

On the multiplicative pair correlations of sums of two squares

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Abstract

We study the pair correlations of the logarithms of the integral values of quadratic norm forms at various scalings, proving the existence of pair correlation measures. We describe a surprising set of asymptotic behaviours when the scaling increases, passing from a punctual measure to a Poissonian behaviour through an exotic behaviour at the transition phase. ¹

1 Introduction

In this paper, we study the asymptotic distribution of the (generalized) pair correlations of the logarithms of the values on integer points of integral quadratic norm forms, including the special case of sums of two squares. See for instance [Hoo, FKR, BeM] and their references for various distributional aspects of sums of two squares.

A framework for this study, that includes the one in [NP, HaZ], is the following one, see also [PP1, PP3, Say1, Say2]. Let $\mathcal{F} = (F_N, \omega_N)_{N \in \mathbb{N}}$ be a nondecreasing sequence of finite subsets F_N of \mathbb{R} , endowed with a *weight function* $\omega_N : F_N \rightarrow]0, +\infty[$. Let $\phi : \mathbb{N} \rightarrow]0, +\infty[$ be a *scaling function* converging to $+\infty$. Let $\psi : \mathbb{N} \rightarrow]0, +\infty[$ be a *renormalizing function*, that will be naturally chosen depending on ϕ . We denote by Δ_z the unit Dirac measure at any point z of any measurable space. We define the *empirical pair correlation measure of \mathcal{F} at time N with scaling $\phi(N)$* as the measure on \mathbb{R} with finite support

$$\mathcal{R}_N^{\mathcal{F}, \phi} = \frac{1}{\psi(N)} \sum_{x, y \in F_N : x \neq y} \omega_N(x) \omega_N(y) \Delta_{\phi(N)(y-x)}.$$

When the sequence of measures $(\mathcal{R}_N^{\mathcal{F}, \phi})_{N \in \mathbb{N}}$ weak-star converges to a measure $m_{\mathcal{F}, \phi}$ on \mathbb{R} , we call $m_{\mathcal{F}, \phi}$ the *asymptotic pair correlation measure of \mathcal{F} for the scaling ϕ* . When $m_{\mathcal{F}, \phi} = \rho_{\mathcal{F}, \phi} \text{Leb}_{\mathbb{R}}$ is absolutely continuous with respect to the Lebesgue measure $\text{Leb}_{\mathbb{R}}$ on \mathbb{R} , the Radon-Nikodym derivative $\rho_{\mathcal{F}, \phi}$ is called the *asymptotic pair correlation density of \mathcal{F} for the scaling ϕ* . When $\rho_{\mathcal{F}, \phi}$ is constant, ones says that the pair correlations exhibit a *Poissonian asymptotic behaviour*.

Let K be a quadratic imaginary number field, with discriminant D_K , ring of integers \mathcal{O}_K and (relative) norm $\mathfrak{n} : z \mapsto z \bar{z}$. For every $a \in \mathbb{N}$, let $r_K(a) = \text{Card}\{z \in \mathcal{O}_K : \mathfrak{n}(z) = a\}$

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be the number of representations of a by the norms of elements of \mathcal{O}_K . We fix $\alpha \in]0, \frac{1}{2}[$ throughout this paper. For all nonzero $a, b, N \in \mathbb{N}$, let

$$r_{K,\alpha}(a, b, N) = \text{Card}\{(w, z) \in \mathcal{O}_K : \mathfrak{n}(w) = a, \mathfrak{n}(z) = b, \mathfrak{n}(z - w) \leq N^{2\alpha}\}, \quad (1)$$

which if N is large enough is equal to the product $r_K(a)r_K(b)$ of the numbers of representations by the norm form \mathfrak{n} of a and of b . In this paper, we study the asymptotic behaviour of the following empirical distribution of pair correlations

$$\mathcal{R}_N = \frac{1}{\psi(N)} \sum_{a,b \in \mathbb{N} : a \neq b, 0 < a, b \leq N^2} r_{K,\alpha}(a, b, N) \Delta_{\phi(N)(\ln a - \ln b)}. \quad (2)$$

In particular, when $K = \mathbb{Q}(i)$, $D_K = -4$, the behaviour of the measures \mathcal{R}_N as $N \rightarrow +\infty$ gives the *multiplicative pair correlation* asymptotics of the sums of two squares, more precisely, the asymptotic of the values $(\frac{a}{b})^{\phi(N)}$ of the ratios $\frac{a}{b}$ of the sums of two squares a and b raised to the power $\phi(N)$, weighted by the number of their representations in sectors.

In addition to the purely arithmetic interest, a geometric motivation in order to study these representations is that the logarithms of the norms of \mathcal{O}_K , when K has class number one, form the ortholength spectrum of geodesic segments from a neighbourhood of the (unique) cusp to itself in the 3-dimensional real hyperbolic Bianchi orbifold of K , see [PP3, Sect. 7] for details.

In order to simplify the statements in this introduction, we only consider the power scalings $\phi : N \mapsto N^\beta$ for $\beta \in]0, 1 + \frac{\alpha}{2}[$, where this upper bound on β corresponds to the restriction in Equation (1) on the pairs of representations. Such a power scaling is a usual choice, see for instance [NP, HaZ]. We denote by $\mathcal{R}_N^{\alpha,\beta}$ the empirical distribution given by Equation (2) for this power scaling. We define a continuous, positive, piecewise real analytic, even function $\rho_{1-\alpha} : \mathbb{R} \rightarrow [0, +\infty[$ by

$$t \mapsto \begin{cases} \frac{8\pi}{3|D_K|} & \text{if } t = 0 \\ \frac{8\pi}{|D_K|t^3} \left(\arcsin\left(\frac{t}{2}\right) - \frac{t}{2}\left(1 - \frac{t^2}{4}\right)\sqrt{1 - \frac{t^2}{4}} \right) & \text{if } 0 < |t| \leq 2 \\ \frac{4\pi^2}{|D_K||t|^3} & \text{if } |t| > 2. \end{cases} \quad (3)$$

Theorem 1.1 *Assume that we have $D_K \equiv 0 \pmod{4}$. Let $\alpha \in]0, \frac{1}{2}[$ and $\beta \in]0, 1 + \frac{\alpha}{2}[$. As $N \rightarrow +\infty$, the empirical pair correlation measures $\mathcal{R}_N^{\alpha,\beta}$ converge, for the weak-star convergence of measures on \mathbb{R} , to the asymptotic pair correlation measure $m_{\alpha,\beta}$ given by*

$$m_{\alpha,\beta} = \begin{cases} \frac{4\pi^2}{|D_K|} \Delta_0 & \text{if } \beta =]0, 1 - \alpha[\text{ and } \psi(N) = N^{2+2\alpha}, \\ \rho_{1-\alpha} \text{ Leb}_{\mathbb{R}} & \text{if } \beta = 1 - \alpha \text{ and } \psi(N) = N^{2+2\alpha} = N^{3+\alpha-\beta}, \\ \frac{8\pi}{3|D_K|} \text{ Leb}_{\mathbb{R}} & \text{if } \beta \in]1 - \alpha, 1 + \frac{\alpha}{2}[, \alpha \leq \frac{1}{6} \text{ and } \psi(N) = N^{3+\alpha-\beta}. \end{cases}$$

In particular, the asymptotic behaviour of the empirical pair correlation measures $\mathcal{R}_N^{\alpha,\beta}$ is Poissonian when $\beta \in]1 - \alpha, 1 + \frac{\alpha}{2}[$. It has a phase transition with an exotic asymptotic pair correlation density $\rho_{1-\alpha}$ with respect to the Lebesgue measure when $\beta = 1 - \alpha$, see Figure 1 below for an example. Below this threshold, the empirical pair correlation measures concentrate on a punctual measure. Such phase transition phenomena as the scaling increases appear frequently, see for instance [PP1, PP3, Say1, Say2].

We refer to Theorem 4.1 for a more complete version of Theorem 1.1, without the restriction on the discriminant D_K , with more general scaling functions, as well as for error terms. These error terms constitute the main technical parts of this paper. It is an interesting feature that even in the Poissonian asymptotic behaviour case (when $\beta \in]1 - \alpha, 1 + \frac{\alpha}{2}[$), the validity of the error terms depend on whether $\beta \in]1 - \alpha, 1 - \frac{\alpha}{2}]$, $\beta \in]1 - \frac{\alpha}{2}, 1[$ or $\beta \in [1, 1 + \frac{\alpha}{2}[$, while the constant value $\frac{8\pi}{3|D_K|}$ of the asymptotic pair correlation density does not change.

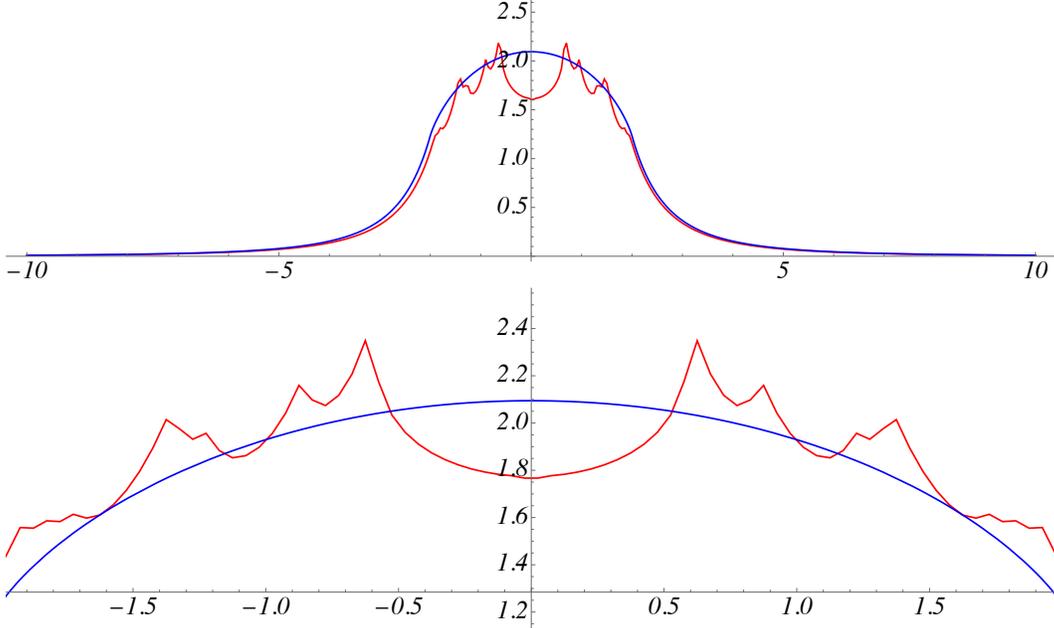


Figure 1 : For $\alpha = 0.15$, $\beta = 0.85$, graph of $\rho_{1-\alpha}$ (in blue) with the density of $\mathcal{R}_N^{\alpha,\beta}$ (in red) on $[-10, 10]$ for $N = 2000$, and on the shorter interval $[-2, 2]$ for $N = 3000$.

The study of pair correlations in a noncompact setting has a rich history, including the seminal paper [Mon] on the zeros of the Riemann zeta function. The lengths of the closed geodesics in negative curvature have a Poissonian pair correlation asymptotics or their empirical distribution converges to an exponential probability measure, depending on the scaling factor, see [PS, PP2]. For real numbers α', β' satisfying some Diophantine condition, the image of \mathbb{Z}^2 by the quadratic form $(x, y) \mapsto (x - \alpha')^2 + (y - \beta')^2$ also exhibits a Poissonian pair correlation asymptotic behaviour, by [Mar]. Similar problems often arise in quantum chaos, including energy level spacings or clusterings, and in statistical physics, including molecular repulsion or interstitial distribution, and in various number-theoretical contexts. See for instance [Ber, RS, Van, Zah, EM, BocZ, MaS, EBMV, LaS, HoK, Wei, LST].

In Section 5, we discuss the results of experiments that indicate a change from the Poissonian asymptotic behaviour of the pair correlations when β is beyond the range of Theorem 1.1, with strong level repulsion phenomena, and finally a total loss of mass for $\beta > 2$. It would also be interesting to study the weighted family

$$\mathcal{F}_K = (F_N = \{\ln n : 0 < n \leq N, r_K(n) \neq 0\}, w_N = r_K \circ \exp)_{N \in \mathbb{N}}$$

with weights given by r_K (removing the zero weights). For instance, when $D_K = -4$, the function r_K is the number of representations of integers as sums of two squares. In [PP3,

Coro. 2.5], we proved that the asymptotic pair correlation density of \mathcal{F}_K with constant scaling function $\phi = 1$, i. e. the distribution function of \mathcal{F}_K , is $t \mapsto \frac{1}{2}e^{-|t|}$. See also [PP2] for a general result when ϕ is constant.

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2 Parametrizing the pair correlation data

Let us fix $N \in \mathbb{N} \setminus \{0\}$. In this section, we give a parametrisation of the set of representations $q \in \mathcal{O}_K$ by the norm form \mathfrak{n} of the positive integers a at most N , under the additional constraints for them to be part of the pair correlation data appearing in the empirical distribution \mathcal{R}_N given by Equation (2).

We consider the (relative) *trace* and *norm* maps from \mathbb{C} to \mathbb{R} defined by $\mathfrak{tr} : z \mapsto z + \bar{z}$ and $\mathfrak{n} : z \mapsto z \bar{z}$ respectively. Let $(1, \omega_K)$ be the usual \mathbb{Z} -basis of \mathcal{O}_K , with

$$\omega_K = \begin{cases} \frac{i\sqrt{|D_K|}}{2} & \text{if } D_K \equiv 0 \pmod{4} \\ \frac{1+i\sqrt{|D_K|}}{2} & \text{otherwise.} \end{cases}$$

We have

$$\mathfrak{tr}(\omega_K) = \begin{cases} 0 & \text{if } D_K \equiv 0 \pmod{4} \\ 1 & \text{otherwise,} \end{cases} \quad \text{and} \quad \mathfrak{n}(\omega_K) = \begin{cases} \frac{|D_K|}{4} & \text{if } D_K \equiv 0 \pmod{4} \\ \frac{1+|D_K|}{4} & \text{otherwise.} \end{cases}$$

The norm of any nonzero element of \mathcal{O}_K is a positive integer. In particular, the area $\text{covol}_{\mathcal{O}_K}$ of the fundamental parallelogram $[0, 1] + \omega_K[0, 1]$ of the \mathbb{Z} -lattice \mathcal{O}_K in \mathbb{C} , its diameter $\text{diam}_{\mathcal{O}_K}$ and the shortest length $\text{Sys}_{\mathcal{O}_K}$ of a nonzero element of \mathcal{O}_K satisfy

$$\text{covol}_{\mathcal{O}_K} = \frac{\sqrt{|D_K|}}{2}, \quad 1 \leq \text{diam}_{\mathcal{O}_K} = O(\sqrt{|D_K|}) \quad \text{and} \quad \text{Sys}_{\mathcal{O}_K} = 1. \quad (4)$$

In this section, we fix $p \in \mathcal{O}_K \setminus \{0\}$ and we write

$$p = x_p + \omega_K y_p$$

with $x_p, y_p \in \mathbb{Z}$. We define

$$x'_p = 2x_p + y_p \mathfrak{tr}(\omega_K) \quad \text{and} \quad y'_p = x_p \mathfrak{tr}(\omega_K) + 2\mathfrak{n}(\omega_K)y_p,$$

which are easily seen to be elements of \mathbb{Z} that are not simultaneously zero. We also denote by $c'_p = (x'_p, y'_p) \in \mathbb{N} \setminus \{0\}$ the (positive) greatest common divisor of x'_p and y'_p , and we define

$$v_p = \frac{1}{c'_p}(y'_p - \omega_K x'_p) \in \mathcal{O}_K \setminus \{0\}.$$

Note that when $D_K \equiv 0 \pmod{4}$, then $\mathfrak{tr}(\omega_K) = 0$ and $\overline{\omega_K} = -\omega_K$, hence $x'_p = 2x_p$, $y'_p = 2\mathfrak{n}(\omega_K)y_p$ and

$$c'_p v_p = 2\mathfrak{n}(\omega_K)y_p - 2x_p\omega_K = -2\omega_K(x_p + \omega_K y_p) = -i\sqrt{|D_K|}p. \quad (5)$$

(2) We have $H_p^+ = \{z \in \mathbb{C} : \mathbf{n}(z) < \mathbf{n}(p+z)\}$, hence

$$J_{p,N} = \{q \in \mathcal{O}_K : 0 < \mathbf{n}(q) < \mathbf{n}(p+q) \leq N^2\}.$$

Furthermore, $J_{p,N}$ is empty if $|p| \geq 2N$.

(3) We have $\Lambda_p = c'_p \mathbb{Z}$.

(4) For every $t \in \mathbb{R}$, the affine (real) line $L_{p,t}$ is perpendicular to the (real) line $\mathbb{R}p$ (hence is parallel to the boundary ∂H_p^+ of H_p^+) and meets $\mathbb{R}p$ exactly at the point

$$z_{p,t} = \frac{t c'_p}{2\mathbf{n}(p)} p \text{ (see Figure 2).}$$

(5) For every $t \in \mathbb{R}$, the intersection $L_{p,t} \cap \mathcal{O}_K$ is nonempty if and only if $t \in \mathbb{Z}$. For every $k \in \mathbb{Z}$, if $w_{p,k}$ is one of the at most two points of $L_{p,k} \cap \mathcal{O}_K$ the closest to $z_{p,k}$ (see Figure 2), we have $L_{p,k} \cap \mathcal{O}_K = w_{p,k} + \mathbb{Z}v_p$. Furthermore, there exists $t_{p,k} \in [-\frac{1}{2}, \frac{1}{2}]$ such that $w_{p,k} - z_{p,k} = t_{p,k} v_p$.

