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Effective equidistribution of lattice points in positive characteristic

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Résumé. Étant donné une place ω d'un corps de functions global K sur un corps fini, d'anneau des fonctions affines associé R_ω et de complétion K_ω, le but de ce texte est de donner un résultat d'équidistribution jointe effectif pour les points entiers primitifs renormalisés $(a,b) \in {R_\omega}^2$ du plan ${K_\omega}^2$, et pour les solutions renormalisées de l'équation du pgcd $ax + by = 1$. Les outils principaux sont les techniques de Gorodnik et Nevo sur le comptage de points entiers dans des familles de parties bien arrondies. Ceci donne un résultat plus précis en caractéristique positive d'un resultat de Nevo et du premier auteur sur l'équidistribution des points entiers primitifs de $\mathbb{Z}^2$.

Abstract. Given a place ω of a global function field K over a finite field, with associated affine function ring R_ω and completion K_ω, the aim of this paper is to give an effective joint equidistribution result for renormalized primitive lattice points $(a,b) \in {R_\omega}^2$ in the plane ${K_\omega}^2$, and for renormalized solutions to the gcd equation $ax + by = 1$. The main tools are techniques of Gorodnik and Nevo for counting lattice points in well-rounded families of subsets. This gives a sharper analog in positive characteristic of a result of Nevo and the first author for the equidistribution of the primitive lattice points in $\mathbb{Z}^2$.

1. Introduction

This paper has two motivations. The first one is the following result of Dinaburg-Sinai [7]. Given two coprime positive integers a, b with $a < b$,

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let (x_0, y_0) be a shortest solution (with respect to the supremum norm $\|(x,y)\|_\infty = \max\{|x|,|y|\})$ to the equation $|ax+by| = 1$ with unknown $(x,y) \in \mathbb{Z}^2$. Dinaburg-Sinai proved that the quotients of norms

$$\frac{\|(x_0,y_0)\|_\infty}{\|(a,b)\|_\infty}$$

equidistribute in the interval $[0,1]$ as $\|(a,b)\|_\infty$ tends to $+\infty$. A key idea in the approach of this paper, as well as the one for [19, 18], is due to Risager-Rudnick [24], who translate the above problem in terms of the equidistribution of the real parts of points of an $\mathrm{SL}_2(\mathbb{Z})$-orbit in the Poincaré upper-half plane, and give a solution different from the one of [15] (which uses spectral theory of automorphic forms).

The second motivation is the well-studied Linnik problem of equidistribution on the unit sphere $\mathbb{S}^{n-1}$ of the directions of integral vectors in the Euclidean space $\mathbb{R}^n$ for $n \geqslant 2$. See for instance [8, 27, 9, 10, 12, 4, 13, 1, 2] as well as the joint works of the first author [19, 18]. Let us denote by $\mathbb{Z}^n_{\rm prim}$ the set of primitive integral vectors, by $\mathrm{Leb}_{\mathbb{S}^{n-1}}$ the spherical measure on $\mathbb{S}^{n-1}$ renormalized to be a probability measure, and by Δ_x the unit Dirac mass at any point x in any measurable space. A simple version of this equidistribution phenomenon is the now well-known fact that, as $N \to +\infty$, we have

$$\frac{1}{\mathrm{Card}\{v \in \mathbb{Z}^n_{\rm prim} : \|v\| \leqslant N\}} \sum_{v \in \mathbb{Z}^n_{\rm prim} \,:\, \|v\| \leqslant N} \Delta_{\frac{v}{\|v\|}} \;\overset{*}{\rightharpoonup}\; \mathrm{Leb}_{\mathbb{S}^{n-1}}\;,$$

where $\overset{*}{\rightharpoonup}$ denotes the weak-star convergence of measures, here on the compact space $\mathbb{S}^{n-1}$. Actually, as considered in the above references and pointed out by the referee, a much stronger result holds when considering the primitive integral vectors on a sphere with appropriate large radius (instead of in a ball with large radius). This will be the case also in this paper, though the ultrametric properties makes this restriction to spheres much easier to handle, and without restrictions on the radius. A connection between the two motivations is that when $n = 2$, an integral vector (a,b) is primitive if and only if there exists an integral vector (x,y) with $|ax+by| = 1$.

The goal of this paper is to address analogous questions in local fields with positive characteristic. In this introduction, we describe our results in the special following case.

Let $\mathbb{F}_q$ be a finite field of order a positive power q of some positive prime, and let $K = \mathbb{F}_q(Y)$ be the field of rational functions in one variable Y over $\mathbb{F}_q$. Let $R = \mathbb{F}_q[Y]$ be the ring of polynomials in Y over $\mathbb{F}_q$, let $\widehat{K} = \mathbb{F}_q((Y^{-1}))$ be the non-Archimedean local field of formal Laurent series in Y^{-1} over $\mathbb{F}_q$ and let $\mathscr{O} = \mathbb{F}_q[[Y^{-1}]]$ be the local ring of $\widehat{K}$ (consisting of formal power series in Y^{-1} over $\mathbb{F}_q$). We denote by $|\cdot|$ the complete non-Archimedean absolute value on $\widehat{K}$ such that $|P| = q^{\deg P}$ for every $P \in R$.

We endow $\widehat{K}$ with its Haar measure $\mu_{\widehat{K}}$ standardly normalized so that $\mu_{\widehat{K}}(\mathscr{O}) = 1$, and the quotient $\widehat{K}/R$ with the induced measure $\mu_{\widehat{K}/R}$ and the quotient distance. We also endow the plane $\widehat{K}^{\,2}$ with the product measure and with the supremum norm. We denote by $\mathbb{S}^1_\infty$ the (compact-open) unit sphere of $\widehat{K}^{\,2}$, that we equip with the restriction $\mu_{\mathbb{S}^1_\infty}$ of the product measure.

Given $v = (a, b) \in \widehat{K}^{\,2} - \{(0, 0)\}$, we denote by $\|v\|_\infty = \max\{|a|, |b|\} \in q^{\mathbb{Z}}$ its supremum norm. We denote by $z_v = a$ if $|a| \geqslant |b|$, and $z_v = b$ otherwise, the component of v with maximum absolute value. We also denote by $\breve{v} = \big(a\, Y^{-\log_q \|v\|_\infty}, b\, Y^{-\log_q \|v\|_\infty}\big)$ the vector v canonically renormalised to be in the unit sphere $\mathbb{S}^1_\infty$, of which we think as the *direction* of v.

We let $R^2_{\rm prim}$ denote the set of elements $v = (a, b)$ in the standard R-lattice R^2 of the plane $\widehat{K}^{\,2}$ that are *primitive*, that is, satisfy $aR+bR = R$. Let $w_v = (-y', x')$ be such that (x', y') is a solution to the gcd equation $ax + by = 1$ of v, with unknown $(x, y) \in R^2$. We could for instance take the *shortest* one, that is, the one with the smallest supremum norm (see Section 5 for the existence and uniqueness). We then think of w_v as a normalized "rotated" version of v (or generating the "orthogonal" R-lattice in analogy with [1, 2]). What follows is actually independent of the choice of w_v.

The following result is a joint equidistribution theorem, with error term, for the direction and renormalized gcd solution of the primitive lattice points in the non-Archimedean plane $\widehat{K}^{\,2}$.

Error terms in equidistribution results usually require smoothness properties on test functions. The appropriate smoothness regularity of functions defined on totally disconnected spaces like $\widehat{K}^N$ for $N \in \mathbb{N}$ is the locally constant one. For every metric space E and $\epsilon > 0$, a bounded map $f : E \to \mathbb{R}$ is *ϵ-locally constant* if it is constant on every closed ball of radius ϵ in E. Its *ϵ-locally constant norm* is $\|f\|_\epsilon = \frac{1}{\epsilon} \sup_{x\in E} |f(x)|$.

Theorem 1.1. *For the weak-star convergence of measures on the compact space $\mathbb{S}^1_\infty \times (\widehat{K}/R)$, we have, as $n \to +\infty$,*

$$\frac{1}{q^2(q-1)}\, q^{-2n} \sum_{v\in R^2_{\rm prim}\,:\, \|v\|_\infty = q^n} \Delta_{\breve{v}} \otimes \Delta_{\frac{z_{w_v}}{z_v}+R} \;\overset{*}{\rightharpoonup}\; \mu_{\mathbb{S}^1_\infty} \otimes \mu_{\widehat{K}/R}\,.$$

Furthermore, there exists $\tau \in \,]0, \frac{1}{8}]$ such that for all $\epsilon, \delta > 0$, there is a mutiplicative error term of the form $1 + \mathrm{O}_\delta(q^{2n(-\tau+\delta)}\, \|f\|_\epsilon\, \|g\|_\epsilon)$ when evaluated on pairs (f, g) for all ϵ-locally constant maps $f : \mathbb{S}^1_\infty \to \mathbb{R}$ and $g : \widehat{K}/R \to \mathbb{R}$.

The factor $\frac{1}{q^2(q-1)}\ q^{-2n}$ in front of the above sum is a renormalization factor, needed in order to have a convergence to the natural finite measure on the right hand side (whose total mass $\frac{q^2-1}{q^3}$ will be computed in Section 2.1). The constant τ is described in terms of representation-theoretic data for the locally compact group $\mathrm{SL}_2(\widehat{K})$, but it is not explicit, as it relies in particular on a nonexplicit spectral constant (see the proof of Theorem 4.1).

We will actually prove a more general version of this result, when K is replaced by any (global) function field in one variable over a finite field and when congruence properties are added, see Theorem 4.5. See also Corollary 4.6 for a counting corollary of primitive lattice points.

We begin in Subsection 2.1 by recalling basic facts about functions fields over finite fields. In Subsection 2.2, we define the various closed subgroups of the totally disconnected locally compact group $\mathrm{SL}_2(\widehat{K})$ which will be useful in order to transfer arithmetic information on lattice points in the plane to group-theoretic information. We will also discuss the properties of their Haar measures. In Section 3, we give a precise correspondence between primitive lattice points and elements in the Nagao-Weyl modular group $\mathrm{SL}_2(\mathbb{F}_q[Y])$. We adapt in Section 4 the results of Gorodnik-Nevo [16] (building on works of [11, 14]) on counting lattice points in well-rounded subsets of semi-simple Lie groups, and check that a family of nice compact-open subsets coming from a mixture of the LU and Iwasawa decompositions of $\mathrm{SL}_2(\widehat{K})$ is indeed well-rounded. Finally, in Section 5, we give an application to the distribution properties of the continued fraction expansions of elements in $\mathbb{F}_q(Y)$, thus giving an analogue to the result of Dinaburg-Sinai in [7] described in the beginning of this introduction.

2. Background on function fields and their modular groups

2.1. Global function fields. We refer for instance to [17, 25] and [5, Chap. 14] for the content of this Section.

Let $\mathbb{F}_q$ be a finite field of order q, where q is a positive power of a positive prime. Let K be a (global) *function field* over $\mathbb{F}_q$, that is, the function field of a geometrically connected smooth projective curve $\mathbf{C}$ over $\mathbb{F}_q$, or equivalently an extension of $\mathbb{F}_q$ of transcendance degree 1, in which $\mathbb{F}_q$ is algebraically closed. We denote by $\mathbf{g}$ the genus of the curve $\mathbf{C}$.

There is a bijection between the set of closed points of $\mathbf{C}$ and the set of (normalised discrete) valuations ω of its function field K, where the valuation of a given element $f \in K$ is the order of the zero or the opposite of the order of the pole of f at the given closed point. We fix such a valuation ω from now on.

We denote by K_ω the completion of K for the valuation ω, and by

$$\mathscr{O}_\omega = \{x \in K_\omega \;:\; \omega(x) \geqslant 0\}$$

the valuation ring of (the unique extension to K_ω) of ω. Let us fix a uniformiser $\pi_\omega \in K_\omega$ of ω, that is, an element in K_ω with $\omega(\pi_\omega) = 1$. We denote by q_ω the order of the residual field $\mathscr{O}_\omega/\pi_\omega\mathscr{O}_\omega$ of ω, which is a (possibly proper) power of q. We normalize the absolute value associated with ω as usual: for every $x \in K_\omega$, we have the equality

$$|\,x\,|_\omega = (q_\omega)^{-\omega(x)}\ .$$

Finally, let R_ω denote the affine algebra of the affine curve $\mathbf{C} - \{\omega\}$, consisting of the elements of K whose only poles (if any) are at the closed point ω of $\mathbf{C}$. Its field of fractions is equal to K.

The case in the introduction corresponds to $\mathbf{C} = \mathbb{P}^1$ (so that $\mathbf{g} = 0$) and $\omega = \omega_\infty$ the valuation associated with the point at infinity $[1:0]$. Then

- $K = \mathbb{F}_q(Y)$ is the field of rational functions in one variable Y over $\mathbb{F}_q$,
- ω_∞ is the valuation defined, for all $P, Q \in \mathbb{F}_q[Y]$, by

$$\omega_\infty(P/Q) = \deg Q - \deg P\ .$$

- $R_{\omega_\infty} = \mathbb{F}_q[Y]$ is the (principal) ring of polynomials in one variable Y over $\mathbb{F}_q$,
- $K_{\omega_\infty} = \mathbb{F}_q((Y^{-1}))$ is the field of formal Laurent series in one variable Y^{-1} over $\mathbb{F}_q$,
- $\mathscr{O}_{\omega_\infty} = \mathbb{F}_q[[Y^{-1}]]$ is the ring of formal power series in one variable Y^{-1} over $\mathbb{F}_q$, $\pi_{\omega_\infty} = Y^{-1}$ is the usual choice of a uniformizer, and $q_{\omega_\infty} = q$.

