INTRODUCTION
TO
COMPACT MATRIX QUANTUM GROUPS
AND THEIR
COMBINATORICS

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## Appendix A: Two theorems on complex matrix algebras

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CHAPTER 1
INTRODUCING QUANTUM GROUPS

The purpose of the first part of this course is to introduce objects called *compact quantum groups* and to deal in full detail with their algebraic aspects and in particular representation theory. It turns out that most interesting examples of compact quantum groups fall into a specific subclass called *compact matrix quantum groups*. This subclass has the advantage of being more intuitive, as well as allowing for a simplified treatment of the whole theory. We will therefore restrict to it, and the bridge with the more non-commutative geometric approach to compact quantum group will be briefly explained in Chapter 5.

We believe that there is no better way of introducing a new concept than giving examples. We will therefore spend some time introducing one of the most important family of examples of compact matrix quantum groups, first defined by S. Wang in [Wan98] and called the *quantum permutation groups*.

1.1 The graph isomorphism game

There are several ways of motivating the definition of quantum permutation groups, because these objects are related to several important notions like quantum isometry groups in the sense of noncommutative geometry (see for instance [Bic03] or [Ban05]) or quantum exchangeability in the sense of free probability (see for instance [KS09]). In this text, we will start from a recent connection, first made explicit in [LMR17], between quantum permutation groups and quantum information theory. This connection appears through a *game* which we now describe.

As always in quantum information theory, the game is played by two players named *Alice* (denoted by $A$) and *Bob* (denoted by $B$). In this so-called *graph isomorphism game*, they cooperate to win against the *Referee* (denoted by $R$) leading the game. The rules are given by two finite graphs\(^1\) $X$ and $Y$ with vertex sets having the same cardinality, which are known to $A$ and $B$. At each round of the game, $R$ sends a vertex $v_A \in V(X)$ to $A$ and a vertex $v_B \in V(X)$ to $B$. Each of them answers with a vertex $w_A \in V(Y)$, $w_B \in V(Y)$ of the other graph and they win the round if the following condition is matched:

The relation\(^2\) between $v_A$ and $v_B$ is the same as the one between $w_A$ and $w_B$.\(^3\)

Now, the crucial point is that once the game starts, $A$ and $B$ cannot communicate in any way. This can be summarized as follows:

---

1. The following discussion concerning graphs is only intended to motivate the introduction of quantum permutation groups, hence we do not give precise definitions. A rigorous treatment will be given in Chapter 8.
2. Here by relation we mean either being equal, being adjacent, or not being adjacent.
3. This is not the most general version of the graph isomorphism game. We refer the reader to [AMR+18] for a more comprehensive exposition.
Chapter 1. Introducing quantum groups

The question one asks is then: under which condition on the graphs $X$ and $Y$ can the players devise a strategy which wins whatever the given vertices are? It is not very difficult to see that the answer is the following (see [AMR+18, Sec 3.1] for a proof):

**Proposition 1.1.1.** There exists a perfect strategy if and only if $X$ and $Y$ are isomorphic.

This settles the problem in classical information theory, but in the quantum world, $A$ and $B$ can refine their strategy without communicating through the use of entanglement. This means that they can set up a quantum mechanical system and then split it into two parts, such that manipulating one part instantly modifies the other one. We will not go into the details, but it turns out that this gives more strategies, which are said to be quantum. Using these quantum strategies, the previous proposition can be improved. Before giving a precise statement, let us fix some notations:

- Given a Hilbert space $H$, we denote by $\mathcal{B}(H)$ the algebra of bounded (i.e. continuous) linear maps from $H$ to $H$.
- Given a graph $X$, we denote by $A_X$ the adjacency matrix of $A$.

The following result is a combination of [AMR+18, Thm 5.8] and [LMR17, Thm 4.4].

**Theorem 1.1.2** (Atserias-Mančinska-Roberson-Šamal-Severini-Varvitsiotis) There exists a perfect quantum strategy if and only if there exists a matrix $P = (p_{ij})_{1 \leq i,j \leq N}$ with coefficients in $\mathcal{B}(H)$ for some Hilbert space $H$, such that:

- $p_{ij}$ is an orthogonal projection for all $1 \leq i,j \leq N$,
- $\sum_{k=1}^{N} p_{kj} = \text{Id}_H = \sum_{k=1}^{N} p_{ik}$ for all $1 \leq i,j \leq N$,
- $A_X P = PA_Y$.

The proof of this result involves several tools, coming from quantum information theory, graph theory and compact quantum group theory. For those reasons, we postpone it to Chapter 8.

**Remark 1.1.3.** From the perspective of quantum physics, this definition is at least reasonable. Indeed, a family of orthogonal projections summing up to one is a particular instance of a measurement system. We are therefore considering a collection of measurement systems with compatibility conditions coming from the graphs.

**Remark 1.1.4.** It is not straightforward to produce a pair of graphs for which there is a perfect quantum strategy but no classical one, but they do exist. The first example, given in [AMR+18, Sec 6.2], has 24 vertices and may be the smallest possible one.

An intriguing point of Theorem 1.1.2 is the operator-valued matrices which appear in the statement. To understand them, let us consider the case $H = \mathbb{C}$. Then, the coefficients are scalars and since they are projections, they all equal either 0 or 1. Moreover, the sum on any row is 1, hence there is exactly one non-zero coefficient on each row. The same being true for the columns, we have a permutation matrix! We should therefore think of the operator-valued matrices as quantum version of permutations and this leads to the following definition:

---

4. The concept of quantum strategy turns out to be quite subtle, depending on the type of operators allowed. We here use the term in a purposely vague sense and refer the reader to the discussion at the beginning of Chapter 8 for more details.
1.2. The quantum permutation algebra

### Definition 1.1.5
Let $H$ be a Hilbert space. A quantum permutation matrix in $H$ is a matrix $P = (p_{ij})_{1 \leq i,j \leq N}$ with coefficients in $\mathcal{B}(H)$ such that:

- $p_{ij}$ is an orthogonal projection for all $1 \leq i,j \leq N$,
- $\sum_{k=1}^{N} p_{kj} = \text{Id}_H = \sum_{k=1}^{N} p_{ik}$ for all $1 \leq i,j \leq N$.

Moreover, with this point of view the last point of Theorem 1.1.2 has a nice interpretation. Indeed, if $\sigma$ is a permutation and the corresponding matrix $P_\sigma$ satisfies $A_X P_\sigma = P_\sigma A_Y$, then this means that the bijection induced by $\sigma$ between the vertices of $X$ and those of $Y$ respects the edges. In other words, it is a graph isomorphism. Therefore, if the conditions of Theorem 1.1.2 are matched, one says that the graphs are quantum isomorphic.

## 1.2 The quantum permutation algebra

### 1.2.1 Universal definition

The brief discussion of Section 1.1 suggests that quantum permutation matrices are interesting objects which require further study. However, their definition lacks several important features of classical permutation matrices. In particular, there is no obvious way to “compose” quantum permutation matrices, especially if they do not act on the same Hilbert space, so that one could recover an analogue of the group structure of permutations. To overcome this problem, it is quite natural from an (operator) algebraic point of view to introduce a universal object associated to quantum permutation matrices. Note that in order to translate the fact that the operators $p_{ij}$ are orthogonal projections, it is convenient to use the natural involution on $\mathcal{B}(H)$ given by taking adjoints. For this purpose, we will consider $*$-algebras, that is to say complex algebras $A$ endowed with an antilinear and antimultiplicative involution $x \mapsto x^*$.

**Definition 1.2.1.** Let $A_\mathfrak{u}(N)$ be the universal $*$-algebra generated by $N^2$ elements $(p_{ij})_{1 \leq i,j \leq N}$ such that

1. $p_{ij}^2 = p_{ij} = p_{ij}^*$,
2. For all $1 \leq i,j \leq N$, $\sum_{k=1}^{N} p_{ik} = 1 = \sum_{k=1}^{N} p_{kj}$,
3. For all $1 \leq i,j,k,\ell \leq N$, $p_{ij} p_{ik} = \delta_{jk} p_{ij}$ and $p_{ij} p_{\ell j} = \delta_{i\ell} p_{ij}$.

This will be called the quantum permutation algebra on $N$ points.

**Remark 1.2.2.** The third condition in the definition may seem redundant since it is automatically satisfied for projections in a Hilbert space. However, a $*$-algebra may not have a faithful representation on a Hilbert space, hence Condition (3) does not follow from the two other ones.

Definition 1.2.1 refers to a so-called universal object and we will give a few details about it for the sake of completeness. This roughly means that we want the “largest possible” algebra generated by elements that we call $p_{ij}$ and such that the relations in the statement are satisfied. Proving that such an object exists and is well-behaved is not very difficult but requires a bit of abstraction. The intuition is to start with a full algebra of noncommutative polynomials and then quotient by the desired relations. As for usual polynomials, it is easier to use a definition based on sequences.

**Definition 1.2.3.** Given a set $I$, we denote by $\mathcal{U}_I$ the vector space of all finitely supported sequences of elements of $I$. It is endowed with the algebra structure induced by the concatenation of sequences.
If we denote by $X_i$ the sequence $(i, 0, \cdots)$, then the elements $(X_i)_{i \in I}$ generate $\mathcal{U}_I$ and any element can therefore be written as a linear combination of products of these generators, the latter products being called monomials. Note that this decomposition is unique up to the commutativity of addition. We therefore may, and should (and will) see $\mathcal{U}_I$ as the algebra of all noncommutative polynomials over the set $I$, and denote it by $C(X_i \mid i \in I)$. For our purpose, we will turn this into a $*$-algebra by setting $X_i^* = X_i$ for all $i \in I$.

Assume now that we have a subset $\mathcal{R} \subset C(X_i \mid i \in I)$ called relations, then we can build our universal object:

**Definition 1.2.4.** The universal $*$-algebra generated by $(X_i)_{i \in I}$ with the relations $\mathcal{R}$ is the quotient of $C(X_i \mid i \in I)$ by the intersection of all the $*$-ideals containing $\mathcal{R}$. We will still denote its generators by $(X_i)_{i \in I}$.

That this is the correct definition is confirmed by the following universal property:

**Exercise 1.** Let $\mathcal{A}$ be a $*$-algebra generated by elements $(x_i)_{i \in I}$ and let $\mathcal{R} \subset C(X_i \mid i \in I)$. Prove that if $P(x_i) = 0$ for all $P \in \mathcal{R}$, then there exists a unique surjective $*$-homomorphism from the universal $*$-algebra generated by $(X_i)_{i \in I}$ with the relations $\mathcal{R}$ to $\mathcal{A}$ mapping $X_i$ to $x_i$.

**Solution.** We first construct a $*$-homomorphism from $C(X_i \mid i \in I)$. The requirements of the statements force $\pi(X_i) = x_i$, and the fact that $\pi$ is a $*$-algebra homomorphism uniquely determines it on the whole of $C(X_i \mid i \in I)$, i.e.

$$\pi(X_{i_1} \cdots X_{i_n}) = x_{i_1} \cdots x_{i_n}.$$ 

Moreover, it is surjective because the $x_i$’s are generators. By assumption ker($\pi$) is a $*$-ideal containing $\mathcal{R}$, hence it also contains the intersection $J$ of all the $*$-ideals containing it. As a consequence, $\pi$ factors through $C(X_i \mid i \in I)/J$, which is precisely the universal algebra.

We now have a nice object to study, but the link to the classical permutation group is somewhat blurred. To clear it, let us consider the functions $c_{ij} : S_N \to C$ defined by

$$c_{ij}(\sigma) = \delta_{j, \sigma(i)}.$$ 

This is nothing but the function sending the permutation matrix of $\sigma$ to its $(i, j)$-th coefficient. In particular, $c_{ij}$ always takes the value 0 or 1, hence $c_{ij} = c_{ij}^2$. Similarly, it is straightforward to check that Conditions (2) and (3) of Definition 1.2.1 are satisfied. Hence, by the universal property of Exercise 1, there is a unique $*$-homomorphism

$$\pi_{ab} : \left\{ \begin{array}{rl} A_s(N) & \to F(S_N) \\ p_{ij} & \mapsto c_{ij} \end{array} \right.$$ 

where $F(S_N)$ is the algebra of all functions from $S_N$ to $C$. Moreover, since the functions $c_{ij}$ obviously generate the whole algebra $F(S_N)$, $\pi_{ab}$ is onto.

We will now use this link to investigate a possible “group-like” structure on $A_s(N)$. At the level of the coefficient functions, the group law of $S_N$ satisfies the following equation:

$$c_{ij}(\sigma_1\sigma_2) = \sum_{k=1}^{N} c_{ik}(\sigma_1)c_{kj}(\sigma_2).$$ 

The trouble here is that the right-hand side is an element of $F(S_N \times S_N)$, which has no analogue in terms of quantum permutations so far. It would be helpful to express the product solely in terms of $F(S_N)$. It turns out that there is an algebraic construction which exactly does this: the tensor product.
1.2.2  THE TENSOR PRODUCT

Our problem is to build the algebra of functions on \( S_N \times S_N \) using only algebraic constructions on \( F(S_N) \). One may try to consider the direct product \( F(S_N) \times F(S_N) \), but it has dimension \( 2n! \) while \( F(S_N \times S_N) \) has dimension \( (n!)^2 \), so that we need something else. Let us nevertheless dwell on the direct product to get some insight. Given two polynomial functions \( P \) and \( Q \) on \( S_N \), we can see \( PQ \) as a two-variable polynomial. However, the set theoretic map \( \Phi : ((P,Q) \in F(S_N) \times F(S_N) \mapsto PQ \in F(S_N \times S_N) \)
fail to be linear. Indeed, we have the two following issues: first,
\[
\Phi(((P,Q) + (P',Q')) = \Phi(P + P', Q + Q') \\
= (P + P')(Q + Q') \\
\neq PQ + P'Q'
\]
and second
\[
\Phi(\lambda(P,Q)) = \Phi(\lambda P, \lambda, Q) \\
= \lambda^2 PQ \\
\neq \lambda \Phi(P, Q).
\]

In order to remedy this, we can use a universal construction as we already did to define \( A_s(N) \). In other words, we will start from the largest vector space on which the map \( \Phi \) can be defined as a linear map:

**Definition 1.2.5.** Given two vector spaces \( V \) and \( W \), the free vector space on \( V \times W \) is the vector space \( F(V \times W) \) of all finite linear combinations of elements of \( V \times W \).

One must be careful that the elements of \( V \times W \) form a basis of \( F(V \times W) \), hence
\[
(v, w) + (v', w') \neq (v + v', w + w')
\]
in that space. The interest of this construction is that the map \( \Phi \), defined on the basis of \( F(F(S_N) \times F(S_N)) \) by \( \Phi(P, Q) = PQ \) has by definition a unique extension to a linear map
\[
\tilde{\Phi} : F(F(S_N) \times F(S_N)) \to F(S_N \times S_N).
\]
The problem is of course that this map is far from injective, and we have to identify its kernel. Here are three obvious ways of building vectors on which \( \tilde{\Phi} \) vanishes:

- \( \tilde{\Phi}((P, Q) + (P', Q')) = PQ + P'Q' = P(Q + Q') = \tilde{\Phi}(P, Q + Q') \),
- \( \tilde{\Phi}((P, Q) + (P', Q)) = PQ + P'Q = (P + P')Q = \tilde{\Phi}(P + P', Q) \),
- \( \tilde{\Phi}(\lambda P, Q) = \lambda PQ = \tilde{\Phi}(P, \lambda Q) \).

The main result of this section is that this is enough to generate the kernel. Before proving this, let us give a formal definition.

**Definition 1.2.6.** Given two vector spaces \( V \) and \( W \), we denote by \( \mathcal{I}(V, W) \) the linear subspace of \( F(V \times W) \) spanned by the vectors
\[
\bullet (v, w) + (v', w') - (v, w + w'),
\]
\[
\bullet (v, w) + (v', w) - (v + v', w),
\]
\[
\bullet (\lambda v, w) - (v, \lambda w)
\]
for all \((v, w) \in V \times W\). Then, the tensor product of \(V\) and \(W\) is the quotient vector space
\[
V \otimes W = F(V \times W)/\mathcal{I}(V,W).
\]
The image of \((v, w)\) in this quotient will be denoted by \(v \otimes w\).

This construction may seem weird at first sight, since we are quotienting a “huge” vector space by a “huge” vector subspace. However, it turns out that the result is very tractable and perfectly fits our requirements. Before proving this, let us elaborate a bit more on the general construction by identifying a basis.

**Proposition 1.2.7.** Let \((e_i)_{i \in I}\) and \((f_j)_{j \in J}\) be bases of \(V\) and \(W\) respectively. Then,
\[
(e_i \otimes f_j)_{(i,j) \in I \times J}
\]
is a basis of \(V \otimes W\).

**Proof.** Let \(v \in V\) and \(w \in W\). By assumption, they can be written as
\[
v = \sum_{i \in I_v} \lambda_i e_i \quad \text{and} \quad w = \sum_{j \in J_w} \mu_j f_j
\]
for some finite subsets \(I_v \subseteq I\) and \(J_w \subseteq J\). Thus,
\[
(v, w) = \sum_{(i,j) \in I_v \times J_w} \lambda_i \mu_j (e_i, f_j) \in \mathcal{I}(V,W)
\]
by definition. In other words, we have in \(V \otimes W\) the equality
\[
v \otimes w = \sum_{(i,j) \in I_v \times J_w} \lambda_i \mu_j e_i \otimes f_j
\]
proving that the family is generating.

To show linear independence, let us consider for some fixed \((i, j) \in I \times J\) the linear map
\[
\varphi_{ij} : F(V \times W) \to \mathbb{C}
\]
sending \((v, w)\) to \(e_i^*(v) \times f_j^*(w)\). By construction, the kernel of \(\varphi_{ij}\) contains \(\mathcal{I}(V,W)\), hence it factors through the canonical quotient map \(\pi : F(V \times W) \to V \otimes W\) in to a linear map \(\psi_{ij} : V \otimes W \to \mathbb{C}\). It then follows that
\[
\psi_{ij}(e_i' \otimes f_j') = \delta_{ii'} \delta_{jj'}
\]
and this clearly implies that the family is linearly independent, concluding the proof. ■

As a consequence, we can elucidate the tensor product construction for finite-dimensional vector spaces:

**Corollary 1.2.8.** Let \(V\) and \(W\) be vector spaces of dimension \(n\) and \(m\) respectively. Then, \(V \otimes W\) has dimension \(n \times m\).

In particular, the dimension issue with the direct product disappears when considering tensor products. Back to our problem now, we want to prove that \(F(S_N \times S_N)\) is isomorphic to \(F(S_N) \otimes F(S_N)\). We will do this in greater generality, since we will take all along inspiration for the general theory of linear affine algebraic groups. In other words, we will consider algebras of the form
\[
\mathcal{O}(X) = \mathbb{C}[X_1, \cdots, X_N]/I
\]
for some ideal \(I\).

---

5. The notation here is the standard one from commutative algebra since we are now considering algebra of commutative polynomials.
Proposition 1.2.9. Let $I \subset C[X_1, \ldots, X_n]$ and $J \subset C[Y_1, \ldots, Y_m]$ be ideals. Then, the map

$$\Phi : (a + I, b + J) \mapsto ab + (I + J)$$

factors through a linear isomorphism

$$C[X_1, \ldots, X_n]/I \otimes C[Y_1, \ldots, Y_m]/J \simeq C[X_1, \ldots, X_n, Y_1, \ldots, Y_m]/(I + J).$$

Proof. To lighten notations, let us denote by $A_n$ the complex polynomial algebra on $n$ indeterminates. If $a, b \in A$, $x \in I$ and $y \in J$, then

$$(a + x)(b + y) = ab + ay + xb + xy$$

and $ay + xb + xy \in I + J$ so that there is a well-defined linear map

$$\bar{\Phi} : F(A_n/I \times A_m/J) \to A_{n+m}/(I + J).$$

One easily checks that $I(A_n/I, A_m/J) \subset \ker(\bar{\Phi})$, hence there is a well-defined induced map

$$\Phi : A_n/I \otimes A_m/J \to A_{n+m}/(I + J).$$

Conversely, any element of $A_{n+m}$ can be written uniquely (up to permutation of the summands) as $\sum_i P_i Q_i$ where $P_i \in A_n$ and $Q_i \in A_m$. Let us set

$$\bar{\Psi} \left( \sum_i P_i Q_i \right) = \sum_i \overline{P_i} \otimes \overline{Q_i} \in A_n/I \otimes A_m/J,$$

where the bar denotes the image in the quotient. Obviously, $\bar{\Psi}$ vanishes on $I$ and $J$, hence on $I + J$, allowing to factor it through a map

$$\Psi : A_{n+m}/(I + J) \to A_n/I \otimes A_m/J.$$ 

Now, applying $\Phi \circ \Psi$ and $\Psi \circ \Phi$ to the basis vectors $X_i^k Y_j^l$ and $X_i^k \otimes Y_j^l$, respectively shows that both compositions are the identity, concluding the proof. \hfill $\blacksquare$

The result is quite satisfying, except that we do not want to deal with vector spaces but with algebras. The construction is however easy to generalize. First note that if $A$ and $B$ are algebras, then there is a unique algebra structure on $A \otimes B$ defined by

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb'.$$

Indeed, one can certainly define a suitable bilinear map on $F(A \times B) \otimes F(A \times B)$, and checking that it vanishes as soon as one of the components is in $I(A,B)$ yields the result. If moreover $A$ and $B$ are $*$-algebras, then there is a unique $*$-algebra structure on $A \otimes B$ given by

$$(a \otimes b)^* = a^* \otimes b^*.$$ 

We can now state and prove our main result :

Theorem 1.2.10 Let $I \subset A_n$ and $J \subset A_m$ be $*$-ideals. Then, the map

$$(a + I, b + J) \mapsto ab + (I + J)$$

factors through an algebra $*$-isomorphism

$$A_n/I \otimes A_m/J \simeq A_{n+m}/(I + J).$$

Proof. One simply has to check that the previous linear isomorphisms $\Phi$ and $\Psi$ are algebra $*$-homomorphisms, which is straightforward. \hfill $\blacksquare$
Corollary 1.2.11. The map \( \iota : F(S_N) \otimes F(S_N) \to F(S_N \times S_N) \) sending \( f \otimes g \) to the map \( (\sigma, \tau) \mapsto f(\sigma)g(\tau) \) extends to a \( * \)-algebra isomorphism.

Proof. Let \( I \) be the ideal of \( A = C[X_{ij} \mid 1 \leq i, j \leq N] \) generated by the polynomials giving the relations of Definition 1.2.1, so that \( A/I = F(S_N) \). Theorem 1.2.10 thus yields an isomorphism

\[
F(S_N) \otimes F(S_N) \to C[X_{ij}, Y_{ij} \mid 1 \leq i, j \leq N]/\tilde{I}
\]

where \( \tilde{I} \) is generated by the two copies of \( I \) and the image of \( P \otimes Q \) is \( P \times Q \). Any element of the right-hand side can be written as a linear combination of products \( P \times Q \) and therefore defines a function on \( S_N \times S_N \). Moreover, any function appears in that ways, hence the result. \( \blacksquare \)

As a conclusion, we can identify canonically \( F(S_N \times S_N) \) with \( F(S_N) \otimes F(S_N) \), so that we have an analogue of functions on pairs of quantum permutation matrices, which is simply \( \mathcal{A}_q(N) \otimes \mathcal{A}_q(N) \).

Now that we are talking about tensor products, let us take the occasion to define the corresponding construction on linear maps, so that it is ready for use in the next chapters.

Exercise 2. Let \( T_i : V_i \to W_i \) be linear maps. Prove that there exists a unique linear map

\[
T_1 \otimes T_2 : V_1 \otimes V_2 \to W_1 \otimes W_2
\]

such that for any \( (v_1, v_2) \in V_1 \otimes V_2 \),

\[
(T_1 \otimes T_2)(v_1 \otimes v_2) = T_1(v_1) \otimes T_2(v_2).
\]

Solution. We can define a map

\[
T_1 \otimes T_2 : \mathcal{F}(V_1 \times V_2) \to \mathcal{F}(W_1 \times W_2) \to W_1 \otimes W_2
\]

by the formula

\[
(T_1 \otimes T_2)(v_1, v_2) = T_1(v_1) \otimes T_2(v_2).
\]

statement just by definition of these two vector spaces. Then, the linearity of \( T_1 \) and \( T_2 \) implies that \( T_1 \otimes T_2 \) vanishes on \( \mathcal{I}(V_1, V_2) \), hence it factors through \( V_1 \otimes V_2 \), yielding the result. \( \blacksquare \)

1.2.3 Coproduct

Back to our formula for the product, we can now write

\[
c_{ij}(\sigma_1 \sigma_2) = \sum_{k=1}^{N}(c_{ik} \otimes c_{kj})(\sigma_1, \sigma_2)
\]

Considering the elements \( p_{ij} \) as “coefficient functions”, this suggests to encode a kind of “group law” through the map

\[
\Delta : p_{ij} \to \sum_{k=1}^{N} p_{ik} \otimes p_{kj}.
\]

Proposition 1.2.12. There exists a unique \( * \)-homomorphism \( \Delta : \mathcal{A}_q(N) \to \mathcal{A}_q(N) \otimes \mathcal{A}_q(N) \) satisfying the formula (1.1).

Proof. Let us set, for \( 1 \leq i, j \leq N \),

\[
q_{ij} = \sum_{k=1}^{N} p_{ik} \otimes p_{kj}.
\]

We claim that the \( q_{ij} \)’s satisfy Conditions (1) to (3) of Definition 1.2.1. The existence of \( \Delta \) then follows from the universal property. \( \blacksquare \)
Exercise 3. Prove the claim in the preceding proof.

Solution. It is clear that \( q_{ij}^* = q_{ij} \). Let us now compute the square:

\[
q_{ij}^2 = \sum_{k,\ell=1}^N p_{ik} p_{\ell\ell} \otimes p_{k\ell} p_{\ell j}
\]

\[
= \sum_{k,\ell=1}^N \delta_{k\ell} p_{ik} \otimes p_{k\ell}
\]

\[
= q_{ij}.
\]

We have therefore checked Condition (1). Moreover,

\[
\sum_{i=1}^N q_{ij} = \sum_{k,i=1}^N p_{ik} \otimes p_{kj}
\]

\[
= \sum_{k=1}^N \left( \sum_{i=1}^N p_{ik} \right) \otimes p_{kj}
\]

\[
= \sum_{k=1}^N 1 \otimes p_{kj}
\]

\[
= 1 \otimes 1
\]

hence Condition (2) is also satisfied. Eventually, for \( j \neq j' \),

\[
q_{ij}q_{ij'} = \sum_{k,\ell=1}^N p_{ik} p_{\ell\ell} \otimes p_{k\ell} p_{\ell j'}
\]

The first tensor in the sum vanishes unless \( k = \ell \), but in that case the second one vanishes and Condition (3) follows. The argument for \( i \neq i' \) is similar. □

The map \( \Delta \) is called the coproduct and is a reasonable substitute for matrix multiplication (i.e. the group law of a matrix group). In particular, it satisfies an analogue of the associativity property of the group law, called coassociativity:

\[
(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta.
\]  \[ (1.2) \]

Exercise 4. Prove that the coproduct on \( A_s(N) \) is indeed coassociative. Check also that the corresponding equation on the coefficient functions in \( S_N \) is equivalent to associativity of the composition of permutations.

Solution. Because \( \Delta \) is a \(*\)-algebra homomorphism, it is enough to check coassociativity on the generators, and this is straightforward:

\[
(\Delta \otimes \text{id}) \circ \Delta(u_{ij}) = \sum_{k=1}^N \Delta(u_{ik}) \otimes u_{kj}
\]

\[
= \sum_{k,\ell=1}^N u_{i\ell} \otimes u_{\ell k} \otimes u_{kj}
\]

\[
= \sum_{\ell=1}^N u_{i\ell} \otimes \Delta(u_{\ell j})
\]

\[
= (\text{id} \otimes \Delta) \circ \Delta(u_{ij}).
\]

As for the second assertion, we have already seen that \( \Delta(c_{ij})(g, h) = c_{ij}(g, h) \). Thus,

\[
(\Delta \otimes \text{id}) \circ \Delta(c_{ij})(\sigma_1, \sigma_2, \sigma_3) = \Delta(c_{ij})(\sigma_1 \sigma_2, \sigma_3) = c_{ij}((\sigma_1 \sigma_2) \sigma_3)
\]
while\[
(id \otimes \Delta) \circ \Delta(c_{ij})(\sigma_1, \sigma_2, \sigma_3) = \Delta(c_{ij})(\sigma_1, \sigma_2\sigma_3) = c_{ij}(\sigma_1\sigma_2\sigma_3)
\]
so that coassociativity is equivalent to \(f((\sigma_1\sigma_2)\sigma_3) = f(\sigma_1(\sigma_2\sigma_3))\) for all \(f \in \mathcal{F}(S_N)\) and \(\sigma_1, \sigma_2, \sigma_3 \in S_N\), which is in turn equivalent to the associativity of the group law. ■

The coproduct certainly indicates that we are on the right track to produce a group-like structure on the quantum permutation algebra. However, we still need a neutral element and an inverse. But instead of trying to translate each of them, we will take advantage of the fact that we are considering a matrix group. Indeed, for any permutation, the corresponding matrix is orthogonal, so that for any permutation \(\sigma\),

\[
\sum_{k=1}^{N} c_{ik}(\sigma)c_{jk}(\sigma) = \delta_{ij} = \sum_{k=1}^{N} c_{ki}(\sigma)c_{kj}(\sigma).
\]

Since this holds for any \(\sigma\), it can be written as an equality of functions in \(\mathcal{F}(S_N)\) and it turns out that the same equality holds in \(A_s(N)\):

**Proposition 1.2.13.** For any \(1 \leq i, j \leq N\),

\[
\sum_{k=1}^{N} p_{ik}p_{kj} = \delta_{ij} = \sum_{k=1}^{N} p_{ki}p_{kj}.
\]

**Proof.** This is a direct consequence of Conditions (1) to (3). ■

This means that the quantum permutation algebra is somehow “made of orthogonal quantum matrices” and this property should contain all information about the unit and the inverse. Another way to state this is that the matrix \(P = (p_{ij})_{1 \leq i, j \leq N} \in M_N(A_s(N))\) is orthogonal in the sense that its inverse equals its transpose. As a conclusion, the algebra \(A_s(N)\) with its generators \((p_{ij})_{1 \leq i, j \leq N}\) seem to have all the properties one can expect for a group-like object. It therefore deserves the name of quantum group that we will define in the next section.

**Remark 1.2.14.** The fact that Condition (1.4) yields a full group-like structure can be encoded in the two following maps:

- The antipode \(S : A_s(N) \rightarrow A_s(N)\), which is the unique \(*\)-antihomomorphism induced by

\[
p_{ij} \mapsto p_{ji}.
\]

Since the transpose of \(P\) is its inverse, this plays the rôle of the inverse map.

- The counit \(\varepsilon : A_s(N) \rightarrow \mathbb{C}\), which is the unique \(*\)-homomorphism induced by

\[
p_{ij} \mapsto \delta_{ij}.
\]

Since the matrix \((\delta_{ij})_{1 \leq i, j \leq N}\) is the identity, this play the rôle of the neutral element.

Equation (1.4) then becomes

\[
m \circ (id \otimes S) = \varepsilon = m \circ (S \otimes id),
\]

where \(m : A_s(N) \otimes A_s(N) \rightarrow A_s(N)\) is the multiplication map. Our focus in this text is on the matricial aspect of quantum groups, and we will therefore never use these maps. Note however that \((A_s(N), \Delta, \varepsilon, S)\) is what is called a Hopf algebra. The theory of Hopf algebras is vast and has many connections to other fields. The reader may consult for instance [Rad11] for a detailed introduction or [Kas95] for more categorical aspects and important applications.
1.3 Compact matrix quantum groups

Our study of the quantum permutation algebras has given us enough motivation to introduce a notion of compact quantum group. There is a nice and complete theory of these objects, which was developed by S.L. Woronowicz in [Wor98]. There are to our knowledge two books explaining this theory in detail, [Tim08] and [NT13] to which the reader may refer for alternative and more comprehensive expositions.

1.3.1 A first definition

The purpose of this text is to give some examples of the interaction between the combinatorics of partitions and the theory of compact quantum groups. The most striking examples involve compact quantum groups which belong to a specific class which is, in a sense, simpler to define and handle. It was introduced by S.L. Woronowicz in [Wor87] as a generalization of compact groups of matrices and as a first attempt to a general definition of compact quantum groups. We will therefore focus on this class for the moment, even though our definition differs from [Wor87, Def 1.1] and is closer to [Wan95a, Def 2.1'].

Definition 1.3.1. An orthogonal compact matrix quantum group of size $N$ is given by an $*$-algebra $A$ generated by $N^2$ elements $(u_{ij})_{1 \leq i,j \leq N}$ such that

1. $u_{ij} = u_{ij}^*$ for all $1 \leq i,j \leq N$,
2. For all $1 \leq i,j \leq N$,
   \[ \sum_{k=1}^{N} u_{ik}u_{kj} = \delta_{ij} = \sum_{k=1}^{N} u_{ki}u_{kj}. \]
3. There exists a $*$-homomorphism $\Delta : A \to A \otimes A$ such that for all $1 \leq i,j \leq N$,
   \[ \Delta(u_{ij}) = \sum_{k=1}^{N} u_{ik} \otimes u_{kj}. \]

Denoting by $u \in M_N(A)$ the matrix with coefficients $(u_{ij})_{1 \leq i,j \leq N}$, we will denote the orthogonal compact matrix quantum group by $(A,u)$.