(6) Let $\kappa_p = \lfloor -\frac{\mathbf{n}(p)}{c'_p} \rfloor + 1$. For every $k \in \mathbb{Z}$, the affine line $L_{p,k}$ is contained in the open halfspace H_p^+ if and only if we have $k \geq \kappa_p$.

(7) The set $J_{p,N}$ is the set of elements $\frac{k c'_p}{2\mathbf{n}(p)} p + (t_{p,k} + \ell)v_p$ such that $k, \ell \in \mathbb{Z}$, $k \geq \kappa_p$ and $(\frac{k c'_p}{2\mathbf{n}(p)} + 1)^2 |p|^2 + (t_{p,k} + \ell)^2 |v_p|^2 \leq N^2$.

Proof. For every $z = x + \omega_K y \in \mathbb{C}$ with $x, y \in \mathbb{R}$, by the definition of x'_p and y'_p , we have

$$\mathbf{tr}(\bar{p}z) = 2(x_p x + \mathbf{n}(\omega_K) y_p y) + (x_p y + y_p x) \mathbf{tr} \omega_K = x'_p x + y'_p y.$$

This proves Assertion (1) by the definition of v_p and by Equation (7). This also proves Assertion (3) since then $\Lambda_p = x'_p \mathbb{Z} + y'_p \mathbb{Z} = c'_p \mathbb{Z}$ by the definition of c'_p .

The set $\{z \in \mathbb{C} : \langle p, z + \frac{p}{2} \rangle > 0\}$ is the halfplane in \mathbb{C} containing 0 with boundary the affine line $-\frac{p}{2} + p^\perp$, hence is equal to H_p^+ . The first claim of Assertion (2) is then immediate using Equation (7) since for every $z \in \mathbb{C}$, we have $\mathbf{n}(z) < \mathbf{n}(p+z)$ if and only if $\mathbf{tr}(\bar{p}(z + \frac{p}{2})) > 0$. The second claim follows since $B(-p, N) = \{z \in \mathbb{C} : |z+p| \leq N\}$ and the norm \mathbf{n} is the square of the absolute value. The intersection $H_p^+ \cap B(-p, N)$ is empty if $|p| \geq 2N$, as the inequalities $\mathbf{n}(z) < \mathbf{n}(p+z) \leq N^2$ imply that $|z| < N$ and that $|z| \geq |p| - N$ by the inverse triangle inequality. The last claim of Assertion (2) follows.

For all $\lambda, t \in \mathbb{R}$, we have $\lambda p \in L_{p,t}$ if and only if $\mathbf{tr}(\bar{p} \lambda p) = t c'_p$, that is, if and only if $\lambda = \frac{t c'_p}{2\mathbf{n}(p)}$. Assertion (4) then follows, since $L_{p,0} = \{z \in \mathbb{C} : \mathbf{tr}(\bar{p}z) = 0\}$ is the vector line orthogonal to $\mathbb{R}p$ and $L_{p,t} = z_{p,t} + L_{p,0}$.

Let us prove Assertion (5). By the definitions of Λ_p and $L_{p,t}$, the intersection $L_{p,t} \cap \mathcal{O}_K$ is nonempty if and only if $t c'_p \in \Lambda_p$, hence the first claim follows from Assertion (3). Let $k \in \mathbb{Z}$. Since the point $w_{p,k}$ belongs by construction to the affine line $L_{p,k}$, which is perpendicular to $\mathbb{R}p$ by Assertion (4), and since the vector v_p is nonzero and perpendicular to p by Assertion (1), we have $L_{p,k} = w_{p,k} + \mathbb{R}v_p$. Since $w_{p,k} \in \mathcal{O}_K$, we have $L_{p,k} \cap \mathcal{O}_K = \{w_{p,k} + s v_p : s v_p \in \mathcal{O}_K\}$. The second claim of Assertion (5) then follows from the fact that the vector $v_p \in \mathcal{O}_K \setminus \{0\}$ is primitive (it has relatively prime integral coordinates in the \mathbb{Z} -basis $(1, \omega_K)$ of \mathcal{O}_K). This proves that two consecutive points of \mathcal{O}_K on the affine line $L_{p,k}$ are at distance exactly $|v_p|$. Hence we have $|w_{p,k} - z_{p,k}| \leq \frac{1}{2} |v_p|$ for every $k \in \mathbb{Z}$, and the last claim of Assertion (5) follows.

Let us now prove Assertion (6). The affine line ∂H_p^+ , which contains $-\frac{p}{2}$ and is perpendicular to $\mathbb{R}p$, is equal to

$$\left\{ z \in \mathbb{C} : \mathbf{tr}(\bar{p}z) = \mathbf{tr} \left(\bar{p} \left(-\frac{p}{2} \right) \right) = -\mathbf{n}(p) \right\} = L_{p, -\frac{\mathbf{n}(p)}{c'_p}}.$$

Therefore for every $k \in \mathbb{Z}$, the affine line $L_{p,k}$ is contained in H_p^+ if and only if $k > -\frac{\mathbf{n}(p)}{c'_p}$, that is, if and only if $k \geq \kappa_p = \lfloor -\frac{\mathbf{n}(p)}{c'_p} \rfloor + 1$. Note that κ_p is nonpositive.

Finally, let us prove Assertion (7). Note that \mathbb{C} is foliated by the affine lines $L_{p,t}$ for $t \in \mathbb{R}$. For every $t \in \mathbb{R}$, since $J_{p,N}$ is contained in \mathcal{O}_K , the intersection $L_{p,t} \cap J_{p,N}$ is empty if $t \notin \mathbb{Z}$ by Assertion (5). Hence $J_{p,N} = \bigcup_{k \in \mathbb{Z}} L_{p,k} \cap J_{p,N}$. By Assertion (6) and the definition of $J_{p,N}$, we hence have $J_{p,N} = \bigcup_{k \geq \kappa_p} L_{p,k} \cap \mathcal{O}_K \cap B(-p, N)$. By Assertion (5), any element $z \in L_{p,k} \cap \mathcal{O}_K$ can be uniquely written as $z = w_{p,k} + \ell v_p$ for some $\ell \in \mathbb{Z}$. Hence by Assertions (4) and (5), we have

$$z = z_{p,k} + w_{p,k} - z_{p,k} + \ell v_p = \frac{k c'_p}{2 \mathbf{n}(p)} p + t_{p,k} v_p + \ell v_p.$$

This proves the result, since p and v_p are orthogonal, hence the inequality $|z + p| \leq N$ is equivalent to the last inequality of Assertion (7). \square

Let us consider the map $j_{p,N} : \mathbb{R}^2 \rightarrow \mathbb{C}$ defined by

$$j_{p,N} : (s, t) \mapsto N \left(s \frac{p}{|p|} + t \frac{v_p}{|v_p|} \right), \quad (8)$$

which is a homothety of Euclidean vector spaces (and in particular a homeomorphism). By Lemma 2.1 (7), we have

$$j_{p,N}^{-1}(J_{p,N}) = \left\{ \left(s = \frac{c'_p}{2|p|N} k, \quad t = \frac{t_{p,k} |v_p|}{N} + \frac{|v_p|}{N} \ell \right) : \right. \\ \left. k, \ell \in \mathbb{Z}, \quad s > -\frac{|p|}{2N}, \quad \left(s + \frac{|p|}{N} \right)^2 + t^2 \leq 1 \right\}. \quad (9)$$

Recall that $\alpha \in]0, \frac{1}{2}[$ has been fixed in the introduction. The finite subset $j_{p,N}^{-1}(J_{p,N})$ of \mathbb{R}^2 is contained in $B(-\frac{|p|}{N}, 1)$ and converges as N tends to $+\infty$ for the Hausdorff distance on the set of closed subsets of the metric space \mathbb{R}^2 to the closed halfdisc

$$B^+(0, 1) = \{(s, t) \in \mathbb{R}^2 : s \geq 0, \quad s^2 + t^2 \leq 1\}, \quad (10)$$

uniformly in $p \in \mathcal{O}_K$ with $0 < |p| \leq N^\alpha$ since $\alpha < 1$. Furthermore, since the horizontal coordinate s in Equation (9) varies by constant steps $\frac{c'_p}{2|p|N}$ as k varies in \mathbb{Z} and, when s is fixed, the vertical coordinate t varies by constant steps $\frac{|v_p|}{N}$ as ℓ varies in \mathbb{Z} , a two-dimensional Riemann sum argument proves that the measure

$$\frac{c'_p}{2|p|N} \frac{|v_p|}{N} \sum_{(s,t) \in j_{p,N}^{-1}(J_{p,N})} \Delta_{(s,t)}$$

weak-star converges as $N \rightarrow +\infty$ (uniformly in $p \in \mathcal{O}_K$ with $0 < |p| \leq N^\alpha$) to the restriction $\text{Leb}_{B^+(0,1)}$ to $B^+(0, 1)$ of the Lebesgue measure of \mathbb{R}^2 . In particular, since the area of $B^+(0, 1)$ is $\frac{\pi}{2}$ and by the right-hand part of Equation (6) for the last equality, as $N \rightarrow +\infty$ and uniformly in $p \in \mathcal{O}_K$ with $0 < |p| \leq N^\alpha$, we have

$$\text{Card } J_{p,N} = \text{Card } j_{p,N}^{-1}(J_{p,N}) \sim \frac{\pi |p| N^2}{c'_p |v_p|} = O(N^2). \quad (11)$$

3 Uniformisation of the empirical distribution

Let us fix again $N \in \mathbb{N} \setminus \{0\}$, that we will assume to tend to $+\infty$ after Equation (15). In this technical section, we represent the empirical pair correlation distribution \mathcal{R}_N defined in Equation (2) on $[0, +\infty[$ as a distribution with an explicit density with respect to the Lebesgue measure, up to a controlled error term, see Equation (33). We fix in this section

$$\alpha \in \left]0, \frac{1}{2}\right[, \quad \gamma \in \left]0, \frac{1-2\alpha}{2}\right[\quad \text{and} \quad \epsilon = \epsilon_N = \frac{1}{N^\gamma} \in]0, 1[, \quad (12)$$

so that γ exists and $\epsilon \rightarrow 0$ as $N \rightarrow +\infty$. We assume in this section that

$$\lim_{N \rightarrow +\infty} \frac{\phi(N)}{N^{2-2\alpha-2\gamma}} = 0. \quad (13)$$

When $\phi : N \mapsto N^\beta$ is a power scaling as in the introduction with $\beta \in]0, 2-2\alpha[$, the above assumption (13) is equivalent to the fact that $\gamma \in]0, \frac{2-2\alpha-\beta}{2}[$.

Let $\log : \mathbb{C}^\times \rightarrow \mathbb{C}/(2\pi i\mathbb{Z})$ be the biholomorphic (complex) logarithm map. Note that the trace map \mathbf{tr} , being constant modulo $2\pi i\mathbb{Z}$, induces a map (called by the same name) $\mathbf{tr} : \mathbb{C}/(2\pi i\mathbb{Z}) \rightarrow \mathbb{R}$.

With the notation of Equation (1), note that for all elements $a, b \in \mathbb{N}$ that are nonzero, we have $r_{K,\alpha}(a, b, N) = r_K(a) r_K(b)$ if N is large enough and $r_{K,\alpha}(a, b, N) = r_{K,\alpha}(b, a, N)$.

Let \mathcal{R}_N^+ be the restriction to $[0, +\infty[$ of the empirical measure \mathcal{R}_N defined in Equation (2). Using Equation (1) and writing $a = \mathbf{n}(q)$ and $b = \mathbf{n}(p+q)$ in the indices of the first sum below for p and q varying in \mathcal{O}_K , since $\mathbf{tr} \circ \log = \ln \circ \mathbf{n}$ and by the multiplicativity of the norm for the second equality below, and by Lemma 2.1 (2) for the third one, we have

$$\begin{aligned} \mathcal{R}_N^+ &= \frac{1}{\psi(N)} \sum_{a, b \in \mathbb{N} : 0 < a < b \leq N^2} r_{K,\alpha}(a, b, N) \Delta_{\phi(N)(\ln b - \ln a)} \\ &= \frac{1}{\psi(N)} \sum_{p, q \in \mathcal{O}_K : 0 < \mathbf{n}(q) < \mathbf{n}(p+q) \leq N^2, |p| \leq N^\alpha} \Delta_{\phi(N) \mathbf{tr} \log \frac{p+q}{q}} \\ &= \frac{1}{\psi(N)} \sum_{p \in \mathcal{O}_K : 0 < |p| \leq N^\alpha} \sum_{q \in J_{p,N}} \Delta_{\phi(N) \mathbf{tr} \log(1 + \frac{p}{q})}. \end{aligned}$$

Let us fix for now $p \in \mathcal{O}_K$ with $0 < |p| \leq N^\alpha$. Let us consider the measure with finite support

$$\nu_{p,N} = \sum_{q \in J_{p,N}} \Delta_{\phi(N) \mathbf{tr} \log(1 + \frac{p}{q})},$$

so that we have

$$\mathcal{R}_N^+ = \frac{1}{\psi(N)} \sum_{p \in \mathcal{O}_K : 0 < |p| \leq N^\alpha} \nu_{p,N}. \quad (14)$$

By Equation (8) and since $j_{p,N}$ is a bijection, we have

$$\nu_{p,N} = \sum_{(s,t) \in j_{p,N}^{-1}(J_{p,N})} \Delta_{\phi(N) \mathbf{tr} \log \left(1 + \frac{p}{N(s \frac{p}{|p|} + t \frac{vp}{|vp|})} \right)}. \quad (15)$$

Let $f \in C_c^1([0, +\infty[)$ be a C^1 -smooth function on $[0, +\infty[$ with compact support. As $N \rightarrow +\infty$, for every $(s, t) \in \mathbb{R}^2$ with $s^2 + t^2 \geq \epsilon^2$ (where ϵ is given by Equation (12)),

- by the expansion of $\log(1+z)$ near $z=0$ since $|s \frac{p}{|p|} + t \frac{v_p}{|v_p|}| = \sqrt{s^2 + t^2} \geq \epsilon$, so that using Equation (12) we have $\left| \frac{p}{N(s \frac{p}{|p|} + t \frac{v_p}{|v_p|})} \right| \leq \frac{|p|}{N\epsilon} \leq \frac{1}{N^{1-\alpha-\gamma}}$, which tends to 0 as $N \rightarrow +\infty$, and by the linearity of the trace, for the first equality,
- since p and v_p are orthogonal, for the third equality,
- by the assumption of Equation (13) so that $\frac{\phi(N)|p|^2}{N^2 \epsilon^2} \leq \frac{\phi(N)}{N^{2-2\alpha-2\gamma}}$ tends to 0 as $N \rightarrow +\infty$, and by the mean value inequality, for the fourth equality, we have

$$\begin{aligned}
& f\left(\phi(N) \operatorname{tr} \log\left(1 + \frac{p}{N(s \frac{p}{|p|} + t \frac{v_p}{|v_p|})}\right)\right) \\
&= f\left(\frac{\phi(N)}{N} \operatorname{tr}\left(\frac{p}{s \frac{p}{|p|} + t \frac{v_p}{|v_p|}}\right) + O\left(\frac{\phi(N)|p|^2}{\epsilon^2 N^2}\right)\right) \\
&= f\left(\frac{\phi(N)}{N(s^2 + t^2)} \operatorname{tr}\left(p\left(s \frac{\bar{p}}{|p|} + t \frac{\bar{v}_p}{|v_p|}\right)\right) + O\left(\frac{\phi(N)|p|^2}{\epsilon^2 N^2}\right)\right) \\
&= f\left(\frac{2\phi(N)|p|s}{N(s^2 + t^2)} + O\left(\frac{\phi(N)|p|^2}{\epsilon^2 N^2}\right)\right) \\
&= f\left(2\frac{\phi(N)}{N}|p|\frac{s}{s^2 + t^2}\right) + O\left(\frac{\phi(N)|p|^2}{N^2 \epsilon^2} \|f'\|_\infty\right). \tag{16}
\end{aligned}$$

As in Section 2, let us denote by $B(z, r)$ the closed ball of center z and radius r in \mathbb{C} . The function

$$\hat{f} = \hat{f}_{p,N} : (s, t) \mapsto f\left(2\frac{\phi(N)}{N}|p|\frac{s}{s^2 + t^2}\right)$$

is well defined and C^1 -smooth on $B(0, 2) \setminus B(0, \frac{\epsilon}{2})$. An easy computation gives that the supremum norm $\|d\hat{f}\|_\infty$ of its differential on $B(0, 2) \setminus B(0, \frac{\epsilon}{2})$ satisfies

$$\|d\hat{f}\|_\infty = O\left(\frac{\phi(N)|p|}{N \epsilon^2} \|f'\|_\infty\right). \tag{17}$$

As $N \rightarrow +\infty$ and uniformly in $p \in \mathcal{O}_K$ with $0 < |p| \leq N^\alpha$, using

- the fact that $\operatorname{Card}\{j_{p,N}^{-1}(J_{p,N}) \cap B(0, \epsilon)\} = O(N^2 \epsilon^2)$ as $N \rightarrow +\infty$, by the same proof as for Equation (11), for the first equality,
 - Equations (15) and (16), as well as Equation (11) for dealing with the error term of Equation (16), for the second equality,
- we have

$$\begin{aligned}
\nu_{p,N}(f) &= \sum_{(s,t) \in j_{p,N}^{-1}(J_{p,N}) \setminus B(0,\epsilon)} f\left(\phi(N) \operatorname{tr} \log\left(1 + \frac{p}{N(s \frac{p}{|p|} + t \frac{v_p}{|v_p|})}\right)\right) + O(N^2 \epsilon^2 \|f\|_\infty) \\
&= \sum_{z \in j_{p,N}^{-1}(J_{p,N}) \setminus B(0,\epsilon)} \hat{f}(z) + O(\phi(N)|p|^2 \epsilon^{-2} \|f'\|_\infty) + O(N^2 \epsilon^2 \|f\|_\infty). \tag{18}
\end{aligned}$$

