Equidistribution of common perpendiculars in negative curvature

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Abstract

Let A^- and A^+ be properly immersed closed locally convex subsets of a Riemannian manifold M with pinched negative sectional curvature. When the Bowen-Margulis measure on T^1M is finite and mixing for the geodesic flow, we prove that the Lebesgue measures along the common perpendiculars of length at most t from A^- to A^+, counted with multiplicities and lifted to T^1M, equidistribute to the Bowen-Margulis measure as $t \to +\infty$. When M is locally symmetric with finite volume and the geodesic flow is exponentially mixing, we give an error term for the asymptotic. When T^1M is endowed with a bounded Hölder-continuous potential, and when the associated equilibrium state is finite and mixing for the geodesic flow, we prove the equidistribution of these Lebesgue measures weighted by the amplitudes of the potential to the equilibrium state. [1]

1 Introduction

Let M be a nonelementary complete connected Riemannian good orbifold with pinched sectional curvature at most -1, and let $(\mathtt{g}^t)_{t\in\mathbb{R}}$ be its geodesic flow on its unit tangent bundle T^1M. Let A^- and A^+ be proper nonempty properly immersed closed locally convex subsets of M. A *common perpendicular* from A^- to A^+ is a locally geodesic path in M starting perpendicularly from A^- and arriving perpendicularly to A^+ (see [PaP1, §2.2] or [BrPP, §2.4] for explanations when the boundary of A^- or A^+ is not smooth). For all $t > 0$, we denote by $\operatorname{Perp}(A^-, A^+, t)$ the set of common perpendiculars from A^- to A^+ with length at most t, considered with multiplicities. We refer to [PaP1, §3.3] or [BrPP, §12.1] for the definition of the multiplicities, which are equal to 1 if M is a manifold and if A^- and A^+ are embedded and disjoint. For every $\alpha \in \operatorname{Perp}(A^-, A^+, t)$, if $\ell(\alpha)$ is its length and if v_α^- and v_α^+ are its initial and terminal unit tangent vectors, we denote by $\operatorname{Leb}_\alpha$ the pushforward measure on T^1M, by the map $t \mapsto \mathtt{g}^t v_\alpha^-$, of the Lebesgue measure on $[0, \ell(\alpha)]$.

We denote the total mass of any measure m by $\|m\|$. Referring to [Rob] (see also Section 2) for definitions, we denote by δ the critical exponent of the fundamental group of M and by $m_{\rm BM}$ a Bowen-Margulis measure on T^1M. By [OP] and [DT], if $m_{\rm BM}$ if finite, then δ is the topological entropy of the geodesic flow $(\mathtt{g}^t)_{t\in\mathbb{R}}$ and $\frac{m_{\rm BM}}{\|m_{\rm BM}\|}$ is its unique measure of maximal entropy. We denote by $\sigma^{\mp}_{A^\pm}$ the skinning measures of $A^\pm$ defined in [PaP1] (see

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also Section 2), generalising [OhS1, OhS2] when M has constant curvature and A^-, A^+ are balls, horoballs or totally geodesic submanifolds.

Theorem 1. *Assume that the Bowen-Margulis measure $m_{\rm BM}$ is finite and mixing for the geodesic flow of M. Assume that the skinning measures $\sigma^+_{A^-}$ and $\sigma^-_{A^+}$ are finite and nonzero. For the narrow convergence of measures on T^1M, we have*

$$\lim_{t\to+\infty} \frac{\delta\,\|m_{\rm BM}\|}{t\;e^{\delta\,t}\;\|\sigma^+_{A^-}\|\;\|\sigma^-_{A^+}\|}\sum_{\alpha\in{\rm Perp}(A^-,\,A^+,\,t)}{\rm Leb}_\alpha=\frac{m_{\rm BM}}{\|m_{\rm BM}\|}\,. \tag{1}$$

If furthermore M is locally symmetric with finite volume, and if the geodesic flow of M is exponentially mixing, then there exists $\ell\in\mathbb{N}$ such that for every compact subset K in T^1M and every C^ℓ-smooth function $\psi: T^1M\to\mathbb{C}$ with support in K and $W^{\ell,2}$-Sobolev norm $\|\psi\|_\ell$, as $s\to+\infty$, we have

$$\frac{\delta\,\|m_{\rm BM}\|}{t\;e^{\delta\,t}\;\|\sigma^+_{A^-}\|\;\|\sigma^-_{A^+}\|}\sum_{\alpha\in{\rm Perp}(A^-,\,A^+,\,t)}{\rm Leb}_\alpha(\psi)=\frac{m_{\rm BM}(\psi)}{\|m_{\rm BM}\|}+{\rm O}_K(\frac{1}{t}\|\psi\|_\ell)\,.$$

This result is an analog of the theorems of Bowen and Margulis on the equidistribution towards the Bowen-Margulis measure of the Lebesgue measures along the periodic orbits of the geodesic flows when M is compact (see [Rob, Theo. 5.1.1] for the extension to the above generality, and [PauPS, §9.3] for the extension to equilibrium states).

As a very special case which is already new and striking, if we choose $A^-=A^+=\{p\}$ for any $p\in M$, Theorem 1 shows that the Lebesgue mesures along geodesic loops based at p, lifted to T^1M, equidistribute to the Bowen-Margulis measure as the upper bound on their lengths increases to $+\infty$.

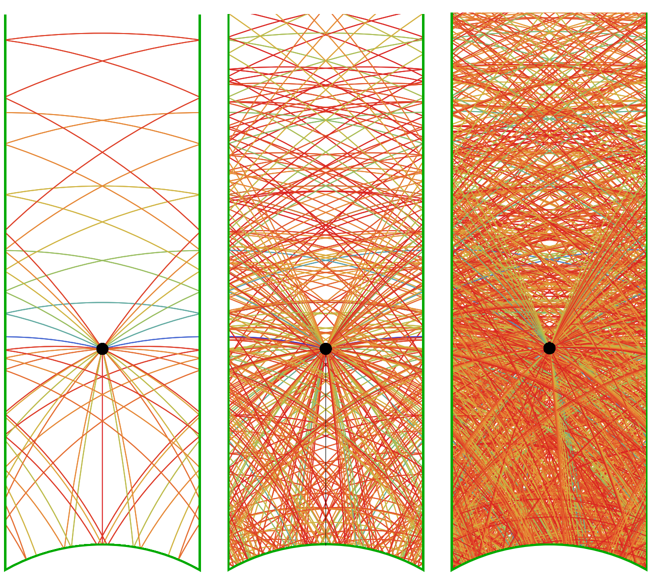

Figure 1: Geodesic loops equidistribute.

The figures on the left show geodesic loops based at the image of the point $2i\in\mathbb{H}^2_\mathbb{R}$ in the modular orbifold $M={\rm PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2_\mathbb{R}$, with $\mathbb{H}^2_\mathbb{R}$ the upper halfspace model of the real hyperbolic plane. These geodesic loops, that are in general not closed geodesics, are shown lifted to the usual fundamental domain of ${\rm PSL}_2(\mathbb{Z})$ with its boundary identifications. The first figure displays the geodesic loops of length at most 2.5, and the second and third figures display the geodesic loops of length at most 4 and 5 restricted to the same part of M. See also [LS] and [PolWar] for related recent work.

Theorem 1 builds on the joint equidistribution theorem [PaP3, Theo. 2] of the authors, stated as Theorem 3 in Section 3, for the pairs (v^-_α, v^+_α) of initial and terminal unit tangent vectors of the elements $\alpha\in{\rm Perp}(A^-,A^+,t)$ in the outer/inner normal bundles of $A^\pm$. But Theorems 1 and 3 are very different in nature, the second one being an equidistribution on $T^1M \times T^1M$ towards the product of two measures whose supports have zero measure for the Bowen-Margulis measure on T^1M. We refer to [PaP1, PaP3, PaP4, BrPP, PaP6, PaP8], and in particular to the surveys [PaP2, PaP5], for other motivations, arithmetic applications and references for the study of common perpendiculars in negative curvature.

We have the following version of Theorem 1 for equilibrium states. We refer to [PauPS, DT] and Subsection 4.1 for the various definitions of the objects in the following statement.

Theorem 2. *Let $F : T^1M \to \mathbb{C}$ be a bounded Hölder-continuous function with positive topological pressure δ_F and amplitudes $\int_\alpha F$ along every $\alpha \in \operatorname{Perp}(A^-, A^+, t)$. Assume that the Gibbs measure m_F of F is finite and mixing for the geodesic flow of M. Assume that the skinning measures $\sigma^+_{A^-}$ and $\sigma^-_{A^+}$ associated with F are finite and nonzero. For the narrow convergence of measures on T^1M, we have*

$$\lim_{t\to+\infty} \frac{\delta_F \,\|m_F\|}{t\, e^{\delta_F t}\, \|\sigma^+_{A^-}\|\, \|\sigma^-_{A^+}\|} \sum_{\alpha\in\operatorname{Perp}(A^-,\,A^+,\,t)} e^{\int_\alpha F}\operatorname{Leb}_\alpha = \frac{m_F}{\|m_F\|}\,.$$

The first claim of Theorem 1 is the particular case when $F = 0$ of Theorem 2. But for the convenience of the reader, we will give a full proof of Theorem 1 in Section 3, and only indicate in Section 4 the quite technical changes brought to that proof by the potential F.

Theorem 1 will be an important tool in two forthcoming papers. In [ParPS] we study the equidistribution of divergent geodesics in M (when M has finite volume), and in [EPP] we study the equidistribution of reciprocal closed geodesics in M.

2 The geometric and measure theoretic background

We refer to [BrH] for background on the geometric content of this section. Let $\widetilde{M}$ be a complete simply connected Riemannian manifold with (dimension at least 2 and) pinched negative sectional curvature $-b^2 \leqslant K \leqslant -1$, and let $x_* \in \widetilde{M}$ be a fixed basepoint.

Let Γ be a nonelementary (not virtually nilpotent) discrete group of isometries of $\widetilde{M}$, let M be the quotient (hence good) Riemannian orbifold $\Gamma\backslash\widetilde{M}$, and let T^1M be the quotient orbifold $\Gamma\backslash T_1\widetilde{M}$. We denote by $\partial_\infty\widetilde{M}$ the boundary at infinity of $\widetilde{M}$, by

$$\delta = \lim_{t\to+\infty} \frac{1}{t} \operatorname{Card}\{\gamma\in\Gamma \,:\, d(x_*, \gamma x_*) \leqslant t\}$$

the critical exponent of Γ, and by $\Lambda\Gamma$ the limit set of Γ. We denote by π the footpoint projections $T^1\widetilde{M} \to \widetilde{M}$ and $T^1M \to M$. Let $(\mathsf{g}^t)_{t\in\mathbb{R}}$ be the geodesic flow on the unit tangent bundle $T^1\widetilde{M}$ of $\widetilde{M}$, as well as its quotient flow on T^1M.