By analogy with our reasoning on $S_N$, $A$ will be thought of as the algebra of functions on a non-existent “quantum space”. However, if we consider general “compact quantum spaces”, we cannot use all the functions like for $S_N$. Here our crucial intuition will be that compact groups of matrices are completely determined by their algebra of regular functions, that is to say functions which are polynomial in the matrix coefficients. The usual notation for this is $\mathcal{O}(G)$, whence the notation $A = \mathcal{O}(G)$ if $G$ denotes the orthogonal compact matrix quantum group. We can now formalize the properties of the quantum permutation algebras established in Section 1.2:

Definition 1.3.2. For any integer $N$, the pair $(A_s(N), P)$ is an orthogonal compact matrix quantum group, where $P = (p_{ij})_{1 \leq i,j \leq N}$. It is called the quantum permutation group on $N$ points and is usually referred to using the notation $S^+_N$.

Consequently, we may from now on write $\mathcal{O}(S^+_N)$ instead of $A_s(N)$. This quantum group was first defined by S. Wang in [Wan98]. It is natural (and crucial for applications to quantum information theory) to wonder whether this is really different from $S_N$.

Exercise 5. Prove that for $N = 1, 2, 3$, $S^+_N = S_N$ in the sense that $\pi_{ab}$ is injective. Prove moreover that for any $N \geq 4$, $\mathcal{O}(S^+_N)$ is noncommutative, hence not isomorphic to $F(S_N)$.

Solution. For $N = 1$, $A_s(1)$ is generated by one self-adjoint projection, hence is isomorphic to $C = F(S_1)$. For $N = 2$, observe that the relations force

\[ P = \begin{pmatrix} p_{11} & 1 - p_{11} \\ 1 - p_{11} & p_{11} \end{pmatrix} \]
making $A_s(2)$ abelian, hence equal to $F(S_2)$.

For $N = 3$, we must again prove that $A_s(3)$ is abelian. Here a simple argument from [LMR17]. It is enough to prove that $p_{11}$ commutes with $p_{22}$ since any independent permutation of the rows and columns of $P$ yields an automorphism of $A_s(N)$ by the universal property. We start by observing that

$$p_{11}p_{22} = p_{11}(p_{12} + p_{13}) + p_{11}p_{22}p_{13}.$$  

But

$$p_{11}p_{22}p_{13} = p_{11} \left( 1 - p_{21} - p_{23} \right) p_{13} = p_{11}p_{13} - p_{11}p_{21}p_{13} - p_{11}p_{23}p_{13} = 0,$$

hence

$$p_{11}p_{22} = p_{11}p_{22}p_{11} = (p_{11}p_{22}p_{11})^* = p_{22}p_{11}.$$

For $N \geq 4$, consider a Hilbert space $H$ and two orthogonal projections $p, q \in B(H)$ which do not commute. Then, consider the matrix

$$
\begin{pmatrix}
    p & 1 - p & 0 & 0 \\
    1 - p & p & 0 & 0 \\
    0 & 0 & q & 1 - q \\
    0 & 0 & 1 - q & q \\
\end{pmatrix}
$$

and complete it to an $N \times N$ matrix by putting it in the upper left corner, setting the other diagonal coefficients to 1 and all the other coefficients to 0. This yields a quantum permutation matrix, hence a $*$-homomorphism $\pi : O(S_N^+ \otimes H) \to B(H)$. Because $\pi(u_{11}) = p$ and $\pi(u_{23}) = q$ do not commute, we infer that $O(S_N^+) \otimes H$ is not commutative.

To get a better understanding of Definition 1.3.1, it is worth working out the link with the classical case. This requires the identification of a tensor product of linear maps, that we give here as a Lemma.

**Lemma 1.3.3.** Let $T_1 : V_1 \to W_1$ and $T_2 : V_2 \to W_2$ be linear maps. Then,

$$\ker(T_1 \otimes T_2) = \ker(T_1) \otimes W_2 + W_1 \otimes \ker(T_2).$$

**Proof.** We may assume without loss of generality that the maps are surjective. Moreover, we have decompositions

$$V_1 = \ker(T_1) \oplus V'_1$$

such that the maps restrict to isomorphisms on $V'_1$. Let us set

$$\bar{T}_i = (\text{id} \oplus T^{-1}_{i|V'_i}) \circ T_i$$

which is the projection onto $V'_i$ parallel to $\ker(T_i)$. It is easy to see that we have a decomposition

$$V_1 \otimes V_2 = (\ker(T_1) \otimes \ker(T_2)) \oplus (\ker(T_1) \otimes V'_2) \oplus (V'_1 \otimes \ker(T_2)) \oplus (V'_1 \otimes V'_2).$$

By definition, $\bar{T}_1 \otimes \bar{T}_2$ vanishes on the first three summands and is the identity is the last one so that its kernel is

$$(\ker(T_1) \otimes \ker(T_2)) \oplus (\ker(T_1) \otimes V'_2) \oplus (V'_1 \otimes \ker(T_2)) = \ker(T_1 \otimes T_2) = \ker(T_1) \otimes W_2 + W_1 \otimes \ker(T_2).$$

Applying $(\text{id} \oplus T_{i|V'_i})^{-1} \otimes (\text{id} \oplus T_{j|V'_j})$ to this equality then yields the result. \hfill \blacksquare

**Exercise 6.** Let $(A, u)$ be an orthogonal compact matrix quantum group such that $A$ is commutative. Prove that there exists a compact subgroup $G$ of $O_N$ and an isomorphism

$$A \simeq O(G)$$

sending $u_{ij}$ to the coefficient function $c_{ij}$ for all $1 \leq i, j \leq N$. 

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**Hint**: If $O_N$ denote the group of $N \times N$ orthogonal matrices, then $\mathcal{O}(O_N)$ is the universal $*$-algebra generated by elements $(c_{ij})_{1 \leq i,j \leq N}$ which pairwise commute and form an orthogonal matrix\(^6\).

**Solution.** By universality, there is a surjective $*$-homomorphism
\[
\pi : \mathcal{O}(O_N) \to A
\]
sending $c_{ij}$ to $u_{ij}$. Set now $I = \ker(\pi)$ and let
\[
G = \{ M \in O_N \mid P(M) = 0 \text{ for all } P \in I \}.
\]
This is a closed, hence compact, subset of $O_N$ and $A \simeq \mathcal{O}(G)$, so that it only remains to prove that $G$ is a subgroup. If $P \in I$, let us write
\[
\Delta(P) = \sum_i P_i \otimes Q_i \in \mathcal{O}(O_N) \otimes \mathcal{O}(O_N)
\]
with linearly independent tensors. Because
\[
(\pi \otimes \pi) \circ \Delta(P) = \Delta \circ \pi(P) = 0,
\]
Lemma 1.3.3 implies that for all $i$, either $P_i$ or $Q_i$ belongs to $I$. Thus, for any $M_1, M_2 \in G$,
\[
P(M_1M_2) = \Delta(P)(M_1, M_2) = \sum_i P_i(M_1)Q_i(M_2) = 0
\]
and $M_1M_2 \in G$. One can conclude with a sledgehammer argument: $G$ is a group because it is a compact bisimplifiable semigroup. It is also possible to exploit the idea used for the coproduct a little more. Indeed, if $S$ is the map induced by $c_{ij} \mapsto c_{ji}$, then
\[
\pi \circ S(P) = S \circ \pi(P) = 0
\]
so that $P(M^{-1}) = S(P)(M) = 0$ for any $M \in G$, hence $M^{-1} \in G$. Eventually, if $\varepsilon$ is the map induced by $c_{ij} \mapsto \delta_{ij}$, one has $\varepsilon \circ \pi = \varepsilon$ so that $I \subset \ker(\varepsilon)$. As a consequence,
\[
P(\text{Id}_N) = \varepsilon(P) = 0
\]
for all $P \in I$, yielding $\text{Id}_N \in G$. \hfill \Box

### 1.3.2 The Quantum Orthogonal Group

Before delving into the general theory of compact quantum groups, let us give another fundamental example which is also due to S. Wang but earlier in [Wan95a]. After a look at Definition 1.3.1, it is natural to wonder about the largest possible orthogonal compact matrix quantum group. Its definition relies on the following simple fact:

**Exercise 7.** Let $N$ be an integer and let $\mathcal{A}_o(N)$ be the universal $*$-algebra generated by $N^2$ self-adjoint elements $(U_{ij})_{1 \leq i,j \leq N}$ such that
\[
\sum_{k=1}^N U_{ik}U_{jk} = \delta_{ij} = \sum_{k=1}^N U_{ki}U_{kj}.
\]
Then, there exists a unique $*$-homomorphism
\[
\Delta: \mathcal{A}_o(N) \to \mathcal{A}_o(N) \otimes \mathcal{A}_o(N)
\]
such that for all $1 \leq i, j \leq N$,
\[
\Delta(U_{ij}) = \sum_{k=1}^N U_{ik} \otimes U_{kj}.
\]
\(^6\) This is straightforward once one notices that the algebra of regular functions on $M_N(\mathbb{C})$ is just $\mathbb{C}[X_{ij} \mid 1 \leq i, j \leq N]$. 

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Solution. The proof is similar to that of Proposition 1.2.12. We set
\[
V_{ij} = \sum_{k=1}^{N} U_{ik} \otimes U_{kj}
\]
and have to check that the corresponding matrix \( V \) is orthogonal. Indeed,
\[
\sum_{k=1}^{N} V_{ik} V_{jk} = \sum_{k,l,m=1}^{N} U_{il} U_{jm} \otimes U_{mk}
\]
\[
= \sum_{\ell,m=1}^{N} U_{il} U_{jm} \otimes \delta_{\ell m}
\]
\[
= \sum_{\ell=1}^{N} U_{il} U_{j\ell} \otimes 1
\]
\[
= \delta_{ij} 1 \otimes 1.
\]
The other equality is proved similarly and it then follows from universality that there exists a *-homomorphism sending \( U_{ij} \) to \( V_{ij} \). \( \blacksquare \)

Denoting by \( U \) the matrix \( (U_{ij})_{1 \leq i,j \leq N} \in M_{N}(A_o(N)) \), this motivates the following definition:

**Definition 1.3.4.** The pair \( (A_o(N),U) \) is an orthogonal compact matrix quantum group called the *quantum orthogonal group*. It is usually referred to using the notation \( O^+_N \).

As the name suggest, \( O^+_N \) is linked to orthogonal groups. Indeed, if \( c_{ij} : O_N \to \mathbb{C} \) are the matrix coefficient functions, then there is a surjective *-homomorphism \( \pi_{ab} : O^+_N \to O(N) \) sending \( U_{ij} \) to \( c_{ij} \). Thus, \( O^+_N \) is a quantum version of \( O_N \) just as \( S^+_N \) is a quantum version of \( S_N \).

Note that it follows from the universal property that there is a surjective *-homomorphism
\[
O(O^+_N) \to O(S^+_N),
\]
so that \( O(O^+_N) \) is not commutative as soon as \( N \geq 4 \). However, more is true in that case.

**Proposition 1.3.5.** The *-algebra \( O(O^+_N) \) is non-commutative as soon as \( N \geq 2 \).

**Proof.** Let \( \mathbb{Z}_2 \) denote the cyclic group of order 2, and consider the free product \( \mathbb{Z}_2^\times N \) with canonical generators \( a_1, \cdots, a_N \). Seeing these elements as unitaries in the complex group algebra \( \mathbb{C} \left[ \mathbb{Z}_2^\times N \right] \), the diagonal matrix with coefficients \( a_1, \cdots, a_N \) satisfies the relations of Definition 1.3.1. Hence, there is a surjective *-homomorphism
\[
O(O^+_N) \to \mathbb{C} \left[ \mathbb{Z}_2^\times N \right]
\]
sending \( U_{ii} \) to \( a_i \) and \( U_{ij} \) to 0 for \( i \neq j \). This proves in particular that \( U_{11} \) and \( U_{22} \) do not commute. \( \blacksquare \)

### 1.3.3 The unitary case

Exercise 6 illustrates the fact that orthogonal compact matrix quantum groups generalize subgroups of \( O_N \). This can be made rigorous in the following way: by universality, for any orthogonal compact matrix quantum group \( G = (\mathcal{O}(G), u) \), there is a surjective *-homomorphism
\[
\pi : O(O^+_N) \to \mathcal{O}(G)
\]
1.3. Compact matrix quantum groups

sending $U_{ij}$ to $u_{ij}$ and therefore satisfying

$$\Delta \circ \pi(x) = (\pi \otimes \pi) \circ \Delta(x)$$

for all $x \in \mathcal{O}(G)$. Thus, orthogonal compact matrix quantum groups are “quantum subgroups” of $O_N^\times$.

One may wonder whether it is possible to consider analogues of closed subgroups of the unitary group $U_N$ instead of the orthogonal one. This is possible, but we will not need it in this text. Moreover, this more general setting involves subtleties which make some arguments tricky. This can already be seen on the following definition:

**Definition 1.3.6.** A unitary compact matrix quantum group of size $N$ is given by a $\ast$-algebra $A$ generated by $N^2$ elements $(u_{ij})_{1 \leq i, j \leq N}$ such that

1. There exist a $\ast$-homomorphism $\Delta : A \to A \otimes A$ such that for all $1 \leq i, j \leq N$,

$$\Delta(u_{ij}) = \sum_{k=1}^{N} u_{ik} \otimes u_{kj},$$

2. For all $1 \leq i, j \leq N$,

$$\sum_{k=1}^{N} u_{ik}^* u_{jk} = \delta_{ij} = \sum_{k=1}^{N} u_{ki}^* u_{kj}$$

and

$$\sum_{k=1}^{N} u_{ki}^* u_{kj} = \delta_{ij} = \sum_{k=1}^{N} u_{ik}^* u_{jk}$$

hold.

**Remark 1.3.7.** The relations in the previous definition mean that both the matrix $u$ and its conjugate $\pi$ (the matrix where each coefficient is replaced with its adjoint) are unitary. The second one does not follow from the first one in general (see [Wan95a, Sec 4.1] for a counter-example), so that both need to be included in the definition.

Once again there is an obvious example obtained by considering the largest possible such quantum group at a fixed size $N$:

**Definition 1.3.8.** Let $A_u(N)$ be the universal $\ast$-algebra generated by $N^2$ self-adjoint elements $(U_{ij})_{1 \leq i, j \leq N}$ such that

$$\sum_{k=1}^{N} U_{ik} U_{jk}^* = \delta_{ij} = \sum_{k=1}^{N} U_{ki}^* U_{kj},$$

and

$$\sum_{k=1}^{N} U_{ik}^* U_{jk} = \delta_{ij} = \sum_{k=1}^{N} U_{ki} U_{kj}^*.$$

One proves exactly as in Exercise 7 that there is a unique $\ast$-homomorphism

$$\Delta : A_u(N) \to A_u(N) \otimes A_u(N)$$

such that

$$\Delta(U_{ij}) = \sum_{k=1}^{N} U_{ik} \otimes U_{kj}$$

and the pair $U_N^\times = (A_u(N), U)$ is called a free unitary quantum group. We leave it as an exercise to the reader to prove that for $N \geq 2$, the quotient of $A_u(N)$ by the $\ast$-ideal generated by the elements $U_{ij}$ for $i \neq j$ is isomorphic to the complex group algebra $\mathbb{C}[\mathcal{F}_N]$ of the free group on $N$ generators.

Even though concrete examples of unitary compact quantum groups are more difficult to deal with than orthogonal ones, most of the general theory is exactly the same (see Chapter 6). As a consequence, we will state and prove general results in the setting of unitary compact matrix quantum groups as soon as this does not entail any additional technicality in the proof.
CHAPTER 2

REPRESENTATION THEORY

Now that we have a definition of a quantum group, it is time to investigate their general structure. It turns out that compact groups are mainly tractable because they have a very nice representation theory. It is therefore natural to start by looking for a suitable notion of representation for compact quantum groups.

2.1 Finite-dimensional representations

2.1.1 Two definitions

Following our now usual strategy, we will restate the notion of representation in terms of functions. Recall that for a group $G$, a representation on a vector space $V$ is a group homomorphism

$$\rho : G \to \mathcal{L}(V).$$

Assume for instance that $V$ is finite-dimensional so that we can identify $\mathcal{L}(V)$ with $M_n(\mathbb{C})$ for some $n$. Composing $\rho$ with the coefficient functions produces a bunch of functions $(\rho_{ij})_{1 \leq i, j \leq n}$ from $G$ to $\mathbb{C}$. Given our setting, it is natural to consider those finite-dimensional representations for which the coefficient functions belong to $\mathcal{O}(G)$. It turns out that for a compact group of matrices, this holds for all finite-dimensional representations. The fact that $\rho$ is a representation translates into two properties:

1. The matrix $[\rho_{ij}(g)]$ is invertible for all $g \in G$,

2. For any $1 \leq i, j \leq n$, $\rho_{ij}(gh) = \sum_{k=1}^{n} \rho_{ik}(g)\rho_{kj}(h)$.

The second point is reminiscent of the discussion around the definition of the coproduct in Section 1.2.3 and we therefore know how to translate it. As for the first one, it means that $\rho$ is an invertible element in $M_n(\mathcal{O}(G))$. As a conclusion, we may give the following definition:

**Definition 2.1.1.** Let $\mathbb{G} = (\mathcal{O}(\mathbb{G}), u)$ be a compact matrix quantum group and let $n$ be an integer. A continuous $n$-dimensional representation of $\mathbb{G}$ is an element $v \in M_n(\mathcal{O}(\mathbb{G}))$ such that

1. $v$ is invertible,

2. $\Delta(v_{ij}) = \sum_{k=1}^{N} v_{ik} \otimes v_{kj}$ for all $1 \leq i, j \leq N$.

If moreover $v$ is unitary in the sense that $v^*v = \text{Id}_{M_n(\mathcal{O}(\mathbb{G}))} = vv^*$, then it is said to be a unitary representation.
Example 2.1.2. The first, extremely important, example is \( u \), which is a unitary representation. Since it defines the quantum group, it ought to determine all the representations. This idea will be made clear by the Tannaka-Krein reconstruction Theorem 3.3.3, and because of this peculiar rôle, \( u \) is called the fundamental representation of \( \mathbb{G} \).

Example 2.1.3. The second important example is the element

\[ \varepsilon = 1 \in M_1(\mathcal{O}(\mathbb{G})) = \mathcal{O}(\mathbb{G}), \]

which is also a representation, called the trivial representation.

The standard operations on representations generalize to this setting and will be crucial for the sequel. For instance, if \( v \) and \( w \) are two finite-dimensional representations of dimension \( n \) and \( m \) respectively, then

\[ v \oplus w = \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix} \in M_{n+m}(\mathcal{O}(\mathbb{G})) \]

is their direct sum, while

\[ v \otimes w = (v_{ij} w_{k\ell})_{1 \leq i,j \leq n, 1 \leq k, \ell \leq m} \in M_n(\mathcal{O}(\mathbb{G})) \otimes M_m(\mathcal{O}(\mathbb{G})) \cong M_{nm}(\mathcal{O}(\mathbb{G})) \]

is their tensor product. Here, our convention is that the rows of \( v \otimes w \) are indexed by the pairs \((i,k)\) and the columns by the pairs \((j,\ell)\).

Exercise 8. Prove that the direct sum and the tensor product of two representations is again a representation.

Solution. Let us start with the direct sum \( z = v \oplus w \). First,

\[ z^* z = \begin{pmatrix} v^* v & 0 \\ 0 & w^* w \end{pmatrix} = \begin{pmatrix} \text{Id} & 0 \\ 0 & \text{Id} \end{pmatrix} = \begin{pmatrix} vv^* & 0 \\ 0 & ww^* \end{pmatrix} = zz^* \]

so that this is a unitary matrix. Second, the property of the coproduct is checked case-by-case depending on whether the indices are smaller or larger than \( n = \text{dim}(v) \). For instance, for \( 1 \leq i,j \leq n \),

\[ \Delta(z_{ij}) = \Delta(v_{ij}) = \sum_{k=1}^n v_{ik} \otimes v_{kj} = \sum_{k=1}^n z_{ik} \otimes z_{kj} = \sum_{k=1}^{n+m} z_{ik} \otimes z_{kj} \]

and similarly for the other three cases.

We now turn to the tensor product \( z = v \otimes w \). Unitarity is again easily checked, following our indexing conventions:

\[ \sum_{a=1}^n \sum_{b=1}^m z^*_{(i,k),(a,b)} z_{(i',k'),(a,b)} = \sum_{b=1}^m w^*_{kb} \left( \sum_{a=1}^n v^*_{ia} v_{ja} \right) w_{k',b} \]

\[ = \sum_{b=1}^m w^*_{kb} \delta_{ia} w_{k',b} \]

\[ = \delta_{(i,k),(i',k')} \]

and similarly the other way round. The other property is a consequence of the multiplicativity of the coproduct:

\[ \Delta \left( z_{(i,k),(j,\ell)} \right) = \Delta(v_{ij}) \Delta(w_{k\ell}) \]

\[ = \sum_{a=1}^n \sum_{b=1}^m v_{ia} w_{kb} \otimes v_{aj} w_{b\ell} \]

\[ = \sum_{a=1}^n \sum_{b=1}^m z_{(i,k),(a,b)} \otimes z_{(a,b),(k,\ell)}. \]

\[ 1 \] The fact that this indeed yields an isomorphism between \( M_n(\mathcal{O}(\mathbb{G})) \otimes M_m(\mathcal{O}(\mathbb{G})) \) and \( M_{nm}(\mathcal{O}(\mathbb{G})) \) follows from the corresponding fact when \( \mathcal{O}(\mathbb{G}) = \mathbb{C} \) (by a dimension argument) and the isomorphism \( M_n(\mathcal{O}(\mathbb{G})) \otimes M_m(\mathcal{O}(\mathbb{G})) \cong (M_n(\mathbb{C}) \otimes M_m(\mathbb{C})) \otimes \mathcal{O}(\mathbb{G}) \) (which follows for instance from Exercise 9).
Another way of thinking about representations of groups is to consider them as a particular kind of action. Indeed, a representation of $G$ on $V$ can be seen as a map

$$\alpha : G \times V \rightarrow V$$

satisfying the axioms of an action and those making the maps $\alpha(g,\cdot)$ linear. To write such a map in terms of $O(G)$, we will need the following result:

**Exercise 9.** Let $G$ be a compact group of matrices and let $V$ be a vector space. Then, the map

$$f \otimes x \mapsto (g \mapsto f(g)x)$$

is well-defined and yields a linear isomorphism between $O(G) \otimes V$ and the space $O(G,V)$ of polynomial functions from $G$ to $V$.

**Solution.** As usual, we first consider the obviously well-defined map

$$\Phi : \mathcal{F}(O(G) \times V) \rightarrow O(G,V)$$

satisfying the same formula as in the statement. A straightforward computation shows that $I(O(G),V) \subset \ker(\Phi)$ so that we have a well-defined quotient map

$$\Phi : O(G) \otimes V \rightarrow O(G,V)$$

as in the statement. Fixing a basis $(e_i)_{1 \leq i \leq \dim(V)}$ of $V$, consider now an element

$$x = \sum_{i=1}^{\dim(V)} f_i \otimes e_i \in \ker(\Phi).$$

By linear independence, it follows that for all $1 \leq i \leq \dim(V)$ and $g \in G$, $f_i(g) = 0$. This implies that $f_i = 0$ as an element of $O(G)$, hence $x = 0$, concluding the proof.

This leads us to an alternative definition, usually called a corepresentation of $O(G)$. We will keep this distinction in order to avoid confusion with our previous definition:

**Definition 2.1.5.** Let $G$ be a compact matrix quantum group. A corepresentation of $O(G)$ is a vector space $V$ together with a linear map

$$\rho : V \rightarrow O(G) \otimes V$$

which is injective and satisfies Equation (2.1).
Given two corepresentations $\rho_1$ and $\rho_2$ on $V_1$ and $V_2$ respectively, their direct sum is simply defined by

$$\rho_1 \oplus \rho_2 : (x_1, x_2) \mapsto \rho_1(x_1) + \rho_2(x_2) \in (\mathcal{O}(G) \otimes (V_1 \oplus V_2)) \simeq (\mathcal{O}(G) \otimes V_1) \oplus (\mathcal{O}(G) \otimes V_2).$$

We used here a distributivity property of the tensor product, the proof of which we leave as an exercise:

**Exercise 10.** Prove that given three vector spaces $V_1$, $V_2$ and $V_3$, there exists a canonical isomorphism

$$V_1 \otimes (V_2 \oplus V_3) \simeq (V_1 \otimes V_2) \oplus (V_1 \otimes V_3).$$

**Solution.** Let us consider the map

$$\Phi : \mathcal{F}(V_1 \times (V_2 \times V_3)) \to \mathcal{F}(V_1 \times V_2) \oplus \mathcal{F}(V_1 \times V_3) \to (V_1 \otimes V_2) \oplus (V_1 \otimes V_3)$$

sending $(x_1, (x_2, x_3))$ to $((x_1 \otimes x_2), (x_1 \otimes x_3))$. Then, $\mathcal{I} \subset \ker(\Phi)$ so that it factors as

$$\tilde{\Phi} : V_1 \otimes (V_2 \oplus V_3) \to (V_1 \otimes V_2) \oplus (V_1 \otimes V_3).$$

In the same way, there exists a linear map

$$\tilde{\Psi} : (V_1 \otimes V_2) \oplus (V_1 \otimes V_3) \to V_1 \otimes (V_2 \oplus V_3)$$

sending $(x_1 \otimes x_2), (x_1 \otimes x_3)$ to $x_1 \otimes (x_2, 0) + x_1 \otimes (0, x_3)$. To conclude, if suffices to check that these two maps are inverse to each other, which is straightforward. ■

The description of the tensor product is a bit more involved: if

$$\rho(x_1) = \sum_i a^i_1 \otimes x^i_1 \text{ and } \rho(x_2) = \sum_j a^j_2 \otimes x^j_2,$$

then

$$\rho_1 \otimes \rho_2 : x_1 \otimes x_2 \mapsto \sum_{i,j} a^i_1 a^j_2 \otimes x^i_1 \otimes x^j_2.$$

The two points of view are of course equivalent. To see this, let us start with a representation $\nu$ of dimension $n$ and let $(e_i)_{1 \leq i \leq N}$ be the canonical basis of $\mathbb{C}^n$. We set

$$\rho_\nu : e_i \mapsto \sum_{j=1}^n v_{ij} \otimes e_j$$

and extend it by linearity to a linear map $\mathbb{C}^n \to \mathcal{O}(G) \otimes \mathbb{C}^n$. Conversely, if $\rho$ is a coaction on $V$ and if a basis is fixed, there exists elements $(v(\rho)_{ij})_{1 \leq i,j \leq \dim(V)}$ in $\mathcal{O}(G)$ such that

$$\rho(e_i) = \sum_{j=1}^{\dim(V)} v(\rho)_{ij} \otimes e_j.$$

**Remark 2.1.6.** There are canonical isomorphisms (whose existence the reader is invited to prove)

$$\mathcal{L}(V, \mathcal{O}(G) \otimes V) \simeq \mathcal{O}(G) \otimes V \otimes V^* \simeq \mathcal{O}(G) \otimes \mathcal{L}(V)$$

providing a basis-free version of the above correspondence between representations and corepresentations.

As one may hope, the direct sum and tensor product constructions behave well with respect to the operations $\nu \to \rho_\nu$ and $\rho \to v(\rho)$.

**Proposition 2.1.7.** Prove that $\nu$ is a representation if and only if $\rho_\nu$ is a corepresentation and that $\rho$ is a corepresentation if and only if $v(\rho)$ is a representation. Prove moreover the following identities:
2.1. Finite-dimensional representations

1. \( \rho_v(\rho) = \rho \) and \( v(\rho_v) = v \),
2. \( \rho_{v \oplus w} = \rho_v \oplus \rho_w \),
3. \( \rho_{v \otimes w} = \rho_v \otimes \rho_w \),
4. \( u(\rho \oplus \delta) = u(\rho) \oplus u(\delta) \),
5. \( u(\rho \otimes \delta) = u(\rho) \otimes u(\delta) \).

**Proof.** We start by noticing that

\[
(id \otimes \rho) \circ \rho(e_i) = \sum_{j=1}^{n} \sum_{k=1}^{n} v(\rho)_{ij} \otimes v(\rho)_{jk} \otimes e_k
\]

while

\[
(\Delta \otimes id) \circ \rho(e_i) = \sum_{j=1}^{n} \Delta(v(\rho)_{ij}) \otimes e_j.
\]

By linear independence, this shows that Equation (2.1) is equivalent to the compatibility of \( u(\rho) \) with the coproduct.

Let us now show that \( \rho \) is injective if \( v(\rho) \) is invertible. For this we will need an analogue of the evaluation map at the identity, which is given by the counit \( \varepsilon : O(G) \to \mathbb{C} \) sending \( v_{ij} \) to \( \delta_{ij} \) (here \( v \) denotes the fundamental representation of \( G \)). It satisfies \( (\varepsilon \otimes id) \circ \Delta = id \) (it is enough to check this on the generators, where it is obvious) so that setting \( P = (\varepsilon \otimes id) \circ \rho \in \mathcal{L}(V) \),

\[
P = (\varepsilon \otimes \varepsilon \otimes id) \circ (\Delta \otimes id) \circ \rho
\]

so that \( P \) is a projection. Since moreover

\[
\rho \circ P = \rho \circ (\varepsilon \otimes id) \circ \rho
\]

we see that if \( P \neq id_V \), then \( \rho \) is not injective. We have therefore proven that \( \varepsilon(v(\rho)_{ij}) = \delta_{ij} \). Let us consider the unique \(*\)-antihomomorphism \( S \) of \( O(G) \) such that \( S(u_{ij}) = u_{ji} \). Then, one can check on the generators that

\[
m \circ (id \otimes S) \circ \Delta = \varepsilon = m \circ (S \otimes id) \circ \Delta
\]

so that \( (S(v(\rho)_{ij}))_{1 \leq i,j \leq n} \) is an inverse for \( v \).

Assume conversely that \( v(\rho) \) is invertible and let \( w \) be its inverse. The matrix

\[
M = (\varepsilon(v(\rho)_{ij}))_{1 \leq i,j \leq n}
\]

is then invertible with inverse \( (\varepsilon(w_{ij}))_{1 \leq i,j \leq n} \). Moreover,

\[
M_{ij} = (\varepsilon \otimes \varepsilon) \circ \Delta(v(\rho)_{ij}) = \sum_{k=1}^{n} M_{ik} M_{kj} = M^2_{ij}
\]

so that \( M \) is an invertible projection, hence the identity. In other words, \( \varepsilon(v(\rho)_{ij}) = \delta_{ij} \), which yields

\[
(\varepsilon \otimes id) \circ \rho = id_V
\]
i.e. the injectivity of $\rho$.

The equalities $v(\rho_v) = v$ and $\rho_v(\rho_v) = \rho$ directly follow from the definition. Moreover, applying the definition, we see that for $x \in V$, $(\rho_v \oplus \rho_w)(x) = \rho_v(x) = \rho_v \oplus w(x)$ and similarly for $x \in W$, yielding the first equality. The second equality in turn follows from

$$(\rho_v \otimes \rho_w)(e_i \otimes e_k) = \sum_{j=1}^{n} \sum_{\ell=1}^{m} v_{ij} w_{k\ell} \otimes e_j \otimes e_\ell = (v \otimes w)(i,k) \otimes e_j \otimes e_\ell.$$  

The rest follows from the first equalities. $lacksquare$

### 2.1.2 Intertwiners

The heart of representation theory is understanding the link between various representations, which is given by intertwiners. In the following definitions, we see scalar matrices as matrices with coefficients in $O(G)$ through the embedding $\mathbb{C} \cdot 1_{O(G)} \subset O(G)$.

#### Definition 2.1.8

Let $G$ be a compact matrix quantum group and let $v, w$ be representations of $G$.

- An intertwiner between $v$ and $w$ is a linear map $T : \mathbb{C}^n \to \mathbb{C}^m$ such that $Tv = wT$.

- The representations $v$ and $w$ are said to be equivalent if there exists an invertible intertwiner between them. If this intertwiner is moreover unitary, then they are said to be unitarily equivalent.

- The representation $w$ is said to be a subrepresentation of $v$ if there exists an injective intertwiner between $v$ and $w$.

- A representation is said to be irreducible if it has no subrepresentation except for 0 and itself.

It turns out that concrete computations with intertwiners are often easier to do using corepresentations. As one may expect, the concept is easily translated:

**Exercise 11.** Prove that $T$ intertwines $v$ and $w$ if and only if

$$(\text{id} \otimes T) \circ \rho_v = \rho_w \circ (\text{id} \otimes T).$$

**Solution.** The fact that $T$ is an intertwiner is equivalent to the fact that for any $1 \leq i \leq \dim(w)$ and $1 \leq j \leq \dim(v)$,

$$\sum_{k=1}^{n} T_{ik} v_{kj} = \sum_{\ell=1}^{m} w_{ik} T_{\ell j}.$$  

Tensoring with $e_i$ and summing over $i$ yields

$$(\text{id} \otimes T) \circ \rho_v(e_j) = \rho_w \circ (\text{id} \otimes T)(e_j),$$

hence the result. $lacksquare$

This correspondence allows for another description of subrepresentation.

#### Proposition 2.1.9

Let $v$ be a representation of dimension $n$. There is, up to equivalence, a one-to-one correspondence between

- Subrepresentations of $v$,

- Invariant subspaces of $\rho_v$. 

---
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Proof. Let $V \subset \mathbb{C}^n$ be an invariant subspace for $\rho_v$ and set $w$ be the representation associated to the restriction of $\rho_v$ to $V$. If $T : V \to \mathbb{C}^n$ denotes the inclusion map, then

$$\rho_v \circ T(x) = (\text{id} \otimes T) \circ \rho_w(x)$$

so that $T$ is an injective intertwiner between $w$ and $v$, i.e. $w$ is a subrepresentation of $v$.

