Introduction to compact quantum groups — Université Paris-Saclay
M2 AAG — 2019-2020

Homework

The homework is due for 07/04/2020. It must be sent to me in pdf format directly by e-mail.

Goal : The aim of this homework is to classify all categories of non-crossing partitions using the representation theory of the associated compact quantum groups. We will not treat all cases in detail, but try to see the main ideas of the strategy. The first step is to observe that any category of partitions belongs to exactly one of the following cases :

- All the blocks of the partitions have size two.
- All the blocks of the partitions have size *at most two* and there is a partition with a block of size one.
- All the blocks of the partitions have even size and there is a partition with a block of size at most four.
- There is a partition with a block of odd size at least three.

1. The easy cases

Let $\mathcal{C}$ be a category of non-crossing partitions.

1.1) We assume that all blocks of partitions in $\mathcal{C}$ have size two. Recall why $\mathcal{C} = NC_2$.

1.2) We assume now that all blocks of partitions in $\mathcal{C}$ have even size and that $\mathcal{C}$ contains a partition with a block of size at least 4.

i) Let p_4 be the partition $\{\{1,2,3,4\}\}$. Prove that $p_4 \in \mathcal{C}$.

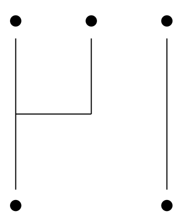
ii) Recall why this implies $\mathcal{C} = NC_{\text{even}}$.

2. One-dimensional representations

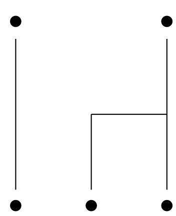
In this section we will prove the following statement, needed in the sequel : the representation u_p is one-dimensional if and only if $t(p) = 0$[1]. So let $p \in \mathcal{C}(k,k)$ be a projective partition. We will use the two following operations on partitions :

- We denote by $\overline{p}$ the partition obtained by rotating p upside-down,
- We denote by $\widetilde{p}$ be the partition obtained by rotating all the lower points of p to the right of the upper points.

Here is an example : for

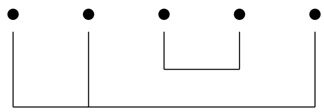
$p =$ 

we have

$\overline{p} =$ 

and

[1] Recall that $t(p)$ is the number of through-blocks of p.

$\widetilde{p} =$ 

2.3) Assume that $t(p) = 0$. Compute the ranks of S_p and P_p and conclude.

2.4) Assume conversely that $\dim(u_p) = 1$ and let us assume by contradiction that $t(p) > 0$.

i) Prove that

$$p\square^{t(p)}\overline{p} = \widetilde{p}^*\widetilde{p}.$$

and that therefore $u_{p\square^{t(p)}\overline{p}}$ is the trivial representation ε.

ii) Using the formula for the fusion rules, deduce that

$$p \otimes \overline{p} = \widetilde{p}^*\widetilde{p}$$

and conclude.

3. An example

Let us denote by $p_{\mathrm{ds}} \in NC(1,1)$ the partition $\{\{1\},\{2\}\}$, called the *double singleton partition*. Let $\mathcal{C}_{\mathrm{ds}}$ be the category of partitions generated by p_{ds}. The aim of this section is to compute the representation theory of $\mathbb{G}_N(\mathcal{C}_{\mathrm{ds}})$ for $N \geqslant 4$. We set

$$s_j = \sum_{i=1}^{N} u_{ij}.$$

3.1) Prove that s_j does not depend on j and that $\Delta(s_j) = s_j \otimes s_j$. We will simply denote it by s from now on.

3.2) Show that $s = u_{p_{\mathrm{ds}}}$ and $s \otimes s = \varepsilon$.

3.3) Let $\mathcal{D}$ be the set of all partitions with blocks of size at most 2 and an even number of singletons. Prove that $\mathcal{D}$ is a category of partitions. Deduce from this that s is not the trivial representation.

3.4) Let $p \in \mathcal{C}_{\mathrm{ds}}(k,k)$ be a projective partition such that $t(p) = 0$. Prove that u_p is either trivial or equal to s.