Recall (see for instance [28, II.2, Notations]) that R_ω is a Dedekind ring, not principal in general. We have (see for instance [5, Eq. (14.2)]) that

$$R_\omega \cap \mathscr{O}_\omega = \mathbb{F}_q\ . \tag{2.1}$$

Lemma 2.1. *For all elements $a, b, c, d \in R_\omega - \mathbb{F}_q$ such that $ad - bc = 1$ and $|\,a\,|_\omega \geqslant |\,b\,|_\omega$, we have $|\,c\,|_\omega \geqslant |\,d\,|_\omega$.*

Proof. The equality $ad - bc = 1$ implies that $\omega(ad - bc) = 0$. We have $\omega(ad) < 0$ and $\omega(bc) < 0$ since the only elements of R_ω which have nonnegative valuations are the elements in the ground field $\mathbb{F}_q$ by Equation (2.1). Therefore $\omega(ad) = \omega(bc)$ and

$$\omega(c) - \omega(d) = \omega(a) - \omega(b)\ .$$

The left hand side is nonpositive, since the right hand side is. This proves the result. □

The (absolute) *norm* of a nonzero ideal I of the ring R_ω is defined by $N(I) = [R_\omega : I] = |R_\omega/I|$. *Dedekind's zeta function* of K is (see for instance [17, §7.8] or [25, §5])

$$\zeta_K(s) = \sum_I \frac{1}{N(I)^s}$$

where the summation runs over the nonzero ideals I of R_ω. By (for instance) [25, §5], it is a rational function of q^{-s} with simple poles at $s = 0, s = 1$. In particular, when $K = \mathbb{F}_q(Y)$, then (see [25, Theo. 5.9] with $\mathbf{g} = 0$)

$$\zeta_{\mathbb{F}_q(Y)}(-1) = \frac{1}{(q-1)(q^2-1)}\ . \tag{2.2}$$

We denote by

$$R^2_{\omega,\mathrm{prim}} = \{(a,b) \in {R_\omega}^2 : aR_\omega + bR_\omega = R_\omega\}$$

the set of *primitive* elements in the lattice ${R_\omega}^2$ in the plane ${K_\omega}^2$. Note that since R_ω is not always principal, not every point of ${R_\omega}^2$ is an R_ω-multiple of an element of $R^2_{\omega,\mathrm{prim}}$.

For every $v \in {K_\omega}^2 - \{(0,0)\}$, we write $v = (x_v, y_v)$, and define

$$z_v = \begin{cases} x_v & \text{if } |\,x_v|_\omega \geqslant |\,y_v|_\omega \\ y_v & \text{if } |\,x_v|_\omega < |\,y_v|_\omega \end{cases} \quad \text{and} \quad z'_v = \begin{cases} y_v & \text{if } |\,x_v|_\omega \geqslant |\,y_v|_\omega \\ x_v & \text{if } |\,x_v|_\omega < |\,y_v|_\omega\ , \end{cases} \tag{2.3}$$

as well as

$$\|v\|_\omega = \max\{|\,x_v|_\omega, |\,y_v|_\omega\}, \quad v^\perp = (y_v, -x_v) \quad \text{and} \quad \breve{v} = {\pi_\omega}^{\log_{q_\omega}(\|v\|_\omega)}\, v\ . \tag{2.4}$$

We denote the unit sphere in the plane ${K_\omega}^2$ endowed with the supremum norm $\|\cdot\|_\omega$ by

$$\mathbb{S}^1_\omega = \{v \in {K_\omega}^2 : \|v\|_\omega = 1\}\ .$$

Note that $v^\perp$ has the same norm as v and belongs to $R^2_{\omega,\mathrm{prim}}$ if v does, and that $\mathbb{S}^1_\omega = \{\,\breve{v} : v \in {K_\omega}^2 - \{(0,0)\}\,\}$. We think of $\breve{v}$ as the *direction* (or renormalisation) of v, it is a preferred element in the intersection of the unit sphere $\mathbb{S}^1_\omega$ with the vector line defined by v.

We denote by $\|\mu\|$ the total mass of any finite measure μ. We denote by μ_{K_ω} the Haar measure of the (abelian) locally compact topological group $(K_\omega, +)$, normalised so that $\mu_{K_\omega}(\mathscr{O}_\omega) = 1$. This measure scales as follows under multiplication: for all $\lambda, x \in K_\omega$, we have

$$d\mu_{K_\omega}(\lambda x) = |\lambda|_\omega\, d\mu_{K_\omega}(x)\ . \tag{2.5}$$

We denote by μ_{K_ω/R_ω} the induced Haar measure on the compact additive topological group K_ω/R_ω. Using the above scaling for the first equation and [5, Lem. 14.4] for the second one, for every $m \in \mathbb{N}$, we have the equality

$$\mu_{K_\omega}(\pi_\omega^m \mathscr{O}_\omega) = q_\omega^{-m} \quad \text{and} \quad \|\, \mu_{K_\omega/R_\omega} \| = q^{\mathbf{g}-1} \;. \tag{2.6}$$

We endow ${K_\omega}^2$ with the product $\mu_{K_\omega} \otimes \mu_{K_\omega}$ of the Haar measures on each factor. Note that the unit ball of ${K_\omega}^2$ is ${\mathscr{O}_\omega}^2$, so that for every $k \in \mathbb{Z}$, the measure of any ball in ${K_\omega}^2$ of radius q_ω^k, which is of the form $v + \pi_\omega^{-k} {\mathscr{O}_\omega}^2$ for some $v \in {K_\omega}^2$, is equal to q_ω^{2k}.

We denote by $\mu_{\mathbb{S}^1_\omega}$ the restriction to the compact-open subset $\mathbb{S}^1_\omega$ of ${K_\omega}^2$ of the product measure. Since

$$\mu_{K_\omega}(\mathscr{O}_\omega^\times) = \mu_{K_\omega}(\mathscr{O}_\omega - \pi_\omega \mathscr{O}_\omega) = 1 - q_\omega^{-1} \tag{2.7}$$

by Equation (2.6), and since $\mathbb{S}^1_\omega = (\mathscr{O}_\omega^\times \times \mathscr{O}_\omega) \cup (\mathscr{O}_\omega \times \mathscr{O}_\omega^\times)$, the total mass of $\mu_{\mathbb{S}^1_\omega}$ is

$$\|\mu_{\mathbb{S}^1_\omega}\| = (1 - q_\omega^{-1}) + (1 - q_\omega^{-1}) - (1 - q_\omega^{-1})^2 = \frac{q_\omega^2 - 1}{q_\omega^2} \;. \tag{2.8}$$

2.2. The modular group. The aim of this section is to introduce the various closed subgroups of the special linear group of the plane ${K_\omega}^2$ that will be useful in order to transfer arithmetic information concerning lattice points in ${R_\omega}^2$ into group-theoretic information. We will also discuss the properties of their Haar measures.

Let $G = \mathrm{SL}_2(K_\omega)$, which is a totally disconnected locally compact topological group. The *modular group* $\Gamma = \mathrm{SL}_2(R_\omega)$ is a non-uniform lattice in G. When $\mathbf{C} = \mathbb{P}^1$ and $\omega = \omega_\infty$ as in the introduction, then up to finite index, it is called *Nagao's lattice* (see [22, 30]). For every nonzero ideal I of R_ω, we denote by $\Gamma_0[I]$ the *Hecke congruence subgroup* of Γ modulo I:

$$\Gamma_0[I] = \left\{ \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) \in \Gamma : b \in I \right\} .$$

By [5, Lem. 16.5], the index of $\Gamma_0[I]$ in Γ is

$$\big[\, \Gamma : \Gamma_0[I] \,\big] = N(I) \prod_{\mathfrak{p} | I} \Big(1 + \frac{1}{N(\mathfrak{p})}\Big) \;. \tag{2.9}$$

where the product ranges over the prime factors $\mathfrak{p}$ of the ideal I.

For every commutative ring S, we denote by $\mathscr{M}_2(S)$ the S-module of 2×2 matrices with coefficients in S. For every closed subgroup H of G, we denote by $H(\mathscr{O}_\omega)$ the compact-open subgroup $H \cap \mathscr{M}_2(\mathscr{O}_\omega)$ of H, and by μ_H the (left) Haar measure of H normalized so that

$$\mu_H(H(\mathscr{O}_\omega)) = 1 \;.$$

Note that G is unimodular. For every lattice Γ' of G, we denote by $\mu_{\Gamma' \backslash G}$ the measure on $\Gamma' \backslash G$ induced by μ_G. By Exercice 2e) in [28, II.2.3] (which normalizes the Haar measure of G so that the mass of $G(\mathscr{O}_\omega)$ is $q_\omega - 1$), the total mass of $\mu_{\Gamma\backslash G}$ is

$$\|\mu_{\Gamma\backslash G}\| = \zeta_K(-1)\ . \tag{2.10}$$

Let Z be the diagonal subgroup of G, let U^- and U^+ be its lower and upper unipotent triangular subgroups, and let $P^- = U^- Z$ be its lower triangular Borel subgroup. We also consider the Cartan subgroup $A = \big\{\big(\begin{smallmatrix}\pi_\omega^n & 0\\ 0 & \pi_\omega^{-n}\end{smallmatrix}\big) : n\in\mathbb{Z}\big\}$ of G, whose centralizer in G is Z.

Since $A(\mathscr{O}_\omega) = \{\mathrm{id}\}$ has measure one for the measure μ_A with the above normalisation, the Haar measure μ_A on A is exactly the counting measure:

$$\mu_A = \sum_{g\in A} \Delta_g\ . \tag{2.11}$$

The maps from K_ω to U^- and U^+, defined by $\alpha\mapsto \mathfrak{u}^-(\alpha) = \big(\begin{smallmatrix}1 & 0\\ \alpha & 1\end{smallmatrix}\big)$ and $\alpha\mapsto\mathfrak{u}^+(\alpha) = \big(\begin{smallmatrix}1 & \alpha\\ 0 & 1\end{smallmatrix}\big)$ respectively, are homeomorphisms (and even abelian group isomorphisms). They send $\mathscr{O}_\omega$ to $U^\pm(\mathscr{O}_\omega)$, and the Haar measure of $(K_\omega,+)$ to the Haar measure of $U^\pm$: namely, for (almost) all $\alpha\in K_\omega$, we have

$$d\mu_{U^\pm}(\mathfrak{u}^\pm(\alpha)) = d\mu_{K_\omega}(\alpha)\ . \tag{2.12}$$

Similarly, the map from the multiplicative group $K_\omega^\times$ to the diagonal group Z, defined by $\alpha\mapsto\big(\begin{smallmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{smallmatrix}\big)$, is a homeomorphism (and even an abelian group isomorphism). It sends $\mathscr{O}_\omega^\times$ to $Z(\mathscr{O}_\omega)$, and the restriction to $K_\omega^\times$ of the Haar measure μ_{K_ω} to a multiple of the Haar measure of Z: namely, for (almost) all $\alpha\in K_\omega^\times$, by Equation (2.7), we have

$$\frac{q_\omega - 1}{q_\omega}\ d\mu_Z\big(\big(\begin{smallmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{smallmatrix}\big)\big) = d\mu_{K_\omega}(\alpha)\ . \tag{2.13}$$

Let

$$\mathbb{S}^{1,\sharp}_\omega = \big\{v\in\mathbb{S}^1_\omega\ :\ |\,x_v|_\omega \geqslant |\,y_v|_\omega\big\} = \mathscr{O}_\omega^\times\times\mathscr{O}_\omega\ ,$$

which is a compact-open subset of the plane ${K_\omega}^2$. The map from $\mathbb{S}^{1,\sharp}_\omega$ to $P^-(\mathscr{O}_\omega)$ defined by $(\alpha,\beta)\mapsto\mathfrak{p}^-(\alpha,\beta) = \big(\begin{smallmatrix}\alpha & 0\\ \beta & \alpha^{-1}\end{smallmatrix}\big)$ is a homeomorphism. Let us prove that it sends the restriction to $\mathbb{S}^{1,\sharp}_\omega$ of the measure $\mu_{\mathbb{S}^1_\omega}$ to a multiple of the Haar measure of $P^-(\mathscr{O}_\omega)$. First note that for (almost) every $\alpha\in\mathscr{O}_\omega^\times$ and $\beta\in\mathscr{O}_\omega$, since $|\alpha|_\omega = 1$, the action by conjugation of $\big(\begin{smallmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{smallmatrix}\big)$ on $U^-(\mathscr{O}_\omega)$, which satisfies $\big(\begin{smallmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{smallmatrix}\big)\mathfrak{u}^-(\beta)\big(\begin{smallmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{smallmatrix}\big)^{-1} = \mathfrak{u}^-(\alpha^2\beta)$, preserves the Haar measure $\mu_{U^-(\mathscr{O}_\omega)}$ by Equations (2.12) and (2.5). Hence the measure $d\nu(\mathfrak{p}^-(\alpha,\beta)) = d\mu_{U^-(\mathscr{O}_\omega)}(\mathfrak{u}^-(\beta))\ d\mu_{Z(\mathscr{O}_\omega)}\big(\big(\begin{smallmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{smallmatrix}\big)\big)$ is a Haar measure on $P^-(\mathscr{O}_\omega)$. Since $\mu_{P^-(\mathscr{O}_\omega)}$, $\mu_{U^-(\mathscr{O}_\omega)}$ and $\mu_{Z(\mathscr{O}_\omega)}$ are probability measures, we have (this will be extended in Lemma 2.2)

$$d\mu_{P^-(\mathscr{O}_\omega)}(\mathfrak{p}^-(\alpha,\beta)) = d\mu_{U^-(\mathscr{O}_\omega)}(\mathfrak{u}^-(\beta))\ d\mu_{Z(\mathscr{O}_\omega)}\big(\big(\begin{smallmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{smallmatrix}\big)\big)\ .$$