Let $\Lambda = \frac{c'_p}{2|p|N} \mathbb{Z} + \frac{|v_p|}{N} \mathbb{Z}i$, which is a \mathbb{Z} -lattice in the Euclidean space $\mathbb{R}^2 = \mathbb{C}$, with fundamental parallelogram $[0, \frac{c'_p}{2|p|N}] \times [0, \frac{|v_p|}{N}]$. Its diameter $\operatorname{diam}_\Lambda$ and area $\operatorname{covol}_\Lambda$ satisfy

respectively by the left part and the right part of Equation (6) that

$$\text{diam}_\Lambda = \left(\frac{c'_p{}^2}{4|p|^2 N^2} + \frac{|v_p|^2}{N^2} \right)^{1/2} = O\left(\frac{|p|}{N}\right) \quad (19)$$

and

$$\text{covol}_\Lambda = \frac{c'_p |v_p|}{2|p| N^2} \in \left[\frac{1}{2c_K N^2}, \frac{c_K}{2N^2} \right]. \quad (20)$$

If N is large enough and uniformly in $p \in \mathcal{O}_K$ with $0 < |p| \leq N^\alpha$, for all integers $k, \ell \in \mathbb{Z}$, since the real number $t_{p,k}$ given by Lemma 2.1 (5) satisfies the inequality $|t_{p,k}| \leq \frac{1}{2}$, every point $z = \left(\frac{c'_p}{2|p|N} k, \frac{t_{p,k}|v_p|}{N} + \frac{|v_p|}{N} \ell \right)$ in $j_{p,N}^{-1}(J_{p,N}) \setminus B(0, \epsilon)$ is at distance at most $\frac{|v_p|}{N}$ from the point $z' = \left(\frac{c'_p}{2|p|N} k, \frac{|v_p|}{N} \ell \right)$ of Λ . By the left part of Equation (6) and since $\gamma < 1 - \alpha$, for every N large enough, we have

$$\frac{|v_p|}{N} \leq \frac{c_K |p|}{N} \leq \frac{c_K}{N^{1-\alpha}} < \frac{1}{2N^\gamma} = \frac{\epsilon}{2}.$$

Recalling that we have $j_{p,N}^{-1}(J_{p,N}) \subset B\left(-\frac{|p|}{N}, 1\right)$, we hence have $z, z' \in B(0, 2) \setminus B(0, \frac{\epsilon}{2})$ for every N large enough. Again by the mean value theorem and by Equation (17), we therefore have

$$|\hat{f}(z) - \hat{f}(z')| = O\left(\frac{|p|}{N} \|d\hat{f}\|_\infty\right) = O\left(\frac{\phi(N)|p|^2}{N^2 \epsilon^2} \|f'\|_\infty\right).$$

Thus by Equations (18) and (11), we have

$$\begin{aligned} \nu_{p,N}(f) &= \sum_{z \in j_{p,N}^{-1}(J_{p,N}) \setminus B(0, \epsilon)} \hat{f}(z') \\ &\quad + O(\phi(N)|p|^2 \epsilon^{-2} \|f'\|_\infty) + O(N^2 \epsilon^2 \|f\|_\infty). \end{aligned}$$

The symmetric difference between the set $\Lambda \cap (B^+(0, 1) \setminus B(0, \epsilon))$ and the set of elements z' such that $z \in j_{p,N}^{-1}(J_{p,N}) \setminus B(0, \epsilon)$ is, by the triangle inequality, contained in the intersection of Λ with the $2\frac{|p|}{N}$ -neighbourhood $\mathcal{N}_{2\frac{|p|}{N}}(\partial B^+(0, 1))$ of the boundary of $B^+(0, 1)$. By the Gauss counting argument and by Equation (20), this intersection has cardinality $O\left(\frac{1}{\text{covol}_\Lambda} \text{Leb}_{\mathbb{C}}\left(\mathcal{N}_{2\frac{|p|}{N}}(\partial B^+(0, 1))\right)\right) = O(N|p|)$. Hence

$$\begin{aligned} \nu_{p,N}(f) &= \sum_{z' \in \Lambda \cap (B^+(0, 1) \setminus B(0, \epsilon))} \hat{f}(z') \\ &\quad + O(\phi(N)|p|^2 \epsilon^{-2} \|f'\|_\infty) + O((N|p| + N^2 \epsilon^2) \|f\|_\infty). \end{aligned} \quad (21)$$

By the usual approximation of two-dimensional integrals by Riemann sums, we have

$$\left| \int_{B^+(0, 1) \setminus B(0, \epsilon)} \hat{f} d\text{Leb}_{\mathbb{C}} - \text{covol}_\Lambda \sum_{z' \in \Lambda \cap (B^+(0, 1) \setminus B(0, \epsilon))} \hat{f}(z') \right| = O(\text{diam}_\Lambda \|d\hat{f}\|_\infty). \quad (22)$$

By Equations (19), (20) and (17), we have

$$\frac{\text{diam}_\Lambda}{\text{covol}_\Lambda} \|d\hat{f}\|_\infty = O(\phi(N)|p|^2 \epsilon^{-2} \|f'\|_\infty).$$

Therefore Equation (21) becomes, using Equations (20) and (22) for the first equality below, and the fact that the area of $B(0, \epsilon)$ is $O(\epsilon^2)$ and Equation (6) for the second one,

$$\begin{aligned} \nu_{p,N}(f) &= \frac{2|p|N^2}{c'_p|v_p|} \int_{B^+(0,1) \setminus B(0,\epsilon)} \widehat{f} \, d\text{Leb}_{\mathbb{C}} \\ &\quad + O(\phi(N)|p|^2\epsilon^{-2}\|f'\|_{\infty}) + O((|p|N + N^2\epsilon^2)\|f\|_{\infty}) \\ &= \frac{2|p|N^2}{c'_p|v_p|} \int_{B^+(0,1)} \widehat{f} \, d\text{Leb}_{\mathbb{C}} \\ &\quad + O(\phi(N)|p|^2\epsilon^{-2}\|f'\|_{\infty}) + O((|p|N + N^2\epsilon^2)\|f\|_{\infty}). \end{aligned} \quad (23)$$

Let us define $\tilde{f} = \tilde{f}_{p,N} \in C_c([0, +\infty[)$ by $\tilde{f} : u \mapsto f(2\frac{\phi(N)}{N}|p|u)$, so that we have $\widehat{f}(s, t) = \tilde{f}(\frac{s}{s^2+t^2})$ for all $(s, t) \in B^+(0, 1) \setminus \{0\}$. Let us now compute the pushforward measure of $\text{Leb}_{B^+(0,1)}$ by the map from \mathbb{R}^2 to \mathbb{R} defined by $(s, t) \mapsto \frac{s}{s^2+t^2}$. Neglecting sets of measure 0 and using

- Equation (10) and the symmetry $t \mapsto -t$, for the first equation,
 - the change of variable (with s fixed) $u = \frac{s}{s^2+t^2} \geq 0$ so that $t = \sqrt{\frac{s}{u} - s^2}$ and $dt = -\frac{\sqrt{s}}{2u^{3/2}\sqrt{1-us}} du$, for the second equation,
 - the fact that $u \in [s, 1/s]$ if and only if $s \in [0, \min\{u, \frac{1}{u}\}]$ and Fubini's theorem, for the third equation,
- we have

$$\begin{aligned} \int_{B^+(0,1)} \tilde{f}\left(\frac{s}{s^2+t^2}\right) ds dt &= 2 \int_{s=0}^1 \int_{t=0}^{\sqrt{1-s^2}} \tilde{f}\left(\frac{s}{s^2+t^2}\right) dt ds \\ &= \int_{s=0}^1 \int_{u=s}^{1/s} \tilde{f}(u) \frac{\sqrt{s}}{u^{3/2}\sqrt{1-us}} du ds = \int_{u=0}^{+\infty} \frac{\tilde{f}(u)}{u^{3/2}} \int_{s=0}^{\min\{u, \frac{1}{u}\}} \sqrt{\frac{s}{1-us}} ds du. \end{aligned} \quad (24)$$

Using successively the changes of variable (with u fixed) $\sigma = us$ and $\theta = \arcsin \sqrt{\sigma}$, and setting $m_u = \min\{1, u^2\}$, an easy computation gives

$$\begin{aligned} \frac{1}{u^{3/2}} \int_{s=0}^{\min\{u, \frac{1}{u}\}} \sqrt{\frac{s}{1-us}} ds &= \frac{1}{u^3} \int_{\sigma=0}^{m_u} \sqrt{\frac{\sigma}{1-\sigma}} d\sigma = \frac{1}{u^3} \int_{\theta=0}^{\arcsin(\sqrt{m_u})} 2 \sin^2 \theta \, d\theta \\ &= \frac{1}{u^3} (\arcsin(\sqrt{m_u}) - \sqrt{m_u}\sqrt{1-m_u}). \end{aligned} \quad (25)$$

Note that $\sqrt{m_u} = \min\{1, u\}$. Consider the function $g :]0, +\infty[\rightarrow \mathbb{R}$ defined by

$$g : u \mapsto \begin{cases} \frac{1}{u^3} (\arcsin(u) - u\sqrt{1-u^2}) & \text{if } u \leq 1 \\ \frac{\pi}{2u^3} & \text{otherwise.} \end{cases} \quad (26)$$

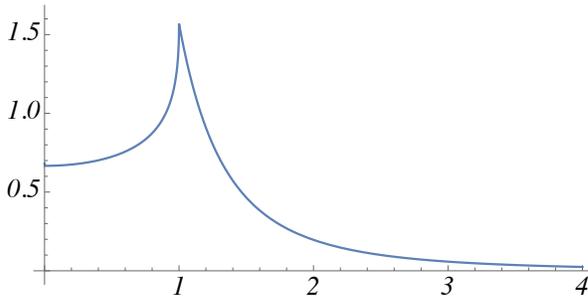


Figure 3 : The graph of the function g .

Let us recall the asymptotic expansions near $u = 0$ of $\arcsin u = u + \frac{u^3}{6} + O(u^5)$ and $\sqrt{1-u^2} = 1 - \frac{u^2}{2} + O(u^4)$. Hence the function g extends continuously at 0 by $g(0) = \frac{2}{3}$. It is continuous on $[0, +\infty[$, positive, with upper bound $\|g\|_\infty = g(1) = \frac{\pi}{2}$. It is integrable, and C^1 -smooth on $[0, +\infty[$ except at $u = 1$, where g is not differentiable on the left. More precisely, since the derivatives of both $u \mapsto -u\sqrt{1-u^2}$ and $u \mapsto \arcsin u$ are Landau-equivalent to $\frac{1}{\sqrt{1-u^2}}$ as u tends to 1 from below, we have

$$g'(u) = O\left(\frac{1}{\sqrt{1-u}}\right) \quad \text{as } u \rightarrow 1^-. \quad (27)$$

By Equation (24) (which remains valid) applied with \tilde{f} the characteristic function $\mathbb{1}_{[0,1]}$ of $[0, 1]$, we have $\int_0^{+\infty} g(s) ds = \int_0^{+\infty} \mathbb{1}_{[0,1]}(s) g(s) ds = 2 \int_{s=0}^1 \int_{t=\sqrt{s-s^2}}^{\sqrt{1-s^2}} dt ds = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$, hence

$$\int_0^{+\infty} g(s) ds = \int_0^1 g(s) ds + \int_1^{+\infty} g(s) ds = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}. \quad (28)$$

Furthermore, we have $g(u) = \frac{2}{3} + O(u^2)$ as u tends to 0. Hence by Equations (24) and (25), and by the definition of \tilde{f} , we have

$$\begin{aligned} \int_{B^+(0,1)} \hat{f} d\text{Leb}_{\mathbb{C}} &= \int_{B^+(0,1)} \tilde{f}\left(\frac{s}{s^2+t^2}\right) ds dt = \int_{u=0}^{+\infty} \tilde{f}(u) g(u) du \\ &= \int_{u=0}^{+\infty} f\left(2 \frac{\phi(N)}{N} |p| u\right) g(u) du. \end{aligned}$$

Equation (23) hence becomes, using the change of variable $t = 2 \frac{\phi(N)}{N} |p| u$ for the second equality,

$$\begin{aligned} \nu_{p,N}(f) &= \frac{2|p|N^2}{c'_p |v_p|} \int_{u=0}^{+\infty} f\left(2 \frac{\phi(N)}{N} |p| u\right) g(u) du \\ &\quad + O(\phi(N) |p|^2 \epsilon^{-2} \|f'\|_\infty) + O(N |p| \|f\|_\infty) + O(N^2 \epsilon^2 \|f\|_\infty) \\ &= \int_{t=0}^{+\infty} f(t) \frac{N^3}{\phi(N) c'_p |v_p|} g\left(\frac{tN}{2\phi(N)|p|}\right) dt \\ &\quad + O(\phi(N) |p|^2 \epsilon^{-2} \|f'\|_\infty) + O(N |p| \|f\|_\infty) + O(N^2 \epsilon^2 \|f\|_\infty). \quad (29) \end{aligned}$$

In order to prove the main result of Section 3, which is Equation (33) below, we will use the following classical Gauss counting result. For every $\beta' \in [-1, +\infty[$, by for instance the proof of [Say1, Lem. 2.10] for the first equality and by Equation (4) for the second one, as $x \geq 1$ tends to $+\infty$, we have

$$\begin{aligned} \sum_{p \in \mathcal{O}_K: 0 < |p| \leq x} |p|^{\beta'} &= \frac{2\pi}{\text{covol}_{\mathcal{O}_K}} \frac{x^{\beta'+2}}{\beta'+2} + O_{\beta'}\left(\frac{1 + \text{diam}_{\mathcal{O}_K}^2}{\text{covol}_{\mathcal{O}_K}} x^{\beta'+1}\right) \\ &= \frac{4\pi}{\sqrt{|D_K|}} \frac{x^{\beta'+2}}{\beta'+2} + O_{\beta'}(\sqrt{|D_K|} x^{\beta'+1}) = O_{\beta', D_K}(x^{\beta'+2}). \quad (30) \end{aligned}$$

Similarly, we have the following analogous result for the case $\beta' = -2$ that we will only use in Section 4: As $x \geq 1$ tends to $+\infty$, we have

$$\sum_{p \in \mathcal{O}_K: 0 < |p| \leq x} \frac{1}{|p|^2} = \frac{4\pi}{\sqrt{|D_K|}} \ln x + O_{D_K}(1). \quad (31)$$

For every $N \in \mathbb{N} \setminus \{0\}$, recalling the definition of the function g in Equation (26), let us define a function $\Theta_N : [0, +\infty[\rightarrow [0, +\infty[$ by

$$\Theta_N : t \mapsto \frac{N^3}{\psi(N) \phi(N)} \sum_{p \in \mathcal{O}_K : 0 < |p| \leq N^\alpha} \frac{1}{c'_p |v_p|} g\left(\frac{t N}{2 \phi(N) |p|}\right). \quad (32)$$

By the assumption $\gamma < \frac{1-\alpha}{2}$ in Equation (12), we have $1 + 3\alpha \leq 2 + 2\alpha - 2\gamma$. By Equations (14) and (29), by using Equation (30) with $\beta' = 2, 1, 0$ and $x = N^\alpha$ in order to control the three error terms in Equation (29), since $\epsilon = N^{-\gamma}$ and since $1 + 3\alpha \leq 2 + 2\alpha - 2\gamma$, we hence have

$$\begin{aligned} \mathcal{R}_N^+(f) &= \frac{1}{\psi(N)} \sum_{p \in \mathcal{O}_K : 0 < |p| \leq N^\alpha} \nu_{p,N}(f) = \int_{t=0}^{+\infty} f(t) \Theta_N(t) dt \\ &+ O\left(\frac{\phi(N) N^{4\alpha+2\gamma}}{\psi(N)} \|f'\|_\infty\right) + O\left(\frac{N^{2+2\alpha-2\gamma}}{\psi(N)} \|f\|_\infty\right). \end{aligned} \quad (33)$$

4 The main result and its proof

Before stating our main result Theorem 4.1, let us give the mild restrictions on the scaling function ϕ that we will use. We keep the notation $\alpha \in]0, \frac{1}{2}[$ of the introduction. We assume in this section that the limits

$$\lambda_\phi = \lim_{N \rightarrow +\infty} \frac{\phi(N)}{N^{1-\alpha}}, \quad \lambda'_\phi = \lim_{N \rightarrow +\infty} \frac{\phi(N)}{N^{1-\frac{\alpha}{2}}}, \quad \lambda''_\phi = \lim_{N \rightarrow +\infty} \frac{\phi(N)}{N} \quad \text{and} \quad \lambda'''_\phi = \lim_{N \rightarrow +\infty} \frac{\phi(N)}{N^{1+\frac{\alpha}{2}}}$$

exist in $[0, +\infty[$, and that there exists $\gamma > 0$ such that Equations (12) and (13) still hold. When ϕ is a power scaling $N \mapsto N^\beta$ as in the introduction and if $\beta \in]0, 2 - 2\alpha[$, these assumptions are satisfied if and only if $\gamma \in]0, \min\{\frac{1-2\alpha}{2}, 1 - \alpha - \frac{\beta}{2}\}[$.

