-},\,\gamma'\Gamma_{\widetilde{D}^+})\in\Gamma\backslash((\Gamma/\Gamma_{\widetilde{D}^-})\times(\Gamma/\Gamma_{\widetilde{D}^+}))\\ \overline{\gamma\widetilde{D}^-}\cap\overline{\gamma'\widetilde{D}^+}=\varnothing,\ d(\gamma\widetilde{D}^-,\gamma'\widetilde{D}^+)\leqslant t\\ \dot\alpha_{\gamma\widetilde{D}^-,\,\gamma'\widetilde{D}^+}(0)\in\gamma\widetilde{\Omega}^-,\ \dot\alpha_{\gamma\widetilde{D}^-,\,\gamma'\widetilde{D}^+}(d(\gamma\widetilde{D}^-,\gamma'\widetilde{D}^+))\in\gamma'\widetilde{\Omega}^+}}m_{\gamma\widetilde{D}^-,\gamma'\widetilde{D}^+}\,. \tag{8}$$

where Γ acts diagonally on $\Gamma/\Gamma_{\widetilde{D}^-}\times\Gamma/\Gamma_{\widetilde{D}^+}$. In order to simplify the notation, let $\mathscr{N}_{D^-,\,D^+}=\mathscr{N}_{\partial^1_+D^-,\,\partial^1_-D^+}$, $\mathscr{N}_{\Omega^-,\,D^+}=\mathscr{N}_{\Omega^-,\partial^1_-D^+}$ and $\mathscr{N}_{D^-,\,\Omega^+}=\mathscr{N}_{\partial^1_+D^-,\,\Omega^+}$.

As mentionned in the introduction, the general purely exponential asymptotic theorem on the counting function $\mathscr{N}_{D^-,\,D^+}(s)$ as $s\to+\infty$ proven in [PP5, Thm. 1], which requires the finiteness of the skinning measures of D^- and D^+, does not apply when D^- or D^+ is the image of a divergent geodesic. Indeed, if ℓ is a divergent geodesic, then the skinning measure of its image has infinite total mass. Theorems 5 and 6 below show that, when D^- or D^+ is a divergent geodesic, the growth of the counting function $\mathscr{N}_{D^-,\,D^+}$ is no longer purely exponential.

Theorem 5. *Let M be a noncompact finite volume complete connected real hyperbolic good orbifold. Let D^+ and D^- be nonempty properly immersed closed locally convex subsets of M. Assume that D^- has nonzero finite (outer) skinning measure and that D^+ is the image of a divergent geodesic in M. Then as $s\to+\infty$, we have*

$$\mathscr{N}_{D^-,D^+}(s)=\frac{\Gamma(\frac{n}{2})\,\iota_{\rm rec}(D^+)\,\|\sigma^+_{D^-}\|}{2^n\sqrt{\pi}\;\Gamma(\frac{n+1}{2})\,m(D^+)\;\operatorname{Vol}M}\;s\,e^{(n-1)\,s}+\operatorname{O}(e^{(n-1)s})\,.$$

Proof. This proof gives more detail than might seem necessary, and uses as much as possible the general notation of the beginning of Sections 2 and 4, in order to serve proving Theorem 6 (also when M is real hyperbolic) and Theorem 9 (when $\widetilde{M}\neq\mathbb{H}^n_{\mathbb{R}}$). We believe that this process will be easier for the reader.

[4]See [PP5, §3.3] and [BPP, §12.1] for precisions.
[5]See [PP5, page 86] for precisions.

Let ℓ be a divergent geodesic in M whose image is D^+. Let $\mathscr{V}_-$ and $\mathscr{V}_+$ be the initial and terminal Margulis neighbourhoods of ℓ. Let t_- be the first exit time of ℓ from $\mathscr{V}_-$ and let t_+ be the last entry time of ℓ into $\mathscr{V}_+$. Let $\widetilde{\ell}$ be a lift of ℓ in $\widetilde{M}$, and let $\widetilde{D}^+$ be the image of $\widetilde{\ell}$. For simplicity, let $m^+ = m(D^+)$ and $\iota^+_{\rm rec} = \iota_{\rm rec}(D^+)$. Let

$$\begin{aligned}
\Omega_- &= \big\{v \in \partial^1_- D^+ : p_\bullet(v) \in \ell(\,]-\infty, t_-[\,)\big\}\,,\\
\Omega_0 &= \big\{v \in \partial^1_- D^+ : p_\bullet(v) \in \ell([t_-, t_+])\big\} \quad \text{and}\\
\Omega_+ &= \big\{v \in \partial^1_- D^+ : p_\bullet(v) \in \ell(\,]t_+, +\infty[\,)\big\}\,.
\end{aligned}$$

We denote by $\mathscr{H} = H_{\widetilde{\ell}(+\infty)}$ the horoball of the equivariant family $(H_\xi)_{\xi\in{\rm Par}_\Gamma}$ with point at infinity $\widetilde{\ell}(+\infty)$, that is a lift of $\mathscr{V}_+$. Let $\widetilde{D}^-$ be a lift of D^-.

Case 1. Let us first assume that ℓ is not weakly reciprocal. Let us prove that the subsets Ω_- and Ω_+ are disjoint. Assume for a contradiction that there exists an element γ of Γ mapping an element in $\partial^1_- \widetilde{D}^+$ with footpoint in $\widetilde{\ell}(\,]-\infty, t_-[\,)$ to another element with footpoint in $\widetilde{\ell}(\,]t_+, +\infty[\,)$. Then γ would map a point $\widetilde{\ell}(s_-)$ with $s_- \in \,]-\infty, t_-[$ to a point $\widetilde{\ell}(s_+)$ with $s_+ \in \,]t_+, +\infty[$. Since the equivariant family $(H_\xi)_{\xi\in{\rm Par}_\Gamma}$ is precisely invariant, the element γ would also map $\widetilde{\ell}(-\infty)$ to $\widetilde{\ell}(+\infty)$. Therefore the restriction of γ to $\widetilde{D}^+$ would be the central symmetry with respect to the unique fixed point of γ on $\widetilde{D}^+$, thereby exchanging its two points at infinity. Thus ℓ would be weakly reciprocal, a contradiction.

In particular, since $\partial^1_- D^+ = \Omega_- \cup \Omega_0 \cup \Omega_+$, for every $s \geqslant 0$, we have

$$\mathscr{N}_{D^-,\Omega_-}(s) + \mathscr{N}_{D^-,\Omega_+}(s) \leqslant \mathscr{N}_{D^-,D^+}(s) \leqslant \mathscr{N}_{D^-,\Omega_-}(s) + \mathscr{N}_{D^-,\Omega_0}(s) + \mathscr{N}_{D^-,\Omega_+}(s)\,. \tag{9}$$

We shall prove that as $s \to +\infty$, we have

$$\mathscr{N}_{D^-,\Omega_+}(s) = \frac{\|\sigma^+_{D^-}\|\ \Xi_{\widetilde{M}}}{2^\delta\, m^+\, \|m_{\rm BM}\|}\ s\, e^{\delta\, s} + {\rm O}(e^{\delta\, s})\,. \tag{10}$$

By the independence property on the horoball in the definition of $\Xi_{\widetilde{M}}$ in Equation (5), the same proof replacing the horoball $\mathscr{H} = H_{\widetilde{\ell}(+\infty)}$ by the horoball $H_{\widetilde{\ell}(-\infty)}$ (that is a lift of $\mathscr{V}_-$) gives that as $s \to +\infty$, we will have

$$\mathscr{N}_{D^-,\Omega_-}(s) = \frac{\|\sigma^+_{D^-}\|\ \Xi_{\widetilde{M}}}{2^\delta\, m^+\, \|m_{\rm BM}\|}\ s\, e^{\delta\, s} + {\rm O}(e^{\delta\, s})\,. \tag{11}$$

Let $D_0 = \ell([t_-, t_+])$, which is a compact nonempty properly immersed locally convex subset of M, hence has a nonzero finite inner skinning measure. By [PP5, Theo. 1], as $s \to +\infty$, we therefore have

$$\mathscr{N}_{D^-,\Omega_0}(s) \leqslant \mathscr{N}_{D^-,D_0}(s) = {\rm O}(e^{\delta\, s})\,. \tag{12}$$

Thus by Equations (9), (10), (11) and (12), and since $\iota^+_{\rm rec} = 2$ when ℓ is not weakly reciprocal, as $s \to +\infty$, we will have

$$\mathscr{N}_{D^-,D^+}(s) = \frac{\iota^+_{\rm rec}\,\|\sigma^+_{D^-}\|\ \Xi_{\widetilde{M}}}{2^\delta\, m^+\, \|m_{\rm BM}\|}\ s\, e^{\delta\, s} + {\rm O}(e^{\delta\, s})\,. \tag{13}$$

By [PP5, Prop. 20 (2)], since M is real hyperbolic in the assumptions of Theorem 5, since the Patterson-Sullivan measures have been normalized in Section 2 to have total mass $\operatorname{Vol}(\mathbb{S}^{n-1})$, the metric measured space $(\partial\mathscr{H}, d_{\mathscr{H}}, (\widetilde{p}_\bullet)_*\widetilde{\sigma}^-_{\mathscr{H}})$ is isomorphic to $\mathbb{R}^{n-1}$ endowed with its usual Euclidean distance and with 2^{n-1} times its Lebesgue measure. Furthermore, the Hamenstädt distance $d_{\mathscr{H}}$ is equal to the cuspidal distance $d'_{\mathscr{H}}$ since M is real hyperbolic. Hence by the definition of $\Xi_{\widetilde{M}}$ in Equation (5), we have

$$\Xi_{\widetilde{M}} = 2^{n-1}\operatorname{Vol}(\mathbb{B}_{n-1}) = \frac{2^{n-1}\,\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n+1}{2})}\,. \tag{14}$$

By [PP5, Prop. 20 (1)], we have

$$\|m_{\rm BM}\| = 2^{n-1}\operatorname{Vol}(\mathbb{S}^{n-1})\operatorname{Vol}(M) = 2^{n-1}\,\frac{2\,\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\operatorname{Vol}(M)\,. \tag{15}$$

We also have $\delta = n-1$. Hence Theorem 5 will follow from Equation (13), once we have proven Equation (10). Up to changing the parametrisation of the geodesic line $\widetilde{\ell}$ by a translation, we may assume that $t_+ = 0$ to simplify the notation. We now start the work on the sum defining $\mathscr{N}_{D^-,\Omega_+}(s)$ given by Equation (8) in order to prove Equation (10).

Step 1. The first step is to simplify the set of indices in this sum.

The map from $\Gamma\backslash((\Gamma/\Gamma_{\widetilde{D}^-})\times(\Gamma/\Gamma_{\widetilde{D}^+}))$ to $\Gamma_{\widetilde{D}^-}\backslash\Gamma/\Gamma_{\widetilde{D}^+}$ defined by

$$\Gamma(\gamma\Gamma_{\widetilde{D}^-},\gamma'\Gamma_{\widetilde{D}^+})\mapsto\Gamma_{\widetilde{D}^-}\gamma^{-1}\gamma'\Gamma_{\widetilde{D}^+}$$

is a bijection, whose inverse is $[\gamma] = \Gamma_{\widetilde{D}^-}\gamma\Gamma_{\widetilde{D}^+}\mapsto\Gamma(\gamma^{-1}\Gamma_{\widetilde{D}^-},\Gamma_{\widetilde{D}^+})$, by an immediate checking. In order to simplify the notation, let

$$z_{\gamma\widetilde{D}^-,\,\widetilde{D}^+} = \alpha_{\gamma\widetilde{D}^-,\,\widetilde{D}^+}(d(\gamma\widetilde{D}^-,\widetilde{D}^+))\,.$$

Since $\widetilde{D}^+$ is not weakly reciprocal, we have $\partial^1_-\widetilde{D}^+\cap\widetilde{p}^{-1}(\Omega_+) = \widetilde{p}_\bullet^{-1}(\widetilde{\ell}(\,]0,+\infty[\,))$. Thus by Equation (8) and by a change of variable $\gamma\mapsto\gamma^{-1}$, for every $s\geqslant 0$, we have

$$\begin{aligned}\mathscr{N}_{D^-,\Omega_+}(s) &= \sum_{\substack{[\gamma]\in\Gamma_{\widetilde{D}^-}\backslash\Gamma/\Gamma_{\widetilde{D}^+}\,:\,0<d(\gamma^{-1}\widetilde{D}^-,\widetilde{D}^+)\leqslant s\\ \dot{\alpha}_{\gamma^{-1}\widetilde{D}^-,\,\widetilde{D}^+}(d(\gamma^{-1}\widetilde{D}^-,\widetilde{D}^+))\in\widetilde{p}_\bullet^{-1}(\widetilde{\ell}(\,]0,+\infty[\,))}} m_{\gamma^{-1}\widetilde{D}^-,\widetilde{D}^+}\\ &= \sum_{[\gamma]\in\Gamma_{\widetilde{D}^+}\backslash\Gamma/\Gamma_{\widetilde{D}^-}\,:\,0<d(\gamma\widetilde{D}^-,\widetilde{D}^+)\leqslant s,\;z_{\gamma\widetilde{D}^-,\,\widetilde{D}^+}\in\widetilde{\ell}(\,]0,+\infty[\,)} m_{\gamma\widetilde{D}^-,\widetilde{D}^+}\,.\end{aligned} \tag{16}$$

Step 2. The second step in the proof of Equation (10) is to prove that the contribution to the above sum defining $\mathscr{N}_{\widetilde{D}^-,\Omega_+}(s)$ of the indices with multiplicities different from 1 is negligible.

By Equation (2), the critical exponent of a positive codimension totally geodesic subspace of $\widetilde{M}$ is at most $\delta-1$. Note that the stabilizer $\Gamma_{\widetilde{D}^+}$ of $\widetilde{D}^+$ in Γ is finite since $\widetilde{\ell}(+\infty)$ is a parabolic fixed point of Γ, hence no loxodromic element in Γ preserves $\widetilde{D}^+$. The fixed point set of a nontrivial isometry with finite order of $\widetilde{M}$ is a totally geodesic subspace with positive codimension. Hence as $[\gamma]$ ranges over $\Gamma_{\widetilde{D}^+}\backslash\Gamma/\Gamma_{\widetilde{D}^-}$ with $d(\gamma\widetilde{D}^-,\widetilde{D}^+)>0$, the common perpendiculars between $\gamma\widetilde{D}^-$ and $\widetilde{D}^+$ with multiplicity $m_{\gamma\widetilde{D}^-,\widetilde{D}^+}\neq 1$ are contained in finitely many positive codimension totally geodesic subspaces of $\widetilde{M}$. By Equation

(7), the multiplicities $m_{\gamma\widetilde{D}^-,\widetilde{D}^+}$ are at most 1. Hence as $s\to+\infty$, by the same proof as the one that follows, we have

$$\begin{aligned}
&\sum_{[\gamma]\in\Gamma_{\widetilde{D}^+}\backslash\Gamma/\Gamma_{\widetilde{D}^-}\,:\,0<d(\gamma\widetilde{D}^-,\widetilde{D}^+)\leqslant s,\; m_{\gamma\widetilde{D}^-,\widetilde{D}^+}\neq 1} m_{\gamma\widetilde{D}^-,\widetilde{D}^+}\\
&\leqslant\ \mathrm{Card}\{[\gamma]\in\Gamma_{\widetilde{D}^+}\backslash\Gamma/\Gamma_{\widetilde{D}^-}: 0<d(\gamma\widetilde{D}^-,\widetilde{D}^+)\leqslant s,\ \ m_{\gamma\widetilde{D}^-,\widetilde{D}^+}\neq 1\}\\
&=\ \mathrm{O}(s\,e^{(\delta-1)s})=\mathrm{O}(e^{\delta\,s})\,. \qquad (17)
\end{aligned}$$

Hence by Equation (16), we have

$$\mathscr{N}_{D^-,\,\Omega_+}(s)=\mathrm{Card}\left\{\gamma\in\Gamma_{\widetilde{D}^+}\backslash\Gamma/\Gamma_{\widetilde{D}^-}:\ \begin{array}{c}0<d(\gamma\widetilde{D}^-,\widetilde{D}^+)\leqslant s\\ z_{\gamma\widetilde{D}^-,\,\widetilde{D}^+}\in\widetilde{\ell}(\,]0,+\infty[\,)\end{array}\right\}+\mathrm{O}(e^{\delta\,s})\,. \qquad (18)$$

The map from $(\Gamma_{\mathscr{H}}\backslash\Gamma/\Gamma_{\widetilde{D}^-})\times\Gamma_{\mathscr{H}}$ to $\Gamma_{\widetilde{D}^+}\backslash\Gamma/\Gamma_{\widetilde{D}^-}$ defined by $([\gamma],\beta)\mapsto\Gamma_{\widetilde{D}^+}\beta\gamma\Gamma_{\widetilde{D}^-}$ is onto, and will be used in Step 6, Equation (24), in order to disintegrate the counting in Equation (18) above a counting in $\Gamma_{\mathscr{H}}\backslash\Gamma/\Gamma_{\widetilde{D}^-}$, the point being that the images of $\mathscr{H}$ and $\widetilde{D}^-$ in M both have finite skinning measures. But this requires some preliminary work.

Step 3. The third step in the proof of Equation (10) is to define two exceptional finite subsets F in $\Gamma_{\mathscr{H}}\backslash\Gamma/\Gamma_{\widetilde{D}^-}$ and F' in $\Gamma_{\mathscr{H}}$ whose contribution to the counting that will occur in Step 6, Equation (24), will be proven to be negligible in Step 5.

The quotient space $\Gamma_{\mathscr{H}}\backslash\partial\mathscr{H}$ is compact, since the point at infinity of the horoball $\mathscr{H}$ is the parabolic fixed point $\widetilde{\ell}(+\infty)$ of Γ. Since the family $(\gamma'\widetilde{D}^-)_{\gamma'\in\Gamma/\Gamma_{\widetilde{D}^-}}$ is locally finite, the subset F of elements $[\gamma]\in\Gamma_{\mathscr{H}}\backslash\Gamma/\Gamma_{\widetilde{D}^-}$ such that the closed convex subsets $\mathscr{H}$ and $\gamma\widetilde{D}^-$ are not disjoint is finite. For every $[\gamma]\in\Gamma_{\mathscr{H}}\backslash\Gamma/\Gamma_{\widetilde{D}^-}\smallsetminus F$, we denote by $p_\gamma\in\partial\mathscr{H}$ and $p'_\gamma\in\partial(\gamma\widetilde{D}^-)$ the two endpoints of the common perpendicular between $\mathscr{H}$ and $\gamma\widetilde{D}^-$ (see the picture below).

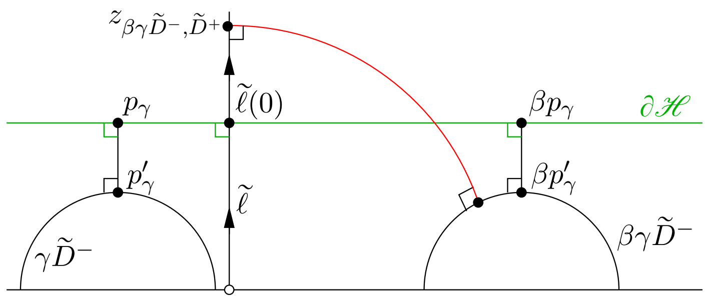
Again since $\Gamma_{\mathscr{H}}\backslash\partial\mathscr{H}$ is compact, there exists a constant $c_1>0$ (for instance the radius of a closed ball in $(\partial\mathscr{H},d'_{\mathscr{H}})$ with center $\widetilde{\ell}(0)$ which maps onto $\Gamma_{\mathscr{H}}\backslash\partial\mathscr{H}$) such that we may fix from now on a representative γ of every $[\gamma]\in\Gamma_{\mathscr{H}}\backslash\Gamma/\Gamma_{\widetilde{D}^-}\smallsetminus F$ (by multiplying it on the left by an element of $\Gamma_{\mathscr{H}}$) so that $d'_{\mathscr{H}}(p_\gamma,\widetilde{\ell}(0))\leqslant c_1$.

As M is assumed to be real hyperbolic in Theorem 5, we define $\mathbb{K}=\mathbb{R}$ and $c_{\mathbb{K}}=2$. Since the isometric action of $\Gamma_{\mathscr{H}}$ on $(\partial\mathscr{H},d'_{\mathscr{H}})$ is discrete, there exists a finite subset F' of $\Gamma_{\mathscr{H}}$ such that for every $\beta\in\Gamma_{\mathscr{H}}\smallsetminus F'$, we have $d'_{\mathscr{H}}(\widetilde{\ell}(0),\beta\widetilde{\ell}(0))\geqslant c_{\mathbb{K}}+c_1$. By the triangle inequality, for all $[\gamma]\in\Gamma_{\mathscr{H}}\backslash\Gamma/\Gamma_{\widetilde{D}^-}\smallsetminus F$ and $\beta\in\Gamma_{\mathscr{H}}\smallsetminus F'$, we thus have

$$d'_{\mathscr{H}}(\widetilde{\ell}(0),\beta p_\gamma)\geqslant d'_{\mathscr{H}}(\widetilde{\ell}(0),\beta\widetilde{\ell}(0))-d'_{\mathscr{H}}(\beta\widetilde{\ell}(0),\beta p_\gamma)\geqslant(c_{\mathbb{K}}+c_1)-c_1=c_{\mathbb{K}}\,. \qquad (19)$$

Therefore since M is real hyperbolic, by the second claim of Lemma 4 (ii) applied with $D=\beta\gamma\widetilde{D}^-$ and $x_0=\beta p_\gamma$, the point $z_{\beta\gamma\widetilde{D}^-,\,\widetilde{D}^+}$ belongs to the positive subray $\widetilde{\ell}(\,]0,+\infty[\,)$.

The picture below represents in red the common perpendicular between $\beta\gamma\widetilde{D}^-$ and $\widetilde{D}^+$ in the generic situation when $[\gamma]\notin F$ and $\beta\notin F'$.



Step 4. In this rather independent fourth step in the proof of Equation (10), we study the orbital growth of the parabolic subgroup $\Gamma_{\mathscr{H}}$.

For all $[\gamma] \in \Gamma_{\mathscr{H}}\backslash\Gamma/\Gamma_{\widetilde{D}^-} \smallsetminus F$ and $t \geqslant 0$, let

$$\Phi_{[\gamma]}(t) = \operatorname{Card}\{\beta \in \Gamma_{\mathscr{H}} : d'_{\mathscr{H}}(\widetilde{\ell}(0), \beta p_\gamma) \leqslant t\}\,.$$

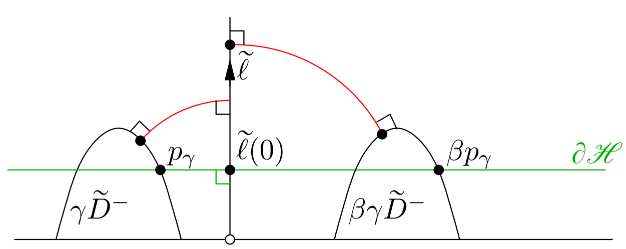
The group of isometries of $\widetilde{M}$ preserving $\mathscr{H}$ acts transitively on $\partial\mathscr{H}$ and preserves the measure $(\widetilde{p}_\bullet)_*\widetilde{\sigma}^-_{\mathscr{H}} = (\widetilde{p}_\bullet)_*\widetilde{\sigma}^+_{\mathscr{H}}$ on $\partial\mathscr{H}$ (see [PP5, Prop. 20 (3)], [PP6, Lem. 12 (iv)], [PP7, Lem. 7·2] for details). Furthermore, using the definition of $\Xi_{\widetilde{M}}$ in Equation (5), it satisfies the following homogeneity property: for every $x \in \partial\mathscr{H}$ and $r > 0$, we have

$$(\widetilde{p}_\bullet)_*\widetilde{\sigma}^-_{\mathscr{H}}(B_{d'_{\mathscr{H}}}(x,r)) = \Xi_{\widetilde{M}}\, r^\delta\,.$$

Recall that $\Gamma_{\mathscr{H}}$ is a uniform lattice in the isometry group of $(\partial\mathscr{H}, d'_{\mathscr{H}})$. By the standard Gauss counting argument (covering the ball with center $\widetilde{\ell}(0)$ and radius r by translates by elements of $\Gamma_{\mathscr{H}}$ of a given compact fundamental domain with measure zero boundary and measure $\|\sigma^-_{\mathscr{V}_+}\|$ for the measure $(\widetilde{p}_\bullet)_*\widetilde{\sigma}^-_{\mathscr{V}_+}$, with a $\mathrm{O}(\cdot)$ which is uniform in $[\gamma]$ since p_γ varies in a compact subset of $\partial\mathscr{H}$, we have

$$\Phi_{[\gamma]}(t) = \frac{\Xi_{\widetilde{M}}}{\|\sigma^-_{\mathscr{V}_+}\|}\, t^\delta + \mathrm{O}(t^{\delta-1})\,. \tag{20}$$

Step 5. In this fifth step in the proof of Equation (10), we prove that the contribution to the counting that will occur in Step 6, Equation (24), of the two exceptional finite subsets F in $\Gamma_{\mathscr{H}}\backslash\Gamma/\Gamma_{\widetilde{D}^-}$ and F' in $\Gamma_{\mathscr{H}}$ defined in Step 3 is negligible.