By Equations (2.12) and (2.13), we thus have, for (almost) every $\alpha \in \mathscr{O}_\omega^\times$ and $\beta \in \mathscr{O}_\omega$

(2.14)
$$d\mu_{P^-(\mathscr{O}_\omega)}(\mathfrak{p}^-(\alpha,\beta)) = \frac{q_\omega}{q_\omega - 1}\, d\mu_{K_\omega}(\alpha)\, d\mu_{K_\omega}(\beta) = \frac{q_\omega}{q_\omega - 1}\, d\mu_{\mathbb{S}^1_\omega}(\alpha,\beta)\ .$$

We will need the following refined LU decomposition of elements of the special linear group G. Let $g = \left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right) \in G$ with $\alpha \neq 0$. Then there are unique elements $\mathfrak{u}_g^\pm \in U^\pm$, $\mathfrak{m}_g \in Z(\mathscr{O}_\omega)$ and $\mathfrak{a}_g \in A$ such that

$$g = \mathfrak{u}_g^-\ \mathfrak{m}_g\ \mathfrak{a}_g\ \mathfrak{u}_g^+\ .$$

Indeed, the existence of such a decomposition follows by taking

(2.15)
$$\mathfrak{u}_g^- = \begin{pmatrix}1 & 0\\ \frac{\beta}{\alpha} & 1\end{pmatrix},\ \mathfrak{u}_g^+ = \begin{pmatrix}1 & \frac{\gamma}{\alpha}\\ 0 & 1\end{pmatrix},$$
$$\mathfrak{m}_g = \begin{pmatrix}\alpha\pi_\omega^{-\omega(\alpha)} & 0\\ 0 & \alpha^{-1}\pi_\omega^{\omega(\alpha)}\end{pmatrix},\ \mathfrak{a}_g = \begin{pmatrix}\pi_\omega^{\omega(\alpha)} & 0\\ 0 & \pi_\omega^{-\omega(\alpha)}\end{pmatrix}.$$

In order to prove the uniqueness of this decomposition, if $g = \mathfrak{u}^-\ \mathfrak{m}\ \mathfrak{a}\ \mathfrak{u}^+$ where $\mathfrak{u}^\pm \in U^\pm$, $\mathfrak{m} \in Z(\mathscr{O}_\omega)$ and $\mathfrak{a} \in A$ is another such writing, then the equality

$$(\mathfrak{u}^-)^{-1}\ \mathfrak{u}_g^- = \mathfrak{m}\ \mathfrak{a}\ \mathfrak{u}^+(\mathfrak{m}_g\ \mathfrak{a}_g\ \mathfrak{u}_g^+)^{-1}$$

between a unipotent lower triangular matrix and an upper triangular matrix implies that $\mathfrak{u}^- = \mathfrak{u}_g^-$ and that $(\mathfrak{m}\ \mathfrak{a})^{-1}\ \mathfrak{m}_g\ \mathfrak{a}_g = \mathfrak{u}^+(\ \mathfrak{u}_g^+)^{-1}$. This last equality between a diagonal matrix and a unipotent upper triangular matrix gives $\mathfrak{u}^+ = \mathfrak{u}_g^+$ and $\mathfrak{m}\ \mathfrak{a} = \mathfrak{m}_g\ \mathfrak{a}_g$, which in turns give $\mathfrak{m} = \mathfrak{m}_g$ and $\mathfrak{a} = \mathfrak{a}_g$ since $A \cap Z(\mathscr{O}_\omega) = \{\mathrm{id}\}$. We also consider

(2.16)
$$\mathfrak{p}_g = \mathfrak{u}_g^-\ \mathfrak{m}_g = \begin{pmatrix}\alpha\pi_\omega^{-\omega(\alpha)} & 0\\ \beta\pi_\omega^{-\omega(\alpha)} & \alpha^{-1}\pi_\omega^{\omega(\alpha)}\end{pmatrix} \in P^-\ .$$

Note that if $\omega(\alpha) \leqslant \omega(\beta)$, or equivalently if $|\,\alpha\,|_\omega \geqslant |\,\beta\,|_\omega$, then we have $\mathfrak{p}_g \in P^-(\mathscr{O}_\omega) = U^-(\mathscr{O}_\omega)Z(\mathscr{O}_\omega)$, so that $\mathfrak{p}_g$ belongs to the maximal compact subgroup $G(\mathscr{O}_\omega)$ of G. In particular, the writing $g = \mathfrak{p}_g\ \mathfrak{a}_g\ \mathfrak{u}_g^+$ is an Iwasawa decomposition of g.

We conclude this section by providing the expression for the Haar measure of G in the refined LU decomposition. The composition map from the product $U^- \times Z(\mathscr{O}_\omega) \times A \times U^+$ to G is an homeomorphism onto an open-dense subset with full Haar measure in G, and the following result says that the Haar measure of G is absolutely continuous with respect to the product of the Haar measures of the factors. The main point of its proof is to compute the Radon-Nikodym derivative. We denote by $\chi : Z \to K_\omega^\times$ the standard character $\left(\begin{smallmatrix}\alpha^{-1} & 0\\ 0 & \alpha\end{smallmatrix}\right) \mapsto \alpha$. It is well known (by the standard action of a split torus on its root groups) that for all $\mathfrak{z} \in Z$ and $\alpha \in K_\omega$, we have

$$\mathfrak{z}\,\mathfrak{u}^-(\alpha)\,\mathfrak{z}^{-1} = \mathfrak{u}^-(\,\chi(\mathfrak{z})^2\,\alpha) \quad\text{and}\quad \mathfrak{z}^{-1}\,\mathfrak{u}^+(\alpha)\,\mathfrak{z} = \mathfrak{u}^+(\,\chi(\mathfrak{z})^2\,\alpha)\,. \tag{2.17}$$

Lemma 2.2. *For μ_G-almost every $g \in G$, we have*

$$d\mu_G(g) = \frac{q_\omega}{q_\omega+1}\;|\chi(\mathfrak{a}_g)|_\omega^{-2}\;d\mu_{U^-}(\mathfrak{u}_g^-)\;d\mu_{Z(\mathscr{O}_\omega)}(\mathfrak{m}_g)\;d\mu_A(\mathfrak{a}_g)\;d\mu_{U^+}(\mathfrak{u}_g^+)\,.$$

Proof. By [20, §III.1], since G and U^+ are unimodular, there exists a constant $c_1 > 0$ such that $d\mu_G(p^-u^+) = c_1\; d\mu_{P^-}(p^-)\; d\mu_{U^+}(u^+)$ for (almost) every $p^- \in P^-$ and $u^+ \in U^+$, using the product map $P^- \times U^+ \to G$. Note that U^- is unimodular and that Z normalizes U^- as made precise in Equation (2.17). Hence there exists a constant $c_2 > 0$ such that, for (almost) every $u^- \in U^-$ and $z \in Z$, we have

$$|\chi(z)|_\omega^{-2}\;d\mu_{U^-}(u^-)\;d\mu_Z(z) = c_2\;d\mu_{P^-}(u^-z)\,.$$

This indeed follows by uniqueness from the fact that the left hand side defines a left Haar measure on P^- using the product map $(u^-, z) \mapsto u^-z$ from $U^- \times Z$ to P^- (which is an homeomorphism), by Equations (2.5) and (2.12). Since $Z = Z(\mathscr{O}_\omega)A$ with A and $Z(\mathscr{O}_\omega)$ abelian and commuting, this proves that there exists a constant $c_3 > 0$ such that

$$d\mu_G(g) = c_3\;|\chi(\mathfrak{a}_g)|_\omega^{-2}\;d\mu_{U^-}(\mathfrak{u}_g^-)\;d\mu_{Z(\mathscr{O}_\omega)}(\mathfrak{m}_g)\;d\mu_A(\mathfrak{a}_g)\;d\mu_{U^+}(\mathfrak{u}_g^+)\,. \tag{2.18}$$

In order to compute the constant c_3, we evaluate the measures on both sides on the compact-open subgroup

$$H = \{\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right) \in G(\mathscr{O}_\omega) : \alpha, \delta \in 1 + \pi_\omega\mathscr{O}_\omega,\; \beta, \gamma \in \pi_\omega\mathscr{O}_\omega\}\,.$$

This group, being the kernel of the reduction modulo $\pi_\omega\mathscr{O}_\omega$, has index $|\operatorname{SL}_2(\mathbb{F}_{q_\omega})| = q_\omega(q_\omega^2 - 1)$ in $G(\mathscr{O}_\omega)$. Since $\mu_G(G(\mathscr{O}_\omega)) = 1$, the group H has Haar measure $\mu_G(H) = \frac{1}{q_\omega(q_\omega^2-1)}$. By Equation (2.15), the refined LU decomposition identifies H with the product space $H_{U^-} \times H_Z \times H_{U^+}$ in $U^- \times Z \times U^+$, where

$$H_{U^-} = \{\left(\begin{smallmatrix}1 & 0\\ \beta & 1\end{smallmatrix}\right) : \beta \in \pi_\omega\mathscr{O}_\omega\},\; H_Z = \{\left(\begin{smallmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{smallmatrix}\right) : \alpha \in 1 + \pi_\omega\mathscr{O}_\omega\},$$

$$H_{U^+} = \{\left(\begin{smallmatrix}1 & \gamma\\ 0 & 1\end{smallmatrix}\right) : \gamma \in \pi_\omega\mathscr{O}_\omega\}\,.$$

These groups have index respectively q_ω, $\big|\mathscr{O}_\omega^\times/(1+\pi_\omega\mathscr{O}_\omega)\big| = |\mathbb{F}_{q_\omega}^\times| = q_\omega - 1$ and q_ω in $U^-(\mathscr{O}_\omega)$, $Z(\mathscr{O}_\omega)$ and $U^+(\mathscr{O}_\omega)$. Hence the measure of H for the measure on the right hand side of Equation (2.18) is equal to $\frac{c_3}{q_\omega^2(q_\omega-1)}$. This implies that $c_3 = \frac{q_\omega}{q_\omega+1}$, as wanted. □

3. Primitive lattice points seen in the modular group

Recalling the relevant notation from Subsection 2.1, let K be a function field over $\mathbb{F}_q$, let ω be a (normalized discrete) valuation of K, let K_ω be the associated completion of K, and let R_ω be the affine function ring associated with ω. The aim of this section is to naturally associate elements in the modular group $\Gamma = \mathrm{SL}_2(R_\omega)$ to primitive lattice points in ${R_\omega}^2$.

We start by introducing subsets of the plane ${K_\omega}^2$ and of the group $G = \mathrm{SL}_2(K_\omega)$ which will be technically useful. Let

$$G^\sharp = \Big\{ \big(\begin{smallmatrix}\alpha & \gamma \\ \beta & \delta\end{smallmatrix}\big) \in G \;:\; |\,\alpha\,|_\omega \geqslant |\,\beta\,|_\omega \Big\} \quad \text{and} \quad \Gamma^\sharp = \Gamma \cap G^\sharp\,,$$

$$K_\omega^{2,\sharp} = \big\{ (a,b) \in {K_\omega}^2 \;:\; |\,a\,|_\omega \geqslant |\,b\,|_\omega \big\} \quad \text{and} \quad R^{2,\sharp}_{\omega,\mathrm{prim}} = R^2_{\omega,\mathrm{prim}} \cap K_\omega^{2,\sharp}\,.$$

We identify any element $v = (x,y) \in {K_\omega}^2$ with the column matrix $v = \big(\begin{smallmatrix} x \\ y\end{smallmatrix}\big)$ of its components, and thus write 2×2 matrices of elements of K_ω as 1×2 matrices of elements of ${K_\omega}^2$. For all measurable subsets Θ of $\mathbb{S}^1_\omega$ and $\mathscr{D}'$ of K_ω, and for every $n \in \mathbb{Z}$, let

$$\begin{aligned} P^-_\Theta &= \big\{ \begin{pmatrix} v' & w'\end{pmatrix} \in P^-(\mathscr{O}_\omega) : v' \in \Theta \big\},\\ A_n &= \big\{ \big(\begin{smallmatrix} \pi_\omega^{-n} & 0 \\ 0 & \pi_\omega^n\end{smallmatrix}\big)\big\} \subset A,\\ U^+_{\mathscr{D}'} &= \big\{ \big(\begin{smallmatrix} 1 & \gamma \\ 0 & 1\end{smallmatrix}\big) \in U^+ : \gamma \in \mathscr{D}' \big\}\,. \end{aligned}$$

By Lemma 2.2 and the various explicitations of Haar measures in Equations (2.14), (2.11) and (2.12), we have

$$\begin{aligned} \mu_G(P^-_\Theta A_n U^+_{\mathscr{D}'}) &= \frac{q_\omega}{q_\omega+1}\,\frac{q_\omega}{q_\omega-1}\,\mu_{\mathbb{S}^1_\omega}(\Theta)\,\big(|\pi_\omega^n|_\omega{}^{-2}\big)\,\mu_{K_\omega}(\mathscr{D}')\\ &= \frac{q_\omega^{2n+2}}{q_\omega^2-1}\,\mu_{\mathbb{S}^1_\omega}(\Theta)\,\mu_{K_\omega}(\mathscr{D}')\,. \end{aligned} \tag{3.1}$$

The following result gives a precise 1-to-1 correspondence between primitive lattice points in $R^{2,\sharp}_{\omega,\mathrm{prim}}$ and appropriate matrices in the modular group Γ.