For all $k \in \mathbb{N}$ and $N \in \mathbb{N} \setminus \{0\}$, let us define

$$S_{N,k} = \frac{|D_K| (k+1)}{4\pi} \sum_{p \in \mathcal{O}_K : 0 < |p| \leq N^\alpha} \frac{|p|^k}{c'_p |v_p|}. \quad (34)$$

By the right part of Equation (6) and by Equation (30) with $\beta' = k - 1$ and $x = N^\alpha$, we have

$$\begin{aligned} S_{N,k} &\leq \frac{c_K |D_K| (k+1)}{4\pi} \sum_{p \in \mathcal{O}_K : 0 < |p| \leq N^\alpha} |p|^{k-1} \\ &= c_K \sqrt{|D_K|} N^{(k+1)\alpha} + O_k(c_K |D_K|^{\frac{3}{2}} N^{k\alpha}). \end{aligned}$$

With the similarly obtained lower bound, there hence exists a constant $c'_{K,k} > 0$ such that for every $N \in \mathbb{N} \setminus \{0\}$, we have

$$\frac{1}{c'_{K,k}} N^{(k+1)\alpha} \leq S_{N,k} \leq c'_{K,k} N^{(k+1)\alpha}. \quad (35)$$

Note that when $D_K \equiv 0 \pmod{4}$, we have more precisely by Equations (5) and (30) with $\beta' = k - 1$ and $x = N^\alpha$ that

$$S_{N,k} = \frac{\sqrt{|D_K|} (k+1)}{4\pi} \sum_{p \in \mathcal{O}_K : 0 < |p| \leq N^\alpha} |p|^{k-1} = N^{(k+1)\alpha} + O_k(|D_K| N^{k\alpha}). \quad (36)$$

We now define the measures m_ϕ that will appear as asymptotic pair correlation measures in Theorem 4.1. If $\lambda_\phi \in]0, +\infty[$, we define an even function $\rho_{1-\alpha} : \mathbb{R} \rightarrow [0, +\infty[$ by

$$\rho_{1-\alpha}(t) = \begin{cases} \frac{8\pi}{3|D_K|} & \text{if } t = 0 \\ \frac{8\pi\lambda_\phi^3}{|D_K|t^3} \left(\arcsin\left(\frac{t}{2\lambda_\phi}\right) - \frac{t}{2\lambda_\phi} \left(1 - \frac{t^2}{2\lambda_\phi^2}\right) \left(1 - \frac{t^2}{4\lambda_\phi^2}\right)^{\frac{1}{2}} \right) & \text{if } 0 < t \leq 2\lambda_\phi \\ \frac{4\pi^2\lambda_\phi^3}{|D_K|t^3} & \text{if } t > 2\lambda_\phi. \end{cases} \quad (37)$$

It is easy to see that $\rho_{1-\alpha}$ is bounded, continuous, positive and piecewise real analytic. See Equation (3) and the graph of $\rho_{1-\alpha}$ when $\lambda_\phi = 1$ in Figure 1 of the Introduction (corresponding to the power scaling $\phi : N \mapsto N^{1-\alpha}$). Let

$$m_\phi = \begin{cases} \frac{4\pi^2}{|D_K|} \Delta_0 & \text{if } \lambda_\phi = 0, \\ \rho_{1-\alpha} \text{ Leb}_{\mathbb{R}} & \text{if } \lambda_\phi \in]0, +\infty[, \\ \frac{8\pi}{3|D_K|} \text{ Leb}_{\mathbb{R}} & \text{if } \lambda_\phi = +\infty. \end{cases}$$

Theorem 4.1 *For every $A \geq 1$ and for every $f \in C_c^1(\mathbb{R})$ with support contained in $[-A, A]$, as $N \rightarrow +\infty$, we have*

$$\mathcal{R}_N(f) = \int_{\mathbb{R}} f(t) dm_\phi(t) + \begin{cases} \begin{aligned} & O\left(\left(\frac{\phi(N)}{N^{1-\alpha}}\right)^{\frac{1}{2}} \|f'\|_\infty\right) && \text{if } \lambda_\phi = 0 \\ & + O\left(\max\left\{\frac{\phi(N)}{N^{1-\alpha}}, \frac{1}{N^{2\gamma}}\right\} \|f\|_\infty\right) && \text{and } \psi(N) = N^2 S_{N,1}, \end{aligned} \\ \begin{aligned} & O\left(N^{-\alpha} \|f'\|_\infty\right) + O\left(\max\left\{\frac{A}{N^{\frac{\alpha}{4}}}, \frac{1}{N^{2\gamma}}\right\} \|f\|_\infty\right) && \text{if } \lambda_\phi \in]0, +\infty[, D_K \equiv 0 \pmod{4}, \\ & + O\left(A \left|\frac{N^{1-\alpha}}{\phi(N)} - \frac{1}{\lambda_\phi}\right|^{\frac{1}{2}} \left|\ln\left|\frac{N^{1-\alpha}}{\phi(N)} - \frac{1}{\lambda_\phi}\right|\right| \|f\|_\infty\right) && \text{and } \psi(N) = \frac{N^{3+\alpha}}{\phi(N)}, \end{aligned} \\ \begin{aligned} & O\left(N^{\frac{5\alpha-2}{4}} \|f'\|_\infty\right) && \text{if } \lambda_\phi = +\infty, \lambda'_\phi < +\infty, \alpha < \frac{2}{5} \\ & + O\left(\max\left\{A^3 \left(\frac{N^{1-\alpha}}{\phi(N)}\right)^{\frac{2}{3}}, N^{\frac{5\alpha-2}{4}}\right\} \|f\|_\infty\right) && \text{and } \psi(N) = \frac{N^3 S_{N,0}}{\phi(N)}, \end{aligned} \\ \begin{aligned} & O\left(\frac{\phi(N)}{N^{1+\frac{\alpha}{2}}} \|f'\|_\infty + \frac{N^{1-\frac{\alpha}{2}}}{\phi(N)} \|f\|_\infty\right) && \text{if } \lambda'_\phi = +\infty, \lambda''_\phi = 0, \alpha \leq \frac{2}{11}, \\ & && \text{and } \psi(N) = \frac{N^3 S_{N,0}}{\phi(N)}, \end{aligned} \\ \begin{aligned} & O\left(\frac{\phi(N)}{N^{1+\frac{\alpha}{2}}} (\|f'\|_\infty + A^3 \|f\|_\infty)\right) && \text{if } \lambda''_\phi > 0, \lambda'''_\phi = 0, \alpha \leq \frac{2}{11}, \\ & && \text{and } \psi(N) = \frac{N^3 S_{N,0}}{\phi(N)}. \end{aligned} \end{cases}$$

Note that we have $\lambda'''_\phi \leq \lambda''_\phi \leq \lambda'_\phi \leq \lambda_\phi$ for the extended order on $[0, +\infty]$. Hence if $\lambda''_\phi = +\infty$, then $\lambda'_\phi = \lambda_\phi = +\infty$ and if $\lambda_\phi < +\infty$, then $\lambda'_\phi = \lambda''_\phi = \lambda'''_\phi = 0$. Hence, for instance when $\alpha \leq \frac{1}{6}$, the list of cases in the above Theorem 4.1 is complete, except that the case $\lambda_\phi \in]0, +\infty[, D_K \not\equiv 0 \pmod{4}$ and the case $\lambda'''_\phi > 0$ are missing. The first one should be handled similarly, though the computational complexity seems to be much higher. For the second one, we refer to Section 5.

Proof. The pushforward of a measure μ by a mapping h is denoted by $h_*\mu$. We denote by $\text{sg} : \mathbb{R} \rightarrow \mathbb{R}$ the change of sign map $t \mapsto -t$. By the change of variables $(a, b) \mapsto (b, a)$ in the summation of Equation (2), we have $\text{sg}_* \mathcal{R}_N = \mathcal{R}_N$ for every $N \in \mathbb{N} \setminus \{0\}$. Since

the above measures m_ϕ are invariant under sg, we hence only have to prove Theorem 4.1 where the empirical measure \mathcal{R}_N is replaced by its restriction \mathcal{R}_N^+ to $[0, +\infty[$.

We fix throughout this proof $A \geq 1$ and $f \in C_c^1([0, +\infty[)$ a C^1 -smooth function on $[0, +\infty[$ with compact support contained in $[0, A]$. We consider $N \in \mathbb{N} \setminus \{0\}$ large enough.

We now separate the proof of Theorem 4.1 into the five cases appearing in its statement, corresponding to an increasing scaling. For reasons that will become clear, we will prove Case 4 after Case 5.

Case 1. Let us assume that $\lambda_\phi = \lim_{N \rightarrow +\infty} \frac{\phi(N)}{N^{1-\alpha}} = 0$.

Let us take ψ to be the function $N \mapsto N^2 S_{N,1}$, with $S_{N,1}$ defined in Equation (34) for $k = 1$. Let γ be any element of $]0, \frac{1-2\alpha}{2}[$. The assumptions (12) and (13) are satisfied since $\gamma \leq \frac{1-\alpha}{2}$ and $\lim_{N \rightarrow +\infty} \frac{\phi(N)}{N^{1-\alpha}} = 0$. Hence we may apply the results of Section 3.

We are going to prove that in Case 1, as $N \rightarrow +\infty$, the measure $\Theta_N \text{Leb}_{[0, +\infty[}$ on $[0, +\infty[$, with Θ_N defined in Equation (32), weak-star converges to the Dirac mass $\frac{2\pi^2}{|D_K|} \Delta_0$ at 0 with weight $\frac{2\pi^2}{|D_K|}$. Below are the graphs of Θ_N for various N , for the power scaling $\phi(N) = N^\beta$, in the Gaussian case $K = \mathbb{Q}[i]$, with $\alpha = 0.15$ and $\beta = 0.8 < 1 - \alpha$.

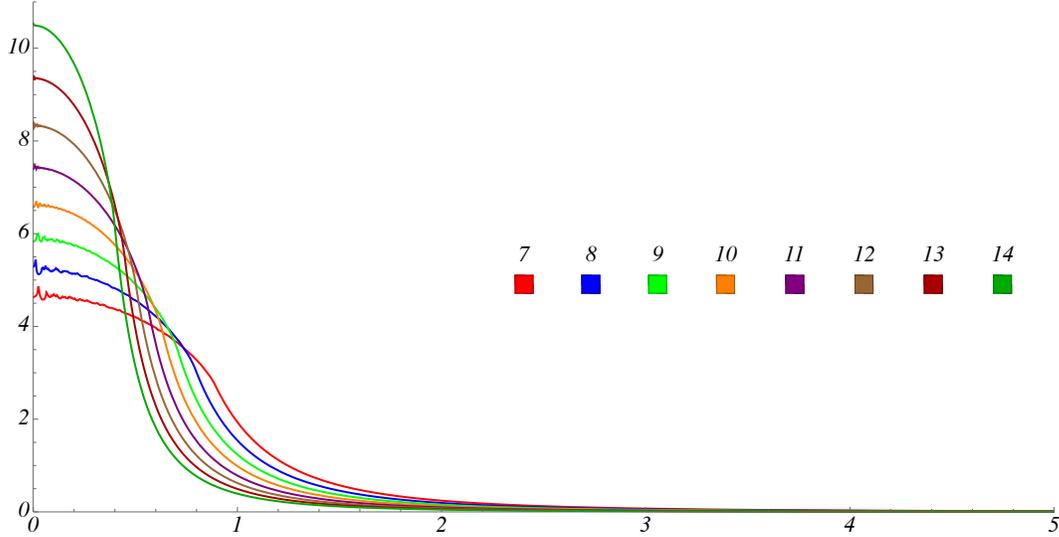


Figure 4 : Graph of Θ_N for $N = 10^m$ with $m \in \{7, 8, 9, 10, 11, 12, 13, 14\}$.

Recall that by Equation (28), the positive function g is integrable over $[0, +\infty[$ with $\int_0^{+\infty} g(s) ds = \frac{\pi}{2}$. For every $N \in \mathbb{N} \setminus \{0\}$, by Equation (32) for the first equality, by the change of variable $s = \frac{tN}{2\phi(N)|p|}$ for the second equality, since by Equation (34) for $k = 1$ we have

$$\sum_{p \in \mathcal{O}_K: 0 < |p| \leq N^\alpha} \frac{|p|}{c'_p |v_p|} = \frac{4\pi S_{N,1}}{2|D_K|},$$

and by the definition of $\psi(N)$ for the last equality, we have

$$\begin{aligned} \int_0^{+\infty} \Theta_N(t) dt &= \frac{N^3}{\psi(N) \phi(N)} \sum_{p \in \mathcal{O}_K: 0 < |p| \leq N^\alpha} \frac{1}{c'_p |v_p|} \int_0^{+\infty} g\left(\frac{tN}{2\phi(N)|p|}\right) dt \\ &= \frac{2N^2}{\psi(N)} \sum_{p \in \mathcal{O}_K: 0 < |p| \leq N^\alpha} \frac{|p|}{c'_p |v_p|} \int_0^{+\infty} g(s) ds = \frac{2\pi^2}{|D_K|}. \end{aligned} \quad (38)$$

Let us now prove that the function Θ_N converges uniformly on compact subsets of $]0, +\infty[$ to 0 as $N \rightarrow +\infty$. With the above centered equation, this will prove that the measure $\Theta_N \text{Leb}_{]0, +\infty[}$ weak-star converges to the Dirac mass $\frac{2\pi^2}{|D_K|} \Delta_0$ as $N \rightarrow +\infty$.

For every $N \in \mathbb{N} \setminus \{0\}$, let us define

$$\eta_N = \left(\frac{\phi(N)}{N^{1-\alpha}} \right)^{\frac{1}{2}},$$

noting that under the assumption of Case 1, we have $\eta_N \rightarrow 0$ as $N \rightarrow +\infty$.

Let $t \in [\eta_N, +\infty[$. For every $p \in \mathcal{O}_K$ such that $0 < |p| \leq N^\alpha$, we have in particular $\frac{tN}{2\phi(N)|p|} \geq \frac{\eta_N N^{1-\alpha}}{2\phi(N)} = \frac{1}{2\eta_N}$, which tends to $+\infty$ as $N \rightarrow +\infty$. In particular, if N is large enough, we have $u = \frac{tN}{2\phi(N)|p|} \geq 1$. Thus if N is large enough, respectively

- by Equation (32) and Equation (26) when $u \geq 1$,
- since $\psi(N) = N^2 S_{N,1}$ and by the definition of $S_{N,3}$ in Equation (34) for $k = 3$,
- by Equation (35) with $k = 1$ and $k = 3$,

we have

$$\begin{aligned} \Theta_N(t) &= \frac{N^3}{\psi(N)\phi(N)} \sum_{p \in \mathcal{O}_K: 0 < |p| \leq N^\alpha} \frac{\pi}{2c'_p |v_p|} \frac{8\phi(N)^3 |p|^3}{t^3 N^3} \\ &= \frac{N^3 \pi}{N^2 S_{N,1} \phi(N)} \frac{4\pi S_{N,3}}{|D_K| 4} \frac{8\phi(N)^3}{2t^3 N^3} = \frac{4\pi^2 \phi(N)^2 S_{N,3}}{|D_K| t^3 N^2 S_{N,1}} \\ &= \frac{1}{t^3} \mathcal{O} \left(\frac{\phi(N)^2 N^{4\alpha}}{N^2 N^{2\alpha}} \right) = \frac{1}{t^3} \mathcal{O} \left(\left(\frac{\phi(N)}{N^{1-\alpha}} \right)^2 \right). \end{aligned} \quad (39)$$

Hence under the assumption of Case 1, the function Θ_N converges uniformly on compact subsets of $]0, +\infty[$ to 0 as $N \rightarrow +\infty$. But we will need more information on the error terms.

As $N \rightarrow +\infty$, respectively by the additivity of the integral, by the mean value theorem, by Equation (38) (that gives $\int_0^{+\infty} \Theta_N(t) dt = \frac{2\pi^2}{|D_K|}$), by Equation (39), and by the value

$\eta_N = \left(\frac{\phi(N)}{N^{1-\alpha}} \right)^{\frac{1}{2}}$, we have

$$\begin{aligned} \int_0^{+\infty} \Theta_N(t) f(t) dt &= \int_0^{\eta_N} \Theta_N(t) f(t) dt + \int_{\eta_N}^{+\infty} \Theta_N(t) f(t) dt \\ &= \int_0^{\eta_N} \Theta_N(t) (f(0) + \mathcal{O}(\eta_N \|f'\|_\infty)) dt + \int_{\eta_N}^{+\infty} \Theta_N(t) f(t) dt \\ &= \frac{2\pi^2}{|D_K|} f(0) + \mathcal{O}(\eta_N \|f'\|_\infty) + \int_{\eta_N}^{+\infty} \Theta_N(t) (f(t) - f(0)) dt \\ &= \frac{2\pi^2}{|D_K|} f(0) + \mathcal{O}(\eta_N \|f'\|_\infty) + \int_{\eta_N}^{+\infty} \frac{dt}{t^3} \mathcal{O} \left(\left(\frac{\phi(N)}{N^{1-\alpha}} \right)^2 \|f\|_\infty \right) \\ &= \frac{2\pi^2}{|D_K|} f(0) + \mathcal{O} \left(\left(\frac{\phi(N)}{N^{1-\alpha}} \right)^{\frac{1}{2}} \|f'\|_\infty \right) + \mathcal{O} \left(\frac{\phi(N)}{N^{1-\alpha}} \|f\|_\infty \right). \end{aligned} \quad (40)$$

As $N \rightarrow +\infty$, since $\psi(N) = N^2 S_{N,1}$, by Equation (35) with $k = 1$, and since we have $2 - 2\alpha - 2\gamma \geq 1 - \alpha$ as $\gamma \leq \frac{1-\alpha}{2}$ by Equation (12), we have

$$\frac{\phi(N) N^{4\alpha+2\gamma}}{\psi(N)} = \frac{\phi(N) N^{4\alpha+2\gamma}}{N^2 S_{N,1}} = \mathcal{O} \left(\frac{\phi(N) N^{4\alpha+2\gamma}}{N^2 N^{2\alpha}} \right) = \mathcal{O} \left(\frac{\phi(N)}{N^{1-\alpha}} \right) = \mathcal{O} \left(\left(\frac{\phi(N)}{N^{1-\alpha}} \right)^{\frac{1}{2}} \right). \quad (41)$$

Similarly, we have

$$\frac{N^{2+2\alpha-2\gamma}}{\psi(N)} = \frac{N^{2+2\alpha-2\gamma}}{N^2 S_{N,1}} = O\left(\frac{N^{2+2\alpha-2\gamma}}{N^{2+2\alpha}}\right) = O\left(\frac{1}{N^{2\gamma}}\right). \quad (42)$$

Therefore, using in Equation (33) the three Equations (40), (41) and (42), we have

$$\mathcal{R}_N^+(f) = \frac{2\pi^2}{|D_K|} f(0) + O\left(\left(\frac{\phi(N)}{N^{1-\alpha}}\right)^{\frac{1}{2}} \|f'\|_\infty\right) + O\left(\max\left\{\frac{\phi(N)}{N^{1-\alpha}}, \frac{1}{N^{2\gamma}}\right\} \|f\|_\infty\right). \quad (43)$$

By symmetry, the restriction of \mathcal{R}_N to $] -\infty, 0]$ also contributes $\frac{2\pi^2}{|D_K|} \Delta_0$ to the limit measure, with an error term as in Equation (43) for every $f \in C_c(\mathbb{R})$. This proves the first case of Theorem 4.1.