For every $v \in T^1\widetilde{M}$, let $v_- \in \partial_\infty\widetilde{M}$ and $v_+ \in \partial_\infty\widetilde{M}$, respectively, be the endpoints at $-\infty$ and $+\infty$ of the geodesic line defined by v. We denote by $p_\pm : T^1\widetilde{M} \to \partial_\infty\widetilde{M}$ the *endpoint maps* $v \mapsto v_\pm$. Let us denote by $\operatorname{diag} = \{(x,x) : x \in X\}$ the diagonal in any Cartesian square $X \times X$ of a set X. *Hopf's parametrisation* with respect to the point x_* of $T^1\widetilde{M}$ is the homeomorphism which identifies $T^1\widetilde{M}$ with $(\partial_\infty\widetilde{M} \times \partial_\infty\widetilde{M} \smallsetminus \operatorname{diag}) \times \mathbb{R}$ by the map $v \mapsto (v_-, v_+, s)$, where s is the signed distance to $\pi(v)$ of the closest point to x_* on the geodesic line defined by v.

For every $\xi \in \partial_\infty\widetilde{M}$, let $\rho_\xi : [0, +\infty[\,\to \widetilde{M}$ be the geodesic ray with origin x_* and point at infinity ξ. The *Busemann cocycle* of $\widetilde{M}$ is the map $\beta : \widetilde{M} \times \widetilde{M} \times \partial_\infty\widetilde{M} \to \mathbb{R}$ defined by

$$(x, y, \xi) \mapsto \beta_\xi(x, y) = \lim_{t\to+\infty} d(\rho_\xi(t), x) - d(\rho_\xi(t), y)\,. \tag{2}$$

The *visual distance* d_{x_*} on $\partial_\infty \widetilde{M}$ seen from x_* is defined by $d_{x_*}(\xi,\eta) = e^{-\frac{1}{2}(\beta_\xi(x_*,y)+\beta_\eta(x_*,y))}$ where y is the closest point to x_* on the geodesic line between two distinct points at infinity ξ and η.

We refer to [Rob] for more background and for the basic properties of the following measures. A family $(\mu_x)_{x\in\widetilde{M}}$ of finite measures on $\partial_\infty\widetilde{M}$, whose support is the limit set $\Lambda\Gamma$ of Γ, is a *Patterson-Sullivan density* for Γ if

$$\gamma_*\mu_x = \mu_{\gamma x}$$

for all $\gamma \in \Gamma$ and $x \in \widetilde{M}$, and if the following Radon-Nikodym derivatives exist for all $x, y \in \widetilde{M}$ and satisfy for (almost) every $\xi \in \partial_\infty\widetilde{M}$

$$\frac{d\mu_x}{d\mu_y}(\xi) = e^{-\delta\,\beta_\xi(x,y)}\,.$$

We fix such a family $(\mu_x)_{x\in\widetilde{M}}$. The *Bowen-Margulis measure* on $T^1\widetilde{M}$ (associated with this Patterson-Sullivan density) is the measure $\widetilde{m}_{\rm BM}$ on $T^1\widetilde{M}$ given by the density

$$d\widetilde{m}_{\rm BM}(v) = e^{-\delta(\beta_{v_-}(\pi(v),x_*)\,+\,\beta_{v_+}(\pi(v),x_*))}\,d\mu_{x_*}(v_-)\,d\mu_{x_*}(v_+)\,dt \tag{3}$$

in Hopf's parametrisation of $T^1\widetilde{M}$ with respect to x_*. The Bowen-Margulis measure $\widetilde{m}_{\rm BM}$ is independent of x_*, and it is invariant under the actions of the group Γ and of the geodesic flow. Thus, it defines a measure $m_{\rm BM}$ on T^1M which is invariant under the quotient geodesic flow, called the *Bowen-Margulis measure* on T^1M. If $m_{\rm BM}$ is finite, then the Patterson-Sullivan densities are unique up to a multiplicative constant; hence the Bowen-Margulis measure is uniquely defined, up to a multiplicative constant.

Babillot [Bab, Theo. 1] showed that if $m_{\rm BM}$ is finite, then it is mixing for the geodesic flow of M if the length spectrum of M is not contained in a discrete subgroup of $\mathbb{R}$, as in particular when M is locally symmetric. For every Riemannian orbifold Y, we denote by $\|\cdot\|_\ell$ the $W^{\ell,2}$-Sobolev norm on the vector space $C^\ell_c(Y)$ of C^ℓ-smooth functions with compact support on Y. We refer for instance to [BrPP, §9.1] for the definition of the exponentially mixing property (for the Sobolev regularity) of the geodesic flow of M. Note that when M is locally symmetric with finite volume, by the work of Li-Pan [LP] when M is real hyperbolic and by the Margulis arithmeticity result with the works of Kleinbock-Margulis and Clozel, see for instance [BrPP, page 182], when M is quaternionic or octonionic hyperbolic, the only case when the geodesic flow of M is not yet known to be exponentially mixing is when M is complex hyperbolic.

Let D be a nonempty proper closed convex subset of $\widetilde{M}$. We refer to [PaP1, §2.2] or [BrPP, §2.4] for the definition of the outer and inner normal bundles ∂^1_+D and ∂^1_-D of D. We refer to [PaP1] for more background and for the basic properties of the following measures. The *(outer) skinning measure* on ∂^1_+D (associated with the Patterson-Sullivan density $(\mu_x)_{x\in\widetilde{M}}$) is the measure $\widetilde{\sigma}^+_D$ on ∂^1_+D defined, using the positive endpoint homeomorphism $p_+ : v \mapsto v_+$ from ∂^1_+D to $\partial_\infty\widetilde{M} - \partial_\infty D$, by

$$d\,\widetilde{\sigma}^+_D(v) = e^{-\delta\,\beta_{v_+}(\pi(v),x_*)}\,d\mu_{x_*}(v_+)\,.$$

The *(inner) skinning measure* $d\widetilde{\sigma}^-_D(v) = e^{-\delta\,\beta_{v_-}(\pi(v),x_*)}\,d\mu_{x_*}(v_-)$ is the similarly defined measure on ∂^1_-D. When D is a singleton, we immediately have

$$\forall\ v \in T^1_{x_*}\widetilde{M}, \qquad d\,\widetilde{\sigma}^\pm_{\{x_*\}}(v) = d\mu_{\{x_*\}}(v_\pm)\,. \tag{4}$$

If the family $(\gamma D)_{\gamma\in\Gamma/\Gamma_D}$ is locally finite in $\widetilde{M}$, and if A is the image of D in M, we say that A is a *proper nonempty properly immersed closed locally convex subset* of M. We denote by $\sigma^{\pm}_{A}$ the locally finite Borel measure on T^1M induced by the Γ-invariant locally finite Borel measure $\sum_{\gamma\in\Gamma/\Gamma_D}\gamma_*\,\widetilde{\sigma}^{\pm}_D$ on $T^1\widetilde{M}$. The support of $\sigma^{\pm}_A$ is the image $\partial^1_{\pm}A$ of $\partial^1_{\pm}D$ by the map $T^1\widetilde{M}\to T^1M$.

We conclude this section by the following new topological construction. In the same way the geometric compactification $\overline{\widetilde{M}}=\widetilde{M}\cup\partial_\infty\widetilde{M}$ of $\widetilde{M}$ compactifies $\widetilde{M}$ by gluing its boundary at infinity $\partial_\infty\widetilde{M}$ (see [BrH]), there exists a *geometric compactification* $\overline{T^1\widetilde{M}}=T^1\widetilde{M}\cup\partial_\infty\widetilde{M}$ of $T^1\widetilde{M}$ by gluing $\partial_\infty\widetilde{M}$, in which $T^1\widetilde{M}$ is open and dense, constructed as follows. The footpoint projection $\pi: T^1\widetilde{M}\to\widetilde{M}$ uniquely extends by the identity map on $\partial_\infty\widetilde{M}$ to a set-theoretic map $\overline{T^1\widetilde{M}}\to\overline{\widetilde{M}}$ again denoted by π. A basis of open subsets for the topology of $\overline{T^1\widetilde{M}}$ consists of either the open subsets of $T^1\widetilde{M}$ or of the preimages by π of the open subsets of $\overline{\widetilde{M}}$.

It is well known that the uniform structure[2] on $\partial_\infty\widetilde{M}$ defined by the visual distance d_{x_*} extends to a uniform structure on $\overline{\widetilde{M}}$ (hence by pullback by π on $\overline{T^1\widetilde{M}}$), as follows. Recall (see [Bourd]) that the visual distance between two distinct points at infinity $\xi,\eta\in\partial_\infty M$ is comparable to the exponential of the opposite of the distance from x_* to the geodesic line between ξ and η: There exists a universal constant $c\geqslant 0$ such that $e^{-d(x_*,\,]\xi,\eta[)}\leqslant d_{x_*}(\xi,\eta)\leqslant e^{-d(x_*,\,]\xi,\eta[)+c}$. A fundamental system of entourages $(\mathscr{E}_\epsilon)_{\epsilon>0}$ for the uniform structure on $\overline{\widetilde{M}}$ consists of the sets $\mathscr{E}_\epsilon$ of pairs (x,y) in $\overline{\widetilde{M}}\times\overline{\widetilde{M}}$ such that the distance between x_* and the geodesic segment, ray or line between x and y is at least $-\ln\epsilon$.

3 Equidistribution of Lebesgue measures along common perpendiculars

This whole Section is devoted to the proof of Theorem 1. We fix D^- and D^+ two nonempty proper closed convex subsets of $\widetilde{M}$, such that the families $(\alpha^{\pm}D^{\pm})_{\alpha^{\pm}\in\Gamma/\Gamma_{D^{\pm}}}$ are locally finite in $\widetilde{M}$, and we denote by A^- and A^+ respectively their images in M. Using the notation of the introduction $v^{\pm}_{\alpha}$ for $\alpha\in\operatorname{Perp}(A^-,\,A^+,\,t)$, one of the tools of the proof of Theorem 1 is the following result. It is proved in [PaP3, Coro. 12] for its first claim and [PaP3, Theo. 15 (2)] for its second claim (with an extra smoothing argument of the boundary of $D^{\pm}$).

