Bijections for tree-decorated map and applications to random maps.

Luis Fredes

(Work in progress with Avelio Sepúlveda (Univ. Lyon 1))

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MAPS
A **planar map** is a proper embedding of a finite connected planar graph in the sphere, considered up to direct homeomorphisms of the sphere.

Same graph, different embeddings on the sphere (sketch by N. Curien)

Maps seen as different objects (sketch by N. Curien)
The **faces** are the connected components of the complement of the edges. It has a distinguished half-edge: the **root edge**.

The face that is at the left of the root-edge will be called the **root-face**.
A planar tree is a map with one face. The set of trees with $a$ edges.

$$C_a = \frac{1}{a+1} \binom{2a}{a}$$
The **degree of a face** is the number of edges adjacent to it.

A **quadrangulation** is a map whose faces have degree 4. Let $Q_f$ be the set of all quadrangulations with $f$ faces, then

$$|Q_f| = 3^f \frac{2}{f+2} \frac{1}{f+1} \binom{2f}{f} C_f.$$ 

Analytic [Tutte ’60] and Bijective [Cori-Vauquelin-Schaeffer ’98].
Quadrangulations with a boundary

A quadrangulation with a boundary is a map where the root-face plays a special role: it has arbitrary degree.

The set of quadrangulations with \( f \) internal faces and a boundary of size \( 2p \) has cardinality

\[
\frac{3^f p}{(f + p + 1)(f + p)} \binom{2f + p - 1}{f} \binom{2p}{p}.
\]

Analytic by [Bender & Canfield '94; Bouttier & Guitter '09] and bijective by [Schaeffer '97; Bettinelli '15]
Quadrangulations with a **simple** boundary

The set of quadrangulations with $f$ internal faces and a **simple boundary** of size $2p$ (root-face of degree $2p$) has cardinality

$$\frac{3^{f-p}2^p}{(f+2p)(f+2p-1)} \binom{2f+p-1}{f-p+1} \binom{3p}{p}.$$ 

Analytic [Bouttier & Guitter ’09]
A spanning tree-decorated map (ST map) is a pair \((m, t)\) where \(m\) is a map and \(t \subset_M m\) is a spanning tree of \(m\).

The family of ST maps with \(a\) edges is counted by

\[
C_a C_{a+1}
\]

Analytic by [Mullin ’67] and bijective by [Walsh and Lehman ’72; Cori, Dulucq & Viennot ’86; Bernardi ’06]
A \((f, a)\) **tree-decorated map** is a pair \((m, t)\) where \(m\) is a map with \(f\) faces, and \(t\) is a tree with \(a\) edges, so that \(t \subset_M m\) **containing the root-edge**.
Proposition (F. & Sepúlveda ’19)

The set of \((f, a)\) tree-decorated maps is in bijection with (the set of maps with a simple boundary of size \(2a\) and \(f\) interior faces) \(\times\) (the set of trees with \(a\) edges).
What do we obtain when the boundary is not simple?

We introduce BUBBLE-MAPS!
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We introduce BUBBLE-MAPS!

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Some remarks and extensions

- From the map with a boundary the bijection preserves:
  1. Internal faces.
  2. Internal vertices.
  3. Internal edges.
- It also preserves attributes on them.
- We can change the bijection in function of edges in the map, instead of faces.
- We can restrict the bijection to q-angulations.
- It works with some subfamilies of trees:
  2. SAW decorated maps (Already done by Caraceni & Curien).
Corollary (F. & Sepúlveda ’19)

The number of \((f, a)\) tree-decorated quadrangulations is

\[
3^{f-a} \frac{(2f + a - 1)!}{(f + 2a)!(f - a + 1)!} \frac{2a}{a + 1} \binom{3a}{a, a, a}
\]
Corollary (F. & Sepúlveda ’19)

The number of $(f, a)$ tree-decorated quadrangulations is

\[
3^{f-a} \frac{(2f + a - 1)!}{(f + 2a)! (f - a + 1)!} \frac{2a}{a + 1} \binom{3a}{a, a, a}
\]