Conversely, let $T : \mathbb{C}^m \to \mathbb{C}^n$ be an injective intertwiner between some representation $w$ and $v$. Up to conjugating $v$ with a unitary matrix, we may assume that $T$ is the inclusion map of the first $m$ coordinates. Let us call $V$ the range of $T$. For any $x \in V$,

$$\rho_v(x) = \rho_v \circ T(x) = (\text{id} \otimes T) \circ \rho_w(x) \in \mathcal{O}(G) \otimes V,$$

hence $V$ is invariant for $\rho_V$. \hfill ■

Let us illustrate this with some simple examples.

Example 2.1.10. Consider the fundamental representation $P$ of $S_N^\mathbb{C}$, let $(e_i)_{1 \leq i \leq N}$ be the canonical orthonormal basis of $\mathbb{C}^N$ and set

$$\xi = \sum_{i=1}^N e_i.$$ 

It is a straightforward consequence of Condition (2) that $\rho_P(\xi) = 1 \otimes \xi$. In other words, $\rho_P$ has a fixed vector, which is equivalent to $P$ having a trivial subrepresentation given by the map $\lambda \in \mathbb{C} \mapsto \lambda \xi$.

This phenomenon is analogous to a well-known fact for permutation groups: the permutation representation $\rho$ of $S_N$ on $\mathbb{C}^N$ decomposes as the direct sum of the trivial representation and an irreducible representation. Let us show that the same holds for $S_N^\mathbb{C}$ by exploiting the idea that $S_N$ is a “subgroup” of $S_N^\mathbb{C}$.

Example 2.1.11. We set $V = \xi^\perp \subset \mathbb{C}^N$ and note that because

$$(\text{id} \otimes \xi^*) \rho_P(e_i - e_j) = \sum_{k=1}^N p_{ik} \langle \xi, e_k \rangle - \sum_{\ell=1}^N p_{j\ell} \langle \xi, e_k \rangle$$

$$= \sum_{k=1}^N p_{ik} - \sum_{\ell=1}^N p_{j\ell}$$

$$= 0,$$

$$\rho_P(V) \subset \mathcal{O}(G) \otimes V$$

so that $V$ is a subrepresentation. Consider now a subspace $W \subset V$ which is invariant under $\rho_P$. Letting

$$\tilde{\rho} : V \to \mathcal{O}(S_N) \otimes V \cong \mathcal{O}(S_N, V)$$

be the map sending $x$ to $g \mapsto \rho(g)(x)$. We have the equality

$$(\pi_{ab} \otimes \text{id}_V) \circ \rho_P = \tilde{\rho},$$

from which it follows that $W$ is invariant under $\tilde{\rho}$. Hence, $W = \{0\}$ or $W = V$ and $V$ is irreducible for $P$.

---

2. Simply notice that $\xi^\perp$ contains the vector $e_1 - e_2$ and that letting $S_N$ act on it one can obtain $e_i - e_j$ for any $i \neq j$, which generate $\xi^\perp$. 

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We can use the same strategy for the fundamental representation of $O_N^+$:

**Exercise 12.** Let $N \geq 2$ be an integer and consider the fundamental representation $U$ of $O_N^+$.

1. Show that $U$ is irreducible.

2. Let $(e_i)_{1 \leq i \leq N}$ be the canonical basis of $\mathbb{C}^N$. Show that the vector

\[ \xi = \sum_{i=1}^{N} e_i \otimes e_i \]

is fixed for $\rho_{U \otimes U}$.

**Solution.** The strategy is the same as for $S_N^+$, so let us denote by $\rho$ the defining representation of $O_N$ on $V = \mathbb{C}^N$ and by $\tilde{\rho}$ the map $x \mapsto (g \mapsto \rho(g)(x))$.

1. Because $(\pi_{ab} \otimes \text{id}) \circ \rho_{U} = \tilde{\rho}$ and the right-hand side is irreducible, we infer that $\rho_{U}$, hence also $U$, is irreducible.

2. We compute

\[
\rho_{U \otimes U}(\xi) = \sum_{i=1}^{N} \sum_{k,\ell=1}^{N} U_{ik} U_{\ell i} \otimes e_k \otimes e_\ell \\
= \sum_{k,\ell=1}^{N} \left( \sum_{i=1}^{N} U_{ik} U_{\ell i} \right) \otimes e_k \otimes e_\ell \\
= \sum_{k,\ell=1}^{N} \delta_{k\ell} \otimes e_k \otimes e_\ell \\
= 1 \otimes \xi.
\]

As one sees from these examples, pushing further the study by considering higher tensor powers of $U$ or $P$, one will have to deal with the full representation theory of $S_N$ and $O_N$. We will see in Chapter 3 that there is another way of investigating the representation theory of these quantum groups which completely avoids the use of classical groups. This is fortunate because the representation theory of $O_N^+$, for instance, turns out to be much simpler than that of $O_N$.

### 2.1.3 Structure of the representation theory

Observe that $O(G)$ is spanned by products of coefficients of $u$, which are nothing but the coefficients of tensor powers of $u$. In other words, the whole orthogonal compact matrix quantum group can be recovered from its finite-dimensional representations. The main theorem of this section, which is fundamental, turns this observation into a tractable tool for the study of orthogonal compact matrix quantum groups. Before getting to this, let us warm up by observing that Schur’s celebrated Lemma still holds in our setting. This will be easier to explain once we define a notion of restriction of a representation:

**Definition 2.1.12.** Let $v$ be a representation of a unitary compact matrix quantum group $G$ and let $V$ be an invariant subspace for $\rho_v$. The restriction of $v$ to $V$ is the representation associated to the restriction of $\rho_v$ to $V$.

**Lemma 2.1.13 (Schur's Lemma).** Let $G$ be a unitary compact matrix quantum group and let $v$ and $w$ be irreducible representations of dimension $n$ and $m$ respectively. If $T$ is an intertwiner between $v$ and $w$, then either $T = 0$ or $T$ is invertible (hence $v$ is equivalent to $w$). Moreover, in the latter case $T$ is a scalar multiple of the identity.
Proof. Consider the subspace \( Z = \ker(T) \subset \mathbb{C}^n \). Then, for any \( x \in Z \),

\[
(id \otimes T) \circ \rho_v(x) = \rho_w \circ T(x) = 0
\]

so that \( \rho_v(Z) \subset \ker(id \otimes T) = \mathcal{O}(G) \otimes Z \) (the latter equality follows from Lemma 1.3.3). Hence, \( Z \) is stable, meaning that the restriction of \( v \) to it is a subrepresentation. By irreducibility, we conclude that either \( Z = \mathbb{C}^n \), in which case \( T = 0 \), or \( Z = \{0\} \). In the second case, applying the same reasoning to \( T^* \) shows that \( \ker(T^*) = \{0\} \), implying that \( T \) is also surjective, hence an equivalence between \( v \) and \( w \).

If \( T \) is invertible, let \( \lambda \in \mathbb{C} \) be one of its eigenvalues. Then, \( T - \lambda \cdot id \) is an intertwiner between \( v \) and \( w \) and is not invertible, thus \( T = \lambda \cdot id \) by the first part of the statement. \( \blacksquare \)

We can now turn to the statement and proof of the main theorem on the structure theory of unitary compact matrix quantum groups. It was first proven (in a slightly different version) by S.L. Woronowicz in [Wor87, Prop 4.6 and Lem 4.8] and is the cornerstone of the study of compact quantum groups.

**Theorem 2.1.14** (Woronowicz) Let \( G = (\mathcal{O}(G), u) \) be a unitary compact matrix quantum group. Then,

1. Any finite-dimensional unitary representation splits as a direct sum of irreducible unitary representations,
2. Coefficients of inequivalent irreducible finite-dimensional representations are linearly independent,
3. Any finite-dimensional representation is equivalent to a unitary one.

**Proof.**

1. Let \( v \) be a finite-dimensional unitary representation of dimension \( n \) and assume that it is not irreducible. By Proposition 2.1.9, \( \rho_v \) then has an invariant subspace \( W \subset \mathbb{C}^n \). Picking an orthonormal basis of \( W \) and completing it into an orthonormal basis of \( V \), we get a unitary matrix \( B \in M_n(\mathbb{C}) \) such that \( w = B^*vB \) is block upper triangular, i.e. has the form

\[
w = \begin{pmatrix}
A_{11} & A_{12} \\
0 & A_{22}
\end{pmatrix}
\]

Let us check that this implies \( A_{12} = 0 \). We compute

\[
w^*w = \begin{pmatrix}
A_{11}^*A_{11} & A_{11}^*A_{12} \\
A_{12}^*A_{11} & A_{12}^*A_{12} + A_{22}^*A_{22}
\end{pmatrix}
\]

and observe that \( A_{11} \) is in fact a representation since it is the restriction of \( w \) to \( W \). Indeed, we have identified \( W \) with the subspace generated by the \( \dim(W) \) first vectors of the canonical basis, and

\[
\rho_w(e_i) = \sum_{j=1}^n w_{ij} \otimes e_j = \sum_{j=1}^{\dim(W)} (A_{11})_{ij} \otimes e_j
\]

In particular, \( A_{11} \) is invertible as an element of \( M_{\dim(W)}(\mathcal{O}(G)) \) and the upper left corner of \( w^*w \) shows that its inverse is \( A_{11}^* \). But then, looking at the upper right corner yields

\[
A_{12} = A_{11}A_{11}^*A_{12} = A_{11}0 = 0.
\]

As a consequence, \( w \) is a direct sum of unitary sub-representations. The result therefore follows by induction.
2. Let \( \{v^{(1)}, \cdots, v^{(n)}\} \) be pairwise inequivalent irreducible representations with \( v^{(i)} \) acting on a finite-dimensional vector space \( V^{(i)} \). For a linear form \( f \in \mathcal{O}(G)^* \), we denote by \( \hat{f}(v) \) the matrix with coefficients \( (f(v_{ij}))_{1 \leq i, j \leq \dim(v)} \) and we set

\[
B = \left\{ \sum_{i=1}^{n} \hat{f}(v^{(i)}) \mid f \in \mathcal{O}(G)^* \right\} \subset \bigoplus_{i=1}^{n} \mathcal{L} \left( V^{(i)} \right) = \mathcal{B}.
\]

We claim that this inclusion is an equality. The proof goes through several steps:

(a) By definition \( B \) is a vector space. Moreover, setting \( f \ast g = (f \otimes g) \circ \Delta \in \mathcal{O}(G) \), we have

\[
\left( \hat{f}(v) \hat{g}(v) \right)_{ij} = \sum_{k=1}^{\dim(v)} f(v_{ik})g(v_{kj}) = \left( (\hat{f} \ast \hat{g})(v) \right)_{ij}
\]

so that \( B \) is an algebra.

(b) Let \( p_i \) be the minimal central projection in \( B \) corresponding to the \( i \)-th summand and consider

\[
\pi_i : x \in B \mapsto p_ixp_i \in \mathcal{L} \left( V^{(i)} \right).
\]

This is an algebra representation which is moreover irreducible. Indeed, if \( T : V^{(i)} \to V^{(i)} \) the equality \( T \circ \pi_i = \pi_i \circ T \) means that for all \( f \in \mathcal{O}(G)^* \),

\[
T \hat{f}(v^{(i)}) = \hat{f}(v^{(i)}).
\]

By linearity, this is equivalent to

\[
\hat{f} \left( T v^{(i)} - v^{(i)} T \right) = 0
\]

and since linear forms separate points, this is also equivalent to \( T v^{(i)} = v^{(i)} T \). Because \( v^{(i)} \) is irreducible, this forces \( T \) to be a multiple of the identity, i.e. \( \pi_i \) is irreducible. Thus, by Burnside’s Theorem (see Theorem A in the Appendix),

\[
\pi_i(B) = \mathcal{L} \left( V^{(i)} \right).
\]

(c) The proof of the irreducibility of \( \pi_i \) adapts easily to show that any interwinder between \( \pi_i \) and \( \pi_j \) must be an intertwiner between \( v^{(i)} \) and \( v^{(j)} \). Because these are assumed to be irreducible and inequivalent, Lemma 2.1.13 forces \( T = 0 \). In other words the representations are pairwise inequivalent.

(d) Noticing that \( \oplus \pi_i = \text{id}_B \), we conclude that \(^3\)

\[
B = \text{End}_B(B) = \text{End}_B \left( \bigoplus_{i=1}^{n} \pi_i(B) \right) = \bigoplus_{i=1}^{n} \text{End}_B(\pi_i(B)) = \bigoplus_{i=1}^{n} \text{End}_{\mathcal{L}(V^{(i)})} \left( \mathcal{L} \left( V^{(i)} \right) \right) = \bigoplus_{i=1}^{n} \mathcal{L} \left( V^{(i)} \right) = B.
\]

\(^3\) For the first and last lines, observe that for any unital algebra \( B \), the map \( b \mapsto (x \mapsto x.b) \) gives an isomorphism \( B \mapsto \text{End}_B(B) \) with inverse \( T \mapsto T(1) \).
Now let, for each \(1 \leq i \leq n\), \(\left(\lambda_{k,\ell}^{(i)}\right)_{1 \leq k,\ell \leq \dim(v^{(i)})}\) be complex numbers and set
\[
x = \sum_{i=1}^{n} \sum_{k,\ell=1}^{\dim(v^{(i)})} \lambda_{k,\ell}^{(i)} v_{k,\ell}^{(i)}.
\]
If we denote by \(\Lambda_i\) the matrix with coefficients \(\lambda_{k,\ell}^{(i)}\), then
\[
\Lambda = (\Lambda_1, \cdots, \Lambda_n) \in \mathcal{B} = \mathcal{B},
\]
thus there exists \(f \in \mathcal{O}(G)\) such that for all \(i\), \(\hat{f}(v^{(i)}) = \Lambda_i\). As a consequence,
\[
f(x) = \sum_{i=1}^{n} \sum_{k,\ell=1}^{\dim(v^{(i)})} |\lambda_{k,\ell}|^2
\]
and \(x\) therefore vanishes if and only if all the coefficients vanish, proving linear independence.

3. Observe that \(\mathcal{O}(G)\) is spanned by the products of coefficients of \(u\), that is to say coefficients of tensor powers of \(u\) which, by point 1, are linear combinations of coefficients of irreducible unitary representations. If now \(v\) is a finite-dimensional representation, its coefficients are in the linear span of coefficients of unitary representations, hence by point 2 it is equivalent to a unitary representation.

We have proved along the way a useful result which we restate here for later reference:

**Lemma 2.1.15.** Let \(v\) and \(w\) be representations of a unitary compact matrix quantum group \(G\) of dimension \(n\) and \(m\) respectively. Then, \(T \in \mathcal{L}(V, W)\) is an intertwiner if and only if, for all \(f \in \mathcal{O}(G)^*\),
\[
T \hat{f}(v) = \hat{f}(w)T.
\]

**Proof.** The fact that \(T\) is an intertwiner reads, for any \(1 \leq i \leq m\) and \(1 \leq j \leq n\),
\[
\sum_{k=1}^{m} T_{ik} v_{kj} = \sum_{k=1}^{m} T_{kj} w_{ik}.
\]
Applying \(f\) to both sides yields the only if condition. The converse follows because linear maps separate the points. \(\square\)

Let us conclude this section by outlining the following crucial consequence of Theorem 2.1.14:

**Corollary 2.1.16.** Let \(G = (\mathcal{O}(G), u)\) be an orthogonal compact matrix quantum group. Then, any irreducible representation is equivalent to a subrepresentation of \(u^{\otimes k}\).

**Proof.** Because \(\mathcal{O}(G)\) is generated by the coefficients of \(u\), it is the linear span of coefficients of irreducible subrepresentations of tensor powers of \(u\). Thus, any finite-dimensional representation is equivalent to one of these by point 2 of Theorem 2.1.14. \(\square\)

2.2 **Interlude : Invariant theory**

Now that we have a nice abstract theory of representations of compact matrix quantum groups, it should be high time we try to compute, say, all irreducible representations up to equivalence for some quantum groups like \(O^+_N\) for instance. This is however not an easy task and requires the introduction of some combinatorial tools. To motivate this and to introduce the main object which will be needed, we start by revisiting the work of R. Brauer who, in [Bra37], studied the orthogonal group \(O_N\) using partitions of finite sets.
2.2.1 The orthogonal group: pair partitions

Let us consider the group $O_N$. The defining representation $\rho$ of $O_N$ as matrices acting on $V = \mathbb{C}^N$ should by definition “contain everything”. The precise meaning of this last expression is that, by Corollary 2.1.16, given any finite-dimensional representation $\pi$ of $O_N$, there exists an integer $k$ such that $\pi$ is unitarily equivalent to a subrepresentation of $\rho^\otimes k$.

This means that we can focus on subrepresentations of tensor powers of $\rho$. Going back to the definition, we see that subrepresentations can be read on the intertwiner spaces. Indeed, if $T$ is an isometric intertwiner between $\pi$ and $\rho^\otimes k$, then $TT^*$ is an orthogonal projection intertwining $\rho^\otimes k$ with itself. In other words, it should in principle be possible to recover the whole representation theory of $O_N$ from the algebra structure of the spaces

$$\text{Mor}_{O_N}(\rho^\otimes k, \rho^\otimes k) = \left\{ T : V^\otimes k \to V^\otimes k \mid T\rho^\otimes k(g) = \rho^\otimes k(g)T \text{ for all } g \in O_N \right\} = \rho^\otimes k(O_N)' .$$

The study of such commutant algebras is usually known under the name of Schur-Weyl duality. In our setting, orthogonality allows us to further reduce the problem thanks to the following elementary result:

**Proposition 2.2.1.** For any integer $k$, there exists a canonical linear isomorphism

$$\Phi_k : \text{Mor}_{O_N}(\rho^\otimes k, \rho^\otimes k) \simeq \text{Mor}_{O_N}(\rho^\otimes 2k, \varepsilon) ,$$

where $\varepsilon$ denotes the trivial representation of $O_N$.

**Proof.** Since the elements of $O_N$ are unitary, they leave the inner product on $V$ invariant. However, this inner product does not yield a linear form on $V \otimes V$ since it is not bilinear on $V \times V$ but only sesquilinear. It can however be made linear using the duality map $D_V : V \otimes V \to \mathbb{C}$ defined by

$$D_V(x \otimes y) = \langle x, \overline{y} \rangle,$$

where $\overline{y}$ is the image of $y$ in the conjugate Hilbert space $\overline{V}$.

The key fact is that because the coefficients of matrices in $O_N$ are real-valued, the map $D_V$ is invariant under the representation $\rho^\otimes 2$. Now, any map

$$T : V^\otimes k \to V^\otimes k$$

can be turned into a map

$$\tilde{T} : V^\otimes 2k \simeq V^\otimes k \otimes V^\otimes k \to \mathbb{C}$$

via the formula

$$\tilde{T} : x_1 \otimes \cdots \otimes x_k \otimes y_1 \otimes \cdots \otimes y_k \to D_{V^\otimes k}(T(x_1 \otimes \cdots \otimes x_k), y_1 \otimes \cdots \otimes y_k).$$

One easily checks that $T$ is an intertwiner if and only if $\tilde{T}$ is, hence the result. \hfill \qed

As a consequence, we are now looking for the invariants of the orthogonal group, that is to say the polynomial maps to $\mathbb{C}$ which are invariant under a given representation. Let us practice a little by computing $\text{Mor}_{O_N}(\rho^\otimes 2, \varepsilon)$. We know that $D$ yields a non-trivial element of this space. Moreover, because $\rho$ is irreducible, it follows from Schur’s Lemma 2.1.13 that

$$\dim \left( \text{Mor}_{O_N}(\rho^\otimes 2, \varepsilon) \right) = \dim \left( \text{Mor}_{O_N}(\rho, \rho) \right) = 1 ,$$

so that $\text{Mor}_{O_N}(\rho^\otimes 2, \varepsilon) = \mathbb{C} \cdot D_{CN}$. We can extend this idea to build non-trivial elements of $\text{Mor}_{O_N}(\rho^\otimes 2k, \varepsilon)$ for any $k \geq 1$ by pairing tensors using the map $D_{CN}$. To do this, we just need a partition $p$ of $\{1, \cdots, 2k\}$ into subsets of size 2. Such a partition is called a pair partition and the set of pair partition of $\{1, \cdots, 2k\}$ is denoted by $\mathcal{P}_2(2k)$. Given such a pair partition $p$, we set

$$f_p : x_1 \otimes \cdots \otimes x_{2k} = \prod_{\{a,b\} \subseteq p} D_{CN}(x_a, x_b).$$

We can then produce many intertwiners by taking linear combinations and one of the main results of R. Brauer’s work [Bra37] is that we indeed get everything:

4. This is just the same abelian group as $V$ but with scalar multiplication given by $\lambda \cdot x = \overline{\lambda} x$. 

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2.2.2 Interlude: Invariant theory

**Theorem 2.2.2** (Brauer) For any integer \(k\), we have

\[
\text{Mor}_{\mathcal{O}_N}(\rho^{\otimes 2k+1}, \varepsilon) = \{0\}
\]

\[
\text{Mor}_{\mathcal{O}_N}(\rho^{\otimes 2k}, \varepsilon) = \text{Vect} \{f_p \mid p \in P_2(2k)\}.
\]

2.2.2 The quantum orthogonal group: noncrossing partitions

Building on the previous discussion, we now want to use the same strategy to investigate the spaces \(\text{Mor}_{\mathcal{O}_N^+}(U^{\otimes 2k}, \varepsilon)\). One easily checks that the map \(D_{\mathcal{C}_N}\) is still an invariant of \(U^{\otimes 4}\):

\[
(id \otimes D_{\mathcal{C}_N}) \circ \rho_{U^{\otimes 2}}(e_i \otimes e_j) = (id \otimes D_{\mathcal{C}_N}) \left( \sum_{k, \ell} U_{ik} U_{j\ell} \otimes e_k \otimes e_\ell \right)
\]

\[
= \sum_{k=1}^{N} U_{ik} U_{jk}
\]

\[
= \delta_{ij}
\]

\[
= \rho_\varepsilon(D_{\mathcal{C}_N}(e_i \otimes e_j)).
\]

So what will be the difference between \(O_N\) and \(O_N^+\)? Let us look at \(U^{\otimes 4}\) and the \(O_N\)-intertwiner of \(\rho^{\otimes 4}\) given by the partition

\(p_{\text{cross}} = \{\{1, 3\}, \{2, 4\}\}\).

Does this yield an interwiner for \(O_N^+\)? Let us compute:

**Exercise 13.** Prove that for any orthogonal compact matrix quantum group \(G = (\mathcal{O}(G), u)\),

\[
(id \otimes f_{p_{\text{cross}}}) \circ \rho_{U^{\otimes 4}}(e_{i_1} \otimes e_{i_2} \otimes e_{i_3} \otimes e_{i_4}) = \sum_{k, \ell=1}^{N} u_{i_1 k} u_{i_2 \ell} u_{i_3 k} u_{i_4 \ell}
\]

and

\[
\rho_\varepsilon \circ (id \otimes f_{p_{\text{cross}}}) = \delta_{i_1 i_3} \delta_{i_2 i_4}.
\]

**Solution.** This is an elementary computation:

\[
(id \otimes f_{p_{\text{cross}}}) \circ \rho_{U^{\otimes 4}}(e_{i_1} \otimes e_{i_2} \otimes e_{i_3} \otimes e_{i_4}) = \sum_{j_1, \cdots, j_4=1}^{N} u_{i_1 j_1} \cdots u_{i_4 j_4} f_p(e_{j_1} \otimes e_{j_2} \otimes e_{j_3} \otimes e_{j_4})
\]

\[
= \sum_{j_1, \cdots, j_4=1}^{N} u_{i_1 j_1} \cdots u_{i_4 j_4} \delta_{j_1 j_3} \delta_{j_2 j_4}
\]

\[
= \sum_{j_1, j_2=1}^{N} u_{i_1 j_1} u_{i_2 j_2} u_{i_3 j_1} u_{i_4 j_2}
\]

and Equation (2.2) follows from the changes of indices \(k = j_1, \ell = j_2\). As for the second one, this is the definition of \(f_{p_{\text{cross}}}\).

Understanding the meaning of this relation is the key to the world of partition quantum groups. We will therefore state it as a proposition:

**Proposition 2.2.3.** Let \(G = (\mathcal{O}(G), u)\) be an orthogonal compact matrix quantum group and let \(p_{\text{cross}} = \{\{1, 3\}, \{2, 4\}\}\). Then, \(f_{p_{\text{cross}}} \in \text{Mor}_G(U^{\otimes 4}, \varepsilon)\) if and only if \(G\) is a classical group.
Proof. We will play around with the equality (2.2) = (2.3) to show that the coefficients of \( u \) pairwise commute. More precisely, multiplying each side by \( u_{i_4j_2}u_{i_3j_1} \), for two arbitrary indices \( 1 \leq j_1, j_2 \leq N \), and summing over \( i_4 \) and \( i_3 \) yields

\[
\sum_{k,\ell,i_3,i_4=1}^{N} u_{i_1k}u_{i_2\ell}u_{i_3k}u_{i_4\ell}u_{i_4j_2}u_{i_3j_1} = \sum_{i_3,i_4=1}^{N} \delta_{i_1i_3}\delta_{i_2i_4}u_{i_4j_2}u_{i_3j_1} = u_{i_2j_2}u_{i_1j_1}.
\]

The left-hand side above can be simplified using the fact that \( u \) is orthogonal. Indeed,

\[
\sum_{i_4=1}^{N} u_{i_4\ell}u_{i_4j_2} = \delta_{\ell j_2}
\]

and similarly for \( i_3 \) so that

\[
\sum_{k,\ell,i_3,i_4=1}^{N} u_{i_1k}u_{i_2\ell}u_{i_3k}u_{i_4\ell}u_{i_4j_2}u_{i_3j_1} = \sum_{k,\ell,i_3=1}^{N} u_{i_1k}u_{i_2\ell}u_{i_3k}\delta_{\ell j_2}u_{i_4j_1} = \sum_{k,\ell=1}^{N} u_{i_1k}u_{i_2\ell}\delta_{k j_1}\delta_{\ell j_2} = u_{i_1j_1}u_{i_2j_2}.
\]

Thus, we have proven that if \( f_p \) is an intertwiner, then the coefficients of \( u \) pairwise commute. Moreover, it is clear that the converse holds. It now follows that \( \mathcal{O}(G) \) is commutative so that by Exercise 6, \( G \) is in fact a classical group. 

As a consequence of Proposition 1.3.5, \( f_p \) is not an intertwiner of \( O_N^+ \) and in the construction of R. Brauer we have used, without noticing it, the commutativity of \( \mathcal{O}(O_N) \). So what is really the smallest space of intertwiners that one can build from \( D_{CN} \) using pair partitions? The answer relies on the following definition:

**Definition 2.2.4.** A partition is said to be **crossing** if there exists \( k_1 < k_2 < k_3 < k_4 \) such that

- \( k_1 \) and \( k_3 \) are in the same block,
- \( k_2 \) and \( k_4 \) are in the same block,
- the four elements are not in the same block.

Otherwise, it is said to be **noncrossing**.

To illustrate this notion we now introduce a very useful pictorial description of partitions: we draw \( k \) points in a row and then connect two points if and only if they belong to the same subset of the partition. It is then clear that for instance

\[
p_{\text{cross}} = \{\{1, 3\}, \{2, 4\}\} = \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet 
\end{array}
\]

cannot be drawn without letting the lines cross. It is therefore a crossing partition.

It is certainly not clear at the moment that non-crossing partitions have to do with the quantum orthogonal groups \( O_N^+ \) and there is indeed still some work needed to see the link. As a motivation, let us state the quantum analogue of R. Brauer’s result, which was proven by T. Banica in [Ban96]. We will denote the set of non-crossing pair partitions on \( 2k \) points by \( \text{NC}_2(2k) \).
Theorem 2.2.5 (Banica) For any integer $k$, we have

\[
\text{Mor}_{\mathcal{O}_N}^G (\rho^{\otimes 2k+1}, \varepsilon) = \{0\} \\
\text{Mor}_{\mathcal{O}_N}^G (\rho^{\otimes 2k}, \varepsilon) = \text{Vect}\{f_p \mid p \in NC_2(2k)\}.
\]

The proof will be a consequence of the general theory that we will develop in the next chapter.
3.1 Linear maps associated to partition

As we have seen in Section 2.2, the idea to use partitions of finite sets to study the representation theory of compact groups dates, at least, to the work of R. Brauer [Bra37]. This connection was used several times and at several degrees of generality to study some specific families of examples of compact quantum groups, but only with the breakthrough article of T. Banica and R. Speicher [BS09] did the systematic formalization and study of the relationship between partitions and compact quantum groups started to spread as a research subject of its own.

T. Banica and R. Speicher were motivated by the question of liberation: how can one find a good presentation of the polynomial algebra of a compact matrix group so that removing the commutation relations yields an interesting compact matrix quantum group? We will however follow a different path based on the following summary of Section 2.2: the representation theory of $O_N$ is determined by the pair partitions $P_2$ while the representation theory of $O_N^+$ is determined by the noncrossing pair partitions $NC_2$. This raises a natural question, namely what other (quantum) groups have their representation theory determined by partitions?

The answer requires enlarging our setting. First, we will from now on consider arbitrary partitions of finite sets, not only those in pairs. To define a linear form associated to such a general partition $p$, we need to extend the definition of the maps $f_p$. For this purpose, let $p \in \mathcal{P}(k)$ and let $1 \leq i_1, \cdots, i_k \leq N$. Place these indices on the points of $p$ from left to right. Then, if whenever two indices are connected, they are equal, we set $\delta_p(i_1, \cdots, i_k) = 1$. Otherwise, we set $\delta_p(i_1, \cdots, i_k) = 0$. For instance, with the following partition

$$p = \begin{array}{cccccc}
i_1 & | & i_2 & | & i_3 & | \vdots & | & i_5 & | & i_6 \end{array}$$

we get

$$\delta_p(i_1, i_2, i_3, i_4, i_5, i_6) = \delta_{i_1i_2i_4} \delta_{i_3i_6}.$$ 

**Definition 3.1.1.** Let $p \in \mathcal{P}(k)$ and let $N$ be an integer. Then, we define a map

$$f_p : \left(\mathbb{C}^N\right)^{\otimes k} \to \mathbb{C}$$

by the formula

$$f_p(e_{i_1} \otimes \cdots \otimes e_{i_k}) = \delta_p(i_1, \cdots, i_k).$$

**Exercise 14.** Check that for pair partitions, this coincides with the previous definition.
In particular, if \( i_1, \ldots, i_{2k} \) and iterating gives isomorphisms
\[
\delta_p(i_1, \ldots, i_{2k}) = \prod_{\{a,b\} \subset p} \delta_{ij}(i_a, i_b) = \prod_{\{a,b\} \subset p} D_{C^N}(e_i \otimes e_i)
\]
\[\blacksquare\]

If \( k \) and \( \ell \) are integers, then we claimed that the space \( \text{Mor}_G(V^\otimes k, V^\otimes \ell) \) should be recoverable from the invariants of \( G \). Let us explain how this follows from extending the construction of the maps \( \Phi_k \) introduced in Proposition 2.2.1. Indeed, using the duality map \( D \) one can build an isomorphism
\[
\Psi_{k,1} : \mathcal{L}(V^\otimes k, V^\otimes \ell) \rightarrow \mathcal{L}(V^\otimes(k+1), V^\otimes(\ell-1))
\]
by the formula
\[
\Psi_{k,1}(T)(x_1 \otimes \cdots \otimes x_k \otimes x_{k+1}) = \left( \text{id}_V^\otimes(\ell-1) \otimes D \right) \left( T(x_1 \otimes \cdots \otimes x_k) \otimes x_{k+1} \right)
\]
and iterating gives isomorphisms
\[
\Phi_{k, \ell} = \Psi_{k+\ell-1,1} \circ \cdots \circ \Psi_{k,1} : \mathcal{L}(V^\otimes k, V^\otimes \ell) \rightarrow \mathcal{L}(V^\otimes(k+\ell), C).
\]

In particular, if \( p \) is a partition of \( \{1, \ldots, k + \ell\} \), we can define an operator
\[
T_p = \Phi_{k, \ell}^{-1}(f_p)
\]
which is an intertwiner as soon as \( f_p \) is.

The drawback of that construction is that we have lost the pictorial description of the operator. To recover it, notice that we can also draw a partition \( p \in \mathcal{P}(k + \ell) \) on two rows instead of one, drawing for instance \( k \) points corresponding to \( \{1, \ldots, k\} \) in the upper row and \( \ell \) points corresponding to \( \{k + 1, \ldots, k + \ell\} \) in the lower row. With this description, the operator \( T_p \) admits an explicit description which generalizes that of \( f_p \). Let \( (e_i)_{1 \leq i \leq N} \) be the canonical basis of \( V = C^N \). For a partition \( p \in \mathcal{P}(k + \ell) \) drawn with \( k \) upper points and \( \ell \) lower points, we extend the definition of the function \( \delta_p \), taking now as arguments a \( k \)-tuple \( i = (i_1, \ldots, i_k) \) and an \( \ell \)-tuple \( j = (j_1, \ldots, j_\ell) \), in the following way:

- We draw the indices of \( i \) on the upper points of the partition (from left to right) and the indices of \( j \) on the lower points of \( p \) (from left to right),
- If whenever two points are connected, their indices are equal, we set \( \delta_p(i, j) = 1 \),
- Otherwise, we set \( \delta_p(i, j) = 0 \).