Theorem 3. *Assume that the Bowen-Margulis measure $m_{\rm BM}$ is finite and mixing for the geodesic flow of M. Assume that the skinning measures $\sigma^+_{A^-}$ and $\sigma^-_{A^+}$ are finite and nonzero. For the narrow convergence of measures on $T^1M\times T^1M$, we have*

$$\lim_{t\to+\infty}\ \delta\,\|m_{\rm BM}\|\,e^{-\delta\,t}\sum_{\alpha\in\operatorname{Perp}(A^-,\,A^+,\,t)}\Delta_{v^-_\alpha}\otimes\Delta_{v^+_\alpha}\ =\ \sigma^+_{A^-}\otimes\sigma^-_{A^+}\,.$$

If furthermore M is locally symmetric with finite volume, and if the geodesic flow of M is exponentially mixing, then there exists $\ell\in\mathbb{N}$ and $\kappa>0$ such that for all $\phi^{\pm}\in C^\ell_c(T^1M)$, as $s\to+\infty$, we have

$$\frac{\delta\,\|m_{\rm BM}\|}{e^{\delta\,t}}\sum_{\alpha\in\operatorname{Perp}(A^-,\,A^+,\,t)}\phi^-(v^-_\alpha)\,\phi^+(v^+_\alpha)=\sigma^+_{A^-}(\phi^-)\,\sigma^-_{A^+}(\phi^+)+\operatorname{O}(e^{-\kappa t}\,\|\phi^-\|_\ell\,\|\phi^+\|_\ell)\,.\quad\Box$$

[2] See [Bourb, Chap. 2] for background on uniform spaces.

We start the proof of Theorem 1 by giving the notation that we will use. Let D' and D'' be nonempty closed convex subsets of $\widetilde{M}$ such that $d(D', D'') > 0$. Let $[D', D'']$ be the common perpendicular arc between D' and D'' (oriented from D' towards D''). Let $u_{D',D''}$ (respectively $v_{D',D''}$) be the initial (respectively terminal) unit tangent vector to $[D', D'']$, so that $v_{D',D''} = \mathsf{g}^{d(D',D'')} u_{D',D''}$. Let $\operatorname{Leb}_{[D',D'']}$ be the measure on $T^1\widetilde{M}$ which is the pushforward measure by the continuous map $t \mapsto \mathsf{g}^t u_{D',D''}$ of the Lebesgue measure on $[0, d(D', D'')]$. Its support is

$$T^1[D', D''] = \left\{\mathsf{g}^t u_{D',D''} : t \in [0, d(D', D'')]\right\}.$$

For every isometry γ of $\widetilde{M}$, we naturally have $\gamma u_{D',D''} = u_{\gamma D',\gamma D''}$, $\gamma v_{D',D''} = v_{\gamma D',\gamma D''}$ and $\gamma_* \operatorname{Leb}_{[D',D'']} = \operatorname{Leb}_{[\gamma D',\gamma D'']}$.

For every $t > 0$, let

$$\nu_{1,t} = \sum_{\alpha^- \in \Gamma/\Gamma_{D^-},\, \alpha^+ \in \Gamma/\Gamma_{D^+} \,:\, 0 < d(\alpha^- D^-,\, \alpha^+ D^+) \leqslant t} \operatorname{Leb}_{[\alpha^- D^-, \alpha^+ D^+]}\,,$$

which is a Γ-invariant Borel measure on $T^1\widetilde{M}$. We claim that this measure $\nu_{1,t}$ is locally finite. Indeed, let K be a nonempty compact subset of $T^1\widetilde{M}$, let t_K be the diameter of the subset $\pi(K)$ of $\widetilde{M}$, and let $x_K \in \pi(K)$. For all $\alpha^\pm \in \Gamma/\Gamma_{D^\pm}$ such that K meets the support of $\operatorname{Leb}_{[\alpha^- D^-, \alpha^+ D^+]}$, we have $d(x_K, \alpha^\pm D^\pm) \leqslant t + t_K$. Since the families $(\alpha^\pm D^\pm)_{\alpha^\pm \in \Gamma/\Gamma_{D^\pm}}$ are locally finite, the sets

$$E^\pm = \{\alpha^\pm \in \Gamma/\Gamma_{D^\pm} : d(x_K, \alpha^\pm D^\pm) \leqslant t + t_K\}$$

are finite. Hence $\nu_{1,t\,|K} = \sum_{\alpha^\pm \in E^\pm \,:\, 0 < d(\alpha^- D^-,\, \alpha^+ D^+) \leqslant t} \operatorname{Leb}_{[\alpha^- D^-,\, \alpha^+ D^+]\,|K}$ is a finite sum of finite measures on K.

The locally finite Γ-invariant Borel measure $\nu_{1,t}$ induces by the orbifold covering map $T^1\widetilde{M} \to T^1M = \Gamma\backslash T^1\widetilde{M}$ a locally finite Borel measure on T^1M, see [PauPS, §2.6]. This measure is exactly $\sum_{\alpha \in \operatorname{Perp}(A^-, A^+, t)} \operatorname{Leb}_\alpha$, which is, up to normalization, the measure in the left hand-side of Equation (1). To see this, note that the Lebesgue measure $\operatorname{Leb}_\alpha$ along every $\alpha \in \operatorname{Perp}(A^-, A^+, t)$ lifts to the Lebesgue measure $\operatorname{Leb}_{[\alpha^- D^-, \alpha^+ D^+]}$ along a common perpendicular between the images (at distance at most t) of D^- and D^+ by two elements α^- and α^+ of Γ well defined modulo the stabilizers of D^- and D^+, and conversely. The multiplicities are defined in [PaP3, §3.3], see also [BrPP, §12.1], precisely in order to deal with the orbifold covering and with the multiple intersections of the locally finite families $(\alpha^\pm D^\pm)_{\alpha^\pm \in \Gamma/\Gamma_{\widetilde{D}^\pm}}$.

Let us fix $\epsilon \in \,]0, \frac{1}{2}[$ and a point $x_0 \in \widetilde{M}$. At the very end of the proof, ϵ will tend to 0 and x_0 will vary in an ϵ-net of $\widetilde{M}$. We will denote by $\epsilon_1, \epsilon_2, \epsilon_3$ positive functions of ϵ (depending only on $(\widetilde{M}, \Gamma)$) that converge to 0 as ϵ tends to 0. Let $B = \pi^{-1}(B(x_0, \frac{\epsilon}{2}))$ be the subset of $T^1\widetilde{M}$ of elements whose footpoints are at distance less than $\frac{\epsilon}{2}$ from x_0. Let $\psi : T^1\widetilde{M} \to \mathbb{R}$ be a nonzero nonnegative continuous function with compact support contained in B. Using Hopf's parametrisation with respect to the given point x_0, that identifies $T^1\widetilde{M}$ with $(\partial_\infty\widetilde{M} \times \partial_\infty\widetilde{M} \smallsetminus \operatorname{diag}) \times \mathbb{R}$, we also assume that ψ is a product of three nonzero nonnegative continuous functions with compact support $\psi^- : \partial_\infty\widetilde{M} \to \mathbb{R}$, $\psi^+ : \partial_\infty\widetilde{M} \to \mathbb{R}$ and $\psi^0 : \partial_\infty\widetilde{M} \to \mathbb{R}$, on each factor of the product $\partial_\infty\widetilde{M} \times \partial_\infty\widetilde{M} \times \mathbb{R}$, so that

$$\psi : (\xi, \eta, s) \mapsto \psi^-(\xi)\; \psi^+(\eta)\; \psi^0(s)\,.$$

The following picture will be useful throughout the proof.

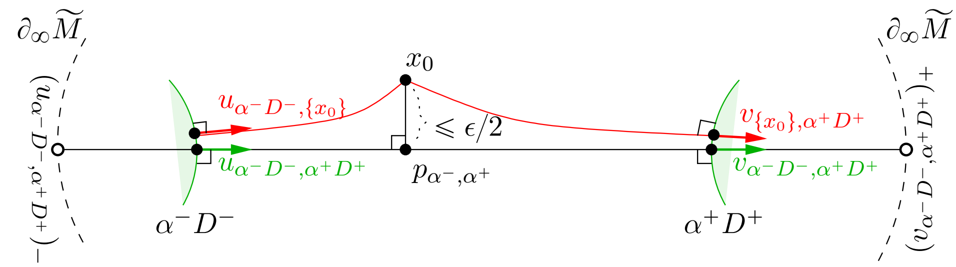

Figure 2: Splitting common perpendiculars.

For every $t > 0$, using Hopf's parametrisation with respect to x_0, we have

$$\nu_{1,t} = \sum_{\substack{\alpha^- \in \Gamma/\Gamma_{D^-},\, \alpha^+ \in \Gamma/\Gamma_{D^+} \\ 0 < d(\alpha^- D^-,\, \alpha^+ D^+) \leqslant t}} \big(\Delta_{(u_{\alpha^- D^-, \alpha^+ D^+})_-} \otimes \Delta_{(v_{\alpha^- D^-, \alpha^+ D^+})_+} \otimes ds\big)_{|\, T^1[\alpha^- D^-, \alpha^+ D^+]}\,.$$

For every $t > 0$, let us consider the Γ-invariant Borel measure $\nu_{2,t}$ on $(T^1\widetilde{M} \times T^1\widetilde{M} \smallsetminus \mathrm{diag}) \times \mathbb{R}$ defined by

$$\nu_{2,t} = \sum_{\substack{\alpha^- \in \Gamma/\Gamma_{D^-},\, \alpha^+ \in \Gamma/\Gamma_{D^+} \\ 0 < d(\alpha^- D^-,\, \alpha^+ D^+) \leqslant t}} \Delta_{u_{\alpha^- D^-, \alpha^+ D^+}} \otimes \Delta_{v_{\alpha^- D^-, \alpha^+ D^+}} \otimes ds\,.$$

The measure $\nu_{2,t}$ is locally finite since the set of initial unit tangent vectors $u_{\alpha^- D^-, \alpha^+ D^+}$ for $\alpha^\pm \in \Gamma/\Gamma_{D^\pm}$ such that $0 < d(\alpha^- D^-,\, \alpha^+ D^+) \leqslant t$ and the corresponding set of terminal unit tangent vectors $v_{\alpha^- D^-, \alpha^+ D^+}$ are locally finite in $T^1\widetilde{M}$, and the support of $\nu_{2,t}$ is the product of these two locally finite sets and $\mathbb{R}$. Both measures $\nu_{1,t}$ and $\nu_{2,t}$ can be seen as measures on $(\overline{T^1\widetilde{M}} \times \overline{T^1\widetilde{M}} \smallsetminus \mathrm{diag}) \times \mathbb{R}$ using the geometric compactification of $T^1\widetilde{M}$ defined at the end of Section 2.

