Invariant measures of discrete interacting particles systems: algebraic aspects

Luis Fredes
(joint work with J.F. Marckert).

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Define a set of $\kappa$ colors $E_\kappa := \{0, 1, \ldots, \kappa - 1\}$ for $\kappa \in \{\infty, 2, 3, \ldots\}$.

An **interacting particle system (IPS)** is a stochastic process $(\eta_t)_{t \in \mathbb{R}^+}$ embedded on a graph $G = (V, E)$ with configuration space in $S^V$. We will work with $S = E_\kappa$ and with $G = \mathbb{Z}, \mathbb{Z}/n\mathbb{Z}$.
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\[ t + \Delta t \]

\[ \Delta t \sim \exp(1) \]
Contact process

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Contact process

\[ t + \Delta t \]

\[ \Delta t \sim \exp(1) \]

\[ \Delta t \sim \exp(2\lambda) \]
General case

$t + \Delta t$

\[
L \xrightarrow{\exp(T)} | \quad t
\]
General case

\[ t + \Delta t \]

\[ t \]

\[ L \]
General case

\[ t + \Delta t \]

\[ \Delta t \sim \exp(T[L]) \]

\[ t \]
Invariant measure of particle system

Definition

A distribution $\mu$ on $E^V_\kappa$ is said to be invariant if $\eta^t \sim \mu$ for any $t \geq 0$, when $\eta^0 \sim \mu$. 

Usual questions in the topic:

Existence?
Uniqueness?
Convergence?
Rate of convergence?
Simple representation? (Integrability)
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A distribution \( \mu \) on \( E^\nu_\kappa \) is said to be *invariant* if \( \eta^t \sim \mu \) for any \( t \geq 0 \), when \( \eta^0 \sim \mu \).

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Some things (not much) are known about I.I.D. random invariant distributions of IPS.
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🤔 What about another type of distribution?
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What about another type of distribution?

MARKOV!!!!!!
Consider a Markov distribution (MD) \((\rho, M)\), with Markov Kernel (MK) \(M\) of memory \(m = 1\) and \(\rho\) the invariant measure of \(M\), i.e. for any \(x \in E_{\kappa}^{[a,b]}\)

\[
P(X[a, b] = x) = \rho_{x_a} \prod_{j=a}^{b-1} M_{x_j, x_{j+1}}.
\]
Consider a Markov distribution (MD) \((\rho, M)\), with Markov Kernel (MK) \(M\) of memory \(m = 1\) and \(\rho\) the invariant measure of \(M\), i.e. for any \(x \in E^{[a,b]}_k\)

\[
\mathbb{P}(X[a, b] = x) = \rho_{x_a} \prod_{j=a}^{b-1} M_{x_j, x_{j+1}} =: \gamma(x).
\]
Denote by $\mu^t$ the measure of the process on $E^\mathbb{Z}_{\kappa}$ at time $t \geq 0$.

$t > 0$

\[ X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow X_4 \rightarrow X_5 \rightarrow X_6 \rightarrow X_7 \]

$Y \sim \mu^t = \gamma$

\[ X \sim \mu^0 = \gamma \]

Evolution under $T$
Definition

A process \((X_k, k \in \mathbb{Z}/n\mathbb{Z})\) taking its values in \(E_{\mathbb{Z}/n\mathbb{Z}}\) is said to have a Gibbs distribution \(G(M)\) characterized by a MK \(M\), if for any \(x \in E_{\mathbb{Z}/n\mathbb{Z}}^{[0,n-1]}\),

\[
\mathbb{P}(X^{[0, n-1]} = x) = \frac{\prod_{j=0}^{n-1} M_{x_j, x_{j+1} \mod n}}{\text{Trace}(M^n)} .
\]
Definition

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\[
\mathbb{P}(X[0, n-1] = x) = \frac{\prod_{j=0}^{n-1} M_{x_j, x_{j+1} \mod n}}{\text{Trace}(M^n)} =: \nu(x).
\]
Evolution under $T$

$t > 0$

$X_{10}$ $X_9$ $X_8$ $X_7$ $X_6$ $X_5$

$Y_{10}$ $Y_9$ $Y_8$ $Y_7$ $Y_6$ $Y_5$

$t = 0$

$X_{10}$ $X_9$ $X_8$ $X_7$ $X_6$ $X_5$

$Y_{10}$ $Y_9$ $Y_8$ $Y_7$ $Y_6$ $Y_5$

$Y \sim \mu^t = \nu$

$X \sim \mu^0 = \nu$
Theorem 1 (F- Marckert ’17)

Let $E_\kappa$ be finite, $L = 2$, $m = 1$. If $M > 0$ then the following statements are equivalent for the couple $(T, M)$:

1. $(\rho, M)$ is invariant by $T$ on $\mathbb{Z}$.
2. $G(M)$ is invariant by $T$ on $\mathbb{Z}/n\mathbb{Z}$, for all $n \geq 3$
3. $G(M)$ is invariant by $T$ on $\mathbb{Z}/7\mathbb{Z}$
4. A finite system of equations of degree 7 in $M$ and linear in $T$.
Suppose $\mu^t$ is described with a MD. For any $x \in E_1^{[1,n]}$ we define

$$\text{Line}_{n,T}^M(x) := \frac{\partial}{\partial t} \mu_{[1,n]}^t(x)$$
Suppose $\mu^t$ is described with a MD. For any $x \in E^{[1,n]}_\kappa$ we define

$$\text{Line}^M_T(x) := \frac{\partial}{\partial t} \mu^t_{[1,n]}(x)$$

**Definition**

A $(\rho, M)$ MD under its invariant distribution is said to be AI by $T$ on the line when $\text{Line}_n \equiv 0$, for all $n \in \mathbb{N}$.
But you are CHEATING!!!
Suppose $\mu^t$ is described with a MD. For any $x \in E^{[1,n]}_\kappa$ we define

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**Definition**

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Suppose $\mu^t$ is described with a MD. For any $x \in E^{[1,n]}_k$ we define

$$\text{Line}_{n}^{M,T}(x) := \frac{\partial}{\partial t} \mu_{[1,n]}^t(x)$$

= Mass creation rate of $x$

– Mass destruction rate of $x$

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Suppose $\mu^t$ is described with a MD. For any $x \in E^{[1,n]}_\kappa$ we define

$$\text{Line}_{n}^{M,T}(x) := \frac{\partial}{\partial t} \mu^t_{[1,n]}(x)$$

$$= \lim_{h \to 0} \sum_{w \in E^{\mathbb{Z}}_\kappa} \mathbb{P}(\eta^{t+h}[1,n] = x | \eta^t = w)$$

- Mass destruction rate of $x$

**Definition**

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Suppose $\mu^t$ is described with a MD. For any $x \in E_{[1,n]}^{1/k}$ we define

$$\text{Line}_{n}^{M,T}(x) := \frac{\partial}{\partial t} \mu^t_{[1,n]}(x)$$

$$= \lim_{h \to 0} \sum_{w \in E_r^Z} \mathbb{P}(\eta^{t+h}[1,n] = x | \eta^t = w)$$

$$- \lim_{h \to 0} \sum_{w \in E_r^Z} \mathbb{P}(\eta^{t+h} = w | \eta^t[1,n] = x)$$

**Definition**

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Suppose $\mu^t$ is described with a MD. For any $x \in E_{\kappa}^{[1,n]}$ we define

$$\text{Line}_{n}^{M,T}(x) := \frac{\partial}{\partial t} \mu_{[1,n]}^t (x)$$

$$= \lim_{h \to 0} \sum_{w \in E_{\kappa}^Z} \mathbb{P}(\eta^{t+h}[1, n] = x | \eta^t = w)$$

$$- \sum_{x \in \mathbb{Z}} \sum_{j=0}^{n} \gamma(x[-1, n+2]) \sum_{(u,v) \in E_{\kappa}^2} T_{[x_j, x_{j+1}] u, v}$$

where $w^k$ differs from $x$ in $w^k[k, k + 1] = (u, v)$.

**Definition**

A $(\rho, M)$ MD under its invariant distribution is said to be AI by $T$ on the line when $\text{Line}_n \equiv 0$, for all $n \in \mathbb{N}$. 
Suppose $\mu^t$ is described with a MD. For any $x \in E_{1, n}^{[1, n]}$ we define

$$
\text{Line}_{n}^{M, T}(x) := \frac{\partial}{\partial t} \mu^{t}_{[1, n]}(x)
$$

$$
= \sum_{x_{-1}, x_0, x_{n+1}, x_{n+2} \in E_{\kappa}} \sum_{j=0}^{n} \sum_{(u, v) \in E_{\kappa}^2} \gamma(w^j[-1, n + 2]) T_{[u, v|x_j, x_{j+1}]}
$$

$$
- \sum_{x_{-1}, x_0, x_{n+1}, x_{n+2} \in E_{\kappa}} \sum_{j=0}^{n} \gamma(x[-1, n + 2]) \sum_{(u, v) \in E_{\kappa}^2} T_{[x_j, x_{j+1}|u, v]}
$$

where $w^k$ differs from $x$ in $w^k[k, k + 1] = (u, v)$.

