A logarithm law for automorphism groups of trees

Sa'ar Hersonsky Frédéric Paulin

Abstract

Let Γ be a geometrically finite tree lattice. We prove a Khintchine-Sullivan type theorem for the Hausdorff measure of the set of points at infinity of the tree that are well approximated by the parabolic fixed points of Γ. Using Bruhat-Tits trees, an application is given for the Diophantine approximation of formal Laurent series in the variable X^{-1} over the finite field $\mathbb{F}_q$ by rational fractions in X over $\mathbb{F}_q$, satisfying some congruence properties. [1, 2]

1 Introduction

Let T be a locally finite tree, ∂T be its space of ends, and $\mathrm{Aut}(T)$ be its locally compact group of automorphisms. To simplify the statements in the introduction, we assume that T has no degree 1 or 2 vertex and that ${}_{\mathrm{Aut}(T)}\backslash T$ is a finite graph. A *lattice* of T is a discrete subgroup with a cofinite Haar measure in $\mathrm{Aut}(T)$. It is *uniform* if $\Gamma\backslash T$ is a finite graph. In [2, chap. 4], H. Bass and A. Lubotzky gave numerous examples of non uniform lattices. In [17], the second author has introduced a special class of discrete subgroups of $\mathrm{Aut}(T)$, where the interesting ones are non uniform lattices, called geometrically finite (see also Section 2). In particular, they contain all the algebraic examples. More precisely, let $\hat{K}$ be a non-archimedian local field, and $\underline{G}$ be a connected semi-simple algebraic group over $\hat{K}$ with $\hat{K}$-rank 1. Let T be the Bruhat-Tits tree of $(\underline{G}, \hat{K})$ (see [6, 20]), endowed with its action of $\underline{G}(\hat{K})$. By the work of J.-P. Serre, M. Raghunathan and A. Lubotzky [15], every lattice contained in the image of $\underline{G}(\hat{K})$ in $\mathrm{Aut}(T)$ is geometrically finite in the sense of [17].

Let Γ be a fixed geometrically finite lattice in $\mathrm{Aut}(T)$. Let Γ_∞ be a parabolic subgroup of Γ, ∞ be its unique fixed point in ∂T, and $(H_r)_{r\in\Gamma\infty}$ be its associated family of horoballs (see [17] and also Section 2 for definitions). For every r which is different from ∞ in the orbit $\Gamma\infty$, define $D(r) = d(H_\infty, H_r)$, where d is the distance on T. Let $d_{\partial T}$ be a usual distance on ∂T, δ be its Hausdorff dimension and let μ be the associated Hausdorff measure in dimension δ (see Section 2).

Theorem 1.1 *Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a map such that $\log\psi$ is Lipschitz. Let $E(\psi)$ be the set of points ξ in $\partial T - \{\infty\}$ such that there exist infinitely many r in $\Gamma\infty$ with*

$$d_{\partial T}(\xi, r) \leq \psi(D(r))\, e^{-D(r)} .$$

Then $\mu(E(\psi)) = 0$ (respectively $\mu({}^cE(\psi)) = 0$) if and only if the integral $\int_1^{+\infty}\psi(t)^\delta dt$ converges (respectively diverges).

[1] **AMS codes:** 20E08, 11J61, 20G25 .

[2] **Keywords:** Tree lattice, logarithm law, Diophantine approximation, Bruhat-Tits tree, formal Laurent series.

A more general statement is given by Theorem 2.2. The main consequence is the following corollary, called the *logarithm law* for Γ.

Define the height function $h : T \to [0, +\infty[$ for Γ by $h(x) = 0$ if $x \in T - \bigcup_{r\in\Gamma\infty} H_r$, and if $x \in H_r$, then $h(x) = d(x, \partial H_r)$.

Corollary 1.2 *For every vertex x in T, let c_η denote the geodesic ray starting from x and ending at $\eta \in \partial T$. Then for μ-almost every η in ∂T,*

$$\limsup_{t\to+\infty} \frac{h(c_\eta(t))}{\log t} = \frac{1}{\delta}\ .$$

Theorem 1.1 and this corollary are analogues, in the setting of trees and in particular for non archimedean rank one lattices, of the similar well-known results for

- $\mathrm{PSL}_2(\mathbb{Z})$ acting on the real hyperbolic plane $\mathbb{H}^2_\mathbb{R}$, due to Khintchine [12],
- a lattice in the isometry group of a real hyperbolic space $\mathbb{H}^n_\mathbb{R}$, due to D. Sullivan [23],
- a lattice in the isometry group of a symmetric space of non compact type, due to D. Kleinbock and G. Margulis [13],
- a geometrically finite group of isometries acting on $\mathbb{H}^n_\mathbb{R}$ due to B. Stratmann and S.L. Velani [22], and
- more general geometrically finite groups of isometries of complete simply connected Riemannian manifolds with negative curvature, due to the authors [11].

A particular case of the theorem was announced by J. Athreya [1], when T is the Bruhat-Tits tree of $(\underline{\mathrm{SL}_2}, \mathbb{F}_q((X^{-1})))$ and $\Gamma = \mathrm{SL}_2(\mathbb{F}_q[X])$.