We endow $T^1\widetilde{M} = (\partial_\infty \widetilde{M} \times \partial_\infty \widetilde{M} \smallsetminus \mathrm{diag}) \times \mathbb{R}$ with a product distance of the visual distance d_{x_0} seen from x_0 on each factor $\partial_\infty \widetilde{M}$ and the Euclidean distance on the factor $\mathbb{R}$. We extend the function $\psi = \psi^- \times \psi^+ \times \psi^0$, for now defined on $(\partial_\infty \widetilde{M} \times \partial_\infty \widetilde{M} \smallsetminus \mathrm{diag}) \times \mathbb{R}$, to a continuous function on $(\overline{T^1\widetilde{M}} \times \overline{T^1\widetilde{M}} \smallsetminus \mathrm{diag}) \times \mathbb{R}$, which is a product of three continuous functions again denoted by ψ^-, ψ^+, ψ^0 on each factor of the product $\overline{T^1\widetilde{M}} \times \overline{T^1\widetilde{M}} \times \mathbb{R}$. We may assume that the supports of the extended functions $\psi^\pm$ are contained in the complementary subset of $\pi^{-1}(B(x_0, -\ln \epsilon))$ without changing their modulus of continuity (up to a constant) for the uniform structure.

Let $\alpha^- \in \Gamma/\Gamma_{D^-}$ and $\alpha^+ \in \Gamma/\Gamma_{D^+}$ be elements that give a nontrivial contribution to the sum $\nu_{2,t}(\psi)$. We then have $x_0 \notin \alpha^- D^- \cup \alpha^+ D^+$, since $d(x_0, \pi(u_{\alpha^- D^-, \alpha^+ D^+}) \geqslant -\ln \epsilon$, hence $d(x_0, \alpha^- D^-) \geqslant -\ln \epsilon - \frac{\epsilon}{2} \geqslant -\ln \frac{1}{2} - \frac{1}{4} > 0$, and similarly $d(x_0, \alpha^+ D^+) > 0$. Furthermore, the vectors $u_{\alpha^- D^-, \alpha^+ D^+}$ and $v_{\alpha^- D^-, \alpha^+ D^+}$ are close in $\overline{T^1\widetilde{M}}$ to the points at infinity $(u_{\alpha^- D^-, \alpha^+ D^+})_-$ and $(v_{\alpha^- D^-, \alpha^+ D^+})_+$ respectively, uniformly in such α^- and α^+. By the uniform continuity of ψ, for t large enough, we have

$$e^{-\epsilon_1} \nu_{1,t}(\psi) \leqslant \nu_{2,t}(\psi) \leqslant e^{\epsilon_1} \nu_{1,t}(\psi)\,. \tag{5}$$

The vectors $u_{\alpha^- D^-, \alpha^+ D^+}$ and $v_{\alpha^- D^-, \alpha^+ D^+}$ are uniformly close in $T^1\widetilde{M}$ to the vectors $u_{\alpha^- D^-, \{x_0\}}$ and $v_{\{x_0\}, \alpha^+ D^+}$ respectively, which themselves are uniformly close in $\overline{T^1\widetilde{M}}$ to the points at infinity $(u_{\alpha^- D^-, \{x_0\}})_-$ and $(v_{\{x_0\}, \alpha^+ D^+})_+$. Thus similarly, if

$$\nu_{3,t} = \sum_{\substack{\alpha^- \in \Gamma/\Gamma_{D^-},\, \alpha^+ \in \Gamma/\Gamma_{D^+} \\ x_0 \notin \alpha^- D^- \cup \alpha^+ D^+,\, 0 < d(\alpha^- D^-, \alpha^+ D^+) \leqslant t}} \Delta_{(u_{\alpha^- D^-, \{x_0\}})_-} \otimes \Delta_{(v_{\{x_0\}, \alpha^+ D^+})_+} \otimes ds\,,$$

then $\nu_{3,t}$ is a Γ-invariant Borel measure on $T^1\widetilde{M} = (\partial_\infty \widetilde{M} \times \partial_\infty \widetilde{M} \smallsetminus \mathrm{diag}) \times \mathbb{R}$ such that

$$e^{-\epsilon_2} \nu_{2,t}(\psi) \leqslant \nu_{3,t}(\psi) \leqslant e^{\epsilon_2} \nu_{2,t}(\psi)\,. \tag{6}$$

For every $t > 1$, let $N = \lceil \frac{t}{\epsilon} \rceil$, so that $(N-1)\epsilon < t \leqslant N\epsilon$. For every $k \in \mathbb{N}$, let

$$\mathscr{A}_k = \big\{(\alpha^-, \alpha^+) \in \Gamma/\Gamma_{D^-} \times \Gamma/\Gamma_{D^+} : x_0 \notin \alpha^\pm D^\pm,\ \max\{0, k\epsilon - 1\} < d(\alpha^- D^-, \alpha^+ D^+) \leqslant k\epsilon\big\}$$

and

$$Z_k = \sum_{(\alpha^-, \alpha^+) \in \mathscr{A}_k} \psi^-\big((u_{\alpha^- D^-, \{x_0\}})_-\big)\ \psi^+\big((v_{\{x_0\}, \alpha^+ D^+})_+\big) \int \psi^0\, ds\,. \tag{7}$$

We then have

$$\sum_{k=1}^{N-1} Z_k \leqslant \nu_{3,t}(\psi) \leqslant \sum_{k=1}^{N} Z_k\,. \tag{8}$$

For all $i, j \in \mathbb{Z}$, let us define

$$\mathscr{A}_i^- = \big\{\alpha^- \in \Gamma/\Gamma_{D^-} : \max\{0, (i-1)\epsilon\} < d(\alpha^- D^-, x_0) \leqslant i\epsilon\big\}\,,$$

$$\underline{\mathscr{A}}_j^+ = \big\{\alpha^+ \in \Gamma/\Gamma_{D^+} : \max\{0, j\epsilon - 1 + \epsilon\} < d(x_0, \alpha^+ D^+) \leqslant j\epsilon\big\}\,, \text{ and}$$

$$\overline{\mathscr{A}}_j^+ = \big\{\alpha^+ \in \Gamma/\Gamma_{D^+} : \max\{0, j\epsilon - 1\} < d(x_0, \alpha^+ D^+) \leqslant j\epsilon + 2\epsilon\big\}\,.$$

Let $\alpha^- \in \Gamma/\Gamma_{D^-}$ and $\alpha^+ \in \Gamma/\Gamma_{D^+}$ be such that we have $d(\alpha^- D^-, \alpha^+ D^+) > 0$, $d(x_0, [\alpha^- D^-, \alpha^+ D^+]) \leqslant \frac{\epsilon}{2}$ and $x_0 \notin \alpha^- D^- \cup \alpha^+ D^+$. Let p_{α^-, α^+} be the closest point to x_0 on the geodesic segment $[\alpha^- D^-, \alpha^+ D^+]$ (see Figure 2). Since $d(x_0, p_{\alpha^-, \alpha^+}) \leqslant \frac{\epsilon}{2}$ and since closest point projections do not increase the distances, we have

$$d(\alpha^- D^-, x_0) - \frac{\epsilon}{2} \leqslant d(\pi(u_{\alpha^- D^-, \alpha^+ D^+}), x_0) - \frac{\epsilon}{2} \leqslant d(\alpha^- D^-, p_{\alpha^-, \alpha^+}) \leqslant d(\alpha^- D^-, x_0)\,.$$

Similarly, we have

$$d(x_0, \alpha^+ D^+) - \frac{\epsilon}{2} \leqslant d(p_{\alpha^-, \alpha^+}, \alpha^+ D^+) \leqslant d(x_0, \alpha^+ D^+)\,.$$

Hence

$$d(\alpha^- D^-, x_0) + d(x_0, \alpha^+ D^+) - \epsilon \leqslant d(\alpha^- D^-, \alpha^+ D^+) \leqslant d(\alpha^- D^-, x_0) + d(x_0, \alpha^+ D^+)\,.$$

Therefore

$$\text{if} \quad \alpha^- \in \mathscr{A}_i^- \quad \text{and} \quad \alpha^+ \in \underline{\mathscr{A}}_{k-i}^+, \quad \text{then} \quad (\alpha^-, \alpha^+) \in \mathscr{A}_k\,. \tag{9}$$

and

$$\text{if} \quad \alpha^- \in \mathscr{A}_i^- \quad \text{and} \quad (\alpha^-, \alpha^+) \in \mathscr{A}_k, \quad \text{then} \quad \alpha^+ \in \overline{\mathscr{A}}_{k-i}^+\,. \tag{10}$$

By integrating on the first factor the first formula in Theorem 3 applied by taking $A^+ = \Gamma\{x_0\}$, and by lifting to $T^1\widetilde{M}$, for the weak-star convergence of measures on $T^1_{x_0}\widetilde{M}$, we have (uniformly on ϵ)

$$\lim_{s\,\epsilon\to+\infty} \frac{\delta\,\|m_{\rm BM}\|}{e^{\delta\,s\,\epsilon}\,\|\sigma^+_{A^-}\|} \sum_{\alpha^-\in\bigcup_{i=1}^s \mathscr{A}_i^-} \Delta_{v_{\alpha^- D^-,\{x_0\}}} = \widetilde{\sigma}^-_{\{x_0\}}\,. \tag{11}$$

For every $i\in\mathbb{N}$, let us define

$$a_i = \sum_{\alpha^-\in\mathscr{A}_i^-} \psi^-\big((u_{\alpha^- D^-,\{x_0\}})_-\big)\,.$$

Since the negative endpoint map $p_- : T^1_{x_0}M \to \partial_\infty\widetilde{M}$ is a homeomorphism, its pushforward map $(p_-)_*$ on finite measures is weak-star continuous. Note that we have the equality $p_-(v_{\alpha^- D^-,\{x_0\}}) = (u_{\alpha^- D^-,\{x_0\}})_-$. Hence by evaluating on ψ^- the image of Equation (11) by $(p_-)_*$, and by Equation (4), we have

$$\begin{aligned}\lim_{s\,\epsilon\to+\infty} \frac{\delta\,\|m_{\rm BM}\|}{e^{\delta\,s\,\epsilon}\,\|\sigma^+_{A^-}\|}\sum_{i=1}^s a_i &= \lim_{s\,\epsilon\to+\infty} \frac{\delta\,\|m_{\rm BM}\|}{e^{\delta\,s\,\epsilon}\,\|\sigma^+_{A^-}\|} \sum_{\alpha^-\in\bigcup_{i=1}^s \mathscr{A}_i^-} \psi^-((u_{\alpha^- D^-,\{x_0\}})_-)\\ &= (p_-)_*\widetilde{\sigma}^-_{\{x_0\}}(\psi^-) = \mu_{x_0}(\psi^-)\,.\end{aligned} \tag{12}$$