Proposition 3.1. *Let $\mathscr{D}$ be a fixed (strict) fundamental domain for the lattice R_ω acting by translations on K_ω. There exists a unique bijection from $R^{2,\sharp}_{\omega,\mathrm{prim}}$ to $\Gamma^\sharp \cap (P^- U^+_{\mathscr{D}})$ of the form $v \mapsto \gamma_v = \begin{pmatrix} v & w_v\end{pmatrix}$ (where w_v will be defined in the following proof) such that for every n in $\mathbb{Z}$, for all measurable subsets Θ of $\mathbb{S}^1_\omega$ and $\mathscr{D}'$ of $\mathscr{D}$, and for every nonzero ideal I of R_ω, the following two assertions are equivalent:*

1. *the lattice point v satisfies $\|v\|_\omega = q_\omega^n$, $y_v \in I$, $\check{v} \in \Theta$ and $\frac{x_{w_v}}{x_v} \in \mathscr{D}'$,*
2. *the modular matrix γ_v belongs to the Hecke congruence subgroup $\Gamma_0[I]$ and satisfies $\gamma_v \in P^-_\Theta\, A_n\, U^+_{\mathscr{D}'}$.*

Proof. Let $v = (a, b) \in R^{2,\sharp}_{\omega,\mathrm{prim}}$. In particular $a \neq 0$ and $\|v\|_\omega = |\,a\,|_\omega$. Let us define

$$\mathrm{Sol}_{a,b} = \{(x, y) \in {R_\omega}^2 : ax + by = 1\}\ ,$$

which is the set of solutions in ${R_\omega}^2$ to the equation $ax + by = 1$.

Given $w_0 = (x_0, y_0) \in \mathrm{Sol}_{a,b}$, we claim that

$$\mathrm{Sol}_{a,b} = \{w_0 + \lambda\, v^\perp : \lambda \in R_\omega\}\ ,$$

where $w \mapsto w^\perp$ is defined in Section 2.1. Indeed, we clearly have

$$\{w_0 + \lambda\, v^\perp\ :\ \lambda \in R_\omega\} \subset \mathrm{Sol}_{a,b}\ .$$

Conversely, let $(x, y) \in \mathrm{Sol}_{a,b}$ be a solution different from (x_0, y_0). We have $a(x - x_0) = b(y_0 - y)$. We may assume that $b \neq 0$, since otherwise $a \in R_\omega^\times$ and $v^\perp = (0, -a)$ so that the result is clear. Then $x \neq x_0$ and $y \neq y_0$, so that the nonzero principal ideal (a), being coprime with the principal ideal (b) in the Dedekind ring R_ω, divides the principal ideal generated by $y_0 - y$, and $y - y_0$ is a multiple of $-a$, which implies that $x - x_0$ is the same multiple of b.

Let w_v be the unique element of ${R_\omega}^2$ such that $(w_v)^\perp$ is the unique element of $\mathrm{Sol}_{a,b}$ with $\frac{x_{w_v}}{a} \in \mathscr{D}$. As $x_{w_v} = -\,y_{(w_v)^\perp}$, this is possible since, by the above, the subset of K_ω consisting of the elements $-\frac{y}{a}$, where y varies over the second components of elements of $\mathrm{Sol}_{a,b}$, is exactly one orbit by translation under R_ω (without repetition).

Let us define $\gamma_v = \begin{pmatrix} v & w_v \end{pmatrix} = \begin{pmatrix} a & x_{w_v} \\ b & y_{w_v} \end{pmatrix}$. We have $\gamma_v \in \Gamma$ since $(w_v)^\perp$ belongs to $\mathrm{Sol}_{a,b}$ so that $\det \gamma_v = 1$. Furthermore, we have $\gamma_v \in \Gamma^\sharp$ since $v \in R^{2,\sharp}_{\omega,\mathrm{prim}}$. Let $g = \gamma_v$. By Equation (2.16), the first column of $\mathfrak{p}_g$ is $(a\pi_\omega^{-\omega(a)}, b\pi_\omega^{-\omega(a)}) = \pi_\omega^{\log_{q_\omega} |a|_\omega}\, v = \check{v}$, so that $\mathfrak{p}_g \in P^-_\Theta$ if and only if $\check{v} \in \Theta$. Since $\|v\|_\omega = |\,a\,|_\omega = q_\omega^{-\omega(a)}$ and by Equation (2.15), we have $\mathfrak{a}_g \in A_n$ if and only if $\|v\|_\omega = q_\omega^n$. Again by Equation (2.15), we have $\mathfrak{u}_g^+ \in U^+_{\mathscr{D}'}$ if and only if $\frac{x_{w_v}}{x_v} = \frac{x_{w_v}}{a} \in \mathscr{D}'$.

The map $v \mapsto \gamma_v$ from $R^{2,\sharp}_{\omega,\mathrm{prim}}$ to $\Gamma^\sharp$ is clearly injective. Its image is $\Gamma^\sharp \cap (P^- U^+_{\mathscr{D}})$, since if $\begin{pmatrix} v & w \end{pmatrix} \in \Gamma^\sharp \cap (P^- U^+_{\mathscr{D}})$ and $v = (a, b)$, then v belongs to $R^{2,\sharp}_{\omega,\mathrm{prim}}$ and $w^\perp$ is an element of $\mathrm{Sol}_{a,b}$ such that by Equation (2.15) we have $-\frac{y_{w^\perp}}{a} = \frac{x_w}{a} \in \mathscr{D}$, hence $w = w_v$ by uniqueness. We clearly have $y_v = b \in I$ if and only if $\gamma_v \in \Gamma_0[I]$. This proves the result. $\square$

4. Joint equidistribution of primitive lattice points

The aim of this section is to prove the main result of this paper, Theorem 4.5, establishing the effective joint equidistribution of directions and renormalized solutions to the associated gcd equations for primitive lattice points, generalizing Theorem 1.1 in the introduction to any function field.

The main tool for this result is an adaptation of two theorems of Gorodnik and Nevo [16], that we now state, after the necessary definitions.

Let $\mathbf{G}'$ be an absolutely connected and simply connected semi-simple algebraic group over K_ω, which is almost K_ω-simple. Let $G' = \mathbf{G}'(K_\omega)$ be the locally compact group of K_ω-points of $\mathbf{G}'$. Let Γ' be a non-uniform[1] lattice in G', and let $\mu_{G'}$ be any (left) Haar measure of G'. Note that $G' = G$ and $\Gamma' = \Gamma_0[I]$ satisfy these assumptions for every nonzero ideal I of R_ω.

Let $\rho > 0$. Let $(\mathscr{V}'_\epsilon)_{\epsilon>0}$ be a fundamental system of neighborhoods of the identity in G', which

- is symmetric (that is, $x \in \mathscr{V}'_\epsilon$ if and only if $x^{-1} \in \mathscr{V}'_\epsilon$),
- is nondecreasing with ϵ (that is, $\mathscr{V}'_\epsilon \subset \mathscr{V}'_{\epsilon'}$ if $\epsilon \leqslant \epsilon'$), and
- has *upper local dimension* ρ, that is, there exist $m_1, \epsilon_1 > 0$ such that $\mu_{G'}(\mathscr{V}'_\epsilon) \geqslant m_1\, \epsilon^\rho$ for every $\epsilon \in\,]0, \epsilon_1[$.

Let $C \geqslant 0$. Let $(\mathscr{B}_n)_{n\in\mathbb{N}}$ be a family of measurable subsets of G'. We define

$$(\mathscr{B}_n)^{+\epsilon} = \mathscr{V}'_\epsilon \mathscr{B}_n \mathscr{V}'_\epsilon = \bigcup_{g,h\in\mathscr{V}'_\epsilon} g\mathscr{B}_n h \quad\text{and}\quad (\mathscr{B}_n)^{-\epsilon} = \bigcap_{g,h\in\mathscr{V}'_\epsilon} g\mathscr{B}_n h\ .$$

The family $(\mathscr{B}_n)_{n\in\mathbb{N}}$ is *C-Lipschitz well-rounded* with respect to $(\mathscr{V}'_\epsilon)_{\epsilon>0}$ if there exists $\epsilon_0 > 0$ and $n_0 \in \mathbb{N}$ such that for all $\epsilon \in\,]0, \epsilon_0[$ and $n \geqslant n_0$, we have

$$\mu_{G'}((\mathscr{B}_n)^{+\epsilon}) \leqslant (1 + C\,\epsilon)\, \mu_{G'}((\mathscr{B}_n)^{-\epsilon})\ .$$

Theorem 4.1. *For every $\rho > 0$, there exists $\tau(\Gamma') \in\,]0, \frac{1}{2(1+\rho)}]$ such that for every $C \geqslant 0$, for every symmetric nonincreasing fundamental system $(\mathscr{V}'_\epsilon)_{\epsilon>0}$ of neighborhoods of the identity in G' with upper local dimension ρ, for every family $(\mathscr{B}_n)_{n\in\mathbb{N}}$ of measurable subsets of G' that is C-Lipschitz well-rounded with respect to $(\mathscr{V}'_\epsilon)_{\epsilon>0}$, and for every $\delta > 0$, we have that, as $n \to +\infty$,*

$$\Big|\mathrm{Card}(\mathscr{B}_n \cap \Gamma') - \frac{1}{\|\mu_{\Gamma'\backslash G'}\|}\, \mu_{G'}(\mathscr{B}_n)\Big| = \mathrm{O}\left(\mu_{G'}(\mathscr{B}_n)^{1-\tau(\Gamma')+\delta}\right)\ ,$$

where the function $\mathrm{O}(\cdot)$ depends only on $G', \Gamma', \delta, C, (\mathscr{V}'_\epsilon)_{\epsilon>0}, \rho$.

Proof. The proof is a simple adaptation of a particular case of results of Gorodnik-Nevo [16], which are phrased for algebraic number fields and not for function fields.

By the assumptions on $\mathbf{G}'$ and Γ', and by [3, Theo. 2.8], the regular representation π^0 of G' on $\mathbb{L}^2_0(G'/\Gamma')$ has a spectral gap. By [6] (see [3, Theo. 2.7]), since π^0 has a spectral gap, there exists $p \geqslant 2$ such that π^0

[1]This implies that $\mathbf{G}'$ is isotropic over K_ω, as part of the assumptions of [16].

is strongly $\mathbb{L}^p$ (called $\mathbb{L}^{p+}$ in [16, Def. 3.1]). We do not know what is the smallest such p. As in [16, Eq. (3.1)], let $n_e(p) = 1$ if $p = 2$ and otherwise let $n_e(p) = \lceil \frac{p}{2} \rceil \in \mathbb{N} - \{0, 1\}$. Since π^0 is strongly $\mathbb{L}^p$, by [16, Theo. 4.5], for every measurable subset B of G' with finite and positive Haar measure, if $\beta = \frac{1}{\mu_{G'}(B)}\,(\mu_{G'})_{|B}$ and $\pi^0(\beta)$ is the operator on $\mathbb{L}^2_0(G'/\Gamma')$ defined by

$$\pi^0(\beta)f(x) = \frac{1}{\mu_{G'}(B)} \int_B f(g^{-1}x)\,d\mu_{G'}(g)$$

for all $f \in \mathbb{L}^2_0(G'/\Gamma')$ and almost all $x \in G'/\Gamma'$, then we have that, for every $\eta > 0$,

$$\|\pi^0(\beta)\| = \mathrm{O}_{G',\Gamma',\eta}\left((\mu_{G'}(B))^{-\frac{1}{2\,n_e(p)}+\eta}\right) .$$

Actually, Theorem 4.5 of [16] is stated in characteristic zero. But its proof has two ingredients, a spectral transfer principle, which is valid for any locally compact second countable group by [6, Theo. 1], and a Kunze-Stein phenomenon, which is valid even in positive characteristic by [29, Theo. 1].