Denote by $|\cdot|_\infty$ the usual absolute value on the field $\widehat{K} = \mathbb{F}_q((X^{-1}))$ of formal Laurent series in X^{-1} over $\mathbb{F}_q$. Endowed $\widehat{K}$ with its Haar measure. Let Q_0 be a fixed non zero element in the ring $A = \mathbb{F}_q[X]$ of polynomials over $\mathbb{F}_q$. By using [16, 17], we obtain the following corollary, concerning the Diophantine approximation of elements of $\widehat{K}$ by rational fractions.

Corollary 1.3 *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a map such that $u \mapsto \log \varphi(e^u)$ is Lipschitz. If the integral $\int_1^{+\infty} \varphi(t)/t\ dt$ diverges (resp. converges), then for almost every (resp. almost no) f in $\widehat{K}$, there exist infinitely many couples (P, Q) in $A \times (A - \{0\})$, where P is coprime with Q and Q is divisible by Q_0, such that $|f - P/Q|_\infty \le \varphi(|Q|_\infty)/|Q|^2_\infty$.*

A more general statement is given by Corollary 2.3. The special case of Corollary 1.3 when Q_0 is the constant polynomial 1 was announced by J. Athreya [1].

Acknowledgement. The second author acknowledges the support of the University of Georgia for a stay during which this paper was written.

2 Notations, statements and applications

We refer to [3, 18] for basic definitions about CAT(-1) geodesic metric spaces (among which are trees), their horospheres, their boundaries and their discrete groups of isometries.

Let T be a locally finite tree, ∂T be its space of ends and $T \cup \partial T$ be its compactification by its ends. Such a tree is called *uniform* if there exists a discrete subgroup Γ in $\mathrm{Aut}(T)$ such that the quotient graph $\Gamma\backslash T$ is finite. The tree T is endowed with the maximal geodesic distance d making each edge isometric to the interval $[0, 1]$. For every x, y in $T \cup \partial T$, we denote by $[x, y]$ the geodesic segment, ray or line between them, with the usual convention concerning the endpoints. Fix a point x_0 in T. Let d_{x_0} be the *visual distance* on ∂T seen from x_0. It is defined by $d_{x_0}(\eta, \eta') = e^{-d(x_0,p)}$ where $[x_0, \eta[\, \cap\, [x_0, \eta'[\, = [x_0, p]$ if $\eta \neq \eta'$. The *Buseman function* $\beta_\xi : X \times X \to \mathbb{R}$ of a point ξ in ∂T is defined by $\beta_\xi(x, y) = d(x, u) - d(y, u)$ for every u in T close enough to ξ. An *horoball* centered at ξ is the preimage by $y \mapsto \beta_\xi(x_0, y)$ of $[t, +\infty[$ for some $t \in \mathbb{R}$.

Let Γ be a discrete subgroup of $\mathrm{Aut}(T)$. We assume that Γ is *non-elementary*, i.e. that Γ preserves no point nor pair of points in $T \cup \partial T$. Let $\Lambda\Gamma$ be its *limit set*, i.e. the smallest non empty closed Γ-invariant subset in ∂T. Let $C\Lambda\Gamma$ be the minimal non empty subtree of T which is invariant by Γ.

The *critical exponent* of Γ is the unique number δ in $[0, +\infty]$ such that the *Poincaré series* $\sum_{\gamma\in\Gamma} e^{-s\, d(x_0,\gamma x_0)}$ of Γ converges for $s > \delta$ and diverges for $s < \delta$. The group Γ is called *of divergent type* if its Poincaré series diverges at $s = \delta$.

If Γ is of divergent type with a finite non zero critical exponent, then there exists (see [8]) a family $(\mu_x)_{x\in T}$ of finite measures on ∂T, with support $\Lambda\Gamma$, called the *Patterson-Sullivan density*, which is unique up to a positive scalar factor, such that

- $\forall\, \gamma \in \Gamma\,,\ \ \gamma_*\mu_x = \mu_{\gamma x}$,
- $\forall\, x, y \in T,\ \forall\, \xi \in \partial T\,,\ \ \frac{d\mu_x}{d\mu_y}(\xi) = e^{-\delta\beta_\xi(x,y)}$.

An element ω in $\Lambda\Gamma$ is a *conical limit point* if there exist a sequence $(\gamma_n)_{n\in\mathbb{N}}$ in Γ, a geodesic ray c in T ending at ω and $A \geq 0$ such that $d(\gamma_n x_0, c) \leq A$ for every n in $\mathbb{N}$. An element ω in $\Lambda\Gamma$ is called a *bounded parabolic point* if its stabilizer Γ_ω in Γ acts properly discontinuously with compact quotient on $\partial T - \{\omega\}$. A *parabolic subgroup* Γ_∞ of Γ is a maximal infinite locally finite subgroup (see [2, 17] for other characterizations and the following properties). In particular, there is one and only one point in $T \cup \partial T$ fixed by Γ_∞ and it belongs to ∂T.