With this in hand, we have:

**Proposition 3.1.2.** For any tuples \( i \) and \( j \), we have
\[
T_p(e_{i_1} \otimes \cdots \otimes e_{i_k}) = \sum_{j_1, \ldots, j_\ell=1}^N \delta_p(i, j)e_{j_1} \otimes \cdots \otimes e_{j_\ell}.
\]

**Proof.** Unwinding the definition of \( T_p \), we have for any \( j_1, \ldots, j_\ell \),
\[
(T_p(e_{i_1} \otimes \cdots \otimes e_{i_k}), e_{j_1} \otimes \cdots \otimes e_{j_\ell}) = f_p(e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes e_{j_1} \otimes \cdots \otimes e_{j_\ell}).
\]
Moreover, it follows from our definition \(^1\) that
\[
\delta_p(i, j) = \delta_p(i \otimes j^{-1}),
\]
where \( \otimes \) denotes the concatenation of tuples and \( j^{-1} \) is the reversed tuple, hence the result. \[\blacksquare\]

\(^1\) Note that we have two definitions of the symbol \( \delta_p \), which should not be confused: one has two arguments corresponding the choice of an upper and a lower row in \( p \), while the original one has only one argument.
For instance, the partition drawn at the beginning of this chapter can also be seen as

\[ p = \begin{array}{ccc}
  i_1 & i_2 & i_3 \\
  j_1 & j_2 & j_3 
\end{array} \]

yielding

\[ T_p(e_{i_1} \otimes e_{i_2} \otimes e_{i_3}) = \delta_{i_1,i_2} \sum_{j_2=1}^{N} e_{i_1} \otimes e_{j_2} \otimes e_{i_3}. \]

### 3.2 Operations on partitions

Assume that we have a collection \( C \) of partitions made of subsets \( C(k,\ell) \) for all integers \( k \) and \( \ell \) and that we want to find an orthogonal compact matrix (quantum) group \( G \) such that for all \( k, \ell \in \mathbb{N} \),

\[ \text{Mor}_G \left( u \otimes^k, u \otimes^\ell \right) = \text{Vect} \left\{ T_p \mid p \in C(k,\ell) \right\}. \]

Obviously, the set \( C \) must satisfy some stability conditions in order for the spaces above to be intertwiner spaces. For instance, when two intertwiners can be composed, their composition must again be an intertwiner so that we need to ensure that \( T_q \circ T_p \) is a linear combination of maps of the form \( T_r \). This condition is linked to the following operation on partitions: given two partitions \( p \in \mathcal{P}(k,\ell) \) and \( q \in \mathcal{P}(\ell,m) \), we can perform their vertical concatenation \( qp \) (or \( q \circ p \) if ambiguity needs to be avoided) by placing \( q \) below \( p \) and connecting the lower points of \( p \) to the corresponding ones in the upper row of \( q \). This process may produce loops, which can be formalised as follows:

**Definition 3.2.1.** For two partitions \( p \in \mathcal{P}(k,\ell) \) and \( q \in \mathcal{P}(\ell,m) \), consider the set \( L \) of elements in \( \{1, \ldots, \ell\} \) which are not connected to an upper point of \( p \) nor to a lower point of \( q \). The lower row of \( p \) and the upper row of \( q \) both induce a partition of the set \( L \). For \( x, y \in L \), let us set \( x \sim y \) if \( x \) and \( y \) belong either to the same block of the partition induced by \( p \) or to the one induced by \( q \). The transitive closure of \( \sim \) is an equivalence relation on \( L \) and the corresponding partition is called the loop partition of \( L \), its blocks are called loops.

When composing two partitions, we erase the loops and only remember their number, denoted by \( rl(q,p) \). Here is an example with \( rl(q,p) = 1 \):

At the level of the operators \( T_p \), this translates into the following formula:

\[ T_q \circ T_p = \dim(V)^{rl(q,p)} T_{qp}. \quad (3.1) \]

**Exercise 15.** Prove Equation (3.1).

**Solution.** Assume that \( p \in \mathcal{P}(k,\ell) \) and \( q \in \mathcal{P}(\ell,m) \). For a tuple \( i = (i_1, \cdots, i_k) \), we have

\[
T_q \circ T_p(e_{i_1} \otimes \cdots \otimes e_{i_k}) = \sum_{j_1, \cdots, j_\ell=1}^{N} \delta_{p(i,j)} T_q(e_{j_1} \otimes \cdots \otimes e_{j_\ell})
\]

\[
= \sum_{j_1, \cdots, j_\ell=1}^{N} \sum_{s_1, \cdots, s_m=1}^{N} \delta_{p(i,j)} \delta_{q(j,s)} (e_{s_1} \otimes \cdots \otimes e_{s_m}).
\]
It follows from the definition that if there is a \( j \) such that \( \delta_p(i, j) \neq 0 \) and \( \delta_q(j, s) \neq 0 \), then \( \delta_{qp}(i, s) \neq 0 \). Conversely, for given tuples \( i \) and \( s \), any tuple \( j \) which coincides with \( i \) on the lower row of \( p \) and with \( s \) on the upper row of \( q \) will do. The points of such a tuple which are connected either to an upper point of \( p \) or to a lower point of \( q \) are already determined, so that the only freedom we have concerns points which are only connected to the lower row in \( p \) and to the upper row of \( q \). The value of \( j \) must coincide on such blocks if the get connected in the composition, thus the number of compatible tuples \( j \) is the number of possible indexing of the loops which get removed in the composition \( qp \). This gives the formula in the statement. \( \blacksquare \)

Stability under vertical concatenation is not sufficient to produce commutants of (quantum) group representations. For instance, if \( T_p \) and \( T_q \) are intertwiners, then \( T_p \otimes T_q \) also is and must therefore come from partitions. This corresponds to the horizontal concatenation: if \( p \) and \( q \) are two partitions of \( \{1, \cdots, 2k\} \) and \( \{1, \cdots, 2\ell\} \) respectively, then we build a partition \( p \otimes q \) of \( \{1, \cdots, 2(k + \ell)\} \) by simply drawing \( q \) on the right of \( p \). For instance

\[
\begin{array}{c}
\begin{array}{c}
\bullet \quad \bullet \\
\otimes \\
\bullet \quad \bullet
\end{array}
\end{array}
= 
\begin{array}{c}
\begin{array}{c}
\bullet \quad \bullet \\
\bullet \quad \bullet
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
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\bullet \quad \bullet \\
\bullet \quad \bullet
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\bullet \quad \bullet \\
\bullet \quad \bullet
\end{array}
\end{array}
\end{array}
\]

As for vertical concatenation, this operation has a nice interpretation at the level of operators, namely

\[
T_p \otimes T_q = T_{p \otimes q}
\]  

(3.2)

**Exercise 16.** Prove Equation (3.2).

**Solution.** Assume that \( p \in \mathcal{P}(k, \ell) \) and \( q \in \mathcal{P}(m, s) \). For a tuple \( i = (i_1, \cdots, i_{k+m}) \), the image of \( e_{i_1} \otimes \cdots \otimes e_{i_{k+m}} \) under \( T_p \otimes T_q \) is

\[
X = \left( \sum_{j_1, \cdots, j_k = 1}^{N} \delta_p((i_1, \cdots, i_k), (j_1, \cdots, j_k)) e_{j_1} \otimes \cdots \otimes e_{j_k} \right) \otimes \left( \sum_{j_{k+1}, \cdots, j_{k+s} = 1}^{N} \delta_p((i_{k+1}, \cdots, i_{k+m}), (j_{k+1}, \cdots, j_{k+s})) e_{j_{k+1}} \otimes \cdots \otimes e_{j_{k+s}} \right)
= \sum_{j_1, \cdots, j_{k+s} = 1}^{N} \delta_{p \otimes q}(i, j) e_{j_1} \otimes \cdots \otimes e_{j_{k+s}}
= T_{p \otimes q}(e_{i_1} \otimes \cdots \otimes e_{i_{k+m}})
\]

where \( p^* \) is the partition obtained by reflecting \( p \) with respect to an horizontal axis between the two rows. Here is an example:
3.3. Tannaka-Krein reconstruction

Do we now have everything needed to produce an orthogonal compact matrix quantum group out of \( \mathcal{C} \)? The answer is yes, but the proof of it requires a detour through a more abstract structure.

### 3.3 Tannaka-Krein reconstruction

Reconstructing a group from its representations is the subject of Tannaka-Krein duality. The main point of this theory is that the representations of the group and their intertwiners can be assembled into one rich algebraic structure called a tensor category. The reader may for instance refer to the book [EGNO15] for a comprehensive introduction to tensor categories. For our purpose, we can restrict to a specific type of such categories for which we will use the following more “down-to-earth” definition.

**Definition 3.3.1.** Let \( V \) be a finite-dimensional Hilbert space. A **concrete baby\(^2\) rigid C*-tensor category** \( \mathcal{C} \) is a collection of spaces \( Mor_{\mathcal{C}}(k, \ell) \subset \mathcal{L}(V^\otimes k, V^\otimes \ell) \) for all \( k, \ell \in \mathbb{N} \) such that

1. If \( T \in Mor_{\mathcal{C}}(k, \ell) \) and \( T' \in Mor_{\mathcal{C}}(k', \ell') \), then \( T \otimes T' \in Mor_{\mathcal{C}}(k + k', \ell + \ell') \),
2. If \( T \in Mor_{\mathcal{C}}(k, \ell) \) and \( T' \in Mor_{\mathcal{C}}(\ell, r) \), then \( T' \circ T \in Mor_{\mathcal{C}}(k, r) \),
3. If \( T \in Mor_{\mathcal{C}}(k, \ell) \), then \( T^* \in Mor(\ell, k) \),
4. \( id : x \mapsto x \in Mor_{\mathcal{C}}(1, 1) \),
5. \( D : x \otimes y \mapsto \langle x, y \rangle \in Mor_{\mathcal{C}}(2, 0) \).

If moreover \( \sigma : x \otimes y \mapsto y \otimes x \in Mor(2, 2) \), then \( \mathcal{C} \) is said to be **symmetric**.

The fundamental example is of course given by (quantum) groups:

**Example 3.3.2.** Let \( G \) be an orthogonal compact matrix quantum group with a fundamental representation \( u \) of size \( N \). Set \( V = \mathbb{C}^N \) and

\[
Mor_{\mathcal{C}}(k, \ell) = Mor_G(u^\otimes k, u^\otimes \ell).
\]

This defines a concrete baby rigid C*-tensor category called the **representation category** of \( G \) and denoted by \( \mathcal{R}(G) \). If moreover \( G \) is classical, then \( \mathcal{R}(G) \) is symmetric.

That this example is in fact fully general is the content of the quantum Tannaka-Krein theorem proved by S.L. Woronowicz in [Wor88]:

**Theorem 3.3.3** (Woronowicz) Let \( \mathcal{C} \) be a concrete baby rigid C*-tensor category associated to a Hilbert space \( V \). Then, there exists an orthogonal compact matrix quantum group \( G \) with a fundamental representation \( u \) of dimension \( \dim(V) \) such that for all \( k, \ell \in \mathbb{N} \),

\[
Mor_G(u^\otimes k, u^\otimes \ell) = Mor_{\mathcal{C}}(k, \ell).
\]

Moreover, the quantum group \( G \) is unique up to isomorphism and is a classical group if and only if \( \mathcal{C} \) is symmetric.

---

\(^2\) This term is of course not standard. We introduce it here to stay rigorous since our definition is more restrictive than that of a general concrete rigid C*-tensor category.
There are several proofs of this result using a different amount of categorical machinery, see for instance [Wor88] or [NT13, Thm 2.3.2]. However, due to our restricted definition of concrete rigid baby C*-tensor category, we can give a very elementary proof which is close to that of [Mal18].

**Proof.** The idea is to build $G = (O(G), u)$ as a quotient of the space of all “noncommutative matrices” by all the relations making the maps $T$ intertwiners of tensor powers of its fundamental representation. So let $N$ be the dimension of $V$ and consider the universal complex algebra $A$ generated by $N^2$ elements $X_{ij}$ with the involution given by $X_{ij}^* = X_{ij}$. Fixing an orthonormal basis $(e_i)_{1 \leq i \leq N}$ of $V$ once and for all, we introduce some useful shorthand notations: if $i = (i_1, \ldots, i_k)$, $j = (j_1, \ldots, j_l)$ and $T \in \mathcal{L}(V^\otimes k, V^\otimes \ell)$, then we set

\[
e_i = e_{i_1} \otimes \cdots \otimes e_{i_k}
\]
\[
T_{ij} = \langle T(e_i), e_j \rangle
\]
\[
X_{ij} = X_{i_1j_1} \cdots X_{i_kj_k} \quad (\text{if } k = \ell)
\]

Consider now the following non-commutative polynomial

\[
P_{T, i, j} = \sum_s X_{is} T_{sj} - T_{is} X_{sj}.
\]

Then, for a compact quantum group $(O(G), u)$, $T$ intertwines $u^\otimes k$ with $u^\otimes \ell$ if and only if

\[
P_{T, i, j} (u) := P_{T, i, j} ((u_{ij})_{1 \leq i, j \leq N}) = 0
\]

for all tuples $i$ and $j$. Let us therefore consider the sets

\[
I_{k, \ell} = \text{ Vect } \{ P_{T, i, j}(X) \mid T \in \text{ Mor}_\mathcal{O}(k, \ell), i = (i_1, \ldots, i_k), j = (j_1, \ldots, j_\ell) \}
\]
\[
I_k = \bigoplus_{i,j=0}^k I_{i,j}
\]
\[
I = \bigcup_{k \in \mathbb{N}} I_k.
\]

According to our basic idea, let us set $\tilde{A} = A/I$ (this makes sense at least as a quotient of vector spaces) and let $u_{ij}$ be the image of $X_{ij}$ in this quotient. If $(\tilde{A}, u)$ is an orthogonal compact matrix quantum group, then it certainly is the one we are looking for. But for this to hold, $I$ must satisfy some extra conditions, namely

1. $I$ is an ideal, so that $\tilde{A}$ is an algebra,
2. $I = I^*$, so that $\tilde{A}$ is a $*$-algebra,
3. $\Delta(I) \subset I \otimes \tilde{A} + \tilde{A} \otimes I$, so that the coproduct is well-defined on $\tilde{A}$. Indeed, if $\pi_I$ denotes the quotient map, then for any $x \in O(O_N^+)$ and $y \in I$,

\[
(\pi_I \otimes \pi_I) \circ \Delta (x + y) = (\pi_I \otimes \pi_I) \circ \Delta (x)
\]

so that the coproduct factors through the quotient.

As one may expect, these properties follow from the axioms of concrete baby rigid C*-tensor categories, and here is how:

1. Observe that if $i', j'$ are $m$-tuples, then

\[
X_{i'j'} P_{T, i, j}(X) = P_{\text{id}^\otimes m \otimes T; i' \otimes i, j' \otimes j}(X),
\]

where $\otimes$ denotes the concatenation of tuples. Thus, $I$ absorbs monomials, hence also polynomials, on the left. The same property on the right follows from a similar argument. As a consequence, $I$ is an ideal.

---

3. As already mentioned, the abelianization of $A$ is nothing but the algebra of regular functions on $M_N(C)$. The algebra $A$ can therefore be thought of as the regular functions on all “noncommutative matrices"
2. Denoting by $i^{-1} = (i_k, \cdots, i_1)$ the reversed tuple, we have

\[ P_{T; i, j}(X) = \sum_{s} X_{i^{-1}s^{-1}} T_{sj} - X_{s^{-1}j^{-1}} T_{is} \]

\[ = \sum_{s} X_{i^{-1}s^{-1}} T_{js}^* - X_{s^{-1}j^{-1}} T_{si}^* \]

\[ = \sum_{s} X_{i^{-1}s^{-1}} S(T^*)_{s^{-1}j^{-1}} - X_{s^{-1}j^{-1}} S(T^*)_i^{-1}s^{-1} \]

\[ = P_{R(T), i^{-1}, j^{-1}} \]

where $R(T)$ is defined to be the operator with coefficients

\[ R(T)_{ij} = T_{j - i - 1}. \]

To see that $R(T)$ is in $\text{Mor}\ell,k$ if $T \in \text{Mor}\ell,k$, let us define inductively operators

\[ D_k : V^\otimes 2k \to C \]

by the formula

\[ D_{k+1} = D_k \circ (\text{id}_V \otimes D \otimes \text{id}_V \otimes k). \]

These are in fact the maps $T_{p_k}$ corresponding to the partition

\[ p_k = \begin{array}{cccc}
\cdot & \cdots & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot
\end{array} \]

obtained by nesting $k$ pairings. An easy induction shows that

\[ D_k(e_i \otimes e_j) = \delta_{i,j-1}. \]

Using the map $D_k$ we define, for a linear map $T$,

\[ \tilde{T} = (\text{id}_V \otimes D_t) (\text{id}_V \otimes T \otimes \text{id}_V \otimes t) (D_k^* \otimes \text{id}_V \otimes t). \]

This can be seen pictorially as a “reversed version” of $T$:

\[ \tilde{T} = \begin{array}{cccc}
\cdot & \cdots & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot
\end{array} \]

\[ T \]

\[ \cdots \]

\[ \cdots \]

Now,

\[ \tilde{T}(e_i) = (\text{id}_V \otimes D_t) (\text{id}_V \otimes T \otimes \text{id}_V \otimes t) \left( \sum_{s} e_s \otimes e_{s-1} \otimes e_i \right) \]

\[ = \sum_s \sum_t (\text{id}_V \otimes D_t) (e_s \otimes T_{s^{-1}t} e_t \otimes e_i) \]

\[ = \sum_s T_{s^{-1}t} e_s \]

\[ = R(T)(e_i) \]

so that $R(T)$ is a morphism as soon as $T$ is.
3. This follows from a straightforward calculation:

\[
\Delta(P_{T,i,j}(X)) = \sum_{t,s} X_{is} \otimes X_{st} T_{ij} - X_{ts} \otimes X_{sj} T_{it}
\]

\[
= \sum_s X_{is} \otimes \left( \sum_t X_{st} T_{ij} - X_{tj} T_{st} \right)
+ \left( \sum_t X_{tj} T_{st} - X_{ts} T_{it} \right) \otimes X_{sj}
\]

\[
= \sum_s X_{is} \otimes P_{T,s,j}(X) + P_{T,i,s}(X) \otimes X_{sj}.
\]

As a consequence, \( G = (\hat{A}, u) \) is an orthogonal compact matrix quantum group (observe that the relations \( P_{D,i,j}(u) = 0 \) are precisely the orthogonality condition in Definition 1.3.1). Moreover, by construction,

\[
\text{Mor}_\mathcal{E}(k, \ell) \subset \text{Mor}_G \left( u^\otimes k, u^\otimes \ell \right)
\]

for all \( k, \ell \in \mathbb{N} \). This implies that setting \( u_k = \bigoplus_{i=0}^k u^\otimes i \) acting on \( V_k = \bigoplus_{i=0}^k H^\otimes i \),

\[
B_k = \bigoplus_{i,j=0}^k \text{Mor}_\mathcal{E}(i, j) \subset \text{Mor}_G (u_k, u_k).
\]

To show the converse inclusion, let us first note that by universality, all the monomials in \( A \) are linearly independent. Hence denoting by \( X_k \) the matrix with coefficients \( X_{ij} \) with indices running over all tuples of length \( k \), there exists for any operator \( T \in \mathcal{L}(V_k) \) a linear map \( f \in A^* \) such that \( \hat{f}(X_k) = T \). Setting

\[
\mathcal{D}_k = \left\{ \hat{f}(X_k) \mid f \in \mathcal{I}_k \right\},
\]

we have

\[
\hat{f}(X_k) \in B_k \Rightarrow \omega \left( [T, \hat{f}(X_k)] \right) = 0, \forall T \in B_k, \forall \omega \in \mathcal{L}(H_k)^* \\
\Rightarrow (\omega \otimes f) (|T, X_k|) = 0, \forall T \in B_k, \forall \omega \in \mathcal{L}(H_k)^* \\
\Rightarrow f \left( (\omega \otimes \text{id}) |T, X_k| \right) = 0, \forall T \in B_k, \forall \omega \in \mathcal{L}(H_k)^* \\
\Rightarrow f \in \mathcal{I}_k^k.
\]

The last line follows from the fact that linear maps on \( \mathcal{L}(H_k) \) are linear combinations of coefficient functions, hence \( \omega(|T, X_k|) \in \mathcal{I}_k \). As a consequence, \( B'_k \subset \mathcal{D}_k \) so that by the Double Commutant Theorem (see Theorem B in the Appendix), \( B'_k \subset B_k \). By Lemma 2.1.15, any \( T \in \text{Mor}_G (u_k, u_\ell) \) commutes with \( \hat{f}(u_k) \) for all \( f \in \mathcal{O}(\mathbb{G})^* \). Equivalently, \( T \) commutes with \( \hat{f}(X_k) \) for all \( f \in \mathcal{I}_k^k \), that is to say with \( \mathcal{D}_k \). In other words, \( T \in \mathcal{D}_k' \subset B_k \) and the equality is proved.

Let us turn to uniqueness. Consider an orthogonal compact matrix quantum group \( \mathbb{H} \) such that

\[
\text{Mor}_\mathcal{E}(k, \ell) = \text{Mor}_\mathbb{H} \left( u^\otimes k, u^\otimes \ell \right)
\]

for all \( k, \ell \in \mathbb{N} \). By definition there exists an ideal \( J \in \mathcal{A} \) such that \( \mathcal{O}(\mathbb{H}) = \mathcal{A}/J \) and we can, up to isomorphism, identify the copies of \( \mathcal{A} \) used for \( \mathbb{G} \) and for \( \mathbb{H} \). Moreover, with the previous notations, \( \mathcal{I} \subset J \). If now \( J_k \) denotes the intersection of \( J \) with the span of coefficients of \( X_k \), then the computations above show that

\[
\left\{ \hat{f}(X_k) \mid f \in J_k \right\}' = \text{Mor}_\mathbb{H} (u_k, u_k) = \mathcal{D}_k' = B_k = \left\{ \hat{f}(X_k) \mid f \in \mathcal{I}_k \right\}'.
\]

so that \( J_k = \mathcal{I}_k \). Thus, \( J = \cup J_k = \cup \mathcal{I}_k = \mathcal{I} \) and \( \mathbb{H} \simeq \mathbb{G} \).

The last point concerns symmetry. But we have already seen that \( \sigma \in \text{Mor}_\mathcal{E}(2, 2) \) is equivalent to having \( f_{\text{cross}} \in \text{Mor}_\mathcal{E}(4, 0) \), and we proved in Proposition 2.2.3 that this is equivalent to \( \mathbb{G} \) being classical. \( \blacksquare \)
Recall that we want to apply Tannaka-Krein reconstruction to build a compact quantum group out of a category of partitions. Considering the axioms in Definition 3.3.1, we see that Axioms (1) to (3) correspond to the three operations defined in Section 3.2. As for the last two axioms, they follow from the following elementary computations:

- \( T_\parallel = \text{id} \),
- \( T_\sqcup = D \).

To simplify later statement, let us give a name to this

**Definition 3.3.4.** A category of partitions \( \mathcal{C} \) is a collection of sets of partitions \( \mathcal{C}(k, \ell) \) for all integers \( k \) and \( \ell \) such that

1. If \( p \in \mathcal{C}(k, \ell) \) and \( q \in \mathcal{C}(k', \ell') \), then \( p \otimes q \in \mathcal{C}(k + k', \ell + \ell') \),
2. If \( p \in \mathcal{C}(k, \ell) \) and \( q \in \mathcal{C}(\ell, r) \), then \( qp \in \mathcal{C}(k, r) \).
3. If \( p \in \mathcal{C}(k, \ell) \), then \( p^* \in \mathcal{C}(\ell, k) \).
4. \( | \in \mathcal{C}(1, 1) \).
5. \( \sqcup \in \mathcal{C}(2, 0) \).

If moreover \( p_{\text{cross}} \in \mathcal{C}(2, 2) \), then \( \mathcal{C} \) is said to be **symmetric**.

We conclude this section by a theorem which is rather a summary and rewriting of the whole construction that we have done:

**Theorem 3.3.5** (Banica-Speicher) Let \( N \) be an integer and let \( \mathcal{C} \) be a category of partitions. Then, there exists an orthogonal compact matrix quantum group \( \mathbb{G} = (\mathcal{O}(\mathbb{G}), u) \), where \( u \) has dimension \( N \), such that for any \( k, \ell \in \mathbb{N} \),

\[
\text{Mor}_\mathbb{G}(u^\otimes k, u^\otimes \ell) = \text{Vect} \{ T_p \mid p \in \mathcal{C}(k, \ell) \}.
\]

Moreover, \( \mathbb{G} \) is a classical group if and only if \( \mathcal{C} \) is symmetric. The compact quantum group \( \mathbb{G} \) will be denoted by \( \mathbb{G}_N(\mathcal{C}) \) and called the **partition quantum group** associated to \( N \) and \( \mathcal{C} \).

**Remark 3.3.6.** The quantum groups associated to a category of partitions as we defined them here were first introduced by T. Banica and R. Speicher in [BS09] under the name of **easy quantum groups**. It was already clear that this setting could be extended to unitary compact matrix quantum groups by adding black and white colours to the points of the partition. The formal description of this extended formalism was made in [TW17]. However, it had by then become apparent that one can even allow arbitrary colourings of the partitions and that idea led to the general setting of **partition quantum groups** introduced in [Fre17].

**Remark 3.3.7.** One the first questions raised by the definition of partition quantum groups is their classification, based on the categories of partitions. For easy quantum groups, there is a complete classification, as a result of many works including [BS09], [Web13], and [RW16]. The classification of the unitary versions of easy quantum groups is more subtle, but many results are now known, starting with [TW17] and followed by several authors like [MW19a, MW19b]. As for the general setting of partition quantum groups, the only known results are restricted to noncrossing partitions with two colours, see [Fre19].

### 3.4 Examples of partition quantum groups

We conclude this chapter by some examples. Here is the basic strategy: to determine \( \mathbb{G}_N(\mathcal{C}) \), one only has to consider generators of \( \mathcal{C} \), that is, a subset \( F \) such that \( \mathcal{C} \) is the smallest category of partitions containing \( F \). Then, \( \mathcal{O}(\mathbb{G}_N(\mathcal{C})) \) is the quotient of \( \mathcal{O}(\mathcal{O}_N^+ \mathcal{C}) \) by the relations given by
the fact that $T_p$ is an intertwiner for $p \in F$. This can be proven by following again all steps of the proof of Theorem 3.3.3. Indeed, starting only with a set of morphisms which generated all the morphisms of the concrete baby C*-tensor category, we can define $I$ as the ideal generated in $\mathcal{A}$ by the corresponding polynomials. The proof then shows that $\mathcal{A}/I$ is an orthogonal compact matrix quantum group with the correct morphism spaces.

In order to apply that strategy, we will have to identify the generators of some specific categories of partitions. To do this, the following elementary lemma will prove useful:

**Lemma 3.4.1.** Let $p$ be a non-crossing partition lying on one line. Then, $p$ contains an interval, that is to say a block consisting of consecutive points.

*Proof.* The proof is done by induction on the number of points of $p$. If it has at most two points, then the result is clear. Assume that the result holds for all partitions on at most $n$ points and consider a partition $p$ on $n + 1$ points. We will focus on its leftmost point, and there are two possibilities:

- If it belongs to an interval, then we are done,
- Otherwise, it is connected to the $i$-th point from the left. By non-crossing, a point between the first and $i$-th one can only be connected to another point between the first and $i$-th one. Here is a picture of the situation:

  ![Partition Diagram](image)

  Thus, restricting to the second to $i - 1$-th points yields a subpartition $q$ of $p$ on at most $n$ points. We then conclude by induction.

What about partitions not lying on one line? The result still holds, once one sees that in fact we can always reduce to one line. This is done through a rotation operation which we now introduce. Let $p$ be a partition and consider the leftmost point of its upper row. Let us rotate it to the left of the lower row, without changing the strings connecting it to other points. We thus obtain a new partition $p'$, called a rotated version of $p$. The key fact is that categories of partitions are invariant under this operation.

**Lemma 3.4.2.** If $C$ is a category of partitions and if $p \in C$, then $p' \in C$. The same holds for rotations of points on the right.

*Proof.* This simply follows from the fact that if $p$ has $k$ upper points, then

$$p' = (\mid \otimes p) \circ (\sqcup \otimes |^{k-1}) \in C.$$
3.4. Examples of partition quantum groups

The proof for rotations on the right is analogous.

We are now ready to give our first explicit example of partition quantum group, which is nothing but our old friend $O_N^+$.

**Proposition 3.4.3.** There are isomorphisms

$$\mathcal{G}_N(\mathcal{NC}_2) \simeq O_N^+$$ and $\mathcal{G}_N(P_2) \simeq O_N$.

**Proof.** This relies on the fact that $\mathcal{NC}_2$ is the smallest category of partitions, in the sense that it is generated by $\sqcup$. This is proven by induction on the number of points of a partition in $\mathcal{NC}_2$ as follows. If the partition has two points, then it is $\sqcup$. If the result holds for all partitions on at most $2n$ points, let $p$ be a partition on $2(n + 1)$ points. Up to rotating, we may assume that $p$ lies on one line and that the first two points belong to an interval. Because we are considering pair partitions, this means that $p = \sqcup \otimes p'$. But then, $p' \in \mathcal{NC}_2$ and has $2n$ points so that by induction it is in the category of partitions generated by $\sqcup$, hence also $p$, concluding the proof.

Because any non-crossing pair partition can be obtained from $\sqcup$ using the category operations, $\mathcal{O}(\mathcal{G}_N(\mathcal{NC}_2))$ is obtained from the universal $\ast$-algebra $\mathcal{A}$ by adding the relations coming from the fact that $T_{\sqcup} = D$ is an intertwiner. But these are exactly the relations making the fundamental representation orthogonal. This proves the first isomorphism.

As a consequence of the beginning of the proof, $P_2$ is generated as a category of partitions by $\sqcup$ and the crossing partition $\{\{1, 3\}, \{2, 4\}\}$. As shown in Proposition 2.2.3, the relations corresponding to the fact that the crossing partition yields an intertwiner are exactly the commutation relations between all the coefficients of the fundamental representation. Thus, the corresponding algebra is the abelianization of $\mathcal{O}(O_N^+)$. This is $\mathcal{O}(\mathcal{O}_N)$.

Note that we can now recover R. Brauer’s Theorem 2.2.2, as well as T. Banica’s Theorem 2.2.5, as immediate corollaries of Theorem 3.3.5. The quantum permutation groups of course also fall into this class, but this requires a bit of work.

**Exercise 17.** Let $\mathbb{G} = (\mathcal{O}(\mathbb{G}), u)$ be an orthogonal compact matrix quantum group. Prove that $\mathcal{O}(\mathbb{G})$ satisfies the defining relations of $\mathcal{A}_s(N)$ in Definition 1.2.1 if and only if it has the following intertwiners:

- Conditions (1) and (3) : $T_p \in \text{Mor}(u^\otimes 2, u)$ where $p$ is the partition with one block in $\mathcal{P}(2, 1)$,
- Condition (2) : $T_q \in \text{Mor}(P, \varepsilon)$ where $q$ is the singleton partition in $\mathcal{P}(1, 0)$.

**Solution.** We compute on the one hand

$$\left(\text{id} \otimes T_p\right) \circ \rho_{u^\otimes 2} (e_{i_1} \otimes e_{i_2}) = \sum_{j_1, j_2}^{N} u_{i_1j_1} u_{i_2j_2} \otimes T_p (e_{j_1} \otimes e_{j_2})$$

$$= \sum_{j_1, j_2}^{N} u_{i_1j_1} u_{i_2j_2} \otimes \delta_{j_1, j_2} e_{j_1}$$

$$= \sum_{j}^{N} u_{i_1j} u_{i_2j} \otimes e_{j}$$

and on the other hand

$$\rho_u \circ T_p (e_{i_1} \otimes e_{i_2}) = \rho_u (\delta_{i_1i_2} e_{i_1})$$

$$= \delta_{i_1i_2} \sum_{j=1}^{N} u_{i_1j} \otimes e_{j}$$

- 43 -
By linear independence, we conclude that for any $i_1, i_2, j$,

$$u_{i_1,j}u_{i_2,j} = \delta_{i_1i_2}u_{i_1,j}.$$ 

Moreover, $T^*_p$ satisfies the corresponding relations with $u^* = u^t$, yielding Conditions (1) and (3). The converse straightforwardly follows from the same computation.

As for the second point,

$$\left(\text{id} \otimes T_q\right) \circ \rho_u(e_i) = \sum_{j=1}^N u_{ij} \otimes T_q(e_j)$$

while $\varepsilon \circ T_q(e_j) = 1 \otimes 1$. Thus,

$$\sum_{j=1}^N u_{ij} = 1$$

and using $u^*$ instead, we see that Condition (2) holds. Once again, the converse is straightforward.

**Remark 3.4.4.** We have included the map $T_q$ in the statement of Exercise 17 because it is a nice and important example of partition map. However, it is redundant because of the orthogonality assumption. Indeed, if $T_\| \parallel$ and $T_p$ are intertwiners, then so is

$$T^*_\| \parallel \circ T^*_p = T^*_{\| \parallel \circ p} = T_q.$$ 

**Proposition 3.4.5.** There are isomorphisms

$$G_N(\mathcal{NC}) \simeq S^+_N \text{ and } G_N(\mathcal{P}) \simeq S_N.$$ 

**Proof.** It follows from Exercise 17 that $S^+_N$ is a partition quantum group and that its category of partitions is generated by $p$ and $q$. Let us show that this category of partitions is in fact the category $\mathcal{NC}$ of all noncrossing partitions. This can be done in two steps:

1. Let $k \geqslant 1$ and consider the partition

$$\left(\left| \otimes \right| \otimes \left(\| \otimes \right) \otimes \left(\right)^{(k-1)} \otimes \left| \otimes \right)\right) \circ p^\otimes k.$$ 

This is a partition on $k + 3$ points with only one block. As a consequence, any one-block partition is in the category of partitions $\langle p \rangle$ generated by $p$.

2. Let us prove by induction on $n$ that any noncrossing partition on at most $n$ points is in $\langle p \rangle$. This is clear for $n \leqslant 3$. If it is true for some $n$, let $p$ be a partition on $n + 1$ points. If $p$ has only one block, then it is in $\langle p \rangle$ by the first point. Otherwise, by Lemma 3.4.1, there is an interval $b$ in $p$. Rotating $p$ we can then write it as $b \otimes p'$ for some partition $p'$ on at most $n$ points. The result now follows form the induction hypothesis.