Hence for every $\eta>0$, there exists $i_0 = i_0(\eta,\epsilon)$ such that if $s\geqslant i_0$, then

$$\frac{\|\sigma^+_{A^-}\|\,(\mu_{x_0}(\psi^-)-\eta)}{\delta\,\|m_{\rm BM}\|}\,e^{\delta\,s\,\epsilon} \leqslant \sum_{i=1}^s a_i \leqslant \frac{\|\sigma^+_{A^-}\|\,(\mu_{x_0}(\psi^-)+\eta)}{\delta\,\|m_{\rm BM}\|}\,e^{\delta\,s\,\epsilon}\,. \tag{13}$$

Note that $e^{2\,\delta\,\epsilon} - e^{-\delta} = (1-e^{-\delta})e^{{\rm O}(\epsilon)}$. Similarly, now integrating on the second factor the first formula in Theorem 3 now applied with $A^- = \Gamma\{x_0\}$, taking the difference between the times $j\epsilon+2\epsilon$ and $j\epsilon-1$, lifting to $T^1\widetilde{M}$, pushing forwards by $(p_+)_*$ and evaluating on ψ^+, we have (uniformly on ϵ)

$$\lim_{j\,\epsilon\to+\infty} \frac{\delta\,\|m_{\rm BM}\|}{(1-e^{-\delta})\,e^{\delta\,j\,\epsilon}\,\|\sigma^-_{A^+}\|} \sum_{\alpha^+\in\mathscr{A}_j^+} \psi^+\big((v_{\alpha^- D^-,\{x_0\}})_+\big) = e^{{\rm O}(\epsilon)}\,\mu_{x_0}(\psi^+)\,. \tag{14}$$

With the above ${\rm O}(\epsilon)$, for every $\eta>0$, let $f_\pm : [0,+\infty[\,\to\mathbb{R}$ be the smooth function

$$f_\pm : s\mapsto \frac{(1-e^{-\delta})\,e^{{\rm O}(\epsilon)}\,\|\sigma^-_{A^+}\|\,(\mu_{x_0}(\psi^+)\pm\eta)}{\delta\,\|m_{\rm BM}\|}\,e^{\delta\,(k-s)\,\epsilon}\,.$$

It follows from Equation (14) that there exists $j_0 = j_0(\eta,\epsilon)$ such that if $j\geqslant j_0$, then

$$f_-(k-j) \leqslant \sum_{\alpha^+\in\mathscr{A}_j^+} \psi^+\big((v_{\alpha^- D^-,\{x_0\}})_+\big) \leqslant f_+(k-j)\,. \tag{15}$$

Let us define

$$C_1 = \frac{\|\sigma^+_{A^-}\|\,(\mu_{x_0}(\psi^-)+\eta)}{\delta\,\|m_{\rm BM}\|} \quad\text{and}\quad C_2 = \frac{(1-e^{-\delta})\,e^{{\rm O}(\epsilon)}\,\|\sigma^-_{A^+}\|\,(\mu_{x_0}(\psi^+)+\eta)}{\delta\,\|m_{\rm BM}\|}\,.$$

By decomposing the sum defining Z_k in Equation (7) into thin slices of width ϵ of the first index α^-, and by using Equation (10), we have

$$\begin{aligned}
Z_k &= \sum_{i\in\mathbb{N}}\;\sum_{\alpha^-\in\mathscr{A}_i^-}\Big(\psi^-\big((u_{\alpha^- D^-,\{x_0\}})_-\big)\sum_{\substack{\alpha^+\in\Gamma/\Gamma_{D^+}\\(\alpha^-,\alpha^+)\in\mathscr{A}_k}}\psi^+\big((v_{\{x_0\},\alpha^+D^+})_+\big)\Big)\int\psi^0\,ds\\
&\leqslant \sum_{i\in\mathbb{N}} a_i\Big(\sum_{\alpha^+\in\overline{\mathscr{A}}^+_{k-i}}\psi^+\big((v_{\{x_0\},\alpha^+D^+})_+\big)\Big)\int\psi^0\,ds\,. \qquad (16)
\end{aligned}$$

Note that $\overline{\mathscr{A}}^+_j$ is empty if $j\leqslant -3$, hence $\overline{\mathscr{A}}^+_{k-i}$ is empty if $i\geqslant k+3$. We decompose the sum $\sum_{i=0}^{k+2}$ into the sum $\sum_{i=0}^{k-j_0}$, where we can use the upper bound in Equation (15) with $j=k-i$, and the sum $\sum_{i=k-j_0+1}^{k+2}$, where we can use the estimate $\sum_{i=k-j_0+1}^{k+3} a_i = \mathrm{O}(e^{\delta\,k\,\epsilon})$ following from Equation (12) and the estimate $\sum_{\alpha^+\in\overline{\mathscr{A}}^+_{k-i}}\psi^+\big((v_{\alpha^-D^-,\{x_0\}})_+\big)=\mathrm{O}(e^{\delta\,(k-i)\,\epsilon})$ following from Equation (14). Hence, using Abel's summation formula and the fact that $a_0=0$ for the equality below, we have

$$\begin{aligned}
Z_k &\leqslant \sum_{i=0}^{k-j_0} a_i\; f_+(i)\int\psi^0\,ds + \mathrm{O}_{j_0}(e^{\delta\,k\,\epsilon})\\
&= \Big(\Big(\sum_{i=0}^{k-j_0} a_i\Big)\,f_+(k-j_0) - \int_{u=0}^{k-j_0}\Big(\sum_{i=0}^{u} a_i\Big) f'_+(u)\,du\Big)\int\psi^0\,ds + \mathrm{O}_{j_0}(e^{\delta\,k\,\epsilon})\,.
\end{aligned}$$

Note that $f_+(k-j_0)=\mathrm{O}_{j_0}(1)$ and $f'_+(u) = -\,\delta\,\epsilon\;C_2\;e^{\delta\,(k-u)\,\epsilon} = \mathrm{O}(e^{\delta\,k\,\epsilon})$ when $u\leqslant k$. Using again $\sum_{i=1}^{k-j_0} a_i = \mathrm{O}(e^{\delta\,k\,\epsilon})$, subdividing the integral $\int_{u=0}^{k-j_0}$ into $\int_{u=0}^{i_0}$ and $\int_{u=i_0}^{k-j_0}$ and using on the second one the inequality $\sum_{i=1}^{u} a_i\leqslant C_1\;e^{\delta\,u\,\epsilon}$ following from the upper bound in Equation (13), we have

$$\begin{aligned}
Z_k &\leqslant \int_{u=i_0}^{k-j_0}(C_1\;e^{\delta u\epsilon})(\delta\,\epsilon\;C_2\;e^{\delta(k-u)\epsilon})\,du\int\psi^0\,ds + \mathrm{O}_{i_0,j_0}(e^{\delta k\epsilon})\\
&= \delta\,k\,\epsilon\,C_1\,C_2\;e^{\delta k\epsilon}\int\psi^0\,ds + \mathrm{O}_{i_0,j_0}(e^{\delta k\epsilon})\,.
\end{aligned}$$

By the upper bound in Equation (8), by expliciting the values of C_1, C_2, by a geometric series argument, and since $t = N\epsilon + \mathrm{O}(\epsilon)$, for t large enough, we hence have

$$\frac{\delta\;\|m_{\rm BM}\|}{t\;e^{\delta\,t}\;\|\sigma^+_{A^-}\|\;\|\sigma^-_{A^+}\|}\nu_{3,t}(\psi)\leqslant\frac{e^{\mathrm{O}(\epsilon)}}{\|m_{\rm BM}\|}(\mu_{x_0}(\psi^-)+\eta)\;(\mu_{x_0}(\psi^+)+\eta)\;\int\psi^0\,ds + \mathrm{O}_{i_0,j_0}\Big(\frac{1}{t}\Big)\,.$$

By taking the upper limit as $t\to+\infty$ and by letting $\eta\to 0$, we thus have

$$\limsup_{t\to+\infty}\frac{\delta\;\|m_{\rm BM}\|}{t\;e^{\delta\,t}\;\|\sigma^+_{A^-}\|\;\|\sigma^-_{A^+}\|}\;\nu_{3,t}(\psi)\leqslant e^{\mathrm{O}(\epsilon)}\frac{1}{\|m_{\rm BM}\|}\,d\mu_{x_0}\otimes d\mu_{x_0}\otimes ds\;(\psi)\,.$$

Since every element $v\in T^1\widetilde{M}$ of the support of ψ satisfies $d(\pi(v),x_0)\leqslant\frac{\epsilon}{2}$ and since by Equation (2) we have $|\,\beta_\xi(x,y)\,|\leqslant d(x,y)$ for all $\xi\in\partial_\infty\widetilde{M}$ and $x,y\in\widetilde{M}$, by Equation (3), we have

$$d\mu_{x_0}\otimes d\mu_{x_0}\otimes ds\;(\psi) = e^{\mathrm{O}(\epsilon)}\;\widetilde{m}_{\rm BM}(\psi)\,. \qquad (17)$$

Therefore by Equations (5) and (6), we have

$$\limsup_{t\to+\infty} \frac{\delta\,\|m_{\rm BM}\|}{t\,e^{\delta\,t}\,\|\sigma^+_{A^-}\|\,\|\sigma^-_{A^+}\|}\;\nu_{1,t}(\psi) \leqslant e^{\epsilon_3}\;\frac{\widetilde{m}_{\rm BM}(\psi)}{\|m_{\rm BM}\|}\,.$$

A similar argument replacing Equation (10) by Equation (9), and the upper bounds in Equations (15), (13), (8) by their lower bounds, gives

$$\liminf_{t\to+\infty} \frac{\delta\,\|m_{\rm BM}\|}{t\,e^{\delta\,t}\,\|\sigma^+_{A^-}\|\,\|\sigma^-_{A^+}\|}\;\nu_{1,t}(\psi) \geqslant e^{-\epsilon_3}\;\frac{\widetilde{m}_{\rm BM}(\psi)}{\|m_{\rm BM}\|}\,.$$

Let ψ' be a continuous function with compact support on $T^1\widetilde{M}$. By covering its support with sets of the form $\pi^{-1}(B(x_0,\frac{\epsilon}{2}))$ for finitely many $x_0\in T^1\widetilde{M}$, by using a partition of unity, by a uniform approximation by product functions, and by letting ϵ tends to 0, we thus obtain that

$$\lim_{t\to+\infty} \frac{\delta\,\|m_{\rm BM}\|}{t\,e^{\delta\,t}\,\|\sigma^+_{A^-}\|\,\|\sigma^-_{A^+}\|}\;\nu_{1,t}(\psi') = \frac{\widetilde{m}_{\rm BM}(\psi')}{\|m_{\rm BM}\|}\,.$$

This proves that the measures $\mu_t = \frac{\delta\,\|m_{\rm BM}\|}{t\,e^{\delta\,t}\,\|\sigma^+_{A^-}\|\,\|\sigma^-_{A^+}\|}\sum_{\alpha\in{\rm Perp}(A^-,\,A^+,\,t)}{\rm Leb}_\alpha$ converge to $\frac{m_{\rm BM}}{\|m_{\rm BM}\|}$ for the weak-star convergence of measures on T^1M.