Now, by [16, Theo. 1.9] where $a = 1$, which is valid for any locally compact second countable group, and whose assumptions we just verified, we have

$$\left|\frac{\mathrm{Card}(\mathscr{B}_n \cap \Gamma')}{\mu_{G'}(\mathscr{B}_n)} - \frac{1}{\|\mu_{\Gamma'\backslash G'}\|}\right| = \mathrm{O}_{G',\Gamma',C,\rho,(\mathscr{V}'_\epsilon)_{\epsilon>0}}\left(\mu_{G'}(\mathscr{B}_n)^{(-\frac{1}{2\,n_e(p)}+\eta)(\frac{1}{\rho+1})}\right) .$$

Theorem 4.1 follows with $\tau(\Gamma') = \frac{1}{2\,n_e(p)(\rho+1)}$. □

The main result that will allow us to use Theorem 4.1 is the following proposition. We will use, as a fundamental system of neighborhoods of the identity element in G, a family of compact-open subgroups of $G(\mathscr{O}_\omega)$ given by the kernels of the morphisms of reduction modulo $\pi_\omega^n \mathscr{O}_\omega$ for $n \in \mathbb{N}$. For every $\epsilon > 0$, let $N_\epsilon = \lfloor -\log_{q_\omega} \epsilon \rfloor$ so that $N_\epsilon \geqslant 1$ if and only if $\epsilon \leqslant \frac{1}{q_\omega}$. Let $\mathscr{V}_\epsilon = G(\mathscr{O}_\omega)$ if $\epsilon > \frac{1}{q_\omega}$ and otherwise let

$$\begin{aligned}\mathscr{V}_\epsilon &= \ker(G(\mathscr{O}_\omega) \to G(\mathscr{O}_\omega/\pi_\omega^{N_\epsilon}\mathscr{O}_\omega))\\ &= \left\{ \begin{pmatrix} 1+\pi_\omega^{N_\epsilon}\alpha & \pi_\omega^{N_\epsilon}\gamma \\ \pi_\omega^{N_\epsilon}\beta & 1+\pi_\omega^{N_\epsilon}\delta \end{pmatrix} \in G(\mathscr{O}_\omega) : \alpha,\beta,\gamma,\delta \in \mathscr{O}_\omega \right\} .\end{aligned}$$

The family $(\mathscr{V}_\epsilon)_{\epsilon>0}$ is indeed nondecreasing and we have $\bigcap_{\epsilon>0} \mathscr{V}_\epsilon = \{\mathrm{id}\}$. Note that for all $\epsilon_1, \dots, \epsilon_k > 0$, we have

$$\begin{aligned}\min\{N_{\epsilon_1}, \cdots, N_{\epsilon_k}\} &\geqslant \min\{-\log_{q_\omega} \epsilon_1, \cdots, -\log_{q_\omega} \epsilon_k\} - 1\\ &\geqslant -\log_{q_\omega}(\epsilon_1 + \cdots + \epsilon_k) - 1 \geqslant N_{q_\omega(\epsilon_1+\cdots+\epsilon_k)} ,\end{aligned}$$

hence

$$\mathscr{V}_{\epsilon_1}\mathscr{V}_{\epsilon_2}\cdots\mathscr{V}_{\epsilon_k} \subset \mathscr{V}_{q_\omega(\epsilon_1+\cdots+\epsilon_k)} . \tag{4.1}$$

Proposition 4.2. *For all metric balls Θ in $\mathbb{S}^1_\omega$ and $\mathscr{D}'$ in K_ω with radius less than 1, the family $\big(P^-_\Theta\, A_n\, U^+_{\mathscr{D}'}\big)_{n\in\mathbb{N}}$ is 0-Lipschitz well-rounded with respect to $(\mathscr{V}_\epsilon)_{\epsilon>0}$.*

Proof. We will actually prove (as allowed by the ultrametric situation) the stronger statement that given Θ and $\mathscr{D}'$ as above, if ϵ is small enough, then for every $n \in \mathbb{N}$, we have

$$\big(P^-_\Theta A_n\, U^+_{\mathscr{D}'}\big)^{-\epsilon} = P^-_\Theta A_n\, U^+_{\mathscr{D}'} = \big(P^-_\Theta A_n\, U^+_{\mathscr{D}'}\big)^{+\epsilon} .$$

We start the proof by some elementary linear algebra considerations. For every subgroup H of G, let $\mathscr{V}^H_\epsilon = \mathscr{V}_\epsilon \cap H$. We endow $\mathscr{M}_2(K_\omega)$ with its supremum norm $\|\cdot\|_\omega$ defined, for every element $X \in \mathscr{M}_2(K_\omega) - \{0\}$, by $\|X\|_\omega = \max\{|X_{i,j}|_\omega : 1 \leqslant i, j \leqslant 2\} \in q_\omega^{\mathbb{Z}}$. The unit ball of $\|\cdot\|_\omega$ is $\mathscr{M}_2(\mathscr{O}_\omega)$. We denote the operator norm of a linear operator ℓ of $\mathscr{M}_2(K_\omega)$ by

$$\|\ell\|_\omega = \max\Big\{\frac{\|\ell(X)\|_\omega}{\|X\|_\omega} : X \in \mathscr{M}_2(K_\omega) - \{0\}\Big\} \in q_\omega^{\mathbb{Z}} \cup \{0\} ,$$

so that $\ell(\mathscr{M}_2(\mathscr{O}_\omega)) \subset \mathscr{M}_2(\pi_\omega^{-\log_{q_\omega}\|\ell\|_\omega}\mathscr{O}_\omega)$. For every $g \in G$, recall that $\operatorname{Ad} g$ is the linear automorphism $x \mapsto gxg^{-1}$ of $\mathscr{M}_2(K_\omega)$.

Lemma 4.3. *For all $\epsilon > 0$ and $g \in G$, we have*

$$g\,\mathscr{V}_\epsilon\, g^{-1} \;\subset\; \mathscr{V}_{\epsilon\,\|\operatorname{Ad} g\,\|_\omega} , \quad \mathscr{V}_\epsilon = \mathscr{V}^{P^-}_\epsilon\; \mathscr{V}^{U^+}_\epsilon \quad \text{and} \quad \mathscr{V}^{P^-}_\epsilon = \mathscr{V}^{U^-}_\epsilon\; \mathscr{V}^{Z}_\epsilon .$$

Furthermore, we have $\mu_G(\mathscr{V}_\epsilon) \geqslant \frac{q_\omega^2}{q_\omega^2-1}\,\epsilon^3$ for every $\epsilon > 0$ small enough, so that $\rho = 3$ is an upper local dimension of the family $(\mathscr{V}_\epsilon)_{\epsilon>0}$.

Proof. Let I_2 be the identity element in G. The first claim follows from the fact that

$$\begin{aligned} g\,\mathscr{V}_\epsilon\, g^{-1} &= I_2 + \pi_\omega^{N_\epsilon} g\mathscr{M}_2(\mathscr{O}_\omega)g^{-1} \\ &\subset I_2 + \pi_\omega^{N_\epsilon - \log_{q_\omega}\|\operatorname{Ad} g\,\|_\omega}\mathscr{M}_2(\mathscr{O}_\omega) = \mathscr{V}_{\epsilon\,\|\operatorname{Ad} g\,\|_\omega} . \end{aligned}$$

The second and third claims follow from the fact that by Equations (2.15) and (2.16), if $g \in \mathscr{V}_\epsilon$ then $\mathfrak{a}_g = I_2$, $\mathfrak{u}^\pm_g \in \mathscr{V}^{U^\pm}_\epsilon$ and $\mathfrak{m}_g \in \mathscr{V}^Z_\epsilon$.

Let us now apply Lemma 2.2 and the decomposition $\mathscr{V}_\epsilon = \mathscr{V}^{U^-}_\epsilon\; \mathscr{V}^Z_\epsilon\; \mathscr{V}^{U^+}_\epsilon$:

$$\mu_G(\mathscr{V}_\epsilon) = \frac{q_\omega}{q_\omega+1}\,\mu_{U^-}(\mathscr{V}^{U^-}_\epsilon)\;\mu_{Z(\mathscr{O}_\omega)}(\mathscr{V}^Z_\epsilon)\;\mu_{U^+}(\mathscr{V}^{U^+}_\epsilon) .$$

By Equation (2.12) applied twice, by the left part of Equation (2.6), and since $N_\epsilon = \lfloor -\log_{q_\omega}\epsilon\rfloor$, we have that for $\epsilon \leqslant \frac{1}{q_\omega}$,

$$\begin{aligned} \mu_G(\mathscr{V}_\epsilon) &= \frac{q_\omega}{q_\omega+1}\,\mu_{K_\omega}(\pi_\omega^{N_\epsilon}\mathscr{O}_\omega)\,\big|\mathscr{O}_\omega^\times/(1+\pi_\omega^{N_\epsilon}\mathscr{O}_\omega)\big|^{-1}\,\mu_{K_\omega}(\pi_\omega^{N_\epsilon}\mathscr{O}_\omega) \\ &= \frac{q_\omega}{q_\omega+1}\;\frac{1}{(q_\omega-1)\,q_\omega^{N_\epsilon-1}}\;q_\omega^{-2\,N_\epsilon} \geqslant \frac{q_\omega^2}{q_\omega^2-1}\,\epsilon^3 . \end{aligned}$$

This proves the final claim of Lemma 4.3. $\Box$

The main ingredient in the proof of Proposition 4.2 is the following effective refined LU decomposition.

Lemma 4.4. *With $c : G \to \,]0, +\infty[$ the continuous function defined by $h \mapsto \|\operatorname{Ad} h\|_\omega$, for every $g \in G$ with $|\chi(\mathfrak{a}_g)|_\omega \leqslant 1$, we have*

$$\mathscr{V}_\epsilon\, g\, \mathscr{V}_\epsilon \;\subset\; \mathfrak{p}_g^-\; \mathscr{V}^{P^-}_{q_\omega(c(\mathfrak{p}_g^-)+c(\mathfrak{u}_g^+))\epsilon}\; \mathfrak{a}_g\; \mathscr{V}^{U^+}_{q_\omega(c(\mathfrak{p}_g^-)+2c(\mathfrak{u}_g^+))\epsilon}\; \mathfrak{u}_g^+ \;.$$

Proof. In order to simplify notation, let $\mathfrak{a} = \mathfrak{a}_g$, $\mathfrak{p} = \mathfrak{p}_g^-$ and $\mathfrak{u} = \mathfrak{u}_g^+$, so that $g = \mathfrak{p}\,\mathfrak{a}\,\mathfrak{u}$. For every $h \in G$, let $c_h = \|\operatorname{Ad} h\|_\omega$. In the following sequence of equalities and inclusions, we use

- the first claim of Lemma 4.3, for the first inclusion,
- the second claim of Lemma 4.3, for the second equality,
- the fact that

$$\mathfrak{a}\,\mathscr{V}^{P^-}_{c_\mathfrak{u}\epsilon} = \mathfrak{a}\,\mathscr{V}^{U^-}_{c_\mathfrak{u}\epsilon}\;\mathscr{V}^{Z}_{c_\mathfrak{u}\epsilon} \subset \mathscr{V}^{U^-}_{c_\mathfrak{u}\epsilon}\;\mathfrak{a}\,\mathscr{V}^{Z}_{c_\mathfrak{u}\epsilon} = \mathscr{V}^{P^-}_{c_\mathfrak{u}\epsilon}\;\mathfrak{a} \subset \mathscr{V}_{c_\mathfrak{u}\epsilon}\,\mathfrak{a}$$

by the third claim of Lemma 4.3, by the left hand side of Equation (2.17) with $\chi(\mathfrak{a}) \in \mathscr{O}_\omega$ and since $\mathfrak{a}$ and Z commute, for the second inclusion,

- the facts that $\mathscr{V}_{c_\mathfrak{u}\epsilon}$ is a normal subgroup of $G(\mathscr{O}_\omega)$ and that the inclusion $\mathscr{V}^{U^+}_{c_\mathfrak{p}\epsilon} \subset G(\mathscr{O}_\omega)$ holds, for the third equality,
- again the second claim of Lemma 4.3, and the right hand side of Equation (2.17) with $\chi(\mathfrak{a}) \in \mathscr{O}_\omega$, for the last inclusion.

We thus have

$$\begin{aligned}
\mathscr{V}_\epsilon\, g\, \mathscr{V}_\epsilon &= \mathfrak{p}\,\mathfrak{p}^{-1}\mathscr{V}_\epsilon\,\mathfrak{p}\,\mathfrak{a}\,\mathfrak{u}\,\mathscr{V}_\epsilon\,\mathfrak{u}^{-1}\mathfrak{u} \;\subset\; \mathfrak{p}\,\mathscr{V}_{c_\mathfrak{p}\epsilon}\,\mathfrak{a}\,\mathscr{V}_{c_\mathfrak{u}\epsilon}\,\mathfrak{u}\\
&= \;\mathfrak{p}\,\mathscr{V}^{P^-}_{c_\mathfrak{p}\epsilon}\;\mathscr{V}^{U^+}_{c_\mathfrak{p}\epsilon}\,\mathfrak{a}\,\mathscr{V}^{P^-}_{c_\mathfrak{u}\epsilon}\mathscr{V}^{U^+}_{c_\mathfrak{u}\epsilon}\;\mathfrak{u} \;\subset\; \mathfrak{p}\,\mathscr{V}^{P^-}_{c_\mathfrak{p}\epsilon}\;\mathscr{V}^{U^+}_{c_\mathfrak{p}\epsilon}\;\mathscr{V}_{c_\mathfrak{u}\epsilon}\,\mathfrak{a}\,\mathscr{V}^{U^+}_{c_\mathfrak{u}\epsilon}\;\mathfrak{u}\\
&= \;\mathfrak{p}\,\mathscr{V}^{P^-}_{c_\mathfrak{p}\epsilon}\;\mathscr{V}_{c_\mathfrak{u}\epsilon}\,\mathscr{V}^{U^+}_{c_\mathfrak{p}\epsilon}\;\mathfrak{a}\,\mathscr{V}^{U^+}_{c_\mathfrak{u}\epsilon}\;\mathfrak{u} \;\subset\; \mathfrak{p}\,\mathscr{V}^{P^-}_{c_\mathfrak{p}\epsilon}\;\mathscr{V}^{P^-}_{c_\mathfrak{u}\epsilon}\;\mathfrak{a}\,\mathscr{V}^{U^+}_{c_\mathfrak{u}\epsilon}\,\mathscr{V}^{U^+}_{c_\mathfrak{p}\epsilon}\;\mathscr{V}^{U^+}_{c_\mathfrak{u}\epsilon}\;\mathfrak{u}\;.
\end{aligned}$$

Lemma 4.4 now follows from Equation (4.1). $\Box$

Now, in order to prove Proposition 4.2, we write $\Theta = v_0 + \pi_\omega^m \mathscr{O}_\omega{}^2$ and $\mathscr{D}' = x_0 + \pi_\omega^{m'}\mathscr{O}_\omega$, for some $m, m' \in \mathbb{N} - \{0\}$, $x_0 \in K_\omega$ and $v_0 \in \mathbb{S}^1_\omega$. Let

$$c = \max\{q_\omega(c(\mathfrak{p}) + 2c(\mathfrak{u})) : \mathfrak{p} \in P^-_\Theta,\; \mathfrak{u} \in U^+_{\mathscr{D}'}\}\,,$$

which is finite since P^-_Θ and $U^+_{\mathscr{D}'}$ are compact. Let $\epsilon_0 = \frac{1}{c}\, q_\omega^{-m'-m} > 0$, so that we have $N_{c\epsilon} > \max\{m, m'\} \geqslant 1$ if $\epsilon < \epsilon_0$.