Adding the crossing partition, we see that $S_N$ corresponds to the category $\mathcal{P}$ of all partitions.

Once again, it follows from this and Theorem 3.3.5 that the intertwiner spaces of $S_N$ and $S^+_N$ are given by partition maps, namely

$$\text{Mor}_{S_N^+} \left(\rho^\otimes k, \rho^\otimes \ell\right) = \text{Vect} \{T_p \mid p \in \mathcal{NC}(k, \ell)\}$$

$$\text{Mor}_{S_N} \left(\rho^\otimes k, \rho^\otimes \ell\right) = \text{Vect} \{T_p \mid p \in \mathcal{P}(k, \ell)\}$$

4. Recall that $u_{ij}^* = u_{ij}$ is part of the axioms of an orthogonal compact matrix quantum group.
In the classical case, this result was proven at the same time by P. Martin in [Mar94] and by V. Jones in [Jon93]. As for the quantum case, it was established by T. Banica in [Ban99b]. We conclude by another example, of a slightly different flavour:

**Exercise 18.** Let $\mathcal{O}(H_N^+)\subset\mathcal{O}(O_N^+)$ be the quotient of $\mathcal{O}(O_N^+)$ by the relations

$$u_{ij} u_{ik} = u_{ik} u_{ij}$$

(3.5)

for all $1 \leq i \leq N$ and $j \neq k$. Prove that this is the partition quantum group corresponding to the category $\text{NC}_{\text{even}}$ of all even noncrossing partitions, meaning noncrossing partitions such that all blocks have even size. Why is this called the hyperoctaedral quantum group?

**Solution.** Let $p \in \text{NC}(2, 2)$ be the partition with only one block. Then,

$$(\text{id} \otimes T_p) \circ \rho_u \otimes e (e_1 \otimes e_2) = \sum_{j_1, j_2=1}^N u_{i_1 j_1} u_{i_2 j_2} \otimes T_p (e_{j_1} \otimes e_{j_2})$$

$$= \sum_{j_1, j_2=1}^N u_{i_1 j_1} u_{i_2 j_2} \otimes \sum_{j=1}^N \delta_{j_1 j_2} e_j \otimes e_j$$

$$= \sum_{j=1}^N u_{i_1 j} u_{i_2 j} \otimes e_j \otimes e_j$$

while

$$\rho_u \otimes e \circ T_p (e_1 \otimes e_2) = \delta_{i_1 i_2} \rho_u \otimes e (e_1 \otimes e_1)$$

$$= \delta_{i_1 i_2} \sum_{j_1, j_2=1}^N u_{i_1 j} u_{i_2 j} \otimes e_{j_1} \otimes e_{j_2}.$$  

Identifying the two sums shows on the one hand that $u_{i_1 j} u_{i_2 j} = 0$ if $i_1 \neq i_2$ and on the other hand that $u_{i_1 j_1} u_{i_2 j_2} = 0$ if $j_1 \neq j_2$, which are exactly the relations in the statement. Conversely, if these relations are satisfied then both expressions collapse to

$$\delta_{i_1 i_2} \sum_{j=1}^N u_{i_1 j}^2 \otimes e_j \otimes e_j.$$  

We have shown that $p$ generates the category of partitions of $H_N^+$. Note that $\text{NC}_{\text{even}}$ is indeed a category of partitions, since all the operations preserve even blocks. Thus, $\langle p \rangle \subset \text{NC}_{\text{even}}$ and the converse inclusion follows from the same argument as in Example 17: the analogue of Equation (3.4) yields that any one-block partition on an even number of points in $\langle p \rangle$, and from then on the proof goes by induction.

Consider the abelianization $G$ of $H_N^+$. Its coefficient functions satisfy Equation (3.5), so that on each row and each column of any element of $G$, there is at most one non-zero coefficient. Since $G$ is moreover made of orthogonal matrices, it follows that there is exactly one non-zero coefficient on each row and column, and that this coefficient is $-1$ or $1$. In other words, $G$ is the group of signed permutation matrices, i.e. permutation matrices where one allows $-1$ instead of $1$ as a coefficient. This group is also known as the hyperoctaedral group $H_N$, hence the name of $H_N^+$.

As before, one can deduce from this a description of the invariants of these compact quantum group. For $H_N$ this recovers a result of K. Tanabe in [Tan97] while for $H_N^+$, this was first established by T. Banica and R. Vergioux in [BV09]:

$$\text{Mor}_{H_N^+} (u^{\otimes k}, u^{\otimes \ell}) = \text{Vect} \{ T_p \mid p \in \text{NC}_{\text{even}} (k, \ell) \}$$

$$\text{Mor}_{H_N} (\rho^{\otimes k}, \rho^{\otimes \ell}) = \text{Vect} \{ T_p \mid p \in \mathcal{P}_{\text{even}} (k, \ell) \}$$

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3.4. Examples of partition quantum groups

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CHAPTER 4
THE REPRESENTATION THEORY OF PARTITION QUANTUM GROUPS

Our goal in this chapter is to describe the representation theory of the quantum groups $O_N^+$ and $S_N^+$. This was first done by T. Banica in [Ban96] and [Ban99b] respectively, using Temperley-Lieb categories and their variants. However, the setting of partition quantum groups allows us to take a more general approach. We will therefore give a description of the representation theory of any partition quantum group associated to noncrossing partitions, and then apply it to our favourite examples. This is the approach developed in [FW16].

Let us fix once and for all a category of partitions $\mathcal{C}$. Our first task is to find all the irreducible representations of $\mathbb{G}_N(\mathcal{C})$. By Corollary 2.1.16, we know that it is enough to find irreducible subrepresentations of $u^k$ for all $k \in \mathbb{N}$, which by definition are given by minimal projections in $\text{Mor}_{\mathbb{G}_N(\mathcal{C})}(u^k, u^k)$.

4.1 PROJECTIVE PARTITIONS

The good news is that we have a nice generating family of the intertwiner spaces, namely the maps $T_p$ for $p \in \mathcal{C}$. However, this generating family may not be linearly independent and this is a source of troubles. We will therefore, for the sake of simplicity, rule out the issue in this text thanks to the following result:

**Theorem 4.1.1** Let $N$ be an integer,

1. If $N \geq 2$, then the linear maps $(T_p)_{p \in NC_2(k,\ell)}$ are linearly independent for all $k, \ell \in \mathbb{N}$,

2. If $N \geq 4$ and $\mathcal{C}$ is any category of non-crossing partitions, then $(T_p)_{p \in C(k,\ell)}$ are linearly independent for all $k, \ell \in \mathbb{N}$.

**Proof.** Our strategy will be to deduce the second point from the first one.

1. It is sufficient to prove that for any $k \in \mathbb{N}$, the vectors $\xi_p = f_p^*$ are linearly independent for all $p \in NC_2(k,0)$. For this, we can try to show that the Gram determinant is non-zero. Given two partitions $p$ and $q$,

$$\langle \xi_p, \xi_q \rangle = \sum_{i_1, \ldots, i_k} \sum_{j_1, \ldots, j_k} (\delta_p(i) e_{i_1} \otimes \cdots \otimes e_{i_k}, \delta_q(j) e_{j_1} \otimes \cdots \otimes e_{j_k}) .$$

$$= \sum_{i_1, \ldots, i_k} \delta_p(i) \delta_q(i).$$

The last expression is the number of tuples which are compatible with both $p$ and $q$. Let us denote by $p \lor q$ the partition obtained by gluing together blocks of $p$ and $q$ having a common point. Then,

$$\delta_p(i) \delta_q(i) = \delta_{p \lor q}(i)$$
so that, denoting by \( b(\cdot) \) the number of blocks of a partition,

\[
\langle \xi_p, \xi_q \rangle = N^{b(p \vee q)}.
\]

The determinant of this matrix is known as the meander determinant and was computed by P. Di Francesco, O. Golinelli and E. Guitter in [DFGG97, Sec 5.2] (see also [BC10, Thm 6.1] for another proof), yielding the following result:

\[
\det \left( N^{b(p \vee q)} \right)_{p,q \in NC_2(k)} = \prod_{i=1}^{k} P_i(N)^{a_{k,i}},
\]

where

\[
a_{k,i} = \binom{2k}{k-i} - \binom{2k}{k-i-1} + \binom{2k}{k-i-2},
\]

and \( P_i \) is the \( i \)-th dilated Chebyshev polynomial of the second kind, defined recursively by \( P_0(X) = 1, P_1(X) = X \) and for any \( i \geq 1 \),

\[
XP_i(X) = P_{i+1}(X) + P_{i-1}(X).
\]

One easily checks that the roots of \( P_i \) are exactly \( \{ \cos(2j\pi/i) \mid 0 \leq j \leq i \} \subset [-2,2] \) and the result follows.

2. The idea will be to reduce the problem to the previous case. For that, notice that given a partition \( p \in NC_2(2k,2\ell) \), one can produce another partition \( \tilde{p} \in NC(k,\ell) \) by gluing points two-by-two. Conversely, if \( p \in NC(k,\ell) \) then it can be “doubled” in the following way:

(a) Assume for simplicity that \( p \) lies on one line. For each point \( a \) of \( p \), draw a point \( a_\ell \) on its left and \( a_r \) on its right.

(b) Then, connect \( a_r \) to \( b_\ell \) if \( a \) and \( b \) are connected and \( b \) is the closest (travelling from left to right cyclically) point of the block.

(c) Here is an example:

\[
\begin{array}{c}
\bullet \quad \bullet \quad \bullet \quad \bullet \\
\end{array}
\quad \rightarrow \quad
\begin{array}{ccc}
\ell & r & \ell & r & \ell & r & \ell & r
\end{array}
\quad \rightarrow \quad
\begin{array}{c}
\bullet \quad \bullet \quad \bullet \quad \bullet \\
\end{array}
\quad \rightarrow \quad
\begin{array}{ccc}
\ell & r & \ell & r & \ell & r & \ell & r
\end{array}
\]

Because of the linear independence proved in the first part, \( \xi_p \mapsto \xi_{\tilde{p}} \) yields a well-defined surjective linear map

\[
\Phi : \text{Vect}\{\xi_p \mid NC_2(2k,0)\} \subset \left( \mathbb{C}^N \right)^{\otimes 2k} \rightarrow \text{Vect}\{\xi_p \mid p \in NC(k,0)\} \subset \left( \mathbb{C}^N \right)^{\otimes k}.
\]

Now, it follows from [KS08, Prop 3.1], that for \( p, q \in NC_2(2k) \),

\[
\langle \xi_p, \xi_q \rangle = N^k \frac{\langle \xi_{\tilde{p}}, \xi_{\tilde{q}} \rangle}{\|\xi_p\|\|\xi_q\|}.
\]

This means that the Gram matrices of the two families are conjugate by the diagonal matrix with coefficients \( N^{k/2}\|\xi_{\tilde{p}}\| \) which is invertible, hence the linear independence.
4.1. Projective partitions

Assuming therefore $C \subset NC$ and $N \geq 4$, how can we build projections in $\text{Mor}_{G \cap (C\setminus\{C\})}(k,k)$? A natural thing to do is to look first for operators $T_p$ which may be projections. The partition $p$ should then satisfy $p \circ p = p = p^*$, but one easily sees that this does not yield a projection in general for normalization reasons:

$$T_p T_p = N_{rl(p,p)} T_p.$$

This is however easy to fix and leads to the following key definition:

**Definition 4.1.2.** A partition $p$ is said to be projective if $pp = p = p^*$. Then, there is a multiple $S_p$ of $T_p$ which is an orthogonal projection.

The set of projective partitions in $C(k,k)$ will be denoted by $\text{Proj}_C(k)$.

Two questions immediately arise:

- Are all $T_p$’s which are proportional to a projection of this form?
- Are there many projective partitions?

The answers to both questions rely on a fundamental fact that we will now explain. Given a partition $p$, we call through-blocks the blocks containing both upper and lower points and we denote their number by $t(p)$.

**Proposition 4.1.3.** Any non-crossing partition $p \in NC(k,\ell)$ can be written in a unique way in the form $p = p_u^* p_d$, where $p_u \in NC(k,t(p))$, $p_d \in NC(\ell,t(p))$ and both satisfy

1. All lower points are in different blocks,
2. Each lower point is connected to at least one upper point,
3. If $i < j$ are lower points and $a(i), a(j)$ are the leftmost upper point connected to $i$ and $j$ respectively, then $a(i) < a(j)$.

This is called the through-block decomposition of $p$.

**Proof.** This statement is obvious pictorially, for instance:

\[
p = \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet
\end{array}
\]

yields

\[
p_u = \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet
\end{array}
\quad \text{and} \quad
p_d = \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet
\end{array}
\]

This suggests the procedure to build the partitions. Let $b_1, \ldots, b_{t(p)}$ be the through-blocks of $p$ ordered by their leftmost point in the upper row. Then, to build $p_u$ we start with the upper row of $p$ and connect $b_i$ to the $i$-th (starting from the left) lower point. The construction of $p_d$ is similar, using the lower parts of the through-blocks $b'_1, \ldots, b'_t$. The crucial thing is that $b_i$ and $b'_i$ are the two parts of the same through-block. Indeed, if there exists $i < j$ and $k < \ell$ such that $b_i$ is connected to $b'_j$ and $b_j$ to $b'_k$, then this would produce a crossing. As a consequence, $p_u^* p_d = p$.

To prove uniqueness, simply notice that if $p'_u^* p'_d$ is another through-block decomposition, then the non-through-block parts must coincide since they are the non-through-blocks of $p$, and the through-blocks are completely determined by the properties in the statement. 

Let us show how useful this is by answering the two previous questions in one shot:
Proposition 4.1.4. A partition $p$ is projective if and only if there exists a partition $r$ such that $p = r^*r$. As a consequence,

- $T_p$ is a multiple of a partial isometry for all $p$,
- $T_p$ is a multiple of a projection if and only if $p$ is projective.

Proof. If $p$ is projective, then $p = p^*p$. Conversely, let $r$ be any partition and let $r = r^*_d r_u$ be its through-block decomposition. The properties of Proposition 4.1.3 imply that

$$r_d r_u^* = |d|d = r^*_u r_u.$$

Thus,

$$r^* r = r^*_u r_d r_u^* r_u = r^*_u r_u$$

and

$$(r^*_u r_u)^* (r^*_u r_u) = r^*_u r_u r^*_u r_u = r^*_u r_u.$$

Using this we can now transfer the usual notions of equivalence and comparison of projections to projective partitions:

Definition 4.1.5. Let $p, q$ be two projective partitions. Then,

- We say that $p$ is dominated by $q$, and write $p \preceq q$, if $pq = pqp$,
- We say that $p$ is equivalent to $q$, and write $p \sim q$, if there exists a partition $r$ such that $r^* r = p$ and $r r^* = q$.

All this is nice and encouraging since it shows that partitions encode a structure comparable to that of matrix algebras. However, for a projective partition $p$, the projection $S_p$ usually fails to be minimal. For instance, $S_{|d|d} = \text{Id}$. To get smaller projections, we can use the comparison relation and substract smaller projections. This yields

Definition 4.1.6. Let $C$ be a category of non-crossing partitions and let $p \in C$ be a projective partition. We set

$$R_p = \sup_{q \in C, q \preceq p} S_q$$

and

$$P_p = S_p - R_p.$$

Note that it is not even clear that $P_p \neq 0$. A little linear algebra argument is needed to show that the supremum in the definition is a linear combination of maps $T_r$ with $t(r) < t(p)$:

Proposition 4.1.7. The projection $R_p$ is a linear combination of maps $T_r$ with $t(r) < t(p)$. As a consequence, $P_p \neq 0$.

Proof. We first claim that if $M$ is a direct sum of matrix algebras and $(P_i)_{i \in I}$ are orthogonal projections, then $R = \sup_{i \in I} P_i$ is a linear combination of projections $(Q_j)_{j \in J}$ such that for any $j \in J$, there exists $i \in I$ with

$$\text{Im}(Q_j) \subset \text{Im}(P_i).$$

Indeed, there is a basis $(e_i)_{1 \leq i \leq s}$ of $\text{Im}(R)$ such that for every $1 \leq \ell \leq s$, there is an index $i \in I$ such that $e_i \in \text{Im}(P_i)$. Complete this basis with an orthonormal basis of the orthogonal complement of $\text{Im}(R)$ and let $B$ be the change-of-basis matrix from this basis to the canonical basis of $C^\omega$. This means that

$$R = B^{-1} \left( \sum_{1 \leq \ell \leq s} E_{\ell} \right) B,$$

where the $(\ell, \ell)$-th coefficient of $E_{\ell}$ is 1 and all the others are 0. Setting $Q_{\ell} = B^{-1} E_{\ell} B$ for $1 \leq \ell \leq s$, we get minimal projections summing up to $R$. Moreover,

$$\text{Im}(Q_{\ell}) = C e_\ell \subset \text{Im}(P_i).$$
for some \( i \). Eventually, up to splitting the \( P_l's \) in direct sums, we may assume that they belong to one of the blocks of \( M \). Then, \( Q_\ell \in M \) for all \( \ell \) and the claim is proven.

Let now \( R_p = \sum_\ell Q_\ell \) be the decomposition given by the previous claim applied to the partitions \( (P_\ell)_{\ell \in I} = (S_q)_{q < p} \) (which is applicable since the span of the partition map is a subalgebra of \( \mathcal{L}(C^N \otimes k) \) stable under taking adjoints, hence a direct sum of matrix algebras by Corollary A). For each \( \ell \),

\[
Q_\ell = \sum_r \lambda_r S_r
\]

but since its range is contained in the range of some \( S_q \) for \( q < p \), we have

\[
Q_\ell = T_q Q_\ell = \sum_r \lambda_r S_{qr}.
\]

Because \( t(qr) \leq t(q) < t(p) \), the proof of the first statement is complete. As for the second one, it follows from the linear independence of Theorem 4.1.1. \( \blacksquare \)

### 4.2 From partitions to representations

We have just constructed orthogonal projections in the space of self-intertwiners of \( u \otimes k \), we can therefore obtain subrepresentations from this in a natural way:

**Definition 4.2.1.** For a projective partition \( p \in \mathcal{C}(k, k) \), we set

\[
\alpha_p = P_p u \otimes k P_p \in \mathcal{L} \left( P_p \left( C^N \right)^{\otimes k} P_p \right) \otimes \mathcal{O}(\mathbb{G}) \cong M_{rk(p)}(C) \otimes \mathcal{O}(\mathbb{G})
\]

The aim of this section is to study these objects in detail. We will use the following simple result several times without further reference:

**Lemma 4.2.2.** The rank of \( T_p \) is \( N^p \). As a consequence,

- For any \( r_1, r_2 \in P \), \( t(r_1 r_2) \leq \min(t(r_1), t(r_2)) \),
- If \( q < p \) are projective partition, then \( q = p \) if and only if \( t(q) = t(p) \).

**Proof.** Upper non-through-blocks in \( p \) yield equations defining the kernel of \( T_p \) and have consequently no influence on the rank. If \( p \in P(k, l) \), the image of \( T_p \) is a subspace of \( (C^N)^{\otimes l} \). Each lower non-through-block implies that some tensor factors reduce to a one-dimensional subspace and each through-block collapses all the tensor factors which are in it to one copy of \( C^N \). Hence, the image is the tensor product of one copy of \( C \) for each lower non-through-block and one copy of \( C^N \) for each through-block, i.e. the rank of \( T_p \) is \( N^p \).

The first consequence now follows from the corresponding inequality for the rank of a composition of linear maps, while the second one is clear on the associated projections. \( \blacksquare \)

#### 4.2.1 Irreducibility

We will proceed to describe the whole representation theory of \( \mathcal{G}_N(C) \) using the representations \( \alpha_p \). The first problem concerning \( \alpha_p \) is irreducibility:

**Theorem 4.2.3** The representation \( \alpha_p \) is irreducible for all projective partitions \( p \in \mathcal{C} \). Moreover, for any irreducible representation \( v \) of \( \mathcal{G}_N(C) \), there exists a projective partition \( p \in \mathcal{C} \) such that \( v \sim \alpha_p \).

**Proof.** By definition, \( \alpha_p \) is irreducible if and only if \( P_p \) is minimal, so that we have to prove that

\[
P_p \text{Mor}_{\mathcal{G}_N(C)}(u \otimes k) \left( u \otimes k \right) P_p = CP_p
\]

and it is of course enough to prove that \( P_p T_r P_p \in CP_p \) for all \( r \in \mathcal{C} \) and since \( P_p = P_p S_p \), we already know that

\[
P_p T_r P_p \in CP_p T_{rp} P_p.
\]
Note that by definition, $prp$ is an equivalence between two projective partitions dominated by $p$. If $t(prp) < t(p)$, then these two partitions are strictly dominated by $p$ by Lemma 4.2.2, hence $T_{prp}$ is dominated by $R_p$. It follows that $P_pT_pP_p = 0$. If $t(prp) = t(p)$, consider the partition $q = p_u p_u^*$. It has $t(p)$ lower points and $t(p)$ upper points. Moreover, $$p_u^* q p_u = prp$$ has $t(p)$ through-blocks so that in $q$, any lower point is connected to exactly one upper point. The only non-crossing partition having this property is $|\otimes t(p)|$, hence $q = |t(p)|$, implying $$prp = p_u^* p_u = p.$$ It follows that $$P_pT_pP_p \in CP_pT_pT_pP_p = CP_pT_{prp}P_p = CP_p.$$ To prove the second part of the statement, it is enough to show that any irreducible subrepresentation of $\mathbb{G}^\otimes k$ is equivalent to some $u_p$. By Theorem 2.1.14, this will follow from the fact that the supremum of the projections $P_p$ is the identity. So let $Q$ be this supremum and let us prove by induction on the number of through-blocks that it dominates $S_p$ for all projective partitions $p \in \mathbb{C}(k,k)$. If the number of through-blocks of $p$ is minimal, then $P_p = S_p$ is dominated by $Q$ by definition. Assume now that the result holds for all partitions with $t(p) \leq n$. Then $R_p$ is dominated by $Q$ by the induction hypothesis, as well as $P_p$ by definition. Thus, $$S_p = P_p + R_p < Q,$$ concluding the proof by induction. In particular, it dominates $T_{\otimes k} = \text{Id}$, hence is the identity. ■

We now have found all the irreducible representations of $\mathbb{G}_N(\mathbb{C})$, but our list is certainly highly redundant. In other words, our second task is to decide whether $u_p$ and $u_q$ are equivalent or not. Once again, this matches perfectly with our previous definitions. Before embarking in the proof, let us clarify a point concerning equivalence.

**Lemma 4.2.4.** Let $\mathcal{C}$ be a category of non-crossing partitions and let $p, q \in \text{Proj}_{\mathcal{C}}(k)$ be projective partitions. If $r \in \mathcal{C}$ is such that $r^* r = p$ and $rr^* = q$, then $r = q_u^* p_u$, where $p = p_u^* p_u$ and $q = q_u^* q_u$ are the through-block decompositions.

**Proof.** First note that by Lemma 4.2.2, if $p \sim q$ then $t(p) = t(q)$. The result then follows from the properties of the through-block decomposition, namely the facts that $p_u p_u^* = |t(p)|$ and $q_u q_u^* = |t(q)|$. ■

We can now characterize the equivalence of representations associated to partitions.

**Proposition 4.2.5.** Let $p$ and $q$ be projective partitions in $\mathcal{C}$. Then, $u_p \sim u_q$ if and only if $p \sim q$.

**Proof.** Assume first that $p \sim q$ and let $r \in \mathcal{C}$ be such that $r^* r = p$ and $rr^* = q$. Let $S_r$ denotes a partial isometry proportional to $T_r$ and set $$V = P_p S_r P_p.$$ By Proposition 4.1.7, $S_r^* R_q S_r$ is a linear combination of maps $T_{r^* \ell r}$ for $\ell < q$. Since $$t(r^* \ell r) \leq t(\ell) < t(q) = t(p),$$ it follows from arguments similar to the proof of Theorem 4.2.3 that $P_p S_r^* R_q S_r P_p = 0$. As a consequence, $$V^* V = P_p S_r^* S_q S_r P_p \in CP_p S_r^* q r P_p = CP_p S_p P_p = CP_p.$$
and because $V$ is a partial isometry, we must have $V^*V = P_p$. A similar argument shows that $VV^* = P_q$, hence $V$ is an equivalence between $u_p$ and $u_q$.

Conversely, assume that there exists a unitary intertwiner

$$V \in \text{Mor}_{\mathbb{G}_N(\mathbb{C})}(u_p, u_q) = P_q \text{Mor}_{\mathbb{G}_N(\mathbb{C})}(u^\otimes k, u^\otimes \ell) P_p$$

and extend it to a partial isometry $W \in \text{Mor}_{\mathbb{G}_N(\mathbb{C})}(u^\otimes k, u^\otimes \ell)$. Then, there exists partitions $r_i \in \mathcal{C}$ such that

$$W = \sum_i \lambda_i T_{r_i}.$$ 

There is at least one index $i$ such that $P_q T_{r_i} P_p \neq 0$. For such an $i$, $P_p T_{r_i^*} P_q \neq 0$. But as we saw in the proof of Theorem 4.2.3, this implies that $p(r_i^* r_i)p = p$ and since $r_i^* r_i$ is a projection, we conclude that $r_i^* r_i = p$. Doing the same reasoning for $P_q W = W$ shows that $r_i r_i^* = q$, hence $p \sim q$. 

Using these results, we can refine Theorem 4.2.3 by giving the exact decomposition of tensor powers of the fundamental representations.

**Proposition 4.2.6.** For any integer $k$,

$$u^\otimes k \sim \bigoplus_{p \in \text{Proj}_\mathcal{C}(k)} u_p.$$

**Proof.** It follows from the proof of Theorem 4.2.3 that $u^\otimes k$ contains all the representations on the right-hand side and nothing more. However, it is not clear that they are all in direct sum. To prove this, let us first denote by $E_k \subset \text{Proj}_\mathcal{C}(k)$ a set of representatives of the equivalence classes of projective partitions and by $n_k(p)$ the cardinality of the equivalence class of $p$. Consider the map

$$f : \begin{cases} \mathcal{C}(k, k) \\ r \end{cases} \to \begin{cases} \text{Proj}_\mathcal{C}(k) \\ r^* r \end{cases}$$

and note that $f^{-1}(\{p\})$ consists in all partitions of the form $r_q^p$ for projective partitions $q \sim p$. Thus, $|f^{-1}(\{p\})| = n_k(p)$ so that summing up yields the equality

$$|\mathcal{C}(k, k)| = \sum_p n_k(p) = \sum_{p \in E_k} n_k(p)^2.$$

On the other hand, $u^\otimes k$ does split into a direct sum of irreducible representation, and each of the them is equivalent to $u_p$ for some $p \in E_k$. Thus, there exists integers $\nu_k(p)$, for $p \in E_k$, such that

$$u^\otimes k \sim \bigoplus_{p \in E_k} \nu_k(p) u_p$$

so that if we prove that $\nu_k(p) = n_k(p)$ for all $p$, then we will be done. To see that, let us compute the dimension of the morphism space of both sides of the equality. The left-hand yields

$$\dim \text{Mor}_{\mathbb{G}_N(\mathbb{C})}(u^\otimes k, u^\otimes k) = \dim (\text{Vect} \{T_p \mid p \in \mathcal{C}(k, k)\}) = |\mathcal{C}(k, k)| = \sum_{p \in E_k} n_k(p)^2,$$

because we are considering non-crossing partitions and $N \geq 4$, while the right-hand is

$$\sum_{p \in E_k} \nu_k(p)^2.$$

Since, $\nu_k(p) \leq n_k(p)$ by definition, we conclude that they are equal.
4.2.2 Fusion rules

The last step to describe the representation theory is to compute the so-called fusion rules. This means that for two irreducible representations \( v \) and \( w \), we will find all the irreducible subrepresentations of \( v \otimes w \). Note that given two projective partitions \( p \in C(k, k) \) and \( q \in C(\ell, \ell) \), the intertwiner space of \( u_p \otimes u_q \) is by definition

\[
\operatorname{Mor}_{C_N(C)}(u_p, u_q) = (P_p \otimes P_q) \operatorname{Mor}_{C_N(C)}(u^{\otimes (k+\ell)}, u^{\otimes (k+\ell)}) (P_p \otimes P_q).
\]

A good starting point is therefore to find the projective partitions \( r \) such that \( (P_p \otimes P_q) P_r \neq 0 \).

**Proposition 4.2.7.** Let \( X_C(p, q) \) be the set of projective partitions \( r \preceq p \otimes q \) such that there is no projective partition \( p' \prec p \) or \( q' \prec q \) satisfying \( r \preceq p' \otimes q \) or \( r \preceq p \otimes q' \). Then, there exists a unitary equivalence

\[
u_p \otimes u_q \sim \sum_{r \in X_C(p, q)} u_r.
\]

**Proof.** We will proceed in two steps, analyzing the operators \((P_p \otimes P_q) P_r \).

1. Let us first prove that if \( r \notin X_C(p, q) \), then \((P_p \otimes P_q) P_r = 0\). Indeed,

\[
P_p \otimes P_q = (S_p - T_p) \otimes (S_q - T_q) = S_p \otimes S_q - (S_p \otimes R_q + R_p \otimes S_q - R_p \otimes R_q) = S_p \otimes S_q - (A + B - C)
\]

where \( A = S_p \otimes R_q, B = R_p \otimes S_q \) and \( C = R_p \otimes R_q \). Noticing that \( AB = BA = C \), we see that

\[
R = A + B - C
\]

is the supremum of the two commuting projections \( A \) and \( B \). In other words, \( R \) is the supremum of \( S_r \) for all \( r \notin X_C(p, q) \). Thus, if \( r \notin X_C(p, q) \) then \( S_r \) is dominated either by \( T_p \otimes R_q \) or by \( R_p \otimes T_q \), hence by \( R \), and the same holds for \( P_r \prec S_r \). It then follows that \((P_p \otimes P_q) P_r = 0\).

2. If now \( r \in X_C(p, q) \), \( P_r \) is a minimal projection by Theorem 4.2.3 hence there exists \( \lambda \geq 0 \) such that

\[
P_r P_r = \lambda P_r
\]

and because \( P_r \) is not dominated by \( R \), \( \lambda < 1 \). Thus, setting

\[
V = (P_p \otimes P_q) P_r,
\]

we have

\[
V^*V = (1 - |\lambda|^2)P_r = \mu^{-2}P_r
\]

for some \( \mu > 0 \). It follows that setting \( W = \mu V \) we get an equivalence between \( P_r \) and a subprojection of \( P_p \otimes P_q \). Therefore, the range of this subprojection yields a subrepresentation equivalent to \( u_r \).

To conclude, simply notice that by the two points of this proof,

\[
\sup_{r \in X_C(p, q)} \sup_{r \leq p \otimes q} (P_p \otimes P_q) P_r (P_p \otimes P_q) = \sup_{r \leq p \otimes q} (P_p \otimes P_q) P_r (P_p \otimes P_q) = (P_p \otimes P_q).
\]

The previous result means that any irreducible subrepresentation of \( u_p \otimes u_q \) is equivalent to \( u_r \) for some \( r \in X_C(p, q) \), and that any \( u_r \) appears at least once. But it does not say that different but equivalent \( r \)'s need both appear. To see this, we first need a better description of the set \( X_C(p, q) \). The intuitive idea is that a partition \( r \in X_C(p, q) \) is obtained by "mixing" some blocks of \( p \) with some blocks of \( q \). Concretely, this mixing is done thanks to the following partitions:
4.2. From partitions to representations

**Definition 4.2.8.** We denote by \( h^k \) the projective partition in \( NC(2k, 2k) \) where the \( i \)-th point in each row is connected to the \((2k - i + 1)\)-th point in the same row (i.e. an increasing inclusion of \( k \) blocks of size 2). If we moreover connect the outer blocks of each row, then we obtain another projective partition in \( NC(2k, 2k) \) denoted by \( h^k \).

Here is a pictorial representation of these so-called mixing partitions:

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\( h^k = \)

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From this, we define binary operations on projective partitions, using \(|\) to denote the identity partition:

\[
p \square^k q = (p_u^* \otimes q_u^*)(\otimes^{t(p)-k} \otimes h^k \otimes \otimes^{t(q)-k})(p_u \otimes q_u)
\]

\[
p \oplus^k q = (p_u^* \otimes q_u^*)(\otimes^{t(p)-k} \otimes h^k \otimes \otimes^{t(q)-k})(p_u \otimes q_u)
\]

where \( p = p_u^* p_u \) and \( q = q_u^* q_u \) are the through-block decompositions. We are now ready to complete the description of the representation theory of partition quantum groups:

**Proposition 4.2.9.** Let \( C \) be a category of non-crossing partitions, let \( N \geq 4 \) be an integer and let \( p, q \in C \) be projective partitions. Then, \( r \in X_C(p, q) \setminus \{p \otimes q\} \) if and only if there exists \( 1 \leq k \leq \min(t(p), t(q)) \) such that \( r = p \square^k q \) or \( r = p \oplus^k q \). Moreover,

\[
u_r = u_r \text{ if } r \in C \text{ and } v_r = 0 \text{ otherwise.}
\]

**Proof.** To prove the first assertion, let us consider \( r \in X_C(p, q) \) and let us denote by \( 2a \) and \( 2b \) the respective number of points of \( p \) and \( q \). We start with the partition \( p' \in \mathcal{P}(a, 0) \) formed by the first \( a \) upper points of \( r \). Then, given a block of \( p' \), we insert a lower point and connect them if one of the following two conditions is matched:

- The corresponding block in \( r \) is a through-block,
- The corresponding block in \( r \) is connected with one of the last \( b \) upper points.
Starting with the leftmost block and proceeding rightwise, we get a non-crossing partition $p^\sharp$ satisfying the conditions of Proposition 4.1.3. The same construction starting with the last $b$ points produces another partition $q^\sharp$ and we can eventually set

$$h = (p^\sharp \otimes q^\sharp)r(p^{\sharp*} \otimes q^{\sharp*}).$$

Note that by definition, $h$ is projective. Moreover, there are integers $\alpha$ and $\beta$ such that $p^{\sharp*}p^\sharp = |^{\otimes \alpha}$ and $q^{\sharp*}q^\sharp = |^{\otimes \beta}$, so that

$$r = (p^{\sharp*} \otimes q^{\sharp*})h(p^\sharp \otimes q^\sharp).$$

We now have two things to prove. First, that $h$ is of the form of Definition 4.2.8 and second that $p^\sharp = p_u$ and $q^\sharp = q_u$.

