Bijections for tree-decorated maps and applications

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Overview

1. Maps
   - Maps families and bijections
     - Planar trees
     - Quadrangulations
     - Quadrangulations with a boundary
     - Spanning tree-decorated maps
   - Tree-decorated map
   - Bijection
     - Counting results

2. Convergences
   - Known limits
     - Uniform trees
     - Uniform quadrangulations
     - Brownian Disk
     - Uniform ST map

3. The shocked map
   - Motivation
   - Limit results
     - Local limit results
     - Scaling limit results
MAPS
A **planar map** is a proper embedding of a finite connected planar graph in the sphere, considered up to direct homeomorphisms of the sphere. The **faces** are the connected components of the complement of the edges. It has a distinguished half-edge: the **root edge**. The face that is at the left of the root-edge will be called the **root-face**.
**Figure:** Same planar graph with different embeddings (sketch by N. Curien).
Figure: Same planar map seen as different objects/codings
(sketch by N. Curien).
Planar trees

A **planar tree** is a map with one face. Denote as $\mathcal{T}_m$ the number of trees with $m$ edges.

$$\mathcal{T}_m = C_m = \frac{1}{m+1} \binom{2m}{m}$$
Quadrangulations

The degree of a face is the number of edges adjacent to it (an edge included in a face is counted twice). A quadrangulation is a map whose faces have degree 4.
Let $Q_f$ be the set of all quadrangulations with $f$ faces, then

$$|Q_f| = 3^f \frac{2}{f+1} \frac{1}{f+1} \binom{2f}{f}.$$
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$$|Q_f| = 3^f \cdot \frac{2}{f+1} \cdot \frac{1}{f+1} \cdot \binom{2f}{f}.$$  

THIS NUMBER ALSO COUNTS GENERAL MAPS WITH $m = f$ EDGES!
A quadrangulation with a boundary is a map where the **root-face** plays a special role: it has **arbitrary degree**. All others faces are called **internal faces** and have degree 4.
The set of quadrangulations with \( f \) internal faces and a boundary of size \( p \) has cardinality

\[
3^f \frac{f}{(f + p + 1)(2f + p)} \binom{2f + p}{f} \binom{2p}{p}.
\]
The set of quadrangulations with $f$ internal faces and a simple boundary of size $p$ (root-face of degree $p$) has cardinality

$$
\frac{3^{f-p}(2f + p - 1)!}{(f + 2p)!(f - p + 1)!} \frac{(3p)!}{p!(2p - 1)!}.
$$
A spanning tree-decorated map (ST map) is a pair $(m, t)$ where $m$ is a map and $t \subset M$ $m$ is a spanning tree of $m$. 
The family of ST maps with $m$ edges is in bijection with a pair of interlaced trees (mating of trees), one of size $m$ and other of size $m + 1$ (lots of bijections for this family). As a consequence this family is counted by

$$C_m C_{m+1}$$

TO OUR KNOWLEDGE ST Q-ANGULATIONS HAVE NOT BEEN COUNTED.
What about a tree decorating a map, but not decorated in a spanning tree?
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\((f, m)\)-tree decorated map!!! where \(m\) denote the number of edges of the tree decorating the map and \(n\) the number of faces of the map.
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What happens when we use \(m = 1\) and \(m = f + 1\)?
What about a tree decorating a map, but not decorated in a spanning tree?

\((f, m)\)-tree decorated map!!! where \(m\) denote the number of edges of the tree decorating the map and \(n\) the number of faces of the map.

What happens when we use \(m = 1\) and \(m = f + 1\)?

We interpolate between the uniform quadrangulation and the ST quadrangulation!!!!
An \((f, m)\) **tree-decorated map** is a pair \((m, t)\) where \(m\) is a map with \(f\) faces, and \(t\) is a tree with \(m\) edges, so that \(t \subset M\) \(m\) containing the root-edge.
An \((f, m)\) **tree-decorated map** is a pair \((m, t)\) where \(m\) is a map with \(f\) faces, and \(t\) is a tree with \(m\) edges, so that \(t \subseteq M m\) containing the root-edge.