Let $c_2 = \max_{[\gamma]\in F} d(\gamma\widetilde{D}^-, \partial\mathscr{H})$. For every $[\gamma] \in F$, let $p_\gamma \in \partial\mathscr{H}$ be such that $d(\gamma\widetilde{D}^-, \partial\mathscr{H}) = d(\gamma\widetilde{D}^-, p_\gamma)$ (see the above picture when $\gamma\widetilde{D}^-$ meets $\partial\mathscr{H}$, though $\gamma\widetilde{D}^-$ could be contained in the interior of $\mathscr{H}$). By the triangle inequality and since closest point projections do not increase the distances, by Lemma 3, and since the Hamenstädt distance and the cuspidal distance are equivalent, there exists a constant $c_3 > 0$ (with actually $c_3 = 1$ when M is real hyperbolic) such that for every $\beta \in \Gamma_{\mathscr{H}}$, we have

$$e^{d(\beta\gamma\widetilde{D}^-,\widetilde{D}^+)} \geqslant e^{d(\beta p_\gamma,\widetilde{D}^+)-2\,c_2} \geqslant e^{-2\,c_2}\, d_{\mathscr{H}}(\beta p_\gamma, \widetilde{\ell}(0)) \geqslant \frac{1}{c_3}\, e^{-2\,c_2}\, d'_{\mathscr{H}}(\beta p_\gamma, \widetilde{\ell}(0))\,.$$

Thus by Step 4, which also works when $[\gamma] \in F$ with the above p_γ, we have the following negligible estimate for F:

$$\begin{aligned}
&\operatorname{Card}\Big\{([\gamma],\beta) \in F \times \Gamma_{\mathscr{H}} : \begin{array}{c} 0 < d(\beta\gamma\widetilde{D}^-, \widetilde{D}^+) \leqslant s \\ z_{\beta\gamma\widetilde{D}^-,\,\widetilde{D}^+} \in \widetilde{\ell}(\,]0,+\infty[\,) \end{array} \Big\}\\
&\leqslant (\operatorname{Card}\ F)\ \max_{[\gamma]\in F} \operatorname{Card}\big\{\beta \in \Gamma_{\mathscr{H}} : d'_{\mathscr{H}}(\beta p_\gamma, \widetilde{\ell}(0)) \leqslant c_3\, e^{s+2\,c_2}\big\}\\
&= (\operatorname{Card}\ F)\ \max_{[\gamma]\in F} \Phi_{[\gamma]}(c_3\, e^{s+2\,c_2}) = \mathrm{O}(e^{\delta s})\,.
\end{aligned} \tag{21}$$

Let us prove an analogous estimate for F'. Let $([\gamma],\beta) \in (\Gamma_{\mathscr{H}}\backslash\Gamma/\Gamma_{\widetilde{D}^-} \smallsetminus F) \times F'$ be such that the closest point $z_{\beta\gamma\widetilde{D}^-,\,\widetilde{D}^+}$ on $\widetilde{D}^+$ to $\beta\gamma\widetilde{D}^-$ belongs to $\widetilde{\ell}(\,]0,+\infty[\,)$. Since $[\gamma] \notin F$, the closed convex subsets $\gamma\widetilde{D}^-$ and $\mathscr{H}$, hence $\beta\gamma\widetilde{D}^-$ and $\mathscr{H}$, are disjoint. By the intermediate value theorem, the common perpendicular between $\beta\gamma\widetilde{D}^-$ and $\widetilde{D}^+$ meets $\partial\mathscr{H}$ (see the picture before Step 4). Hence we have

$$d(\gamma\widetilde{D}^-, \mathscr{H}) = d(\beta\gamma\widetilde{D}^-, \mathscr{H}) \leqslant d(\beta\gamma\widetilde{D}^-, \widetilde{D}^+)\,.$$

Since the Riemannian orbifold M is locally symmetric with finite volume, the group Γ contains finitely many conjugacy classes of finite subgroups. Hence there exists a constant $c_4 > 0$ such that for every $[\gamma] \in \Gamma_{\mathscr{H}}\backslash\Gamma/\Gamma_{\widetilde{D}^-} \smallsetminus F$, we have $m_{\gamma\widetilde{D}^-,\mathscr{H}} \geqslant c_4$.

Since both D^- and $\mathscr{V}_+$ have finite skinning measures, by [PP5, Theo. 1], we have $\mathscr{N}_{D^-,\,\mathscr{V}_+}(s) = \mathrm{O}(e^{\delta s})$. Thus

$$\begin{aligned}
&\operatorname{Card}\Big\{([\gamma],\beta) \in (\Gamma_{\mathscr{H}}\backslash\Gamma/\Gamma_{\widetilde{D}^-} \smallsetminus F) \times F' : \begin{array}{c} 0 < d(\beta\gamma\widetilde{D}^-, \widetilde{D}^+) \leqslant s \\ z_{\beta\gamma\widetilde{D}^-,\,\widetilde{D}^+} \in \widetilde{\ell}(\,]0,+\infty[\,) \end{array} \Big\}\\
&\leqslant (\operatorname{Card}\ F')\ \operatorname{Card}\{[\gamma] \in \Gamma_{\mathscr{H}}\backslash\Gamma/\Gamma_{\widetilde{D}^-} : 0 < d(\gamma\widetilde{D}^-, \mathscr{H}) \leqslant s\}\\
&\leqslant \frac{\operatorname{Card}\ F'}{c_4} \mathscr{N}_{D^-,\,\mathscr{V}_+}(s) = \mathrm{O}(e^{\delta s})\,.
\end{aligned} \tag{22}$$

Step 6. The sixth step in the proof of Equation (10) is to disintegrate the counting defining $\mathscr{N}_{\widetilde{D}^-,\Omega_+}(s)$ in Equation (18) along the orbits of the parabolic subgroup $\Gamma_{\mathscr{H}}$ of Γ fixing $\widetilde{\ell}(+\infty)$.

Let $\Gamma'_{\widetilde{D}^+} = \Gamma_{\widetilde{D}^+} \cap \Gamma_{\mathscr{H}}$. Since ℓ is not weakly reciprocal in Case 1, we have $\Gamma'_{\widetilde{D}^+} = \Gamma_{\widetilde{D}^+}$, and in particular Equation (18) can be rewritten

$$\mathscr{N}_{D^-,\Omega_+}(s) = \operatorname{Card}\Big\{\gamma \in \Gamma'_{\widetilde{D}^+}\backslash\Gamma/\Gamma_{\widetilde{D}^-} : \begin{array}{c} 0 < d(\gamma\widetilde{D}^-, \widetilde{D}^+) \leqslant s \\ z_{\gamma\widetilde{D}^-,\,\widetilde{D}^+} \in \widetilde{\ell}(\,]0,+\infty[\,) \end{array} \Big\} + \mathrm{O}(e^{\delta\, s})\,. \tag{23}$$

But what follows will also be useful for Case 2, hence the generality. Since the point at infinity $\widetilde{\ell}(+\infty)$ is a parabolic fixed point of Γ, the group $\Gamma'_{\widetilde{D}^+}$ is the pointwise stabilizer of $\widetilde{D}^+$, hence has order $m^+ = m(D^+)$.

We use the representatives of double classes in $\Gamma_{\mathscr{H}}\backslash\Gamma/\Gamma_{\widetilde{D}^-}$ defined in Step 3, though any choice would work as well in this Step 6. The map from $\Gamma'_{\widetilde{D}^+}\backslash\Gamma/\Gamma_{\widetilde{D}^-}$ to $\Gamma_{\mathscr{H}}\backslash\Gamma/\Gamma_{\widetilde{D}^-}$ given by $\Gamma'_{\widetilde{D}^+}\gamma'\Gamma_{\widetilde{D}^-}\mapsto\Gamma_{\mathscr{H}}\gamma'\Gamma_{\widetilde{D}^-}$ is well defined since $\Gamma'_{\widetilde{D}^+}$ is contained in $\Gamma_{\mathscr{H}}$. Its fiber over the element $[\gamma]\in\Gamma_{\mathscr{H}}\backslash\Gamma/\Gamma_{\widetilde{D}^-}$ is the subset $\{\Gamma'_{\widetilde{D}^+}\beta\gamma\Gamma_{\widetilde{D}^-}:\beta\in\Gamma_{\mathscr{H}}\}$ of $\Gamma'_{\widetilde{D}^+}\backslash\Gamma/\Gamma_{\widetilde{D}^-}$. Let us fix $[\gamma]\in\Gamma_{\mathscr{H}}\backslash\Gamma/\Gamma_{\widetilde{D}^-}\smallsetminus F$, so that $\mathscr{H}$ and $\gamma\widetilde{D}^-$ have a common perpendicular. Given two distinct elements $\Gamma'_{\widetilde{D}^+}\beta,\Gamma'_{\widetilde{D}^+}\beta'$ in $\Gamma'_{\widetilde{D}^+}\backslash\Gamma_{\mathscr{H}}$, we have $\Gamma'_{\widetilde{D}^+}\beta\gamma\Gamma_{\widetilde{D}^-}=\Gamma'_{\widetilde{D}^+}\beta'\gamma\Gamma_{\widetilde{D}^-}$ if and only if there exists $\alpha\in\Gamma_{\widetilde{D}^-}$ such that $\beta'\gamma\alpha\gamma^{-1}\beta^{-1}\in\Gamma'_{\widetilde{D}^+}$, hence if and only if $\gamma\Gamma_{\widetilde{D}^-}\gamma^{-1}\cap(\beta')^{-1}\Gamma'_{\widetilde{D}^+}\beta$ is nonempty. Since the classes $\Gamma'_{\widetilde{D}^+}\beta$ and $\Gamma'_{\widetilde{D}^+}\beta'$ are distinct, this implies that $\Gamma_{\gamma\widetilde{D}^-}\cap\Gamma_{\mathscr{H}}\neq\{\mathrm{id}\}$. Since $[\gamma]\notin F$, the multiplicity $m_{\gamma\widetilde{D}^-,\mathscr{H}}$ defined in Equation (7) of the common perpendicular between $\gamma\widetilde{D}^-$ and $\mathscr{H}$ is different from 1. By Equation (17) applied with $(\mathscr{H},\widetilde{D}^-)$ instead of $(\widetilde{D}^+,\widetilde{D}^-)$, outside a number of elements $[\gamma]\in\Gamma_{\mathscr{H}}\backslash\Gamma/\Gamma_{\widetilde{D}^-}\smallsetminus F$ that is a $\mathrm{O}(e^{\delta s})$, the map $\Gamma'_{\widetilde{D}^+}\beta\mapsto\Gamma'_{\widetilde{D}^+}\beta\gamma\Gamma_{\widetilde{D}^-}$ is injective. Note that the canonical map $\Gamma_{\mathscr{H}}\to\Gamma'_{\widetilde{D}^+}\backslash\Gamma_{\mathscr{H}}$ is m^+-to-1.

By Equation (21) that controls the contribution of the double classes $[\gamma]\in\Gamma_{\mathscr{H}}\backslash\Gamma/\Gamma_{\widetilde{D}^-}$ that are in F, and since the two conditions below on $([\gamma],\beta)$ are invariant under multiplying β on the left by any element of $\Gamma'_{\widetilde{D}^+}$, Equation (23) hence becomes

$$\begin{aligned}\mathscr{N}_{D^-,\Omega_+}(s)=\frac{1}{m^+}\operatorname{Card}\Big\{([\gamma],\beta)\in(\Gamma_{\mathscr{H}}\backslash\Gamma/\Gamma_{\widetilde{D}^-})\times\Gamma_{\mathscr{H}}:\ &\begin{array}{l}0<d(\beta\gamma\widetilde{D}^-,\widetilde{D}^+)\leqslant s\\ z_{\beta\gamma\widetilde{D}^-,\,\widetilde{D}^+}\in\widetilde{\ell}(\,]0,+\infty[\,)\end{array}\Big\}\\ &+\mathrm{O}(e^{\delta\,s})\,.\end{aligned}\tag{24}$$

Step 7. In this final step in the proof of Equation (10), we compute the contribution to the counting in Equation (24) of the elements in the main domain $(\Gamma_{\mathscr{H}}\backslash\Gamma/\Gamma_{\widetilde{D}^-}\smallsetminus F)\times(\Gamma_{\mathscr{H}}\smallsetminus F')$, and we conclude the proof of Equation (10). Most of the technical work is devoted to getting an error term.

For every $s>1$, let

$$\Sigma_s=\operatorname{Card}\Big\{([\gamma],\beta)\in(\Gamma_{\mathscr{H}}\backslash\Gamma/\Gamma_{\widetilde{D}^-}\smallsetminus F)\times(\Gamma_{\mathscr{H}}\smallsetminus F'):\ \begin{array}{l}s-1<d(\beta\gamma\widetilde{D}^-,\widetilde{D}^+)\leqslant s\\ z_{\beta\gamma\widetilde{D}^-,\,\widetilde{D}^+}\in\widetilde{\ell}(\,]0,+\infty[\,)\end{array}\Big\}\,.$$

The second assumption above is superfluous, since by the definition of the set F' in Step 3, we have $z_{\beta\gamma\widetilde{D}^-,\,\widetilde{D}^+}\in\widetilde{\ell}(\,]0,+\infty[\,)$ whenever $\beta\in\Gamma_{\mathscr{H}}\smallsetminus F'$ and $[\gamma]\in\Gamma_{\mathscr{H}}\backslash\Gamma/\Gamma_{\widetilde{D}^-}\smallsetminus F$.

Let $\eta\in\,]0,1[$ (that will tend to 0 at the end of the proof). Recall that $d(\gamma\widetilde{D}^-,\mathscr{H})>0$ as $[\gamma]\in\Gamma_{\mathscr{H}}\backslash\Gamma/\Gamma_{\widetilde{D}^-}\smallsetminus F$. By summing over thin slices with width η of the first factor elements, we have

$$\Sigma_s=\sum_{k=1}^{+\infty}\ \sum_{\substack{[\gamma]\in\Gamma_{\mathscr{H}}\backslash\Gamma/\Gamma_{\widetilde{D}^-}\smallsetminus F\\ (k-1)\eta<d(\gamma\widetilde{D}^-,\mathscr{H})\leqslant k\eta}}\ \sum_{\substack{\beta\in\Gamma_{\mathscr{H}}\smallsetminus F'\\ s-1<d(\beta\gamma\widetilde{D}^-,\widetilde{D}^+)\leqslant s}}1\,.\tag{25}$$

Let $k\in\mathbb{N}\smallsetminus\{0\}$ and $([\gamma],\beta)\in(\Gamma_{\mathscr{H}}\backslash\Gamma/\Gamma_{\widetilde{D}^-}\smallsetminus F)\times(\Gamma_{\mathscr{H}}\smallsetminus F')$ be such that

$$(k-1)\eta<d(\gamma\widetilde{D}^-,\mathscr{H})\leqslant k\eta\quad\text{and}\quad s-1<d(\beta\gamma\widetilde{D}^-,\widetilde{D}^+)\leqslant s\,.\tag{26}$$

Since M is real hyperbolic (so that $d_{\mathscr{H}}=d'_{\mathscr{H}}$), by the first claim of Lemma 4 (ii) applied with $D=\beta\gamma\widetilde{D}^-$ and $x_0=\beta p_\gamma$, whose assumptions are satisfied by Equation (19), and since $\mathscr{H}$ is invariant under $\beta^{-1} \in \Gamma_{\mathscr{H}}$, we have

$$d(\beta\gamma\widetilde{D}^-, \widetilde{D}^+) = d(\gamma\widetilde{D}^-, \mathscr{H}) + \ln(2\, d'_{\mathscr{H}}(\beta p_\gamma, \widetilde{\ell}(0))) \\ + \operatorname{O}\big(d'_{\mathscr{H}}(\beta p_\gamma, \widetilde{\ell}(0))^{-2}\, e^{-2\, d(\gamma\widetilde{D}^-, \mathscr{H})}\big)\,. \tag{27}$$

In particular, up to increasing the finite set F', we may assume that $d'_{\mathscr{H}}(\beta p_\gamma, \widetilde{\ell}(0)))$ is large enough so that $d(\gamma\widetilde{D}^-, \mathscr{H}) \leqslant d(\beta\gamma\widetilde{D}^-, \widetilde{\ell})$. Let $N = \lfloor \frac{s}{\eta} \rfloor$, so that we have $N\eta \leqslant s < (N+1)\eta$ and in the summation (25), we may restrict k to vary between 1 and $N+1$ for a majoration and between 1 and N for a minoration.

Since $d'_{\mathscr{H}}(\widetilde{\ell}(0), \beta p_\gamma) \geqslant c_{\mathbb{K}}$ by Equation (19), and since $e^{-2\, d(\gamma\widetilde{D}^-, \mathscr{H})} \leqslant 1$, Equations (27) and (26) give

$$d'_{\mathscr{H}}(\beta p_\gamma, \widetilde{\ell}(0)) = \frac{1}{2}\; e^{d(\beta\gamma\widetilde{D}^-,\, \widetilde{D}^+) - d(\gamma\widetilde{D}^-,\, \mathscr{H}) + \operatorname{O}(1)} = e^{s - k\eta + \operatorname{O}(1)}\,.$$

Thus $d'_{\mathscr{H}}(\beta p_\gamma, \widetilde{\ell}(0))^{-2}\, e^{-2\, d(\gamma\widetilde{D}^-, \mathscr{H})} = \operatorname{O}(e^{-2\,s+2k\eta})\, \operatorname{O}(e^{-2k\eta}) = \operatorname{O}(e^{-2\,s})$. Bootstrapping this in Equation (27), and by Equation (26), we have

$$\frac{1}{2}\; e^{s-1-k\eta+\operatorname{O}(e^{-2\,s})} \leqslant d'_{\mathscr{H}}(\beta p_\gamma, \widetilde{\ell}(0)) \leqslant \frac{1}{2}\; e^{s-(k-1)\eta+\operatorname{O}(e^{-2\,s})}\,. \tag{28}$$

Conversely (this will be used only at the end of Step 7), if we had

$$\frac{1}{2}\; e^{s-1-(k-1)\eta+\operatorname{O}(e^{-2\,s})} \leqslant d'_{\mathscr{H}}(\beta p_\gamma, \widetilde{\ell}(0)) \leqslant \frac{1}{2}\; e^{s-k\eta+\operatorname{O}(e^{-2\,s})}\,, \tag{29}$$

for an appropriate function $\operatorname{O}(\cdot)$ that is independent of η, while still having the inequalities $(k-1)\eta < d(\gamma\widetilde{D}^-, \mathscr{H}) \leqslant k\eta$, then by Equation (27), we would have the inequalities $s-1 < d(\beta\gamma\widetilde{D}^-, \widetilde{D}^+) \leqslant s$. Note that the right and left hand sides of Equations (28) and (29) differ by a multiplicative factor $e^{\operatorname{O}(\eta)} = 1 + \operatorname{O}(\eta)$ as η tends to 0.

It follows from Equation (28), by Step 4 and Equation (20), that we have (with a function $\operatorname{O}(\cdot)$ that is independent of η and $[\gamma]$)

$$\begin{aligned} &\operatorname{Card}\{\beta \in \Gamma_{\mathscr{H}} \smallsetminus F' : s-1 < d(\beta\gamma\widetilde{D}^-, \widetilde{D}^+) \leqslant s\} \\ &\leqslant \Phi_{[\gamma]}\big(\frac{1}{2}\; e^{s-(k-1)\eta+\operatorname{O}(e^{-2\,s})}\big) - \Phi_{[\gamma]}\big(\frac{1}{2}\; e^{s-1-k\eta+\operatorname{O}(e^{-2\,s})}\big) \\ &= \frac{\Xi_{\widetilde{M}}}{2^\delta\, \|\sigma^-_{\mathscr{V}_+}\|} e^{\delta s - \delta k\eta}\big(e^{\delta\eta+\operatorname{O}(e^{-s})} - e^{-\delta+\operatorname{O}(e^{-s})}\big) + \operatorname{O}\big(e^{(s-k\eta)(\delta-1)}\big)\,. \end{aligned} \tag{30}$$

Let $C_1 = \frac{\Xi_{\widetilde{M}}}{2^\delta\, \|\sigma^-_{\mathscr{V}_+}\|}\, (e^{\delta\eta+\operatorname{O}(e^{-s})} - e^{-\delta+\operatorname{O}(e^{-s})})$ and let $f : [0, +\infty[\; \to \mathbb{R}$ be an appropriate function such that

$$f : t \mapsto C_1\, e^{\delta s - \delta t\eta} + \operatorname{O}(e^{(s-t\eta)(\delta-1)})\,.$$

Its derivative can be chosen to be $f' : t \mapsto -\delta\, \eta\, C_1\, e^{\delta s - \delta t\eta} + \operatorname{O}(\eta\, e^{(s-t\eta)(\delta-1)})$. Since $s = N\eta + \operatorname{O}(\eta)$, we have $f(N+1) = \operatorname{O}(1)$. For every $k \in \mathbb{N}$, let

$$a_k = \operatorname{Card}\{[\gamma] \in \Gamma_{\mathscr{H}}\backslash\Gamma/\Gamma_{\widetilde{D}^-} \smallsetminus F : (k-1)\eta < d(\gamma\widetilde{D}^-, \mathscr{H}) \leqslant k\eta\}\,.$$

Note that $a_0 = 0$ by the definition of F. By [PP5, Theo. 15 (2)], which can be applied since its assumption on the exponential decay of correlations is satisfied by [LP] since M

is real hyperbolic with finite volume, and by a further regularisation process in order to remove the smoothness assumption on ∂D^-, there exists $\kappa''>0$ and a function $\operatorname{O}(\cdot)$ that is independent of η such that for every $t\geqslant 0$, we have

$$\sum_{k=0}^{t} a_k = \frac{\|\sigma^+_{D^-}\|\,\|\sigma^-_{\mathscr{V}_+}\|}{\delta\,\|m_{\rm BM}\|}\,e^{\delta t\eta}+\operatorname{O}(e^{(\delta-\kappa'')t\eta})\,. \tag{31}$$

Let $C_2=\frac{\|\sigma^+_{D^-}\|\,\|\sigma^-_{\mathscr{V}_+}\|}{\delta\,\|m_{\rm BM}\|}$. By Equations (25) and (30), for an appropriate function f, by Abel's summation formula, by Equation (31) and again since $N\eta=s+\operatorname{O}(\eta)$, we have

$$\begin{aligned}
\Sigma_s&\leqslant\sum_{k=0}^{N+1}a_k\,f(k)=\big(\sum_{k=0}^{N+1}a_k\big)f(N+1)-\int_0^{N+1}\big(\sum_{k=0}^{t}a_k\big)f'(t)\,dt\\
&=\operatorname{O}(e^{\delta\,s})+\int_0^{N+1}\big(C_2\,e^{\delta t\eta}+\operatorname{O}(e^{(\delta-\kappa'')t\eta})\big)\big(\delta\,\eta\,C_1\,e^{\delta s-\delta t\eta}+\operatorname{O}(\eta\,e^{(s-t\eta)(\delta-1)})\big)\,dt\\
&=\operatorname{O}(e^{\delta\,s})+\delta\,C_2\,C_1\,(N+1)\,\eta\,e^{\delta s}\\
&\qquad\qquad+\operatorname{O}\big(e^{\delta\,s}\int_0^{N+1}\eta\,e^{-\kappa''t\eta}\,dt\big)+\operatorname{O}\big(e^{s(\delta-1)}\int_0^{N+1}\eta\,e^{t\eta}\,dt\big)\\
&=\delta\,C_2\,C_1\,(s+\operatorname{O}(\eta))\,e^{\delta s}+\operatorname{O}(e^{\delta\,s})\,.
\end{aligned}$$

Replacing C_1 and C_2 by their values, and letting η tend to 0, we hence have

$$\Sigma_s\leqslant\frac{\|\sigma^+_{D^-}\|\;\Xi_{\widetilde{M}}}{2^\delta\,\|m_{\rm BM}\|}\;s\,e^{\delta\,s}(1-e^{-\delta})+\operatorname{O}(e^{\delta\,s})\,.$$

The same lower bound is obtained similarly, replacing Equation (28) by Equation (29). By a summation, we have

$$\begin{aligned}
&\operatorname{Card}\Big\{([\gamma],\beta)\in(\Gamma_{\mathscr{H}}\backslash\Gamma/\Gamma_{\widetilde{D}^-}\smallsetminus F)\times(\Gamma_{\mathscr{H}}\smallsetminus F'):\;\begin{array}{l}0<d(\beta\gamma\widetilde{D}^-,\widetilde{D}^+)\leqslant s\\ z_{\beta\gamma\widetilde{D}^-,\,\widetilde{D}^+}\in\widetilde{\ell}(\,]0,+\infty[\,)\end{array}\Big\}\\
&=\frac{\|\sigma^+_{D^-}\|\;\Xi_{\widetilde{M}}}{2^\delta\,\|m_{\rm BM}\|}\;s\,e^{\delta\,s}+\operatorname{O}(e^{\delta s})\,.
\end{aligned}\tag{32}$$

By separating the counting domain $(\Gamma_{\mathscr{H}}\backslash\Gamma/\Gamma_{\widetilde{D}^-})\times\Gamma_{\mathscr{H}}$ as the disjoint union of $F\times\Gamma_{\mathscr{H}}$, of $(\Gamma_{\mathscr{H}}\backslash\Gamma/\Gamma_{\widetilde{D}^-}\smallsetminus F)\times F'$ and of $(\Gamma_{\mathscr{H}}\backslash\Gamma/\Gamma_{\widetilde{D}^-}\smallsetminus F)\times(\Gamma_{\mathscr{H}}\smallsetminus F')$, Equation (10) finally follows from Equations (24), (21), (22) and (32).

Case 2. Let us now assume that ℓ is weakly reciprocal. We then have $\Omega_-=\Omega_+$. Hence $\mathscr{N}_{D^-,\Omega_-}(t)=\mathscr{N}_{D^-,\Omega_+}(t)$ and

$$\mathscr{N}_{D^-,\Omega_+}(t)\leqslant\mathscr{N}_{D^-,D^+}(t)\leqslant\mathscr{N}_{D^-,\Omega_+}(t)+\mathscr{N}_{D^-,\Omega_0}(t) \tag{33}$$

for every $t\geqslant 0$. Let us prove that Equation (10) is still satisfied. Since $\iota^+_{\rm rec}=1$ when ℓ is weakly reciprocal, this will prove as in Case 1, replacing the call to Equation (9) by a call to Equation (33), that Equation (13) is still satisfied. Then by the same computations as in Case 1, Theorem 5 when ℓ is weakly reciprocal will follow.