A discrete subgroup Γ of $\mathrm{Aut}(T)$ is *geometrically finite* if it is non elementary and if every point in $\Lambda\Gamma$ is either a conical limit point or a bounded parabolic point. A characterization in terms of the quotient graph of groups $\Gamma\backslash\backslash T$ (see [2] for a definition when Γ acts without inversion) and the basic properties of geometrically finite subgroups Γ of $\mathrm{Aut}(T)$ are given in [17]. In particular, the point fixed by a parabolic subgroup of Γ is then a bounded parabolic point.

From now on, we fix a geometrically finite subgroup Γ in $\mathrm{Aut}(T)$ and a parabolic subgroup Γ_∞ of Γ. Let ∞ be the unique fixed point of Γ_∞ in ∂T and let $\Gamma\infty$ be its orbit under Γ. In [17], it is proved that there exists a unique Γ-invariant family of horoballs $(H_r)_{r\in\Gamma\infty}$, with pairwise disjoint interiors, with H_r centered at r and maximal with respect to inclusion.

Let d_∞ be the *Hamenstädt distance* on $\partial T - \{\infty\}$ associated to H_∞ (see [10, Appendix], [17]). It is defined by $d_\infty(\eta, \eta') = e^{-d_\pm(p,q)}$ where $\{p\} = \partial H_\infty \cap]\infty, \eta[,\ ]\infty, q] =]\infty, \eta[\cap]\infty, \eta'[$ and $d_\pm(p, q)$ is the signed distance between p and q on $]\infty, \eta[$, if $\eta \neq \eta'$. The Hausdorff dimension and the Hausdorff measure class of $\Lambda\Gamma - \{\infty\}$ with respect to the distance d_{x_0} coincides with the Hausdorff dimension and the Hausdorff measure class of $\Lambda\Gamma - \{\infty\}$ with respect to the Hamenstädt distance d_∞ respectively, as locally on $\partial T - \{\infty\}$, the distances d_{x_0} and d_∞ differ by a positive scalar factor.

If ρ is the geodesic ray in T starting from x_0 and converging to ∞, then the measures $e^{\delta t}\mu_{\rho(t)}$ on $\partial T - \{\infty\}$ converge, as $t \to +\infty$, to a measure μ_∞ on $\partial T - \{\infty\}$, called the *Patterson-Sullivan measure of* Γ *on* $\partial T - \{\infty\}$, supported on $\Lambda\Gamma - \{\infty\}$. The measure μ_∞ has the same measure class as μ_{x_0} on $\partial T - \{\infty\}$ (see [11]).

The following theorem summarizes the known results on Γ that we will use throughout this paper.

Theorem 2.1 *Let Γ be a geometrically finite subgroup of $Aut(T)$ such that $C\Lambda\Gamma$ is a uniform tree without degree 2 vertices. Let Γ_∞ be a parabolic subgroup of Γ. Then the following assertions hold.*

(1) The group Γ has a finite non zero critical exponent, and is of divergent type. Its action on $\Lambda\Gamma$ is ergodic with respect to the measure μ_{x_0}. This measure has no atom.

(2) There exists a constant $c_1 > 0$ such that for every n in $\mathbb{N}$,

$$\frac{1}{c_1} e^{n\delta} \leq \mathrm{Card}\{\gamma \in \Gamma \ : \ d(x_0, \gamma x_0) \leq n\} \leq c_1 e^{n\delta} \ .$$

(3) The critical exponent δ equals the Hausdorff dimension of $\Lambda\Gamma$ with respect to the distance d_{x_0}.

(4) The critical exponent of Γ_∞ is equal to $\delta/2$.

(5) There exists a constant $c_2 > 0$ such that for every n in $\mathbb{N}$,

$$\frac{1}{c_2} e^{n\delta/2} \leq \mathrm{Card}\{\gamma \in \Gamma_\infty \ : \ d(x_0, \gamma x_0) \leq n\} \leq c_2 e^{n\delta/2} \ .$$

Proof. Let $T' = C\Lambda\Gamma$. Then T', endowed with the restricted action of Γ, satisfies the hypotheses of the theorem if and only if T, endowed with the action of Γ, does. If the assertions (1)-(5) are true for T', then they are also true for T. Hence we may assume that $T = C\Lambda\Gamma$. Up to minor changes in the following geometric series argument when replacing T by its barycentric subdivision, we may assume that Γ acts without inversion.

Recall that the quotient graph of groups $\Gamma\backslash\backslash T$ is the union of a finite graph of finite groups and of finitely many cuspidal rays of groups (see [2, 17] for the definitions and [17] for the proof). The covolume of Γ in $\mathrm{Aut}(T)$ is (see [2]) a multiple of the sum of the inverses of the cardinals of the vertex groups in $\Gamma\backslash\backslash T$. By an easy geometric series argument, the fact that T has no vertex of degree 2 (here and below, the assumption that there are no degree 2 vertices inside an horoball of an equivariant family of pairwise disjoint horoballs centered at all parabolic fixed points is sufficient) implies in particular that Γ is a lattice in T. (See for instance [7], or simply note that if Γ_n is the stabilizer of the

n-th vertex of a cuspidal ray, then $[\Gamma_{n+1} : \Gamma_n] \geq 2$ as T has no vertex of degree 2, hence $\sum_n 1/|\Gamma_n| \leq \sum_n 1/2^n$ is finite.)