In what follows, a **Uniform \((f, m)\) tree-decorated quadrangulations** is a random variable chosen in the family of all \((f, m)\) tree-decorated quadrangulations.
Proposition (F. & Sepúlveda ’18+)

The set of \((f, m)\) tree-decorated maps is in bijection with the Cartesian product between the set of maps with a simple boundary of size \(2m\) and \(f\) interior faces and the set of trees with \(m\) edges.
Bijection

Proposition (F. & Sepúlveda '18+)

The set of \((f, m)\) tree-decorated maps is in bijection with the Cartesian product between the set of maps with a simple boundary of size \(2m\) and \(f\) interior faces and the set of trees with \(m\) edges.

\[
\begin{align*}
\mathcal{M}_b^m & \quad + \quad t' \quad \leftrightarrow \quad (m, t)
\end{align*}
\]

Figure: Sketch of the bijection. Left: Map with boundary and planted tree representing this bijection. Right: Tree decorated map. We plot it being embedded in the sphere. The arrows are root-edges and the grid lines represent the inner faces.
Figure: Left: Zoom of the tree decorated map. In green the decoration and in black the edges that do not belong to the decoration.
Right: Map with boundary and planted tree. Transformation obtained from the corners (green points) of the decoration.
Why does the boundary need to be simple?

If not the gluing produces BUBBLES!
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If not the gluing produces BUBBLES!

Figure: Left: Map with a non-simple boundary (interior faces filled with lines) and a tree. Right: Bubbles (3D plot) form by the gluing of a map with non-simple boundary and a tree.
The number of \((f, m)\) tree-decorated triangulations are

\[
2^{f-2m}(3f/2 + m - 2)!! \frac{2^m (4m)}{(2m)} \frac{1}{m+1} \binom{2m}{m},
\]

where \(n!! = \prod_{i=0}^{\lfloor (n-1)/2 \rfloor} (n-2i)\).

The number of \((f, m)\) tree-decorated quadrangulations is

\[
3^{f-m} \frac{(2f + m - 1)!}{(f + 2m)! (f - m + 1)!} \frac{(3m)!}{m!(2m-1)!} \frac{1}{m+1} \binom{2m}{m}.
\]
Counting results

**Corollary (F. & Sepúlveda ’18+)**

The number of \((f, m)\) tree-decorated triangulations are

\[
2^{f-m}(3f/2 + m - 2)!! \frac{2m}{(f/2 - m + 1)!(f/2 + 3m)!!} \frac{4m}{m + 1} \binom{2m}{m},
\]

where \(n!! = \prod_{i=0}^{\lfloor (n-1)/2 \rfloor} (n - 2i)\).

The number of \((f, m)\) tree-decorated quadrangulations is

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\]

We also count

- Maps (triangulations and quadrangulations) with a simple boundary decorated in a spanning tree.
- Maps (triangulations and quadrangulations) with a simple boundary decorated in a subtree.
- Forest-decorated maps.
CONVERGENCE
RESULTS
Local Limit (Benjamini-Schramm Topology ’01)

For a map $m$ and $r \in \mathbb{N}$, define $B_r(m)$ as the ball of radius $r$ from the root-vertex. Consider $\mathcal{M}$ a family of finite maps. The local topology on $\mathcal{M}$ is the metric space $(\mathcal{M}, d_{\text{loc}})$, where

$$d_{\text{loc}}(m_1, m_2) = \left(1 + \sup\{r \geq 0 : B_r(m_1) = B_r(m_2)\}\right)^{-1}$$

Meaning that a sequence of maps $(m_i)_{i \in \mathbb{N}}$ converges if for all $r \in \mathbb{N}$, $B_r(m_i)$ is constant from certain point on.
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**Proposition**

The space $(\overline{\mathcal{M}}, d_{\text{loc}})$ is Polish (metric, separable and complete).
Recall that if \((E, d_E)\) is a metric space and \(A, B \subset Z\), the Hausdorff distance between \(A\) and \(B\) is given by

\[
d_H(A, B) = \max \left\{ \max_{x \in B} d_E(x, A), \max_{y \in A} d_E(y, B) \right\}
\]
Gromov-Hausdorff topology

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\]

Consider the set \(S\) of compact metric spaces up to isometry classes. The Gromov-Hausdorff distance between two metric spaces \((X, d)\) and \((X', d')\) is defined as

\[
d_{GH}((X, d), (X', d')) = \inf d_H(\phi(X), \phi'(X'))
\]

where the infimum is taken over all metric spaces \((E, d_E)\) and all isometric embeddings \(\phi, \phi'\) from \(X, X'\) respectively into \(E\).
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**Proposition**

The function \(d_{GH}\) induces a metric on \(S\). The space \((S, d_{GH})\) is separable and complete.
Uniform Trees

Let $t_m$ be a tree uniformly chosen in $T_m$. 