Since ℓ is weakly reciprocal, there exists an element $\iota_{\widetilde{D}^+} \in \Gamma$ such that we have $\iota_{\widetilde{D}^+}\,\widetilde{\ell}(\,]0,+\infty[\,) = \widetilde{\ell}(\,]-\infty,t_-[\,)$. By the definition of Ω_+, by the commutativity of the diagram (1) and since the family $(H_\xi)_{\xi\in\mathrm{Par}_\Gamma}$ is precisely invariant, we have

$$\begin{aligned}\partial^1_-\widetilde{D}^+\cap\widetilde{p}^{-1}(\Omega_+) &= \partial^1_-\widetilde{D}^+\cap\widetilde{p}^{-1}\big(p_\bullet{}^{-1}(p\circ\widetilde{\ell}(\,]0,+\infty[\,))\big) = \partial^1_-\widetilde{D}^+\cap\Gamma\,\widetilde{p}_\bullet{}^{-1}(\widetilde{\ell}(\,]0,+\infty[\,))\\ &= \partial^1_-\widetilde{D}^+\cap\big(\widetilde{p}_\bullet{}^{-1}(\widetilde{\ell}(\,]0,+\infty[\,))\cup\iota_{\widetilde{D}^+}\widetilde{p}_\bullet{}^{-1}(\widetilde{\ell}(\,]0,+\infty[\,))\big)\\ &= \partial^1_-\widetilde{D}^+\cap\widetilde{p}_\bullet{}^{-1}(\widetilde{\ell}(\,]-\infty,t_-[\,\cup\,]0,+\infty[\,))\,.\end{aligned}$$

Hence, as in Step 1, we now have

$$\mathscr{N}_{D^-,\,\Omega_+}(s) = \sum_{[\gamma]\in\Gamma_{\widetilde{D}^+}\backslash\Gamma/\Gamma_{\widetilde{D}^-}\,:\,0<d(\gamma\widetilde{D}^-,\widetilde{D}^+)\leqslant s,\;z_{\gamma\widetilde{D}^-,\,\widetilde{D}^+}\in\widetilde{\ell}(\,]-\infty,t_-[\,\cup\,]0,+\infty[\,)} m_{\gamma\widetilde{D}^-,\widetilde{D}^+}\,. \tag{34}$$

Since ℓ is weakly reciprocal, the intersection $\Gamma'_{\widetilde{D}^+} = \Gamma_{\widetilde{D}^+}\cap\Gamma_{\mathscr{H}}$ now has index 2 in $\Gamma_{\widetilde{D}^+}$. Given a double class $[\gamma]\in\Gamma_{\widetilde{D}^+}\backslash\Gamma/\Gamma_{\widetilde{D}^-}$ such that $0<d(\gamma\widetilde{D}^-,\widetilde{D}^+)\leqslant s$, its preimage by the canonical projection $\Gamma'_{\widetilde{D}^+}\backslash\Gamma/\Gamma_{\widetilde{D}^-}\to\Gamma_{\widetilde{D}^+}\backslash\Gamma/\Gamma_{\widetilde{D}^-}$ consists in the set $\{\,\Gamma'_{\widetilde{D}^+}\gamma\Gamma_{\widetilde{D}^-},\Gamma'_{\widetilde{D}^+}\iota_{\widetilde{D}^+}\gamma\Gamma_{\widetilde{D}^-}\,\}$. We have $\Gamma'_{\widetilde{D}^+}\gamma\Gamma_{\widetilde{D}^-} = \Gamma'_{\widetilde{D}^+}\iota_{\widetilde{D}^+}\gamma\Gamma_{\widetilde{D}^-}$ if and only if there exists $\alpha\in\Gamma_{\widetilde{D}^-}$ such that $\iota_{\widetilde{D}^+}\gamma\alpha\gamma^{-1}\in\Gamma'_{\widetilde{D}^+}$, hence if and only if $\Gamma_{\gamma\widetilde{D}^-}\cap\iota^{-1}_{\widetilde{D}^+}\Gamma'_{\widetilde{D}^+}$ is nonempty. Since $\iota^{-1}_{\widetilde{D}^+}\Gamma'_{\widetilde{D}^+}$ is not the trivial class in $\Gamma_{\widetilde{D}^+}/\Gamma'_{\widetilde{D}^+}$, this implies that $m_{\gamma\widetilde{D}^-,\widetilde{D}^+}$ is different from 1. Hence by Equation (17), the canonical map $\Gamma'_{\widetilde{D}^+}\backslash\Gamma/\Gamma_{\widetilde{D}^-}\to\Gamma_{\widetilde{D}^+}\backslash\Gamma/\Gamma_{\widetilde{D}^-}$ is 2-to-1 outside a number $\mathrm{O}(e^{\delta s})$ of elements, and exactly one $\Gamma'_{\widetilde{D}^+}\gamma\Gamma_{\widetilde{D}^-}$ of the two preimages satisfies that $z_{\gamma\widetilde{D}^-,\widetilde{D}^+}$ belongs to $\widetilde{\ell}(\,]0,+\infty[\,)$. Thus, as in Steps 1 and 2, we have

$$\mathscr{N}_{D^-,\,\Omega_+}(s) = \mathrm{Card}\Big\{\gamma\in\Gamma'_{\widetilde{D}^+}\backslash\Gamma/\Gamma_{\widetilde{D}^-}\,:\,\begin{array}{l}0<d(\gamma\widetilde{D}^-,\widetilde{D}^+)\leqslant s\\ z_{\gamma\widetilde{D}^-,\,\widetilde{D}^+}\in\widetilde{\ell}(\,]0,+\infty[\,)\end{array}\Big\}+\mathrm{O}(e^{\delta\,s})\,, \tag{35}$$

that is, Equation (23) is still valid. As in Step 6, we therefore have

$$\begin{aligned}\mathscr{N}_{D^-,\,\Omega_+}(s) = \frac{1}{m^+}\,\mathrm{Card}\Big\{([\gamma],\beta)\in(\Gamma_{\mathscr{H}}\backslash\Gamma/\Gamma_{\widetilde{D}^-})\times\Gamma_{\mathscr{H}}\,:\,&\begin{array}{l}0<d(\beta\gamma\widetilde{D}^-,\widetilde{D}^+)\leqslant s\\ z_{\beta\gamma\widetilde{D}^-,\,\widetilde{D}^+}\in\widetilde{\ell}(\,]0,+\infty[\,)\end{array}\Big\}\\ +\,\mathrm{O}(e^{\delta\,s})\,,&\end{aligned} \tag{36}$$

that is, Equation (24) is still valid. The remainder of the proof of Equation (10), that is, its Step 7, now proceeds exactly as in Case 1. □

Theorem 6. *Let M be a noncompact finite volume complete connected real hyperbolic good orbifold of dimension n. Let D^+ and D^- be the images of two divergent geodesics in M. Then, as $s\to+\infty$, we have*

$$\mathscr{N}_{D^-,D^+}(s) = \frac{(n-1)\,\pi^{\frac{n}{2}-1}\,\Gamma(\frac{n}{2})\,\iota_{\mathrm{rec}}(D^-)\,\iota_{\mathrm{rec}}(D^+)}{2^{n+1}\,\Gamma(\frac{n+1}{2})^2\,m(D^-)\,m(D^+)\,\;\mathrm{Vol}\,M}\;s^2\,e^{(n-1)\,s}+\mathrm{O}\left(s\,e^{(n-1)\,s}\right).$$

Proof. The strategy is similar to the one we used in the proof of Theorem 5, except that we will now disintegrate the study of $\mathscr{N}_{D^-,D^+}$ over the study of the number $\mathscr{N}_{\mathscr{V}_\pm,D^+}$ of common perpendiculars starting from a Margulis neighbourhood $\mathscr{V}_\pm$ of an end of D^- and arriving at D^+, and replace the call to [PP5] in Equations (12), (22) and (31) by a call to Theorem 5 that we just proved.

The notation is now the following one (and differs from the one at the beginning of the proof of Theorem 5). Let ℓ be a divergent geodesic in M whose image is D^-. Let $\mathscr{V}_-$ and $\mathscr{V}_+$ be the initial and terminal Margulis neighbourhoods of ℓ. Let t_- be the first exit time of ℓ from $\mathscr{V}_-$ and let t_+ be the last entry time of ℓ into $\mathscr{V}_+$. We may assume that $t_+ = 0$. Let $\widetilde{\ell}$ be a lift of ℓ in $\widetilde{M}$, and let $\widetilde{D}^-$ be the image of $\widetilde{\ell}$. For simplicity, let $m^\pm = m(D^\pm)$ and $\iota_{\rm rec}^\pm = \iota_{\rm rec}(D^\pm)$. Let

$$\begin{aligned}\Omega_- &= \big\{v \in \partial^1_+ D^- : p_\bullet(v) \in \ell(\,]-\infty, t_-[\,)\big\}\,,\\ \Omega_0 &= \big\{v \in \partial^1_+ D^- : p_\bullet(v) \in \ell([t_-, 0])\big\} \quad \text{and}\\ \Omega_+ &= \big\{v \in \partial^1_+ D^- : p_\bullet(v) \in \ell(\,]0, +\infty[\,)\big\}\,.\end{aligned}$$

We denote by $\mathscr{H} = H_{\widetilde{\ell}(+\infty)}$ the horoball of the family $(H_\xi)_{\xi\in{\rm Par}_\Gamma}$ with point at infinity $\widetilde{\ell}(+\infty)$, that is a lift of $\mathscr{V}_+$. Let $\widetilde{D}^+$ be a geodesic line in $\widetilde{M}$ whose image in M is D^+.

Case 1. Let us first assume that ℓ is not weakly reciprocal. As for Equation (9), the subsets Ω_- and Ω_+ are disjoint and for every $s \geqslant 0$, we have

$$\mathscr{N}_{\Omega_-,D^+}(s) + \mathscr{N}_{\Omega_+,D^+}(s) \leqslant \mathscr{N}_{D^-,D^+}(s) \leqslant \mathscr{N}_{\Omega_-,D^+}(s) + \mathscr{N}_{\Omega_0,D^+}(s) + \mathscr{N}_{\Omega_+,D^+}(s)\,. \tag{37}$$

We shall prove that as $s \to +\infty$, we have

$$\mathscr{N}_{\Omega_+,D^+}(s) = \frac{\delta\ \iota_{\rm rec}^+\ \Xi_{\widetilde{M}}^{\ 2}}{2^{2\delta+1}\ m^-\ m^+\ \|m_{\rm BM}\|}\ s^2\, e^{\delta\, s} + {\rm O}(s\, e^{\delta\, s})\,. \tag{38}$$

By the same argument as in the proof of Equation (11), we will also have

$$\mathscr{N}_{\Omega_-,D^+}(s) = \frac{\delta\ \iota_{\rm rec}^+\ \Xi_{\widetilde{M}}^{\ 2}}{2^{2\delta+1}\ m^-\ m^+\ \|m_{\rm BM}\|}\ s^2\, e^{\delta\, s} + {\rm O}(s\, e^{\delta\, s})\,. \tag{39}$$

Let $D_0 = \ell([t_-, t_+])$, which is a compact nonempty properly immersed locally convex subset of M, hence has a nonzero finite outer skinning measure. By Theorem 5, as $s \to +\infty$, we therefore have

$$\mathscr{N}_{\Omega_0,D^+}(s) \leqslant \mathscr{N}_{D_0,D^+}(s) = {\rm O}(s\, e^{\delta\, s})\,. \tag{40}$$

Thus by Equations (37), (38), (39) and (40), since $\iota_{\rm rec}^- = 2$ as ℓ is not weakly reciprocal, as $s \to +\infty$, we will have

$$\mathscr{N}_{D^-,D^+}(s) = \frac{\delta\ \iota_{\rm rec}^-\ \iota_{\rm rec}^+\ \Xi_{\widetilde{M}}^{\ 2}}{2^{2\delta+1}\ m^-\ m^+\ \|m_{\rm BM}\|}\ s^2\, e^{\delta\, s} + {\rm O}(s\, e^{\delta\, s})\,. \tag{41}$$

As M is finite volume real hyperbolic, we have $\delta = n - 1$, and Theorem 6 will follow from Equation (41) using Equations (14) and (15), once we have proven Equation (38).

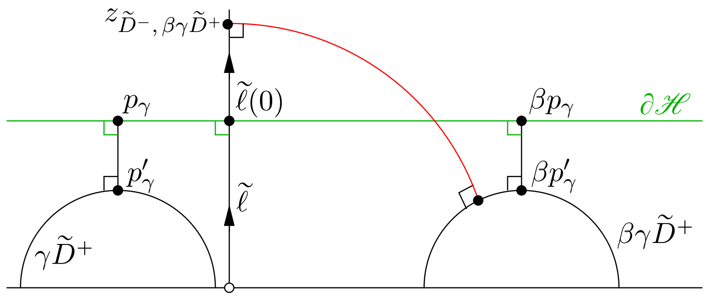
The remainder of the proof is devoted to proving Equation (38). For every element $\gamma \in \Gamma$ such that $d(\widetilde{D}^-, \gamma\widetilde{D}^+) > 0$, we now denote by $z_{\widetilde{D}^-,\gamma\widetilde{D}^+} \in \widetilde{D}^+$ the origin of the common perpendicular from $\widetilde{D}^-$ to $\gamma\widetilde{D}^+$. As in Steps 1 and 2 in the proof of Theorem 5, since ${\rm O}(s^2 e^{(\delta-1)s}) = {\rm O}(e^{\delta\, s})$, for every $s \geqslant 0$, we have

$$\begin{aligned}\mathscr{N}_{\Omega_+,D^+}(s) &= \sum_{[\gamma]\in\Gamma_{\widetilde{D}^-}\backslash\Gamma/\Gamma_{\widetilde{D}^+}\,:\,0<d(\widetilde{D}^-,\gamma\widetilde{D}^+)\leqslant s\,,\ z_{\widetilde{D}^-,\gamma\widetilde{D}^+}\in\widetilde{\ell}(\,]0,+\infty[\,)} m_{\widetilde{D}^-,\gamma\widetilde{D}^+}\\ &= {\rm Card}\left\{\gamma \in \Gamma_{\widetilde{D}^-}\backslash\Gamma/\Gamma_{\widetilde{D}^+}\ :\ \begin{array}{l} 0 < d(\widetilde{D}^-, \gamma\widetilde{D}^+) \leqslant s\\ z_{\widetilde{D}^-,\gamma\widetilde{D}^+} \in \widetilde{\ell}(\,]0, +\infty[\,)\end{array}\right\} + {\rm O}(e^{\delta\, s})\,.\end{aligned} \tag{42}$$

As in the first part of Step 3 in the proof of Theorem 5, we now define

$$F = \{[\gamma] \in \Gamma_{\mathscr{H}} \backslash \Gamma / \Gamma_{\widetilde{D}^+} : \mathscr{H} \cap \gamma \widetilde{D}^+ \neq \varnothing\}$$

and for every $[\gamma] \in \Gamma_{\mathscr{H}} \backslash \Gamma / \Gamma_{\widetilde{D}^+} \smallsetminus F$, we now denote by $p_\gamma \in \partial \mathscr{H}$ and $p'_\gamma \in \partial(\gamma \widetilde{D}^+)$ the two endpoints of the common perpendicular between $\mathscr{H}$ and $\gamma \widetilde{D}^+$.



As in the second part of Step 3 in the proof of Theorem 5 (recalling that $c_{\mathbb{K}} = 2$ in the real hyperbolic case), there exists $c_5 > 0$ and a choice of representatives γ in $[\gamma] \in \Gamma_{\mathscr{H}} \backslash \Gamma / \Gamma_{\widetilde{D}^+}$ such that $d'_{\mathscr{H}}(p_\gamma, \widetilde{\ell}(0)) \leqslant c_5$. We now define

$$F' = \big\{\beta \in \Gamma_{\mathscr{H}} : d'_{\mathscr{H}}(\widetilde{\ell}(0), \beta\, \widetilde{\ell}(0)) < c_{\mathbb{K}} + c_5\big\}\,.$$

Since M is real hyperbolic and by the second claim of Lemma 4 (ii) applied with $D = \beta\gamma\widetilde{D}^+$ and $x_0 = \beta p_\gamma$, if $[\gamma] \in \Gamma_{\mathscr{H}} \backslash \Gamma / \Gamma_{\widetilde{D}^+} \smallsetminus F$ and $\beta \in \Gamma_{\mathscr{H}} \smallsetminus F'$, then $z_{\widetilde{D}^-,\, \beta\gamma\widetilde{D}^+} \in \widetilde{\ell}(\,]0, +\infty[\,)$ (see the above picture). For all $[\gamma] \in \Gamma_{\mathscr{H}} \backslash \Gamma / \Gamma_{\widetilde{D}^+} \smallsetminus F$ and $t \geqslant 0$, let us now define

$$\Phi_{[\gamma]}(t) = \operatorname{Card}\{\beta \in \Gamma_{\mathscr{H}} : d'_{\mathscr{H}}(\widetilde{\ell}(0), \beta p_\gamma) \leqslant t\}\,.$$

As in Step 4 in the proof of Theorem 5, we have

$$\Phi_{[\gamma]}(t) = \frac{\Xi_{\widetilde{M}}}{\|\sigma^+_{\mathscr{V}_+}\|}\, t^\delta + \mathrm{O}(t^{\delta-1})\,. \tag{43}$$

As in the first part of Step 5 in the proof of Theorem 5, we have

$$\operatorname{Card}\left\{([\gamma], \beta) \in F \times \Gamma_{\mathscr{H}} : \begin{array}{l} 0 < d(\widetilde{D}^-, \beta\gamma\widetilde{D}^+) \leqslant s \\ z_{\widetilde{D}^-,\, \beta\gamma\widetilde{D}^+} \in \widetilde{\ell}(\,]0, +\infty[\,) \end{array}\right\} = \mathrm{O}(e^{\delta s})\,. \tag{44}$$

Since $\mathscr{V}_+$ has finite outer skinning measure and since D^+ is a divergent geodesic, by Theorem 5, we have

$$\mathscr{N}_{\mathscr{V}_+, D^+}(s) = \mathrm{O}(s e^{\delta\, s})\,. \tag{45}$$

Hence as in the second part of Step 5 in the proof of Theorem 5, we have

$$\operatorname{Card}\left\{([\gamma], \beta) \in (\Gamma_{\mathscr{H}} \backslash \Gamma / \Gamma_{\widetilde{D}^+} \smallsetminus F) \times F' : \begin{array}{l} 0 < d(\widetilde{D}^-, \beta\gamma\widetilde{D}^+) \leqslant s \\ z_{\widetilde{D}^-,\, \beta\gamma\widetilde{D}^+} \in \widetilde{\ell}(\,]0, +\infty[\,) \end{array}\right\}$$
$$= \mathrm{O}(s\, e^{\delta s})\,. \tag{46}$$

As in Step 6 in the proof of Theorem 5, since ℓ is not weakly reciprocal, the stabilizer $\Gamma_{\widetilde{D}^-}$ of $\widetilde{D}^-$ coincides with its pointwise stabilizer $\Gamma'_{\widetilde{D}^-} = \Gamma_{\widetilde{D}^-} \cap \Gamma_{\mathscr{H}}$, hence has order m^- and is contained in $\Gamma_{\mathscr{H}}$. As for Equation (24), disintegrating the counting function of Equation (42) by the canonical map $\Gamma'_{\widetilde{D}^-}\backslash\Gamma/\Gamma_{\widetilde{D}^+} \to \Gamma_{\mathscr{H}}\backslash\Gamma/\Gamma_{\widetilde{D}^+}$, Equation (42) gives

$$\begin{aligned}\mathscr{N}_{\Omega_+,D^+}(s) = \frac{1}{m^-}\,\mathrm{Card}\left\{([\gamma],\beta)\in(\Gamma_{\mathscr{H}}\backslash\Gamma/\Gamma_{\widetilde{D}^+})\times\Gamma_{\mathscr{H}}:\ \begin{array}{l}0<d(\widetilde{D}^-,\beta\gamma\widetilde{D}^+)\leqslant s\\ z_{\widetilde{D}^-,\,\beta\gamma\widetilde{D}^+}\in\widetilde{\ell}(\,]0,+\infty[\,)\end{array}\right\}\\ +\,\mathrm{O}(e^{\delta\, s})\,.\end{aligned}\tag{47}$$

As in the beginning of Step 7 in the proof of Theorem 5, with $s\in\,]1,+\infty[$ large enough, $\eta\in\,]0,1[$ small enough and $N=\lfloor\frac{s}{\eta}\rfloor$, if

$$\Sigma_s = \mathrm{Card}\left\{([\gamma],\beta)\in(\Gamma_{\mathscr{H}}\backslash\Gamma/\Gamma_{\widetilde{D}^+}\smallsetminus F)\times(\Gamma_{\mathscr{H}}\smallsetminus F'):\ \begin{array}{l}s-1<d(\widetilde{D}^-,\beta\gamma\widetilde{D}^+)\leqslant s\\ z_{\widetilde{D}^-,\,\beta\gamma\widetilde{D}^+}\in\widetilde{\ell}(\,]0,+\infty[\,)\end{array}\right\},$$

then

$$\Sigma_s = \sum_{k=1}^{N+1}\ \sum_{\substack{[\gamma]\in\Gamma_{\mathscr{H}}\backslash\Gamma/\Gamma_{\widetilde{D}^+}\smallsetminus F\\ (k-1)\eta<d(\gamma\widetilde{D}^+,\,\mathscr{H})\leqslant k\eta}}\ \sum_{\substack{\beta\in\Gamma_{\mathscr{H}}\smallsetminus F'\\ s-1<d(\widetilde{D}^-,\,\beta\gamma\widetilde{D}^+)\leqslant s}} 1\,.\tag{48}$$

Let $k\in[\![1,N+1]\!]$ and $([\gamma],\beta)\in(\Gamma_{\mathscr{H}}\backslash\Gamma/\Gamma_{\widetilde{D}^+}\smallsetminus F)\times(\Gamma_{\mathscr{H}}\smallsetminus F')$ be such that

$$(k-1)\eta<d(\gamma\widetilde{D}^+,\mathscr{H})\leqslant k\eta\quad\text{ and }\quad s-1<d(\widetilde{D}^-,\beta\gamma\widetilde{D}^+)\leqslant s\,.$$

Then since M is real hyperbolic, by the first claim of Lemma 4 (ii) applied with $D=\beta\gamma\widetilde{D}^+$ and $x_0=\beta p_\gamma$, we have

$$\begin{aligned}d(\widetilde{D}^-,\beta\gamma\widetilde{D}^+) = d(\gamma\widetilde{D}^+,\mathscr{H}) + \ln(2\,d'_{\mathscr{H}}(\beta p_\gamma,\widetilde{\ell}(0)))\\ +\,\mathrm{O}\left(d'_{\mathscr{H}}(\beta p_\gamma,\widetilde{\ell}(0))^{-2}\,e^{-2\,d(\gamma\widetilde{D}^+,\mathscr{H})}\right).\end{aligned}\tag{49}$$

As in the middle part of Step 7 in the proof of Theorem 5, up to increasing F', we have

$$\frac{1}{2}\,e^{s-1-k\eta+\mathrm{O}(e^{-2\,s})}\leqslant d'_{\mathscr{H}}(\beta p_\gamma,\widetilde{\ell}(0))\leqslant\frac{1}{2}\,e^{s-(k-1)\eta+\mathrm{O}(e^{-2\,s})}\,.\tag{50}$$

By Equations (50) and (43), with functions $\mathrm{O}(\cdot)$ independent of η and $[\gamma]$, we have

$$\begin{aligned}&\mathrm{Card}\{\beta\in\Gamma_{\mathscr{H}}\smallsetminus F' : s-1<d(\widetilde{D}^-,\beta\gamma\widetilde{D}^+)\leqslant s\}\\ &\leqslant\frac{\Xi_{\widetilde{M}}}{2^\delta\|\sigma^+_{\mathscr{V}_+}\|}e^{\delta\, s-\delta k\eta}\big(e^{\delta\eta+\mathrm{O}(e^{-s})}-e^{-\delta+\mathrm{O}(e^{-s})}\big)+\mathrm{O}(e^{(s-k\eta)(\delta-1)})\,.\end{aligned}\tag{51}$$

Since M is real hyperbolic and since $\mathscr{V}_+$ has finite outer skinning measure, let us apply Theorem 5 with $D^-=\mathscr{V}_+$, and more precisely Equation (13) with the help of Step 2 in the proof of Theorem 5 in order to deal with the multiplicities $m_{\gamma\mathscr{V}_+,\gamma'\widetilde{D}^+}$ not equal to 1. Then, with a function $\mathrm{O}(\cdot)$ that is independent of η, if we now define, for every $k\in\mathbb{N}$,

$$a_k = \mathrm{Card}\big\{[\gamma]\in\Gamma_{\mathscr{H}}\backslash\Gamma/\Gamma_{\widetilde{D}^+}\smallsetminus F : (k-1)\eta<d(\gamma\widetilde{D}^+,\mathscr{H})\leqslant k\eta\big\}\,,$$

then for every $t\geqslant 0$, we have

$$\sum_{k=0}^{t} a_k = \frac{\iota^+_{\mathrm{rec}}\,\|\sigma^+_{\mathscr{V}_+}\|\ \Xi_{\widetilde{M}}}{2^\delta\ m^+\ \|m_{\mathrm{BM}}\|}\ t\,\eta\ e^{\delta t\eta}+\mathrm{O}\left(e^{\delta t\eta}\right).\tag{52}$$

Let us now define

$$C_1 = \frac{\Xi_{\widetilde{M}}}{2^\delta \, \|\sigma^+_{\mathscr{V}_+}\|} \big(e^{\delta\eta + \mathrm{O}(e^{-s})} - e^{-\delta + \mathrm{O}(e^{-s})}\big)\,, \qquad C_2 = \frac{\iota^+_{\mathrm{rec}} \, \|\sigma^+_{\mathscr{V}_+}\| \, \Xi_{\widetilde{M}}}{2^\delta \, m^+ \, \|m_{\mathrm{BM}}\|}\,,$$

and a function $f : t \mapsto C_1 \, e^{\delta s - \delta t\eta} + \mathrm{O}(e^{(s-t\eta)(\delta-1)})$ with an appropriately chosen $\mathrm{O}(\cdot)$. As in the middle part of Step 7 in the proof of Theorem 5, by Equations (48) and (51), by Abel's summation formula, since $f(N+1) = \mathrm{O}(1)$ and by Equation (52), using again that $N\eta = s + \mathrm{O}(\eta)$ and since $\int_0^{(N+1)\eta} u \, e^u \, du = \mathrm{O}(s \, e^s)$, we have

$$\begin{aligned}
\Sigma_s &\leqslant \sum_{k=0}^{N+1} a_k \, f(k) = \big(\sum_{k=0}^{N+1} a_k\big) f(N+1) - \int_0^{N+1} \big(\sum_{k=0}^{t} a_k\big) f'(t) \, dt \\
&= \mathrm{O}(s \, e^{\delta\, s}) + \int_0^{N+1} \big(C_2 \, t \, \eta \, e^{\delta t\eta} + \mathrm{O}(e^{\delta t\eta})\big)\big(\delta \, \eta \, C_1 \, e^{\delta s - \delta t\eta} + \mathrm{O}(\eta \, e^{(s-t\eta)(\delta-1)})\big) \, dt \\
&= \mathrm{O}(s \, e^{\delta\, s}) + \delta \, C_2 \, C_1 \, \frac{(N+1)^2}{2} \, \eta^2 \, e^{\delta s} + \mathrm{O}\left((N+1) \, \eta \, e^{\delta\, s}\right) \\
&\qquad + \mathrm{O}\left(e^{s(\delta-1)} \int_0^{N+1} \eta \, t \, e^{t\eta} \, \eta \, dt\right) + \mathrm{O}\left(e^{s(\delta-1)} \int_0^{N+1} e^{t\eta} \, \eta \, dt\right) \\
&= \frac{\delta \, C_2 \, C_1}{2} \, (s + \mathrm{O}(\eta))^2 \, e^{\delta s} + \mathrm{O}(s \, e^{\delta\, s})\,.
\end{aligned}$$

Replacing C_1 and C_2 by their values, and letting η tend to 0, we hence have

$$\Sigma_s \leqslant \frac{\delta \, \iota^+_{\mathrm{rec}} \, {\Xi_{\widetilde{M}}}^2}{2^{2\delta+1} \, m^+ \, \|m_{\mathrm{BM}}\|} \, s^2 \, e^{\delta\, s}(1 - e^{-\delta}) + \mathrm{O}(s \, e^{\delta\, s})\,.$$

The same lower bound is obtained as at the end of Step 7 in the proof of Theorem 5, and by a summation, we have

$$\begin{aligned}
&\mathrm{Card}\left\{([\gamma], \beta) \in (\Gamma_{\mathscr{H}} \backslash \Gamma / \Gamma_{\widetilde{D}^+} \smallsetminus F) \times (\Gamma_{\mathscr{H}} \smallsetminus F') : \begin{array}{l} 0 < d(\widetilde{D}^-, \beta\gamma\widetilde{D}^+) \leqslant s \\ z_{\widetilde{D}^-,\, \beta\gamma\widetilde{D}^+} \in \widetilde{\ell}(\,]0, +\infty[\,) \end{array} \right\}. \\
&= \frac{\delta \, \iota^+_{\mathrm{rec}} \, {\Xi_{\widetilde{M}}}^2}{2^{2\delta+1} \, m^+ \, \|m_{\mathrm{BM}}\|} \, s^2 \, e^{\delta\, s} + \mathrm{O}\left(s \, e^{\delta\, s}\right). \qquad (53)
\end{aligned}$$

By separating the counting domain $(\Gamma_{\mathscr{H}} \backslash \Gamma / \Gamma_{\widetilde{D}^+}) \times \Gamma_{\mathscr{H}}$ as the disjoint union of $F \times \Gamma_{\mathscr{H}}$, of $(\Gamma_{\mathscr{H}} \backslash \Gamma / \Gamma_{\widetilde{D}^+} \smallsetminus F) \times F'$ and of $(\Gamma_{\mathscr{H}} \backslash \Gamma / \Gamma_{\widetilde{D}^+} \smallsetminus F) \times (\Gamma_{\mathscr{H}} \smallsetminus F')$, Equation (38) finally follows from Equations (47), (44), (46) and (53).