When $\phi : N \rightarrow N^\beta$ is a power function, the assumption of Case 1 on ϕ are satisfied if and only if $\beta \in]0, 1 - \alpha[$. If furthermore $D_K \equiv 0 \pmod{4}$, then by Equation (36) with $k = 1$, as $N \rightarrow +\infty$, we have $\psi(N) = N^2 S_{N,1} \sim N^{2+2\alpha}$. This proves the first case of Theorem 1.1.

Case 2. Let us assume that $\lambda_\phi = \lim_{N \rightarrow +\infty} \frac{\phi(N)}{N^{1-\alpha}} \in]0, +\infty[$ and that $D_K \equiv 0 \pmod{4}$.

Let us take ψ to be the function $N \mapsto \frac{N^{3+\alpha}}{\phi(N)}$. Let γ be any element of $]0, \frac{1-2\alpha}{2}[$. The assumptions (12) and (13) are satisfied since $\gamma < \frac{1-\alpha}{2}$ and $\lim_{N \rightarrow +\infty} \frac{\phi(N)}{N^{1-\alpha}} < +\infty$. Hence we may apply the results of Section 3.

Below are the graphs of Θ_N for various N , for the power scaling $\phi : N \mapsto N^\beta$, in the Gaussian case $K = \mathbb{Q}[i]$, with $\alpha = 0.15$ and $\beta = 1 - \alpha = 0.85$.

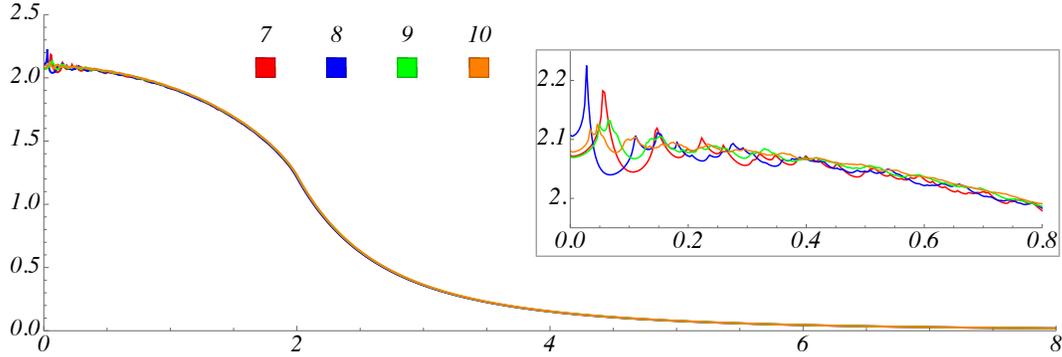


Figure 5 : Graph of Θ_N for $N = 10^m$ where m is 7 (red), 8 (blue), 9 (green), 10 (orange).

Let $t \in [0, A]$ and $N \in \mathbb{N} \setminus \{0\}$. Since $D_K \equiv 0 \pmod{4}$ under the assumptions of Case 2, since for every $p \in \mathcal{O}_K \setminus \{0\}$ we then have $c'_p |v_p| = \sqrt{|D_K|} |p|$ by Equation (5), and since $\psi(N) = \frac{N^{3+\alpha}}{\phi(N)}$, Equation (32) becomes

$$\begin{aligned} \Theta_N(t) &= \frac{N^3}{\sqrt{|D_K|} \psi(N) \phi(N)} \sum_{p \in \mathcal{O}_K: 0 < |p| \leq N^\alpha} \frac{1}{|p|} g\left(\frac{tN}{2\phi(N)|p|}\right) \\ &= \frac{1}{\sqrt{|D_K|} N^{2\alpha}} \sum_{p \in \mathcal{O}_K: 0 < |p| \leq N^\alpha} \frac{N^\alpha}{|p|} g\left(\frac{N^{1-\alpha} t N^\alpha}{\phi(N) 2|p|}\right). \end{aligned} \quad (44)$$

Let us define a decomposition of the summation in Equation (44) corresponding to the subdivision $[0, +\infty[= [0, 1[\cup [1, +\infty[$ where the function g given by Equation (26) has two different expressions, plus some safety zone around 1. Let us define

$$\varepsilon_N = \max \left\{ 8\lambda_\phi^2 \left| \frac{N^{1-\alpha}}{\phi(N)} - \frac{1}{\lambda_\phi} \right|, \frac{1}{N^{\frac{\alpha}{4}}} \right\} > 0. \quad (45)$$

Note that under the assumptions of Case 2, we have $\lim_{N \rightarrow +\infty} \varepsilon_N = 0$. With the usual convention on empty sums, let

$$\Theta_N^-(t) = \frac{1}{\sqrt{|D_K|} N^\alpha} \sum_{p \in \mathcal{O}_K: 0 < |p| \leq N^\alpha, |p| < \frac{t}{2\lambda_\phi} N^\alpha} \frac{1}{|p|} g\left(\frac{tN}{2\phi(N)|p|}\right), \quad (46)$$

$$\Theta_N^0(t) = \frac{1}{\sqrt{|D_K|} N^\alpha} \sum_{p \in \mathcal{O}_K: 0 < |p| \leq N^\alpha, \frac{t}{2\lambda_\phi} N^\alpha \leq |p| < \frac{t+\varepsilon_N}{2\lambda_\phi} N^\alpha} \frac{1}{|p|} g\left(\frac{tN}{2\phi(N)|p|}\right), \quad (47)$$

$$\text{and } \Theta_N^+(t) = \frac{1}{\sqrt{|D_K|} N^\alpha} \sum_{p \in \mathcal{O}_K: \frac{t+\varepsilon_N}{2\lambda_\phi} N^\alpha \leq |p| \leq N^\alpha} \frac{1}{|p|} g\left(\frac{N^{1-\alpha}}{\phi(N)} \frac{tN^\alpha}{2|p|}\right). \quad (48)$$

So that by Equation (44) we have

$$\Theta_N(t) = \Theta_N^-(t) + \Theta_N^0(t) + \Theta_N^+(t). \quad (49)$$

Step 1 of Case 2. Let us first estimate $\Theta_N^-(t)$ as $N \rightarrow +\infty$ uniformly in $t \in [0, +\infty[$.

Since $\frac{tN^\alpha}{2\lambda_\phi|p|} > 1$ (and in particular $t > 0$) whenever the index $p \in \mathcal{O}_K$ occurs in the summation defining $\Theta_N^-(t)$, for N large enough, under the assumptions of Case 2, we also have $\frac{tN}{2\phi(N)|p|} = \frac{N^{1-\alpha}}{\phi(N)} \frac{tN^\alpha}{2|p|} > 1$. By the value $g(u) = \frac{\pi}{2u^3}$ on $u \in [1, +\infty[$ given by Equation (26), and by Equation (30) applied with $\beta' = 2$ and $x = \min\{N^\alpha, \frac{t}{2\lambda_\phi} N^\alpha\}$, Equation (46) gives, as $N \rightarrow +\infty$,

$$\begin{aligned} \Theta_N^-(t) &= \frac{4\pi\phi(N)^3}{\sqrt{|D_K|} t^3 N^{3+\alpha}} \sum_{p \in \mathcal{O}_K: 0 < |p| \leq N^\alpha, |p| < \frac{t}{2\lambda_\phi} N^\alpha} |p|^2 \\ &= \frac{4\pi\phi(N)^3}{\sqrt{|D_K|} t^3 N^{3+\alpha}} \left(\frac{4\pi}{4\sqrt{|D_K|}} \min\left\{N^\alpha, \frac{tN^\alpha}{2\lambda_\phi}\right\}^4 + O\left(\min\left\{N^\alpha, \frac{tN^\alpha}{2\lambda_\phi}\right\}^3\right) \right) \\ &= \frac{4\pi^2}{|D_K|} \left(\frac{\phi(N)}{N^{1-\alpha}} \right)^3 \min\left\{\frac{1}{t^3}, \frac{t}{16\lambda_\phi^4}\right\} + O\left(\frac{1}{N^\alpha} \left(\frac{\phi(N)}{N^{1-\alpha}}\right)^3\right). \end{aligned}$$

Since the function $t \mapsto \min\left\{\frac{1}{t^3}, \frac{t}{16\lambda_\phi^4}\right\}$ is bounded on $]0, +\infty[$, since $\frac{\phi(N)}{N^{1-\alpha}}$ converges to λ_ϕ (hence is bounded, uniformly in t) under the assumptions of Case 2, and since we have $|\frac{1}{x} - \frac{1}{y}| = O(|x - y|)$ when x, y remain in a compact subset of $]0, +\infty[$, we have

$$\begin{aligned} \Theta_N^-(t) &= \frac{4\pi^2}{|D_K|} \min\left\{\frac{\lambda_\phi^3}{t^3}, \frac{t}{16\lambda_\phi}\right\} + O\left(\frac{1}{N^\alpha}\right) + O\left(\left|\frac{\phi(N)}{N^{1-\alpha}} - \lambda_\phi\right|\right) \\ &= \frac{4\pi^2}{|D_K|} \min\left\{\frac{\lambda_\phi^3}{t^3}, \frac{t}{16\lambda_\phi}\right\} + O\left(\frac{1}{N^\alpha}\right) + O\left(\left|\frac{N^{1-\alpha}}{\phi(N)} - \frac{1}{\lambda_\phi}\right|\right). \end{aligned} \quad (50)$$

Note that we have $\frac{\lambda_\phi^3}{t^3} \leq \frac{t}{16\lambda_\phi}$ if and only if $t \geq 2\lambda_\phi$.

Step 2 of Case 2. Let us now estimate $\Theta_N^+(t)$ as $N \rightarrow +\infty$ uniformly in $t \in [\varepsilon_N, +\infty[$.

Let $t \in [\varepsilon_N, +\infty[$ and $N \in \mathbb{N} \setminus \{0\}$. Note that if $t > 2\lambda_\phi$, then $\Theta_N^+(t) = 0$ (see Equation (48)). Hence we assume that $t \in [\varepsilon_N, 2\lambda_\phi]$ from now on in Step 2. Let us define two subsets in \mathbb{C} by

$$\mathcal{A}_t = \left\{ z \in \mathbb{C} : \frac{t}{2\lambda_\phi} \leq |z| \leq 1 \right\} \quad \text{and} \quad \mathcal{A}_{t,N} = \left\{ z \in \mathbb{C} : \frac{t + \varepsilon_N}{2\lambda_\phi} \leq |z| \leq 1 \right\}. \quad (51)$$

The subset \mathcal{A}_t (which is an annulus if $t < 2\lambda_\phi$ and a circle otherwise) contains $\mathcal{A}_{t,N}$ (which is an annulus if $t < 2\lambda_\phi - \varepsilon_N$, a circle if $t = 2\lambda_\phi - \varepsilon_N$, and is empty otherwise).

Note that the map $s \mapsto \frac{s}{s + \varepsilon_N}$ is increasing on $[0, +\infty[$. Hence since $\lim_{N \rightarrow +\infty} \varepsilon_N = 0$, for N large enough, for every element $p \in \mathcal{O}_K$ occurring in the summation defining $\Theta_N^+(t)$ in Equation (48), we have

$$\frac{t N^\alpha}{2\lambda_\phi |p|} \leq \frac{t}{t + \varepsilon_N} \leq \frac{2\lambda_\phi}{2\lambda_\phi + \varepsilon_N} \leq 1 - \frac{\varepsilon_N}{4\lambda_\phi} < 1.$$

Hence for such elements N and p , by the definition of ε_N in Equation (45), we have

$$\begin{aligned} \frac{N^{1-\alpha}}{\phi(N)} \frac{t N^\alpha}{2|p|} &\leq \left| \frac{N^{1-\alpha}}{\phi(N)} - \frac{1}{\lambda_\phi} \right| \frac{t N^\alpha}{2|p|} + \frac{t N^\alpha}{2\lambda_\phi |p|} \leq \left| \frac{N^{1-\alpha}}{\phi(N)} - \frac{1}{\lambda_\phi} \right| \lambda_\phi + \frac{t N^\alpha}{2\lambda_\phi |p|} \\ &\leq \frac{\varepsilon_N}{8\lambda_\phi} + 1 - \frac{\varepsilon_N}{4\lambda_\phi} = 1 - \frac{\varepsilon_N}{8\lambda_\phi} < 1. \end{aligned}$$

Since $g'(u) = O\left(\frac{1}{\sqrt{1-u}}\right)$ for $u \in [0, 1[$ by Equation (27), by the mean value theorem, and since $\varepsilon_N \geq 8\lambda_\phi^2 \left| \frac{N^{1-\alpha}}{\phi(N)} - \frac{1}{\lambda_\phi} \right|$ by Equation (45), for every $t \in [\varepsilon_N, 2\lambda_\phi]$ and for every element $p \in \mathcal{O}_K$ occurring in the summation defining $\Theta_N^+(t)$, as $N \rightarrow +\infty$, we therefore have

$$\begin{aligned} \left| g\left(\frac{N^{1-\alpha}}{\phi(N)} \frac{t N^\alpha}{2|p|}\right) - g\left(\frac{t N^\alpha}{2\lambda_\phi |p|}\right) \right| &= O\left(\frac{t N^\alpha}{|p|} \left| \frac{N^{1-\alpha}}{\phi(N)} - \frac{1}{\lambda_\phi} \right| \frac{1}{\sqrt{\varepsilon_N}}\right) \\ &= O\left(\frac{N^\alpha}{|p|} \left| \frac{N^{1-\alpha}}{\phi(N)} - \frac{1}{\lambda_\phi} \right|^{\frac{1}{2}}\right). \end{aligned} \quad (52)$$

Let us define

$$\tilde{\Theta}_N^+(t) = \frac{1}{\sqrt{|D_K|} N^\alpha} \sum_{p \in \mathcal{O}_K \cap (N^\alpha \mathcal{A}_{t,N})} \frac{1}{|p|} g\left(\frac{t N^\alpha}{2\lambda_\phi |p|}\right). \quad (53)$$

Since $t \geq \varepsilon_N$ in this Step 2 and $\varepsilon_N \geq 8\lambda_\phi^2 \left| \frac{N^{1-\alpha}}{\phi(N)} - \frac{1}{\lambda_\phi} \right|$, we have

$$\frac{1}{t + \varepsilon_N} \leq \frac{1}{2\varepsilon_N} \leq \frac{1}{16\lambda_\phi^2} \left| \frac{N^{1-\alpha}}{\phi(N)} - \frac{1}{\lambda_\phi} \right|^{-1}.$$

Hence by the definition of $\mathcal{A}_{t,N}$ in Equation (51) and by Equation (31) applied twice with $x = N^\alpha$ and with $x = \frac{t + \varepsilon_N}{2\lambda_\phi} N^\alpha$, as $N \rightarrow +\infty$, we have

$$\sum_{p \in \mathcal{O}_K \cap (N^\alpha \mathcal{A}_{t,N})} \frac{1}{|p|^2} = O\left(\left| \ln \frac{N^\alpha}{(t + \varepsilon_N) N^\alpha} \right|\right) + O(1) = O\left(\left| \ln \left| \frac{N^{1-\alpha}}{\phi(N)} - \frac{1}{\lambda_\phi} \right| \right|\right).$$

By Equations (48), (53) and (52), for every $t \in [\varepsilon_N, 2\lambda_\phi]$, as $N \rightarrow +\infty$, we hence have

$$\begin{aligned} |\Theta_N^+(t) - \tilde{\Theta}_N^+(t)| &= O\left(\frac{1}{N^\alpha} \sum_{p \in \theta_K \cap (N^\alpha \mathcal{A}_{t,N})} \frac{1}{|p|} \frac{N^\alpha}{|p|} \left| \frac{N^{1-\alpha}}{\phi(N)} - \frac{1}{\lambda_\phi} \right|^{\frac{1}{2}}\right) \\ &= O\left(\left| \frac{N^{1-\alpha}}{\phi(N)} - \frac{1}{\lambda_\phi} \right|^{\frac{1}{2}} \left| \ln \left| \frac{N^{1-\alpha}}{\phi(N)} - \frac{1}{\lambda_\phi} \right| \right|\right). \end{aligned} \quad (54)$$