Theorem (Kesten '86)

$\lim_{m \to \infty} t_m \to \text{local}$

Proposition

$t_\infty$ is an infinite tree. Each vertex has bounded degree. It has one infinite branch (the spine) which divides the tree in independent critical geometric Galton-Watson trees.

Theorem (Aldous '91)

$(t_m, d) \to \text{GH CRT}$

Proposition

The CRT is a tree. Almost every point is a leaf. Hausdorff dimension 2. Its geodesics are represented in the coding.
Let $t_m$ be a tree uniformly chosen in $\mathcal{T}_m$.

**Theorem (Kesten ’86)**

$$
\begin{align*}
t_m & \xrightarrow{(d)} t_\infty \\
& \underset{\text{local}}{\xrightarrow{}}
\end{align*}
$$
Let $t_m$ be a tree uniformly chosen in $T_m$.

**Theorem (Kesten ’86)**

$$t_m \xrightarrow{(d)}_{\text{local}} t_{\infty}$$

**Proposition**

- $t_{\infty}$ is an infinite tree.
- Each vertex has bounded degree.
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Let $t_m$ be a tree uniformly chosen in $\mathcal{T}_m$.

**Theorem (Kesten '86)**

$$t_m \xrightarrow{\text{(d)}_{\text{local}}} t_\infty$$

**Theorem (Aldous '91)**

$$\left( t_m, \frac{d_{\text{Tree}}}{m^{1/2}} \right) \xrightarrow{\text{(d)}_{\text{GH}}} \text{CRT}$$

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- Its geodesics are represented in the coding.
$t_\infty$: the critical geometric GW tree conditioned to survive.
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Figure: Geometry of $t_\infty$. 
$t_\infty$: the critical geometric GW tree conditioned to survive.

Figure: Geometry of $t_\infty$. 
Figure: Uniform random tree 50k edges.
Uniform quadrangulations

Consider \( q_f \) a uniformly of the set of quadrangulations with \( f \) faces. The Brownian map, was defined by Marckert and Mokkadem in 2006.
Uniform quadrangulations

Consider $q_f$ a uniformly chosen element of the set of quadrangulations with $f$ faces. The Brownian map, was defined by Marckert and Mokkadem in 2006.

**Theorem (Krikun ’06)**

$$q_f \xrightarrow{\text{(d)}}_{\text{local}} \text{UIPQ}$$

**Properties**
- The UIPQ is an infinite quadrangulation.
- Properties about the volume and perimeter of the exploration of the UIPQ are known.
- Locally finitely many faces.

**Theorem (Miermont ’13, Le Gall ’13)**

$$q_f, d \xrightarrow{\text{map}}_{f/4} \text{GH Brownian map}$$

**Properties**
- Hausdorff dimension is 4 (Le Gall ’07).
- Homeomorphic to the two-dimensional sphere (Le Gall & Paulin ’08).
- Its geodesics are described by the coding (well labeled trees).
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**Theorem (Miermont '13, Le Gall '13)**

$$\left(q_f, \frac{d_{\text{map}}}{f^{1/4}}\right) \xrightarrow{(d)}_{GH} \text{Brownian map}$$

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Properties

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- Homeomorphic to the two dimensional sphere (Le Gall & Paulin '08).
- Its geodesics are described by the coding (well labeled trees).
Figure: UIPT representation.
(Sketch by N. Curien)
Figure: Brownian map 30k faces.
Uniform quadrangulation with a boundary

Let $q^p_f$ be a map uniformly chosen in the set of all quadrangulations with a boundary of size $p$ with $f$ faces.
Uniform quadrangulation with a boundary

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Theorem (Curien & Miermont '12)

$$ q^P_f \xrightarrow{(d)}_{\text{local}(f \to \infty)} q^P_\infty \xrightarrow{(d)}_{\text{local}(p \to \infty)} \text{UIHPQ} $$

They also obtain the convergences above conditioned to have simple boundary.