We also count

- $(f, a)$ tree-decorated triangulations.
- Maps (triangulations and quadrangulations) with a simple boundary decorated in a subtree.
- Forest-decorated maps.
- "Tree-decorated general maps".
Re-rooting

$Q_T$  \hspace{2cm} $Q_M$

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Re-rooting

$Q_T$

$Q_M$

$Q_{T,M}$
Re-rooting

\[ |Q_T| \times 2|E| = |Q_M| \times 2|T| \]
Counting results

In the case of spanning tree decorated quadrangulations rooted in the tree we obtain

\[ C_{2,f} = \frac{2}{(f + 1)(f + 2)} \binom{3f}{f, f, f} \]
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and remember that the Catalan numbers are

\[ C_{1,f} = \frac{1}{(f + 1)} \binom{2f}{f} \]
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A possible generalization of Catalan numbers:

\[ C_{m,n} = m! \left( \prod_{i=1}^{m} \frac{1}{(n + i)} \right) \binom{(m + 1)n}{n, n, \ldots, n} = \left( m + n \right)^{-1} \binom{(m + 1)n}{n, n, \ldots, n} \text{ \text{ m+1 times}} \]
Counting results

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Proposition (Vincent Jugé)

\[ C_{m,n} \text{ is an integer } \forall n, m. \]
CONVERGENCE
RESULTS
For a map $m$ and $r \in \mathbb{N}$, let $B_r(m)$ denote the ball of radius $r$ from the root-vertex. Consider $\mathcal{M}$ a family of finite maps. The **local topology** on $\mathcal{M}$ is the metric space $(\mathcal{M}, d_{\text{loc}})$, where

$$d_{\text{loc}}(m_1, m_2) = (1 + \sup\{r \geq 0 : B_r(m_1) = B_r(m_2)\})^{-1}$$

**Proposition**
The space $(\mathcal{M}, d_{\text{loc}})$ is Polish (metric, separable and complete).
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**Proposition**

The space $(\mathcal{M}, d_{\text{loc}})$ is Polish (metric, separable and complete).
Gromov-Hausdorff topology

Let \((E, d_E)\) be a metric space and \(A, B \subset E\). The **Hausdorff distance** is

\[
d_H(A, B) = \inf \left\{ \varepsilon > 0 : A \subset B_\varepsilon, B \subset A_\varepsilon \right\}
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\]
Consider the set $S$ of compact metric spaces up to isometry classes. The **Gromov-Hausdorff distance** between two metric spaces $(X, d)$ and $(X', d')$ is defined as

$$d_{GH}((X, d), (X', d')) = \inf d_H(\phi(X), \phi'(X'))$$

where the infimum is taken over all metric spaces $(E, d_E)$ and all isometric embeddings $\phi, \phi'$ from $X$, $X'$ respectively into $E$. 
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Uniform Trees

$t_a = \text{Unif. tree with } a \text{ edges.}$

**Theorem (Kesten '86)**

\[ t_a \xrightarrow{(d)}_{\text{local}} t_\infty \]

**Properties**

- $t_\infty$ is an infinite tree.
- It has one infinite branch (the spine) which divides the tree in independent critical geometric Galton-Watson trees.

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Theorem (Aldous '91)

\[ (t_a, d) \xrightarrow{(d)} \text{CRT} \]

**Properties**

- The CRT is a tree.
- Almost every point is a leaf.
- Hausdorff dimension 2. (Duquesne & Le Gall '05)

Uniform random tree 50k edges.
Uniform Trees

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**Theorem (Aldous ’91)**

$$\left( t_a, \frac{d_{\text{Tree}}}{a^{1/2}} \right) \xrightarrow{(d) \text{ GH}} \text{CRT}$$

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$t_\infty$ construction.

Uniform random tree 50k edges.
Uniform quadrangulations

$q_f = \text{Unif. quadrangulation with } f \text{ faces.}$

**Theorem (Krikun '06)**

$$q_f \xrightarrow{(d)}_{\text{local}} \text{UIPQ}$$

**Properties**

- The UIPQ is an infinite quad.
- The vol. and per. of the exploration on it have been studied (Curien & Le Gall '14).