The total mass of the measure μ_t converges to 1, since by [PaP3, Theo. 1] we have ${\rm Card\,Perp}(A^-,\,A^+,\,t)\sim\frac{\|\sigma^+_{A^-}\|\,\|\sigma^-_{A^+}\|}{\delta\,\|m_{\rm BM}\|}\,e^{\delta\,t}$ as $t\to+\infty$, and since by this exponential growth property, most of the mass of the sum $\sum_{\alpha\in{\rm Perp}(A^-,\,A^+,\,t)}{\rm Leb}_\alpha$ is obtained when the length of α is close to t. Since $\frac{m_{\rm BM}}{\|m_{\rm BM}\|}$ is a probability measure, there is hence no loss of mass in the above weak-star convergence, and it is well-known that this implies the narrow convergence of μ_t to $\frac{m_{\rm BM}}{\|m_{\rm BM}\|}$. This concludes the proof of the first claim of Theorem 1.

The proof of the second claim of Theorem 1 proceeds very similarly, we only indicate the changes. Let ℓ and κ be as in Theorem 3. Let K be a compact subset of T^1M, and $\widetilde{K}$ a compact subset of $T^1\widetilde{M}$ mapping to K. Let $x_0\in\widetilde{K}$ and $\epsilon\in\,]0,\frac{1}{2}[$. Since M is locally symmetric with finite volume, the geometric compactification $\overline{\widetilde{M}}=\widetilde{M}\cup\partial_\infty\widetilde{M}$ is a smooth manifold with boundary, and Hopf's parametrisation with respect to x_0 is a smooth diffeomorphism. Furthermore, the Patterson-Sullivan measure μ_{x_0} is up to a multiplicative constant the smooth measure on $\partial_\infty\widetilde{M}$ invariant under the group of isometries of $\widetilde{M}$ fixing x_0, and the Bowen-Margulis measure on T^1M is up to a multiplicative constant the Liouville measure of T^1M.

We now start with $\psi\in C^\ell(T^1\widetilde{M})$ with support in $\pi^{-1}(B(x_0,\frac{\epsilon}{2}))$, which is, in Hopf's parametrisation with respect to x_0, a product $(\xi,\eta,s)\mapsto\psi^-(\xi)\;\psi^+(\eta)\;\psi^0(s)$ of three smooth functions $\psi^-:\partial_\infty\widetilde{M}\to\mathbb{C}$, $\psi^+:\partial_\infty\widetilde{M}\to\mathbb{C}$, $\psi^0:\mathbb{R}\to\mathbb{C}$ with compact support. The Bowen-Margulis measure is absolutely continuous with respect to the product measure $d\mu_{x_0}\otimes d\mu_{x_0}\otimes ds$, with a smooth density, which is bounded from above and from below by a positive constant on the compact subset $\widetilde{K}$. Hence by Fubini's theorem, we have

$$\|\psi^-\|_\ell\;\|\psi^+\|_\ell\;\|\psi^0\|_\ell = {\rm O}_{\widetilde{K}}(\|\psi\|_\ell)\,. \tag{18}$$

We extend ψ^- and ψ^+ to smooth functions on $\widetilde{M}$ with support in the complementary subset of $B(x_0,-\ln\epsilon)$, and then on $T^1\widetilde{M}$ by being constant on the fibers of the footpoint projection $\pi:T^1\widetilde{M}\to\widetilde{M}$, with the same Sobolev norms up to a multiplicative constant.

By the error term in Theorem 3, Equation (11) now becomes, for all $\phi \in C^\ell(T^1_{x_0}\widetilde{M})$ and $s > 0$,

$$\frac{\delta\,\|m_{\rm BM}\|}{e^{\delta\,s\,\epsilon}\,\|\sigma^+_{A^-}\|}\sum_{\alpha^-\in\bigcup_{i=1}^s\mathscr{A}_i^-}\phi(v_{\alpha^-D^-,\{x_0\}}) = \widetilde{\sigma}^-_{\{x_0\}}(\phi) + {\rm O}(e^{-\kappa s\epsilon}\,\|\phi\|_\ell)\,.$$

Since $p_- : T^1_{x_0}M \to \partial_\infty\widetilde{M}$ is now a smooth diffeomorphism between compact manifolds, applying the above result to $\phi = \psi^-\circ p_-$, with $C_1' = \frac{\|\sigma^+_{A^-}\|\,\mu_{x_0}(\psi^-)}{\delta\,\|m_{\rm BM}\|}$, Equation (12) becomes

$$\sum_{i=1}^s a_i = C_1'\,e^{\delta\,s\,\epsilon} + {\rm O}(e^{(\delta-\kappa)s\epsilon}\,\|\psi^-\|_\ell)\,. \tag{19}$$

For appropriately choosen functions ${\rm O}(\cdot)$, and $C_2' = \frac{(1-e^{-\delta})\,e^{{\rm O}(\epsilon)}\,\|\sigma^-_{A^+}\|\,\mu_{x_0}(\psi^+)}{\delta\,\|m_{\rm BM}\|}$, we now define a smooth function $f:\mathbb{R}\to\mathbb{R}$ by

$$f : s\mapsto C_2'\,e^{\delta(k-s)\epsilon} + {\rm O}(e^{(\delta-\kappa)(k-s)\epsilon}\,\|\psi^+\|_\ell)\,,$$

so that $f' : s\mapsto -\,\delta\,\epsilon\,C_2'\,e^{\delta(k-s)\epsilon} + {\rm O}(\epsilon\,e^{(\delta-\kappa)(k-s)\epsilon}\,\|\psi^+\|_\ell)$. Equations (14) and (15) become, for every $j\in\mathbb{Z}$,

$$\sum_{\alpha^+\in\overline{\mathscr{A}}^+_j}\psi^+\big((v_{\alpha^-D^-,\{x_0\}})_+\big) = f(k-j)\,. \tag{20}$$

Since $\overline{\mathscr{A}}^+_{k-i}$ is empty if $i\geqslant k+2$, Equation (16) gives, by Abel's summation formula, that

$$\begin{aligned}Z_k &\leqslant \sum_{i=0}^{k+2}\;a_i\;f(i)\int\psi^0\,ds\\ &= \Big(\int\psi^0\,ds\Big)\;\bigg(\Big(\sum_{i=0}^{k+2}\;a_i\Big)\;f(k+2) - \int_{u=0}^{k+2}\Big(\sum_{i=0}^{u}\;a_i\Big)f'(u)\,du\bigg)\,.\end{aligned}$$

We have $f(k+2) = {\rm O}(\|\psi^+\|_\ell)$ by the definition of f and since by the Cauchy-Schwarz inequality, we have $\mu_{x_0}(\psi^+)\leqslant\|\mu_{x_0}\|\|\psi^+\|_0\leqslant\|\mu_{x_0}\|\|\psi^+\|_\ell$. Similarly, $\int\psi^0\,ds = {\rm O}_{\widetilde{K}}(\|\psi^0\|_\ell)$ and by Equation (19), we have $\sum_{i=1}^{k+2}a_i = {\rm O}(e^{\delta k\epsilon}\,\|\psi^-\|_\ell)$. Hence, again by Equation (19) and by the computation of f', we have

$$\begin{aligned}Z_k &\leqslant {\rm O}_{\widetilde{K}}(e^{\delta k\epsilon}\,\|\psi^-\|_\ell\,\|\psi^+\|_\ell\,\|\psi^0\|_\ell)\\ &\quad+\int\psi^0\,ds\int_{u=0}^{k+2}\big(C_1'\,e^{\delta u\epsilon} + {\rm O}(e^{(\delta-\kappa)u\epsilon}\,\|\psi^-\|_\ell)\big)\big(\delta\,\epsilon\,C_2'\,e^{\delta(k-u)\epsilon} + {\rm O}(\epsilon\,e^{(\delta-\kappa)(k-u)\epsilon}\,\|\psi^+\|_\ell)\big)\,du\,.\end{aligned}$$

By Equation (18) and by expliciting the values of C_1', C_2', we hence have

$$\begin{aligned}Z_k &\leqslant {\rm O}_{\widetilde{K}}(e^{\delta k\epsilon}\,\|\psi\|_\ell) + (k+2)\,\delta\,\epsilon\,C_1'\,C_2'\,e^{\delta k\epsilon}\int\psi^0\,ds\\ &\quad+{\rm O}_{\widetilde{K}}\Big(\int_{u=0}^{k+2}(e^{(\delta-\kappa)k\epsilon+\kappa u\epsilon} + e^{\delta k\epsilon-\kappa u\epsilon})\,\epsilon\,du\,\|\psi\|_\ell\Big)\\ &= \frac{(1-e^{-\delta})\,e^{{\rm O}(\epsilon)}\,\|\sigma^+_{A^-}\|\,\|\sigma^-_{A^+}\|}{\delta\,\|m_{\rm BM}\|^2}\;k\,\epsilon\,e^{\delta k\epsilon}\;\mu_{x_0}(\psi^-)\,\mu_{x_0}(\psi^+)\int\psi^0\,ds + {\rm O}_{\widetilde{K}}(e^{\delta k\epsilon}\,\|\psi\|_\ell)\,.\end{aligned}$$

Therefore, by the upper bound in Equation (8), by a geometric series argument, and since $t = N\epsilon + \mathrm{O}(\epsilon)$, we have

$$\nu_{3,t}(\psi) \leqslant \frac{e^{\mathrm{O}(\epsilon)}\ \|\sigma^+_{A^-}\|\ \|\sigma^-_{A^+}\|}{\delta\ \|m_{\rm BM}\|}\ t\,e^{\delta t}\int \psi\ \frac{1}{\|m_{\rm BM}\|}d\mu_{x_0}\otimes d\mu_{x_0}\otimes ds + \mathrm{O}_{\widetilde{K}}(e^{\delta t}\ \|\psi\|_\ell)\ .$$

The same lower bound is proved by using the lower bound in Equation (8) and by replacing Equation (10) by Equation (9). By Equations (5), (6) and (17), we thus have

$$\nu_{1,t}(\psi) = \frac{e^{\epsilon_4}\ \|\sigma^+_{A^-}\|\ \|\sigma^-_{A^+}\|}{\delta\ \|m_{\rm BM}\|}\ t\,e^{\delta t}\ \frac{m_{\rm BM}(\psi)}{\|m_{\rm BM}\|}\ + \mathrm{O}_{\widetilde{K}}(e^{\delta t}\ \|\psi\|_\ell)\ .$$

Let $\psi' : T^1\widetilde{M} \rightarrow \mathbb{R}$ be a continuous function with support in $\widetilde{K}$. By covering $\widetilde{K}$ with sets of the form $\pi^{-1}(B(x_0, \frac{\epsilon}{2}))$ for finitely many $x_0 \in T^1\widetilde{M}$, by using a smooth partition of unity and a smooth approximation by product functions, and by letting ϵ tend to 0, we thus obtain the second claim of Theorem 1.