Let us consider two connected upper points of $h$. They cannot both belong to the first $\alpha$ points, because they are connected to different lower blocks in $p^{\sharp*}$ and hence not connected in $r$ by definition. The same argument works for the last $\beta$ points. This has two immediate consequences:

- Non-trough-blocks in $h$ have size at most two,
- Through-block have size two or four.

Moreover, by construction, any upper point of $p^\sharp$ gets connected when composing with $r$ either with a lower point or another upper point. As a consequence, it cannot yield a singleton in $h$. The same works for $q^\sharp$, so that we have the following possible blocks:

- Non-through-blocks of length 2 connecting one of the first $a$ points to one of the last $b$ points,
- Identity partitions,
- A through-block with four points connecting one of the first $a$ points to one of the last $b$ points on each row.

Let $n$ be the largest integer such that the point $n$ is not in a through-block. By non-crossingness, no point between $\alpha - n$ and $\beta + n$ can be in a through-block and a straightforward induction on $n$ shows that the restriction of $h$ to $\{\alpha - n, \ldots, \beta + n\}$ is the only possible nesting of pairings. Moreover, $\alpha - n - 1$ belongs to a through-block, so that again by non-crossingness, no point $\alpha - n - t$ can be connected to another upper point, forcing them to be connected by an identity partition to the lower row. As a consequence, $h$ is of the desired form.

Observe that by construction,

$$r \preceq (p^{\sharp*} \otimes q^{\sharp*})(p^\sharp \otimes q^\sharp) = (p^{\sharp*}p^\sharp) \otimes (q^{\sharp*}q^\sharp)$$

and consider the composition $(p^{\sharp*}p^\sharp)p$. Assume that two unconnected points of $p^{\sharp*}p^\sharp$ get connected. Then, the same happens in the composition $r(p \otimes q)$ so that the result cannot be $r$, contradicting the assumption that $r \preceq p \otimes q$. As a consequence,

$$(p^{\sharp*}p^\sharp)p = (p^{\sharp*}p^\sharp),$$

i.e. $p^{\sharp*}p^\sharp \preceq p$. The same reasoning shows that $q^{\sharp*}q^\sharp \preceq q$. If we can prove that both these partitions belong to $C$, then it will follow by the definition of $X_C(p, q)$ that $p^{\sharp*}p^\sharp = p$ and $q^{\sharp*}q^\sharp = q$, concluding the proof of the first assertion. Indeed, consider a nesting $s$ of $b$ pairings and consider $|^{\otimes a} \otimes s)(r \otimes |^{\otimes b})(|^{\otimes a} \otimes s)$:
4.3. Examples

The resulting partition, which belongs to \( C \) by construction, is then precisely \( p^* \circ p^* \cdot q^* \). The same argument works, mutatis mutandis, for \( q^* \circ q^* \).

As for the second part of the statement, first notice that as a consequence of the first part, the partitions in \( X_C(p, q) \) all have a different number of through-blocks. But it follows from the definition of equivalence that two equivalent projective partitions must have the same number of through-blocks. Thus, the irreducible subrepresentations of \( u_p \otimes u_q \) are pairwise non-equivalent and this forces the direct sum decomposition.

4.3 Examples

We have now done the hard work and it is high time to be repaid of our efforts by easily deducing from the previous results the representation theory of our main examples.

4.3.1 Quantum Orthogonal Group

As already mentioned, the representation theory of quantum orthogonal groups was first computed by T. Banica in [Ban96]. The proof relied on the identification of the category \( \mathcal{R}(O^+_N) \) of representations of \( O^+_N \) with the Temperley-Lieb category \( TL(N) \). With our setting, the result follows from elementary calculations.

**Theorem 4.3.1** (Banica) For \( N \geq 2 \), the irreducible representations of \( O^+_N \) can be labelled by the non-negative integers in such a way that \( u^0 = \varepsilon \), \( u^1 = U \) and for any \( n \in \mathbb{N} \),

\[
 u^1 \otimes u^n = u^{n+1} \oplus u^{n-1}
\]

**Proof.** Recall that if two projective partitions \( p \) and \( q \) are equivalent, then \( t(p) = t(q) \). Assume now that \( p \) and \( q \) are non-crossing projective pair partitions with \( t(p) = t(q) \). Denoting by \( p = p_u p_a \) and \( q = q_u q_a \) their through-block decompositions, observe that \( p_u, q_u \in NC_2 \), so that \( r = q_u p_a \in NC(2) \). Since \( r^* r = p \) and \( r r^* = q \), we have proven that the equivalence class of \( p \) is given by its number of through blocks. Setting \( u^n = u_{| \otimes n} \) therefore gives all irreducible representations.

Since the empty partition corresponds to the trivial representation, \( u^0 = \varepsilon \). Since \( NC_2(1, 1) \) only contains the identity partition, \( P_1 = S \) = Id so that \( u^1 = U \). Eventually, \( t([\square] \otimes n) = n - 1 \) and \( [\square] \otimes n \notin NC_2 \), hence the fusion rules.
4.3.2 Quantum permutation group

The case of quantum permutation groups was studied by T. Banica in [Ban99b]. Once again, his strategy relied on a variant of the Temperley-Lieb category where the number of strands is doubled. Using non-crossing partition, the proof is almost the same as for \( O_N^+ \).

**Theorem 4.3.2 (Banica)** For \( N \geq 4 \), the irreducible representations of \( S_N^+ \) can be labelled by the non-negative integers in such a way that \( u^0 = \varepsilon \), \( P = \varepsilon \oplus u^1 \) and for any \( n \in \mathbb{N} \),

\[
 u^1 \otimes u^n = u^{n+1} \oplus u^n \oplus u^{n-1}
\]

**Proof.** The same argument as for \( O_N^+ \) shows that equivalence classes of projective partitions in \( NC \) correspond to the number of through-blocks, so that we set \( u^n = u_{\otimes n} \) and \( u^0 = \varepsilon \). However, this time \( NC(1,1) \) has two elements, namely the identity partition \( | \) and the double singleton partition \( \{\{1\}, \{2\}\} \). The second one gives a copy of the trivial representation since the singleton partition is an equivalence between it and the empty partition. The announced decomposition of \( P \) then follows. Eventually, \( t(| \otimes^n) = n - 1 \) and \( t(| \otimes | \otimes^n) = n \), hence the fusion rules. \( \blacksquare \)

4.3.3 Quantum hyperoctaedral group

The two preceding examples may have given the impression that the number of through-blocks is the only important data of a projective partition. To show that this is not the case, let us consider the quantum hyperoctaedral group \( H_N^+ \). Recall that the corresponding category of partitions consists in all even partitions. Let \( p \in NC_{\text{even}}(2,2) \) be the partition with only one-block. Then, \( t(p) = 1 \)

\[
 r_p^u | \in NC(1,2)
\]

is a block of size three. Thus, \( u_p \) is not equivalent to \( u_1 \) and this turns out to be the only obstruction to equivalence:

**Exercise 19.** Prove that the irreducible representations of \( H_N^+ \) can be indexed by words on \( \{0,1\} \), with \( \varepsilon = u^0 \), \( u = u^1 \) and \( u_p = u^0 \).

**Solution.** Let \( p \in NC(k,k) \) and \( q \in NC(k',k') \) have only one block. Then, \( p^*q_a \in NC(k',k) \) is even if and only if \( k \) and \( k' \) have the same parity. If now \( p \in NC_{\text{even}} \), it follows from a straightforward induction that we can write

\[
p = p_1 \otimes \cdots \otimes p_t(p)
\]

where for all \( 1 \leq i \leq t(p) \), \( p_i \) is a projective partitions with \( t(p_i) = 1 \). If \( \tilde{p}_i \) is the one-block partitions on the same number of points as \( p_i \), then

\[
p \sim \tilde{p}_1 \otimes \cdots \otimes \tilde{p}_t(p)
\]

and we conclude that we can label the irreducible representations as in the statement. The last thing to check is that different labels do not yield equivalent partitions. Let \( w \neq w' \) be different words and set \( p^w = p_{w_1} \otimes \cdots \otimes p_{w_n} \), where \( p_1 = | \) and \( p_0 = p \). Let \( i \) be an integer such that \( w_i \neq w'_i \). Then, \( p_d^w p_a^{w'} \) contains as a subpartition \( p_{w_{id}} q_{w_{ia}} \) which is a block of size three. Thus, the partitions are not equivalent. \( \blacksquare \)

We still have to compute the fusion rules, and here again a surprise awaits us. Applying Proposition 4.2.9, we see that

\[
 u^1 \otimes u^0 = u^{10} \oplus u^1 \oplus \varepsilon
\]

\[
 u^0 \otimes u^1 = u^{01} \oplus u^1 \oplus \varepsilon
\]

Because \( u^{10} \neq u^{01} \), the fusion rules are non-commutative! This is a purely quantum phenomenon. The complete description of the representation theory of \( H_N^+ \) was established by T. Banica and R. Vergnioux in [BV09]. To state it conveniently we need some notations. Let \( W \) be the set of words on \( \{0,1\} = \mathbb{Z}_2 \) and endow it with the following operations:
\[ w_1 \cdots w_n = w_n^{-1} \cdots w_1^{-1}, \]
\[ w_1 \cdots w_n w'_1 \cdots w'_m = w_1 \cdots w_n w'_1 \cdots w'_m, \]
\[ w_1 \cdots w_n \ast w'_1 \cdots w'_m = w_1 \cdots w_{n-1}(w_n + w'_1)w'_2 \cdots w'_m, \]

**Theorem 4.3.3** (Banica-Vergnioux) The irreducible representations of \( H_\mathbb{N}^+ \) can be indexed by \( W \) in such a way that \( \varepsilon = u^\emptyset, u = u^1 \) and \( u_p = u^0 \). Moreover, given two words \( w, w' \in W \), we have

\[
 u^w \otimes u^{w'} = \bigoplus_{w = a \cdot z, w' = z \cdot b} u^{a \cdot b} \oplus u^{a \ast b}.
\]

**Proof.** The first part of the statement was proven in Exercise 19. As for the fusion rules, let us consider two words \( w = w_1 \cdots w_n \) and \( w' = w'_1 \cdots w'_m \) and let \( k \leq \max(m, n) \). Setting \( p_0 = p, p_1 = \varepsilon \) and

\[
 p_w = p_{w_0} \otimes \cdots \otimes p_{w_n},
\]

we see that in \( p_w \Box^k p_{w'} \), \( p_{w_n-i+1} \) is glued to \( p_{w'_i} \). But if \( w_{n-i+1} \neq w'_i \), then this yields a block of size three, which therefore does not belong to \( NC_{\text{even}} \). Thus, \( p_w \Box^k p_{w'} \in NC_{\text{even}} \) if and only if the last \( k \) letters of \( w \) match the first \( k \) letters of \( w' \), i.e. \( w = a \cdot z \) and \( w' = z \cdot b \) with \( z \) of length \( \varepsilon \). Moreover, the through blocks of \( p_w \Box p_{w'} \) are just the first \( n - i \) through-blocks of \( p_w \) followed by the last \( m - i \) through-blocks of \( p_{w'} \) and this corresponds to \( p_{a \cdot b} \). Considering now \( p_w \Box p_{w'} \), the same decomposition is needed for it to be in \( NC_{\text{even}} \), but this time the result contains an extra through-block obtained by gluing \( p_{w_n-k+1} \) and \( p_{w'_i} \). If they have the same parity, then the result has eight points hence is equivalent to \( p_0 \) while it is equivalent to \( p_1 \) if their parity differ. Hence, this is equivalent to \( p_{a \ast b} \).

Note eventually that for \( u^w \otimes u^{w'} \) to contain the trivial representation \( u^\emptyset \), one must have \( w' = \overline{w} \). Hence, \( \overline{w^w} = u^{\overline{w}} \). \( \square \)
CHAPTER 5
MEASURABLE AND TOPOLOGICAL ASPECTS

We have taken the bias, throughout the previous chapters, to work exclusively in an algebraic setting. This may seem odd given that the name “compact quantum groups” suggest that they are compact objects. Of course, we are focusing on compact matrix groups which can be described algebraically by polynomial equations, but there should still be some kind of topological structure hidden somewhere. Making sense of this topological structure is a non-trivial matter, mainly because it is a non-commutative topological structure. More precisely, to any compact matrix quantum group one can associate a non-commutative topological space in the sense of non-commutative geometry, as well as a non-commutative measure space. To explain this, we will therefore need to introduce some elements from the theory of operator algebras and non-commutative geometry.

5.1 SOME CONCEPTS FROM NON-COMMUTATIVE GEOMETRY

Let us start by thinking about the case of a compact matrix group $G$. The topology is essentially witnessed by the phenomenon of continuity, hence one may look at the algebra $C(G) = C^0(G, \mathbb{C})$ of continuous complex-valued functions on $G$. The first remark is that $C(G)$ contains $O(G)$ as a $\ast$-subalgebra and that the latter is dense with respect to the uniform norm $\| \cdot \|_\infty$. As a consequence, the normed algebra $(C(G), \| \cdot \|_\infty)$ at least contains all algebraic information about the group $G$, plus some topological data.

There are two important things to note concerning this normed algebra. First, it is “nice” in the sense that it is a Banach algebra\(^1\). Second, the involution on $O(G)$ (given by complex conjugation) extends to $C(G)$, turning it into a Banach $\ast$-algebra. In order to see why this suffices to unravel the topology of $G$, let us describe an alternate way of recovering the group.

Any point $g \in G$ yields through evaluation a character, that is to say a continuous $\ast$-homomorphism $ev_g : C(G) \to \mathbb{C}$. Conversely, if $\sigma$ is a character on $C(G)$, it is completely determined by the image of the coefficient functions $u_{ij}$. Let us denote by $g_\sigma$ the matrix of the images, i.e.

$$(g_\sigma)_{ij} = \sigma(u_{ij}).$$

Because $G$ is a matrix group, there is an ideal $I \subset \mathbb{C}[X_{ij}, 1 \leq i, j \leq N]$, such that $O(G)$ is the quotient by $I$. Because $\sigma$ is an algebra homomorphism, for any polynomial $P \in I$,

$$P((g_\sigma)_{ij}) = P(\sigma(u_{ij})) = \sigma(P(u_{ij})) = 0$$

hence $g_\sigma$ is annihilated by $I$. As we have seen in Exercise 6, this means that $g_\sigma \in G$. In other words, we have a bijective correspondence between

\(^1\) Recall that a Banach algebra is a complete normed algebra $A$ such that $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in A$. 

- The set \( \text{Char}(G) \) of characters on \( C(G) \),
- The set \( \text{Char}^*(G) \) of characters on \( \mathcal{O}(G) \),
- The elements of \( G \).

The previous construction has a bonus: characters are in particular linear maps of norm at most one (this is true for any C*-algebra, see below), i.e. points of the unit ball of \( C(G)^* \). Clearly, \( \text{Char}(G) \) is a closed subset for the weak-\( * \) topology, hence a compact topological space by the Banach-Alaoglu Theorem (see for instance [Con94b, Thm V.3.1]).

We are now ready for the coup de grâce: the topology induced on \( \text{Char}(G) \) by the weak-\( * \) topology coincides, through the set theoretic isomorphism constructed above, with the original topology on \( G \! \)!

5.1.1 C*-algebras

Our purpose now is to abstract the crucial features of the Banach \( * \)-algebra \((C(G), \| \cdot \|_\infty)\) which enabled to recover the topology of \( G \). It has long been recognised that the correct notion to do so is that of a C*-algebra, which we will now define.

**Definition 5.1.1.** A C*-algebra is a complex Banach algebra \( A \) endowed with an involution \( x \mapsto x^* \) such that for all \( x, y \in A \),

- \((x + y)^* = x^* + y^* \),
- \((\lambda x)^* = \overline{\lambda} x^* \) for all \( \lambda \in \mathbb{C} \),
- \((xy)^* = y^* x^* \),
- \(\|x^* x\| = \|x\|^2 \).

Here are the two basic examples of C*-algebras:

**Example 5.1.2.** Let \( X \) be a compact topological space. Then, \( C(X) \) endowed with the uniform norm \( \| \cdot \|_\infty \) and with the involution such that \( f^*(x) = \overline{f(x)} \) is a commutative unital C*-algebra.

**Example 5.1.3.** Let \( H \) be a Hilbert space and let \( A \subset B(H) \) be a closed subalgebra which is stable under taking adjoints. Then, \( A \) is a C*-algebra when endowed with the operator norm and the involution given by adjunction.

The two fundamental results of the theory is that the general case reduces to the above examples. Since this is not our main purpose here, we will not give proofs and refer to the vast literature instead. Of utmost important for us is the commutative case, since it is inspirational for the field of non-commutative geometry (see for instance the book [Con94a] for numerous motivation and results).

**Theorem 5.1.4** (Gelfand-Naimark) Let \( A \) be a commutative unital C*-algebra. Then, there is a compact topological space \( X \) such that \( A \simeq C(X) \).

**Proof.** We refer the reader, among other possible sources, to [Con00, Thm 2.1] for a complete proof. Let us however mention that the basic strategy is exactly the one outlined for compact matrix groups \( G \). One considers the weak-\( * \) compact set \( X \) of characters on \( A \). Evaluation yields a map \( A \to C(X) \) and the job is to prove that it is an isomorphism. 

**Remark 5.1.5.** The result is even stronger: \( X \mapsto C(X) \) is a contravariant equivalence between the category of compact topological spaces and that of commutative unital C*-algebras. As a consequence, one can think of general unital C*-algebras as non-commutative compact spaces and this is the intuition behind non-commutative geometry. It is for instance possible to adapt using this idea tools from algebraic topology, the prominent example being K-theory.

The second main result is that Example 5.1.3 is in fact completely general. We will not need it in the sequel but it has such a strong theoretical interest that we feel compelled to state it.
5.1. Some concepts from non-commutative geometry

**Theorem 5.1.6** Let $A$ be a $C^*$-algebra. Then, there exists a Hilbert space $H$ such that $A$ embeds into $B(H)$.

*Proof.* See [Con00, Thm 10.7]. The proof relies on the so-called Gelfand-Naimark-Segal (GNS in short) construction which will be outlined in the end of Section 5.1.2. ■

5.1.2 Von Neumann algebras

We now have a notion of non-commutative topological space, but what about measure theory? It turns out that the same circle of ideas leads to a satisfying notion of a non-commutative measure space. More precisely, consider again the algebra $C(X)$ for a compact space $X$. If $\mu$ is a Borel probability measure on $X$, then we can define a $\ast$-homomorphism

$$L_\mu : C(X) \to B\left(L^2(X,\mu)\right)$$

given by left multiplication. It is easy to see that this is an embedding. Moreover, it is an exercise to prove that the closure of $L_\mu(C(X))$ with respect to the weak operator topology\(^2\) is exactly the algebra $L^\infty(X,\mu)$ of essentially bounded $\mu$-measurable functions on $X$.

Now, there are two important remarks to make. First, $L^\infty(X,\mu)$ does not depend on $\mu$ but only on the corresponding $\sigma$-algebra, which is itself determined by the topology of $X$. Hence, it reflects the measure space structure of $X$. Second, $L^\infty(X)$ is a $C^*$-algebra, but it is even better since it is also closed with respect to the weak operator topology.

**Definition 5.1.7.** A von Neumann algebra is a $C^*$-algebra $A$ embedded into $B(H)$ for some Hilbert space $H$ such that it is closed with respect to the weak operator topology.

As for $C^*$-algebras, the commutative case is as general as can be (see [Con00, Thm 14.5] for a proof), provided we add some non-degeneracy assumptions on the way the algebra is embedded into $B(H)$.

**Theorem 5.1.8** (Gelfand-Naimark) Let $A \subset B(H)$ be a commutative von Neumann algebra and assume that there is a vector $\xi \in H$ which is

- Cyclic in the sense that the closure of $A.\xi$ equals $H$,
- Separating in the sense that there is no non-zero operator $T \in A$ such that $T(\xi) = 0$.

Then, there exists a measure space $(X,\mu)$ and an isomorphism

$$\Phi : B(H) \to B\left(L^2(X,\mu)\right)$$

such that $\Phi(A) = L^\infty(X,\mu)$.

**Remark 5.1.9.** It may help intuition to know that the above isomorphism is the conjugation by an isomorphism of the corresponding Hilbert spaces, the latter sending the vector $\xi$ to the indicator function $1_X$ of $X$.

The drawback of von Neumann algebras is of course that they require an ambient Hilbert space, which is obtained in the commutative case by choosing a measure. To understand what the analogue of a measure is in the non-commutative case, let us replace $\mu$ by the linear form $\varphi_\mu : C(X) \to \mathbb{C}$ given by

$$\varphi_\mu(f) = \int_X f d\mu.$$  

Here are three basic observations:

1. The measure $\mu$ is positive if and only if $\varphi_\mu(f^*f) = \varphi_\mu(|f|^2) \geq 0$ for all $f \in C(X)$,

\(^2\) This is the topology for which a sequence $(T_n)_{n \in \mathbb{N}}$ of operators converges to $T$ if and only if for any $g,h \in L^2(X,\mu)$, $(T_n(g),h)$ converges to $(T(g),h)$. 

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2. The measure $\mu$ has total mass 1 if and only if $\varphi_\mu(1) = 1$,
3. The measure $\mu$ has full support if and only if $\varphi_\mu(f^*f) > 0$ for all $f \neq 0$.

Based on these, we can define suitable properties of linear maps to mimic measure theory:

**Definition 5.1.10.** Let $A$ be a $*$-algebra. A linear map $\varphi : A \to \mathbb{C}$ is said to be
1. **Positive** if $\varphi(x^*x) \geq 0$ for all $x \in A$,
2. **Unital** if $\varphi(1) = 1$,
3. **Faithful** if $\varphi(x^*x) > 0$ for all $x \neq 0$.

A positive unital linear map is called a **state**.

Using this, we can build a non-commutative measure space from a $*$-algebra and a state. This is the celebrated **GNS construction**, which we will now explain without proofs (see [Con00, Thm 7.7] for details). Starting with a $*$-algebra $A$ and a state $\varphi : A \to \mathbb{C}$, we first define a pre-inner product on $A$ through the formula

$\langle x, y \rangle = \varphi(xy^*)$.

The problem is that it may be degenerate. To remedy this, consider the annihilator subspace

$N_\varphi = \{ x \in A \mid \varphi(xy^*) = 0, \forall y \in A \}$.

Quotienting by this space and completing then yields a **bona fide** Hilbert space, denoted by $L^2(A, \varphi)$. Moreover, left multiplication yields a $*$-homomorphism

$L_\varphi : A \to \mathcal{B}(L^2(A, \varphi))$ and taking the weak operator topology closure of the image yields a von Neumann algebra which deserves to be denoted by $L^\infty(A, \varphi)$.

### 5.2 The quantum Haar measure

Our goal in the present section is to construct a suitable non-commutative topological and measurable structure associated to a compact matrix quantum group $\mathbb{G}$. As outlined in the previous section, the construction of a von Neumann algebra can be done using a state playing the rôle of a probability measure. Since compact groups have a canonical probability measure, namely the Haar measure, it is natural to wonder whether there is a similar canonical state on a compact matrix quantum group.

#### 5.2.1 A discrete detour

A first idea to produce a non-commutative topological structure on a compact matrix quantum group $\mathbb{G}$, could be to consider simply the largest possible $C^*$-algebra containing $\mathcal{O}(\mathbb{G})$ as a dense $*$-subalgebra, which is called the **universal enveloping $C^*$-algebra** of $\mathcal{O}(\mathbb{G})$. The existence of such an object is routine to prove (see for instance [Bla06, II.8.3]) and the coproduct extends to it by universality. To understand why this is unsatisfying, let us give examples of compact matrix quantum group of a different type than those encountered up to now.

Let $\Gamma$ be a finitely generated discrete group\(^3\) with a generating set $S = \{g_1, \ldots, g_N\}$. Its **group algebra** is the vector space $\mathbb{C}[\Gamma]$ freely spanned by elements $a_g$ for $g \in \Gamma$ with the unique algebra structure such that

$a_g a_h = a_{gh}$

and the unique involution given by

$a_g^* = a_{g^{-1}}$.

---

3. By this we simply mean a group equipped with the discrete topology.
5.2. The quantum Haar measure

To see the point of considering that algebra in our context, assume moreover that $\Gamma$ is abelian, and let $\hat{\Gamma}$ be the set of all homomorphisms from $\Gamma$ to the circle $\mathbb{T}$, called the Pontryagin dual of $\Gamma$. This is obviously a group, and it is easily checked to be compact when equipped with the compact-open topology. Let us denote it by $\hat{\Gamma}$. We claim that this is a compact group of matrices. Indeed, consider the group homomorphism

$$\varphi \in \hat{\Gamma} \mapsto \text{diag}(\varphi(a_{g_1}), \ldots, \varphi(a_{g_N})) \in M_N(\mathbb{C}).$$

It is injective because $S$ is generating, hence an isomorphism onto its image. Moreover, it follows that the algebra $\mathcal{O}(\hat{\Gamma})$ is isomorphic to the group algebra $C[\Gamma]$.

Based on this, we can define a compact matrix quantum group out of an arbitrary discrete group. Let us denote by $u_S$ the matrix

$$u = \text{diag}(g_1, \ldots, g_N) \in M_N(C[\Gamma]).$$

Proposition 5.2.1. The pair $(C[\Gamma], u_S)$ forms a compact unitary matrix quantum group in the sense of Definition 1.3.6.

Proof. First, it is clear that the coefficients of $u_S$ generate $C[\Gamma]$. Second, we have $u_S^{-1} = u_S^*$ by definition. Eventually, the map $\Delta : C[\Gamma] \to C[\Gamma] \otimes C[\Gamma]$ such that $\Delta(a_g) = a_g \otimes a_g$

satisfies the conditions for the coproduct. ■

Remark 5.2.2. Note that such a compact matrix quantum group is orthogonal if and only if the generators can be chosen to have order 2. Thus, we need the theory of unitary compact matrix quantum group to deal with these objects in general.

By analogy with the abelian case, we will denote this compact matrix quantum group by $\hat{\Gamma}$ and call it the dual of $\Gamma$. The enveloping C*-algebra $C[\Gamma]$ is nothing but the universal group C*-algebra $C^*(\Gamma)$. It is well-known that this is not the correct operator algebra to look at when investigating geometric or probabilistic properties of $\Gamma$ (like amenability, property (T) and so on). One should instead consider the reduced C*-algebra $C^*_r(\Gamma)$ and its associated von Neumann algebra $L(\Gamma)$.

To construct these, let us consider the Hilbert space $\ell^2(\Gamma)$ of square summable sequences of complex numbers indexed by the elements of $\Gamma$. Denoting by $\delta_g$ the sequence which is 1 at $g$ and 0 elsewhere, we can define for any $g \in \Gamma$ a bounded operators $\lambda_g \in \mathcal{B}(\ell^2(\Gamma))$ by

$$\lambda_g(\delta_h) = \delta_{gh}.$$ 

This yields a *-homomorphism

$$\lambda : C[\Gamma] \to \mathcal{B}(\ell^2(\Gamma))$$

which is easily seen to be injective. Taking completions of $\lambda(C[\Gamma])$ for the operator norm and the weak operator topology yield respectively $C^*_r(\Gamma)$ and $L(\Gamma)$.

For a general compact matrix quantum group, we do not have a priori an analogue of $\ell^2(\Gamma)$ to build a reduced version. This is where the Haar measure enters the game by providing an alternate construction. Let us first find the expression of the Haar integral of $\hat{\Gamma}$.

Proposition 5.2.3. The Haar integral of $\hat{\Gamma}$ is the the linear form $\delta_e : C[\Gamma] \to \mathbb{C}$ given by

$$\delta_e(a_g) = \delta_{e,g}.$$ 

Proof. This is a straightforward computation :

$$(\delta_e \otimes \text{id}) \circ \Delta(a_g) = \delta_e(a_g)a_g = \delta_{eg,a_e} = \delta_e(a_g).1$$

and similarly the other way round. ■
The key fact now is that the GNS representation of \( \delta_e \) exactly yields the previous reduced operator algebras acting on \( l^2(\Gamma) \). To find an analogue of this, first note that the irreducible representations of \( \widehat{\Gamma} \) are all one-dimensional and given (up to equivalence) by the elements of \( g \). Since \( \delta_e \) vanishes by definition on all elements \( g \neq e \), the following definition is natural:

**Definition 5.2.4.** Let \( G \) be a unitary compact matrix quantum group, let \( \text{Irr}(G) \) be the set of equivalence classes of irreducible representations of \( G \) and let \( u^\alpha \) be a representative of \( \alpha \in \text{Irr}(G) \). The **Haar integral** of \( G \) is the linear map \( h : \mathcal{O}(G) \to \mathbb{C} \) defined on the basis of coefficients of irreducible representations by

\[
h\left(u^\alpha_{ij}\right) = \delta_e(\alpha).
\]

The name needs some explanations. To understand where it comes from, first note that for any \( \alpha \in \text{Irr}(G) \) and any \( 1 \leq i, j \leq \dim(\alpha) \),

\[
(h \otimes \text{id}) \circ \Delta\left(u^\alpha_{ij}\right) = h\left(u^\alpha_{ij}\right) \cdot 1 = (\text{id} \otimes h) \circ \Delta\left(u^\alpha_{ij}\right).
\]

This is an invariance property which recalls that of the Haar measure on a classical compact group. In particular if \( h' \) is another linear form on \( \mathcal{O}(G) \) satisfying Equation (5.1), then

\[
h(x) = h(x)h'(1) = h'((h \otimes \text{id}) \circ \Delta(x)) = (h \otimes h') \circ \Delta(x) = h'(x)h(1) = h'(x).
\]

With this we can give a better justification of the name “Haar integral”:

**Exercise 20.** Prove that if \( G \) is a compact matrix group, then the Haar integral on \( \mathcal{O}(G) \) coincides with the integration with respect to the Haar measure.

**Solution.** Notice first that for any \( f \in \mathcal{O}(G) \), \( \Delta(f)(x,y) = f(xy) \). If now \( \varphi \) denotes the linear map given by integration with respect to the Haar measure \( \mu \), then

\[
(\varphi \otimes \text{id}) \circ \Delta(f)(y) = \int_G f(xy)d\mu(x) = \int_G f(x)d\mu(x) = \varphi(f).1
\]

and the same works for \( (\text{id} \otimes \varphi) \circ \Delta \). As explained above, this implies that \( \varphi = h \).

### 5.2.2 Positivity of the Haar Integral

The only non-trivial fact about \( h \) is that it is positive. This will indeed be the first important result of this chapter. To prove it, let us first give a straightforward, though important, corollary of Theorem 2.1.14:

**Corollary 5.2.5.** Let \( G \) be an orthogonal compact matrix quantum group. For any irreducible unitary representation \( v, \overline{v} = v^{\ast t} \) is also an irreducible unitary representation.

**Proof.** By Corollary 2.1.16, \( v \) is unitarily equivalent to a subrepresentation of \( u^\otimes k \) for some \( k \in \mathbb{N} \). Thus, there exist a unitary matrix \( B \) and a unitary representation \( w \) satisfying

\[
B^*u^\otimes kB = \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix}.
\]

As a consequence, \( v^{\ast t} \) is a direct summand of \( B^* u^\otimes k B^{*t} \), hence is a unitary representation. If \( V \) is the subspace of \( \left(C^N\right)^{\otimes k} \) on which \( v \) acts, then \( \overline{v} \) naturally acts on the conjugate Hilbert space \( \overline{V} \). Moreover, \( T \) intertwines \( v \) with itself if and only if \( \overline{T} = T^{\ast t} \) intertwines \( \overline{v} \) with itself. Thus,

\[
\dim(\text{Mor}_G(v,v)) = \dim(\text{Mor}_G(v,v)) = 1
\]

and \( \overline{v} \) is irreducible.
**Definition 5.2.6.** Let $G$ be a compact matrix quantum group and let $v = (v_{ij})_{1 \leq i,j \leq n}$ be an irreducible representation acting on $V = \mathbb{C}^n$. Then, $\overline{v} = (v^*_{ij})_{1 \leq i,j \leq n}$ is called the **conjugate representation** of $v$. If $v$ is any unitary representation and

$$v = \bigoplus_{i=1}^{n} v^i$$

is a decomposition into irreducible subrepresentations, then its conjugate representation is

$$\overline{v} = \bigoplus_{i=1}^{n} \overline{v}^i.$$

Let us now proceed to prove that the Haar integral is a state. The Haar state for compact matrix quantum groups was first constructed by S.L. Woronowicz in [Wor87] in a completely different way: he directly proved the existence of a state satisfying Equation (5.1). Our more algebraic approach is due to [DK94].

**Theorem 5.2.7** (Woronowicz) Let $(G, u)$ be an orthogonal compact matrix quantum group. Then, the Haar integral $h$ is a state on $O(G)$. Moreover, it is faithful.

From now on, $h$ will be called the **Haar state** of $G$.

**Proof.** It turns out that positivity and faithfulness will be proved at the same time. For the sake of clarity, we proceed in several steps.

1. Let $v$ be any finite-dimensional representation acting on a Hilbert space $V$. We claim that the matrix

$$\hat{v} = (h(v_{ij}))_{1 \leq i,j \leq \dim(V)}$$

is the matrix of the orthogonal projection onto the subspace of fixed vectors of $\rho_v$. Indeed, let $W$ be the subspace of fixed vectors. The proof of the first point in Theorem 2.1.14 shows that $v$ decomposes along $W \oplus W^\perp$. Since $v_{W^\perp}$ does not contain the trivial representation, $\hat{h}(v)$ vanishes on $W^\perp$, while it acts by the identity on $W$ by definition. Thus it is the orthogonal projection onto the subspace of fixed vectors.