Case 2. Let us now assume that ℓ is weakly reciprocal. As in Case 2 of Theorem 5, we then have $\Omega_- = \Omega_+$ and

$$\mathscr{N}_{\Omega_+, D^+}(t) \leqslant \mathscr{N}_{D^-, D^+}(t) \leqslant \mathscr{N}_{\Omega_+, D^+}(t) + \mathscr{N}_{\Omega_0, D^+}(t) \qquad (54)$$

for every $t \geqslant 0$. Let us prove that Equation (38) is still satisfied. Since $\iota^-_{\mathrm{rec}} = 1$ as ℓ is weakly reciprocal, this will prove that Equation (41) is still satisfied, hence Theorem 6 when ℓ is weakly reciprocal will follow.

As in Case 2 of Theorem 5, with $\Gamma'_{\widetilde{D}^-} = \Gamma_{\widetilde{D}^-} \cap \Gamma_{\mathscr{H}}$, which has index 2 in $\Gamma_{\widetilde{D}^-}$ and order m^-, as $s \to +\infty$, we have

$$\begin{aligned}
\mathscr{N}_{\Omega_+,\,D^+}(s) &= \sum_{[\gamma]\in\Gamma_{\widetilde{D}^-}\backslash\Gamma/\Gamma_{\widetilde{D}^+}\,:\,0<d(\widetilde{D}^-,\gamma\widetilde{D}^+)\leqslant s,\ z_{\widetilde{D}^-,\,\gamma\widetilde{D}^+}\in\widetilde{\ell}(\,]-\infty,t_-[\,\cup\,]0,+\infty[\,)} m_{\widetilde{D}^-,\gamma\widetilde{D}^+} \\
&= \operatorname{Card}\Big\{\gamma\in\Gamma'_{\widetilde{D}^-}\backslash\Gamma/\Gamma_{\widetilde{D}^+}\,:\ \begin{array}{l} 0<d(\widetilde{D}^-,\gamma\widetilde{D}^+)\leqslant s \\ z_{\widetilde{D}^-,\,\gamma\widetilde{D}^+}\in\widetilde{\ell}(\,]0,+\infty[\,)\end{array}\Big\} + \operatorname{O}(e^{\delta\, s}) \\
&= \frac{1}{m^-}\operatorname{Card}\Big\{([\gamma],\beta)\in(\Gamma_{\mathscr{H}}\backslash\Gamma/\Gamma_{\widetilde{D}^+})\times\Gamma_{\mathscr{H}}\,:\ \begin{array}{l} 0<d(\widetilde{D}^-,\beta\gamma\widetilde{D}^+)\leqslant s \\ z_{\widetilde{D}^-,\,\beta\gamma\widetilde{D}^+}\in\widetilde{\ell}(\,]0,+\infty[\,)\end{array}\Big\} + \operatorname{O}(e^{\delta\, s})\,, \qquad (55)
\end{aligned}$$

that is, Equation (47) is still valid. The remainder of the proof of Equation (38) now proceeds exactly as in Case 1. □

5 Common perpendiculars of divergent geodesics in non-real hyperbolic geometry

In this section, we prove Theorem 9, which is a complex and quaternionic hyperbolic version of Theorems 5 and 6. This result will be applied in [PP8] to study the distribution of Heisenberg Farey neighbours.

In what follows, we denote by $\mathbb{K}$ either the field of complex numbers $\mathbb{C}$ endowed with the conjugation $x = x_0 + ix_1 \mapsto \overline{x} = x_0 - ix_1$ or the skew field of Hamiltonian numbers $\mathbb{H}$ (with standard basis $1, i, j, k$ over $\mathbb{R}$) endowed with the conjugation $x = x_0 + x_1 i + x_2 j + x_3 k \mapsto \overline{x} = x_0 - x_1 i - x_2 j - x_3 k$. We refer to [Vig] for background on $\mathbb{H}$. We denote by $\operatorname{Re} : x \mapsto \frac{1}{2}(x + \overline{x})$ and $\operatorname{Im} : x \mapsto \frac{1}{2}(x - \overline{x})$ the real and imaginary part maps of $\mathbb{K}$,[6] so that $\operatorname{Im}\mathbb{C} = i\mathbb{R}$ and $\operatorname{Im}\mathbb{H} = \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ are the imaginary subspaces of $\mathbb{C}$ and $\mathbb{H}$ respectively. We endow $\mathbb{K}$ and $\operatorname{Im}\mathbb{K}$ with the Euclidean scalar product that makes their canonical basis orthonormal and with its associated Lebesgue measure. Let $n \in \mathbb{N} \smallsetminus \{0, 1\}$. We endow $\mathbb{K}^{n-1}$ with the product Euclidean scalar product and product Lebesgue measure.

For all w, w' in the right vector space $\mathbb{K}^{n-1}$ over $\mathbb{K}$, we denote by $\overline{w} \cdot w' = \sum_{i=1}^{n-1} \overline{w_i}\, w'_i$ their standard Hermitian product, and we define $|w| = \sqrt{\overline{w} \cdot w}$. Recall that the *Siegel domain* model of the hyperbolic n-space $\mathbb{H}^n_{\mathbb{K}}$ over $\mathbb{K}$ is the open subset

$$\big\{(w_0, w) \in \mathbb{K} \times \mathbb{K}^{n-1} \ :\ 2\operatorname{Re} w_0 - |w|^2 > 0\big\}\,,$$

endowed with the Riemannian metric

$$ds^2_{\mathbb{H}^n_{\mathbb{K}}} = \frac{1}{(2\operatorname{Re} w_0 - |w|^2)^2}\big(\,|\,dw_0 - \overline{dw} \cdot w\,|^2 + (2\operatorname{Re} w_0 - |w|^2)\,|dw|^2\,\big)\,. \qquad (56)$$

The metric is normalized so that its sectional curvatures are in $[-4, -1]$, instead of in $[-1, -\frac{1}{4}]$ as in [Gol] when $\mathbb{K} = \mathbb{C}$. The boundary at infinity of $\mathbb{H}^n_{\mathbb{K}}$ is

$$\partial_\infty\mathbb{H}^n_{\mathbb{K}} = \big\{(w_0, w) \in \mathbb{K} \times \mathbb{K}^{n-1} \ :\ 2\operatorname{Re} w_0 - |w|^2 = 0\big\} \cup \{\infty\}\,.$$

As in [PP6, §3] when $\mathbb{K} = \mathbb{C}$ and [PP7, §6] when $\mathbb{K} = \mathbb{H}$, the *horospherical coordinates* $(\zeta, u, t) \in \mathbb{K}^{n-1} \times \operatorname{Im}\mathbb{K} \times [0, +\infty[$ of a point $(w_0, w) \in \mathbb{H}^n_{\mathbb{K}} \cup (\partial_\infty\mathbb{H}^n_{\mathbb{K}} \smallsetminus \{\infty\})$ are

$$(\zeta, u, t) = (w,\ w_0 - \overline{w_0},\ 2\operatorname{Re} w_0 - |w|^2) \quad \text{hence} \quad (w_0, w) = \Big(\frac{|\zeta|^2 + t + u}{2}, \zeta\Big)\,. \qquad (57)$$

[6] Note the nonstandard definition of Im when $\mathbb{K} = \mathbb{C}$.

In horospherical coordinates, the subset

$$\mathscr{H}_\infty = \big\{(\zeta, u, t) \in \mathbb{H}^n_{\mathbb{K}} \;:\; t \geqslant 1\big\},$$

is a horoball in $\mathbb{H}^n_{\mathbb{K}}$ centred at $\infty \in \partial_\infty \mathbb{H}^n_{\mathbb{K}}$. The geodesic line in $\mathbb{H}^n_{\mathbb{K}}$ from ∞ to the point at infinity $(\zeta, u, 0) \in \partial_\infty \mathbb{H}^n_{\mathbb{K}} \smallsetminus \{\infty\}$, through $\partial \mathscr{H}_\infty$ at time $s = 0$, is the map $s \mapsto (\zeta, u, e^{-2s})$.

The following lemma is a convenient replacement in hyperbolic geometry over $\mathbb{K}$ of the classical angle of parallelism formula in real hyperbolic geometry. See also [Gol, §3.2.4] for a different presentation. The proof follows ideas from [PP6, Lemma 10] when $\mathbb{K} = \mathbb{C}$ and [PP7, Lemma 6·2] when $\mathbb{K} = \mathbb{H}$. See also [Par, Prop. 7.1] for an expression of the distance from a point to a geodesic line in the projective model of $\mathbb{H}^n_{\mathbb{C}}$.

Lemma 7. *The orthogonal projection map from $\mathbb{H}^n_{\mathbb{K}}$ to the geodesic line $]0, \infty[$ in $\mathbb{H}^n_{\mathbb{K}}$ with points at infinity $(0, 0)$ and ∞ is, in horospherical coordinates, the map*

$$(\zeta, u, t) \mapsto (0, 0, \big|\, |\zeta|^2 + t + u \,\big|)\,.$$

The distance from $(\zeta, u, t) \in \mathbb{H}^n_{\mathbb{K}}$ to $]0, \infty[$ is $\frac{1}{2}\operatorname{arcosh}\big(\frac{|\zeta|^2 + |\,|\zeta|^2 + t + u\,|}{t}\big)$.

Proof. Let $\mathbb{B}^n$ be the open unit sphere in the standard right Hermitian space $\mathbb{K}^n$ over $\mathbb{K}$. The Cayley transform $\Phi : \mathbb{B}^n \to \mathbb{H}^n_{\mathbb{K}}$, defined by

$$\Phi : (z_1, \dots, z_n) \mapsto \Big(\frac{1 - z_n}{2}, z_1, z_2, \dots, z_{n-1}\Big)(1 + z_n)^{-1}\,,$$

is easily seen to be a smooth bijection, with inverse

$$(w_0, w) \mapsto (2w, 1 - 2w_0)(1 + 2w_0)^{-1}\,. \tag{58}$$

The *ball model* of the hyperbolic n-space over $\mathbb{K}$ is the open subset $\mathbb{B}^n$ endowed with the pull-back of the Riemannian metric (56) by Φ.

Let $\rho > 0$. In this ball model, the metric sphere $S(0, \rho)$ of radius ρ centered at the origin 0 coincides with the Euclidean sphere of radius $\tanh \rho$ centered at 0 by [Gol, page 78, see also §3.3.4] when $\mathbb{K} = \mathbb{C}$, taking into account the different normalization of the curvatures.

The isometry Φ maps $0 \in \mathbb{B}^n$ to $(0, 0, 1) \in \mathbb{H}^n_{\mathbb{K}}$ in the horospherical coordinates. By Equation (58), for all $z' \in \mathbb{K}^{n-1}$ and $z_n \in \mathbb{K}$, writing (w_0, w) the point $\Phi(z', z_n)$ and denoting by (ζ, u, t) its horospherical coordinates, we have $|z'|^2 + |z_n|^2 = \tanh^2 \rho$ if and only if

$$|\, 2w \,|^2 + |\, 1 - 2w_0 \,|^2 = |\, 1 + 2w_0 \,|^2 \tanh^2 \rho$$

that is, using Equation (57) and an easy computation, if and only if

$$|\, 1 + |\zeta|^2 + t + u \,|^2 - 4t \cosh^2 \rho = 0\,. \tag{59}$$

The Riemannian metric of $\mathbb{H}^n_{\mathbb{K}}$ given by Equation (56) becomes in the horospherical coordinates

$$ds^2_{\mathbb{H}^n_{\mathbb{K}}} = \frac{1}{4\,t^2}\big(dt^2 + |\, du - 2\operatorname{Im} \overline{d\zeta} \cdot \zeta \,|^2 + 4\,t\,|\, d\zeta \,|^2\big)\,. \tag{60}$$

Hence for all $\lambda > 0$ and $(\zeta', u') \in \mathbb{K}^{n-1} \times \operatorname{Im} \mathbb{K}$, the *Heisenberg dilation*

$$h_\lambda : (\zeta, u, t) \mapsto (\lambda \zeta, \lambda^2 u, \lambda^2 t)\,,$$

whose inverse is $h_{\lambda^{-1}}$, and the *Heisenberg translation*

$$\tau_{(\zeta',u')} : (\zeta, u, t) \mapsto (\zeta' + \zeta,\, u' + u + 2\,\mathrm{Im}\,\overline{\zeta'} \cdot \zeta,\, t)\,,$$

whose inverse is $\tau_{(-\zeta',-u')}$, are isometries of $\mathbb{H}^n_{\mathbb{K}}$ fixing ∞.

Using the horospherical coordinates, let us fix $(\zeta_0, u_0, t_0) \in \mathbb{H}^n_{\mathbb{K}}$, and let us compute its orthogonal projection on the geodesic line $]0, \infty[$. Let us consider $\lambda = \sqrt{t_0}$, $\zeta' = \frac{\zeta_0}{\sqrt{t_0}}$ and $u' = \frac{u_0}{t_0}$. The isometry $h_\lambda \circ \tau_{(\zeta',u')}$ maps $(0, 0, 1)$ to (ζ_0, u_0, t_0), hence maps the sphere of radius ρ centered at $(0, 0, 1)$ to the sphere of radius ρ centered at (ζ_0, u_0, t_0) in $\mathbb{H}^n_{\mathbb{K}}$. For every $(\zeta, u, t) \in \mathbb{H}^n_{\mathbb{K}}$, using Equation (59) (multiplied by t_0^2) for the last equivalence, we have

$$\begin{aligned}
& (\zeta, u, t) \in S((\zeta_0, u_0, t_0), \rho) \\
\Leftrightarrow\ & \tau^{-1}_{(\zeta',u')} \circ h^{-1}_\lambda(\zeta, u, t) \in S((0, 0, 1), \rho) \\
\Leftrightarrow\ & \big(-\zeta' + \lambda^{-1}\zeta, -u' + \lambda^{-2}u + 2\,\mathrm{Im}(\,\overline{-\zeta'}\,) \cdot (\lambda^{-1}\zeta), \lambda^{-2}t \big) \in S((0, 0, 1), \rho) \\
\Leftrightarrow\ & \big|\, t_0 + |\zeta - \zeta_0|^2 + t + (u - u_0 - 2\,\mathrm{Im}\,\overline{\zeta_0} \cdot \zeta)\,\big|^2 = 4 t_0 t \cosh^2 \rho\,. \qquad (61)
\end{aligned}$$

The closest point to (ζ_0, u_0, t_0) on $]0, \infty[$ is attained when the parameter ρ gives a double point of intersection $(0, 0, t)$ between this sphere and $]0, \infty[$. Taking $\zeta = 0$ and $u = 0$ in Equation (61) gives the following quadratic equation in t

$$t^2 + 2t(|\zeta_0|^2 + t_0 - 2t_0 \cosh^2 \rho) + \big|\, |\zeta_0|^2 + t_0 + u_0 \,\big|^2 = 0\,.$$

It has a double solution if and only if its reduced discriminant

$$(|\zeta_0|^2 + t_0 - 2t_0 \cosh^2 \rho)^2 - \big|\, |\zeta_0|^2 + t_0 + u_0 \,\big|^2$$

is equal to zero, that is, if and only if

$$|\zeta_0|^2 + t_0 - 2t_0 \cosh^2 \rho = -\big|\, |\zeta_0|^2 + t_0 + u_0 \,\big|\,. \qquad (62)$$

The double solution of the above quadratic equation is then $t = \big|\, |\zeta_0|^2 + t_0 + u_0 \,\big|$. This proves the first claim of Lemma 7. The second claim follows from Equation (62) by using the fact that $2 \cosh^2 \rho = \cosh(2\rho) + 1$. □

Let us now make explicit the Hamenstädt distance and cuspidal distance on horospheres in $\mathbb{H}^n_{\mathbb{K}}$. The *Heisenberg group* $\mathrm{Heis}_{n,\mathbb{K}}$ is the real Lie group $\mathbb{K}^{n-1} \times \mathrm{Im}\,\mathbb{K}$ with law

$$(\zeta', u')(\zeta, u) = (\zeta' + \zeta,\, u' + u + 2\,\mathrm{Im}\,\overline{\zeta'} \cdot \zeta)\,.$$

As defined for instance in [Gol, page 160] when $\mathbb{K} = \mathbb{C}$, the *Cygan distance* d_{Cyg} on $\mathrm{Heis}_{n,\mathbb{K}}$ is the unique left-invariant distance on $\mathrm{Heis}_{n,\mathbb{K}}$ such that

$$d_{\mathrm{Cyg}}\big((\zeta, u), (0, 0)\big) = \sqrt{\big|\, |\zeta|^2 + u \,\big|} = \sqrt[4]{|\zeta|^4 + |u|^2}\,. \qquad (63)$$

As introduced in [PP1, page 372] when $\mathbb{K} = \mathbb{C}$, the *modified Cygan distance* d'_{Cyg} on $\mathrm{Heis}_{n,\mathbb{K}}$ is the unique left-invariant distance on $\mathrm{Heis}_{n,\mathbb{K}}$ such that

$$d'_{\mathrm{Cyg}}\big((\zeta, u), (0, 0)\big) = \sqrt{|\zeta|^2 + \big|\, |\zeta|^2 + u \,\big|} = \sqrt{|\zeta|^2 + \sqrt{|\zeta|^4 + |u|^2}}\,. \qquad (64)$$

It is easy to check that $d_{\rm Cyg} \leqslant d'_{\rm Cyg} \leqslant \sqrt{2}\, d_{\rm Cyg}$. As in [HP3, page 216] and [PP1, page 370] both when $\mathbb{K} = \mathbb{C}$, for every $t' > 0$, using Equation (57) for the second equality, let

$$H_{t'} = \{(w_0, w) \in \mathbb{H}^n_{\mathbb{K}} : 2\,\mathrm{Re}\, w_0 - |w|^2 \geqslant t'\} = \{(\zeta, u, t) \in \mathbb{H}^n_{\mathbb{K}} : t \geqslant t'\}\,, \qquad (65)$$

which is a horoball in $\mathbb{H}^n_{\mathbb{K}}$ centered at ∞, so that $H_1 = \mathscr{H}_\infty$. The Heisenberg group $\mathrm{Heis}_{n,\mathbb{K}}$ acts, by the map $(\zeta', u') \mapsto \tau_{(\zeta',u')}{}_{|H_{t'}}$, simply transitively on the horosphere $H_{t'}$ for every $t' > 0$, as well as on $\partial_\infty \mathbb{H}^n_{\mathbb{K}} \smallsetminus \{\infty\}$. Let us prove the claim that for all $t'' \geqslant t'$ and $(\zeta, u), (\zeta', u') \in \mathbb{K}^{n-1} \times \mathrm{Im}\,\mathbb{K}$, we have

$$d_{H_{t'}}((\zeta, u, t'), (\zeta', u', t')) = \sqrt{\frac{t''}{t'}}\; d_{H_{t''}}((\zeta, u, t''), (\zeta', u', t''))\,. \qquad (66)$$

Proof. Since $t'' \geqslant t'$, the horoball $H_{t''}$ is contained in $H_{t'}$. By an immediate computation using the geodesic line $s \mapsto (0, 0, e^{2s})$, we have $d(\partial H_{t''}, \partial H_{t'}) = \frac{1}{2} \ln \frac{t''}{t'}$. By the definition of the Hamenstädt distance in Equation (3), we have as wanted

$$d_{H_{t'}}((\zeta, u, t'), (\zeta', u', t')) = e^{d(\partial H_{t''}, \partial H_{t'})}\; d_{H_{t''}}((\zeta, u, t''), (\zeta', u', t''))\,. \quad \Box$$

By Equation (66) applied with $t' = 1$ and $t'' = 2$ and by [HP3, Prop. 3.12] (which uses the horosphere ∂H_2 instead of the horosphere $\partial H_1 = \partial \mathscr{H}_\infty$) when $\mathbb{K} = \mathbb{C}$, and by a similar computation when $\mathbb{K} = \mathbb{H}$, we have

$$d_{\mathscr{H}_\infty}((\zeta, u, 1), (\zeta', u', 1)) = \sqrt{2}\, d_{H_2}((\zeta, u, 2), (\zeta', u', 2)) = d_{\rm Cyg}((\zeta, u), (\zeta', u'))\,. \qquad (67)$$

By [PP1, Prop. 6.2] applied with $s_0 = 1$ (so that the horoball H_1 of loc. cit. is equal to our horoball $\mathscr{H}_\infty$) when $\mathbb{K} = \mathbb{C}$, and by a similar computation when $\mathbb{K} = \mathbb{H}$, we have

$$d'_{\mathscr{H}_\infty}((\zeta, u, 1), (\zeta', u', 1)) = \frac{1}{\sqrt{2}}\, d'_{\rm Cyg}((\zeta, u), (\zeta', u'))\,. \qquad (68)$$

Lemma 8. *Let $\mathscr{H}$ be a horoball in $\mathbb{H}^n_{\mathbb{K}}$ and let $\widetilde{\ell}$ be a geodesic line in $\mathbb{H}^n_{\mathbb{K}}$ that enters $\mathscr{H}$ perpendicularly at $\widetilde{\ell}(0) \in \partial \mathscr{H}$.*
(i) Let $\widetilde{\ell}'$ be a geodesic line in $\mathbb{H}^n_{\mathbb{K}}$ that exits $\mathscr{H}$ perpendicularly at $\widetilde{\ell}'(0) \in \partial \mathscr{H}$ such that $d_{\mathscr{H}}(\widetilde{\ell}'(0), \widetilde{\ell}(0)) \geqslant 1$. For every $s \geqslant 0$, we have

$$d(\widetilde{\ell}'(s), \widetilde{\ell}\,) = s + \ln d'_{\mathscr{H}}(\widetilde{\ell}'(0), \widetilde{\ell}(0)) + \ln 2 + \mathrm{O}\big(d'_{\mathscr{H}}(\widetilde{\ell}'(0), \widetilde{\ell}(0))^{-2} e^{-2s}\big)\,.$$

(ii) Let D be a closed convex subset of $\mathbb{H}^n_{\mathbb{K}}$ disjoint from $\mathscr{H}$ and let $x_0 \in \partial \mathscr{H}$ be the closest point to D in $\mathscr{H}$. There exists a constant $c_{\mathbb{K}} \geqslant 1$ such that if $d_{\mathscr{H}}(x_0, \widetilde{\ell}(0)) \geqslant c_{\mathbb{K}}$, then

$$d(D, \widetilde{\ell}\,) = d(D, \mathscr{H}) + \ln d'_{\mathscr{H}}(x_0, \widetilde{\ell}(0)) + \ln 2 + \mathrm{O}\big(d'_{\mathscr{H}}(x_0, \widetilde{\ell}(0))^{-2} e^{-2\, d(D, \mathscr{H})}\big)\,,$$

and furthermore, the closest point to D on the image of $\widetilde{\ell}$ belongs to $\mathscr{H}$.

Proof. We use the horospherical coordinates of $\mathbb{H}^n_{\mathbb{K}}$. The isometry group of $\mathbb{H}^n_{\mathbb{K}}$ acts transitively on the set of horoballs of $\mathbb{H}^n_{\mathbb{K}}$ and the stabilizer of each horoball acts transitively on its boundary horosphere. Hence we may assume that $\mathscr{H} = \mathscr{H}_\infty$ and that $\widetilde{\ell}(0) = (0, 0, 1)$. Therefore the geodesic line $\widetilde{\ell}$ is the map $s \mapsto (0, 0, e^{2s})$ and its image is $]0, \infty[$ with the notation of Lemma 7.

(i) Let us define $(\zeta, u) \in \mathbb{K}^{n-1} \times \operatorname{Im} \mathbb{K}$ such that the geodesic line $\widetilde{\ell}'$ is the map given by $s \mapsto (\zeta, u, e^{-2s})$. Using Equations (63) and (67), let

$$D = |\,|\zeta|^2 + u\,| = d_{\rm Cyg}((\zeta,u),(0,0))^2 = d_{\mathscr{H}}(\widetilde{\ell}'(0), \widetilde{\ell}(0))^2\,.$$

Using Equations (64) and 68, let

$$D' = |\zeta|^2 + |\,|\zeta|^2 + u\,| = d'_{\rm Cyg}((\zeta,u),(0,0))^2 = 2\, d'_{\mathscr{H}}(\widetilde{\ell}'(0), \widetilde{\ell}(0))^2\,. \tag{69}$$

Since $|\zeta|^2 \leqslant \sqrt{|\zeta|^4 + |u|^2} = D$ and since $D \geqslant 1$ by the assumption of Assertion (i), we have

$$\begin{aligned}
|\,|\zeta|^2 + e^{-2s} + u\,| &= \sqrt{|\zeta|^4 + 2e^{-2s}|\zeta|^2 + e^{-4s} + |u|^2}\\
&= \sqrt{|\zeta|^4 + |u|^2}\,\sqrt{1 + \frac{2e^{-2s}|\zeta|^2}{|\zeta|^4+|u|^2} + \frac{e^{-4s}}{|\zeta|^4+|u|^2}}\\
&= D\sqrt{1 + \mathrm{O}(e^{-2s}D^{-1})} = D(1 + \mathrm{O}(e^{-2s}D^{-1}))\,.
\end{aligned}$$

Since $D \leqslant |\zeta|^2 + D = D' \leqslant 2D$, we hence have

$$\begin{aligned}
x &= \frac{|\zeta|^2 + |\,|\zeta|^2 + e^{-2s} + u\,|}{e^{-2s}} = e^{2s} D' \frac{|\zeta|^2 + D + D\,\mathrm{O}(e^{-2s}D^{-1})}{D'}\\
&= e^{2s} D' \big(1 + \mathrm{O}(e^{-2s} D'^{-1})\big)\,.
\end{aligned}$$

Recall that as $x \in [1, +\infty[$ tends to $+\infty$, we have

$$\operatorname{arcosh} x = \ln(x + \sqrt{x^2 - 1}\,) = \ln(2\,x) + \mathrm{O}\,\Big(\frac{1}{x^2}\Big)\,.$$

By the last claim of Lemma 7, we therefore have

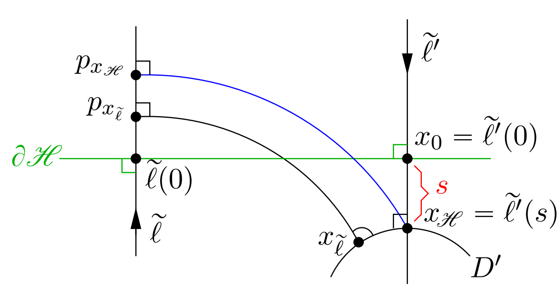
$$d(\widetilde{\ell}'(s), \widetilde{\ell}\,) = \frac{1}{2} \operatorname{arcosh}\Big(\frac{|\zeta|^2 + |\,|\zeta|^2 + e^{-2s} + u\,|}{e^{-2s}}\Big) = s + \frac{1}{2}\log(2D') + \mathrm{O}(e^{-2s}D'^{-1})\,.$$

Assertion (i) then follows from Equation (69).