This concludes the proof of Theorem 1 in the Introduction.

4 Potentials and equidistribution of common perpendiculars

In this section, we prove Theorem 2, after defining the various objects that appear in its statement. We keep the notation $\widetilde{M}$, x_*, Γ, M, π, $(\mathrm{g}^t)_{t\in\mathbb{R}}$, $p_\pm$, $v \mapsto v_\pm$ of Section 2.

4.1 Potentials and Gibbs measures

In this subsection, we briefly recall the thermodynamic formalism of geodesic flows in negative curvature, referring to [PauPS, BrPP, DT] for precisions and further developments.

Let $\widetilde{F} : T^1\widetilde{M} \rightarrow \mathbb{R}$ be a *potential* on $T^1\widetilde{M}$, that is, a Γ-invariant, bounded[3] Hölder-continuous real-valued map on $T^1\widetilde{M}$, and let $F : T^1M = \Gamma\backslash T^1\widetilde{M} \rightarrow \mathbb{R}$ be its induced function. For all $x, y \in \widetilde{M}$, let us define the *amplitude* of $\widetilde{F}$ between x and y to be $\int_x^y \widetilde{F} = 0$ if $x = y$ and otherwise $\int_x^y \widetilde{F} = \int_0^{d(x,y)} \widetilde{F}(\mathrm{g}^t v)\ dt$ where v is the tangent vector at x to the geodesic segment from x to y. If $\alpha : [a, b] \rightarrow M$ is a locally geodesic segment in the orbifold M, and if $\widetilde{\alpha} : [a, b] \rightarrow \widetilde{M}$ is any lift of α, we define the *amplitude* of F along α to be $\int_\alpha F = \int_{\widetilde{\alpha}} \widetilde{F} = \int_{\widetilde{\alpha}(a)}^{\widetilde{\alpha}(b)} \widetilde{F}$.

The *critical exponent* of F is the weighted (by the exponential amplitudes of F) orbital growth rate of the group Γ, defined by

$$\delta_F = \lim_{n\rightarrow+\infty} \frac{1}{n}\ \ln\Big(\sum_{\gamma\in\Gamma,\ n-1<d(x_*,\gamma x_*)\leqslant n} \exp\big(\int_{x_*}^{\gamma x_*} \widetilde{F}\,\big)\Big)\ .$$

This limit exists and is independent of the choice of x_*. We have $\delta_F \in\ ]-\infty, +\infty[$, $\delta_{F+c} = \delta_F + c$ for every constant $c \in \mathbb{R}$ and $\delta_{F\circ\iota} = \delta_F$ for $\iota : v \mapsto -v$ the opposite map on $T^1\widetilde{M}$. See [PauPS, Chap. 4, 6] for equivalent definitions (in particular for the equality with the topological pressure of F).

[3]See [BrPP, §3.2] for a weakening of this assumption.

The (normalised) *Gibbs cocycle* of the potential $\widetilde{F}$ (as for instance defined by Hamenstädt) is the function $C^F : \partial_\infty \widetilde{M} \times \widetilde{M} \times \widetilde{M} \to \mathbb{R}$, defined by the following limit of differences of amplitudes for the renormalised potential

$$(\xi, x, y) \mapsto C^F_\xi(x, y) = \lim_{t \to +\infty} \int_y^{\xi_t} (\widetilde{F} - \delta_F) - \int_x^{\xi_t} (\widetilde{F} - \delta_F) \ ,$$

where $t \mapsto \xi_t$ is any geodesic ray in $\widetilde{M}$ converging to ξ. The Gibbs cocycle is Γ-invariant (for the diagonal action) and locally Hölder-continuous.

A *Patterson density* for $(\Gamma, \widetilde{F})$ is a Γ-equivariant family $(\mu^F_x)_{x \in \widetilde{M}}$ of pairwise absolutely continuous (positive, Borel) measures on $\partial_\infty \widetilde{M}$, whose support is $\Lambda\Gamma$, such that

$$\gamma_* \mu^F_x = \mu^F_{\gamma x} \quad \text{and} \quad \frac{d\mu^F_x}{d\mu^F_y}(\xi) = e^{-C^F_\xi(x,y)}$$

for all $\gamma \in \Gamma$ and $x, y \in \widetilde{M}$, and for (almost) every $\xi \in \partial_\infty \widetilde{M}$. Patterson's classical construction gives its existence, see [PauPS, Prop. 3.9].

The *Gibbs measure* (or *equilibrium state*) on $T^1\widetilde{M}$ associated with a pair $(\mu^{F\circ\iota}_x)_{x \in \widetilde{M}}$ and $(\mu^F_x)_{x \in \widetilde{M}}$ of Patterson densities for $(\Gamma, \widetilde{F} \circ \iota)$ and $(\Gamma, \widetilde{F})$ is the σ-finite nonzero measure $\widetilde{m}_F$ on $T^1\widetilde{M}$ defined in [PauPS, Eq. (43)], using the Hopf parametrisation $v \mapsto (v_-, v_+, s)$ with respect to x_*, by

$$d\widetilde{m}_F(v) = e^{C^{F\circ\iota}_{v_-}(x_*, \pi(v)) + C^F_{v_+}(x_*, \pi(v))} \, d\mu^{F\circ\iota}_{x_*}(v_-) \, d\mu^F_{x_*}(v_+) \, ds \ . \tag{21}$$

It is independent of the choice of x_*. It is Γ-invariant and $(\mathtt{g}^t)_{t \in \mathbb{R}}$-invariant. Therefore it induces[4] a σ-finite nonzero $(\mathtt{g}^t)_{t \in \mathbb{R}}$-invariant measure on $T^1M = \Gamma \backslash T^1\widetilde{M}$, called the *Gibbs measure* on $\Gamma \backslash T^1\widetilde{M}$ for the potential F and denoted by m_F. If m_F is finite, then the above Patterson densities and the Gibbs measure are unique up to a scalar multiple, see [PauPS, §5.3].

Let D be a nonempty proper closed convex subset of $\widetilde{M}$, and let $A = \Gamma D$ be its image in $M = \Gamma \backslash \widetilde{M}$. Using the endpoint homeomorphisms $p_\pm : v \mapsto v_\pm$ from $\partial^1_\pm D$ to $\partial_\infty \widetilde{M} - \partial_\infty D$, the inner and outer *skinning measures* $\widetilde{\sigma}^-_D$ on $\partial^1_- D$ and $\widetilde{\sigma}^+_D$ on $\partial^1_+ D$ associated with the Patterson densities $(\mu^{F\circ\iota}_x)_{x \in \widetilde{M}}$ and $(\mu^F_x)_{x \in \widetilde{M}}$ for $(\Gamma, \widetilde{F} \circ \iota)$ and $(\Gamma, \widetilde{F})$ respectively, are defined in [BrPP, Eq. (7.1)] by

$$d\,\widetilde{\sigma}^-_D(v) = e^{C^{F\circ\iota}_{v_-}(x_*, \pi(v))} \, d\mu^{F\circ\iota}_{x_*}(v_-) \quad \text{and} \quad d\,\widetilde{\sigma}^+_D(v) = e^{C^F_{v_+}(x_*, \pi(v))} \, d\mu^F_{x_*}(v_+) \ . \tag{22}$$

They do not depend on x_*. If the Γ-equivariant family $\mathscr{D} = (\gamma D)_{\gamma \in \Gamma/\Gamma_D}$ is locally finite, the *inner and outer skinning measures* $\widetilde{\sigma}^-_{\mathscr{D}}$ and $\widetilde{\sigma}^+_{\mathscr{D}}$ of D on $T^1\widetilde{M}$ are the Γ-invariant locally finite measures on $T^1\widetilde{M}$ defined by

$$\widetilde{\sigma}^\pm_{\mathscr{D}} = \sum_{\gamma \in \Gamma/\Gamma_D} \widetilde{\sigma}^\pm_{\gamma D} \ .$$

They induce locally finite measures on T^1M, denoted by σ^-_A and σ^+_A, called the *inner and outer skinning measures* of A on T^1M associated with the potential F.

[4] See for instance [PauPS, §2.6] for details on the definition of the induced measure since Γ might have torsion, hence it does not act freely on $T^1\widetilde{M}$.

4.2 Proof of Theorem 2

The proof proceeds similarly to the proof of the first claim of Theorem 1, we only indicate the modifications. Let $D^\pm, A^\pm$ be as in the beginning of Section 3. We will use the following analog of Theorem 3, see [BrPP, Theo. 1.4].