3.5) Let $p \in \mathcal{C}_{\mathrm{ds}}$ be an arbitrary projective partition. Prove that p decomposes up to equivalence as

$$p \sim p_{\mathrm{ds}}^{\epsilon_0} \otimes | \otimes p_{\mathrm{ds}}^{\epsilon_1} \otimes | \otimes \cdots \otimes | \otimes p_{\mathrm{ds}}^{\epsilon_k}$$

where $\epsilon_i \in \{0,1\}$ for all $1 \leqslant i \leqslant k$ and $p_{\mathrm{ds}}^0 = \emptyset$ by convention.

3.6) Conclude that if we set, for a tuple $\epsilon = (\epsilon_0, \cdots, \epsilon_k)$ with $\epsilon_i \in \{0,1\}$,

$$p_\epsilon = p_{\mathrm{ds}}^{\epsilon_0} \otimes | \otimes p_{\mathrm{ds}}^{\epsilon_1} \otimes \cdots \otimes | \otimes p_{\mathrm{ds}}^{\epsilon_k},$$

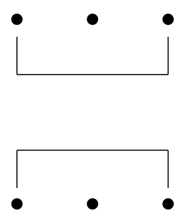
then any irreducible representation of $\mathbb{G}_N(\mathcal{C}_{\mathrm{ds}})$ is equivalent to some u_{p_ϵ}.

4. Blocks of size at most two

We consider now an arbitrary category of non-crossing partitions $\mathcal{C}$ such that all the blocks of the partitions have size *at most two* and there is a partition with a block of size one.

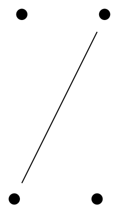
4.1) Prove that $\mathcal{C}_{\mathrm{ds}} \subset \mathcal{C}$.

Let us consider the *projective commutation partition*

$p_{\mathrm{com}} =$ 

4.2) Prove that $p_{\mathrm{com}} \in \mathcal{C}(3,3)$ if and only if

$$q_{\text{com}} =$$



is in $\mathcal{C}(2,2)$.

4.3) Check that the *commutation partition* q_{com} corresponds to the commutation relations

$$su_{ij} = u_{ij}s$$

for all $1 \leqslant i, j \leqslant N$.

4.4) Deduce that if we set

$$p_{n,\epsilon} = |^{\otimes n} \otimes p_{\text{ds}}^{\epsilon},$$

then the representations $(u_{p_{n,\epsilon}})_{n\in\mathbb{N},\epsilon\in\{0,1\}}$ form a complete system of pairwise inequivalent irreducible representations of $\mathbb{G}_N(\langle\mathcal{C}_{\text{ds}}, p_{\text{com}}\rangle)$.

For the remainder of this section, we will assume that the following statements is true : let $\mathcal{C}' \subset \mathcal{C}$ be the smallest category of partitions containing all the projective partitions in $\mathcal{C}$. Then, either $\mathcal{C}' = \mathcal{C}_{\text{ds}}$ or $\langle\mathcal{C}_{\text{ds}}, p_{\text{com}}\rangle$.

4.5) Let $r \in \mathcal{C} \setminus \mathcal{C}'$ which we may assume without loss of generality to be on one line. Prove that $\mathcal{O}(\mathbb{G}_N(\langle\mathcal{C}', r\rangle))$ is the quotient of $\mathcal{O}(\mathbb{G}_N(\mathcal{C}'))$ by the relations making a one-dimensional representation trivial.

4.6) Conclude that there are only three possibilities for $\mathcal{C}$, namely $\mathcal{C}_{\text{ds}}$, $\langle\mathcal{C}_{\text{ds}}, q_{\text{com}}\rangle$ and $\langle\bullet\rangle$ where $\bullet$ denotes the singleton partition.

5. The complete classification

5.1) Based on the two previous sections, sketch a proof that if $\mathcal{C}$ is a category of non-crossing partitions containing a partition with a block of odd size at least three, then either $\mathcal{C} = NC$ or $\mathcal{C} = \langle\{\{1,2,3\},\{4\}\}\rangle$.

5.2) Conclude that there exist exactly **seven** categories of non-crossing partitions.

5.3) Deduce from this that there are exactly **six** categories of partitions whose associated compact quantum group is a classical group.