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A $(\rho, M)$ MD under its invariant distribution is said to be AI by $T$ on the line when $\text{Line}_n \equiv 0$, for all $n \in \mathbb{N}$. 
Suppose $\mu^t$ is described with a MD. For any $x \in E_{\kappa}^{[1,n]}$ we define

$$\text{Line}_{n}^{M,T}(x) := \frac{\partial}{\partial t} \mu_{[1,n]}^t(x)$$

$$= \sum_{x=1}^{x_0, x_n+1, x_{n+2} \in E_{\kappa}} \sum_{j=0}^{n} \left( \sum_{(u,v) \in E_{\kappa}^2} \gamma(w_j([-1, n+2])) T_{[u,v|x_j,x_{j+1}]} \right)$$

$$- \gamma(x([-1, n+2])) \sum_{(u,v) \in E_{\kappa}^2} T_{[x_j,x_{j+1}|u,v]}$$

where $w^k$ differs from $x$ in $w^k[k, k+1] = (u, v)$.

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Suppose $\mu^t$ is described with a MD. For any $x \in E_k^{[1,n]}$ we define

$$\text{Line}_{n}^{M,T}(x) := \frac{\partial}{\partial t} \mu^t_{[1,n]}(x)$$

$$= \sum_{x_0, x_1, x_n, x_{n+1}, x_{n+2} \in E_k} \left( \sum_{j=0}^{n} \left( \sum_{(u,v) \in E_k^2} (\rho_{x_{j-1}} \prod_{-1 \leq k \leq n+1}^{n+1} M_{x_k,x_{k+1}}) M_{x_{j-1}, u} M_{u,v} M_{v,x_{j+2}} T[u,v|x_j,x_{j+1}] \right) \right)$$

$$- \left( \rho_{x_{-1}} \prod_{k=-1}^{n+1} M_{x_k,x_{k+1}} \right) T^{\text{out}}[x_j,x_{j+1}]$$

**Definition**

A $(\rho, M)$ MD is said to be invariant by $T$ on the line when $\text{Line}_n \equiv 0$, for all $n \in \mathbb{N}$. 
Suppose $\mu^t$ is described with a MD. For any $x \in E_{k_{[1,n]}}$ we define

$$\text{Line}_n^{M,T}(x) := \frac{\partial}{\partial t} \mu^t_{[1,n]}(x)$$

$$= \sum_{x_{-1}, x_0, x_{n+1}, x_{n+2} \in E_{\kappa}} \sum_{j=0}^{n} \left( \rho_{x_{j-1}} \prod_{k=-1}^{n+1} M_{x_k, x_{k+1}} \right) \times$$

$$\left( \left( \sum_{(u,v) \in E_{\kappa}^2} T_{[u,v|x_j,x_{j+1}]} \frac{M_{x_{j-1}, u} M_{u, v} M_{v, x_{j+2}}}{M_{x_{j-1}, x_j} M_{x_j, x_{j+1}} M_{x_{j+1}, x_{j+2}}} \right) - T_{\text{out}[x_j,x_{j+1}]} \right)$$

**Definition**

A $(\rho, M)$ MD is said to be invariant by $T$ on the line when $\text{Line}_n \equiv 0$, for all $n \in \mathbb{N}$. 
Suppose $\mu^t$ is described with a MD. For any $x \in \mathbb{E}_{\kappa}^{[1,n]}$ we define

\[
\text{Line}_{n}^{M,T}(x) := \frac{\partial}{\partial t} \mu_{[1,n]}^{t}(x)
\]

\[
= \sum_{x_{-1}, x_0, x_{n+1}, x_{n+2} \in \mathbb{E}_{\kappa}} \sum_{j=0}^{n} \left( \rho_{x_{-1}} \prod_{k=-1}^{n+1} M_{x_k, x_{k+1}} \right) \times
\]

\[
\left( \sum_{(u,v) \in \mathbb{E}_{\kappa}^2} T_{[u,v|x_j, x_{j+1}]} \frac{M_{x_{j-1}, u} M_{u, v} M_{v, x_{j+2}}}{M_{x_{j-1}, x_j} M_{x_j, x_{j+1}} M_{x_{j+1}, x_{j+2}}} \right) - T_{\text{out}}^{[x_j, x_{j+1}]} \right) \]

\[
Z_{x_{j-1}, x_j, x_{j+1}, x_{j+2}}
\]

**Definition**

A $(\rho, M)$ MD is said to be invariant by $T$ on the line when Line$_n \equiv 0$, for all $n \in \mathbb{N}$. 
Definitions

Define for every $a, b, c, d \in E_{\kappa}$

$$Z_{a,b,c,d}^{M,T} := \left( \sum_{(u,v) \in E_{\kappa}^2} T_{[u,v|b,c]} \frac{M_{a,u} M_{u,v} M_{v,d}}{M_{a,b} M_{b,c} M_{c,d}} \right) - T_{[b,c]}.$$
Definition