Theorem 4. *Assume that $\delta_F > 0$, that m_F is finite and mixing for the geodesic flow of M, and that $\sigma^+_{A^-}$ and $\sigma^-_{A^+}$ are finite and nonzero. For the weak-star convergence of measures on $T^1M \times T^1M$, we have*

$$\lim_{t\to+\infty} \delta_F\,\|m_F\|\,e^{-\delta_F\,t} \sum_{\alpha\in\mathrm{Perp}(A^-,\,A^+,\,t)} e^{\int_\alpha F}\,\Delta_{v_\alpha^-}\otimes\Delta_{v_\alpha^+} \;=\; \sigma^+_{A^-}\otimes\sigma^-_{A^+}\,. \qquad \square$$

Let $\epsilon, x_0, \psi, \psi^\pm, \psi^0$ be as in the proof of the first claim of Theorem 1. Let $t > 1$ and $N = \lceil\frac{t}{\epsilon}\rceil$. We now consider the Γ-invariant Borel measures on $T^1\widetilde M$ defined by

$$\nu_{1,t} = \sum_{\substack{\alpha^-\in\Gamma/\Gamma_{D^-},\,\alpha^+\in\Gamma/\Gamma_{D^+}\\ 0<d(\alpha^-D^-,\,\alpha^+D^+)\leqslant t}} e^{\int_{[\alpha^-D^-,\,\alpha^+D^+]}\widetilde F}\;\mathrm{Leb}_{[\alpha^-D^-,\alpha^+D^+]} \quad\text{and}$$

$$\nu_{2,t} = \sum_{\substack{\alpha^-\in\Gamma/\Gamma_{D^-},\,\alpha^+\in\Gamma/\Gamma_{D^+}\\ 0<d(\alpha^-D^-,\,\alpha^+D^+)\leqslant t}} e^{\int_{[\alpha^-D^-,\,\alpha^+D^+]}\widetilde F}\;\Delta_{u_{\alpha^-D^-,\alpha^+D^+}}\otimes\Delta_{v_{\alpha^-D^-,\alpha^+D^+}}\otimes ds\,.$$

Let $\alpha^-\in\Gamma/\Gamma_{D^-}$ and $\alpha^+\in\Gamma/\Gamma_{D^+}$ such that $x_0\notin\alpha^-D^-\cup\alpha^+D^+$ and $d(\alpha^-D^-,\alpha^+D^+)>0$. We keep the notation of Figure 2 in the proof of Theorem 1. By the additivity of the amplitudes, we have

$$\int_{[\alpha^-D^-,\,\alpha^+D^+]}\widetilde F = \int_{\pi(u_{\alpha^-D^-,\,\alpha^+D^+})}^{p_{\alpha^-,\,\alpha^+}}\widetilde F + \int_{p_{\alpha^-,\,\alpha^+}}^{\pi(v_{\alpha^-D^-,\,\alpha^+D^+})}\widetilde F\,.$$

Recall that $d(x_0,p_{\alpha^-,\alpha^+})\leqslant\frac{\epsilon}{2}$ and that $d(\pi(u_{\alpha^-D^-,\alpha^+D^+}),\pi(u_{\alpha^-D^-,\{x_0\}}))\leqslant\frac{\epsilon}{2}$ since closest point maps do not increase the distance. Hence by [PauPS, Lem. 3.2], since $\widetilde F$ is bounded, there exists a constant $c_2\in\,]0,1]$ such that if ϵ is small enough, we have

$$\Big|\int_{\pi(u_{\alpha^-D^-,\,\{x_0\}})}^{x_0}\widetilde F - \int_{\pi(u_{\alpha^-D^-,\,\alpha^+D^+})}^{p_{\alpha^-,\,\alpha^+}}\widetilde F\Big| = \mathrm{O}(\epsilon^{c_2})\,,$$

and similarly

$$\Big|\int_{x_0}^{\pi(v_{\{x_0\},\alpha^+D^+})}\widetilde F - \int_{p_{\alpha^-,\,\alpha^+}}^{\pi(v_{\alpha^-D^-,\,\alpha^+D^+})}\widetilde F\Big| = \mathrm{O}(\epsilon^{c_2})\,.$$

Hence with $\nu_{3,t}$ the Γ-invariant Borel measure on $T^{\widetilde M}$ now defined by

$$\nu_{3,t} = \sum_{\substack{\alpha^-\in\Gamma/\Gamma_{D^-},\,\alpha^+\in\Gamma/\Gamma_{D^+}\\ x_0\notin\alpha^-D^-\cup\alpha^+D^+\\ 0<d(\alpha^-D^-,\,\alpha^+D^+)\leqslant t}} e^{\int_{[\alpha^-D^-,\,\{x_0\}]}\widetilde F+\int_{[\{x_0\},\,\alpha^+D^+]}\widetilde F}\;\Delta_{(u_{\alpha^-D^-,\{x_0\}})_-}\otimes\Delta_{(v_{\{x_0\},\alpha^+D^+})_+}\otimes ds\,,$$

Equations (5) and (6) are still satisfied.

For all $i, j, k \in \mathbb{N}$, we define $\mathscr{A}_k, \mathscr{A}_i^-, \underline{\mathscr{A}}_j^+, \overline{\mathscr{A}}_j^+$ exactly as previously, so that Equations (9) and (10) are still satisfied. Now, let

$$Z_k = \sum_{(\alpha^-,\alpha^+)\in\mathscr{A}_k} e^{\int_{[\alpha^- D^-,\,\{x_0\}]}\widetilde{F}}\;\psi^-\big((u_{\alpha^- D^-,\{x_0\}})_-\big)\times \\ e^{\int_{[\{x_0\},\,\alpha^+ D^+]}\widetilde{F}}\;\psi^+\big((v_{\{x_0\},\alpha^+ D^+})_+\big)\int\psi^0\,ds\,,$$

so that Equation (8) is still satisfied. Using Theorem 4 with $A^+ = \Gamma\{x_0\}$ instead of Theorem 3, Equation (11) becomes

$$\lim_{s\,\epsilon\to+\infty}\frac{\delta_F\,\|m_F\|}{e^{\delta_F s\,\epsilon}\,\|\sigma^+_{A^-}\|}\sum_{\alpha^-\in\bigcup_{i=1}^s\mathscr{A}_i^-} e^{\int_{[\alpha^- D^-,\,\{x_0\}]}\widetilde{F}}\;\Delta_{v_{\alpha^- D^-,\{x_0\}}} = \widetilde{\sigma}^-_{\{x_0\}}\,.$$

For every $i \in \mathbb{N}$, let us now define

$$a_i = \sum_{\alpha^-\in\mathscr{A}_i^-} e^{\int_{[\alpha^- D^-,\,\{x_0\}]}\widetilde{F}}\;\psi^-\big((u_{\alpha^- D^-,\{x_0\}})_-\big)\,.$$

By the left hand side of Equation (22), we have $(p_-)_*\widetilde{\sigma}^-_{\{x_0\}} = \mu^{F\circ\iota}_{x_0}$, hence Equation (12) becomes

$$\lim_{s\,\epsilon\to+\infty}\frac{\delta_F\,\|m_F\|}{e^{\delta_F s\,\epsilon}\,\|\sigma^+_{A^-}\|}\sum_{i=1}^{s} a_i = \mu^{F\circ\iota}_{x_0}(\psi^-)\,.$$

Similarly, using Theorem 4 with $A^- = \Gamma\{x_0\}$ instead of Theorem 3, since $(p_+)_*\widetilde{\sigma}^+_{\{x_0\}} = \mu^F_{x_0}$ by the right hand side of Equation (22), Equation (14) becomes

$$\lim_{j\,\epsilon\to+\infty}\frac{\delta_F\,\|m_F\|}{(1-e^{-\delta_F})\,e^{\delta_F j\,\epsilon}\,\|\sigma^-_{A^+}\|}\sum_{\alpha^+\in\overline{\mathscr{A}}_j^+} e^{\int_{[\{x_0\},\,\alpha^+ D^+]}\widetilde{F}}\,\psi^+\big((v_{\alpha^- D^-,\{x_0\}})_+\big) = e^{\mathrm{O}(\epsilon)}\,\mu^F_{x_0}(\psi^+)\,.$$

We now define

$$f_\pm : s \mapsto \frac{(1-e^{-\delta_F})\,e^{\mathrm{O}(\epsilon)}\,\|\sigma^-_{A^+}\|\,(\mu^F_{x_0}(\psi^+)\pm\eta)}{\delta_F\,\|m_F\|}\,e^{\delta_F(k-s)\epsilon}\,,$$

$$C_1 = \frac{\|\sigma^+_{A^-}\|\,(\mu^{F\circ\iota}_{x_0}(\psi^-)+\eta)}{\delta_F\,\|m_F\|} \quad\text{and}\quad C_2 = \frac{(1-e^{-\delta_F})\,e^{\mathrm{O}(\epsilon)}\,\|\sigma^-_{A^+}\|\,(\mu^F_{x_0}(\psi^+)+\eta)}{\delta_F\,\|m_F\|}\,.$$

As in the proof of the first claim of Theorem 1, we have

$$Z_k \leqslant \sum_{i\in\mathbb{N}} a_i\Big(\sum_{\alpha^+\in\overline{\mathscr{A}}_{k-i}^+} e^{\int_{[\{x_0\},\,\alpha^+ D^+]}\widetilde{F}}\,\psi^+\big((v_{\{x_0\},\alpha^+ D^+})_+\big)\Big)\int\psi^0\,ds \\ \leqslant \delta_F\,k\,\epsilon\,C_1\,C_2\,e^{\delta_F k\epsilon}\int\psi^0\,ds + \mathrm{O}_{i_0,j_0}(e^{\delta_F k\epsilon})\,,$$

$$\text{and}\quad \limsup_{t\to+\infty}\frac{\delta_F\,\|m_F\|}{t\,e^{\delta_F t}\,\|\sigma^+_{A^-}\|\,\|\sigma^-_{A^+}\|}\,\nu_{3,t}(\psi) \leqslant e^{\mathrm{O}(\epsilon)}\,\frac{1}{\|m_F\|}\,d\mu^{F\circ\iota}_{x_0}\otimes d\mu^F_{x_0}\otimes ds\,(\psi)\,.$$

By [PauPS, Lem. 3.4 (i)], since $\widetilde{F}$ is bounded, there exists a constant $c_2 \in \,]0,1]$ such that for all $\xi \in \partial_\infty \widetilde{M}$ and $x, y \in \widetilde{M}$ with $d(x,y) \leqslant 1$, we have

$$\max\big\{\big|\,C_\xi^{F\circ\iota}(x,y)\,\big|,\ \big|\,C_\xi^{F}(x,y)\,\big|\big\} = \mathrm{O}(d(x,y)^{c_2})\,.$$

Since every element $v \in T^1\widetilde{M}$ of the support of ψ satisfies $d(\pi(v), x_0) \leqslant \frac{\epsilon}{2}$ and by Equation (21) with $x_* = x_0$, we thus have

$$d\mu_{x_0}^{F\circ\iota} \otimes d\mu_{x_0}^{F} \otimes ds\ (\psi) = e^{\mathrm{O}(\epsilon^{c_2})}\ \widetilde{m}_F(\psi)\,. \tag{23}$$

The remainder of the proof of Theorem 4 proceeds as the one of the first claim of Theorem 1, using [BrPP, Theo. 1.5 (1)] instead of [PaP3, Theo. 1] to pass from the weak-star convergence to the narrow convergence. This concludes the proof of Theorem 2.

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