Let us fix $\epsilon \in \,]0, \epsilon_0[$. We claim that

$$P^-_\Theta\,\mathscr{V}^{P^-}_{c\epsilon} = P^-_\Theta \quad\text{and}\quad \mathscr{V}^{U^+}_{c\epsilon}\,U^+_{\mathscr{D}'} = U^+_{\mathscr{D}'}\;. \tag{4.2}$$

Indeed, the inclusion of the right hand sides into the left hand sides of these equalities are immediate. If $\mathfrak{p}\in P^-_\Theta$ and $\mathfrak{p}'\in\mathscr{V}^{P^-}_{c\epsilon}$, we may write

$$\mathfrak{p}=\begin{pmatrix}x_{v_0}+\pi_\omega^m\alpha & 0\\ y_{v_0}+\pi_\omega^m\beta & (x_{v_0}+\pi_\omega^m\alpha)^{-1}\end{pmatrix}$$

$$\text{and}\quad \mathfrak{p}'=\begin{pmatrix}1+\pi_\omega^{N_{c\epsilon}}\alpha' & 0\\ \pi_\omega^{N_{c\epsilon}}\beta' & (1+\pi_\omega^{N_{c\epsilon}}\alpha')^{-1}\end{pmatrix}$$

for some $\alpha,\beta,\alpha',\beta'\in\mathscr{O}_\omega$, so that

$$\mathfrak{p}\,\mathfrak{p}'=\begin{pmatrix}x_{v_0}+\pi_\omega^m\alpha+\pi_\omega^{N_{c\epsilon}}\alpha'' & 0\\ y_{v_0}+\pi_\omega^m\beta+\pi_\omega^{N_{c\epsilon}}\beta'' & (x_{v_0}+\pi_\omega^m\alpha+\pi_\omega^{N_{c\epsilon}}\alpha'')^{-1}\end{pmatrix}$$

for some $\alpha'',\beta''\in\mathscr{O}_\omega$ (since $x_{v_0},y_{v_0}\in\mathscr{O}_\omega$). The first claim of Equation (4.2) then follows from the fact that $N_{c\epsilon}>m$. The inclusion $\mathscr{V}^{U^+}_{c\epsilon}\,U^+_{\mathscr{D}'}\subset U^+_{\mathscr{D}'}$ follows from a similar and even easier computation.

Now for every $n\in\mathbb{N}$, we have by Lemma 4.4 and Equation (4.2) that

$$\big(P^-_\Theta A_n\,U^+_{\mathscr{D}'}\big)^{+\epsilon}=\mathscr{V}_\epsilon\,P^-_\Theta A_n\,U^+_{\mathscr{D}'}\mathscr{V}_\epsilon\subset P^-_\Theta\,\mathscr{V}^{P^-}_{c\epsilon}\,A_n\,\mathscr{V}^{U^+}_{c\epsilon}\,U^+_{\mathscr{D}'}=P^-_\Theta A_n\,U^+_{\mathscr{D}'}\;.$$

Since the converse inclusion is immediate, we have

$$\big(P^-_\Theta A_n\,U^+_{\mathscr{D}'}\big)^{+\epsilon}=P^-_\Theta A_n\,U^+_{\mathscr{D}'}\;.$$

Since $\mathscr{V}_\epsilon$ is symmetric, this implies that $g\,P^-_\Theta A_n\,U^+_{\mathscr{D}'}\,h\supset P^-_\Theta A_n\,U^+_{\mathscr{D}'}$ for all $g,h\in\mathscr{V}_\epsilon$ so that $\big(P^-_\Theta A_n\,U^+_{\mathscr{D}'}\big)^{-\epsilon}\supset P^-_\Theta A_n\,U^+_{\mathscr{D}'}$. Since the converse inclusion is immediate, this concludes the proof of Proposition 4.2. □

The main result of this paper is the following one. Recall that z_v, z'_v and $\check{v}$ for v in ${K_\omega}^2-\{(0,0)\}$ have been defined in Equations (2.3) and (2.4). If $v=(a,b)\in R^2_{\omega,\mathrm{prim}}$, we denote by w_v any element of $R^2_{\omega,\mathrm{prim}}$ such that $(w_v)^\perp=(x,y)$ is a solution to the equation $ax+by=1$. As seen in the proof of Proposition 3.1 if $|\,a\,|_\omega\geqslant|\,b\,|_\omega$, and by symmetry otherwise, the class $\frac{z_{w_v}}{z_v}+R_\omega$ of $\frac{z_{w_v}}{z_v}$ in the quotient K_ω/R_ω does not depend on the choice of w_v. For every nonzero ideal I of R_ω, let

$$c_I=\frac{(q_\omega^2-1)\,\zeta_K(-1)\,N(I)\,\prod_{\mathfrak{p}|I}\big(1+\frac{1}{N(\mathfrak{p})}\big)}{q_\omega^2}\tag{4.3}$$

Theorem 4.5. *For every nonzero ideal I of R_ω, for the weak-star convergence on the compact space $\mathbb{S}^1_\omega\times(K_\omega/R_\omega)$, we have, as $n\to+\infty$,*

$$c_I\;q_\omega^{-2n}\sum_{v\in R^2_{\omega,\mathrm{prim}}\,:\,\|v\|_\omega=q_\omega^n,\;z'_v\in I}\Delta_{\check{v}}\otimes\Delta_{\frac{z_{w_v}}{z_v}+R_\omega}\;\overset{*}{\rightharpoonup}\;\mu_{\mathbb{S}^1_\omega}\otimes\mu_{K_\omega/R_\omega}\;.$$

Furthermore, there exists $\tau\in\,]0,\frac{1}{8}]$ such that for all $\epsilon,\delta>0$, there is a multiplicative error term in the above equidistribution claim of the form

$1 + \mathrm{O}_{\omega,\delta,I}\big(q_\omega^{2n(-\tau+\delta)}\,\|f\|_\epsilon\,\|g\|_\epsilon\big)$ *when evaluated on* (f,g) *for all* ϵ*-locally constant maps* $f : \mathbb{S}^1_\omega \to \mathbb{R}$ *and* $g : K_\omega/R_\omega \to \mathbb{R}$*:*

$$c_I\; q_\omega^{-2n} \sum_{v\in R^2_{\omega,\mathrm{prim}}\;:\;\|v\|_\omega = q_\omega^n,\; z'_v\in I} f(\,\breve{v}\,)\; g\Big(\frac{z_{w_v}}{z_v} + R_\omega\Big)$$

$$= \Big(\int_{\mathbb{S}^1_\omega} f\, d\mu_{\mathbb{S}^1_\omega}\Big)\Big(\int_{K_\omega/R_\omega} g\, d\mu_{K_\omega/R_\omega}\Big)\Big(1 + \mathrm{O}_{\omega,\delta,I}\big(q_\omega^{2n(-\tau+\delta)}\,\|f\|_\epsilon\,\|g\|_\epsilon\big)\Big)\;.$$

When $\mathbf{C} = \mathbb{P}^1$, $\omega = \omega_\infty$ and $I = R_{\omega_\infty}$, we recover Theorem 1.1 in the introduction by using Equations (4.3), (2.9) and (2.2), as well as the fact that $q_\omega = q$. Note that up to changing the constant c_I, the same result holds when v ranges over the elements in $R^2_{\omega,\mathrm{prim}}$ with $\|v\|_\omega \leqslant q_\omega^n$ rather than $\|v\|_\omega = q_\omega^n$ and $z'_v \in I$. But as said in the introduction, ranging on spheres rather than balls gives much stronger result, in fit adequation with the number theoretic results on Linnik's problem. Also note that in the statement of Theorem 4.5, the measures $\mu_{\mathbb{S}^1_\omega}$ and μ_{K_ω/R_ω} are not normalized to be probability measures, see Equations (2.8) and (2.6) if a normalization is useful, as for instance in Corollary 4.6.

Given a nonzero (possibly nonprincipal) ideal J of R_ω, an effective joint equidistribution result similar to the one of Theorem 4.5 is possible when the elements $v = (a,b) \in {R_\omega}^2$ are not assumed to be primitive, but instead to satisfy the property that a and b generate the ideal J.

Proof. Let I be a nonzero ideal of R_ω. Let $\tau = \tau(\Gamma_0[I]) \in \,]0, \frac{1}{8}]$ be as in Theorem 4.1 applied with $G' = G$ and $\Gamma' = \Gamma_0[I]$, and with $(\mathscr{V}'_\epsilon)_{\epsilon>0} = (\mathscr{V}_\epsilon)_{\epsilon>0}$ which has upper local dimension $\rho = 3$ according to the final claim of Lemma 4.3. Let $\delta \in \,]0,\tau]$. Fix a compact-open strict fundamental domain $\mathscr{D}$ for the action by translations of R_ω on K_ω, such that for all $x_0 \in \mathscr{D}$ and $m' \in \mathbb{N} - \{0\}$, we have $B(x_0, q_\omega^{-m'}) = x_0 + \pi_\omega^{m'}\mathscr{O}_\omega \subset \mathscr{D}$. This is possible since $R_\omega \cap \pi_\omega\mathscr{O}_\omega = \{0\}$ by Equation (2.1). Note that for all $v_0 \in \mathbb{S}^1_\omega$ (respectively $v_0 \in \mathbb{S}^{1,\sharp}_\omega$) and $m \in \mathbb{N} - \{0\}$, the ball $B(v_0, q_\omega^{-m}) = v_0 + \pi_\omega^m {\mathscr{O}_\omega}^2$ is contained in $\mathbb{S}^1_\omega$ (respectively $\mathbb{S}^{1,\sharp}_\omega$).

Let us prove that for all $m, m' \in \mathbb{N} - \{0\}$, $x_0 \in \mathscr{D}$ and $v_0 \in \mathbb{S}^1_\omega$, if $\Theta = v_0 + \pi_\omega^m {\mathscr{O}_\omega}^2$ and $\mathscr{D}' = x_0 + \pi_\omega^{m'}\mathscr{O}_\omega$, then, as $n \to +\infty$

$$\mathrm{Card}\Big\{v \in R^2_{\omega,\mathrm{prim}}\;:\;\|v\|_\omega = q_\omega^n,\; z'_v \in I,\; \breve{v} \in \Theta,\; \frac{z_{w_v}}{z_v} \in \mathscr{D}'\Big\}$$

$$= \frac{1}{c_I}\; q_\omega^{2n}\;\mu_{\mathbb{S}^1_\omega}(\Theta)\;\mu_{K_\omega}(\mathscr{D}')\big(1 + \mathrm{O}_{\omega,\delta,I}\big(q_\omega^{2n(-\tau+\delta)} q_\omega^{m+m'}\big)\big)\;. \tag{4.4}$$

Since the characteristic functions $\mathbb{1}_\Theta$ and $\mathbb{1}_{\mathscr{D}'}$ of Θ and $\mathscr{D}'$ are respectively q_ω^{-m}- and $q_\omega^{-m'}$-locally constant, and by a finite additivity argument, this proves Theorem 4.5.