(Sketch by N. Curien)
Uniform quadrangulations

$q_f = \text{Unif. quadrangulation with } f \text{ faces.}$

**Theorem (Krikun '06)**

\[ q_f \xrightarrow{(d)_{\text{local}}} \text{UIPQ} \]

**Theorem (Miermont '13, Le Gall '13)**

\[ \left( q_f, \frac{d_{\text{map}}}{f^{1/4}} \right) \xrightarrow{(d)_{\text{GH}}} \text{Brownian map} \]

**Properties**

- The UIPQ is an infinite quad.
- The vol. and per. of the exploration on it have been studied (Curien & Le Gall '14).

**Properties**

- Hausdorff dim. is 4 (Le Gall '07).
- Homeomorphic to $\mathbb{S}^2$ (Le Gall & Paulin '08).

(Sketched by N. Curien)

Unif. quadrangulation 30k faces.
Uniform quadrangulation with a boundary

$q_{f,p}$ = Unif. quadrangulations with a boundary of size $2p$ and $f$ faces.

**Theorem (Curien & Miermont ’12)**

$$q_{f,p} \xrightarrow{(d)}_{local(f \to \infty)} q_{\infty,p} \xrightarrow{(d)}_{local(p \to \infty)} UIHPQ$$
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**Properties (Curien & Miermont '12)**

- \(q_{\infty}^p = \text{Uniform Infinite Planar Quadrangulation with perimeter } 2p.\)
- They also obtain the convergences for the simple boundary case.

UIHPQ (sketch by N. Curien & A. Caraceni).
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UIHPQ (sketch by N. Curien & A. Caraceni).
Brownian Disk

\( q^p_f = \text{Unif. quadrangulations with boundary } 2p \text{ and } f \text{ faces.} \)
For a sequence \( (p(f))_{f \in \mathbb{N}} \), define \( \bar{p} = \lim p(f) f^{-1/2} \) as \( f \to \infty \).

**Theorem (Scaling limit (Bettinelli ’15))**

\[
\left( q^p_f, \frac{d_{\text{map}}}{s(f, p(f))} \right) \xrightarrow{(d) \text{ GH}} \begin{cases} 
\text{Brownian map} & \text{if } s(f, p(f)) = f^{1/4} \text{ and } \bar{p} = 0 \\
\text{Brownian disk} & \text{if } s(f, p(f)) = f^{1/4} \text{ and } \bar{p} \in (0, \infty) \\
\text{CRT} & \text{if } s(f, p(f)) = 2p(f)^{1/2} \text{ and } \bar{p} = \infty 
\end{cases}
\]
Brownian Disk

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\end{cases}
\]

**Properties (Bettinelli & Miermont '15)**

- **Brownian disk properties**
  - The boundary is simple.
  - Hausdorff dim. 4 in the interior, 2 in the boundary.
  - Homeomorphic to the disk $2d$.

Unif. quad. with 30k interior faces and boundary 173.
Uniform ST map

- Expected diameter is of order $n^\chi$ for $0.275 \leq \chi \leq 0.288$ (Ding & Gwynne ’18, Gwynne, Holden & Sun ’16).
- The limit (if it exists) seems not to be the Brownian map.
- Convergence for the local topology (Sheffield ’11).

![Uniform ST map 100k edges.](image)
$q_f^a = \text{Unif. tree-decorated map with } f \text{ faces and a tree of size } a.$

Why it is interesting to study this family??
Uniform tree-decorated maps

$q^a_f = \text{Unif. tree-decorated map with } f \text{ faces and a tree of size } a.$

Why it is interesting to study this family??

- **New statistical mechanic family**

$$P(q^a_f = (m, \cdot)) \propto \#\{\text{trees of size } a \text{ in } m\}$$
Uniform tree-decorated maps

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Why it is interesting to study this family??