(ii) As in the proof of Lemma 4 (ii), let $x_{\widetilde{\ell}} \in D$ be the closest point in D to $\widetilde{\ell}$, and let $x_{\mathscr{H}} \in D$ be the closest point in D to $\mathscr{H}$. Let $\widetilde{\ell}'$ be the geodesic line exiting $\mathscr{H}$ perpendicularly at x_0 at time 0. Let $s = d(D, \mathscr{H})$, so that $x_0 = \widetilde{\ell}'(0)$ and $x_{\mathscr{H}} = \widetilde{\ell}'(s)$, and let $D'' = d'_{\mathscr{H}}(x_0, \widetilde{\ell}(0))$. Finally, let $p_{x_{\widetilde{\ell}}}$ (respectively $p_{x_{\mathscr{H}}}$) be the closest point to D (respectively to $x_{\mathscr{H}}$) on the image of $\widetilde{\ell}$. We have the upper bound

$$d(x_{\widetilde{\ell}}, \widetilde{\ell}\,) = d(D, \widetilde{\ell}\,) \leqslant d(x_{\mathscr{H}}, \widetilde{\ell}\,) = s + \ln D'' + \ln 2 + \mathrm{O}(D''^{-2} e^{-2s})$$

by Assertion (i) In order to obtain the similar lower bound on $d(x_{\widetilde{\ell}}, \widetilde{\ell}\,)$, as in the proof of Lemma 4 (ii) (except that the union of the geodesic lines perpendicular to $\widetilde{\ell}'$ at $x_{\mathscr{H}}$ is no longer totally geodesic), we may replace D by a geodesic line D' through $x_{\widetilde{\ell}}$ and $x_{\mathscr{H}}$ perpendicular to $\widetilde{\ell}'$ at $x_{\mathscr{H}}$. See the picture below.



By Assertion (i), we have $d(x_{\mathscr{H}}, \widetilde{\ell}\,) = s + \ln D'' + \ln 2 + \mathrm{O}(e^{-2(s+\ln D'')})$. Since the closest point projection to a convex subset does not increase the distances, we have the inequality $d(p_{x_{\widetilde{\ell}}}, p_{x_{\mathscr{H}}}) \leqslant d(x_{\widetilde{\ell}}, x_{\mathscr{H}})$. By the triangle inequality, we have

$$d(x_{\mathscr{H}}, p_{x_{\mathscr{H}}}) - d(p_{x_{\widetilde{\ell}}}, p_{x_{\mathscr{H}}}) - d(x_{\widetilde{\ell}}, x_{\mathscr{H}}) \leqslant d(D, \widetilde{\ell}) = d(x_{\widetilde{\ell}}, p_{x_{\widetilde{\ell}}}) \leqslant d(x_{\mathscr{H}}, p_{x_{\mathscr{H}}})\,.$$

To obtain Assertion (ii), we thus only have to prove that $d(x_{\widetilde{\ell}}, x_{\mathscr{H}}) = \mathrm{O}(e^{-2\,d(x_{\mathscr{H}}, \widetilde{\ell}\,)})$. In $\mathbb{H}^n_{\mathbb{R}}$, this is the new input of Lemma 4 (ii) with respect to Lemma 4 (i). In $\mathbb{H}^n_{\mathbb{K}}$, the result follows by comparison using the ideal quadrangle with vertices ∞, $p_{x_{\widetilde{\ell}}}$, $x_{\widetilde{\ell}}$ and $x_{\mathscr{H}}$ with right angles at $p_{x_{\widetilde{\ell}}}$ and $x_{\mathscr{H}}$, and angle at least $\frac{\pi}{2}$ at $x_{\widetilde{\ell}}$. □

We now prove analogs of Theorems 5 and 6 in the complex or quaternionic hyperbolic case. In order to simplify the notation, we define $d_{\mathbb{K}} = \dim_{\mathbb{R}} \mathbb{K}$ for $\mathbb{K} = \mathbb{C}$ and $\mathbb{K} = \mathbb{H}$.

Theorem 9. *Let M be a noncompact finite volume complete connected complex or quaternionic hyperbolic good orbifold, with dimension $n \geqslant 2$ over $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{H}$, with exponentially mixing geodesic flow if $\mathbb{K} = \mathbb{C}$.*

(1) Let D^- be a nonempty properly immersed closed locally convex subset of M with nonzero finite outer skinning measure and let D^+ be the image of a divergent geodesic in M. As $s \to +\infty$, we have

$$\mathscr{N}_{D^-,D^+}(s) = \frac{\prod_{i=1}^{\frac{d_{\mathbb{K}}}{2}} (\frac{n d_{\mathbb{K}}}{2} - i)\ \iota_{\rm rec}(D^+)\ \|\sigma^+_{D^-}\|}{4^{d_{\mathbb{K}}-1}\ \sqrt{\pi}\ \Gamma(\frac{d_{\mathbb{K}}-1}{2})\, m(D^+)\,\mathrm{Vol}(M)}\ s\, e^{(d_{\mathbb{K}}(n+1)-2)\,s} + \mathrm{O}(e^{(d_{\mathbb{K}}(n+1)-2)s})\,.$$

(2) Let D^+ and D^- be the images of two divergent geodesics in M. As $s \to +\infty$, we have

$$\begin{aligned}\mathscr{N}_{D^-,D^+}(s) =&\frac{(d_{\mathbb{K}}(n+1)-2)\ \pi^{\frac{n d_{\mathbb{K}}}{2}-1} \prod_{i=1}^{\frac{d_{\mathbb{K}}}{2}} (\frac{n d_{\mathbb{K}}}{2} - i)\ \iota_{\rm rec}(D^-)\ \iota_{\rm rec}(D^+)}{2^{d_{\mathbb{K}}(n+3)-4}\ \Gamma(\frac{d_{\mathbb{K}}-1}{2})^2\ (\frac{d_{\mathbb{K}}(n-1)}{2} - 1)!\ m(D^-)\ m(D^+)\,\mathrm{Vol}\,M}\ s^2\, e^{(d_{\mathbb{K}}(n+1)-2)\,s}\\ &+ \mathrm{O}\left(s\, e^{(d_{\mathbb{K}}(n+1)-2)s}\right).\end{aligned}$$

We believe that a similar statement is valid also for the octonionic hyperbolic plane case, but we leave the proof to the readers.

Proof. This proof follows closely the proofs of Theorem 5 for Assertion (1) and of Theorem 6 for Assertion (2), that were written for this purpose. We only indicate the changes, that are the ones involving specifically the fact that M was assumed to be real hyperbolic, and no longer is. We start with a lemma, that will replace Equation (14).

Lemma 10. *We have $\Xi_{\widetilde{M}} = \frac{2\,\pi^{\frac{n\,d_{\mathbb{K}}-1}{2}}}{\Gamma(\frac{d_{\mathbb{K}}-1}{2})\,(\frac{d_{\mathbb{K}}(n-1)}{2}-1)!}$.*

Proof. Since the definition (5) of $\Xi_{\widetilde{M}}$ is independent of the choices of a horoball $\mathscr{H}$ and of a point $x \in \partial\mathscr{H}$, we may assume that $\mathscr{H} = \mathscr{H}_\infty$ and that $x = (0, 0, 1)$ in the horospherical coordinates of $\mathbb{H}^n_{\mathbb{K}}$. By [PP6, Lemma 12 (iv)] when $\mathbb{K} = \mathbb{C}$ and by [PP7, Lemma 7·2 (iv)] when $\mathbb{K} = \mathbb{H}$, the measure $(\widetilde{p}_\bullet)_*\, \widetilde{\sigma}^-_{\mathscr{H}_\infty}$ is $2^{d_{\mathbb{K}}-1}$ times the Riemannian measure $\mathrm{vol}_{\partial\mathscr{H}_\infty}$ of the induced Riemannian metric on $\partial\mathscr{H}_\infty$. Since $\mathscr{H}_\infty = H_1$ with the notation of Equation (65), by [PP6, Equation (15)] when $\mathbb{K} = \mathbb{C}$ and by [PP7, Equation (7·11)] when $\mathbb{K} = \mathbb{H}$, we have

$$d\,\mathrm{vol}_{\partial\mathscr{H}_\infty}(\zeta, u, 1) = \frac{1}{2^{d_{\mathbb{K}}-1}}\ d\zeta\, du\,.$$

By Equations (68) and (64), we have

$$\begin{aligned}B_{d'_{\mathscr{H}_\infty}}((0,0,1),1) &= \{(\zeta,u,1)\in\partial\mathscr{H}_\infty : d'_{\rm Cyg}((\zeta,u),(0,0))\leqslant\sqrt{2}\}\\ &= \{(\zeta,u,1)\in\partial\mathscr{H}_\infty : |\zeta|^2+\sqrt{|\zeta|^4+|u|^2}\leqslant 2\}\,.\end{aligned}$$

Hence by the definition (5) of $\Xi_{\widetilde{M}}$ and by using the formulas $du=\rho\,d\rho\,d\,{\rm vol}_{\mathbb{S}^{d_{\mathbb{K}}-2}}$ and $d\zeta=\rho\,d\rho\,d\,{\rm vol}_{\mathbb{S}^{d_{\mathbb{K}}(n-1)-1}}$ of the Lebesgue measures in polar coordinates of the Euclidean spaces $\operatorname{Im}\mathbb{K}$ and then $\mathbb{K}^{n+1}$, we have

$$\begin{aligned}\Xi_{\widetilde{M}} &= (\widetilde{p}_\bullet)_*\,\widetilde{\sigma}^-_{\mathscr{H}_\infty}\big(B_{d'_{\mathscr{H}_\infty}}((0,0,1),1)\big)\big) = \int_{(\zeta,u,1)\in B_{d'_{\mathscr{H}_\infty}}((0,0,1),1)} d\zeta\,du\\ &= \int_{|\zeta|\leqslant 1}\Big(\int_{|u|\leqslant 2\sqrt{1-|\zeta|^2}} du\Big)d\zeta = {\rm Vol}(\mathbb{S}^{d_{\mathbb{K}}-2})\int_{|\zeta|\leqslant 1} 2(1-|\zeta|^2)\,d\zeta\\ &= \frac{1}{2}\,{\rm Vol}(\mathbb{S}^{d_{\mathbb{K}}-2})\,{\rm Vol}(\mathbb{S}^{d_{\mathbb{K}}(n-1)-1})\,.\end{aligned}$$

Lemma 10 follows as ${\rm Vol}(\mathbb{S}^{m-1})=\frac{2\,\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})}$ and $\Gamma(m')=(m'-1)!$ for all $m,m'\in\mathbb{N}\smallsetminus\{0\}$. □

By [PP6, Lemma 12 (iii)] when $\mathbb{K}=\mathbb{C}$ and by [PP7, Lemma 7·2 (iii)] when $\mathbb{K}=\mathbb{H}$, we now have

$$\|m_{\rm BM}\| = \frac{1}{2^{d_{\mathbb{K}}(n-1)}}\;{\rm Vol}(\mathbb{S}^{n\,d_{\mathbb{K}}-1})\;{\rm Vol}(M) = \frac{\pi^{\frac{n\,d_{\mathbb{K}}}{2}}}{2^{d_{\mathbb{K}}(n-1)-1}(\frac{n\,d_{\mathbb{K}}}{2}-1)!}\;{\rm Vol}(M)\,. \tag{70}$$

In order to prove Assertion (1) of Theorem 9, we use the same notation as in the beginning of the proof of Theorem 5. The discussion on whether ℓ is weakly reciprocal or not is the same one as in the proof of Theorem 5. If Equation (10) is still valid, then Equation (13) is also still valid, by the same proof. Assertion (1) of Theorem 9 follows from Equation (13) by using Lemma 10 instead of Equation (14), by using Equation (70) instead of Equation (15), and by using Equation (2) instead of $\delta=n-1$.

The proof of Equation (10) when $\widetilde{M}=\mathbb{H}^n_{\mathbb{K}}$ follows the same seven steps as in the real hyperbolic case, except that

- in Steps 3 and 7, the use of Lemma 4 (ii) with the constant $c_{\mathbb{R}}=2$ is replaced by the use of Lemma 8 (ii) with the constant $c_{\mathbb{K}}$, and
- the use of [LP] in Step 7 is replaced by the exponentially mixing assumption of Theorem 9 when $\mathbb{K}=\mathbb{C}$ and by the exponentially mixing consequence of the arithmeticity property of M recalled in Section 2 when $\mathbb{K}=\mathbb{H}$.

In order to prove Assertion (2) of Theorem 9, we use the same notation as in the beginning of the proof of Theorem 6. The discussion on whether ℓ is weakly reciprocal or not is the same one as in the proof of Theorem 6. If Equation (38) is still valid, then Equation (41) is also still valid, by the same proof. Assertion (2) of Theorem 9 follows from Equation (41) by using Lemma 10 instead of Equation (14), by using Equation (70) instead of Equation (15), and by using Equation (2) instead of $\delta=n-1$.

The proof of Equation (41) when $\widetilde{M}=\mathbb{H}^n_{\mathbb{K}}$ is similar to its proof when $\widetilde{M}=\mathbb{H}^n_{\mathbb{R}}$, replacing the call to Theorem 5 in the proofs of Equations (40), (45) and (52) by a call to Equation (13) that we just proved during the proof of Assertion (1) of Theorem 9. □

6 Ambiguous geodesics

In this section, we first show that the ambiguous conjugacy classes of hyperbolic elements of the modular group $\mathrm{PSL}_2(\mathbb{Z})$ discussed in [Sar] correspond to common perpendiculars of divergent geodesics in the modular orbifold $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2_{\mathbb{R}}$. We then use Theorem 5 and Theorem 6 to recover, by hyperbolic geometry methods, asymptotic counting results of special conjugacy classes in $\mathrm{PSL}_2(\mathbb{Z})$, due to Sarnak [Sar] by arithmetic methods. We start by recalling standard facts on the modular orbifold, and on the images of the imaginary axis by the modular group.

As in Section 3, let $\mathbb{H}^2_{\mathbb{R}} \subset \mathbb{C}$ be the upper halfplane model of the real hyperbolic plane, so that $\partial_\infty\mathbb{H}^2_{\mathbb{R}} = \mathbb{P}_1(\mathbb{R}) = \mathbb{R}\cup\{\infty\}$. Given $\xi \neq \eta$ in $\partial_\infty\mathbb{H}^2_{\mathbb{R}}$, we denote by $]\xi,\eta[$ the geodesic line in $\mathbb{H}^2_{\mathbb{R}}$ with points at infinity ξ and η. We denote by $\left[\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right]$ the image in $\mathrm{PGL}_2(\mathbb{C})$ of $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{C})$. The group $\mathrm{PSL}_2(\mathbb{R})$ acts isometrically and faithfully by homographies on $\mathbb{H}^2_{\mathbb{R}}$, by the map $(\gamma,z)\mapsto \gamma\cdot z = \frac{az+b}{cz+d}$ for $z\in\mathbb{H}^2_{\mathbb{R}}$ and $\gamma = \left[\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right]\in\mathrm{PSL}_2(\mathbb{R})$. Let $\Gamma_{\mathbb{Z}} = \mathrm{PSL}_2(\mathbb{Z})$ be the *modular group*, which is a nonuniform arithmetic lattice in $\mathrm{PSL}_2(\mathbb{R})$ with set of parabolic fixed points $\mathrm{Par}_{\Gamma_{\mathbb{Z}}} = \Gamma_{\mathbb{Z}}\cdot\infty = \mathbb{Q}\cup\{\infty\}$. Let $M_{\mathbb{Z}} = \Gamma_{\mathbb{Z}}\backslash\mathbb{H}^2_{\mathbb{R}}$ be the *modular orbifold*, which is a noncompact complete connected real hyperbolic good orbifold of volume $\frac{\pi}{3}$ with only one cusp.

Let $\widetilde{\ell} : t\mapsto i\,e^t$ be the geodesic line in $\mathbb{H}^2_{\mathbb{R}}$ through $i\in\mathbb{H}^2_{\mathbb{R}}$ at time $t=0$, with endpoints at infinity 0 and ∞, and let $\widetilde{\Delta} = \widetilde{\ell}(\mathbb{R}) = i\mathbb{R}\cap\mathbb{H}^2_{\mathbb{R}} = \;]0,\infty[$ be its image. Then $\ell = \Gamma_{\mathbb{Z}}\widetilde{\ell}$ is a divergent geodesic in $M_{\mathbb{Z}}$, converging at $\pm\infty$ to the only cusp $\Gamma_{\mathbb{Z}}\cdot\infty$ of $M_{\mathbb{Z}}$. Note that ℓ is reciprocal since $\widetilde{\Delta}$ is preserved by the involution $\iota = \left[\begin{smallmatrix} 0 & -1\\ 1 & 0\end{smallmatrix}\right]\in\Gamma_{\mathbb{Z}}$ with fixed point set $\{i\}$ in $\mathbb{H}^2_{\mathbb{R}}$. The stabilizer in $\Gamma_{\mathbb{Z}}$ of $\widetilde{\Delta}$ is $\Gamma_{\widetilde{\Delta}} = \{\mathrm{id},\iota\}$.

Similarly, let $\widetilde{\ell}_1 : t\mapsto \frac{2}{1-2ie^{-t}}$, which is the geodesic line in $\mathbb{H}^2_{\mathbb{R}}$ image of $\widetilde{\ell}$ by $\left[\begin{smallmatrix} 1 & 0\\ 1/2 & 1\end{smallmatrix}\right]$, with endpoints at infinity 0 and 2, and let $\widetilde{\Delta}_1 = \widetilde{\ell}_1(\mathbb{R}) = \;]0,2[$ be its image. Then $\ell_1 = \Gamma_{\mathbb{Z}}\widetilde{\ell}_1$ is a also a divergent geodesic in $M_{\mathbb{Z}}$, which is also reciprocal since $1+i\in(\Gamma_{\mathbb{Z}}\cdot i)\cap\widetilde{\Delta}_1$. The stabilizer in $\Gamma_{\mathbb{Z}}$ of $\widetilde{\Delta}_1$ is $\Gamma_{\widetilde{\Delta}_1} = \big\{\mathrm{id},\left[\begin{smallmatrix} -1 & 2\\ -1 & 1\end{smallmatrix}\right]\big\}$. But $\widetilde{\Delta}_1$ is not the image of $\widetilde{\Delta}$ by any element of $\Gamma_{\mathbb{Z}}$.

Let $\Delta = \ell(\mathbb{R})$ be the image of ℓ and $\Delta_1 = \ell_1(\mathbb{R})$ the one of ℓ_1. Let $\widetilde{D}^-,\widetilde{D}^+\in\{\widetilde{\Delta},\widetilde{\Delta}_1\}$ and D^-,D^+ their images in $M_{\mathbb{Z}}$. The action of $\Gamma_{\mathbb{Z}}$ on $T^1\mathbb{H}^2_{\mathbb{R}}$ is free, hence the multiplicities of the common perpendiculars from D^- to D^+ (defined in Equation (7)) are all equal to 1. Thus Equation (8) gives that

$$\mathscr{N}_{D^-,D^+}(t) = \mathrm{Card}\{[\gamma]\in\Gamma_{\widetilde{D}^-}\backslash\Gamma_{\mathbb{Z}}/\Gamma_{\widetilde{D}^+} : 0 < d(\widetilde{D}^-,\gamma\widetilde{D}^+)\leqslant t\}$$

is the number of images under $\Gamma_{\mathbb{Z}}$ of $\widetilde{D}^+$ that are at positive distance at most t from $\widetilde{D}^-$, modulo the left action of $\Gamma_{\widetilde{D}^-}$. Let ${\mathbb{H}^2_{\mathbb{R}}}^{\pm} = \{z\in\mathbb{H}^2_{\mathbb{R}} : \pm\,\mathrm{Re}(z) > 0\}$. Noting that $\iota{\mathbb{H}^2_{\mathbb{R}}}^{\pm} = {\mathbb{H}^2_{\mathbb{R}}}^{\mp}$, we see that $\mathscr{N}_{\Delta,\Delta}(t)$ equals the number of images under $\Gamma_{\mathbb{Z}}$ of $\widetilde{\Delta}$ that are contained in ${\mathbb{H}^2_{\mathbb{R}}}^{+}$ and at a positive distance at most t from the imaginary axis.

Lemma 11. *Let* $\gamma = \left[\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right]\in\Gamma_{\mathbb{Z}}$.

(1) The geodesic lines $\widetilde{\Delta}$ *and* $\gamma\widetilde{\Delta}$ *have a common point at infinity if and only if* $a\,b\,c\,d = 0$. *They have a common perpendicular if and only if* $a\,b\,c\,d\neq 0$.

(2) The geodesic line $\gamma\widetilde{\Delta}$ *is contained in the right halfplane* ${\mathbb{H}^2_{\mathbb{R}}}^{+}$ *and has no common point at infinity with* $\widetilde{\Delta}$ *if and only if* $\gamma\Gamma_{\widetilde{\Delta}}$ *has a representative* $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$ *with* $a,b,c,d>0$. *This representative is then unique.*

Proof. (1) Note that $\gamma\cdot\infty = \frac{a}{c}$ and $\gamma\cdot 0 = \frac{b}{d}$. The intersection $\partial_\infty(\gamma\widetilde{\Delta}\,)\cap\partial_\infty\widetilde{\Delta}$ is nonempty if and only if $\gamma\cdot\infty=\infty$ or $\gamma\cdot 0 = 0$ or $\gamma\cdot 0=\infty$ or $\gamma\cdot\infty = 0$, that is, if and only if $c=0$ or $b=0$ or $d=0$ or $a=0$ respectively, which proves the first part of Assertion (1).

The geodesic lines $\widetilde{\Delta}$ and $\gamma\widetilde{\Delta}$ have a common perpendicular if and only if they do not have a common point at infinity and do not have a common point inside $\mathbb{H}^2_\mathbb{R}$. Let Γ_∞ be the stabilizer in $\Gamma_\mathbb{Z}$ of the horoball $\mathscr{H}_\infty$ defined in Equation (6) for $n=2$. It is well known that the $\Gamma_\mathbb{Z}$-equivariant family $(\gamma\mathscr{H}_\infty)_{\gamma\in\Gamma_\mathbb{Z}/\Gamma_\infty}$ is precisely invariant. The only horoballs in this family containing i are $\mathscr{H}_\infty$ and $\iota\mathscr{H}_\infty$, and $\widetilde{\Delta}$ is contained in $\mathscr{H}_\infty\cup\iota\mathscr{H}_\infty$. Hence for every $\gamma\in\Gamma_\mathbb{Z}$, the geodesic lines $\widetilde{\Delta}$ and $\gamma\widetilde{\Delta}$ meet if and only if $\gamma\in\Gamma_{\widetilde{\Delta}}=\{\mathrm{id},\iota\}$, in which case $\widetilde{\Delta}$ and $\gamma\widetilde{\Delta}$ have a common point at infinity.

Thus $\widetilde{\Delta}$ and $\gamma\widetilde{\Delta}$ have a common perpendicular if and only if the points $\gamma\cdot\infty=\frac{a}{c}$ and $\gamma\cdot 0=\frac{b}{d}$ are in the same connected component of $\mathbb{R}\smallsetminus\{0\}$. This yields the inequality $\frac{a}{c}\frac{b}{d}>0$, which proves the second part of Assertion (1) by multiplying by $(cd)^2>0$.

(2) The above computations show that $\gamma\widetilde{\Delta}$ is contained in the right halfplane ${\mathbb{H}^2_\mathbb{R}}^+$ and has no common point at infinity with $\widetilde{\Delta}$ if and only if $ac>0$ and $bd>0$. Since $a\neq 0$, the element γ has a unique representative in $\mathrm{SL}_2(\mathbb{Z})$ such that $a>0$, and then $c>0$. Assume from now on that $a>0$. Note that $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1\\ -1 & 0\end{smallmatrix}\right)=\left(\begin{smallmatrix} -b & a\\ -d & c\end{smallmatrix}\right)$. Hence if $b>0$, then $d>0$ and $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ is the unique representative in $\mathrm{SL}_2(\mathbb{Z})$ with positive coefficients of an element of $\gamma\Gamma_{\widetilde{\Delta}}$. And if conversely $b<0$, then $d<0$ and $\left(\begin{smallmatrix} -b & a\\ -d & c\end{smallmatrix}\right)$ is the unique representative in $\mathrm{SL}_2(\mathbb{Z})$ with positive coefficients of an element of $\gamma\Gamma_{\widetilde{\Delta}}$. $\square$

Now, let us define the special conjugacy classes in $\Gamma_\mathbb{Z}$ that we will study. Let

$$w=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\quad\text{and}\quad w_1=\begin{bmatrix}1 & 0\\ 1 & -1\end{bmatrix},$$

which are involutions (elements of order 2) in $\mathrm{PGL}_2(\mathbb{Z})$. Recall that $\mathrm{PGL}_2(\mathbb{Z})$ acts by conjugation on its normal subgroup $\Gamma_\mathbb{Z}=\mathrm{PSL}_2(\mathbb{Z})$. An element $\gamma\in\Gamma_\mathbb{Z}$ is *ambiguous of the first kind*, respectively *ambiguous of the second kind*, if

$$w\gamma w=\gamma^{-1}\,,\qquad\text{respectively}\qquad w_1\gamma w_1=\gamma^{-1}\,,\tag{71}$$

and *ambiguous* if it is conjugated in $\Gamma_\mathbb{Z}$ to an element in $\Gamma_\mathbb{Z}$ which is ambiguous of the first kind or ambiguous of the second kind. Such elements, when hyperbolic, are automorphs of Gauss' ambiguous integral binary quadratic forms, see [Sar] and [Cas, Sect. 14.4] for details and background. Recall that an hyperbolic element $\gamma\in\Gamma_\mathbb{Z}$ has a unique *root*, i.e. an element $\gamma_0\in\Gamma_\mathbb{Z}$ such that there exists $n\in\mathbb{N}\smallsetminus\{0\}$ with $\gamma=\gamma_0^n$, and that γ is *primitive* if $\gamma=\gamma_0$. For a hyperbolic element of $\Gamma_\mathbb{Z}$, being ambiguous, ambiguous of the first kind or ambiguous of the second kind is invariant by taking nonzero powers and roots.