A Gibbs measure with kernel $M$ is said to be invariant by $T$ on $\mathbb{Z}/n\mathbb{Z}$ when $\text{Cycle}_n \equiv 0$, where

$$\text{Cycle}_n(x) := \sum_{j=0}^{n-1} \sum_{u,v \in E_\kappa} \left( \nu(w^j) T[u,v|x_j,x_{j+1} \text{mod} \ n] - \nu(x) T^{\text{out}}[x_j,x_{j+1} \text{mod} \ n] \right)$$

where $w^k$ differs from $x$ in $w^k[k, k + 1 \text{mod} \ n] = (u, v)$.
Definition

A Gibbs measure with kernel $\mathcal{M}$ is said to be invariant by $T$ on $\mathbb{Z}/n\mathbb{Z}$ when $\text{Cycle}_n \equiv 0$, where

$$\text{Cycle}_n(x) := \nu(x) \times \sum_{j=0}^{n-1} \mathbb{Z}_{x_{j-1},x_j,x_{j+1},x_{j+2}}$$

where $w^k$ differs from $x$ in $w^k\left[k, k + 1 \mod n\right] = (u, v)$. 
Extensions
Theorem 1 (F- Marckert)

Let $E_\kappa$ be finite, $L = 2$, $m = 1$. If $M > 0$ then the following statements are equivalent:

1. $(\rho, M)$ is invariant by $T$ on $\mathbb{Z}$.
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3. $G(M)$ is invariant by $T$ on $\mathbb{Z}/7\mathbb{Z}$
4. A finite system of equations of degree 7 in $M$ and linear in $T$. 
Memory and amplitude

**Theorem 1- Strongest form (F- Marckert ’17)**

Let $E_k$ be finite, $L \geq 2$, $m \in \mathbb{N}$. If $M > 0$ then the following statements are equivalent:

1. $(\rho, M)$ is invariant by $T$ on $\mathbb{Z}$.
2. $G(M)$ is invariant by $T$ on $\mathbb{Z}/n\mathbb{Z}$, for all $n \geq m + L$
3. $G(M)$ is invariant by $T$ on $\mathbb{Z}/h\mathbb{Z}$
4. A finite system of equations of degree $h$ in $M$ and linear in $T$.

\[ h := 4m + 2L - 1 \]
Other extensions:

- Theorem 1 when $\kappa = \infty$. 
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- Theorem 1 when $\kappa = \infty$.
- I.I.D. invariant measures on $\mathbb{Z}^d$.

\[
\begin{array}{c}
11 \\
00
\end{array}
\quad \rightarrow 
\begin{array}{c}
01 \\
10
\end{array}
\]
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$$
\begin{array}{c}
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$$

We link our results with the TASEP's matrix ansatz.
Other extensions:

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\[
\begin{pmatrix}
11 \\
00
\end{pmatrix}
\xrightarrow{T}
\begin{pmatrix}
11 & 01 \\
00 & 10
\end{pmatrix}
\xrightarrow{11}
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01 \\
10
\end{pmatrix}
\]

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We link our results with the TASEP’s matrix ansatz.

**Problem:** MK with zero entries.
Applications
Theorem 3 (F.- Marckert ’17)

Consider $\kappa < \infty$. Consider an IRM $T$ with amplitude $L$, which is not identically 0. If for infinitely many integers $n$ the IPS with IRM $T$ possesses an absorbing subset $S_n$ of $E_{\kappa}^{\mathbb{Z}/n\mathbb{Z}}$, with $\emptyset \subsetneq S_n \subsetneq E_{\kappa}^{\mathbb{Z}/n\mathbb{Z}}$. Then, there does not exist any MD with any memory $m$ with full support, invariant by $T$ on the line.
Theorem 3 (F.- Marckert ’17)

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Corollary

The contact process do not have a MD of any memory \( m \geq 0 \) as invariant distribution.
Summary of other applications

- The case \( \kappa = 2, \ m = 1 \) and \( L = 2 \) is totally explicitly solved.
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For $\kappa < \infty$, $L = 2$ and $m = 1$ we have an algorithm to find the set of all possible $M$ MK which are invariant for a given $T$. 
Summary of other applications

- The case $\kappa = 2$, $m = 1$ and $L = 2$ is totally explicitly solved.
- For $\kappa < \infty$, $L = 2$ and $m = 1$ we have an algorithm to find the set of all possible $M$ which are invariant for a given $T$.
- Examples of I.I.D. invariant measures on $\mathbb{Z}^2$. 
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- Zero range, voter model, etc. Also when we make mild changes on these models we have some results.
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- Examples of I.I.D. invariant measures on $\mathbb{Z}^2$.
- Zero range, voter model, etc. Also when we make mild changes on these models we have some results.
- We find an IRM $T$ which possesses some hidden Markov chain as invariant distributions. It is done using a projection from $E_3$ to $E_2$. 
Thank you!