Let us consider the function $G_t : \mathcal{A}_t \rightarrow [0, +\infty[$ defined by

$$G_t : z \mapsto \frac{1}{|z|} g\left(\frac{t}{2\lambda_\phi |z|}\right).$$

By the change of variables $z = N^{-\alpha}p$ in Equation (53), we have

$$\tilde{\Theta}_N^+(t) = \frac{1}{\sqrt{|D_K|} N^{2\alpha}} \sum_{z \in (N^{-\alpha} \theta_K) \cap \mathcal{A}_{t,N}} G_t(z). \quad (55)$$

Let us estimate the upper bound $\|dG_t|_{\mathcal{A}_{t,N}}\|_\infty$ on the operator norm of the (linear) differential $d_z G_t$ of G_t at every point $z \in \mathcal{A}_{t,N}$. Writing the elements of the annulus \mathcal{A}_t in polar coordinates $z = \rho e^{i\theta}$, the map G_t does not depend on the argument θ . Since the function $\rho \mapsto (1 - \frac{t}{2\lambda_\phi \rho})^{-\frac{1}{2}}$ is nonincreasing and since $\varepsilon_N \leq t$ in this Step 2, for every $\rho \in [\frac{t+\varepsilon_N}{2\lambda_\phi}, 1]$, we have

$$\left(1 - \frac{t}{2\lambda_\phi \rho}\right)^{-\frac{1}{2}} \leq \left(1 - \frac{t}{2\lambda_\phi \frac{t+\varepsilon_N}{2\lambda_\phi}}\right)^{-\frac{1}{2}} = \left(\frac{t+\varepsilon_N}{\varepsilon_N}\right)^{\frac{1}{2}} \leq \left(\frac{2t}{\varepsilon_N}\right)^{\frac{1}{2}}.$$

Note that the map $s \mapsto \frac{s^{\frac{3}{2}}}{(s+\varepsilon_N)^3}$ is nonincreasing on $[\varepsilon_N, +\infty[$, hence is bounded by $\frac{1}{8}\varepsilon_N^{-\frac{3}{2}}$. Hence, since the function g is bounded and by Equation (27), as $N \rightarrow +\infty$, for every $z = \rho e^{i\theta}$ in the smaller subset $\mathcal{A}_{t,N}$ so that $\rho \geq \frac{t+\varepsilon_N}{2\lambda_\phi}$, we have

$$\begin{aligned} \frac{\partial G_t}{\partial \rho}(z) &= -\frac{1}{\rho^2} g\left(\frac{t}{2\lambda_\phi \rho}\right) - \frac{t}{2\lambda_\phi \rho^3} g'\left(\frac{t}{2\lambda_\phi \rho}\right) \\ &= O\left(\frac{1}{(t+\varepsilon_N)^2}\right) + O\left(\frac{t}{(t+\varepsilon_N)^3} \frac{1}{\left(1 - \frac{t}{2\lambda_\phi \rho}\right)^{\frac{1}{2}}}\right) \\ &= O\left(\frac{1}{\varepsilon_N^2}\right) + O\left(\frac{t^{\frac{3}{2}}}{(t+\varepsilon_N)^3 \varepsilon_N^{\frac{1}{2}}}\right) = O\left(\frac{1}{\varepsilon_N^2}\right). \end{aligned}$$

Therefore we have

$$\|dG_t|_{\mathcal{A}_{t,N}}\|_\infty = \left\| \frac{\partial G_t}{\partial \rho} \Big|_{\mathcal{A}_{t,N}} \right\|_\infty = O\left(\frac{1}{\varepsilon_N^2}\right). \quad (56)$$

By Equation (4), the \mathbb{Z} -lattice $N^{-\alpha} \theta_K$ of \mathbb{C} has covolume

$$\text{covol}_{N^{-\alpha} \theta_K} = N^{-2\alpha} \text{covol}_{\theta_K} = \frac{\sqrt{|D_K|}}{2N^{2\alpha}}, \quad (57)$$

and its diameter satisfies $\text{diam}_{N^{-\alpha}\mathcal{O}_K} = O(N^{-\alpha})$. By the well-known approximation of integrals by averages of sums over lattices points, we have

$$\int_{\mathcal{A}_{t,N}} G_t d\text{Leb}_{\mathbb{C}} = \text{covol}_{N^{-\alpha}\mathcal{O}_K} \sum_{z \in (N^{-\alpha}\mathcal{O}_K) \cap \mathcal{A}_{t,N}} G_t(z) + O(\text{diam}_{N^{-\alpha}\mathcal{O}_K} \|dG_t|_{\mathcal{A}_{t,N}}\|_{\infty}).$$

Hence by Equations (57) and (56), and since $\varepsilon_N \geq \frac{1}{N^{\frac{3}{4}}}$ by Equation (45), we have

$$\begin{aligned} \sum_{z \in (N^{-\alpha}\mathcal{O}_K) \cap \mathcal{A}_{t,N}} G_t(z) &= \frac{2N^{2\alpha}}{\sqrt{|D_K|}} \int_{\mathcal{A}_{t,N}} G_t d\text{Leb}_{\mathbb{C}} + O\left(\frac{N^{\alpha}}{\varepsilon N^2}\right) \\ &= \frac{2N^{2\alpha}}{\sqrt{|D_K|}} \int_{\mathcal{A}_{t,N}} G_t d\text{Leb}_{\mathbb{C}} + O(N^{\frac{3\alpha}{2}}). \end{aligned} \quad (58)$$

By the definition of the function g in Equation (26) and by an elementary computation, a primitive of the map $s \mapsto \frac{g(s)}{s^2}$ on $]0, 1]$ is the map $h :]0, 1] \rightarrow \mathbb{R}$ defined by

$$\forall u \in]0, 1], \quad h(u) = \frac{1}{4u^4} (u(1-2u^2)\sqrt{1-u^2} - \arcsin(u)). \quad (59)$$

Furthermore, we have

$$h(1) = -\frac{\pi}{8} \quad \text{and} \quad h(u) = -\frac{2}{3u} + O(u) \text{ as } u \rightarrow 0.$$

Since $d\text{Leb}_{\mathbb{C}}(z) = \rho d\rho d\theta$ and by the change of variable $s = \frac{t}{2\lambda_{\phi}\rho}$, we have

$$\begin{aligned} \int_{\mathcal{A}_t} G_t d\text{Leb}_{\mathbb{C}} &= 2\pi \int_{\frac{t}{2\lambda_{\phi}} \leq \rho \leq 1} \frac{1}{\rho} g\left(\frac{t}{2\lambda_{\phi}\rho}\right) \rho d\rho = \frac{\pi t}{\lambda_{\phi}} \int_{\frac{t}{2\lambda_{\phi}} \leq s \leq 1} \frac{g(s)}{s^2} ds \\ &= \frac{\pi t}{\lambda_{\phi}} \left(h(1) - h\left(\frac{t}{2\lambda_{\phi}}\right) \right). \end{aligned} \quad (60)$$

Since the function $s \mapsto sg(s)$ is increasing on $[0, 1]$, the function G_t on \mathcal{A}_t is uniformly bounded by $\frac{2\lambda_{\phi}}{t}g(1) = \frac{\pi\lambda_{\phi}}{t}$. Hence, by Equation (51) and since $t \geq \varepsilon_N$ in this Step 2, when $N \rightarrow +\infty$, we have

$$\begin{aligned} \int_{\mathcal{A}_t \setminus \mathcal{A}_{t,N}} G_t d\text{Leb}_{\mathbb{C}} &= O\left(\frac{1}{t} \text{Area}(\mathcal{A}_t \setminus \mathcal{A}_{t,N})\right) = O\left(\frac{\pi}{t} \left(\frac{(t+\varepsilon_N)^2}{4\lambda_{\phi}^2} - \frac{t^2}{4\lambda_{\phi}^2} \right)\right) \\ &= O\left(\varepsilon_N + \frac{\varepsilon_N^2}{t}\right) = O(\varepsilon_N). \end{aligned} \quad (61)$$

Therefore by Equations (58), (60) and (61), we have

$$\begin{aligned} \sum_{z \in (N^{-\alpha}\mathcal{O}_K) \cap \mathcal{A}_{t,N}} G_t(z) &= \sum_{z \in (N^{-\alpha}\mathcal{O}_K) \cap \mathcal{A}_{t,N}} G_t(z) - \frac{2N^{2\alpha}}{\sqrt{|D_K|}} \int_{\mathcal{A}_{t,N}} G_t d\text{Leb}_{\mathbb{C}} \\ &\quad + \frac{2N^{2\alpha}}{\sqrt{|D_K|}} \int_{\mathcal{A}_t} G_t d\text{Leb}_{\mathbb{C}} - \frac{2N^{2\alpha}}{\sqrt{|D_K|}} \int_{\mathcal{A}_t \setminus \mathcal{A}_{t,N}} G_t d\text{Leb}_{\mathbb{C}} \\ &= O(N^{\frac{3\alpha}{2}}) + \frac{2N^{2\alpha}\pi t}{\sqrt{|D_K|}\lambda_{\phi}} \left(h(1) - h\left(\frac{t}{2\lambda_{\phi}}\right) \right) + O(N^{2\alpha}\varepsilon_N). \end{aligned}$$

Thus by Equations (54) and (55), by the definition of ε_N in Equation (45), as $N \rightarrow +\infty$, for every $t \in [\varepsilon_N, 2\lambda_\phi]$, we have

$$\begin{aligned}
\Theta_N^+(t) &= \tilde{\Theta}_N^+(t) + O\left(\left|\frac{N^{1-\alpha}}{\phi(N)} - \frac{1}{\lambda_\phi}\right|^{\frac{1}{2}} \left|\ln\left|\frac{N^{1-\alpha}}{\phi(N)} - \frac{1}{\lambda_\phi}\right|\right|\right) \\
&= \frac{1}{\sqrt{|D_K|} N^{2\alpha}} \sum_{z \in (N^{-\alpha}\mathcal{C}_K) \cap \mathcal{A}_{t,N}} G_t(z) + O\left(\left|\frac{N^{1-\alpha}}{\phi(N)} - \frac{1}{\lambda_\phi}\right|^{\frac{1}{2}} \left|\ln\left|\frac{N^{1-\alpha}}{\phi(N)} - \frac{1}{\lambda_\phi}\right|\right|\right) \\
&= \frac{2\pi t}{|D_K| \lambda_\phi} \left(h(1) - h\left(\frac{t}{2\lambda_\phi}\right)\right) + O(\varepsilon_N) + O\left(\frac{1}{N^{\frac{\alpha}{2}}}\right) + O\left(\left|\frac{N^{1-\alpha}}{\phi(N)} - \frac{1}{\lambda_\phi}\right|^{\frac{1}{2}} \left|\ln\left|\frac{N^{1-\alpha}}{\phi(N)} - \frac{1}{\lambda_\phi}\right|\right|\right) \\
&= \frac{2\pi t}{|D_K| \lambda_\phi} \left(h(1) - h\left(\frac{t}{2\lambda_\phi}\right)\right) + O\left(\frac{1}{N^{\frac{\alpha}{4}}}\right) + O\left(\left|\frac{N^{1-\alpha}}{\phi(N)} - \frac{1}{\lambda_\phi}\right|^{\frac{1}{2}} \left|\ln\left|\frac{N^{1-\alpha}}{\phi(N)} - \frac{1}{\lambda_\phi}\right|\right|\right). \quad (62)
\end{aligned}$$

Step 3 of Case 2. Let us finally estimate $\Theta_N^0(t)$ as $N \rightarrow +\infty$ uniformly in $t \in [0, +\infty[$.

Let $N \in \mathbb{N} \setminus \{0\}$. Note that if $t > 2\lambda_\phi$, then $\Theta_N^0(t) = 0$ (see Equation (47)). Hence we assume from now on in Step 3 that $t \in [0, 2\lambda_\phi]$. Using the notation of Equation (51), Equation (47) may be written

$$\Theta_N^0(t) = \frac{1}{\sqrt{|D_K|} N^\alpha} \sum_{p \in \mathcal{C}_K \cap (N^\alpha \mathcal{A}_t \setminus N^\alpha \mathcal{A}_{t,N})} \frac{1}{|p|} g\left(\frac{tN}{2\phi(N)|p|}\right).$$

Since the nonnegative function g is uniformly bounded from above (by $\frac{\pi}{2}$), using twice Equation (30) with $\beta' = -1$ and $x = \frac{(t+\varepsilon_N)}{2\lambda_\phi} N^\alpha$, $x = \frac{t}{2\lambda_\phi} N^\alpha$, as $N \rightarrow +\infty$, we have

$$\begin{aligned}
\Theta_N^0(t) &= O\left(\frac{1}{N^\alpha} \sum_{p \in \mathcal{C}_K \cap (N^\alpha \mathcal{A}_t \setminus N^\alpha \mathcal{A}_{t,N})} \frac{1}{|p|}\right) \\
&= O\left(\frac{1}{N^\alpha} \left(\frac{(t+\varepsilon_N)}{2\lambda_\phi} N^\alpha - \frac{t}{2\lambda_\phi} N^\alpha\right)\right) = O(\varepsilon_N). \quad (63)
\end{aligned}$$

Conclusion of Case 2. To conclude, we gather the estimates of Steps 1, 2 and 3.

Let us first compute $\Theta_N(t)$ for $t \in [\varepsilon_N, +\infty[$. We separate the computation into the case $t > 2\lambda_\phi$ and the case $\varepsilon_N \leq t \leq 2\lambda_\phi$.

If $t > 2\lambda_\phi$, then $\Theta_N^0(t) = \Theta_N^+(t) = 0$ as seen in Steps 2 and 3. By Equations (49) and (50), and by the value of $\rho_{1-\alpha}(t)$ when $t > 2\lambda_\phi$ given in Equation (37), we hence have

$$\begin{aligned}
\Theta_N(t) &= \Theta_N^-(t) = \frac{4\pi^2 \lambda_\phi^3}{|D_K| t^3} + O\left(\frac{1}{N^\alpha}\right) + O\left(\left|\frac{N^{1-\alpha}}{\phi(N)} - \frac{1}{\lambda_\phi}\right|\right) \\
&= \rho_{1-\alpha}(t) + O\left(\frac{1}{N^{\frac{\alpha}{4}}}\right) + O\left(\left|\frac{N^{1-\alpha}}{\phi(N)} - \frac{1}{\lambda_\phi}\right|^{\frac{1}{2}} \left|\ln\left|\frac{N^{1-\alpha}}{\phi(N)} - \frac{1}{\lambda_\phi}\right|\right|\right). \quad (64)
\end{aligned}$$

If $\varepsilon_N \leq t \leq 2\lambda_\phi$, by plugging in Equation (49) the three Equations (50), (63) and (62), by the definition of ε_N in Equation (45), by the value of the function h given in Equation (59) (with $h(1) = -\frac{\pi}{8}$) and by the value of $\rho_{1-\alpha}(t)$ when $t \leq 2\lambda_\phi$ given in Equation (37),

as $N \rightarrow +\infty$, we have

$$\begin{aligned}
\Theta_N(t) &= \Theta_N^-(t) + \Theta_N^0(t) + \Theta_N^+(t) \\
&= \left(\frac{\pi^2 t}{4|D_K|\lambda_\phi} + O\left(\frac{1}{N^\alpha}\right) + O\left(\left|\frac{N^{1-\alpha}}{\phi(N)} - \frac{1}{\lambda_\phi}\right|\right) \right) + O(\varepsilon_N) \\
&\quad + \left(\frac{2\pi t}{|D_K|\lambda_\phi} \left(h(1) - h\left(\frac{t}{2\lambda_\phi}\right) \right) + O\left(\frac{1}{N^{\frac{\alpha}{4}}}\right) + O\left(\left|\frac{N^{1-\alpha}}{\phi(N)} - \frac{1}{\lambda_\phi}\right|^{\frac{1}{2}} \left| \ln \left| \frac{N^{1-\alpha}}{\phi(N)} - \frac{1}{\lambda_\phi} \right| \right| \right) \right) \\
&= \frac{8\pi\lambda_\phi^3}{|D_K|t^3} \left(\arcsin\left(\frac{t}{2\lambda_\phi}\right) - \frac{t}{2\lambda_\phi} \left(1 - \frac{t^2}{2\lambda_\phi^2}\right) \left(1 - \frac{t^2}{4\lambda_\phi^2}\right)^{\frac{1}{2}} \right) \\
&\quad + O\left(\frac{1}{N^{\frac{\alpha}{4}}}\right) + O\left(\left|\frac{N^{1-\alpha}}{\phi(N)} - \frac{1}{\lambda_\phi}\right|^{\frac{1}{2}} \left| \ln \left| \frac{N^{1-\alpha}}{\phi(N)} - \frac{1}{\lambda_\phi} \right| \right| \right) \\
&= \rho_{1-\alpha}(t) + O\left(\frac{1}{N^{\frac{\alpha}{4}}}\right) + O\left(\left|\frac{N^{1-\alpha}}{\phi(N)} - \frac{1}{\lambda_\phi}\right|^{\frac{1}{2}} \left| \ln \left| \frac{N^{1-\alpha}}{\phi(N)} - \frac{1}{\lambda_\phi} \right| \right| \right). \tag{65}
\end{aligned}$$