2. Let us show that for any irreducible representation $v$ acting on $V$, the vector

$$\xi = \sum_{i=1}^{\dim(V)} e_i \otimes \overline{v}_i \in V \otimes \overline{V}$$

is fixed for $v \otimes \overline{v}$. Indeed,

$$\rho_v \otimes \overline{v}(\xi) = \sum_{i=1}^{\dim(V)} \sum_{j,k=1}^{\dim(V)} v_{ij} v^*_kj e_j \otimes \overline{v}_k$$

$$= \sum_{i=1}^{\dim(V)} \sum_{j,k=1}^{\dim(V)} \delta_{j,k} e_j \otimes \overline{v}_k$$

$$= 1 \otimes \xi.$$ 

3. We now claim that for any $\alpha, \beta \in \text{Irr}(G)$,

$$\text{Mor}_G \left( \varepsilon, v^\alpha \otimes v^\beta \right) \simeq \text{Mor}_G \left( v^\beta, v^\alpha \right).$$

Indeed, if $T \in B \left( V^\beta, V^\alpha \right)$, then

$$\xi_T = \sum_{i=1}^{\dim(v^\beta)} T(e_i) \otimes \overline{e}_i$$
is fixed for \( v^\alpha \otimes \overline{v^\beta} \) if and only if \( T \) is an intertwiner, where \((e_i)_{1 \leq i \leq \dim(v^{(\beta)})}\) is an orthonormal basis. This is a simple computation:

\[
\left( \rho_{v^\alpha \otimes \overline{v^\beta}} \right) (\xi_T) = \sum_{i=1}^{\dim(v^{(\beta)})} \rho_{v^\alpha} \circ T(e_i) \otimes \rho_{\overline{v^\beta}}(e_i) = \sum_{i=1}^{\dim(v^{(\beta)})} (\id \otimes T) \circ \rho_{v^\beta}(e_i) \otimes \rho_{\overline{v^\beta}}(e_i)
\]

\[
= (\id \otimes T) \circ \rho_{v^\beta \otimes \overline{v^\beta}} \left( \sum_{i=1}^{\dim(v^{(\beta)})} e_i \otimes \overline{e_i} \right) = (\id \otimes T) \left( \sum_{i=1}^{\dim(v^{(\beta)})} 1 \otimes e_i \otimes \overline{e_i} \right) = 1 \otimes \xi_T.
\]

Conversely, \( \xi \in V^\alpha \otimes \overline{V^\beta} \) is a fixed vector for \( v^\alpha \otimes \overline{v^\beta} \), if and only if

\[
T_\xi : r \mapsto (\id \otimes \overline{\rho^*})\rho^*(\xi)
\]

is an intertwiner between \( v^\beta \) and \( v^\alpha \), because \( \xi_{T_\xi} = \xi \) and \( T_{\xi_T} = T \).

4. It follows from Schur’s Lemma (Lemma 2.1.13) that \( \text{Mor}_G \left( v^{(\beta)}, v^{(\alpha)} \right) \{ 0 \} \) if \( \alpha \neq \beta \). In particular,

\[
\hat{h} \left( v^{(\alpha)} \otimes \overline{v^{(\beta)}} \right) = 0
\]

in that case.

5. For \( \alpha = \beta \), the space \( \text{Mor}_G \left( \varepsilon, v^{(\alpha)} \otimes \overline{v^{(\alpha)}} \right) \) is one-dimensional (again by Schur’s Lemma) and is spanned by

\[
\xi = \sum_{i=1}^{\dim(v^{(\alpha)})} e_i \otimes \overline{e_i}.
\]

Thus, if \( P_\xi \) denotes the orthogonal projection onto \( \mathbb{C}\xi \), then

\[
\hat{h} \left( v_{ij}^{(\alpha)} \otimes v_{kl}^{(\alpha)\ast} \right) = \langle e_j \otimes \overline{e_l}, P_\xi (e_i \otimes \overline{e_k}) \rangle = \delta_{ik} \delta_{jl} \frac{1}{\dim(v^{(\alpha)})}.
\]

6. If now

\[
x = \sum_{\alpha \in \text{Irr}(G)} \sum_{i,j=1}^{\dim(v^{(\alpha)})} \lambda_{ij}^{\alpha} v_{ij}^{(\alpha)},
\]

then

\[
\hat{h}(xx^\ast) = \sum_{\alpha \in \text{Irr}(G)} \sum_{i,j=1}^{\dim(v^{(\alpha)})} |\lambda_{ij}^{\alpha}|^2 \geq 0
\]

and this vanishes if and only if \( x = 0 \).

Remark 5.2.8. It follows from the proof above that \( h \) is tracial in the sense that \( h(xy) = h(yx) \) for any \( x, y \in \mathcal{O}(G) \).

Before commenting on the consequences of Theorem 5.2.7, let us highlight an important fact concerning conjugate representations which were established along the way:
Proposition 5.2.9. Let $v$ and $w$ be irreducible representations of an orthogonal compact quantum group $G$. Then, the representation $w \sim v$ if and only if $\text{Mor}_G(\varepsilon, v \otimes w) \neq \{0\}$. Moreover, for any representations $w, w_1$ and $w_2$,

$$\text{Mor}_G(v, w_1 \otimes w_2) \simeq \text{Mor}_G(v \otimes w_2, w_1).$$

The latter property is called Frobenius reciprocity.

**Proof.** The first assertion is a restatement of Point (3) in the proof of Theorem 5.2.7. As for the second one, the proof was done in the particular case where $v = \varepsilon$ in Point (3) in the proof of Theorem 5.2.7. In the general case the proof is similar. If $V, W_1$ and $W_2$ denote the carrier Hilbert spaces of $v, w_1$ and $w_2$ respectively, and if $T : V \to W_1 \otimes W_2$, define $\tilde{T} : V \otimes \overline{W}_2 \to W_1$ by

$$\langle \tilde{T}(x \otimes y_2), y_1 \rangle = \langle T(x), y_1 \otimes y_2 \rangle.$$ 

It is a straightforward computation to check that $T$ is an intertwiner if and only if $\tilde{T}$ is. □

Applying the GNS construction to $\mathcal{O}(G)$ and the state $h$, we get a *-homomorphism

$$L_h : \mathcal{O}(G) \to \mathcal{B}(L^2(G))$$

which is faithful because $h$ is faithful. The weak closure of the range of $L_h$ is then a von Neumann algebra denoted by $L_k^\infty(G)$.

**Remark 5.2.10.** It is possible to start the theory of compact quantum groups from the non-commutative topological point of view. Starting with a C*-algebra endowed with a coproduct, one then seeks a condition ensuring that this “compact quantum semigroup” is indeed a group. This was done by S.L. Woronowicz in [Wor98]. The reader may also refer to [MVD98] for alternate proofs or to [Tim08, Chap II.4 and II.5] and [NT13, Chap 1] for more detailed accounts.

### 5.3 A GLIMPSE OF NON-COMMUTATIVE PROBABILITY THEORY

#### 5.3.1 Weingarten calculus

The definition of the Haar state given in the proof of Theorem 5.2.7 is in a sense explicit since it is given in the basis of coefficients of irreducible representations. However, when doing computations, one sometimes has to compute the Haar state for arbitrary polynomials in the coefficients of the fundamental representation $u$. There is no practical general formula for this, but if we restrict to partition quantum groups, then it is possible to express the image of these polynomials using the corresponding category of partitions.

When trying to compute $\widehat{h}(u_{i_1,j_1} \cdots u_{i_k,j_k})$ for arbitrary indices, the basic idea is to recall that $\widehat{h}(u^{\otimes k})$ is the orthogonal projection onto the subspace of fixed points of $u^{\otimes k}$. Therefore, if we have enough data on this subspace, we may be able to derive information on its orthogonal projector. And it turns out that being a partition quantum group precisely means that we have a combinatorial description of the space $\text{Mor}_G(\varepsilon, u^{\otimes k})$, which is the space of fixed points.

More precisely, if $\mathcal{C}$ is a category of partitions and $G = G_N(\mathcal{C})$, then writing $\xi_p = f_p^*$ we have

$$\text{Mor}_G(\varepsilon, u^{\otimes k}) = \text{Vect} \{ \xi_p | p \in \mathcal{C}(0,k) \}.$$ 

Now, to describe the orthogonal projection, a convenient tool is the Gram matrix of a basis. This one was already computed in the proof of Theorem 4.1.1 and we summarize this in a definition:

**Definition 5.3.1.** The Gram matrix associated to $\mathcal{C},k$ and $N$ is the $|\mathcal{C}(0,k)| \times |\mathcal{C}(0,k)|$-matrix

$$\text{Gr}_N(\mathcal{C},k)$$

with coefficients

$$\text{Gr}(\mathcal{C},k)_{p,q} = N^{b(p;q)}$$

for $p, q \in \mathcal{C}(0,k)$. The corresponding Weingarten matrix is, if it exists,

$$W_N(\mathcal{C},k) = \text{Gr}_N(\mathcal{C},k)^{-1}.$$ 

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As one may expect from this definition, the final formula for the Haar state will involve $W_N$ rather than $\text{Gr}_N$, so that it will not be completely explicit. We will however show in the next section that it is possible to obtain asymptotic estimates on the coefficients of the Weingarten matrix which give free probabilistic information on $O(G)$. The first proof of this result was done by T. Banica and B. Collins for $O_N^+$ and $U_N^+$ in [BC07] and then extended to arbitrary partition quantum groups with the same argument in [BS09, Thm 5.4].

**Theorem 5.3.2** (Banica-Collins) Let $\mathcal{C}$ be a category of partitions and let $N \geq 4$ be an integer. For any $k \in \mathbb{N}$, if $\{\xi_p \mid p \in \mathcal{C}(0, k)\}$ is linearly independent, then

$$h(u_{i_1j_1} \cdots u_{i_kj_k}) = \sum_{p,q \in \mathcal{C}(0,k)} \delta_p(i_1, \cdots, i_k) \delta_q(j_1, \cdots, j_k) W_N(\mathcal{C}, k)_{pq}. \tag{5.2}$$

*Proof.* Let us denote by $W \subset (\mathbb{C}^N)^{\otimes k}$ the subspace of fixed vectors. As already mentioned, the left-hand side in Equation (5.2) is a coefficient of the orthogonal projection $P_W$ onto $W$. Consider the map $\Phi : (\mathbb{C}^N)^{\otimes k} \to W$ given by

$$\Phi(x) = \sum_{p \in \mathcal{C}(0,k)} \langle x, \xi_p \rangle \xi_p.$$ 

This is a surjective map but it is not idempotent. Indeed,

$$\Phi(\xi_p) = \sum_{q \in \mathcal{C}(0,k)} \langle \xi_p, \xi_q \rangle \xi_q = \text{Gr}_N(\mathcal{C}, k) \xi_p.$$ 

In other words, $\Phi = \text{Gr}_N(\mathcal{C}, k) \circ P_W$, which readily yields $P_W = W_N(\mathcal{C}, k) \circ \Phi$. The proof now ends with an easy computation:

$$\langle h(u^{\otimes k}) e_{i_1} \otimes \cdots \otimes e_{i_k}, e_{j_1} \otimes \cdots \otimes e_{j_k} \rangle = \langle W_N(\mathcal{C}, k) \circ \Phi(e_{i_1} \otimes \cdots \otimes e_{i_k}), e_{j_1} \otimes \cdots \otimes e_{j_k} \rangle$$

$$= \sum_{p \in \mathcal{C}(0,k)} \langle e_{i_1} \otimes \cdots \otimes e_{i_k}, \xi_p \rangle \langle W_N(\xi_p), e_{j_1} \otimes \cdots \otimes e_{j_k} \rangle$$

$$= \sum_{p \in \mathcal{C}(0,k)} \delta_p(i) \langle W_N(\xi_p), e_{j_1} \otimes \cdots \otimes e_{j_k} \rangle$$

$$= \sum_{p, q \in \mathcal{C}(0,k)} \delta_p(i) \delta_q(j) W_N(\mathcal{C}, k)_{pq}.$$ 

\[ \blacksquare \]

We will now give applications of Theorem 5.3.2 with a probabilistic flavour. This will only be a glimpse of a research direction of its own, which owes much to the combinatorial approach to free probability theory (see the book [NS06]). In particular, the use of *free cumulants* leads to spectacular applications of Weingarten calculus to *noncommutative de Finetti theorems* as in [BCS12] or to asymptotics of random matrices with noncommutative entries *à la Diaconis-Shahshahani* like in [BCS11].

### 5.3.2 Spectral Measures

To be more precise, consider an element $x \in L^\infty(G)$ which is self-adjoint. Then, it generates a von Neumann subalgebra $\langle x \rangle \subset L^\infty(G)$ which is commutative, hence isomorphic to $L^\infty(\text{Sp}(x))$ by Borel functional calculus (see for instance [Con00, Thm 15.10]). The restriction of $h$ to this algebra is still a state, hence coincides with integration with respect to a Borel probability...
measure $\mu_x$. We can therefore see $x$ as a random variable and wonder about its spectral measure $\mu_x$. We mainly have access to the moments of $\mu_x$, which are given by

$$m_k(\mu_x) = \int_{\text{Sp}(x)} t^k d\mu_x(t) = h(x^k).$$

This means that we will compute moments and then try to reconstruct the probability measure. This is not always possible for an arbitrary probability measure, but it will work on our cases because our measures are supported on the spectrum of bounded operators, and such spectra are always compact\(^4\). For later use, let us recall some facts about one of the most important probability distributions in free probability.

**Definition 5.3.3.** The semicircle distribution (or Wigner distribution) $\mu_{sc}$ is the probability distribution on $[-2, 2]$ with density

$$\frac{1}{\pi \sqrt{4 - x^2}}$$

with respect to the Lebesgue measure.

The computation of the moments of $\mu_{sc}$ is a standard exercise in undergraduate integration.

**Exercise 21.** Prove that the moments of the semicircle distribution are given by $m_{2k+1}(\mu_{sc}) = 0$ and

$$m_{2k}(\mu_{sc}) = \frac{1}{k+1} \binom{2k}{k}.$$

**Solution.** Observe that because $\mu_{sc}$ has an even density, all its odd moments vanish. We can therefore focus on even moments and compute

$$m_{2k}(\mu_{sc}) = \frac{1}{\pi} \int_{-2}^{2} x^{2k} \sqrt{4 - x^2} dx$$

$$= \frac{2^{2k+2}}{\pi} \int_0^{\pi/2} \sin^2(\theta) \cos^2(\theta) d\theta$$

$$= \frac{2^{2k+2}}{\pi} \left( \frac{\pi}{2} \frac{(2k)!}{(2^k(k!)^2} - \frac{\pi}{2} \frac{(2k+2)!}{(2^{k+1}(k+1)!)^2} \right)$$

$$= \frac{1}{k+1} \binom{2k}{k}.$$

\(^4\) It follows from the Stone-Weierstrass Theorem that a finite Borel measure on a compact space is determined by its evaluation on polynomials, hence by its moments.

### 5.3.3 Truncated characters

We will now investigate the character of the fundamental representation, that is to say the element

$$\chi = \sum_{i=1}^{N} u_{ii}.$$

Based on our probabilistic intuition, we will use from now on the following fancy but suggestive notation for the Haar state $\int_G$. The moments of $\chi$ are easy to compute and do not require Weingarten calculus:

**Proposition 5.3.4.** Let $C$ be a category of noncrossing partitions and let $N \geq 4$. Then,

$$m_k(\chi) = |C(k, 0)|.$$
Proof. Denoting by $\text{Fix}(v)$ the subspace of fixed points of $\rho_v$, we have by definition

$$m_k(\chi) = \int_{G/N} \chi^k$$

$$= \text{Tr} \left( \hat{h}(u^\otimes k) \right)$$

$$= \dim \left( \text{Fix} \left( u^\otimes k \right) \right)$$

$$= \dim (\text{Vect} \{ \xi_p \mid p \in \mathcal{C}(k,0) \})$$

$$= |\mathcal{C}(k,0)|$$

where the last lines comes from the linear independence of the partition vectors proven in Theorem 4.1.1. ■

Let us illustrate this result in the case of the quantum orthogonal groups:

Example 5.3.5. Assume that $\mathcal{C} = NC_2$ and first observe that for odd $k$, $NC_2(k,0) = 0$. We now have to compute the number of noncrossing pairings on $2k$ points and this can be done by induction on $k$. Denoting by $(C_k)_{k \in \mathbb{N}}$ the numbers that we are looking for, we have $C_1 = 1$. Moreover, consider a noncrossing pair partition $p$ on $2k$ points and let $\ell$ be the point connected to 1. Then, $p$ induces noncrossing pairings on $\{2, \cdots, \ell-1\}$ and $\{\ell+1, \cdots, 2k\}$. Conversely, given such pairings one can reconstruct $p$ with the condition that 1 should be connected to $\ell$. In other words, we have

$$C_k = \sum_{i=2}^{2k} C_{i-1} C_{2k-i+1}.$$ 

This uniquely defines the sequence $(C_k)_{k \in \mathbb{N}}$, and it turns out that the moments of the semicircle distribution satisfy this recursion relation. Thus, $\chi$ is what is called a *semicircular element*.

For quantum permutation groups, we can resort to the “doubling trick” explained in the proof of Theorem 4.1.1:

Example 5.3.6. For $S_N^+$, we have to compute the number of noncrossing partitions on $k$ points. The bijection $p \mapsto \hat{p}$ used in the proof of Theorem 4.1.1 shows that

$$|NC(k,0)| = |NC_2(2k,0)|.$$

As a consequence, $\chi_{S_N^+}$ has the same distribution as $\chi_{O_N^+}^2$. This is known as the *free Poisson distribution* (or the *Marschenko-Pastur distribution*) with parameter 1. It is supported on $[0,4]$ and has density

$$\frac{1}{2\pi} \sqrt{\frac{4}{x} - 1}$$

with respect to the Lebesgue measure.

The quantum hyperoctaedral group is more involved and the distribution of $\chi_{H_N^+}$ is the *free Bessel distribution* introduced in [BBCC11]. Instead of proving this we will, following the work of T. Banica and R. Speicher in [BS09], try to get more understanding on the previous results by considering *truncated characters* in the following sense:

Definition 5.3.7. Let $\mathcal{C}$ be a category of noncrossing partitions and let $N \geq 4$ be an integer. The truncated characters are the elements

$$\chi_t = \sum_{i=1}^{|tN|} u_{ti}.$$

for $t \in [0,1]$.

The previous result can then be refined thanks to the Weingarten formula. This first requires an estimate on the Gram and Weingarten matrices. In the sequel, $O(N^{-1/2})$ means a matrix all of whose coefficients are dominated by $N^{-1/2}$.
Lemma 5.3.8. Let $\mathcal{C}$ be a category of partitions, let $N, k$ be integers and let $\Gamma_N(\mathcal{C}, k)$ be the diagonal of $\text{Gr}_N(\mathcal{C}, k)$. Then,

$$\text{Gr}_N(\mathcal{C}, k) = \Gamma_N(\mathcal{C}, k)^{1/2} (\text{Id} + O \left( N^{-1/2} \right)) \Gamma_N(\mathcal{C}, k)^{1/2}$$

$$W_N(\mathcal{C}, k) = \Gamma_N(\mathcal{C}, k)^{-1/2} (\text{Id} + O \left( N^{-1/2} \right)) \Gamma_N(\mathcal{C}, k)^{-1/2}$$

Proof. To lighten notations we will omit from now on $\mathcal{C}$ and $k$ in the computations. The trick is to consider the coefficients of $\Gamma_N^{-1/2} \text{Gr}_N \Gamma_N^{-1/2}$:

$$(\Gamma_N^{-1/2} \text{Gr}_N \Gamma_N^{-1/2})_{pq} = (\Gamma_N^{-1/2})_{pp} (\text{Gr}_N)_{pq} (\Gamma_N^{-1/2})_{qq} = N^{b(p/q) - (b(p) + b(q))/2}.$$

If $p = q$, the result is 1. Otherwise, there is at least two blocks of $p$ or of $q$ which are merged in $p \lor q$, hence the result is less than $N^{-1/2}$. In other words, the matrix

$$B_N = \Gamma_N^{-1/2} \text{Gr}_N \Gamma_N^{-1/2} - \text{Id}$$

has all its coefficients dominated by $N^{-1/2}$, yielding the first part of the statement.

As for the second part, we have

$$\Gamma_N^{1/2} W_N^{-1} \Gamma_N^{1/2} = (\text{Id} + B_N)^{-1}$$

$$= \text{Id} + \sum_{n=1}^{+\infty} (-1)^n B_N^n$$

$$= \text{Id} + C_N$$

and each term in the sum defining $C_N$ is dominated by $N^{-1/2}$, yielding the second part. \[\square\]

Theorem 5.3.9 (Banica-Speicher) Let $N \geq 4$, let $\mathcal{C}$ be a category of partitions and let $u$ be the fundamental representation of $\mathbb{G}_N(\mathcal{C})$. Then,

$$\lim_{N \to +\infty} \int_{\mathbb{G}_N(\mathcal{C})} \chi^k_f = \sum_{\ell \in \mathcal{C}(k)} \ell \delta^{(p)}.$$ 

Proof. For clarity, we will omit $k$ and $\mathcal{C}$ in the notations since no confusion is possible. Let us first consider the sum of the first $\ell$ diagonal coefficients. We claim that

$$\int_{\mathbb{G}_N(\mathcal{C})} (u_{i1} + \cdots + u_{i\ell})^k = \text{Tr} \left( W_N \text{Gr}_\ell \right).$$

Indeed, by Theorem 5.3.2,

$$\int_{\mathbb{G}_N(\mathcal{C})} \left( \sum_{i=1}^\ell u_{i\ell} \right)^k = \sum_{i_1, \cdots, i_k = 1}^\ell \int_{\mathbb{G}} u_{i_1 i_2} \cdots u_{i_k i_k}$$

$$= \sum_{i_1, \cdots, i_k = 1}^\ell \sum_{p, q \in \mathcal{C}} \delta_p(i) \delta_q(i) W_N(p, q)$$

$$= \sum_{p, q \in \mathcal{C}} \left( \sum_{i_1, \cdots, i_k = 1}^\ell \delta_p(i) \delta_q(i) \right) W_N(p, q).$$

If both $\delta_p$ and $\delta_q$ do not vanish on $i$, then this means that $i$ matches $p \lor q$. Because the indices run over a set of cardinality $\ell$, there are therefore $\ell \delta(p/q) = (\text{Gr}_\ell)_{pq}$ tuples yielding a non-zero contribution, hence the result.
Using Lemma 5.3.8, we can now write
\[
\int_{G_N(C)} \chi^k_t = \text{Tr} (W_N \Gamma_t) \\
= \text{Tr} \left( \Gamma_N^{-1} \Gamma_t \right) + \text{Tr} \left( \Gamma_N^{-1/2} C_N \Gamma_N^{-1/2} \Gamma_t \right) + \text{Tr} \left( \Gamma_N^{-1} \Gamma_t^{-1/2} B_N \Gamma_t^{1/2} \right) + \text{Tr}(C_N B_N) \\
= \text{Tr} \left( \Gamma_N^{-1} \Gamma_t \right) \left( 1 + 2O \left( N^{-1/2} \right) + O \left( N^{-1} \right) \right) \\
= \left( \sum_{p \in C(k)} b(p) N^{-b(p)} \right) \left( 1 + O \left( N^{-1/2} \right) \right) \\
= \left( \sum_{p \in C(k)} \left( \frac{tN}{N} \right)^{b(p)} \right) \left( 1 + O \left( N^{-1/2} \right) \right)
\]
and this converges to the announced limit as \( N \) goes to infinity. \( \square \)

**Example 5.3.10.** For \( C = NC_2 \), the limit of the odd moments vanish since there is no pair partition on an odd number of points, and we have
\[
\lim_{N \to +\infty} \int_{O_N^+} \chi^{2k}_t = \sum_{p,q \in NC_2(2k)} t^k = t^k \frac{1}{k + 1} \binom{2k}{k}.
\]
This is the same as the semi-circle distribution, except that the radius of the circle has been changed to \( 2t \) instead of \( 2 \). We therefore see that the distribution of \( \chi_t \) smoothly approximates the distribution of \( \chi \).

**Example 5.3.11.** For \( C = NC \), the computation requires the theory of *free cumulants* (see the book [NS06]) which we will not introduce here. Let us simple mention that using it, one can prove that \( \chi_t \) is asymptotically free Poisson with parameter \( t \), i.e. its distribution has density
\[
\frac{1}{2\pi} \sqrt{\frac{4}{x^2 + \left( 1 - \frac{1 + t}{x} \right)^2}}
\]
with respect to the Lebesgue measure.

### 5.3.4 Single coefficients and freeness

We will end with another result of the same type, but focusing on the joint behaviour of several elements. More precisely, let us consider all the coefficients \( u_{ij} \) of the fundamental representation of \( O_N^{\vee} \). Contrary to the character, the distribution of \( u_{ij} \) is very complicated and depends on \( N \) (see [BCZJ09] for an explicit computation). But the asymptotics can be easily obtained by the Weingarten formula :

**Proposition 5.3.12.** For \( N \geq 2 \), set \( x_{ij} = \sqrt{N} u_{ij} \), where \( U \) is the fundamental representation of \( O_N^{\vee} \). Then,
\[
\lim_{N \to +\infty} \int_{O_N^+} x_{ij}^k = \begin{cases} 
0 & \text{if } k \text{ odd} \\
\frac{1}{k + 1} \binom{k}{k/2} & \text{if } k \text{ even}
\end{cases}
\]
In other words, the coefficients are asymptotically semicircular.

**Proof.** Because there is no pair partition on a odd number of points, Theorem 5.3.2 implies that the odd moments vanish. As for the even ones, using again Theorem 5.3.2 we have
\[
\int_{O_N^+} x_{ij}^{2k} = N^k \sum_{p,q \in NC_2(2k)} \delta_p(i, \cdots , i) \delta_q(j, \cdots , j) (W_N)_{pq} \\
= \sum_{p,q \in NC_2(2k)} N^k (W_N)_{pq}.
\]
By Lemma 5.3.8,

\[ N^k (W_N)_{pq} = N^k N^{-(b(p)+b(q))/2} + N^{-(b(p)+b(q))/2} O \left( N^{-1/2} \right). \]

If \( p = q \), since we are considering pair partitions on \( 2k \) points the number of blocks is \( k \) so that the right-hand side equals \( 1 + O(N^{-1/2}) \). If \( p \neq q \), the right-hand side equals

\[ N^k (\Gamma_N)^{-1}_{pp} (1 + O(N^{-1/2}))_{pq} = O(N^{-1/2})_{pq} \]

which goes to 0 as \( N \) goes to infinity. Summing up,

\[ \lim_{N \to +\infty} \int_{O_N^+} x_{ij}^k = |NC_2(k, 0)| \]

and the result follows.

Remark 5.3.13. For \( S^+_N \), the computation is trivial. Indeed, since \( p_{ij} \) is a projection, \( p^k_{ij} = p_{ij} \) for all \( k \) so that all the moments are equal to \( N^{-1} \).

We can even go further and compute the limit of mixed moments, that is to say arbitrary monomials involving the coefficients \( x_{ij} \). In general in probability theory, it is difficult to compute such mixed moments because of the correlations between the variables. However, the situation greatly simplifies if these correlation vanish, i.e. if the variables are independent. In our setting, since the variables do not commute with one another, independence is not the correct notion. Nevertheless, there is a concept of free independence which translates the fact that there is “no correlation”. We will not define this concept here because this would take us too far (but we once again highly recommend reading the book [NS06]) but simply give a formula for arbitrary joint moments. This requires some terminology.

Definition 5.3.14. Given a monomial \( X = x_{i_1j_1} \cdots x_{i_kj_k} \) and a partition \( p \in \mathcal{P}(k) \), we will say \( p \) matches \( X \) if for any points \( \ell_1 \) and \( \ell_2 \) connected by \( p \), we have \( i_{\ell_1} = i_{\ell_2} \) and \( j_{\ell_1} = j_{\ell_2} \). The set of matching partitions will be denoted by \( \mathcal{P}(X) \).

Proposition 5.3.15. We have, for any monomial \( X = x_{i_1j_1} \cdots x_{i_kj_k} \),

\[ \lim_{N \to +\infty} \int_{O_N^+} X = \begin{cases} 0 & \text{if } k \text{ odd} \\ |\mathcal{P}(X) \cap NC_2(k)| & \text{if } k \text{ even} \end{cases} \]

Proof. The proof starts as for Proposition 5.3.12 using Theorem 5.3.2. The odd moments vanish so let us consider a moment of length \( 2k \) :

\[ \int_{O_N^+} X = \sum_{p,q \in NC(2k)} \delta_p(i) \delta_q(j) (W_N)_{p,q}. \]

The same computation as in the proof of Proposition 5.3.12 shows that as \( N \) goes to infinity, all the terms in the right-hand side vanish except for those with \( p = q \). We therefore have

\[ \lim_{N \to +\infty} \int_{O_N^+} X = \sum_{p \in NC_2(k)} \delta_p(i) \delta_p(j). \]

To conclude, simply notice that both \( \delta \)-functions do not vanish, if and only if \( p \) matches \( X \).
CHAPTER 6

A UNITARY EXCURSION

Now that we have a firm grasp on the theory of orthogonal compact matrix quantum group, we may start to feel frustrated by the restrictions that the orthogonality assumption entails. For instance, combining the results of [BS09] and [Web13] yields

**Theorem 6.0.1** (Banica-Speicher, Weber) There are exactly seven categories of non-crossing partitions.

This leaves few examples of compact quantum groups to study. The obvious way to go further is of course to consider unitary quantum groups, which were briefly alluded to in Definition 1.3.6. It turns out that the results of the previous chapters carry on to this more general setting with little modifications, including the combinatorial aspects.

### 6.1 Coloured partitions

In this section we will give the unitary generalizations of all the main results of the past chapters. Because the proofs are very similar, we will only indicate the important differences with the orthogonal case when needed. Let us first note that Theorem 2.1.14 was already stated for unitary compact matrix quantum group because the proof indeed works in that general case.

Now, in order to understand how the partition description should be modified to cover some unitary quantum groups, let us go back to the duality operator. Recall that for a Hilbert space $V$, we defined it as the map $D_V : V \otimes V \to \mathbb{C}$ given by

$$D_V(x \otimes y) = \langle x, y \rangle.$$

If $G$ is an *orthogonal* compact matrix quantum group with fundamental representation $u \in \mathcal{O}(G) \otimes \mathcal{L}(V)$, we have seen that $D_V$ is an intertwiner between $u \otimes u$ and the trivial representation $\varepsilon$. However, the proof crucially relies on the fact that $u_{ij}^* = u_{ij}$. If this is not the case, then the result fails.

To see what should be the correct statement in general, let us write the duality operator differently as $D_V : V \otimes \overline{V} \to \mathbb{C}$

$$D_V(x \otimes \overline{y}) = \langle x, y \rangle.$$

In this way, the question is, given a representation of $G$ on $V$, whether there is a natural representation acting on $\overline{V}$. This was answered in Definition 5.2.6 : one should consider the conjugate representation $\overline{\pi}$ whose coefficients are $u_{ij}^*$. By the axioms of a unitary compact matrix quantum group, this is again a unitary representation and

$$D_V \in \text{Mor}_G(u \otimes \overline{\pi}, \varepsilon).$$
Remark 6.1.1. Another way to understand this is that in the orthogonal case, we implicitly used the fact that the anti-linear isomorphism \( j : V \to V^\ast \) sending \( x \) to \( \pi \) commutes with \( u \), so that we can equivariantly identify \( V \) with its conjugate space. This is not true any more in general.

The main point of the above discussion is to stress that since we no longer assume \( u = \overline{u} \), we should not focus tensor products of \( u \) alone, but on tensor products of \( u \) and \( \overline{u} \). In order to make this precise, we need some shorthand notations. Let \( n \in \mathbb{N} \) be an integer and let \( w = w_1 \cdots w_n \) be a word on \( \{\circ, \cdot\} \). Setting \( u^\circ = u \) and \( u^\cdot = \overline{u} \), we write

\[
 u^w = u^{w_1} \otimes \cdots \otimes u^{w_n}
\]

acting on \( V^w = V^{w_1} \otimes \cdots \otimes V^{w_n} \). We can now state the corollary of Theorem 2.1.14 which is crucial for the partition approach.

**Corollary 6.1.2.** Let \( G = (\mathcal{O}(G), u) \) be a unitary compact matrix quantum group. Then, any irreducible representation is equivalent to a subrepresentation of \( u^w \) for some word \( w \) on \( \{\circ, \cdot\} \).

Based on this, it is natural to try to extend the partition picture by distinguishing points corresponding to \( u \) and \( \overline{u} \). This is easily done as follows:

**Definition 6.1.3.** A **coloured partition** is a partition with the extra data of a colour on each point. The set of coloured partitions is denoted by \( P^\circ \cdot \). A coloured partition is said to be non-crossing if the underlying uncoloured partition is non-crossing. The set of non-crossing coloured partitions is denoted by \( NC^\circ \cdot \).

Here are two examples, the first one being crossing while the second one is not.

\[
\begin{array}{c}
\circ & \bullet & \circ \\
\circ & \circ & \circ \\
\end{array}
\quad
\begin{array}{c}
\circ & \bullet & \circ \\
\circ & \circ & \circ \\
\end{array}
\]

To define the linear map corresponding to a coloured partition, let us call the word formed by the colours of the upper points of \( p \) (read from left to right) its **upper colouring**, while the colours of the lower points (again read from left to right) is its **lower colouring**. Then, for a partition \( p \) with upper colouring \( w \) and lower colouring \( w' \), we define a linear map

\[
T_p : V^w \to V^{w'}
\]

by the same formula as in the uncoloured case.