The normalizer of $\Gamma_\mathbb{Z}$ in the full isometry group of $\mathbb{H}^2_\mathbb{R}$ contains the reflexion

$$W: z\mapsto -\overline{z}$$

in the geodesic line $\widetilde{\Delta}$. The *extended modular group* $\Gamma^+_\mathbb{Z}$ is the group generated by $\Gamma_\mathbb{Z}$ and W. It contains $\Gamma_\mathbb{Z}$ as a normal subgroup of index 2. The two extensions $\mathrm{PGL}_2(\mathbb{Z})$ and $\Gamma^+_\mathbb{Z}$ of $\Gamma_\mathbb{Z}$ are actually isomorphic, see [Bea2] for a detailed discussion. Let $\overline{\cdot}: z\mapsto\overline{z}$ be the complex conjugation. The map $\Phi:\mathrm{PGL}_2(\mathbb{Z})\to\Gamma^+_\mathbb{Z}$, which is the identity on $\Gamma_\mathbb{Z}$ and maps $\eta\in\mathrm{PGL}_2(\mathbb{Z})\smallsetminus\Gamma_\mathbb{Z}$ to the anti-homography $\eta\circ\overline{\cdot}:\mathbb{H}^2_\mathbb{R}\to\mathbb{H}^2_\mathbb{R}$, is a group isomorphism, that is compatible with the actions on $\Gamma_{\mathbb{Z}}$ by conjugation of the two groups: For all $\gamma \in \Gamma_{\mathbb{Z}}$ and $\eta \in \mathrm{PGL}_2(\mathbb{Z})$, we have

$$\Phi(\eta)\,\gamma\,\Phi(\eta)^{-1} = \eta\,\gamma\,\eta^{-1}\,.$$

If $\eta \in \mathrm{PGL}_2(\mathbb{Z}) \smallsetminus \Gamma_{\mathbb{Z}}$ is an involution, then $\Phi(\eta)$ is a reflexion in the geodesic line whose endpoints are the two fixed points of η in $\mathbb{P}_1(\mathbb{R})$. In particular, we have $\Phi(w) = W$, and $W_1 = \Phi(w_1)$ is the reflexion in the geodesic line $\widetilde{\Delta}_1 = \,]0,2[$. The group $\Gamma_{\mathbb{Z}}^{+}$ has exactly three conjugacy classes of involutions, which are the conjugacy classes of W, of W_1 and of the orientation-preserving involution ι. Given an involution $\tau \in \Gamma_{\mathbb{Z}}^{+}$, as in [ErPP], we say that an element $\gamma \in \Gamma_{\mathbb{Z}}^{+}$ is *τ-reciprocal* in $\Gamma_{\mathbb{Z}}^{+}$ if $\tau\gamma\tau = \gamma^{-1}$. We denote by Ax_γ the translation axis of every hyperbolic element $\gamma \in \Gamma_{\mathbb{Z}}$.

Lemma 12. *Let $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_{\mathbb{Z}}$.*

(i) The element γ is ambiguous if and only if there exists an involution $\tau \in \Gamma_{\mathbb{Z}}^{+} \smallsetminus \Gamma_{\mathbb{Z}}$ such that γ is τ-reciprocal in $\Gamma_{\mathbb{Z}}^{+}$.

(ii) The element $\gamma \in \Gamma_{\mathbb{Z}}$ is ambiguous of the first kind (respectively ambiguous of the second kind) if and only if $a = d$ (respectively $a + b = d$). If $\gamma \in \Gamma_{\mathbb{Z}}$ is hyperbolic, then γ is ambiguous of the first kind (respectively ambiguous of the second kind) if and only if Ax_γ meets perpendicularly $\widetilde{\Delta} = \mathrm{Fix}(W)$ (respectively $\widetilde{\Delta}_1 = \mathrm{Fix}(W_1)$), and γ is ι-reciprocal if and only if Ax_γ contains $\{i\} = \mathrm{Fix}(\iota)$.

(iii) The only elements of $\Gamma_{\mathbb{Z}}$ that are both ambiguous of the first kind and ambiguous of the second kind are $\left[\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right]$ for $c \in \mathbb{Z}$, which are not hyperbolic. There are hyperbolic elements that are both conjugated to an element ambiguous of the first kind and conjugated to an element ambiguous of the second kind.[7]

(iv) A primitive hyperbolic element of $\Gamma_{\mathbb{Z}}$ that is conjugated to an element ambiguous of a given kind and not conjugated to an element ambiguous of the other kind has exactly 4 conjugates that are ambiguous of the given kind. A primitive hyperbolic element of $\Gamma_{\mathbb{Z}}$ that is conjugated both to an element ambiguous of the first kind and to an element ambiguous of the second kind has exactly 2 conjugates that are ambiguous of the first kind and 2 conjugates that are ambiguous of the second kind.

Proof. (i) Let $\gamma \in \Gamma_{\mathbb{Z}}$ be ambiguous. Then there exist $\nu \in \Gamma_{\mathbb{Z}}$ and $w' \in \{w, w_1\}$ such that $w'(\nu\gamma\nu^{-1})w' = (\nu\gamma\nu^{-1})^{-1}$. Thus, we have

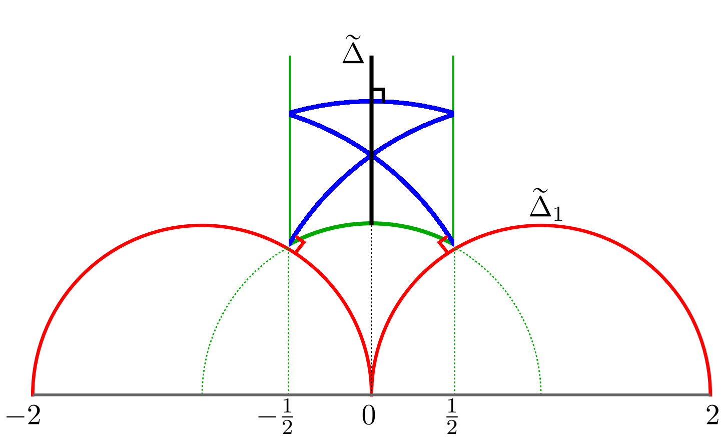
$$\gamma^{-1} = (\nu^{-1}w'\nu)^{-1}\gamma(\nu^{-1}w'\nu) = \Phi(\nu^{-1}w'\nu)^{-1}\gamma\,\Phi(\nu^{-1}w'\nu)\,,$$

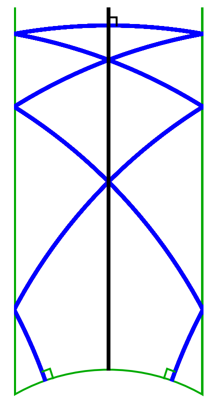
and γ is τ-reciprocal, with $\tau = \Phi(\nu^{-1}w'\nu) \in \Gamma_{\mathbb{Z}}^{+} \smallsetminus \Gamma_{\mathbb{Z}}$. The converse is proven similarly.

(ii) The first claim follows by an easy computation. An involution $\tau \in \Gamma_{\mathbb{Z}}^{+}$ preserves a geodesic line L in $\mathbb{H}^2_{\mathbb{R}}$ if and only if either $\tau \in \Gamma_{\mathbb{Z}}$ and L contains the singleton $\mathrm{Fix}(\tau)$, or $\tau \in \Gamma_{\mathbb{Z}}^{+} \smallsetminus \Gamma_{\mathbb{Z}}$ and either L intersects the geodesic line $\mathrm{Fix}(\tau)$ perpendicularly or $L = \mathrm{Fix}(\tau)$. The second claim follows since for all $\beta, \gamma \in \Gamma_{\mathbb{Z}}^{+}$ with γ hyperbolic, we have $\mathrm{Ax}_{\beta\gamma\beta^{-1}} = \beta\,\mathrm{Ax}_\gamma$, and γ and γ^{-1} translate in opposite directions on $\mathrm{Ax}_{\gamma^{-1}} = \mathrm{Ax}_\gamma$.

[7] The existence of such elements is not immediate from the arithmetic definitions. We will compute the asymptotic growth of the number of the conjugacy classes of these elements with translation length at most $s \to +\infty$ during the proof of Theorem 14, see Lemma 16 (i) and Equation (72): It is equal to $\frac{3}{8\,\pi^2}\; s^2\, e^{\frac{s}{2}} + \mathrm{O}\big(s\; e^{\frac{s}{2}}\big)$, hence it is not negligible with respect to the number of those containing only ambiguous elements of a given kind.

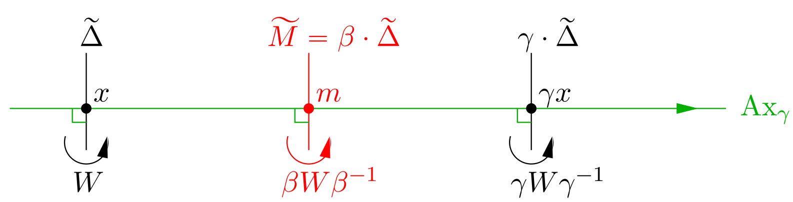
(iii) The first claim follows easily from the first claim of Assertion (ii). We prove the second claim by giving an explicit example. Let $p = \left[\begin{smallmatrix} 2 & 1 \\ 3 & 2 \end{smallmatrix}\right] \in \Gamma_{\mathbb{Z}}$. Then p is hyperbolic, and ambiguous of the first kind by the first claim of Assertion (ii). The element $\nu = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 3 \end{smallmatrix}\right] \in \Gamma_{\mathbb{Z}}$ maps the geodesic line $\widetilde{\Delta}_1 = \,]0, 2[$ to $]\frac{1}{3}, 1[$. The reflexion in the geodesic line $]\frac{1}{3}, 1[$ is hence $\nu W_1 \nu^{-1} \in \Gamma_{\mathbb{Z}}^{+} \smallsetminus \Gamma_{\mathbb{Z}}$. We have $(\nu W_1 \nu^{-1})\, p\, (\nu W_1 \nu^{-1}) = p^{-1}$ by an easy computation, so that $\nu^{-1} p\, \nu$ is ambiguous of the second kind.





The above picture on the left shows in blue the image in M_Γ of the translation axis of p, lifted to the standard fundamental domain of $\Gamma_{\mathbb{Z}}$ with its usual boundary identifications, that is orthogonal in M_Γ both to Δ_1 (at the cone point of angle $\frac{2\pi}{3}$ of the orbifold M_Γ) and to Δ. The above picture on the right shows similarly the image in M_Γ of the translation axis of $p' = \left[\begin{smallmatrix} 3 & 8 \\ 1 & 3 \end{smallmatrix}\right]$, which is the union of two common perpendiculars between $\widetilde{\Delta}$ and $]-1, +1[$. Noting that $]-1, +1[$ is the image of $\widetilde{\Delta}_1$ by the element $z \mapsto z - 1$ of $\Gamma_{\mathbb{Z}}$, the element p' is hence also both (conjugated to) an ambiguous element of the first kind and conjugated to an ambiguous element of the second kind.

(iv) Let $\gamma \in \Gamma_{\mathbb{Z}}$ be primitive hyperbolic, ambiguous of the first kind and not conjugated in $\Gamma_{\mathbb{Z}}$ to an element ambiguous of the second kind. By Lemma 12 (ii), let x be the perpendicular intersection point of $\widetilde{\Delta}$ with Ax_γ. The element $\gamma W \in \Gamma_{\mathbb{Z}}^{+} \smallsetminus \Gamma_{\mathbb{Z}}$ is the reflexion in the mediatrix $\widetilde{M}$ of the segment $[x, \gamma \cdot x]$. Let m be the midpoint of $[x, \gamma \cdot x]$. Since the involutions of $\Gamma_{\mathbb{Z}}^{+} \smallsetminus \Gamma_{\mathbb{Z}}$ are conjugated by elements of $\Gamma_{\mathbb{Z}}$ to either W or W_1, let $\beta \in \Gamma_{\mathbb{Z}}$ be such that $\widetilde{M} = \beta \cdot \widetilde{\Delta}$ or $\widetilde{M} = \beta \cdot \widetilde{\Delta}_1$. The second possibility does not occur, otherwise $\beta^{-1}\,\mathrm{Ax}_\gamma$ would meet perpendicularly $\widetilde{\Delta}_1$, and $\beta^{-1}\gamma\beta$ would be ambiguous of the second kind. Note that β is unique up to right multiplication by ι since $\Gamma_{\widetilde{\Delta}} = \{\mathrm{id}, \iota\}$.



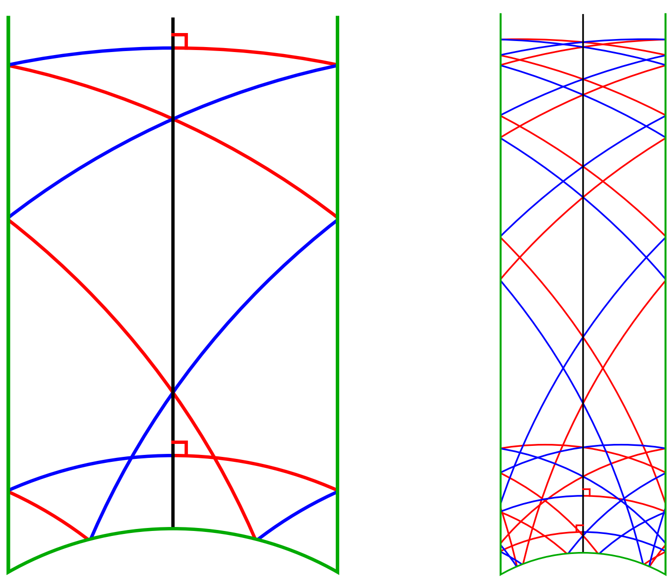
Let $\alpha \in \Gamma_{\mathbb{Z}}$ be such that $\alpha\gamma\alpha^{-1}$ is ambiguous of the first kind. Up to replacing α by a right multiple by a power of γ, which does not change $\alpha\gamma\alpha^{-1}$, we may assume that $\alpha^{-1} \cdot \widetilde{\Delta}$ meets perpendicularly Ax_γ in $z \in [x, \gamma \cdot x[$. We claim that $\alpha^{-1} \cdot \widetilde{\Delta} = \widetilde{\Delta}$ or $\alpha^{-1} \cdot \widetilde{\Delta} = \beta \cdot \widetilde{\Delta}$. Otherwise, if $z \in \,]x, m[$, then $(\alpha^{-1} W \alpha) W$ would be an hyperbolic element with same translation axis as γ, but with translation length $2\,d(x,z) < d(x, \gamma \cdot x)$, contradicting the fact that γ is primitive. And similarly if $z \in \,]m, \gamma \cdot x[$, then $(\alpha^{-1} W \alpha)(\beta W \beta^{-1})$ would be an hyperbolic element with same translation axis as γ and translation length $2\,d(m,z) < d(x, \gamma \cdot x)$. Therefore $\alpha^{-1} \in \{\mathrm{id}, \iota, \beta, \beta\iota\}$, which proves the first claim, upon checking that γ, $\iota\gamma\iota$, $\beta^{-1}\gamma\beta$ and $\iota\beta^{-1}\gamma\beta\iota$ are pairwise distinct, since the centralizer of γ in $\Gamma_{\mathbb{Z}}$ is $\gamma^{\mathbb{Z}}$.

The second claim is proven similarly, except that now $\widetilde{M} = \beta \cdot \widetilde{\Delta}_1$, and the conjugates of γ that are ambiguous of the first kind are γ and $\iota\gamma\iota$, and the conjugates of γ that are ambiguous of the second kind are $\beta^{-1}\gamma\beta$ and $\iota\beta^{-1}\gamma\beta\iota$. $\square$

Remark 13. Let $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_{\mathbb{Z}}$ with $a, b, c, d > 0$. The composition

$$[\gamma, W] = (\gamma W \gamma^{-1}) W = \begin{bmatrix} ad + bc & 2ab \\ 2cd & ad + bc \end{bmatrix} \in \Gamma_{\mathbb{Z}}$$

of the reflexions W in $\widetilde{\Delta}$ and $\gamma W \gamma^{-1}$ in $\gamma\widetilde{\Delta}$ is ambiguous of the first kind by Lemma 12 (ii) or since $W[\gamma, W]W = [W, \gamma] = [\gamma, W]^{-1}$. By Lemma 11, the geodesic lines $\widetilde{\Delta}$ and $\gamma\widetilde{\Delta}$ are disjoint, hence $[\gamma, W]$ is hyperbolic. Its translation axis is the geodesic line that contains the common perpendicular from $\widetilde{\Delta}$ to $\gamma\widetilde{\Delta} = \,]\frac{b}{d}, \frac{a}{c}[$, which is $\Big]-\sqrt{\frac{ab}{cd}}, \sqrt{\frac{ab}{cd}}\Big[$ by an easy computation (see the picture in the proof of Lemma 18 with $x = \frac{b}{d}$ and $y = \frac{a}{c}$). Geometrically, this implies that any common perpendicular from Δ to itself in $M_{\mathbb{Z}}$ can be extended to a closed geodesic in $M_{\mathbb{Z}}$, of length twice the length of the common perpendicular.



The figure on the left (respectively right) shows the common perpendicular from Δ to itself in $M_{\mathbb{Z}}$, lifted to the standard fundamental domain of $\Gamma_{\mathbb{Z}}$ with its usual boundary identifications, constructed as above with $\gamma = \left[\begin{smallmatrix} 3 & 1 \\ 2 & 1 \end{smallmatrix}\right]$ (respectively $\gamma = \left[\begin{smallmatrix} 31 & 23 \\ 35 & 26 \end{smallmatrix}\right]$), and its reflexion along Δ, doubling it to a closed geodesic of $M_{\mathbb{Z}}$.

The figure on page 2 shows the common perpendiculars from Δ to itself constructed as above by elements γ with $bc \leqslant 300$ and $2.05 \leqslant \sqrt{\frac{ab}{cd}} \leqslant 2.1$. The initial tangent vectors of these common perpendiculars have their footpoints in the standard fundamental domain contained in the interval $[2.05\,i, 2.1\,i]$ in the positive imaginary axis. The mustard yellow geodesic defined as above by $\gamma = \left[\begin{smallmatrix} 25 & 23 \\ 12 & 11 \end{smallmatrix}\right]$ meets $]-\frac{1}{2}, \infty[\; = \iota\widetilde{\Delta}_1$ perpendicularly at its highest point, indicating that the element $[\gamma, W]$ is also conjugate to an ambiguous element of the second kind by Lemma 12 (ii).

The following theorem is the main result of this section. A closed geodesic (oriented but not pointed, understood in the orbifold sense in particular concerning its length) in $M_{\mathbb{Z}}$ is *ambiguous* if it meets perpendicularly Δ or Δ_1, and *reciprocal* if it contains $\Gamma_{\mathbb{Z}} i$. Geometrically, such a closed geodesic then "reflects" in Δ, Δ_1, or $\Gamma_{\mathbb{Z}} i$ when mapped to $\Gamma_{\mathbb{Z}}^+ \backslash \mathbb{H}^2_{\mathbb{R}}$.

Theorem 14. *The number $\mathscr{N}_A(s)$ of ambiguous primitive closed geodesics of length at most s satisfies, as $s \to +\infty$, that*

$$\mathscr{N}_A(s) = \frac{3}{4\,\pi^2}\, s^2\, e^{\frac{s}{2}} + \mathrm{O}(s\, e^{\frac{s}{2}})\,.$$

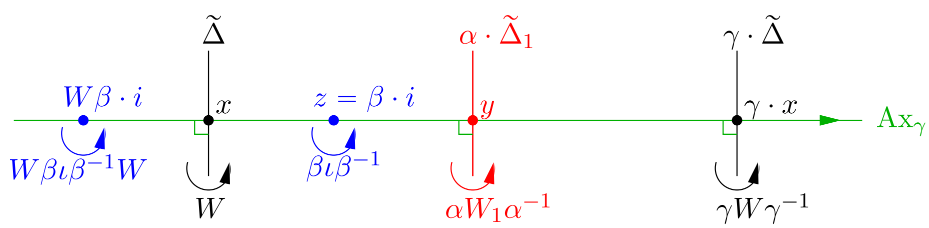
The number $\mathscr{N}_{AR}(s)$ of primitive closed geodesics of length at most s that are both ambiguous and reciprocal satisfies, as $s \to +\infty$, that

$$\mathscr{N}_{AR}(s) = \frac{3}{8\,\pi}\, s\, e^{\frac{s}{4}} + \mathrm{O}(e^{\frac{s}{4}})\,.$$

The map from the set of conjugacy classes of primitive hyperbolic elements of $\Gamma_{\mathbb{Z}}$ to the set of primitive (oriented but not pointed) closed geodesic of the modular orbifold $M_{\mathbb{Z}}$, which maps such a conjugacy class $[\![\gamma]\!]$ to the image in $M_{\mathbb{Z}}$ of the oriented geodesic segment $[a, \gamma a]$ for any $a \in \mathrm{Ax}_\gamma$, is well-known to be a bijection. By Lemma 12 (ii), it sends ambiguous/reciprocal conjugacy classes to ambiguous/reciprocal closed geodesics. Recall that the absolute value x of the trace of any representative in $\mathrm{SL}_2(\mathbb{Z})$ of any element of the class $[\![\gamma]\!]$ and the length s of the associated closed geodesic satisfy, when large, that $x = 2\cosh\frac{s}{2} \sim e^{\frac{s}{2}}$. We hence recover, in the two claims of Theorem 14, respectively Equation (12) and Equation (15) of [Sar], up to a multiplicative constant, possibly coming from the fact that the equality in Equation (61) in [Sar] seems incorrect by Lemma 12 (iii).

Proof. For every $s > 0$, let us denote by $\mathrm{AC}(s)$ the set of ambiguous conjugacy classes of primitive hyperbolic elements of $\Gamma_{\mathbb{Z}}$, by $\mathrm{AC}^{1\smallsetminus 2}(s)$ the ones containing an ambiguous element of the first kind, but no ambiguous element of the second kind, $\mathrm{AC}^{2\smallsetminus 1}(s)$ the ones containing an ambiguous element of the second kind, but no ambiguous element of the first kind, and $\mathrm{AC}^{1\&2}(s)$ the ones containing both an ambiguous element of the first kind and an ambiguous element of the second kind. Let $\mathrm{ARC}(s)$, $\mathrm{ARC}^{1\smallsetminus 2}(s)$, $\mathrm{ARC}^{2\smallsetminus 1}(s)$, $\mathrm{ARC}^{1\&2}(s)$ be the intersection of these sets with the set of conjugacy classes of reciprocal primitive hyperbolic elements of $\Gamma_{\mathbb{Z}}$.

Lemma 15. *The set $\mathrm{ARC}^{1\&2}(s)$ is empty.*



Proof. Assume for a contradiction that there exists a primitive hyperbolic element $\gamma \in \Gamma_{\mathbb{Z}}$ which is ambiguous of the first kind, conjugated to an element ambiguous of the second kind and reciprocal. By Lemma 12 (ii), there exists $\alpha, \beta \in \Gamma_{\mathbb{Z}}$ such that Ax_γ meets perpendicularly $\widetilde{\Delta}$ at a point x, meets perpendicularly $\alpha \cdot \widetilde{\Delta}_1$ at a point y and contains the point $z = \beta \cdot i$. Up to multiplying α and β on the left by powers of γ, we may assume that $y, z \in [x, \gamma \cdot x[$. Note that $x \neq y$ since $\widetilde{\Delta}$ and $\widetilde{\Delta}_1$ are not in the same $\Gamma_{\mathbb{Z}}$ orbit, that $x \neq z$ since $(\Gamma_{\mathbb{Z}} \cdot i) \cap \widetilde{\Delta} = \{i\}$ and the orthogonal geodesic line $]-1, 1[$ to $\widetilde{\Delta}$ at i, whose endpoints are rational, cannot be a translation axis of an element of $\Gamma_{\mathbb{Z}}$, and similarly that $y \neq z$. Let $\ell(\gamma)$ be the translation length of γ, which is the minimal translation length of an hyperbolic element of $\Gamma_{\mathbb{Z}}$ whose translation axis is Ax_γ since γ is primitive. We claim that $d(x, y) \geqslant \frac{\ell(\gamma)}{2}$. Otherwise, the element $(\alpha W_1 \alpha^{-1})W$, composition of the reflexions W in $\widetilde{\Delta}$ and $\alpha W_1 \alpha^{-1}$ in $\alpha\widetilde{\Delta}_1$, would belong to $\Gamma_{\mathbb{Z}}$, and be hyperbolic with translation axis Ax_γ and translation distance $2\,d(x, y) < \ell(\gamma)$, a contradiction. Similarly, we have $d(\gamma \cdot x, y) \geqslant \frac{\ell(\gamma)}{2}$ and hence y is the midpoint of the segment $[x, \gamma \cdot x]$.

As W fixes $i \in \widetilde{\Delta}$ and W_1 fixes $1 + i \in \widetilde{\Delta}_1$, and since W and W_1 normalize $\Gamma_{\mathbb{Z}}$, we have $W\Gamma_{\mathbb{Z}} \cdot i = \Gamma_{\mathbb{Z}} \cdot i$ and $W_1\Gamma_{\mathbb{Z}} \cdot i = W_1\Gamma_{\mathbb{Z}} \cdot (1 + i) = \Gamma_{\mathbb{Z}} \cdot (1 + i) = \Gamma_{\mathbb{Z}} \cdot i$. Since $\alpha W_1 \alpha^{-1}$ is the reflexion along $\alpha \cdot \Delta_1$, both segments $]x, y[$ and $]y, \gamma \cdot x[$ contain a point of $\Gamma_{\mathbb{Z}} \cdot i$. Hence we may assume that $z = \beta \cdot i \in]x, y[$, so that $d(x, z) < d(x, y)$. Consider $(\beta\iota\beta^{-1})(W\beta\iota\beta^{-1}W)$, the composition of the angle π hyperbolic rotations $W\beta\iota\beta^{-1}W$ around $W \cdot z = W\beta \cdot i$ and $\beta\iota\beta^{-1}$ around $z = \beta \cdot i$. It belongs to $\Gamma_{\mathbb{Z}}$, and is hyperbolic with translation axis Ax_γ and translation distance $2\,d(x, z) < 2\,d(x, y) = \ell(\gamma)$, a contradiction. $\square$

Let $\widetilde{D}^-, \widetilde{D}^+ \in \{\widetilde{\Delta}, \widetilde{\Delta}_1, \{i\}\}$ and let D^-, D^+ be their images in $M_{\mathbb{Z}}$. For every $s > 0$, we denote by $\mathrm{Perp}'(D^-, D^+, s)$ the set of common perpendiculars in $M_{\mathbb{Z}}$ between D^- and D^+ that are *primitive*, i.e. that do not meet perpendicularly in their interior $\Delta, \Delta_1, \{\Gamma_{\mathbb{Z}} \cdot i\}$ (with the convention that an open geodesic segment meets perpendicularly a point if and only if it contains it).

Lemma 16. *(i) The map $\Psi^{1\&2}$ from $\mathrm{AC}^{1\&2}(s)$ to $\mathrm{Perp}'(\Delta, \Delta_1, \frac{s}{2})$ which maps the conjugacy class of an ambiguous primitive hyperbolic element of the first kind $\gamma \in \Gamma_{\mathbb{Z}}$ to the image in $M_{\mathbb{Z}}$ of the oriented geodesic segment $[x, m]$ where x is the perpendicular intersection point of Ax_γ with $\widetilde{\Delta}$, and where m is the midpoint of $[x, \gamma \cdot x]$, is a bijection.*

(ii) The map $\Phi^{1\searrow 2}$ from $\mathrm{Perp}'(\Delta, \Delta, \frac{s}{2})$ to $\mathrm{AC}^{1\searrow 2}(s) \smallsetminus \mathrm{ARC}^{1\searrow 2}(s)$ which maps the image in $M_{\mathbb{Z}}$ of the common perpendicular between $\widetilde{\Delta}$ and a disjoint image $\beta \cdot \widetilde{\Delta}$ with $\beta \in \Gamma_{\mathbb{Z}}$ to the conjugacy class of $\beta W \beta^{-1} W$ is a 2-to-1 map.