(1) The finiteness (and non vanishing) of the critical exponent follows for instance from [7, Prop. 1.7]. By [7, Coro. 6.5] for instance, the group Γ is of divergent type. By [18, Theorem 1.7] and [18, Corollary 1.8] for instance, the measure μ_{x_0} is ergodic and has no atom.

(4) See for instance [5, Prop. 3.1].

(2) As every vertex of T has degree at least 3, and as there exists at least one parabolic subgroup, the subgroup of $\mathbb{Z}$ generated by the hyperbolic translation lengths of the elements of Γ equals to $\mathbb{Z}$ (see [5]). By (1), (4) and [18, Théorème 1.11], the Bowen-Margulis measure of Γ (see for instance [18] for a definition of this measure) is finite. Hence by corollaire 2 following théorème 4.1.1 in [18], the result holds.

(3) This follows for instance from [7, Coro. 6.5].

(5) This follows for instance from [5, Prop. 3.1]. □

Before providing a more general statement than Theorem 1.1, we will introduce the class of functions we will work with.

A map $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called *slowly varying* (see [23]) if it is measurable and if there exist constants $B > 0$ and $A \geq 1$ such that for every x, y in $\mathbb{R}_+$, if $|x - y| \leq B$, then $\psi(y) \leq A\,\psi(x)$. Recall (see for instance [11, Sec. 5]) that this implies that ψ is locally bounded, hence it is locally integrable; also, if $\log\psi$ is Lipschitz, then ψ is slowly varying.

Theorem 2.2 *Let Γ be a geometrically finite subgroup of $Aut(T)$ such that $C\Lambda\Gamma$ is a uniform tree without degree 2 vertices. Let Γ_∞ be a parabolic subgroup whose fixed point will be denoted by ∞. Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a slowly varying map. Let $E(\psi)$ be the set of points ξ in $\partial T - \{\infty\}$ such that there exist infinitely many r in $\Gamma\infty$ with $d_\infty(\xi, r) \leq \psi(D(r))e^{-D(r)}$. Then $\mu_\infty(E(\psi)) = 0$ (respectively $\mu_\infty({}^cE(\psi)) = 0$) if and only if the integral $\int_1^{+\infty} \psi(t)^\delta dt$ converges (respectively diverges).*

The proof of this theorem will be given in Section 3. Note that for some of the Bruhat-Tits trees (the ones that are biregular of type $(2, q)$, see [6]), we have to remove the vertices of degree 2 in order to verify the hypothesis of the above theorem. In the new tree, the critical exponent is multiplied by 2, the complexities $D(r)$'s are divided by 2 and the distance d_∞ is replaced by its square root. Hence the conclusion of Theorem 2.2 is also true for these Bruhat-Tits trees.

Let us show now that Theorem 1.1 and Corollary 1.2 in the introduction follows from Theorem 2.2.

Proof of Theorem 1.1. Since Γ in the statement of Theorem 1.1 is assumed in addition to be a lattice, we have $\Lambda\Gamma = \partial T$. Hence, since T has no terminal vertex, $C\Lambda\Gamma = T$. A tree whose quotient by its automorphism group is a finite graph, and whose automorphism group contains a (non-uniform) lattice is unimodular, hence uniform (see [2, para. 1.2, 0.5 (6)]). As we noticed above, a map whose logarithm is Lipschitz is slowly varying. Hence, if the hypotheses of Theorem 1.1 are satisfied, then so are the hypotheses of Theorem 2.2. The critical exponent of Γ coincides with the Hausdorff dimension of $\Lambda\Gamma$ with respect to the d_{x_0} metric. The distances d_∞ and d_{x_0} are locally equivalent, and the measure classes of the measure μ_∞ and of the Hausdorff measure of d_{x_0} coincide. Hence Theorem 1.1 does follow from Theorem 2.2. □

Proof of Corollary 1.2. The proof follows from Theorem 1.1 along the same line as in the other cases we recalled in the introduction (see for instance [11, Sect. 6]). We use the maps $\psi(t) = t^{-\kappa}$, and the fact that for a geodesic line starting from ∞, ending at ξ, and entering in H_r for some $r \in \Gamma\infty - \{\infty\}$ with highest penetration point ξ_r, we have $-\log d_\infty(\xi, r) = D(r) + d(\xi_r, \partial H_r)$. $\square$

We now give some applications of our results. We refer for instance to [14, 19] for nice surveys of the Diophantine approximation properties of elements in $\widehat{K} = \mathbb{F}_q((X^{-1}))$ by elements in $K = \mathbb{F}_q(X)$. Recall the definition of the absolute value of $f \in \widehat{K} - \{0\}$. Let $f = \sum_{i=n}^{\infty} a_i X^{-i}$ where $a_n \neq 0$. Then we define $\nu(f) = n \in \mathbb{Z}$ and $|f|_\infty = q^{-\nu(f)}$. Endow the locally compact additive group $\widehat{K}$ with its (unique up to a constant factor) Haar measure. Let $A = \mathbb{F}_q[X]$.