We first claim that in order to prove the counting result of elements in $R^2_{\omega,\rm prim}$ stated in Equation (4.4), we only have to prove an analogous counting result of elements in $R^{2,\sharp}_{\omega,\rm prim}$, namely that for all $m, m' \in \mathbb{N} - \{0\}$, for all $x_0 \in \mathscr{D}$ and $v_0 \in \mathbb{S}^{1,\sharp}_\omega$, if $\Theta = v_0 + \pi_\omega^m {\mathscr{O}_\omega}^2$ and $\mathscr{D}' = x_0 + \pi_\omega^{m'} \mathscr{O}_\omega$, then, as $n \to +\infty$

$$\begin{aligned} &\operatorname{Card}\big\{v \in R^{2,\sharp}_{\omega,\rm prim} \;:\; \|v\|_\omega = q_\omega^n,\; y_v \in I,\; \check{v} \in \Theta,\; \frac{x_{w_v}}{x_v} \in \mathscr{D}'\big\} \\ &\qquad = \frac{1}{c_I}\, q_\omega^{2n}\, \mu_{\mathbb{S}^1_\omega}(\Theta)\, \mu_{K_\omega}(\mathscr{D}')\big(1 + \mathrm{O}_{\omega,\delta,I}\,\big(q_\omega^{2n(-\tau+\delta)} q_\omega^{m+m'}\big)\big)\;. \end{aligned} \tag{4.5}$$

Indeed, by Lemma 2.1 and Equation (2.3), since $\det\begin{pmatrix} v & w_v\end{pmatrix} = 1$, we have $\frac{z_{w_v}}{z_v} = \frac{x_{w_v}}{x_v}$ when v belongs to $R^{2,\sharp}_{\omega,\rm prim}$ except finitely many of them. The involutive linear map $\iota = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0\end{smallmatrix}\right)$ of exchange of coordinates

- preserves the subsets $R^2_{\omega,\rm prim}$ and $\mathbb{S}^1_\omega$ of the plane ${K_\omega}^2$,
- sends the compact-open set $\mathbb{S}^1_\omega - \mathbb{S}^{1,\sharp}_\omega$ into $\mathbb{S}^{1,\sharp}_\omega$,
- sends an element v in $R^2_{\omega,\rm prim} - R^{2,\sharp}_{\omega,\rm prim}$ to the element $\iota(v)$ in $R^{2,\sharp}_{\omega,\rm prim}$ such that $z'_v = z_{\iota(v)} = y_{\iota(v)}$ and $\frac{z_{w_v}}{z_v} = \frac{x_{w_{\iota(v)}}}{x_{\iota(v)}}$ again by Lemma 2.1 and Equation (2.3), and
- sends $v_0 + \pi_\omega^m {\mathscr{O}_\omega}^2$ to $\iota(v_0) + \pi_\omega^m {\mathscr{O}_\omega}^2$.

Hence Equation (4.4) follows from Equation (4.5).

Now according to Proposition 4.2, the family $\big(P^-_\Theta\, A_n\, U^+_{\mathscr{D}'}\big)_{n\in\mathbb{N}}$ is 0-Lipschitz well-rounded in G with respect to $(\mathscr{V}_\epsilon)_{\epsilon>0}$. Note that

$$\Gamma \cap (P^-_\Theta\, A_n\, U^+_{\mathscr{D}'}) = \Gamma^\sharp \cap (P^-_\Theta\, A_n\, U^+_{\mathscr{D}'})$$

since Θ is contained in $\mathbb{S}^{1,\sharp}_\omega$. In the following sequence of equalities, we use respectively

- Proposition 3.1,
- Theorem 4.1 applied with $G' = G$, $\Gamma' = \Gamma_0[I]$ and $(\mathscr{B}_n)_{n\in\mathbb{N}} = (P^-_\Theta\, A_n\, U^+_{\mathscr{D}'})_{n\in\mathbb{N}}$,
- Equation (3.1),
- the fact that Θ is a metric ball of radius q_ω^{-m} in the plane ${K_\omega}^2$ and $\mathscr{D}'$ a metric ball of radius $q_\omega^{-m'}$ in the line K_ω.

We thus have

$$\begin{aligned}
&\mathrm{Card}\big\{v\in R^{2,\sharp}_{\omega,\mathrm{prim}}\;:\;\|v\|_\omega=q_\omega^n,\ y_v\in I,\ \check v\in\Theta,\ \frac{x_{w_v}}{x_v}\in\mathscr{D}'\big\}\\
&\quad=\mathrm{Card}\big(\Gamma_0[I]\cap(P^-_\Theta\,A_n\,U^+_{\mathscr{D}'})\big)\\
&\quad=\frac{\mu_G(P^-_\Theta\,A_n\,U^+_{\mathscr{D}'})}{\|\mu_{\Gamma_0[I]\backslash G}\|}+\mathrm{O}_{\omega,\delta,I}\left(\big(\mu_G(P^-_\Theta\,A_n\,U^+_{\mathscr{D}'})\big)^{1-\tau+\delta}\right)\\
&\quad=\frac{q_\omega^{2n+2}}{(q_\omega^2-1)\,\|\mu_{\Gamma_0[I]\backslash G}\|}\;\mu_{\mathbb{S}^1_\omega}(\Theta)\;\mu_{K_\omega}(\mathscr{D}')\\
&\qquad+\mathrm{O}_{\omega,\delta,I}\left(\big(q_\omega^{2n}\mu_{\mathbb{S}^1_\omega}(\Theta)\;\mu_{K_\omega}(\mathscr{D}')\big)^{1-\tau+\delta}\right)\\
&\quad=\frac{q_\omega^{2n+2}}{(q_\omega^2-1)\,\|\mu_{\Gamma_0[I]\backslash G}\|}\;\mu_{\mathbb{S}^1_\omega}(\Theta)\;\mu_{K_\omega}(\mathscr{D}')\\
&\qquad\times\left(1+\mathrm{O}_{\omega,\delta,I}\left(q_\omega^{2n(-\tau+\delta)}q_\omega^{2m(\tau-\delta)}q_\omega^{m'(\tau-\delta)}\right)\right)\,.
\end{aligned}$$

Since by Equations (2.10) and (2.9) we have

$$\|\mu_{\Gamma_0[I]\backslash G}\|=\|\mu_{\Gamma\backslash G}\|\,[\,\Gamma:\Gamma_0[I]\,]=\zeta_K(-1)\,N(I)\prod_{\mathfrak{p}|I}\big(1+\frac{1}{N(\mathfrak{p})}\big)$$

and since $\tau\leqslant\frac18$ (so that $2m(\tau-\delta)\leqslant m$ and $m'(\tau-\delta)\leqslant m'$), this proves Equation (4.5) and completes the proof of Theorem 4.5. □

We conclude this section by stating a counting result, which follows from the equidistribution claim of Theorem 4.5 by integrating on the pairs of constant functions with value 1 on $\mathbb{S}^1_\omega$ and on K_ω/R_ω, and by using Equations (2.8) and (2.6).

Corollary 4.6. *There exists $\tau\in\,]0,\frac18]$ such that for every $\delta>0$, we have*

$$\begin{aligned}
&\mathrm{Card}\;\{v\in R^2_{\omega,\mathrm{prim}}\;:\;\|v\|_\omega=q_\omega^n,\ z'_v\in I\}\\
&\qquad=\frac{q^{\mathbf{g}-1}}{\zeta_K(-1)\;N(I)\;\prod_{\mathfrak{p}|I}\left(1+\frac{1}{N(\mathfrak{p})}\right)}\;q_\omega^{2n}+\mathrm{O}_{\delta,I}\left(q_\omega^{2n(1-\tau+\delta)}\right)\,.
\end{aligned}$$

5. Application to the distribution of continued fraction expansions

In this section, we assume that $\mathbf{C}=\mathbb{P}^1$ and $\omega=\omega_\infty$, so that the notation in Section 2.1 coincides with the notation of the introduction: $K=\mathbb{F}_q(Y)$, $R_{\omega_\infty}=R=\mathbb{F}_q[Y]$, $K_{\omega_\infty}=\widehat{K}=\mathbb{F}_q((Y^{-1}))$, $\mathscr{O}_{\omega_\infty}=\mathscr{O}=\mathbb{F}_q[[Y^{-1}]]$ and $|\cdot|_{\omega_\infty}=|\cdot|$.

Let us recall elementary facts on the continued fraction expansions in $\widehat{K}$, similar to the ones in $\mathbb{R}$, see for instance the surveys [21, 26], and [23] for a geometric interpretation. Any element $f\in\widehat{K}$ may be uniquely written

$f = [f] + \{f\}$ with $[f] \in R$ (called the *integral part* of f) and $\{f\} \in Y^{-1}\mathscr{O}$ (called the *fractional part* of f). The *Artin map* $\Psi : Y^{-1}\mathscr{O} - \{0\} \to Y^{-1}\mathscr{O}$ is defined by $f \mapsto \big\{\frac{1}{f}\big\}$. Any $f \in K - R$ has a unique finite continued fraction expansion

$$f = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_n}}}}\,,$$

with $a_0 = [f] \in R$ and $a_i = \Big[\frac{1}{\Psi^{i-1}(f-a_0)}\Big]$ a nonconstant polynomial for $1 \leqslant i \leqslant n$ (called the *coefficients* of the continued fraction expansion of f), where $n \in \mathbb{N} - \{0\}$ is such that $\Psi^n(f - a_0) = 0$.

Two finite sequences of polynomials $(P_i)_{-1 \leqslant i \leqslant n}$ and $(Q_i)_{-1 \leqslant i \leqslant n}$ in R are defined inductively as follows

$$\begin{array}{lll} P_{-1} = 1 & P_0 = a_0, & P_i = P_{i-1}a_i + P_{i-2} \\ Q_{-1} = 0 & Q_0 = 1, & Q_i = Q_{i-1}a_i + Q_{i-2} \end{array}$$

for $1 \leqslant i \leqslant n$. The elements P_i/Q_i for $0 \leqslant i \leqslant n-1$ are called the *convergents* of f, and $P_n/Q_n = f$. The convergents have the following characterisation (see for instance [26, p. 140]): for all $P, Q \in R$ such that $\deg Q < \deg Q_n$

$$\text{if } |f - P/Q| < \frac{1}{|Q|^2} \text{ then } P/Q \text{ is a convergent.} \tag{5.1}$$

For $0 \leqslant i \leqslant n-1$, we have

$$\Big|\, f - \frac{P_i}{Q_i} \,\Big| = \frac{1}{|Q_i|\;|Q_{i+1}|} \tag{5.2}$$

by for instance [26, Eq. (1.12)]), and

$$Q_{i+1}P_i - P_{i+1}Q_i = (-1)^{i+1}\ . \tag{5.3}$$

Since $\deg a_i \geqslant 1$ if $i \geqslant 1$, we have $\deg Q_i > \deg Q_{i-1}$ for $1 \leqslant i \leqslant n$. If $f \in Y^{-1}\mathscr{O}$, then $a_0 = 0$ and $P_i/Q_i \in Y^{-1}\mathscr{O}$, or equivalently $|P_i| < |Q_i|$, for $1 \leqslant i \leqslant n$.

The following result relates the shortest solutions to an equation of the form $ax + by = 1$ with the continued fraction expansion of a/b.

Lemma 5.1. *Let $a, b \in R - \{0\}$ be two coprime polynomials such that $a/b \in Y^{-1}\mathscr{O}$. Let $(P_i/Q_i)_{0 \leqslant i \leqslant n}$ be the sequence of convergents of a/b. Then there exists a unique $\lambda \in \mathbb{F}_q^\times$ such that $(a, b) = (\lambda P_n, \lambda Q_n)$, and $(-(-1)^n\lambda^{-1} Q_{n-1}, (-1)^n\lambda^{-1} P_{n-1})$ is the unique shortest solution to the equation $ax + by = 1$.*

Note that this result implies that for all $a, b \in R - \{0\}$, the equation $ax + by = 1$ has one and only one shortest solution, up to exchanging a and b if $|a| > |b|$ and to replacing (a, b) by $(a - \lambda' b, b)$ for the unique $\lambda' \in \mathbb{F}_q^\times$ such that $\deg(a - \lambda' b) < \deg b$ if $|a| = |b|$.

Proof. We may assume that $a \notin \mathbb{F}_q^\times$, otherwise the result is immediate with $\lambda = a$ since $a_0 = 0$, $a_1 = a^{-1}b$, $n = 1$, $P_0 = 0$, $Q_0 = 1$, $P_1 = 1$ and $Q_1 = a^{-1}b$.

Since P_n and Q_n are coprime polynomials by Equation (5.3) and since $P_n/Q_n = a/b$, there exists $\lambda \in \mathbb{F}_q^\times$ such that $a = \lambda P_n$ and $b = \lambda Q_n$. Let $\widetilde{a} = \frac{a}{(-1)^n \lambda}$ and $\widetilde{b} = \frac{a}{(-1)^n \lambda}$. Let $\mathrm{Sol}_{\widetilde{a}, \widetilde{b}} = \{(x, y) \in R^2 : \widetilde{a}\, x + \widetilde{b}\, y = 1\}$.

We have $\widetilde{a} = (-1)^n P_n$ and $\widetilde{b} = (-1)^n Q_n$. Again by Equation (5.3), this implies that $(-Q_{n-1}, P_{n-1}) \in \mathrm{Sol}_{\widetilde{a}, \widetilde{b}}$.

Let $(\widetilde{x}_0, \widetilde{y}_0)$ be another element of $\mathrm{Sol}_{\widetilde{a}, \widetilde{b}}$. Since we have $|\,\widetilde{a}\,| < |\,\widetilde{b}\,|$, it follows from Lemma 2.1 that $|\widetilde{x}_0| \geqslant |\widetilde{y}_0|$, so that $\|(\widetilde{x}_0, \widetilde{y}_0)\|_\infty = |\widetilde{x}_0|$. We have $\|(-Q_{n-1}, P_{n-1})\|_\infty = |Q_{n-1}|$ since $P_{n-1}/Q_{n-1} \in Y^{-1}\mathscr{O}$. In order to prove that $(-Q_{n-1}, P_{n-1})$ is the unique shortest element of $\mathrm{Sol}_{\widetilde{a}, \widetilde{b}}$, let us assume that $|\widetilde{x}_0| \leqslant |Q_{n-1}|$, and prove that $(\widetilde{x}_0, \widetilde{y}_0) = (-Q_{n-1}, P_{n-1})$.