By Equation (44), since the function g is bounded on $[0, +\infty[$ and by Equation (30) with $\beta' = -1$ and $x = N^\alpha$, as $N \rightarrow +\infty$, uniformly in $t \in [0, +\infty[$, we have

$$\Theta_N(t) = O\left(\frac{1}{N^\alpha} \sum_{p \in \mathcal{O}_K: 0 < |p| \leq N^\alpha} \frac{1}{|p|}\right) = O(1).$$

We have already seen after Equation (37) that the function $\rho_{1-\alpha}$ is bounded on $[0, +\infty[$, hence $\int_0^{\varepsilon_N} \rho_{1-\alpha}(t) f(t) dt = O(\varepsilon_N \|f\|_\infty)$. Therefore, since the support of f is contained in $[0, A]$, by Equation (65), and by the definition of ε_N in Equation (45), we have

$$\begin{aligned}
\int_0^{+\infty} \Theta_N(t) f(t) dt &= \int_0^{\varepsilon_N} \Theta_N(t) f(t) dt + \int_{\varepsilon_N}^A \Theta_N(t) f(t) dt \\
&= O(\varepsilon_N \|f\|_\infty) + \int_{\varepsilon_N}^A \rho_{1-\alpha}(t) f(t) dt \\
&\quad + O\left(\left(\frac{1}{N^{\frac{\alpha}{4}}} + \left|\frac{N^{1-\alpha}}{\phi(N)} - \frac{1}{\lambda_\phi}\right|^{\frac{1}{2}} \left| \ln \left| \frac{N^{1-\alpha}}{\phi(N)} - \frac{1}{\lambda_\phi} \right| \right| \right) \int_{\varepsilon_N}^A |f(t)| dt\right) \\
&= \int_0^{+\infty} \rho_{1-\alpha}(t) f(t) dt \\
&\quad + O\left(\left(\frac{1}{N^{\frac{\alpha}{4}}} + \left|\frac{N^{1-\alpha}}{\phi(N)} - \frac{1}{\lambda_\phi}\right|^{\frac{1}{2}} \left| \ln \left| \frac{N^{1-\alpha}}{\phi(N)} - \frac{1}{\lambda_\phi} \right| \right| \right) A \|f\|_\infty\right). \tag{66}
\end{aligned}$$

As $N \rightarrow +\infty$, since $\psi(N) = \frac{N^{3+\alpha}}{\phi(N)}$, since we have $\frac{\phi(N)}{N^{1-\alpha}} = O(1)$ under the assumptions of Case 2, and since $\gamma < \frac{1-2\alpha}{2}$, we have

$$\frac{\phi(N) N^{4\alpha+2\gamma}}{\psi(N)} = \frac{\phi(N)^2}{N^{3-3\alpha-2\gamma}} = O\left(\frac{1}{N^{1-\alpha-2\gamma}}\right) = O\left(\frac{1}{N^\alpha}\right). \tag{67}$$

Similarly, we have

$$\frac{N^{2+2\alpha-2\gamma}}{\psi(N)} = \frac{\phi(N)}{N^{1-\alpha+2\gamma}} = O\left(\frac{1}{N^{2\gamma}}\right). \tag{68}$$

Therefore, using in Equation (33) the three Equations (66), (67) and (68), we have

$$\begin{aligned} \mathcal{R}_N^+(f) &= \int_0^{+\infty} \rho_{1-\alpha}(t) f(t) dt + O\left(\frac{1}{N^\alpha} \|f'\|_\infty\right) \\ &+ O\left(\max\left\{\frac{A}{N^{\frac{\alpha}{4}}}, A\left|\frac{N^{1-\alpha}}{\phi(N)} - \frac{1}{\lambda_\phi}\right|^{\frac{1}{2}} \left|\ln\left|\frac{N^{1-\alpha}}{\phi(N)} - \frac{1}{\lambda_\phi}\right|\right|, \frac{1}{N^{2\gamma}}\right\} \|f\|_\infty\right). \end{aligned} \quad (69)$$

By symmetry, for every $f \in C_c(\mathbb{R})$, this proves the second case of Theorem 4.1.

When $\phi : N \rightarrow N^\beta$ is a power function, the assumptions of Case 2 on ϕ are satisfied if and only if $\beta = 1 - \alpha$ and $D_K \equiv 0 \pmod{4}$, and we have $\lambda_\phi = 1$, so that the value of $\rho_{1-\alpha}$ given by Equation (37) becomes the one given in Equation (3) of the Introduction. This proves the second case of Theorem 1.1.

Case 3. Let us assume that $\lim_{N \rightarrow +\infty} \frac{N^{1-\alpha}}{\phi(N)} = 0$, that $\limsup_{N \rightarrow +\infty} \frac{\phi(N)}{N^{1-\frac{\alpha}{2}}} < +\infty$ and that $\alpha < \frac{2}{5}$.

Let us take ψ to be the function $N \mapsto \frac{N^3 S_{N,0}}{\phi(N)}$, with $S_{N,0}$ defined in Equation (34) for $k = 0$. Let $\gamma = \frac{2-3\alpha}{8}$. The assumption (12) is then satisfied since $\alpha < \frac{2}{5}$. Since $\frac{\phi(N)}{N^{1-\frac{\alpha}{2}}} = O(1)$ under the assumptions of Case 3, and since $\alpha < \frac{2}{3}$, we have

$$\frac{\phi(N)}{N^{2-2\alpha-2\gamma}} = O\left(\frac{1}{N^{1-\frac{3}{2}\alpha-2\gamma}}\right) = O\left(\frac{1}{N^{\frac{2-3\alpha}{4}}}\right),$$

which tends to 0 as $N \rightarrow +\infty$. Hence the assumption (13) is satisfied and we may apply the results of Section 3.

We are going to prove that in Case 3, as $N \rightarrow +\infty$, the function Θ_N defined in Equation (32) converges uniformly on compact subsets of $[0, +\infty[$ to the constant function $\frac{8\pi}{3|D_K|}$. Below are the graphs of Θ_N for various N , for the power scaling $\phi(N) = N^\beta$, in the Gaussian case $K = \mathbb{Q}[i]$, with $\alpha = 0.15$ and $\beta = 0.9 \in]1 - \alpha, 1 - \frac{\alpha}{2}]$, and in dashed black the horizontal line with height $\frac{8\pi}{3|D_K|} \simeq 2.094$.

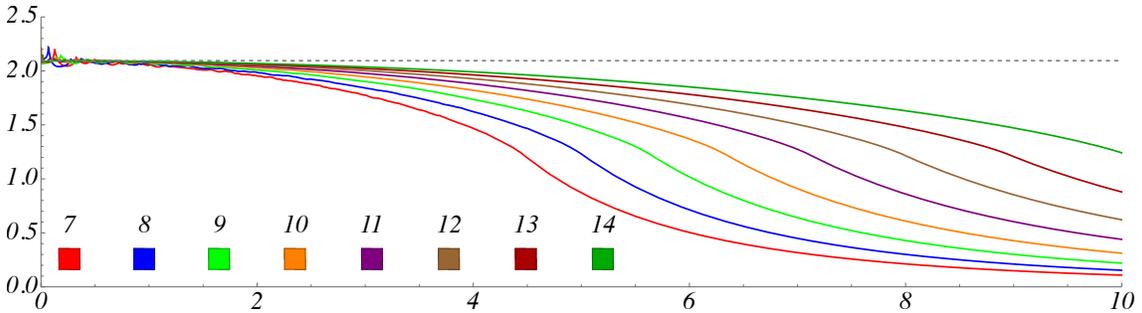


Figure 6 : Graph of Θ_N for $N = 10^m$ where m is 7 (red), 8 (blue) up to 14 (dark green).

For every $N \in \mathbb{N} \setminus \{0\}$, let us define

$$M_N = N^\alpha \left(\frac{N^{1-\alpha}}{\phi(N)} \right)^{\frac{2}{3}}. \quad (70)$$

Since $M_N = \left(\frac{N^{1-\frac{\alpha}{2}}}{\phi(N)} \right)^{\frac{2}{3}} N^{\frac{2\alpha}{3}}$ and since $\frac{N^{1-\frac{\alpha}{2}}}{\phi(N)}$ is bounded from below by a positive constant under the assumptions of Case 3, we have $\lim_{N \rightarrow +\infty} M_N = +\infty$. Let

$$S(M_N) = \frac{|D_K|}{4\pi} \sum_{p \in \mathcal{O}_K: 0 < |p| \leq M_N} \frac{1}{c'_p |v_p|}.$$

As in order to obtain Equation (35) with $k = 0$, we have

$$S(M_N) = O(M_N).$$

Let $t \in [0, A]$. Since $\psi(N) = \frac{N^3 S_{N,0}}{\phi(N)}$, since the function g is bounded, by Equation (35) with $k = 0$, and by the definition of M_N in Equation (70), we hence have

$$\begin{aligned} \frac{N^3}{\psi(N) \phi(N)} \sum_{p \in \mathcal{O}_K: 0 < |p| \leq M_N} \frac{1}{c'_p |v_p|} g\left(\frac{tN}{2\phi(N)|p|}\right) &= O\left(\frac{S(M_N)}{S_{N,0}}\right) = O\left(\frac{M_N}{N^\alpha}\right) \\ &= O\left(\left(\frac{N^{1-\alpha}}{\phi(N)}\right)^{\frac{2}{3}}\right). \end{aligned} \quad (71)$$

Note that if $p \in \mathcal{O}_K$ satisfies $|p| \geq M_N$, then we have $\frac{tN}{2\phi(N)|p|} \leq \frac{AN}{2\phi(N)M_N}$. Since $\psi(N) = \frac{N^3 S_{N,0}}{\phi(N)}$, since we have $g(u) = \frac{2}{3} + O(u^2)$ for $u \in [0, +\infty[$, by the definition of M_N in Equation (70) and since $A \geq 1$, we hence have

$$\begin{aligned} &\frac{N^3}{\psi(N) \phi(N)} \sum_{p \in \mathcal{O}_K: M_N < |p| \leq N^\alpha} \frac{1}{c'_p |v_p|} g\left(\frac{tN}{2\phi(N)|p|}\right) \\ &= \frac{8\pi}{3|D_K|} \frac{S_{N,0} - S(M_N)}{S_{N,0}} + O\left(\left(\frac{AN}{\phi(N)M_N}\right)^2\right) \\ &= \frac{8\pi}{3|D_K|} + O\left(\frac{M_N}{N^\alpha}\right) + O\left(A^2 \left(\frac{N^{1-\alpha}}{\phi(N)} \frac{N^\alpha}{M_N}\right)^2\right) \\ &= \frac{8\pi}{3|D_K|} + O\left(A^2 \left(\frac{N^{1-\alpha}}{\phi(N)}\right)^{\frac{2}{3}}\right). \end{aligned} \quad (72)$$

By decomposing the index set of the sum defining the function Θ_N as

$$\{p \in \mathcal{O}_K : 0 < |p| \leq N^\alpha\} = \{p \in \mathcal{O}_K : 0 < |p| \leq M_N\} \sqcup \{p \in \mathcal{O}_K : M_N < |p| \leq N^\alpha\},$$

it follows from Equations (32), (71) and (72) that as $N \rightarrow +\infty$, for every $t \in [0, A]$, we have

$$\Theta_N(t) = \frac{8\pi}{3|D_K|} + O\left(A^2 \left(\frac{N^{1-\alpha}}{\phi(N)}\right)^{\frac{2}{3}}\right).$$

Therefore, since the support of f is contained in $[0, A]$, we have

$$\int_0^{+\infty} \Theta_N(t) f(t) dt = \int_0^{+\infty} \frac{8\pi}{3|D_K|} f(t) dt + O\left(A^3 \left(\frac{N^{1-\alpha}}{\phi(N)}\right)^{\frac{2}{3}} \|f\|_\infty\right). \quad (73)$$

As $N \rightarrow +\infty$, since $\psi(N) = \frac{N^3 S_{N,0}}{\phi(N)}$, by Equation (35) with $k = 0$, since we have $\frac{\phi(N)}{N^{1-\frac{\alpha}{2}}} = O(1)$ under the assumptions of Case 3, and since $\gamma = \frac{2-3\alpha}{8}$, we have

$$\begin{aligned} \frac{\phi(N) N^{4\alpha+2\gamma}}{\psi(N)} &= \frac{\phi(N)^2 N^{4\alpha+2\gamma}}{N^3 S_{N,0}} = O\left(\frac{\phi(N)^2}{N^{3-3\alpha-2\gamma}}\right) \\ &= O\left(\frac{1}{N^{1-2\alpha-2\gamma}}\right) = O\left(\frac{1}{N^{\frac{2-5\alpha}{4}}}\right). \end{aligned} \quad (74)$$

Similarly, we have

$$\frac{N^{2+2\alpha-2\gamma}}{\psi(N)} = \frac{N^{2+2\alpha-2\gamma} \phi(N)}{N^3 S_{N,0}} = \mathcal{O}\left(\frac{\phi(N)}{N^{1-\alpha+2\gamma}}\right) = \mathcal{O}\left(\frac{1}{N^{2\gamma-\frac{\alpha}{2}}}\right) = \mathcal{O}\left(\frac{1}{N^{\frac{2-5\alpha}{4}}}\right). \quad (75)$$

Therefore, using in Equation (33) the three Equations (73), (74) and (75), we have

$$\begin{aligned} \mathcal{R}_N^+(f) &= \int_0^{+\infty} \frac{8\pi}{3|D_K|} f(t) dt \\ &+ \mathcal{O}\left(\frac{1}{N^{\frac{2-5\alpha}{4}}} \|f'\|_\infty\right) + \mathcal{O}\left(\max\left\{A^3\left(\frac{N^{1-\alpha}}{\phi(N)}\right)^{\frac{2}{3}}, \frac{1}{N^{\frac{2-5\alpha}{4}}}\right\} \|f\|_\infty\right). \end{aligned} \quad (76)$$

By symmetry, for every $f \in C_c(\mathbb{R})$, this proves the third case of Theorem 4.1.

When $\phi : N \rightarrow N^\beta$ is a power function, the assumptions of Case 3 on ϕ are satisfied if and only if $\beta \in]1 - \alpha, 1 - \frac{\alpha}{2}]$. If furthermore $D_K \equiv 0 \pmod{4}$, then by Equation (36) with $k = 0$, as $N \rightarrow +\infty$, we have $\psi(N) = \frac{N^3 S_{N,0}}{\phi(N)} \sim N^{3+\alpha-\beta}$. This proves the third case of Theorem 1.1 when $\beta \in]1 - \alpha, 1 - \frac{\alpha}{2}]$.

Case 5. Let us assume that $\limsup_{N \rightarrow +\infty} \frac{N}{\phi(N)} < +\infty$, that $\lambda_N''' = \lim_{N \rightarrow +\infty} \frac{\phi(N)}{N^{1+\frac{\alpha}{2}}} = 0$ and that $\alpha \leq \frac{2}{11}$.

Let us take $\psi : N \mapsto \frac{N^3 S_{N,0}}{\phi(N)}$ and $\gamma = \frac{1-4\alpha}{2}$. The assumptions (12) and (13) are satisfied (since $\frac{\phi(N)}{N^{2-2\alpha-2\gamma}} = \frac{\phi(N)}{N^{1+\frac{\alpha}{2}}} N^{-\frac{3\alpha}{2}}$ tends to 0 as $N \rightarrow +\infty$ under the assumptions of Case 5), hence we may apply the results of Section 3. For every $N \in \mathbb{N} \setminus \{0\}$, let

$$\eta_N = \frac{N}{\phi(N)},$$

and note that η_N remains bounded as $N \rightarrow +\infty$ under the assumptions of Case 5.

We are going to prove that in this case, the function Θ_N defined in Equation (32) converges uniformly on compact subsets of $[0, +\infty[$ to the constant function $\frac{8\pi}{3|D_K|}$. Below are the graphs of Θ_N for various N , for the power scaling $\phi(N) = N^\beta$, in the Gaussian case $K = \mathbb{Q}[i]$, with $\alpha = 0.15$ and $\beta = 1.05 \in [1, 1 + \frac{\alpha}{2}[$.

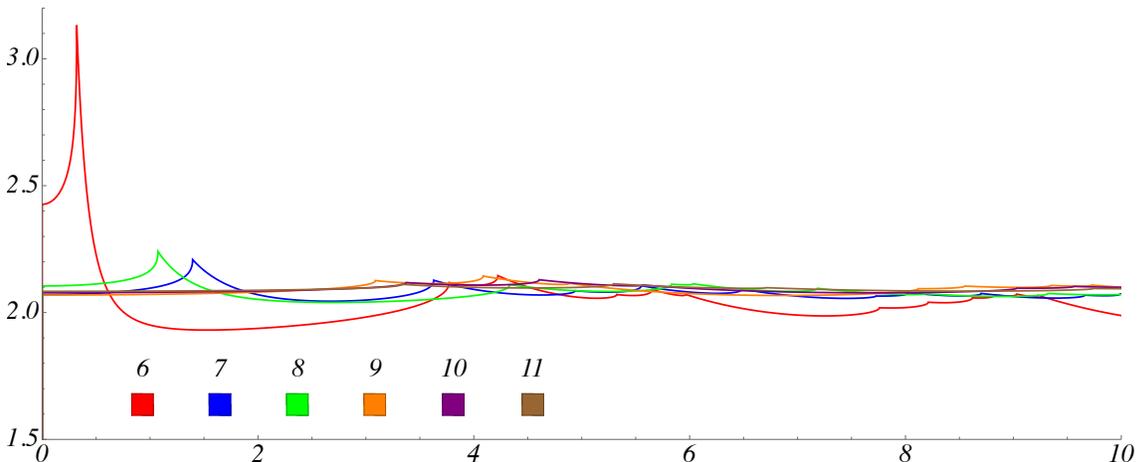


Figure 7 : Graph of Θ_N for $N = 10^m$ where m is 6 (red), 7 (blue) up to 11 (brown).