Corollary 2.3 *Let Γ be a finite index subgroup of $\mathrm{SL}_2(A)$. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a map with $u \mapsto \varphi(e^u)$ slowly varying. If the integral $\int_1^{+\infty} \varphi(t)/t\ dt$ diverges (resp. converges), then for almost every (resp. almost no) f in $\widehat{K}$, there exist infinitely many couples (P, Q) in $A \times (A - \{0\})$, where P is coprime with Q and $P/Q \in \Gamma\infty$, such that $|f - P/Q|_\infty \leq \varphi(|Q|_\infty)/|Q|_\infty^2$.*

Proof. In [16], a geometric interpretation of the above Diophantine approximation is given in terms of the Bruhat-Tits tree $\mathbb{T}_q$ of $(\underline{\mathrm{SL}}_2, \widehat{K})$ (see also [17, 4]). Identify as usual $\partial\mathbb{T}_q$ and $\widehat{K} \cup \{\infty\}$, and let x_0 be the standard base point in $\mathbb{T}_q$ (see [20]). Note that the Hausdorff dimension of d_{x_0} is $\log q$, as $\mathbb{T}_q$ is a regular tree of degree $q+1$ (see [20]). Recall that the kernel of the action of $\mathrm{SL}_2(\widehat{K})$ on $\mathbb{T}_q$ is finite, hence denoting by the same letter a lattice in $\mathrm{SL}_2(\widehat{K})$ and its image in $\mathrm{Aut}(\mathbb{T}_q)$ causes no problem.

Now Γ is a lattice of $\mathbb{T}_q$, hence $C\Lambda\Gamma = \mathbb{T}_q$ is a uniform tree without degree 2 vertices. Let Γ_∞ be the stabilizer in Γ of the point ∞ in $\partial\mathbb{T}_q = \widehat{K} \cup \{\infty\}$, which is a parabolic subgroup of Γ. Since Γ is contained in $\mathrm{SL}_2(A)$, the subset $\Gamma\infty - \{\infty\}$ is contained in K. By [16, 17], the following assertions hold.

- There exists a constant $c > 0$ such that the Hamenstädt distance on $\partial\mathbb{T}_q - \{\infty\} = \widehat{K}$ satisfies $d_\infty(f, f') = c|f - f'|_\infty^{\frac{1}{\log q}}$, for every f, f' in $\widehat{K}$ (see [16, Coro. 5.2]).
- There exists a constant $c' > 0$ such that for every P/Q in $\Gamma\infty - \{\infty\}$, where P and Q are coprime polynomials such that $\nu(P) \geq \nu(Q)$, we have $D(P/Q) = -2\nu(Q) + c' = (2\log|Q|_\infty)/\log q + c'$ (see [16, Coro. 6.1]).

The constants are due to the fact that for $\gamma \in \Gamma$, the horoball centered at a point $\gamma\infty$ in the family associated to Γ contains, but is not necessarily equal to, the horoball centered at a point $\gamma\infty$ in the family associated to $\mathrm{SL}_2(A)$. Hence, for every $c'' > 0$, we have $d_\infty(f, P/Q) \leq c'' e^{-D(P/Q)}$ if and only if $|f - P/Q|_\infty \leq (\frac{c'' e^{-c'}}{c})^{\log q}/|Q|_\infty^2$. Define $\psi(t) = ce^{c'}\ \varphi(q^{(t-c')/2})^{\frac{1}{\log q}}$, for every $t > 0$. So that ψ is slowly varying by the assumption on φ. And $d_\infty(f, P/Q) \leq \psi(D(P/Q))e^{-D(P/Q)}$ if and only if $|f - P/Q|_\infty \leq \varphi(|Q|_\infty)/|Q|_\infty^2$.

By an easy change of variables, the integral $\int_1^{+\infty} \varphi(t)/t\ dt$ diverges if and only if $\int_1^{+\infty} \psi(t)^{\log q}\, dt$ diverges. By invariance, the Haar measure of $\widehat{K}$ is equal to the Hausdorff measure of d_∞, up to a constant positive factor. In particular, their measure classes are the same. The result follows. $\square$

Proof of Corollary 1.3. Apply Corollary 2.3 to the congruence subgroup Γ of $\mathrm{SL}_2(A)$, consisting of the elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that c is divisible by Q_0. The subgroup Γ has finite index in $\mathrm{SL}_2(A)$, as it is the preimage of the upper triangular subgroup by the reduction modulo Q_0 map:

$$\mathrm{Ker}(\mathrm{SL}_2(\mathbb{F}_q[X]) \to \mathrm{SL}_2(\mathbb{F}_q[X]/Q_0\,\mathbb{F}_q[X]))\ ,$$

with the convention that $\mathrm{SL}_2(R)$ is the trivial group if R is the trivial one-point ring. Note that $\Gamma\infty - \{\infty\}$ is the set of fractions P/Q where P is coprime with Q and Q is divisible by Q_0. □

Similar results to Corollary 1.3 can be obtained for instance by varying the congruence subgroup of $\mathrm{SL}_2(\mathbb{F}_q[X])$.