Since $|\widetilde{x}_0| \leqslant |Q_{n-1}| < |Q_n|$, we have

$$\Big|\, \frac{\widetilde{y}_0}{-\widetilde{x}_0} - \frac{P_n}{Q_n} \,\Big| = \frac{1}{|\widetilde{x}_0|\;|Q_n|} < \frac{1}{|-\widetilde{x}_0|^2}\;.$$

Hence by Equation (5.1), $\frac{\widetilde{y}_0}{-\widetilde{x}_0}$ is a convergent of $\frac{P_n}{Q_n}$, that is, there exists $i \in \{0, \ldots, n-1\}$ such that $\frac{\widetilde{y}_0}{-\widetilde{x}_0} = \frac{P_i}{Q_i}$. This implies in particular that there exists $\lambda' \in \mathbb{F}_q^\times$ such that $(\widetilde{y}_0, -\widetilde{x}_0) = (\lambda' P_i, \lambda' Q_i)$. Using Equation (5.2) for the last equality, we have

$$\frac{1}{|Q_i|\;|Q_n|} = \frac{1}{|\widetilde{x}_0|\;|Q_n|} = \Big|\, \frac{\widetilde{y}_0}{-\widetilde{x}_0} - \frac{P_n}{Q_n} \,\Big| = \Big|\, \frac{P_i}{Q_i} - \frac{P_n}{Q_n} \,\Big| = \frac{1}{|Q_i|\;|Q_{i+1}|}\;.$$

Since $|Q_{i+1}| < |Q_n|$ if $i < n-1$, this implies that $i = n-1$. Since $(\widetilde{x}_0, \widetilde{y}_0)$ belongs to $\mathrm{Sol}_{\widetilde{a}, \widetilde{b}}$ and by Equation (5.3), we have $\lambda' = 1$. Hence $(\widetilde{y}_0, -\widetilde{x}_0) = (P_{n-1}, Q_{n-1})$ as wanted.

Since the pair (x_0, y_0) is a solution to the equation $ax + by = 1$ if and only if the pair $((-1)^n \lambda\, x_0, (-1)^n \lambda\, y_0)$ is a solution to the equation $\widetilde{a}\, x + \widetilde{b}\, y = 1$, the result follows. □

The following result is an analogue in the field of formal Laurent series to the main result of [7] in the real field. It gives an application of Theorem 1.1 to the distribution properties of the continued fraction expansions of elements of K. For every $v = (a, b) \in R^2_{\mathrm{prim}}$, we denote by $\Big(\frac{P_i(v)}{Q_i(v)}\Big)_{-1 \leqslant i \leqslant n_v}$ the sequence of convergents of $\frac{a}{b}$ and by $\lambda_v \in \mathbb{F}_q^\times$ the unique element such that $v = (\lambda_v P_{n_v}(v), \lambda_v Q_{n_v}(v))$. We denote by $\mu_{Y^{-1}\mathscr{O}}$ the Haar measure of the compact additive group $Y^{-1}\mathscr{O}$, normalized to be a probability measure.

Corollary 5.2. *Let $P_* = \prod_{i=1}^k \pi_i$ be a nonzero polynomial in the Euclidean ring R, with prime factors $\pi_1, \dots, \pi_k$. Let $c_{P_*} = \frac{q^{1+\deg P_*}\prod_{i=1}^k\left(1-\frac{1}{q^{\deg \pi_i}}\right)}{(q-1)^2}$. For the weak-star convergence of measures on $Y^{-1}\mathscr{O}$, we have, as $n \to +\infty$,*

$$c_{P_*}\, q^{-2n} \sum_{v=(P,Q)\in R^2_{\rm prim}\,:\,\deg P<\deg Q = n,\; P_*\,|\,P} \Delta_{\frac{(-1)^{n_v} Q_{n_v-1}(v)}{\lambda_v^2\, Q_{n_v}(v)}} \;\overset{*}{\rightharpoonup}\; \mu_{Y^{-1}\mathscr{O}}\ .$$

Furthermore, there exists $\tau \in \,]0, \frac{1}{8}]$ such that for all $\epsilon, \delta > 0$, there is a mutiplicative error term in the above equidistribution claim of the form $1 + \mathrm{O}_{\delta,\,P_*}(q^{2n(-\tau+\delta)}\,\|g\|_\epsilon)$ when evaluated on g for every ϵ-locally constant map $g : Y^{-1}\mathscr{O} \to \mathbb{R}$.

Proof. The result follows by applying the joint equidistribution Theorem 4.5 with $\mathbf{C} = \mathbb{P}^1$, $\omega = \omega_\infty$ and $I = P_* R$ (so that $c_I = \frac{q^{\deg P_*}\prod_{i=1}^k\left(1-\frac{1}{q^{\deg \pi_i}}\right)}{q^2(q-1)}$ by Equations (4.3) and (2.2)) to the characteristic function of the set

$$S^1_\infty - S^{1,\sharp}_\infty = \{(x,y) \in \widehat{K}^{\,2} : |x| < |y| = 1\}$$

on the left factor, using the following remarks.

- Let $v = (a, b) \in R^2_{\rm prim}$ be such that $|a| < |b|$ (or equivalently such that $\breve{v} \in S^1_\infty - S^{1,\sharp}_\infty$), and let $(P_i/Q_i)_{-1\leqslant i\leqslant n_v}$ be the sequence of convergents of a/b. Lemma 5.1 (actually Equation (5.3) is sufficient) says that we may take $w_v = (-(-1)^{n_v}\lambda_v^{-1}P_{n_v-1}, -(-1)^{n_v}\lambda_v^{-1}Q_{n_v-1})$. Since $|P_i| < |Q_i|$ for $0 \leqslant i \leqslant n_v$, we have

$$\frac{z_{w_v}}{z_v} = \frac{y_{w_v}}{y_v} = \frac{-(-1)^{n_v}\lambda_v^{-1}Q_{n_v-1}}{\lambda_v Q_{n_v}} = \frac{-(-1)^{n_v}Q_{n_v-1}}{\lambda_v^2 Q_{n_v}}\ .$$

- The map from $Y^{-1}\mathscr{O}$ to $\widehat{K}/R$ defined by $f \mapsto f+R$ is a homeomorphism and an isomorphism of additive groups, which maps the probability measure $\mu_{Y^{-1}\mathscr{O}}$ to $q\,\mu_{\widehat{K}/R}$, since $\mu_{\widehat{K}/R}$ has total mass $\frac{1}{q}$ by Equation (2.6).

- The map from $Y^{-1}\mathscr{O}$ to itself defined by $f \mapsto -f$ is an homeomorphism preserving $\mu_{Y^{-1}\mathscr{O}}$.

- We have $\mu_{S^1_\infty}(S^1_\infty - S^{1,\sharp}_\infty) = \mu_{\widehat{K}} \otimes \mu_{\widehat{K}}(Y^{-1}\mathscr{O} \times \mathscr{O}^\times) = \frac{1}{q}\,(1-\frac{1}{q}) = \frac{q-1}{q^2}$, so that

$$\mu_{S^1_\infty}(S^1_\infty - S^{1,\sharp}_\infty)\, d\mu_{\widehat{K}/R}(-f+R) = \frac{q-1}{q^3}\, d\mu_{Y^{-1}\mathscr{O}}(f)$$

for (almost) every $f \in Y^{-1}\mathscr{O}$. $\square$

Remark. A change of variable (by multiplication by an appropriate element of $\mathbb{F}_q^\times$, which preserves the measures) in the finitely many clopen subsets of $v \in S^1_\infty - S^{1;\sharp}_\infty$ such that the data $(n_v \bmod 2, \lambda_v)$, varying in the finite set $(\mathbb{Z}/2\mathbb{Z}) \times \mathbb{F}_q^\times$, is constant, allows to prove that

$$c_{P_*}\; q^{-2n} \sum_{v=(P,Q)\in R^2_{\rm prim}\,:\,\deg P<\deg Q=n,\; P_*\,|\,P} \Delta_{\frac{Q_{n_v-1}(v)}{Q_{n_v}(v)}} \;\overset{*}{\rightharpoonup}\; \mu_{Y^{-1}\mathscr{O}}\;.$$

References

[1] M. Aka, M. Einsiedler, and U. Shapira. *Integer points on spheres and their orthogonal grids*. J. Lond. Math. Soc. **93** (2016) 143–158.

[2] M. Aka, M. Einsiedler, and U. Shapira. *Integer points on spheres and their orthogonal lattices*. Invent. Math. **206** (2016) 379–396.

[3] J. Athreya, A. Ghosh, and A. Prasad. *Ultrametric logarithm laws, II*. Monat. Math. **167** (2012) 333–356.

[4] Y. Benoist and H. Oh. *Effective equidistribution of S-integral points on symmetric varieties*. Ann. Inst. Fourier **62** (2012) 1889–1942.

[5] A. Broise-Alamichel, J. Parkkonen, and F. Paulin. *Equidistribution and counting under equilibrium states in negative curvature and trees. Applications to non-Archimedean Diophantine approximation*. With an Appendix by J. Buzzi. Prog. Math. **329**, Birkhäuser, 2019.

[6] M. Cowling, U. Haagerup, and R. Howe. *Almost L^2 matrix coefficients*. J. reine angew. Math. **387** (1988) 97–110.

[7] E. Dinaburg and Y. G. Sinai. *The statistics of the solutions of the integer equation $ax-by=$ ± 1*. Funk. Analiz Prilo. **24** (1990), 1–8.

[8] W. Duke. *Hyperbolic distribution problems and half-integral weight Maass forms*. Invent. Math. **92** (1988) 73–90.

[9] W. Duke. *Rational points on the sphere*. Rankin memorial issues. Ramanujan J. **7** (2003) 235–239.

[10] W. Duke. *An introduction to the Linnik problems*. In "Equidistribution in number theory, an introduction", pp 197–216, NATO Sci. Ser. II Math. Phys. Chem. **237**, Springer 2007.

[11] W. Duke, Z. Rudnick and P. Sarnak. *Density of integer points on affine homogeneous varieties*. Duke Math. J. **71** (1993) 143–179.

[12] M. Einsiedler, E. Lindenstrauss, P. Michel, and A. Venkatesh. *Distribution of periodic torus orbits and Duke's theorem for cubic fields*. Ann. Math. **173** (2011) 815–885.

[13] J. Ellenberg, P. Michel, and A. Venkatesh. *Linnik's ergodic method and the distribution of integer points on spheres*. In "Automorphic representations and L-functions", Tata Inst. Fundam. Res. Stud. Math. **22** (2013) 119–185.

[14] A. Eskin and C. McMullen. *Mixing, counting, and equidistribution in Lie groups*. Duke Math. J. **71** (1993) 181–209.

[15] A. Good. *On various means involving the Fourier coefficients of cusp forms*. Math. Z. **183** (1983) 95–129.

[16] A. Gorodnik and A. Nevo. *Counting lattice points*. J. reine. angew. Math. **663** (2012) 127–176.

[17] D. Goss. *Basic structures of function field arithmetic*. Erg. Math. Grenz. **35**, Springer Verlag 1996.

[18] T. Horesh and Y. Karasik. *Equidistribution of primitive vectors in $\mathbb{Z}^n$*. Preprint [arXiv: 1903.01560].

[19] T. Horesh and A. Nevo. *Horospherical coordinates of lattice points in hyperbolic space: effective counting and equidistribution*. Preprint [arXiv:1612.08215].

[20] S. Lang. $\mathrm{SL}_2(\mathbb{R})$. Addison-Wesley, 1975.

[21] A. Lasjaunias. *A survey of Diophantine approximation in fields of power series*. Monat. Math. **130** (2000) 211–229.

[22] H. Nagao. *On* $\mathrm{GL}(2, K[x])$. J. Inst. Polytech. Osaka City Univ. Ser. A **10** (1959) 117–121.
[23] F. Paulin. *Groupe modulaire, fractions continues et approximation diophantienne en caractéristique p*. Geom. Dedi. **95** (2002) 65–85.
[24] M. S. Risager and Z. Rudnick. *On the statistics of the minimal solution of a linear Diophantine equation and uniform distribution of the real part of orbits in hyperbolic spaces*. In "Spectral analysis in geometry and number theory", 187–194, Contemp. Math. **484**, Amer. Math. Soc. 2009.
[25] M. Rosen. *Number theory in function fields*. Grad. Texts Math. **210**, Springer Verlag, 2002.
[26] W. M. Schmidt. *On continued fractions and diophantine approximation in power series fields*. Acta Arith. **XCV** (2000) 139–166.
[27] W. M. Schmidt. *The distribution of sub-lattices of* $\mathbb{Z}^m$. Monatshefte Math. **125** (1998) 37–81.
[28] J.-P. Serre. *Arbres, amalgames,* SL_2. 3ème éd. corr., Astérisque **46**, Soc. Math. France, 1983.
[29] A. Veca. *The Kunze-Stein phenomenon*. Ph. D. Thesis, Univ. of New South Wales, 2002.
[30] A. Weil. *On the analogue of the modular group in characteristic p*. In "Functional Analysis and Related Fields" (Chicago, 1968), pp. 211–223, Springer Verlag, 1970.

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