Recall that the function g defined in Equation (26) satisfies $g(u) = \frac{2}{3} + O(u^2)$ near $u = 0$. Hence for all $t \in [0, A]$ and $p \in \mathcal{O}_K \setminus \{0\}$, we have

$$g\left(\frac{tN}{2\phi(N)|p|}\right) = \frac{2}{3} + O\left(\frac{A^2\eta_N^2}{|p|^2}\right).$$

By the right part of Equation (6) and by the fact that the series $\sum_{p \in \mathcal{O}_K \setminus \{0\}} \frac{1}{|p|^3}$ converges, we have

$$\sum_{p \in \mathcal{O}_K: 0 < |p| \leq N^\alpha} \frac{1}{c'_p |v_p| |p|^2} = O\left(\sum_{p \in \mathcal{O}_K: 0 < |p| \leq N^\alpha} \frac{1}{|p|^3}\right) = O(1).$$

Hence with $S_{N,0}$ the sum defined in Equation (34) for $k = 0$, for every $t \in [0, A]$, Equation (32) gives

$$\begin{aligned} \Theta_N(t) &= \frac{2N^3}{3\psi(N)\phi(N)} \sum_{\substack{p \in \mathcal{O}_K \\ 0 < |p| \leq N^\alpha}} \frac{1}{c'_p |v_p|} + O\left(\frac{A^2\eta_N^2 N^3}{\psi(N)\phi(N)} \sum_{\substack{p \in \mathcal{O}_K \\ 0 < |p| \leq N^\alpha}} \frac{1}{c'_p |v_p| |p|^2}\right) \\ &= \frac{8\pi N^3 S_{N,0}}{3|D_K|\psi(N)\phi(N)} + O\left(\frac{A^2\eta_N^3 N^2}{\psi(N)}\right). \end{aligned}$$

Since the support of f is contained in $[0, A]$, it follows from Equation (33) that

$$\begin{aligned} \mathcal{R}_N^+(f) &= \frac{N^3 S_{N,0}}{\psi(N)\phi(N)} \int_0^{+\infty} f(t) \frac{8\pi}{3|D_K|} dt + O\left(\frac{A^3\eta_N^3 N^2}{\psi(N)} \|f\|_\infty\right) \\ &\quad + O\left(\frac{\phi(N) N^{4\alpha+2\gamma}}{\psi(N)} \|f'\|_\infty\right) + O\left(\frac{N^{2+2\alpha-2\gamma}}{\psi(N)} \|f\|_\infty\right). \end{aligned} \quad (77)$$

Since we have $\psi(N) = \frac{N^3 S_{N,0}}{\phi(N)}$ in this Case 5, since $\eta_N = \frac{N}{\phi(N)}$ remains bounded, and by Equation (35) with $k = 0$, we have

$$\frac{\eta_N^3 N^2}{\psi(N)} = \frac{N^5 \phi(N)}{\phi(N)^3 N^3 S_{N,0}} = O\left(\frac{N^2}{\phi(N)^2 N^\alpha}\right) = O\left(\frac{N^3}{\phi(N)^3 N^{1+\alpha}}\right) = O\left(\frac{\phi(N)}{N^{1+\frac{\alpha}{2}}}\right). \quad (78)$$

Similarly, since $\gamma = \frac{1-4\alpha}{2}$, $\lim_{N \rightarrow +\infty} \frac{\phi(N)}{N^{1+\frac{\alpha}{2}}} = 0$ and $\alpha \leq \frac{2}{11}$ in this Case 5, we have

$$\frac{\phi(N) N^{4\alpha+2\gamma}}{\psi(N)} = \frac{\phi(N)^2 N}{N^3 S_{N,0}} = O\left(\frac{\phi(N)^2}{N^{2+\alpha}}\right) = O\left(\frac{\phi(N)}{N^{1+\frac{\alpha}{2}}}\right)^2 = O\left(\frac{\phi(N)}{N^{1+\frac{\alpha}{2}}}\right) \quad (79)$$

$$\text{and} \quad \frac{N^{2+2\alpha-2\gamma}}{\psi(N)} = \frac{\phi(N) N^{1+6\alpha}}{N^3 S_{N,0}} = O\left(\frac{\phi(N)}{N^{2-5\alpha}}\right) = O\left(\frac{\phi(N)}{N^{1+\frac{\alpha}{2}}}\right). \quad (80)$$

Therefore, putting together Equations (77) to (80), we have

$$\mathcal{R}_N^+(f) = \int_0^{+\infty} f(t) \frac{8\pi}{3|D_K|} dt + O\left(\frac{\phi(N)}{N^{1+\frac{\alpha}{2}}} (\|f'\|_\infty + A^3 \|f\|_\infty)\right). \quad (81)$$

By symmetry, for every $f \in C_c(\mathbb{R})$, this proves the last case of Theorem 4.1.

When $\phi : N \rightarrow N^\beta$ is a power function, the assumptions of Case 5 on ϕ are satisfied if and only if $\beta \in [1, 1 + \frac{\alpha}{2}]$. If furthermore $D_K \equiv 0 \pmod{4}$, then by Equation (36) with

$k = 0$, as $N \rightarrow +\infty$, we have $\psi(N) = \frac{N^3 S_{N,0}}{\phi(N)} \sim N^{3+\alpha-\beta}$. This proves the last case of Theorem 1.1 when $\beta \in [1, 1 + \frac{\alpha}{2}[$.

Case 4. Let us assume that we have $\lim_{N \rightarrow +\infty} \frac{N^{1-\frac{\alpha}{2}}}{\phi(N)} = 0$, that $\lambda_N'' = \lim_{N \rightarrow +\infty} \frac{\phi(N)}{N} = 0$ and that $\alpha \leq \frac{2}{11}$.

Note that the second assumption implies that $\lambda_N''' = \lim_{N \rightarrow +\infty} \frac{\phi(N)}{N^{1+\frac{\alpha}{2}}} = 0$. As in Case 5, we take $\psi : N \mapsto \frac{N^3 S_{N,0}}{\phi(N)}$ and $\gamma = \frac{1-4\alpha}{2}$. The assumptions (12) and (13) are satisfied (since $\frac{\phi(N)}{N^{2-2\alpha-2\gamma}} = \frac{\phi(N)}{N} N^{-2\alpha}$ tends to 0 as $N \rightarrow +\infty$ under the assumptions of Case 4), hence we may apply the results of Section 3. We are going to prove that also in this case, the function Θ_N defined in Equation (32) converges uniformly on compact subsets of $[0, +\infty[$ to the constant function $\frac{8\pi}{3|D_K|}$. Below are the graphs of Θ_N for various N , in the Gaussian case $K = \mathbb{Q}[i]$, with $\alpha = 0.15$ and $\beta = 0.95 \in]1 - \frac{\alpha}{2}, 1[$.

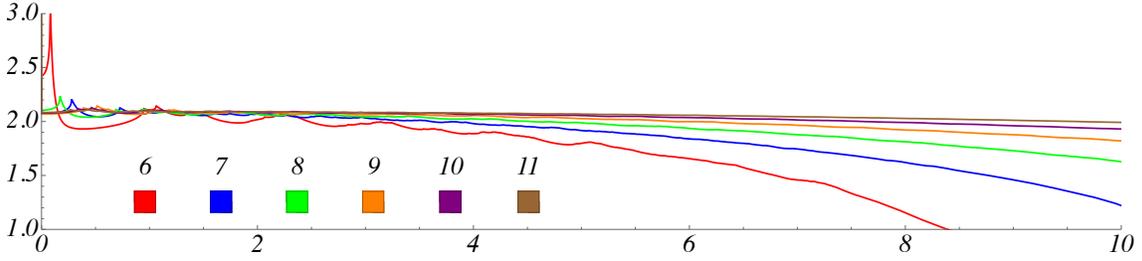


Figure 8 : Graph of Θ_N for $N = 10^m$ where m is 6 (red), 7 (blue) up to 11 (brown).

The proof of Case 4 is almost the same one as the one of Case 5. Note that the quantity $\eta_N = \frac{N}{\phi(N)}$ is now no longer bounded. Equation (78) needs to be replaced by

$$\frac{\eta_N^3 N^2}{\psi(N)} = \frac{N^5 \phi(N)}{\phi(N)^3 N^3 S_{N,0}} = O\left(\frac{N^2}{\phi(N)^2 N^\alpha}\right) = O\left(\frac{N^{1-\frac{\alpha}{2}}}{\phi(N)}\right)^2 = O\left(\frac{N^{1-\frac{\alpha}{2}}}{\phi(N)}\right). \quad (82)$$

This is possible by the first assumption of Case 4 requiring that $\lim_{N \rightarrow +\infty} \frac{N^{1-\frac{\alpha}{2}}}{\phi(N)} = 0$. Under the second assumption of Case 4, we have

$$\frac{\phi(N)}{N^{1+\frac{\alpha}{2}}} = \frac{N^{1-\frac{\alpha}{2}}}{\phi(N)} \left(\frac{\phi(N)}{N}\right)^2 = O\left(\frac{N^{1-\frac{\alpha}{2}}}{\phi(N)}\right).$$

Hence Equation (79) (which is still valid) gives

$$\frac{\phi(N) N^{4\alpha+2\gamma}}{\psi(N)} = O\left(\frac{N^{1-\frac{\alpha}{2}}}{\phi(N)}\right). \quad (83)$$

The limit distribution is still Poissonian with the same constant asymptotic pair correlation density given by $\frac{8\pi}{3|D_K|}$. Only the error term that involves the norm $\|f\|_\infty$ is weakened, corresponding to the new Equations (82) and (83), so that Equation (81) becomes

$$\mathcal{R}_N^+(f) = \int_0^{+\infty} f(t) \frac{8\pi}{3|D_K|} dt + O\left(\frac{\phi(N)}{N^{1+\frac{\alpha}{2}}} \|f'\|_\infty + \frac{N^{1-\frac{\alpha}{2}}}{\phi(N)} A^3 \|f\|_\infty\right).$$

By symmetry, for every $f \in C_c(\mathbb{R})$, this proves the penultimate case of Theorem 4.1.

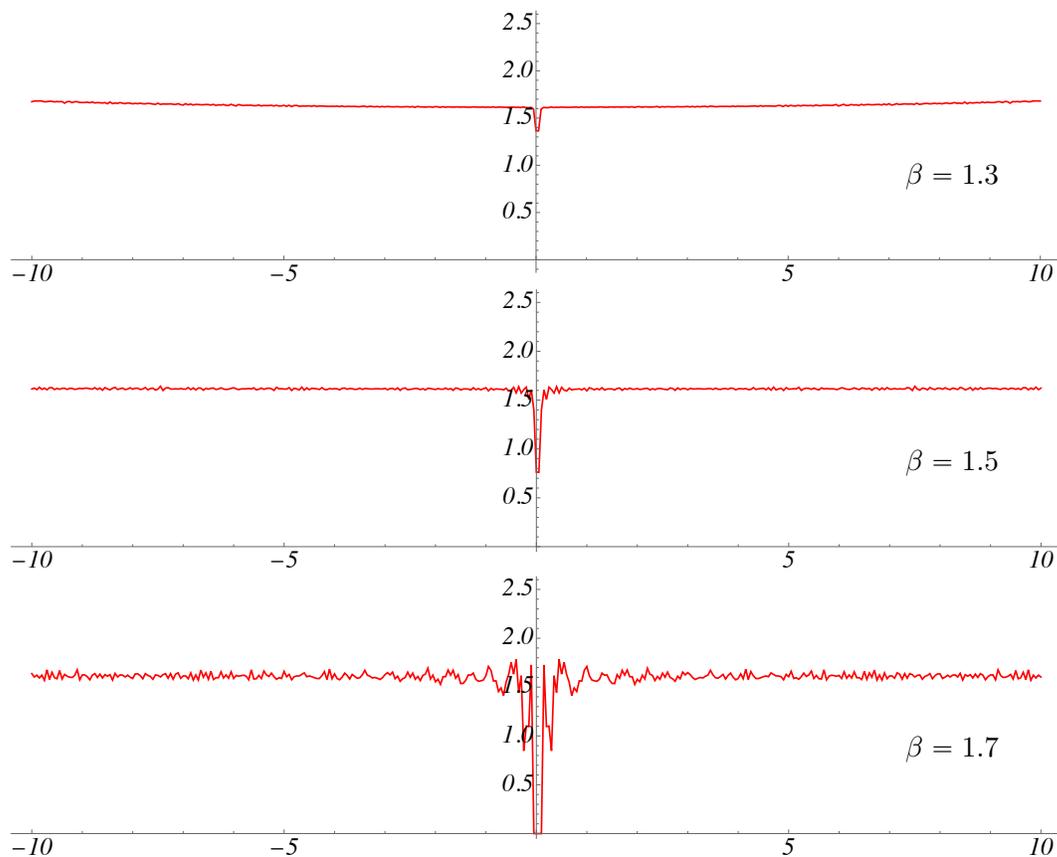
When $\phi : N \rightarrow N^\beta$ is a power function, the assumptions of Case 4 on ϕ are satisfied if and only if $\beta \in]1 - \frac{\alpha}{2}, 1[$. If furthermore $D_K \equiv 0 \pmod{4}$, then by Equation (36) with $k = 0$, as $N \rightarrow +\infty$, we have $\psi(N) = \frac{N^3 S_{N,0}}{\phi(N)} \sim N^{3+\alpha-\beta}$. This proves the last case of Theorem 1.1 when $\beta \in]1 - \frac{\alpha}{2}, 1[$.

This concludes the proof of Theorem 4.1, and along the way, we proved Theorem 1.1 in the Introduction. \square

5 Experiments for bigger scalings

The method used in order to prove the Poissonian asymptotic behaviour of the pair correlations in Theorem 4.1 does not work when $\lambda_\phi''' > 0$ (with the notation of the beginning of Section 4). Recall that this means that $\beta > 1 + \frac{\alpha}{2}$ for power scalings $\phi : N \mapsto N^\beta$. We only proved in Theorem 1.1 that we have a Poissonian asymptotic behaviour when $\beta \in]1 - \alpha, 1 + \frac{\alpha}{2}[$.

Figure 9 below shows the empirical pair correlation measures $\mathcal{R}_{2000}^{\alpha,\beta}$ (defined just before Theorem 1.1 in the Introduction) in the Gaussian case $K = \mathbb{Q}(i)$ with $\alpha = 0.15$ for values $\beta \in \{1.3, 1.5, 1.7, 1.9, 2\}$ outside the range of Theorem 1.1. Again the distributions are renormalized by $\psi(N) = N^{3+\alpha-\beta}$. These graphs indicate that, for some parameters in the range from $1 + \frac{\alpha}{2}$ to 2, possibly including the interval $[1.7, 2]$, the asymptotic pair correlation densities should exist and should exhibit a *strong level repulsion*, that is, they should vanish on an open interval around 0.



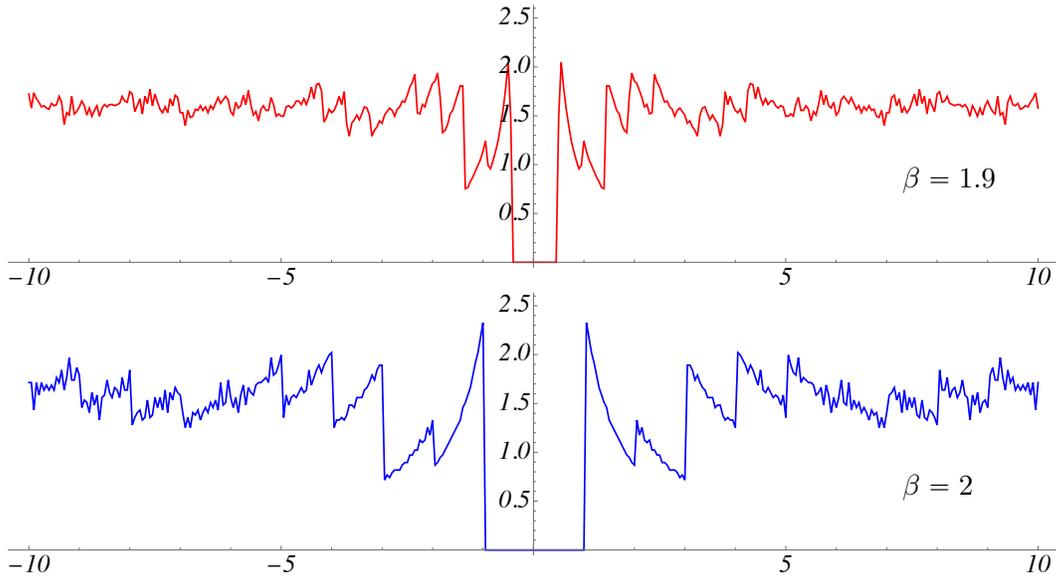


Figure 9 : The empirical pair correlation distributions $\mathcal{R}_{2000}^{0.15,\beta}$ with $\beta \in \{1.3, 1.5, 1.7, 1.9, 2\}$.

Note that the minimal difference of two sums of squares $a < b \leq N^2$ is 1, hence we have

$$N^2(\ln b - \ln a) = N^2 \ln \left(1 + \frac{b-a}{a}\right) \geq N^2 \ln \left(1 + \frac{1}{N^2}\right) \rightarrow_{N \rightarrow +\infty} 1.$$

Thus, if a nonzero asymptotic pair correlation density exists for the quadratic scaling (and we conjecture that it does), it will exhibit a strong level repulsion, since it will vanish on $] -1, 1[$. An analogous computation indicates that there should be a total loss of mass for larger power scalings $\phi : N \mapsto N^\beta$ with $\beta > 2$.

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