3 Proof of the main result

Throughout this section, we will assume that the hypotheses of Theorem 2.2 are satisfied. As $E(\psi)$ is contained in $\Lambda\Gamma$, and as μ_∞ has support in $\Lambda\Gamma$, we may assume, up to replacing T by $C\Lambda\Gamma$, that $T = C\Lambda\Gamma$.

We first claim that for every constant $\eta > 0$, in order to prove Theorem 2.2, it is sufficient to prove the assertion if $\psi(t) \le \eta$ for every t in $\mathbb{R}_+$. Indeed, since ∞ is a bounded parabolic point and Γ acts discretely on T, there exist, modulo the action of the parabolic subgroup Γ_∞, only finitely many elements r in $\Gamma\infty - \{\infty\}$ with $D(r)$ less than some constant. The above reduction is then justified by the same arguments as in the proof of the similar reduction in [11, Lem. 5.2]. In particular, from now on, we will assume that $\psi \le 1$.

Let Δ be a compact subset of $\partial T - \{\infty\}$ whose images under Γ_∞ cover $\partial T - \{\infty\}$, and whose interior contains a representative of every element of $\Gamma\infty - \{\infty\}$ modulo Γ_∞. The existence of such a subset Δ follows from the fact that ∞ is a bounded parabolic point. The following proposition gives counting estimates on the number of points in $(\Gamma_\infty) \cap \Delta$.

Proposition 3.1 *There exists an integer $N \ge 1$ and a constant $c_3 > 0$ such that for every n in $\mathbb{N}$, if $\mathcal{N}(n)$ denotes the number of elements r in $(\Gamma\infty)\cap\Delta$ such that $n \le D(r) < n+N$, then*

$$\frac{1}{c_3} e^{\delta n} \le \mathcal{N}(n) \le c_3 e^{\delta n}\ .$$

Proof. Let $\mathcal{N}'(n) = \mathrm{Card}\{[\gamma] \in \Gamma_\infty\backslash(\Gamma - \Gamma_\infty)/\Gamma_\infty \ :\ d(H_\infty, \gamma H_\infty) \le n\}$. We first claim that there exists a constant $c_1' > 0$ such that $e^{\delta n}/c_1' \le \mathcal{N}'(n) \le c_1' e^{\delta n}$. The proof of this claim is the same (with simplifications due to the tree structure) as the proof of [11, Theo. 3.4]. The crucial ingredients we used in that proof were that ∞ is a bounded parabolic point, that the orbit Γx_0 satisfies some growth property (which is given here by Theorem 2.1 (2)) and that the critical exponent of Γ_∞ is strictly less than δ (this is given here by Theorem 2.1 (4)).

By the properties of Δ, there exists a constant $c > 0$ such that

$$\mathcal{N}'(n) \le \mathrm{Card}\{r \in (\Gamma\infty) \cap \Delta \ :\ D(r) \le n\} \le c\,\mathcal{N}'(n)\ .$$

Hence there exists $c_1'' > 0$ such that $e^{\delta n}/c_1'' \leq \mathrm{Card}\{r \in (\Gamma\infty) \cap \Delta \;:\; D(r) \leq n\} \leq c_1'' e^{\delta n}$. Finally, Proposition 3.1 follows from this, and from [11, Lem. 3.3], as in the proof of [11, Theo. 3.4]. $\square$

If H is an horoball and $t \geq 0$, define $H(t)$ to be the horoball contained in H, whose boundary is at distance t from the boundary of H. To simplify the notation, for every r in $\Gamma\infty - \{\infty\}$, let $H_{r,\psi} = H_r(-\log \psi \circ D(r))$.

For every subset E of T, let $\mathcal{O}_\infty E$ be the set of endpoints of the geodesic lines starting from ∞ and passing through E. Let A_n be the subset of $\partial T - \{\infty\}$, which is the union of the $\mathcal{O}_\infty H_{r,\psi}$'s where r ranges over the elements in $(\Gamma\infty) \cap \Delta$ with $Nn \leq D(r) < (n+1)N$ and N is as in Proposition 3.1.

The proof of Theorem 2.2 is based on the following three propositions. The first one is an adaptation for trees of the fluctuating density property due to Sullivan [23] in the case of finite volume, constant curvature manifolds, to Stratmann-Velani [22] in the geometrically finite, constant curvature manifold case, and to [11] for more general geometrically finite manifolds with variable negative curvature. The proof of the following proposition greatly simplifies in the case of homogeneous trees, but the general case requires some care.