- **New statistical mechanic family**

$$\mathbb{P}(q_f^a = (m, \cdot)) \propto \#\{\text{trees of size } a \text{ in } m\}$$

- **It interpolates**
  - $a = 1 =$ Uniform quadrangulations.
  - $a = f + 1 =$ Uniform ST quadrangulations.
Local limit results

Is there any local limit for the gluing of \( q_{\infty,p} \) and \( t_p \) as \( p \to \infty \)?
Local limit results

Is there any local limit for the gluing of $q_{\infty,p}^S$ and $t_p$ as $p \to \infty$?

Proposition (F. & Sepúlveda ’19+)

There exists a local limit for this gluing and it is the gluing root to root of $t_\infty$ with a $UIHPQ_S$, seeing from the root.
Is there any local limit for the gluing of $q^S_{\infty,p}$ and $t_p$ as $p \to \infty$?

**Proposition (F. & Sepúlveda ’19+)**

There exists a local limit for this gluing and it is the gluing root to root of $t_{\infty}$ with a $UIHPQ_S$, seeing from the root.

**Remark**

We obtain more local limits.
Corollary (F. & Sepúlveda ’19+)

Let \((m, t)\) be a Unif. tree-decorated map with \(f\) faces and boundary of size \(a(f)\) with \(a(f) \leq f + 1\). Then as \(a(f) \to \infty\),

\[
\left( t, \frac{d_{\text{Tree}}}{a(f)^{1/2}} \right) \xrightarrow{GH} CRT.
\]
Scaling limit conjecture

Conjecture (F. & Sepúlveda ’19+)

Let \((m, t)\) be a Unif. tree-decorated map with \(f\) faces and boundary of size \(a(f) = O(f^\alpha)\). Depending on \(\alpha\) as \(f \to \infty\)

\[
\left( (m, t), \frac{d_{\text{map}}}{f^\beta} \right) \xrightarrow{(d)} \begin{cases} 
\text{Brownian map} & \text{if } \alpha < 1/2, \beta = 1/4 \text{(Proved)} \\
\textbf{Shocked map} & \text{if } \alpha = 1/2, \beta = 1/4 \text{(In progress)} \\
\text{Tree-decorated map} & \text{if } \alpha > 1/2,
\end{cases}
\]

\[
\beta = (2\chi - \frac{1}{2}) \alpha - \chi + \frac{1}{2}
\]
Scaling limit conjecture

**Conjecture (F. & Sepúlveda ’19+)**

Let \((m, t)\) be a Unif. tree-decorated map with \(f\) faces and boundary of size \(a(f) = O(f^{\alpha})\). Depending on \(\alpha\) as \(f \to \infty\)

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\text{Tree-decorated map} & \text{if } \alpha > 1/2, \\
\quad \beta = \left(2\chi - \frac{1}{2}\right) \alpha - \chi + \frac{1}{2}
\end{cases}
\]

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Tree-decorated maps
Shocked map properties:

- **It is not degenerated** (Proved).
- It should be the gluing of a Brownian disk and a CRT.
- Hausdorff dim. 4 (Proved).
- The tree has Hausdorff dim. 2 (In progress, $\leq 2$ proved).
- Homeomorphic to $S^2$. (Proved).

**Figure:** Unif. (90k,500) treedecorated quadrangulation.
Why shocked?
Boundary with bridges + tree?
Boundary with bridges + tree?

Degenerate map!
Boundary with bridges + tree?

Degenerate map!
Boundary with bridges + tree?

Degenerate map!

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It is not degenerated.

To prove it we do a sequential gluing, tool used to define a peeling.

Then we use the estimates in [Caraceni & Curien, Self-Avoiding Walks on the UIPQ] and the properties of the contour of a tree, to show that distances do not create big shortcuts.
Homeomorphic to $S^2$.

In discrete

Quad. bord $p$
Homeomorphic to $S^2$.

In discrete

Quad. bord $p$ — $-\bar{p}$

Tree
Homeomorphic to $S^2$.

In discrete

Quad. bord $p$ glued with a tree