$q^P_\infty$ has one infinite irreducible component, called the core. Moreover, $\partial \text{Core}(q^P_\infty)$ is (prob) $\xrightarrow{p \to \infty} \frac{2}{3}$. 
Uniform quadrangulation with a boundary

Let $q^p_f$ be a map uniformly chosen in the set of all quadrangulations with a boundary of size $p$ with $f$ faces.

**Theorem (Curien & Miermont '12)**

$$ q^p_f \xrightarrow{(d)}_{local(f \to \infty)} q^p_\infty \xrightarrow{(d)}_{local(p \to \infty)} UIHPQ $$

**Properties**

- $q^p_\infty$ is called the Uniform Infinite Planar Quadrangulation with a boundary of perimeter $p$.
- They also obtain the convergences above conditioned to have simple boundary.
- The $q^p_\infty$ has one infinite irreducible component, called the core. Moreover,

$$ \frac{\partial \text{Core}(q^p_\infty)}{2p} \xrightarrow{(\text{prob})_{p \to \infty}} \frac{1}{3} $$
Figure: UIHPQ
(Sketch by N. Curien and A. Caraceni).
Brownian Disk

Let $q^p_f$ be a map uniformly chosen in the set of quadrangulations with $f$ faces and boundary of size $p$. For a sequence $(p_n)_{n \in \mathbb{N}}$, define $\bar{p} = \lim p_n n^{-1/2}$ as $n \to \infty$. 

Theorem (Scaling limit (Betinelli '15))

$\left( q^p_n, f, p_f \right)$ $\overset{d}{\longrightarrow}$ GH

\begin{align*}
\text{Brownian map} & \quad \text{if } s(f, p_f) = f^{1/4} \text{ and } \bar{p} = 0 \\
\text{Brownian disk} & \quad \text{if } s(f, p_f) = f^{1/4} \text{ and } \bar{p} \in (0, +\infty) \\
\text{CRT} & \quad \text{if } s(f, p_f) = 2p^{1/2} \text{ and } \bar{p} = +\infty
\end{align*}

Properties (Betinelli & Miermont '15)

- The boundary is simple.
- Hausdorff dimension 4 in the interior, 2 in the boundary.
- Homeomorphic to the two dimensional disk.
- Links with the Brownian map.
Brownian Disk

Let $q^p_f$ be a map uniformly chosen in the set of quadrangulations with $f$ faces and boundary of size $p$. For a sequence $(p_n)_{n \in \mathbb{N}}$, define $\bar{p} = \lim p_n n^{-1/2}$ as $n \to \infty$.

**Theorem (Scaling limit (Betinelli ’15))**

\[
\left(q_n, \frac{d_{\text{map}}}{s(f, p_f)}\right) \xrightarrow{(d)_{GH}} \begin{cases} 
\text{Brownian map} & \text{if } s(f, p_f) = f^{1/4} \text{ and } \bar{p} = 0 \\
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\end{cases}
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**Properties (Betinelli & Miermont ’15)**

**Brownian disk properties**

- The boundary is simple.
- Hausdorff dimension 4 in the interior, 2 in the boundary.
- Homeomorphic to the two dimensional disk.
- Links with the Brownian map.
Figure: Uniform quadrangulation with a boundary 30k interior faces, 173 edges in the boundary.
Let $q_f^{ST}$ be uniformly chosen in the set of ST quadrangulations with $f$ faces.

- The conjectured scaling limit of these objects should be related to continuum Liouville quantum gravity.
- Recently it has been shown that there exists a constant $0.275 \leq \chi \leq 0.288$, such that the expected diameter is of order $n^\chi$ (Ding & Gwynne ’18, Gwynne, Holden & Sun ’16).
- In the case of convergence as a metric space, there is evidence that the limit is not the Brownian map.
- There exists a local limit for this object and other decorated-families (Sheffield ’11).
Figure: Uniform spanning tree-decorated quadrangulation 30k faces.
Uniform \((f, m)\) tree-decorated quadrangulation model is that it interpolates between uniform quadrangulation with \(f\) faces and the uniform ST-decorated quadrangulation with \(f\) faces. In light of this effect, we hope to give a phase transition between these objects obtaining an insight about the metric scaling limit of uniform ST map.
Uniform \((f, m)\) tree-decorated quadrangulation model is that it interpolates between uniform quadrangulation with \(f\) faces and the uniform ST-decorated quadrangulation with \(f\) faces.