Proposition 3.2 *There exists a constant $c > 0$ such that for every r in $\Gamma\infty - \{\infty\}$ and every $t \geq 0$, one has*

$$\frac{1}{c} e^{-\delta(D(r)+t)} \leq \mu_\infty(\mathcal{O}_\infty H_r(t)) \leq c e^{-\delta(D(r)+t)} \; .$$

Proof. It is the same as the proof of [11, Theo. 4.1], except for the second step of the proof of [11, Prop. 4.3]. To check the hypotheses of [11, Theo. 4.1], we use the following claims: the point ∞ is a bounded parabolic point, Γ is of divergent type (see Theorem 2.1 (1)) and for every n in $\mathbb{N}$,

$$\frac{1}{c_2} e^{n\delta/2} \leq \mathrm{Card}\{\gamma \in \Gamma_\infty \;:\; d(x_0, \gamma x_0) \leq n\} \leq c_2 e^{n\delta/2},$$

(see Theorem 2.1 (5)), so that the constant δ_0 that appears in the statement of [11, Theo. 4.1] is now $\delta/2$. Furthermore, Proposition 4.2 in [11], which is needed in the proof of Theorem 4.1 in [11], is a bit simpler in the setting of trees.

Step 2 in the proof of [11, Prop. 4.3] (which is also needed for the proof of [11, Theo. 4.1]) needs to be adjusted. The arguments that used the hypothesis of pinched curvature are no longer valid. But, with the notation of Step 2 in the proof of [11, Prop. 4.3], the convergence of $(X_i, *_i, d_i, G_i)_{i\in\mathbb{N}}$ follows from the fact that these spaces are uniformly proper (see [9]), as T is a uniform tree. The limit is again a locally finite tree, and the rest of the proof is unchanged. $\square$

Proposition 3.3 *The sum $\sum_{n=0}^{\infty} \mu_\infty(A_n)$ diverges if and only if the integral $\int_1^\infty \psi^{\,\delta}$ diverges.*

Proof. The proof of this proposition is completely similar to the proof of [11, Prop. 5.3]. Lemma 5.4 used in that proof is first proved for the case of trees. Proposition 3.2 above replaces [11, Theo. 4.1]. $\square$

Proposition 3.4 *There exists a constant $c > 0$ such that if n, m are distinct integers, then*

$$\mu_\infty(A_n \cap A_m) \leq c\mu_\infty(A_n)\mu_\infty(A_m).$$

Proof. The proof of this proposition is completely similar to the proof of [11, Prop. 5.5]. Lemma 5.6 used in that proof is even first proved for the case of trees. Proposition 3.2 above replaces [11, Theo. 4.1]. □

We now end the proof of Theorem 2.2. Recall that in any tree T', given an horoball H with point at infinity ξ, and distinct points $\eta, \eta' \in \partial T'$ different from ξ, if x is the intersection with ∂H of the geodesic line $]\eta, \xi[$, then η' belongs to $\mathcal{O}_\eta H$ is and only if the geodesic line $]\eta, \eta'[$ goes through x.

In particular, as the distance between H_∞ and $H_{r,\psi}$ is equal to $D(r) - \log \psi \circ D(r)$, a point ξ in $\partial T - \{\infty\}$ belongs to $\mathcal{O}_\infty H_{r,\psi}$ if and only if $d_\infty(\xi, r) \leq \psi \circ D(r)\, e^{-D(r)}$.

Define $A_\infty = \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k$, which is the set of points in $\partial T - \{\infty\}$ belonging to infinitely many A_n's. With the notation in the statement of Theorem 2.2, as the images of Δ by Γ_∞ cover $\partial T - \{\infty\}$, up to enlarging Δ, we then have $E(\psi) = \Gamma_\infty A_\infty$. As Γ_∞ is countable, we have $\mu_\infty(E(\psi)) > 0$ if and only if $\mu_\infty(A_\infty) > 0$.

The following result is well known (see for instance [21]).

Proposition 3.5 *Let (Y, ν) be a measurable space with a finite measure. Let $(B_n)_{n\in\mathbb{N}}$ be a sequence of measurable subsets of Y such that there exists a constant $c > 0$ with $\nu(B_n \cap B_m) \leq c\nu(B_n)\nu(B_m)$ for every distinct integers n, m. Let $B_\infty = \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} B_k$. Then $\nu(B_\infty) > 0$ if and only if $\sum_{n=0}^{\infty} \nu(B_n)$ diverges.* □

Let Y be a big enough compact subset of $\partial T - \{\infty\}$ which contains $\bigcup_{n\in\mathbb{N}} A_n$. Let ν be the restriction of μ_∞ to Y, and let $B_n = A_n$. We now apply Proposition 3.4, Proposition 3.3, and the result above, to obtain that $\mu_\infty(E(\psi)) > 0$ if and only if $\int_1^\infty \psi^\delta$ diverges. This gives the first conclusion of Theorem 2.2.

Conversely, assume that $\int_1^\infty \psi^\delta$ diverges. Let us prove that $\mu_\infty({}^cE(\psi)) = 0$.