Similarly, $\mathrm{AC}^{2\searrow 1}(s) \smallsetminus \mathrm{ARC}^{2\searrow 1}(s)$ has half the cardinality of $\mathrm{Perp}'(\Delta_1, \Delta_1, \frac{s}{2})$.

Proof. (i) If $\gamma \in \Gamma_{\mathbb{Z}}$ is primitive hyperbolic, ambiguous of the first kind, with conjugacy class in $\mathrm{AC}^{1\&2}(s)$, let x (which exists by Lemma 12 (ii)) and m be as in the statement. As seen in the proof of the second claim of Lemma 12 (iv), note that m is the perpendicular intersection point of Ax_γ with $\beta \cdot \widetilde{\Delta}_1$ for some $\beta \in \Gamma_{\mathbb{Z}}$. Hence $c_\gamma = \Gamma_{\mathbb{Z}} \cdot [x, m]$ is indeed a common perpendicular between Δ and Δ_1. By Lemma 15, the interior of $[x, m]$ contains no point of the orbit $\Gamma_{\mathbb{Z}} \cdot i$. If the interior of $[x, m]$ was meeting perpendicularly the image of Δ or of Δ_1 by some $\beta' \in \Gamma_{\mathbb{Z}}$, then the element $\beta' W (\beta')^{-1} W$ or $\beta' W_1 (\beta')^{-1} W$, which belongs to $\Gamma_{\mathbb{Z}}$ and is hyperbolic, would have the same translation axis as γ, and a strictly shorter translation length, contradicting the fact that γ is primitive. Hence c_γ is primitive. Since $d(x, m) = \frac{1}{2} d(x, \gamma \cdot x) \leqslant \frac{s}{2}$, we have $c_\gamma \in \mathrm{Perp}'(\Delta, \Delta_1, \frac{s}{2})$.

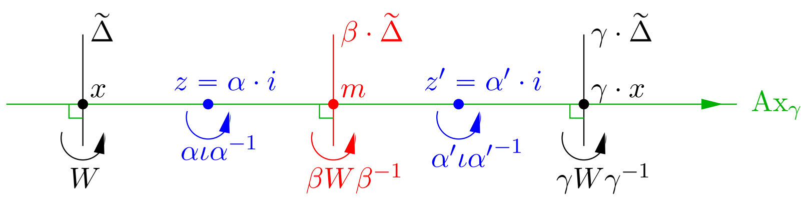
If $\gamma' \in \Gamma_{\mathbb{Z}}$ is primitive hyperbolic, ambiguous of the first kind, conjugated to γ and different from γ, then by the second claim of Lemma 12 (iv), we have $\gamma' = \iota\gamma\iota$. Hence the perpendicular intersection point of $\mathrm{Ax}_{\gamma'}$ with $\widetilde{\Delta}$ is $x' = \iota \cdot x$, and the midpoint of $[x', \gamma' \cdot x'] = \iota \cdot [x, \gamma \cdot x]$ is $\iota \cdot m$. Therefore $c_{\gamma'} = \Gamma_{\mathbb{Z}} \cdot [x', m'] = \Gamma_{\mathbb{Z}}\, \iota \cdot [x, m] = c_\gamma$, and the map $\Psi^{1\&2}$ is well defined. The map $\Phi^{1\&2}$ from $\mathrm{Perp}'(\Delta, \Delta_1, \frac{s}{2})$ to $\mathrm{AC}^{1\smallsetminus 2}(s)$ which maps the image in $M_{\mathbb{Z}}$ of the common perpendicular between $\widetilde{\Delta}$ and a disjoint image $\beta \cdot \widetilde{\Delta}_1$ for some $\beta \in \Gamma_{\mathbb{Z}}$ to the conjugacy class of $(\beta W_1 \beta^{-1})W$ is easily seen to be an inverse of $\Psi^{1\&2}$.

(ii) Let $\beta \in \Gamma_{\mathbb{Z}}$ be such that the intersection $\widetilde{\Delta} \cap \beta \cdot \widetilde{\Delta}$ is empty. Let $\widetilde{c} = [x, y]$ be the common perpendicular between $\widetilde{\Delta}$ and $\beta \cdot \widetilde{\Delta}$ with $x \in \widetilde{\Delta}$, and assume that its interior does not meet perpendicularly an image of $\widetilde{\Delta}$, $\widetilde{\Delta}_1$ or $\{i\}$ by an element of $\Gamma_{\mathbb{Z}}$. Then the composition $\gamma_{\widetilde{c}} = (\beta W \beta^{-1})W$ of the reflexion W in $\widetilde{\Delta}$ and the reflexion $\beta W \beta^{-1}$ in $\beta \cdot \widetilde{\Delta}$ is a hyperbolic element of $\Gamma_{\mathbb{Z}}$, with translation axis containing $\widetilde{c}$, and y is the midpoint of x and $\gamma_{\widetilde{c}} \cdot x$. Hence $d(x, \gamma_{\widetilde{c}} \cdot x) = 2\, d(x, y) \leqslant 2\, \frac{s}{2} = s$. If $\gamma_{\widetilde{c}}$ is not primitive, let γ_0 and $k \geqslant 2$ be such that $\gamma_{\widetilde{c}} = \gamma_0^k$. Then $\gamma_0 \cdot x \in [x, y]$, and $\gamma_0 W$ would be a reflexion in $\Gamma_{\mathbb{Z}}^+$ fixing the mediatrix of $[x, \gamma_0 \cdot x]$. Therefore the interior of $[x, y]$ would meet perpendicularly an image of $\widetilde{\Delta}$ or $\widetilde{\Delta}_1$ by an element of $\Gamma_{\mathbb{Z}}$, a contradiction. Furthermore, $\gamma_{\widetilde{c}}$ is ambiguous of the first kind and not reciprocal nor conjugated to an element ambiguous of the second kind, by Lemma 12 (ii), since $\mathrm{Ax}_{\gamma_{\widetilde{c}}}$ meets perpendicularly $\widetilde{\Delta}$ at x, and does not meet perpendicularly an image of $\{i\}$ or $\widetilde{\Delta}_1$. If $\alpha \in \Gamma_{\mathbb{Z}} \smallsetminus \{\mathrm{id}\}$ is such that $\alpha \cdot \widetilde{c} = [x', y']$ is a common perpendicular between $\widetilde{\Delta}$ and $\beta' \cdot \widetilde{\Delta}$ for some $\beta' \in \Gamma_{\mathbb{Z}}$, then α preserves $\widetilde{\Delta}$, hence $\alpha = \iota$, hence $x' = \iota \cdot x$, $y' = \iota \cdot y$, and $\gamma_{\alpha\, \widetilde{c}} = \iota\, \gamma_{\widetilde{c}}\, \iota$ is conjugated to $\gamma_{\widetilde{c}}$. Hence the map $\Phi^{1\smallsetminus 2}$ is well defined.

Let us prove that the map $\Phi^{1\smallsetminus 2}$ is onto. Let $\gamma \in \Gamma_{\mathbb{Z}}$ be primitive hyperbolic, ambiguous of the first kind and not reciprocal nor conjugated to an element ambiguous of the second kind. Let x be the perpendicular intersection point of $\widetilde{\Delta}$ and Ax_γ, and m the midpoint of $[x, \gamma \cdot x]$. Then $\widetilde{c} = [x, m]$ is a common perpendicular between $\widetilde{\Delta}$ and $\beta \cdot \widetilde{\Delta}$ for some $\beta \in \Gamma_{\mathbb{Z}}$, that does not meet perpendicularly in its interior an image of $\widetilde{\Delta}$ or $\widetilde{\Delta}_1$ or $\{i\}$ by an element of $\Gamma_{\mathbb{Z}}$, since γ is primitive. By construction, we have $\gamma = \gamma_{\widetilde{c}}$. By the proof of the first claim of Lemma 12 (iv), the preimages by $\Phi^{1\smallsetminus 2}$ of the conjugacy class of γ are the images of $[x, m]$ and $[m, \gamma \cdot x]$ in $M_{\mathbb{Z}}$. Note that the two segments $[x, m]$ and $[m, \gamma \cdot x]$ are not in the same orbit under $\Gamma_{\mathbb{Z}}$, since γ is primitive. This proves that $\Phi^{1\smallsetminus 2}$ is 2-to-1. $\square$

Lemma 17. *The map Φ^{1R} from $\mathrm{Perp}'(\Delta, \{\Gamma_{\mathbb{Z}} \cdot i\}, \frac{s}{4})$ to $\mathrm{ARC}^{1\smallsetminus 2}(s)$ which maps the image in $M_{\mathbb{Z}}$ of the common perpendicular between $\widetilde{\Delta}$ and $\alpha \cdot \{i\}$, for some $\alpha \in \Gamma_{\mathbb{Z}}$, to the conjugacy class of $(\alpha\iota\alpha^{-1}W)^2$ is a 2-to-1 map.*

Similarly, the set $\mathrm{ARC}^{2\smallsetminus 1}(s)$ has half the cardinality of $\mathrm{Perp}'(\Delta_1, \{\Gamma_{\mathbb{Z}} \cdot i\}, \frac{s}{4})$.



Proof. Every element $\Gamma_{\mathbb{Z}} \cdot c$ in $\mathrm{Perp}'(\Delta, \{\Gamma_{\mathbb{Z}} \cdot i\}, \frac{s}{4})$ is the image in $M_{\mathbb{Z}}$ of the common perpendicular $[x, z]$ between $\widetilde{\Delta}$ and $\alpha \cdot \{i\}$, for some $\alpha \in \Gamma_{\mathbb{Z}}$ with $z = \alpha \cdot i$. Since $z \notin \widetilde{\Delta}$, the element $\gamma = (\alpha\iota\alpha^{-1}W)^2$, which belongs to $\Gamma_{\mathbb{Z}}$, is a hyperbolic element with translation axis containing $[x,z]$ and translation length $4\,d(x,z)\leqslant 4\,\frac{s}{4}=s$. Since Ax_γ meets perpendicularly $\widetilde{\Delta}$ and contains $\alpha\cdot i$, the element γ is reciprocal and ambiguous of the first kind by Lemma 12 (ii). As in the previous proof, γ is primitive and the conjugacy class of γ in $\Gamma_{\mathbb{Z}}$ does not depend on the above choice of a representative $[x,z]$ of $\Gamma_{\mathbb{Z}}\cdot c$.

Let us prove that Φ^{1R} is onto. Let $\gamma\in\Gamma_{\mathbb{Z}}$ be primitive hyperbolic, reciprocal and ambiguous of the first kind (hence not conjugated to an element ambiguous of the second kind by Lemma 15). As in the proof of the second claim of Lemma 16, let x be the perpendicular intersection point of $\widetilde{\Delta}$ and Ax_γ, let m be the midpoint of $[x,\gamma\cdot x]$ and let $\beta\in\Gamma_{\mathbb{Z}}$ be such that $\beta\cdot\widetilde{\Delta}$ is the mediatrix of $[x,\gamma\cdot x]$. By Lemma 12 (ii), since γ is reciprocal, the translation axis Ax_γ meets the orbit $\Gamma_{\mathbb{Z}}\cdot i$. By translating by powers of γ, there is an orbit point $z=\alpha\cdot i$ in $[x,\gamma\cdot x[$. Since $\Gamma_{\mathbb{Z}}^+$ preserves $\Gamma_{\mathbb{Z}}\cdot i$, up to replacing z by its image by the reflexion $\beta W\beta^{-1}$, we may assume that $z\in[x,m]$. Since Ax_γ has irrational endpoints, we have $z\neq x,m$. If z is not the midpoint of $[x,m]$, say $d(x,z)<\frac{1}{2}d(x,m)$, then $(\alpha\iota\alpha^{-1}W)^2$ would be an hyperbolic element in $\Gamma_{\mathbb{Z}}$ with translation axis Ax_γ and translation length $4\,d(x,z)<2\,d(x,m)=d(x,\gamma\cdot x)$, contradicting the fact that γ is primitive. Hence $d(x,z)=\frac{1}{4}\,d(x,\gamma\cdot x)\leqslant\frac{s}{4}$. Since γ is primitive, the interior of $[x,z]$, which is a common perpendicular between $\widetilde{\Delta}$ and $\alpha\cdot\{i\}$, does not meet perpendicularly an image of $\widetilde{\Delta},\widetilde{\Delta}_1$ or $\{i\}$. Hence Φ^{1R} is onto. Furthermore, using the reflexion $\beta W\beta^{-1}$, there also exists $\alpha'\in\Gamma_{\mathbb{Z}}$ such that the midpoint z' of $[m,\gamma\cdot x]$ is $z'=\alpha'\cdot i$. By the proof of the first claim of Lemma 12 (iv), the preimages by Φ^{1R} of the conjugacy class of γ are the images of $[x,z]$ and $[m,z']$ in $M_{\mathbb{Z}}$. Note that these two segments are not in the same orbit under $\Gamma_{\mathbb{Z}}$, since γ is primitive. This proves that Φ^{1R} is 2-to-1. $\square$

As Δ and Δ_1 are reciprocal, we have $\iota_{\rm rec}(\Delta)=\iota_{\rm rec}(\Delta_1)=1$. Note that the order of the pointwise stabilizer in $\Gamma_{\mathbb{Z}}$ of $\widetilde{\Delta}$ and $\widetilde{\Delta}_1$ is $m(\Delta)=m(\Delta_1)=1$, and the one of $\{i\}$ is $m(\{\Gamma_{\mathbb{Z}}\cdot i\})=2$. Furthermore, by the normalisation of the Patterson-Sullivan measures in Section 2, we have $\|\sigma^+_{\{\Gamma_{\mathbb{Z}}\cdot i\}}\|=\frac{1}{m(\{\Gamma_{\mathbb{Z}}\cdot i\})}\|\mu_i\|=\frac{1}{2}\operatorname{Vol}(\mathbb{S}^1)=\pi$. Recall that $\operatorname{Vol}(M_{\mathbb{Z}})=\frac{\pi}{3}$. By the standard argument comparing the growth of primitive closed geodesics and the nonprimitive ones, see for instance Step 2 of the proof of [PPS, Theorem 9.11], as $s\to+\infty$, if $D^-,D^+\in\{\Delta,\Delta_1\}$, by Theorem 6 applied with $n=2$ and $M=M_{\mathbb{Z}}$, we have then

$$\operatorname{Card}\operatorname{Perp}'(D^-,D^+,s)=\frac{3}{2\pi^2}\,s^2\,e^s+\operatorname{O}\big(s\,e^s\big)\,. \tag{72}$$

Similarly, by Theorem 5, if $D^+\in\{\Delta,\Delta_1\}$, then

$$\operatorname{Card}\operatorname{Perp}'(\{\Gamma_{\mathbb{Z}}\cdot i\},D^+,s)=\frac{3}{2\pi}\,s\,e^s+\operatorname{O}(e^s)\,, \tag{73}$$

Now, by Lemmas 15 and 17, and Equation (73), we have

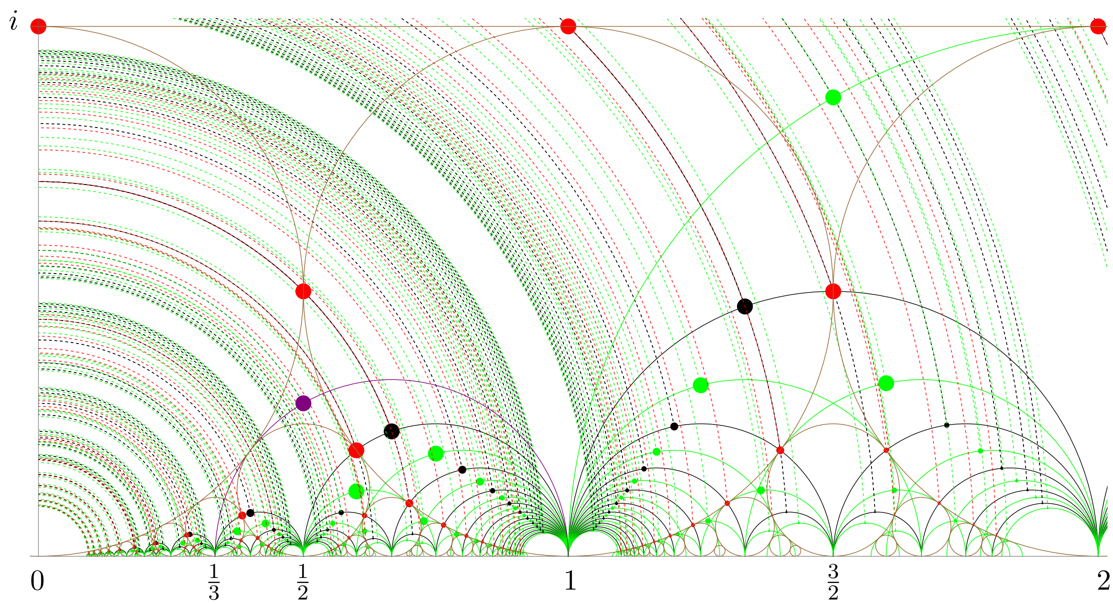
$$\begin{aligned}&\operatorname{Card}\operatorname{ARC}(s)=\operatorname{Card}\operatorname{ARC}^{1\searrow 2}(s)+\operatorname{Card}\operatorname{ARC}^{2\searrow 1}(s)\\&=\frac{1}{2}\operatorname{Card}\operatorname{Perp}'(\Delta,\{\Gamma_{\mathbb{Z}}\cdot i\},\frac{s}{4})+\frac{1}{2}\operatorname{Card}\operatorname{Perp}'(\Delta_1,\{\Gamma_{\mathbb{Z}}\cdot i\},\frac{s}{4})=\frac{3}{8\,\pi}\,s\,e^{\frac{s}{4}}+\operatorname{O}(e^{\frac{s}{4}})\,.\end{aligned}$$

Similarly, using Lemmas 15 and 16, the previous computation that implies that we have

$\operatorname{Card} \mathrm{ARC}(s) = \mathrm{O}(s\, e^{\frac{s}{4}})$, and Equation (72), we have

$$\begin{aligned}
\operatorname{Card} \mathrm{AC}(s) &= \operatorname{Card}\big(\mathrm{AC}^{1\searrow 2}(s) \smallsetminus \mathrm{ARC}^{1\searrow 2}(s)\big) + \operatorname{Card}\big(\mathrm{AC}^{2\searrow 1}(s) \smallsetminus \mathrm{ARC}^{2\searrow 1}(s)\big)\\
&\quad + \operatorname{Card} \mathrm{AC}^{1\&2}(s) + \operatorname{Card} \mathrm{ARC}(s)\\
&= \frac{1}{2} \operatorname{Card} \operatorname{Perp}'(\Delta, \Delta, \frac{s}{2}) + \frac{1}{2} \operatorname{Card} \operatorname{Perp}'(\Delta_1, \Delta_1, \frac{s}{2})\\
&\quad + \operatorname{Card} \operatorname{Perp}'(\Delta, \Delta_1, \frac{s}{2}) + \mathrm{O}(s\, e^{\frac{s}{4}})\\
&= 2\Big(\frac{3}{2\pi^2}\, \big(\frac{s}{2}\big)^2 e^{\frac{s}{2}} + \mathrm{O}\big(\frac{s}{2}\, e^{\frac{s}{2}}\big)\Big) = \frac{3}{4\,\pi^2}\, s^2\, e^{\frac{s}{2}} + \mathrm{O}(s\, e^{\frac{s}{2}})\,.
\end{aligned}$$

This concludes the proof of Theorem 14. □



The above figure shows images in the right halfplane $\mathbb{H}^{2\,+}_{\mathbb{R}}$ by elements of $\Gamma_{\mathbb{Z}}$ of the positive imaginary axis $\widetilde{\Delta}$ in continuous black, and images of $\widetilde{\Delta}_1$ in continuous green (except the geodesic line $]\frac{1}{3}, 1[\, = \nu\widetilde{\Delta}_1$ used in the proof of Lemma 12 (iii) which is drawn in purple). Images by elements of $\Gamma_{\mathbb{Z}}$ of the horosphere $\partial\mathscr{H}_\infty$ are drawn in brown. Images by elements of $\Gamma_{\mathbb{Z}}$ of i are drawn as red points. The common perpendiculars starting from $\widetilde{\Delta}$ and ending at images of $\widetilde{\Delta}$, $\widetilde{\Delta}_1$ or $\{i\}$ by elements of $\Gamma_{\mathbb{Z}}$ are drawn with dashed lines in black, green and red correspondingly to the color of their arrival point, that are marked by black, green and red dots respectively (except the purple one on $]\frac{1}{3}, 1[$). Note that there are (nonprimitive) common perpendiculars passing through black and green points (again giving examples for Lemma 12 (iii)), or through black and red points (corresponding to elements of $\mathrm{ARC}^{1\searrow 2}(s)$ for some large enough s, with the notation of the proof of Theorem 14), but none through black and green and red dots (accordingly to Lemma 15).

7 On the binary additive divisor problem for integers

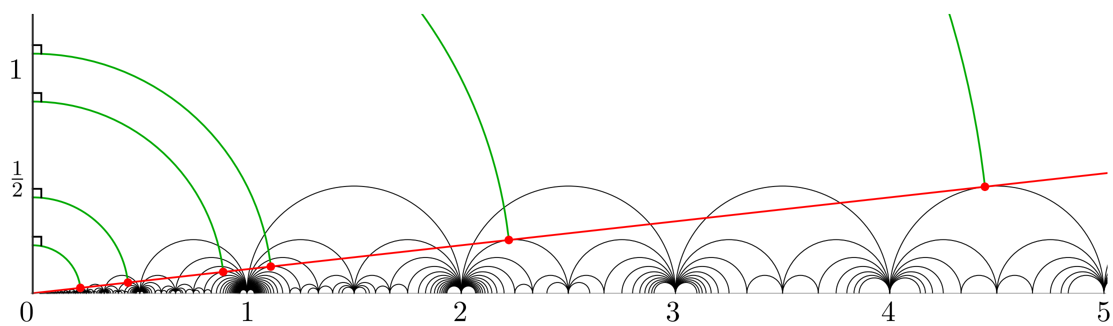
In this section, we discuss the connection of Theorem 6 with the binary additive divisor problem in $\mathbb{Z}$ and use this connection to show that the error term obtained in Theorem 6 is optimal.

Let $\mathbf{d} : \mathbb{N} \smallsetminus \{0\} \to \mathbb{N} \smallsetminus \{0\}$ be the *number of divisors function* of the natural numbers, defined by $n \mapsto \operatorname{Card}\{d \in \mathbb{N} \smallsetminus \{0\} : d \mid n\} = \frac{1}{2} \operatorname{Card}\{d \in \mathbb{Z} \smallsetminus \{0\} : d \mid n\}$. The *binary additive divisor problem*[8] in $\mathbb{Z}$ studies the asymptotic properties as $n \to +\infty$ of the sums $\sum_{k=1}^{n} \mathbf{d}(k)\,\mathbf{d}(k+f)$ for any positive integer f. The link between our counting problem of common perpendiculars between divergent geodesics and the binary additive divisor problem in $\mathbb{Z}$ will be given by Proposition 19 below.

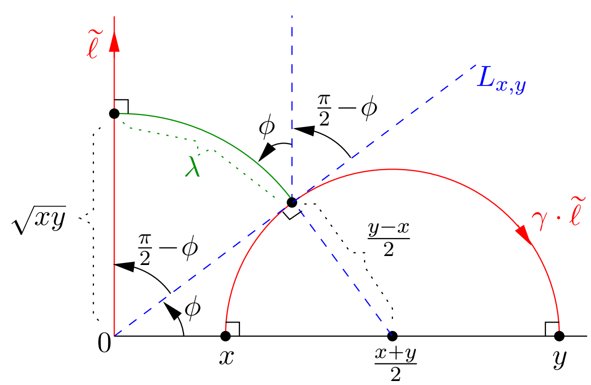
We keep the notation $\Gamma_{\mathbb{Z}}$, $M_{\mathbb{Z}}$, $\widetilde{\ell}$, $\widetilde{\Delta}$, ℓ, Δ and ${\mathbb{H}^2_{\mathbb{R}}}^{\pm}$ of Section 6. We start by proving a quantitative complement to Lemma 11.

Lemma 18. *Let $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_{\mathbb{Z}}$ with $a, b, c, d > 0$. Then the length of the common perpendicular between $\widetilde{\Delta}$ and $\gamma\widetilde{\Delta}$ is $\lambda = \operatorname{arcosh}(1 + 2bc)$.*

The following figure shows in black some of the $\Gamma_{\mathbb{Z}}$-translates of $\widetilde{\Delta}$ in the right halfplane ${\mathbb{H}^2_{\mathbb{R}}}^{+}$, and the six closest points to the vertical geodesic $\widetilde{\Delta}$ on the $\Gamma_{\mathbb{Z}}$-translates at distance $\operatorname{arcosh}(9)$, corresponding to $bc = 4$ with the above notation, and in green the corresponding common perpendiculars.



Proof. Let $x = \frac{b}{d}$ and $y = \frac{a}{c}$, which are the two endpoints at infinity of $\gamma \cdot \widetilde{\Delta}$. Since $a, b, c, d > 0$ and $ad - bc = 1$, we have $ad > bc$, hence $0 < x < y$. The common perpendicular between $\widetilde{\Delta}$ and $\gamma \cdot \widetilde{\Delta}$ is a segment of the Euclidean halfcircle centered at 0 that intersects $\gamma \cdot \widetilde{\Delta}$ at a right angle, see the figure below.



The intersection point is the unique point on $\gamma \cdot \widetilde{\Delta}$ where a Euclidean line $L_{x,y}$ through the origin is tangent to $\gamma \cdot \widetilde{\Delta}$. If $\phi \in \,]0, \pi[$ is the angle that $L_{x,y}$ makes with the positive real line at the origin, then $\sin\phi = \frac{\frac{y-x}{2}}{\frac{y+x}{2}} = \frac{y-x}{y+x}$. By the angle of parallelism formula[9] already

[8] See for instance [Ing, Est, HeB, Mot1].

[9] See Equation (78) in Section 8 for a different computation using complex length.

used in Section 3 and since $ad - bc = 1$, the length λ of the common perpendicular from $\widetilde{\Delta}$ to $\gamma \cdot \widetilde{\Delta}$ is

$$\lambda = \operatorname{arcosh}\Big(\frac{1}{\sin\phi}\Big) = \operatorname{arcosh}\Big(\frac{y+x}{y-x}\Big) = \operatorname{arcosh}\Big(\frac{\frac{a}{c}+\frac{b}{d}}{\frac{a}{c}-\frac{b}{d}}\Big) = \operatorname{arcosh}(1+2bc)\ . \qquad \square$$

Proposition 19. *For every $s > 0$, we have* $\mathscr{N}_{\Delta,\Delta}(s) = \displaystyle\sum_{k=1}^{\lfloor \frac{1}{2}(\cosh s - 1)\rfloor} \mathbf{d}(k)\,\mathbf{d}(k+1)$.