In light of this effect, we hope to give a phase transition between these objects obtaining an insight about the metric scaling limit of uniform ST map.

This model could encode two different statistical mechanic objects, one on the tree and one on the map without considering the tree.
Local limit results

What happens if we glue the UIHPQ$_S$ (local limit of quadrangulations with a simple boundary) with $t_\infty$ (local limit of uniform trees)?
Local limit results

What happens if we glue the UIHPQ$_S$ (local limit of quadrangulations with a simple boundary) with $t_\infty$ (local limit of uniform trees)? Sequential gluing, tool used to define a peeling.

**Figure:** First step in the sequential gluing procedure. The second step is sketched with the next edges in the contour to glue in blue.
Local limit results

What happens if we glue the UIHPQₜ (local limit of quadrangulations with a simple boundary) with $t_\infty$ (local limit of uniform trees)? Sequential gluing, tool used to define a peeling.

![Diagram](image)

**Figure:** First step in the sequential gluing procedure. The second step is sketched with the next edges in the contour to glue in blue.

**Proposition (F. & Sepúlveda ’18+)**

There exists a local limit for the gluing of an infinite tree $t$ with a UIHPQₜ.
Local limit results

What happens if we glue the UIHPQ$_S$ (local limit of quadrangulations with a simple boundary) with $t_\infty$ (local limit of uniform trees)? Sequential gluing, tool used to define a peeling.

Figure: First step in the sequential gluing procedure. The second step is sketched with the next edges in the contour to glue in blue.

**Proposition (F. & Sepúlveda ’18+)**
There exists a local limit for the gluing of an infinite tree $t$ with a UIHPQ$_S$.

**Remark**
We obtain more local limits.
Scaling limit results

**Corollary (F. & Sepúlveda ’18+)**

Let \((m_f, t_{mf})\) be a uniform \((f, m_f)\) tree-decorated quadrangulations with \(m_f \leq f + 1\). Then as \(m_f \to \infty\),

\[
\left( t_{mf}, \frac{d_{tmf}}{m_f^{1/2}} \right) \overset{(d)}{\longrightarrow} \text{CRT}.
\]
Scaling limit conjecture

Conjecture (F. & Sepúlveda ’18+)

Let \((m_f, t_{m_f})\) be a uniform \((f, m_f)\) tree-decorated quadrangulation with \(m_f = O(f^\alpha)\). Depending on \(\alpha\)

\[
\left(\left( m_f, t_{m_f}, \frac{d_{m_f}}{f^\beta} \right) \right) \xrightarrow{(d) \text{ GH}} \begin{cases}
\text{Brownian map} & \text{if } \alpha \leq 1/2, \beta = 1/4 (Proved) \\
\text{Shocked map} & \text{if } \alpha = 1/2, \beta = 1/4 (In progress) \\
\text{Tree-decorated map} & \text{if } \alpha \geq 1/2, \\
\quad \beta = (2\chi - \frac{1}{2}) \alpha - \chi + \frac{1}{2}
\end{cases}
\]

The Shocked map is not trivial (Proved).

The Shocked map should be the gluing between a Brownian disk with perimeter \(p\) and a CRT.

The Shocked map has Hausdorff dimension 4 outside the tree (Proved).

It should have dimension 2 on the decoration (In progress).

Homeomorphic to the two dimensional sphere (In progress).
Conjecture (F. & Sepúlveda ’18+)

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\text{Tree-decorated map} & \text{if } \alpha \geq 1/2, \\
& \beta = (2\chi - \frac{1}{2}) \alpha - \chi + \frac{1}{2}
\end{cases}
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- The Shocked map is not trivial (Proved).
- The Shocked map should be the gluing between a Brownian disk with perimeter \(p\) and a CRT.
- The Shocked map has Hausdorff dimension 4 outside the tree (Proved).
- It should have dimension 2 on the decoration (In progress).
- Homeomorphic to the two dimensional sphere (In progress).
Figure: Uniform tree-decorated quadrangulation 90k faces decorated on a tree of size 500.
Why shocked?
Figure: Golf field struck by lightning.
Thank you!