Let $g : [0, +\infty[\to]0, +\infty[$ be a map decreasing to 0 such that $\int_1^\infty (g\psi)^\delta$ diverges. Let $E'(\psi)$ be the set of ξ in $\partial T - \{\infty\}$ such that there exist $c > 0$ and infinitely many r in $\Gamma\infty - \{\infty\}$ with $d_\infty(\xi, r) \leq c\psi(D(r))e^{-D(r)}$. Since $E(g\psi) \subset E'(g\psi)$, the first conclusion of Theorem 2.2 implies that $\mu_\infty(E'(g\psi)) > 0$. It is clear that $E'(g\psi)$ is a measurable subset of ∂T which is invariant under Γ.

By Theorem 2.1 (1), the action of Γ on ∂T for the measure μ_{x_0} is ergodic and without atom. By ergodicity, $\mu_\infty({}^cE'(g\psi)) = 0$. But $E'(g\psi) \subset E(\psi)$ since g is decreasing to 0. Hence $\mu_\infty({}^cE(\psi)) = 0$. This ends the proof of Theorem 2.2. □

References

[1] J. Athreya, *A logarithm law in positive characteristic*, preprint Univ. Chicago, 2005 (arXiv:math.GT/0506531).

[2] H. Bass, A. Lubotzky, *Tree lattices*, Prog. Math. **176**, Birkhäuser, 2001.

[3] M. R. Bridson, A. Haefliger, *Metric spaces with non-positive curvature*, Grund. math. Wiss. **319**, Springer Verlag, 1998.

[4] A. Broise, F. Paulin, *Dynamique sur le rayon modulaire et fractions continues en caractéristique p*, Preprint, Univ. Orsay, 2002 (hal-00014043).

[5] A. Broise, F. Paulin, *Sur le codage du flot géodésique dans un arbre*, to appear in Annales de la Faculté des Sciences de Toulouse (hal-00014071).

[6] F. Bruhat, J. Tits, *Groupes réductifs sur un corps local (données radicielles valuées)*, Pub. Math. I.H.E.S. **41** (1972) 5-252.

[7] M. Burger, S. Mozes, CAT(-1) *spaces, divergence groups and their commensurators*, J. Amer. Math. Soc. **9** (1996) 57-94.

[8] M. Coornaert, *Mesures de Patterson-Sullivan sur le bord d'un espace hyperbolique au sens de Gromov*, Pacific J. Math. **159** (1993) 241-270.

[9] M. Gromov, *Groups of polynomial growth and expanding maps*, Pub. Math. I.H.E.S. **53** (1981) 53-78.

[10] S. Hersonsky, F. Paulin, *On the rigidity of discrete isometry groups of negatively curved spaces*, Comm. Math. Helv. **72** (1997) 349-388.

[11] S. Hersonsky, F. Paulin, *Counting orbit points in coverings of negatively curved manifolds and Hausdorff dimension of cusp excursions*, Erg. Theo. Dyn. Sys. **24** (2004) 803-824.

[12] A. Khinchine, *Continued Fractions*, Univ. Chicago Press, Chicago, 1964.

[13] D. Kleinbock, G. Margulis, *Logarithm laws for flows on homogeneous spaces*, Invent. Math. **138** (1999) 451–494.

[14] A. Lasjaunias, *A survey of Diophantine approximation in fields of power series*, Monat. Math. **130** (2000) 211-229.

[15] A. Lubotzky, *Lattices in rank one Lie groups over local fields*, GAFA **1** (1991) 405-431.

[16] F. Paulin, *Groupe modulaire, fractions continues et approximation diophantienne en caractéristique p*, Geo. Dedicata **95** (2002) 65-85.

[17] F. Paulin, *Groupes géométriquement finis d'automorphismes d'arbres et approximation diophantienne dans les arbres*, Manuscripta Math. **115** (2004) 1-23.

[18] T. Roblin, *Ergodicité du feuilletage horocyclique, mélange du flot géodésique et équidistribution pour les groupes discrets en courbure négative*, Mém. Soc. Math. France **95**, Soc. Math. France, 2003.

[19] W. Schmidt, *On continued fractions and Diophantine approximation in power series fields*, Acta Arith. **XCV** (2000) 139-166.

[20] J.-P. Serre, *Arbres, amalgames,* SL_2, Astérisque **46**, Soc. Math. France, 1983.

[21] V. G. Sprindzŭk, *Mahler's problem in metric number theory*, Transl. Math. Mono **25**, Amer. Math. Soc., 1969.

[22] B. Stratmann, S. L. Velani, *The Patterson measure for geometrically finite groups with parabolic elements, new and old*, Proc. London Math. Soc. **71** (1995) 197-220.

[23] D. Sullivan, *Disjoint spheres, approximation by imaginary quadratic numbers, and the logarithm law for geodesics*, Acta Math. **149** (1982) 215-237.

University of Georgia
Department of Mathematics
Athens, GA 30602, USA
e-mail: saarh@math.uga.edu

and

Ben-Gurion University
Department of Mathematics
Beer-Sheva, Israel
e-mail: saarh@math.bgu.ac.il

Département de Mathématique et Applications, UMR 8553 CNRS
École Normale Supérieure, 45 rue d'Ulm
75230 PARIS Cedex 05, FRANCE
e-mail: Frederic.Paulin@ens.fr