Proof. By the comment just before Lemma 11, since the set of $\Gamma_{\mathbb{Z}}$-translates of $\widetilde{\Delta}$ is in bijection with the set of right cosets $\Gamma_{\mathbb{Z}}/\Gamma_{\widetilde{\Delta}}$, it follows from Lemmas 11 and 18 that $\mathscr{N}_{\Delta,\Delta}(s)$ is the number of quadruples (a, b, c, d) of positive integers such that $ad - bc = 1$ and $\operatorname{arcosh}(1 + 2bc) \leqslant s$, or equivalently $bc \leqslant n = \lfloor \frac{1}{2}(\cosh s - 1)\rfloor$ since bc is a positive integer. This number is $\sum_{k=1}^{n} \mathbf{d}(k)\,\mathbf{d}(k + 1)$. $\square$

Let us now relate Theorem 6 (in the special case $n = 2$ and $\Gamma_{\mathbb{Z}} = \mathrm{PSL}_2(\mathbb{Z})$) with the known asymptotic result on the binary additive divisor problem in $\mathbb{Z}$. After the work of Ingham [Ing, p. 205], a major input by Estermann [Est], and various improvements on the error term by for instance [HeB, Thm. 2] and [Mot1, Coro. 1], we now know that there exist $a_1, a_2 \in \mathbb{R}$ such that (with a simplified version of the best known error term)

$$\sum_{k=1}^{n} \mathbf{d}(k)\,\mathbf{d}(k+1) = \frac{6}{\pi^2}\, n(\ln n)^2 + a_1\, n \ln n + a_2\, n + \mathrm{O}(n^{\frac{5}{6}})\,. \tag{74}$$

Using [Est, Eq. (36)], we can compute the estimate $a_1 \simeq 1.574 > 0$. By Proposition 19 and since $\lfloor \frac{1}{2}(\cosh s - 1)\rfloor = \frac{1}{4}e^s + \mathrm{O}(1)$, we thus have

$$\mathscr{N}_{\Delta,\Delta}(s) = \sum_{k=1}^{\lfloor \frac{1}{2}(\cosh s - 1)\rfloor} \mathbf{d}(k)\,\mathbf{d}(k+1) = \frac{3}{2\,\pi^2}\, s^2 e^s + b_1\, s\, e^s + b_2\, e^s + \mathrm{O}(e^{\frac{5}{6}s})\,, \tag{75}$$

where $b_1 = \frac{a_1}{4} - \frac{6\ln 2}{\pi^2} \simeq -0,028 \neq 0$. Equation (75) agrees with the asymptotic

$$\mathscr{N}_{\Delta,\Delta}(s) = \frac{3}{2\pi^2}\, s^2 e^s + \mathrm{O}(s\, e^s)$$

given by Theorem 6, as seen for Equation (72). Furthermore, Equation (75) gives an explicit nonzero term of the order $s\, e^s$ and an error term of strictly smaller order. This shows that the size of the error term in Theorem 6 is optimal.

8 The binary additive divisor problem for imaginary quadratic integers

In this section, we use our asymptotic counting of common perpendiculars between divergent geodesics proven in Theorem 6 in order to study the asymptotic binary additive divisor problem for imaginary quadratic integers, confirming a particular case of a conjecture of Motohashi [Mot2, p. 277].

Let $K, D_K, \mathscr{O}_K, \zeta_K$ and $\mathbf{d}_K$ be as in the introduction. Recall that the order $|\mathscr{O}_K^\times|$ of the group of units $\mathscr{O}_K^\times$ of $\mathscr{O}_K$ is equal to 4 when $D_K = -4$, to 6 when $D_K = -3$, and to 2 otherwise.

Proof of Theorem 2. We start the proof by describing the relevant geometric framework. As in Section 3 with $n = 3$, let $\mathbb{H}^3_\mathbb{R} \subset \mathbb{C} \times \mathbb{R}$ be the upper halfspace model of the real hyperbolic 3-space $\mathbb{H}^3_\mathbb{R}$. Its boundary at infinity is $\partial_\infty \mathbb{H}^3_\mathbb{R} = (\mathbb{C} \times \{0\}) \cup \{\infty\}$, that we identify with $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$. The group $\mathrm{PSL}_2(\mathbb{C})$ acts isometrically and faithfully on $\mathbb{H}^3_\mathbb{R}$, by the Poincaré extensions of the (complex) homographies. The *Bianchi group* $\Gamma_{\mathscr{O}_K} = \mathrm{PSL}_2(\mathscr{O}_K)$ is a nonuniform arithmetic lattice in $\mathrm{PSL}_2(\mathbb{C})$, with set of parabolic fixed points $\mathrm{Par}_{\Gamma_{\mathscr{O}_K}} = \mathbb{P}^1(K) = K \cup \{\infty\}$. The *Bianchi orbifold* $M_{\mathscr{O}_K} = \Gamma_{\mathscr{O}_K} \backslash \mathbb{H}^3_\mathbb{R}$ is a noncompact finite volume complete connected real hyperbolic good orbifold. Its number of cusps is equal to the class number of K, see for instance [EGM, §7.2].

Let $\widetilde{\ell} : t \mapsto (0, e^t)$ be the geodesic line in $\mathbb{H}^3_\mathbb{R}$ through $(0,1) \in \mathbb{H}^3_\mathbb{R}$ at time $t = 0$, with endpoints at infinity 0 and ∞. Then $\ell = \Gamma_{\mathscr{O}_K} \widetilde{\ell}$ is a divergent geodesic in $M_{\mathscr{O}_K}$, converging at $\pm\infty$ to the cusp $\Gamma_{\mathscr{O}_K} \cdot \infty$ of $M_{\mathscr{O}_K}$. Note that ℓ is reciprocal since the image $\widetilde{\Delta} = \widetilde{\ell}(\mathbb{R})$ of $\widetilde{\ell}$ is preserved by the involution $\iota = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right] \in \Gamma_{\mathscr{O}_K}$, whose fixed point set is the geodesic line with points at infinity $-i$ and i, that meets $\widetilde{\Delta}$ perpendicularly at $(0,1)$. The pointwise stabilizer of $\widetilde{\Delta}$ in $\Gamma_{\mathscr{O}_K}$ is the group consisting of the diagonal elements $\left[\begin{smallmatrix} u & 0 \\ 0 & \overline{u} \end{smallmatrix}\right]$ for $u \in \mathscr{O}_K^\times$, and the (global) stabilizer $\Gamma_{\widetilde{\Delta}}$ of $\widetilde{\Delta}$ in $\Gamma_{\mathscr{O}_K}$ is the binary dihedral group generated by ι and the pointwise stabilizer. Hence, with $\Delta = \ell(\mathbb{R}) = \Gamma_{\mathscr{O}_K} \widetilde{\Delta}$ the image of ℓ, we have

$$m(\Delta) = \frac{|\mathscr{O}_K^\times|}{2} \quad \text{and} \quad |\Gamma_{\widetilde{\Delta}}| = |\mathscr{O}_K^\times|\,. \tag{76}$$

In particular, $m(\Delta) = 1$ unless $D_K = -3$ or $D_K = -4$.

For every $k \in \mathscr{O}_K \smallsetminus \{0, 1\}$, the product $\mathbf{d}_K(k)\, \mathbf{d}_K(k-1)$ is the number of representations of 1 as the difference $ad - bc$ for a quadruple (a, b, c, d) of elements of $\mathscr{O}_K \smallsetminus \{0\}$ such that $ad = k$ and $bc = k - 1$. Hence

$$\mathbf{d}_K(k)\, \mathbf{d}_K(k-1) = \mathrm{Card}\left\{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathscr{O}_K) : a\,b\,c\,d \neq 0,\ ad = k\right\}. \tag{77}$$

Let $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_{\mathscr{O}_K}$. The geodesic lines $\widetilde{\Delta}$ and $\gamma\widetilde{\Delta}$ have no common endpoint at infinity if and only if $a\,b\,c\,d \neq 0$, by the same argument as the one at the beginning of the proof of Lemma 11. As in that proof, considering now the horoball $\mathscr{H}_\infty = \{(z, v) \in \mathbb{H}^3_\mathbb{R} : v \geqslant 1\}$ and replacing $i \in \mathbb{H}^2_\mathbb{R}$ by $(0,1) \in \mathbb{H}^3_\mathbb{R}$, we see that if $a\,b\,c\,d \neq 0$, then the geodesic lines $\widetilde{\Delta}$ and $\gamma\widetilde{\Delta}$ have empty intersection, since the Γ-equivariant family $(\gamma\mathscr{H}_\infty)_{\gamma \in \Gamma/\Gamma_\infty}$ is again precisely invariant, the only horoballs in this family containing $(0,1)$ are $\mathscr{H}_\infty$ and $\iota\mathscr{H}_\infty$ and $\widetilde{\Delta}$ is contained in $\mathscr{H}_\infty \cup \iota\mathscr{H}_\infty$. In particular, $\widetilde{\Delta}$ and $\gamma\widetilde{\Delta}$ have a common perpendicular if $a\,b\,c\,d \neq 0$.

Let $\lambda_\gamma = d(\widetilde{\Delta}, \gamma\widetilde{\Delta}) = d(p, q) > 0$ be the length of the common perpendicular $[p, q]$ between $\widetilde{\Delta}$ and $\gamma\widetilde{\Delta}$, with $p \in \widetilde{\Delta}$. Let θ_γ be the angle at p between the parallel transport of the oriented geodesic line $\gamma\widetilde{\Delta}$ along $[p, q]$ and the oriented geodesic line $\widetilde{\Delta}$. By [PP2, Lemma 2.2] and since $ad - bc = 1$, we have

$$\cosh \lambda_\gamma + \cos \theta_\gamma = 2\, \frac{|\gamma \cdot \infty|}{|\gamma \cdot \infty - \gamma \cdot 0|} = 2\,|ad|\,. \tag{78}$$

For all $\lambda \geqslant 0$ and $\theta \in \mathbb{R}$, we have $|\frac{e^{-\lambda}}{2} + \cos\theta| \leqslant 2$, so that as $\lambda \to +\infty$, a first order approximation gives

$$\ln(\cosh\lambda + \cos\theta) = \ln\Big(\frac{e^{\lambda}}{2} + \frac{e^{-\lambda}}{2} + \cos\theta\Big) = \lambda - \ln 2 + \mathrm{O}(e^{-\lambda})\,. \tag{79}$$

The only unit normal vectors to $\widetilde{\Delta}$ that have a nontrivial stabilizer in $\Gamma_{\mathscr{O}_K}$ are the finitely many tangent vectors v_u at $(0,1)$ of the oriented geodesic lines with points at infinity u and $-u$ for $u \in \mathscr{O}_K^\times$, stabilized by the element $\big[\begin{smallmatrix} 0 & u \\ -u & 0\end{smallmatrix}\big]$. The number of common perpendiculars between $\widetilde{\Delta}$ and its images by the elements of $\Gamma_{\mathscr{O}_K}$, whose initial tangent vectors are fixed to be v_u, grows at most linearly in their length. In particular, all multiplicities $m_{\widetilde{\Delta},\gamma\widetilde{\Delta}}$ of these common perpendiculars are equal to 1, except for a number of them that is linear in their length. A formula due to Humbert for the volume of $M_{\mathscr{O}_K} = \Gamma_{\mathscr{O}_K}\backslash\mathbb{H}^3_{\mathbb{R}}$ gives

$$\mathrm{Vol}(M_{\mathscr{O}_K}) = \frac{1}{4\pi^2}\,|D_K|^{3/2}\,\zeta_K(2)\ ,$$

see for instance Sections 8.8, 9.6 of [EGM]. With $n = 3$, let us define

$$c_K = \frac{(n-1)\,\pi^{\frac{n}{2}-1}\,\Gamma(\frac{n}{2})\,\iota_{\rm rec}(\Delta)^2}{2^{n+1}\,\Gamma(\frac{n+1}{2})^2\,m(\Delta)^2\ \mathrm{Vol}\,M_{\mathscr{O}_K}} = \frac{\pi^3}{|\mathscr{O}_K^\times|^2\,|D_K|^{\frac{3}{2}}\,\zeta_K(2)}\,. \tag{80}$$

By Theorem 6 applied with $n = 3$, $M = M_{\mathscr{O}_K}$ and $D^- = D^+ = \Delta$, we have

$$\begin{aligned}\mathscr{N}_{\Delta,\Delta}(s) &= \mathrm{Card}\,\big\{[\gamma] \in \Gamma_{\widetilde{\Delta}}\backslash\Gamma_{\mathscr{O}_K}/\Gamma_{\widetilde{\Delta}} : 0 < d(\widetilde{\Delta}, \gamma\,\widetilde{\Delta}) \leqslant s\big\} + \mathrm{O}(s)\\ &= c_K\ s^2 e^{2s} + \mathrm{O}(s\,e^{2s})\,.\end{aligned} \tag{81}$$

Using, in the following computations, respectively

- Equations (77) and (78) for the first equality,
- the facts that the kernel of the isometric action of $\mathrm{SL}_2(\mathscr{O}_K)$ on $\mathbb{H}^3_{\mathbb{R}}$ is the subgroup $\{\pm\,\mathrm{id}\}$ with order 2, that $\Gamma_{\mathscr{O}_K} = \mathrm{SL}_2(\mathscr{O}_K)/\{\pm\,\mathrm{id}\}$, that the assumptions on γ in the second line depend only on its double class $[\gamma]$ in $\Gamma_{\widetilde{\Delta}}\backslash\Gamma_{\mathscr{O}_K}/\Gamma_{\widetilde{\Delta}}$ and that $|\Gamma_{\widetilde{\Delta}}| = |\mathscr{O}_K^\times|$ by Equation (76) for the second equality,
- Equation (79) for the third equality,
- a partition of the set of $[\gamma] \in \Gamma_{\widetilde{\Delta}}\backslash\Gamma_{\mathscr{O}_K}/\Gamma_{\widetilde{\Delta}}$ with $\lambda_\gamma + \mathrm{O}(e^{-\lambda_\gamma}) \leqslant \ln(4N)$ into on the one hand the ones with $\ln(4\sqrt{N}\,) < \lambda_\gamma + \mathrm{O}(e^{-\lambda_\gamma}) \leqslant \ln(4N)$, so that $\lambda_\gamma \geqslant \ln(4\sqrt{N}\,) + \mathrm{O}(1)$ hence $e^{-\lambda_\gamma} = \mathrm{O}(N^{-\frac{1}{2}})$ thus by bootstrap $\lambda_\gamma \leqslant \ln(4N) + \mathrm{O}(N^{-\frac{1}{2}}) = \ln(4N + \mathrm{O}(\sqrt{N}))$, and on the other hand the ones with $\lambda_\gamma + \mathrm{O}(e^{-\lambda_\gamma}) \leqslant \ln(4\sqrt{N}\,)$ so that $\lambda_\gamma \leqslant \ln(4\sqrt{N}\,) + \mathrm{O}(1)$, for the fourth equality,
- Equation (81) for the fifth equality and Equation (80) for the last one,

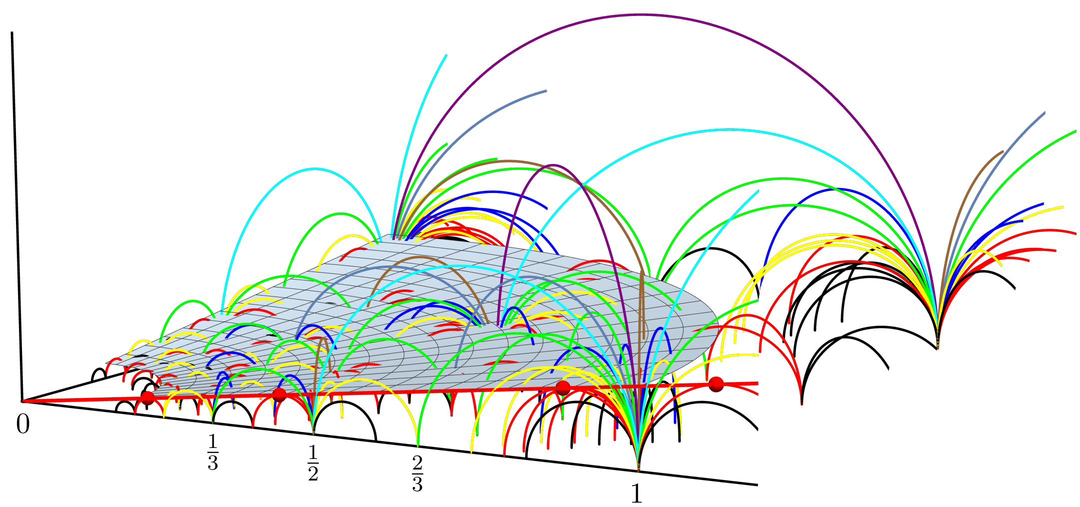
we have

$$
\begin{aligned}
&\frac{1}{|\mathscr{O}_K^\times|^2}\sum_{\substack{k\in\mathscr{O}_K\smallsetminus\{0,1\}\\ |k|\leqslant N}} \mathbf{d}_K(k)\,\mathbf{d}_K(k-1)\\
&=\frac{1}{|\mathscr{O}_K^\times|^2}\operatorname{Card}\left\{\gamma\in\mathrm{SL}_2(\mathscr{O}_K):\ \begin{array}{l}\partial_\infty(\gamma\widetilde{\Delta})\cap\partial_\infty\widetilde{\Delta}=\varnothing\\ \cosh\lambda_\gamma+\cos\theta_\gamma\leqslant 2N\end{array}\right\}\\
&=2\operatorname{Card}\left\{[\gamma]\in\Gamma_{\widetilde{\Delta}}\backslash\Gamma_{\mathscr{O}_K}/\Gamma_{\widetilde{\Delta}}:\ \begin{array}{l}\partial_\infty(\gamma\widetilde{\Delta})\cap\partial_\infty\widetilde{\Delta}=\varnothing\\ \cosh\lambda_\gamma+\cos\theta_\gamma\leqslant 2N\end{array}\right\}\\
&=2\operatorname{Card}\left\{[\gamma]\in\Gamma_{\widetilde{\Delta}}\backslash\Gamma_{\mathscr{O}_K}/\Gamma_{\widetilde{\Delta}}:\ \begin{array}{l}\partial_\infty(\gamma\widetilde{\Delta})\cap\partial_\infty\widetilde{\Delta}=\varnothing\\ \lambda_\gamma+\mathrm{O}(e^{-\lambda_\gamma})\leqslant \ln(4N)\end{array}\right\}\\
&=2\,\mathscr{N}_{\Delta,\Delta}\big(\ln(4N+\mathrm{O}(\sqrt{N}\,))\big)+\mathrm{O}\left(\mathscr{N}_{\Delta,\Delta}(\ln(4\sqrt{N}\,)+\mathrm{O}(1))\right)\\
&=2\,c_K\,\ln^2(4N+\mathrm{O}(\sqrt{N}\,))(4N+\mathrm{O}(\sqrt{N}\,))^2+\mathrm{O}\left(\ln(4N+\mathrm{O}(\sqrt{N}\,))(4N+\mathrm{O}(\sqrt{N}\,))^2\right)\\
&\quad+\mathrm{O}\left((\ln(4\sqrt{N}\,)+\mathrm{O}(1))^2(4\sqrt{N}\,)^2\right)\\
&=\frac{32\pi^3}{|\mathscr{O}_K^\times|^2\,|D_K|^{\frac{3}{2}}\,\zeta_K(2)}(\ln N)^2N^2+\mathrm{O}((\ln N)N^2)\,. \qquad (82)
\end{aligned}
$$

As the sums $\sum_{\substack{k\in\mathscr{O}_K\smallsetminus\{0,-1\}\\ |k|\leqslant N}}\mathbf{d}_K(k)\,\mathbf{d}_K(k+1)$ and $\sum_{\substack{k\in\mathscr{O}_K\smallsetminus\{0,1\}\\ |k|\leqslant N}}\mathbf{d}_K(k)\,\mathbf{d}_K(k-1)$ have the same asymptotic behaviour, Theorem 2 in the introduction follows by taking $N=\lfloor\sqrt{X}\rfloor$ and canceling the first and last factors $\frac{1}{|\mathscr{O}_K^\times|^2}$ from Equation (82). □

Remark 20. We take $K=\mathbb{Q}(i)$ in these remarks. In this case, $D_K=-4$ and $|\mathscr{O}_K^\times|=4$, and Equation (82) becomes

$$\sum_{\substack{k\in\mathscr{O}_K\smallsetminus\{0,1\}\\ |k|\leqslant N}}\mathbf{d}_K(k)\,\mathbf{d}_K(k-1)=\frac{4\pi^3}{\zeta_K(2)}(\ln N)^2N^2+\mathrm{O}((\ln N)N^2)\,.$$



(1) The above figure shows $\Gamma_{\mathscr{O}_K}$-translates of $\widetilde{\Delta}$ with points at infinity in the sector of $\mathbb{C}$ defined by the inequalities $\operatorname{Re} z \geqslant 0$ and $\operatorname{Im} z \geqslant 0$, that are at distance at most $\operatorname{arcosh}(9)$ from $\widetilde{\Delta}$. These translates correspond to $|ad| \leqslant 5$ in the notation of the above proof of Theorem 2. The surface in the figure is a truncated sector of the boundary of the $\operatorname{arcosh}(9)$-neighbourhood of $\widetilde{\Delta}$. Four of the six $\mathrm{PSL}_2(\mathbb{Z})$-translates in $\mathbb{H}^2_{\mathbb{R}}$ at distance $\operatorname{arcosh}(9)$ from $\widetilde{\Delta}$ shown in the figure after Lemma 18 are now visible as the red arcs in the foreground of the present figure with the points of intersection with the surface marked with red points.

(2) Let us relate Theorem 2 with Motohashi's conjecture stated in the third centered formula page 277 of [Mot2], starting by recalling the relevant definitions.[10] Let $\mathscr{I}_K^+$ be the set of nonzero (integral) ideals of $\mathscr{O}_K$, and $\mathtt{N} : \mathscr{I}_K^+ \to \mathbb{N} \smallsetminus \{0\}$ the ideal norm, defined for every $\mathfrak{a} \in \mathscr{I}_K^+$ by $\mathtt{N}(\mathfrak{a}) = [\mathscr{O}_K : \mathfrak{a}]$. Let $d_K : \mathscr{I}_K^+ \to \mathbb{N}$ be the *number of divisors function* of nonzero ideals of $\mathscr{O}_K$, defined by $d_K(\mathfrak{a}) = \operatorname{Card}\{\mathfrak{b} \in \mathscr{I}_K^+ : \mathfrak{b} \mid \mathfrak{a}\}$, so that $\zeta_K(s)^2 = \sum_{\mathfrak{a} \in \mathscr{I}_K^+} \frac{d_K(\mathfrak{a})}{\mathtt{N}(\mathfrak{a})^s}$ for $\operatorname{Re} s > 1$. For every $x \in \mathscr{O}_K \smallsetminus \{0\}$, let $\mathtt{N}(x) = \mathtt{N}(x\mathscr{O}_K) = |x|^2$ be the algebraic norm and $d_K(x) = d_K(x\mathscr{O}_K)$. Note that when $\mathscr{O}_K$ is principal, and in particular when $K = \mathbb{Q}(i)$, we have $d_K(x) = \frac{\mathbf{d}_K(x)}{|\mathscr{O}_K^\times|}$. Theorem 2 when $K = \mathbb{Q}(i)$ becomes

$$\sum_{x \in \mathscr{O}_K \smallsetminus \{0,-1\} \,:\, \mathtt{N}(x) \leqslant X} d_K(x)\, d_K(x+1) = \frac{1}{16} \frac{\pi^3}{\zeta_K(2)} \, X (\ln X)^2 + \mathrm{O}(X \ln X)\,. \tag{83}$$

This confirms Motohashi's conjecture up to the usual multiplicative factor $\frac{1}{16} = \frac{1}{|\mathscr{O}_K^\times|^2}$ in the particular case when $f = 1$ in his notation. See also [SaV] for a similar result on

$$\sum_{k \in \mathscr{O}_K \smallsetminus \{0,-f\}\,,\ \mathtt{N}(k) \leqslant X} d_K(k)\, d_K(k+f)$$

for all f, where the constant term in front of $X(\ln X)^2$ in Equation (83) appears in a more complicated form than above. See for instance [GN] for related counting problems of integral points on homogeneous affine algebraic varieties, as the one in $\mathscr{M}_2(\mathbb{R})$ defined by the equation $\det Y = f$ in the variable $Y \in \mathscr{M}_2(\mathbb{R})$ for a fixed $f \in \mathbb{Z}$.

(3) Numerical computations of the ratio

$$R(N) = \frac{4\,\zeta_K(2)}{\pi^3\; N^2 (\ln N)^2} \sum_{k \in \mathscr{O}_K \smallsetminus \{0,-1\},\ |k| \leqslant N} d_K(k)\, d_K(k+1)\,,$$

for $K = \mathbb{Q}(i)$ show that

$$R(N) \simeq 1.213, 1.195, 1.18 \text{ and } 1.167 \quad \text{when} \quad N = 2000, 4000, 8000 \text{ and } 16000$$

respectively. This slow convergence of $N(R)$ to 1 as $N \to +\infty$ is similar to the case of integers: In Equation (74), the ratio

$$\frac{\sum_{k=1}^n \mathbf{d}(k)\, \mathbf{d}(k+1)}{\frac{6}{\pi^2}\, n (\ln n)^2}$$

[10] See the line after Equation (9.7) in loc. cit., that says that Motohashi's division function d is exactly our $d_{\mathbb{Q}(i)}$ that we define above.

is approximately 1.18 when $n = 10^6$, and the ratio decreases closer to 1 very slowly, since the second term order in the development differs from the main term order only by a logarithmic term. On the other hand, the ratio

$$\frac{\sum_{k=1}^{n} \mathbf{d}(k)\, \mathbf{d}(k+1)}{\frac{6}{\pi^2}\, n(\ln n)^2 + a_1\, n \ln n} \tag{84}$$

is approximately 0.997 when $n = 10^6$, giving already a much better approximation.

The values of the ratio

$$\frac{\sum_{k\in\mathscr{O}_K\smallsetminus\{0,-1\},\ |k|\leqslant N} d_K(k)\, d_K(k+1)}{\frac{\pi^3}{4\,\zeta_K(2)}\, N^2(\ln N)^2 + 8.37\, N^2 \ln N}$$

analogous to the one in Equation (84) at $N = 2000, 4000, 8000, 16000$ are 1.00016, 1.0001, 1.00001 and 0.99938. This leads one to speculate a development similar to Equation (74)

$$\sum_{k\in\mathscr{O}_K\smallsetminus\{0,-1\},\ |k|\leqslant N} d_K(k)\, d_K(k+1) = \frac{\pi^3}{4\,\zeta_K(2)}\, N^2(\ln N)^2 + A_1 N^2 \ln N + \mathrm{o}(N^2 \ln N)\,,$$

with $A_1 \approx 8.4$.

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