Based on this, we can clarify the notion of a category of coloured partitions. There are two differences with the uncoloured case. The first one is that it is only possible to compose \( q \) with \( p \) (in that order) if the upper colouring of \( q \) matches the lower colouring of \( p \). The second one is that the partition corresponding to the duality map for \( C^N \) is, in this setting, the partition

\[
D_{\circ \cdot} =
\begin{array}{c}
\circ \\
\end{array}
\]

while the duality map \( D_{\circ \cdot} \) for \( \overline{C}^N \) is the same with the colour exchanged. Moreover, the identity map of \( C^N \) is given by the partition

\[
\begin{array}{c}
\circ \\
\end{array}
\]

\[
\begin{array}{c}
\circ \\
\end{array}
\]
6.1. Coloured partitions

and the identity id• of \(\mathcal{C}^N\) is given by the same partition but coloured in black. For convenience, given a set \(\mathcal{C}\) of partitions, we will denote by \(\mathcal{C}(w, w')\) the subset consisting in all partitions in \(\mathcal{C}\) with upper colouring \(w\) and lower colouring \(w'\).

**Definition 6.1.4.** A category of coloured partitions \(\mathcal{C}\) is a collection of sets of partitions \(\mathcal{C}(w, w')\) for all words \(w\) and \(w'\) on \(\{\circ, \bullet\}\) such that

1. If \(p \in \mathcal{C}(w, w')\) and \(q \in \mathcal{C}(w'', w''')\), then \(p \otimes q \in \mathcal{C}(w.w'', w'.w''')\),
2. If \(p \in \mathcal{C}(w, w')\) and \(q \in \mathcal{C}(w'', w')\), then \(q \circ p \in \mathcal{C}(w'', w)\),
3. If \(p \in \mathcal{C}(w, w')\), then \(p^* \in \mathcal{C}(w', w)\),
4. \(\text{id}_\circ \in \mathcal{C}(\circ, \circ)\) and \(\text{id}_\bullet \in \mathcal{C}(\bullet, \bullet)\),
5. \(D\circ_\bullet \in \mathcal{C}(\bullet \circ, \emptyset)\) and \(D\bullet_\circ \in \mathcal{C}(\circ \bullet, \emptyset)\).

We have not mentioned crossing partitions so far. One has to be a little careful here, since there are several ways of colouring the crossing partition \(\{\{1, 3\}, \{2, 4\}\}\). Considering the corresponding relations, we see that if we colour all the points in white, then this yields

\[ u_{ij}u_{kl} = u_{kl}u_{ij} \]

for all \(1 \leq i, j, k, l\). This of course implies the corresponding relation for the adjoints, but does not tell us anything about commutation between \(u_{ij}\) and \(u_{ij}^*\) for instance. In order to get everything, we must include all possible colourings.

**Definition 6.1.5.** A category of partitions \(\mathcal{C}\) is said to be symmetric if it contains all coloured versions of the crossing partition. We can now give the coloured analogue of Theorem 3.3.5.

**Theorem 6.1.6** Let \(N\) be an integer and let \(\mathcal{C}\) be a category of coloured partitions. Then, there exists a unitary compact matrix quantum group \(G = (\mathcal{O}(G), u)\), where \(u\) has dimension \(N\), such that for any words \(w, w'\) on \(\{\circ, \bullet\}\),

\[ \text{Mor}_G(u^w, u^{w'}) = \text{Vect}\{T_p \mid p \in \mathcal{C}(w, w')\} \]

Moreover, \(G\) is a classical group if and only if \(\mathcal{C}\) is symmetric. The compact quantum group \(G\) will be denoted by \(G_N(\mathcal{C})\) and called the partition quantum group associated to \(N\) and \(\mathcal{C}\).

To complete the picture, let us first clarify the rotation operation in the coloured setting. Assume that we want to rotate a white point in the upper row to the lower row. We can certainly use the partition \(D\bullet_\circ\) to do this in the following way:

\[ p \mapsto (\text{id}_\circ \otimes p) \circ (D\bullet_\circ^* \otimes |^k) \]

but the point has been changed to a black one in the process. As a consequence, rotations are allowed in categories of coloured partitions, on condition that the colour of the rotated point is changed. Here is the pictorial description of this operation:

```
  *  *  *  *
*  *  *  *
  *  *  *  *
```

```
+---+---+---+
|   |   |   |
|\circ|\bullet|   |
+---+---+---+
```

```
  *  *  *  *
```

```
  *  *  *  *
```

```
```

```
```

```
As one may expect, this setting contains the one without colours that we have been working in up to now. Let us state this formally as a proposition.

**Proposition 6.1.7.** Let \( \mathcal{C} \) be a category of coloured partitions and assume that the partition \( \text{id}_\circ \bullet \), which is the coloured version of \( | \) in \( P(\circ \bullet) \), is in \( \mathcal{C} \). Then, the set \( \mathcal{C}' \) of uncoloured partitions obtained by removing the colours from \( \mathcal{C} \) is a category of partitions, and

\[
\mathbb{G}_N(\mathcal{C}) = \mathbb{G}_N(\mathcal{C}').
\]

**Proof.** First note that applying \( \text{id}_\circ \bullet \) and its rotated version \( \text{id} \bullet \circ \), we can turn black points of any partition in \( \mathcal{C} \) into white points. Thus, if \( p \in \mathcal{C}' \) is a partition such that one of its coloured version is in \( \mathcal{C} \), then \( p \) with all points coloured in white is also in \( \mathcal{C} \). As a consequence, given \( p, q \in \mathcal{C}' \), there are coloured versions in \( \mathcal{C} \) which can be composed, hence \( q \circ p \in \mathcal{C}' \). The other axioms being trivially satisfied, \( \mathcal{C}' \) is a category of partitions.

Applying the definition, we see that \( T_{\text{id}_\circ \bullet} \) intertwines \( u \) with \( u \) if and only if \( u_{ij} = u_{ij}^* \) for all \( i, j \), hence \( \mathbb{G}_N(\mathcal{C}) \) is an orthogonal compact matrix quantum groups. In particular, it is completely determined by the sets

\[
\text{Mor}_{\mathbb{G}_N(\mathcal{C})}(u^\otimes k, u^\otimes \ell) = \text{Vect} \{ T_p | p \in \mathcal{C}(\circ^k, \circ^\ell) \}
\]

and the result follows from the uniqueness in Theorem 3.3.3. \( \blacksquare \)

Note that conversely, if \( \mathcal{C}' \) is a category of uncoloured partition, then the category of coloured partitions obtained by taking all the possible colourings of elements of \( \mathcal{C}' \) satisfies \( \mathbb{G}_N(\mathcal{C}) = \mathbb{G}_N(\mathcal{C}') \).

Let us end with a comment concerning conjugate representations. The proof of Corollary 5.2.5 used the orthogonality of the fundamental representation \( u \), but all that is really needed is the fact that \( \pi \) is unitary, which holds for any unitary compact matrix quantum group\(^1\). Thus, we have the following:

**Corollary 6.1.8.** Let \( \mathbb{G} \) be a unitary compact matrix quantum group. For any unitary representation \( v \), the representation

\[
\overline{v} = (v_{ij}^*)_{1 \leq i, j \leq \dim(v)}
\]

is again unitary and is called the conjugate representation of \( v \). Moreover, \( v \) is irreducible if and only if \( \overline{v} \) is irreducible.

### 6.2 The Quantum Unitary Group

Let us explore the new features exhibited by unitary compact matrix quantum groups, by working out in details the case of the quantum unitary group \( U_N^+ \), whose \(*\)-algebra was given in Definition 1.3.8. This first requires identifying its category of partitions. Because \( U_N^+ \) is the largest unitary compact matrix quantum group, in the sense that all the others are quotients of it, its corresponding category of partitions must be the smallest possible one. Proving this is a good exercise for practicing with coloured partitions.

**Exercise 22.** Let \( \mathcal{U} \) be the set of all non-crossing coloured partitions in pairs such that:

- If the two points of a block are in the same row, then they have different colours,
- If the two points of a block are in different rows, then they have the same colour.

Prove that this is a category of partitions and that it is generated by \( D_\circ \bullet \).

**Solution.** The fact that \( \mathcal{U} \) is a stable under reflections and horizontal concatenation is straightforward. As for vertical concatenation, simply observe that the non-through-blocks of \( q \circ p \) are

---

\(^1\) This is precisely the reason behind that extra assumption in the definition.
non-through-blocks of $p$ or $q$, while the through-blocks are obtained by concatenating non-through-blocks of both partitions with matching colours. Thus, $\mathcal{U}$ is a category of partitions.

Let us now prove that any partition in $\mathcal{U}$ is in the category of partitions $\mathcal{U}'$ generated by $D_{e} \bullet$. For this we will first prove by induction on the number of points that for any partition in $NC_{2}^{\bullet}$ lying on one line, there are two consecutive points which are connected. This is clear for a partition on two points. Let us assume that it is also true for all partitions on at most $2n$ points for some $n \in \mathbb{N}$, and let $p$ be a partition on $2(n+1)$ points. Numbering the points from 1 to $2n+2$ from left to right, the point 1 must be connected to another one, say $i$.

- If $i = 2$, then we are done,
- Otherwise, the points $2, \cdots, i-1$ can only be connected to another point in the same interval by non-crossingness. Thus, the restriction of $p$ to these points is a sub-partition which is in $NC_{2}^{\bullet}$. By induction, there are two consecutive points connected there, concluding the proof.

Back now to the original problem, let $p \in \mathcal{U}$ and assume, up to rotation, that the first two points are connected. Then, either $p = D_{e} \bullet \otimes q$ or $p = D_{e} \otimes q$. In the first case, we have

$$q = (D_{e} \bullet \otimes | \otimes \cdots |) \circ p \in \mathcal{U}$$

with $|$ being suitably coloured to match the colours of $p$, and similarly in the second case. A straightforward induction based on this observation then yields the result.

The second thing we need is the generalization of the results of Chapter 4. The notion of projective partition still makes sense, and the only thing which must be adapted is the through-block decomposition of Proposition 4.1.3. The question here simply is : what should be the colour of the lower points of $p_{d}$ and $p_{u}$? An examination of the proofs shows that this does not matter as long as they are all the same. Indeed, the point is only to be able to compose $p_{d}^{*}$ with $p_{u}$, or to compose these with mixing partitions. As a consequence, we can simply decide once and for all to colour all those points in white, as well as those of the mixing partitions. Then, the proofs carry on verbatim to yield:

**Theorem 6.2.1** Let $\mathcal{C}$ be a category of coloured partitions and let $N \geq 4$ be an integer. Then, the irreducible representations of $G_{N}^{*}(\mathcal{C})$ can be indexed by the projective partitions of $\mathcal{C}$ in such a way that

- $u_{p} \sim u_{q}$ if and only if $p \sim q$,
- $u^{w} = \bigoplus_{p \in \mathcal{C}(w, w)} u_{p}$.

Moreover, the fusion rules are given by

$$u_{p} \otimes u_{q} = u_{p \otimes q} \oplus \bigoplus_{k=1}^{\min(t(p), t(q))} v_{p \cong k} \oplus v_{p \sqsupset k}$$

where $v_{r} = u_{r}$ if $r \in \mathcal{C}$ and $v_{r} = 0$ otherwise.

We can now describe the representation theory of $U_{N}^{\uparrow}$. For convenience, we will first define the abstract object indexing the equivalence classes of irreducible representations. Let $M$ be the set of all words on $\{\circ, \bullet\}$. The monoid structure is simply given by the concatenation of words. In order to deal with conjugate representations, we also need a conjugation operation $w \mapsto \overline{w}$ on $M$. Setting $\overline{\circ} = \bullet$ and $\overline{\bullet} = \circ$, we define

$$\overline{w}_{1} \cdots \overline{w}_{k} = \overline{w}_{k} \cdots \overline{w}_{1}.$$
**Theorem 6.2.2** (Banica) For $N \geq 2$, the irreducible representations of $U_N^+$ can be labelled by the elements of $M$ in such a way that $u^\emptyset = \varepsilon$, $u^\emptyset = V$ and for any $w \in M$,

$$\overline{uw} = u^w.$$  
Moreover, for any $w, w' \in M$,

$$u^w \otimes u^{w'} = \sum_{w = az, w' = zb} u^{azb}.$$  

**Proof.** We define a map $\Phi : M \to \text{Irr}(U_N^+)$ sending $w = w_1 \cdots w_k$ to the equivalence class of

$$p_{\text{id}_{w_1}} \otimes \cdots \otimes \text{id}_{w_k}.$$  

By construction, $u_{\text{id}^w} = u$ and $u_{\text{id}^w} = \overline{u}$ and we will prove that it is a bijection

1. We first prove that $\Phi$ is onto. Let $p \in U$ be a projective partition. A straightforward induction shows that it can be written as

$$p = (b_0^* b_0) \otimes p_1 \otimes (b_1^* b_1) \otimes p_2 \otimes \cdots \otimes p_k \otimes (b_k^* b_k),$$

where $p_i \in \{\text{id}_o, \text{id}_e\}$ with some colouring and $b_i$ lies on one line. Now, any two connected points in $b_i$ have different colours because $p \in U$, hence $b_i \in U$. Composing with

$$r = b_0^* \otimes | \otimes b_1^* \otimes | \otimes \cdots \otimes | \otimes b_k^*,$$

therefore yields an equivalence in $U$ between $p$ and

$$q = p_1 \otimes p_2 \otimes \cdots \otimes p_k.$$  

Setting $w_i$ to be the colour of $p_i$, $u_p$ is then in the equivalence class of $\Phi(w_1 \cdots w_k)$, proving surjectivity.

2. Consider now $w, w' \in M$ such that $\Phi(w) \sim \Phi(w')$ and notice first that they must have the same number of through-blocks, hence $w$ and $w'$ must have the same number of letters. Let $r_i$ be the identity partition with the colour $w_i$ on the upper row and $w'_i$ on the lower row. Then,

$$r = r_1 \otimes \cdots \otimes r_k,$$

implements the equivalence between $\Phi(w)$ and $\Phi(w')$ so that $r \in U$. It then follows from the definition of $U$ that $w_i = w'_i$ for all $i$, hence $w = w'$. In other words, $\Phi$ is injective.

3. As for the fusion rules, first note that $\Phi(w) \Box \Phi(w')$ is never in $U$ since it has a block of size four. Moreover, in $\Phi(w) \Box^n \Phi(w')$, the last point of $\Phi(w)$ is connected to the first one of $\Phi(w')$, hence they must have different colours. Similarly, the last but one of $\Phi(w)$ to the second one of $\Phi(w')$, hence they must have different colours, and so on. Therefore, for this partition to be in $U$, the last $n$ letters of $w$ must coincide (in reversed order) to opposite of the first $n$ ones of $w'$. In other words, we must have $w = az$ and $w' = zb$ with $z$ of length $n$. In that case, all the non-through-blocks can be removed up to equivalence, yielding a representation equivalent to $\Phi(ab)$, as claimed.

4. Eventually that for $u^w \otimes u^{w'}$ to contain the trivial representation $u^\emptyset$, one must have $w' = \overline{w}$. Hence, by Proposition 5.2.9, $\overline{uw} = u^w$ and the proof is complete.
6.3 Making things complex

Classically, one may think of $U_N$ as a complex version of $O_N$, even though this is not the complexification in the sense of Lie or algebraic groups\(^2\). At least, they play the same rôle of largest possible compact matrix group respectively in the unitary and orthogonal setting. In the quantum case, there is a precise way of describing $U_N^+$ as a complexification of $O_N^+$. This relies on an important procedure called the unital free product construction. Since it is interesting in itself, we will devote a bit of time to introducing it in details.

6.3.1 The unital free product construction

Usually, a free product construction takes a family of objects and builds the “largest possible one” generated by the data. In particular, no relation between the original objects should hold in the free product except for those required by the structure of the objects themselves.

The algebra case

Let us focus for the moment on algebras. There is a standard way to build a free product of algebras using tensor algebras\(^3\), but it is not suited for our purpose. Indeed, if the original algebras are unital, then their free product will also be, but the inclusions of the original algebras into the free product will not be unital algebra homomorphisms! One could say in a sense that this is a free product in the category of algebras but that it does not restrict to a free product of unital algebras. In our case, we would like to modify the constructions so that in the free product, the units of the inputs are identified with the unit of the output. This can be done, but makes the construction a bit more involved, and also a bit awkward at first sight. Let us do it nevertheless.

We start with two unital algebras $A_1$ and $A_2$ and fix a direct sum decomposition of vector spaces

$$A_1 = C.1A_1 \oplus A_1' \text{ and } A_2 = C.1 \oplus A_2'.$$

Associated to this decomposition are projections $p_1$ and $p_2$ on $C.1A_1$ and $C.1A_2$ respectively. The idea is that we have taken out the units to be able to easily identify them, so that we can focus on the rest. Intuitively, an element of the free product should be a sum of a multiple of the unit and a polynomial in the elements of $A_1'$ and $A_2'$. Here is how to make this precise: we set

$$A = C.1A \bigoplus_{k=1}^{+\infty} \bigoplus_{i_1 \neq \cdots \neq i_k \in \{1,2\}} A_{i_1}' \otimes A_{i_2}' \otimes \cdots \otimes A_{i_k}' .$$

This makes sense as a vector space, and we claim that it can be endowed with a unital algebra structure in a natural way. For the unitarity, we simply declare that the first summand is a multiplicative unit. The rest of the definition of the product is subtler. For convenience, let us set, for $1 \leq k < +\infty$,

$$A^{(k)} = \bigoplus_{i_1 \neq \cdots \neq i_k \in \{1,2\}} A_{i_1}' \otimes A_{i_2}' \otimes \cdots \otimes A_{i_k}' .$$

**Definition 6.3.1.** Given $a = a_1 \otimes \cdots \otimes a_k \in A^{(k)}$ and $b = b_1 \otimes \cdots \otimes b_\ell \in A^{(\ell)}$, we define their product inductively by

- If $i_k \neq i_\ell$ (i.e. if $a_k$ and $b_1$ do not belong both to $A_1$ or to $A_2$), then

$$ab = a_1 \otimes \cdots \otimes a_k \otimes b_1 \otimes \cdots \otimes b_\ell \in A^{(k+\ell)},$$

\(^2\) The complexification of $O_N$ in that sense is the complex orthogonal group $O(N, \mathbb{C})$, i.e. the group of all complex matrices with inverse equal to its transpose. This is an interesting locally compact matrix group, but it is not compact, hence not suited for us.

\(^3\) This is simply the quotient of the tensor algebra of the direct sum $A_1 \oplus A_2$ by the ideal generated by the elements $a \otimes a' - aa'$ and $b \otimes b' - bb'$ for all $a, a' \in A_1$ and $b, b' \in A_2$. 

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• Otherwise, if \( i \in \{1, 2\} \) be such that \( a_k, b_1 \in A_i \), we set

\[
ab = a_1 \otimes \cdots \otimes a_{k-1} \otimes (a_k b_1 - p_i(a_k b_i)) \otimes b_2 \otimes \cdots \otimes b_\ell \\
+ p_i(a_k b_1)(a_1 \otimes \cdots \otimes a_{k-1})(b_2 \otimes \cdots \otimes b_\ell)
\]

where the second term is already defined by induction.

It is straightforward to check that this bilinear operation endows \( A \) with the structure of a unital algebra and that the inclusions \( A_1 \to \mathbb{C}.1 \oplus A'_1 \) and \( A_2 \to \mathbb{C}.1 \oplus A'_2 \) are unital algebra homomorphisms. But to convince the reader that this definition is sound, the best way is to establish the unavoidable universal property that such a construction should satisfy.

**Exercise 23.** Let \( B \) be a unital algebra and let \( \pi_1 : A_1 \to B \) and \( \pi_2 : A_2 \to B \) be unital algebra homomorphisms. Then, there exists a unique unital algebra homomorphism \( \pi : A \to B \) such that \( \pi|_{A_1} = \pi_1 \) and \( \pi|_{A_2} = \pi_2 \).

**Solution.** Because \( \pi_1 \) and \( \pi_2 \) are in particular linear maps, we can extend them to \( A^{(k)} \) for each \( k \geq 1 \) by taking \( \pi_{i_1} \otimes \cdots \otimes \pi_{i_k} \) on each summand. As for the unit, since \( \pi_1 \) and \( \pi_2 \) are both unital, it is coherent to set \( \pi(1_A) = 1_B \). The map \( \pi \) defined in that way certainly restricts correctly to the summands and we only have to check that it is an algebra homomorphism. This directly follows by induction from the definitions and the fact that \( \pi_1 \) and \( \pi_2 \) are algebra homomorphisms, using and induction on the total length of the two elements we multiply. ■

**Remark 6.3.2.** The above universal property implies that \( A \) does not depend, up to unital isomorphism, on the decompositions of \( A_1 \) and \( A_2 \) which were fixed in the beginning.

We will denote the algebra \( A \) by \( A_1 \ast A_2 \) and call it the **unital free product** of \( A_1 \) and \( A_2 \). Note that if \( A_1 \) and \( A_2 \) are *-algebras, then there is a unique associated *-algebra structure on \( A_1 \ast A_2 \) which turns the inclusions into *-homomorphisms, namely

\[
(a_1 \otimes \cdots \otimes a_k)^* = a_k^* \otimes \cdots \otimes a_1^*
\]

We can now use this definition to extend the notion to the setting of quantum groups.

**The compact matrix quantum group case**

Let \((G, u)\) and \((H, v)\) be two unitary compact matrix quantum groups. Form the unital free product \( A = \mathcal{O}(G) \ast \mathcal{O}(H) \) using the decompositions

\[
\mathcal{O}(G) = \mathbb{C}.1_{\mathcal{O}(G)} \oplus \ker(h_G) \quad \text{and} \quad \mathcal{O}(H) = \mathbb{C}.1_{\mathcal{O}(H)} \oplus \ker(h_H),
\]

where \( h_G \) and \( h_H \) denote the Haar states of \( G \) and \( H \) respectively, and consider the matrix

\[
w = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \in M_{\dim(u) + \dim(v)}(A).
\]

Applying Exercise 23 to the coproducts

\[
\Delta_G : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G) \subset A \otimes A \\
\Delta_H : \mathcal{O}(H) \to \mathcal{O}(H) \otimes \mathcal{O}(H) \subset A \otimes A
\]

yields a *-homomorphism \( \Delta : A \to A \otimes A \) such that

\[
\Delta(w_{ij}) = \sum_{k=1}^{\dim(u) + \dim(v)} w_{ik} \otimes w_{kj}.
\]

Since moreover both \( w \) and \( w \) are unitary by construction, and their coefficients generate \( A \), we have built a new unitary compact matrix quantum group.
6.3. Making things complex

Definition 6.3.3. The unitary compact matrix quantum group \((O(G) \ast O(H), w)\) is called the free product of \((G, u)\) and \((H, v)\) and denoted by \(G \ast H\).

Remark 6.3.4. It is an easy exercise to check that if \(\Gamma_1, \Gamma_2\) are discrete groups, then

\[
\hat{\Gamma}_1 \ast \hat{\Gamma}_2 \simeq \hat{\Gamma}_1 \ast \hat{\Gamma}_2,
\]

where \(\Gamma_1 \ast \Gamma_2\) denotes the free product of groups (see for instance [BO08, Def E.8]. For that reason, many authors write \(G \ast H\) for the free product of two compact quantum groups. We have however chosen to drop the hat to lighten notations because we will not consider the discrete setting hereafter.

In the remainder of this section, we want to prove that \(U_N^+\) is a complexification of \(O_N^+\) in a free product sense. The proof of this requires the knowledge of the representation theory of free products in terms of the representation theory of the factors. This was first described by S. Wang in [Wan95a]. For practical reasons, we will first define a complete family of representatives of the irreducible representations.

Definition 6.3.5. Let \((G, u)\) and \((H, v)\) by unitary compact matrix quantum groups. A word \(w = \alpha_1 \alpha_2 \cdots \alpha_k\) on \(\text{Irr}(G) \setminus \{\varepsilon_G\} \sqcup \text{Irr}(H) \setminus \{\varepsilon_H\}\), where \(\sqcup\) denotes the disjoint union of sets, is said to be reduced if

- \(\alpha_{i+1} \in \text{Irr}(G)\) if \(\alpha_i \in \text{Irr}(H)\),
- \(\alpha_{i+1} \in \text{Irr}(H)\) if \(\alpha_i \in \text{Irr}(G)\).

If \(w\) is a reduced word, then

\[
u^w = u^{\alpha_1} \otimes \cdots \otimes u^{\alpha_k}
\]

is said to be a reduced representation.

Theorem 6.3.6 (Wang) Let \((G, u)\) and \((H, v)\) by unitary compact matrix quantum groups. Then, reduced representations are irreducible and pairwise inequivalent. Moreover, any non-trivial irreducible representations of \(G \ast H\) is equivalent to a reduced representation.

Proof. We will proceed step-wise to prove by induction on the length of \(w\) that

\[H_k : \text{"All reduced representations given by words of length less than or equal to } k \text{ define pairwise inequivalent irreducible representations."}\]

1. Words of length one are just representations of \(G\) and \(H\) so that words of length one yield irreducible representations. Moreover, a representation of \(G\) cannot be equivalent to a representation of \(H\). Indeed, if this was the case then the coefficients would have the same linear span by Point 2 of Theorem 2.1.14. But the decomposition we took to define the unital free product implies that coefficients of non-trivial irreducible representations, which are in \(\ker(h)\), are linearly independent. Thus, \(H_1\) holds.

2. Assume now that \(H_k\) holds and consider a reduced word \(w = \alpha_1 \cdots \alpha_{k+1}\). Then, by Frobenious reciprocity (see Proposition 5.2.9),

\[
\text{Mor}_{G \ast H}\left(u^w, u^w\right) \simeq \text{Mor}_{G \ast H}\left(u^w \otimes \pi^{\alpha_{k+1}}, u^{\alpha_1} \otimes u^{\alpha_2} \otimes \cdots \otimes u^{\alpha_k}\right).
\]

We know again by Proposition 5.2.9 that there is a decomposition

\[
u^{\alpha_{k+1}} \otimes \pi^{\alpha_{k+1}} = v^{\beta_1} \oplus \cdots \oplus v^{\beta_n}
\]

into irreducible subrepresentations with \(v^{\beta_i} \neq v\) for all \(1 \leq i \leq n\). Now, the coefficients of \(u^{\alpha_1} \otimes u^{\alpha_2} \otimes \cdots \otimes u^{\beta_i}\) belong to \(\mathcal{A}^{(k+1)}\), hence are linearly independent from those of \(u^{\alpha_1} \otimes u^{\alpha_2} \otimes \cdots \otimes u^{\alpha_k}\) who lie in \(\mathcal{A}^{(k)}\). As a consequence,

\[
\text{Mor}_{G \ast H}\left(u^{\alpha_1} \otimes u^{\alpha_2} \otimes \cdots \otimes u^{\beta_i}, u^{\alpha_1} \otimes u^{\alpha_2} \otimes \cdots \otimes u^{\alpha_k}\right) = \{0\}.
\]
Thus,
\[ \text{Mor}_{G \ast H}(u^w, u^{w'}) \simeq \text{Mor}_{G \ast H}(u^{\alpha_1} \otimes u^{\alpha_2} \otimes \cdots \otimes u^{\alpha_k}, u^{\alpha'_1} \otimes u^{\alpha'_2} \otimes \cdots \otimes u^{\alpha'_k}) \]
which is one-dimensional by \( H_k \), hence \( u^w \) is irreducible.

3. Consider now two words \( w \) and \( w' \) of length \( k + 1 \). There are two cases.

- If \( \alpha_i \) and \( \alpha'_i \) are not representations of the same group for some \( 1 \leq i \leq k + 1 \), then the coefficients of \( u^w \) and \( u^{w'} \) belong to different summands of \( A^{(k+1)} \), hence the representations are inequivalent.
- Otherwise, we have
\[ \text{Mor}_{G \ast H}(u^w, u^{w'}) \simeq \text{Mor}_{G \ast H}(u^w \otimes \pi^{w_{k+1}}, u^{\alpha'_1} \otimes \cdots \otimes u^{\alpha'_{k+1}}). \]

Proceeding as for irreducibility, we split \( u^{w_{k+1}} \otimes \pi^{w_{k+1}} \) into a sum of irreducibles. All the terms yield null morphisms spaces except perhaps the trivial representation, which appears in the decomposition if and only if \( w'_{k+1} = w_k \). This then reduces the problem to two words of length \( k \) and we conclude by \( H_k \).

To conclude the proof of the statement, simply observe that the coefficients of reduced representations, together with 1 which is the coefficient of the trivial representation, span \( O(G \ast H) \). Thus, by Theorem 2.1.14, any irreducible representation is equivalent to a reduced one.

### 6.3.2 Free complexification

We can now define a general notion of complexification of an orthogonal compact matrix quantum group, based in the unital free product construction. Let us denote from now by \( \mathbb{T} \) the circle group, i.e. the group of complex numbers of modulus one. If \( z \) denotes the inclusion \( \mathbb{T} \hookrightarrow \mathbb{C} \), then \( z \) is a fundamental (one-dimensional) representation of \( \mathbb{T} \) since it generates the \( * \)-algebra \( O(\mathbb{T}) \) of trigonometric polynomials. We will always use this description of \( \mathbb{T} \) as a unitary compact matrix quantum group.

**Proposition 6.3.7.** Let \((G, u)\) be an orthogonal compact matrix quantum group and let \((O(\mathbb{T}), z)\) be the circle group. Then, the subalgebra \( A \) of \( O(G \ast \mathbb{T}) \) generated by \( u_{ij} z \) yields a unitary compact matrix quantum group when endowed with the fundamental representation \( uz = (u_{ij} z) \).

**Proof.** First observe that \((uz)^* uz = u^t u = \text{Id} = uz(uz^*)\) and that the same holds for \( \pi z = u^t z \). Moreover, by definition of the coproduct on the free product,
\[ \Delta(u_{ij} z) = \Delta(u_{ij})\Delta(z) = \left( \sum_{k=1}^{N} u_{ik} \otimes u_{kj} \right) (z \otimes z) = \sum_{k=1}^{N} u_{ik} z \otimes u_{kj} z. \]

Since by construction the coefficients of \( uz \) generate \( A \), the proof is complete.

We will set \( A = O(\hat{G}) \), write \( \hat{u} = uz \) and call \((\hat{G}, \hat{u})\) the free complexification of \( G \). We finally have everything in hand to make sense of \( U_N^+ \) as a complexification of \( O_N^+ \). This is originally due to T. Banica in [Banica97].

**Theorem 6.3.8** (Banica) There is an isomorphism
\[ \hat{O}_N^+ \simeq U_N^+. \]

**Proof.** Because \( \hat{O}_N^+ \) is a unitary compact matrix quantum group generated by a fundamental representation \( W = Uz \) of dimension \( N \), there exists by the universal property of \( O(U_N^+) \) a surjective \( * \)-homomorphism
\[ \pi : O(U_N^+) \rightarrow O(\hat{O}_N^+) \]
sending \( V_{ij} \) to \( W_{ij} \). Our strategy will be to prove that \( \pi \) induces a bijection between equivalence classes of irreducible representations of both compact matrix quantum groups, and then use it to conclude that \( \pi \) itself must be an isomorphism. We will proceed step-wise.
1. We will use the notations of Theorem 4.3.1 and write $u_n = u_{[0,n]}$ for a representative of the $n$-th equivalence class of irreducible representations of $O_N^+$. Recall in particular that $U = u_1$. Let us denote by $S \subset \text{Irr}(O_N^+ \ast T)$ the set of all irreducible representations of the form

$$z^{[\ell_n]} - u_1\zeta \cdots z^{[\ell_p]} u_{[n]}z^{[\ell_p]}$$

with $[-1]_- = -1$, $[-1]_+ = 0$, $[1]_- = 0$, $[1]_+ = 1$ and $\epsilon_{i+1} = (-1)^{\nu_i+1} \epsilon_i$. We claim that $S$ is stable under taking tensor products with $u_1z$ and its conjugate, it contains representatives of all the equivalence classes of irreducible representations of $O_N^+$.

2. We will now use the indexing of irreducible representations of $U_N^+$ from Theorem 6.2.2, which is given by the set $M$ of all words on $\{\circ, \bullet\}$. We define a map $\Phi : M \to S$ inductively on the length of the words in the following way:

- $\Phi(\circ) = u_1z$, $\Phi(\bullet) = z^{-1}u_1$;
- If $w = (\circ)^i (\bullet)^j \circ w'$, then $\Phi(w) = u_{2i} \Phi(\bullet w')$;
- If $w = (\circ)^i \circ \circ w'$, then $\Phi(w) = z^{-1} u_{2i} z \Phi(\circ w')$;
- If $w = (\circ)^i \circ \bullet w'$, then $\Phi(w) = u_{2i+1} \Phi(\bullet w')$;
- If $w = (\circ)^i \circ \bullet w'$, then $\Phi(w) = z^{-1} u_{2i+1} \Phi(\bullet w')$.

This is indeed a bijection, with inverse inductively defined by

- $\Phi^{-1}(u_1z) = \circ$, $\Phi^{-1}(z^{-1}u_1) = \bullet$;
- If $w = u_{2i} w'$, then $\Phi^{-1}(w) = (\circ)^i \Phi^{-1}(w')$;
- If $w = z^{-1} u_{2i} z w'$, then $\Phi^{-1}(w) = (\circ)^i \Phi^{-1}(w')$;
- If $w = u_{2i+1} w'$, then $\Phi^{-1}(w) = (\circ)^i \circ \Phi^{-1}(w')$;
- If $w = z^{-1} u_{2i+1} w'$, then $\Phi^{-1}(w) = (\circ)^i \circ \Phi^{-1}(w')$.

3. One easily checks by induction that the following properties hold:

- $\Phi(\overline{w}) = \overline{\Phi(w)}$;
- $\Phi(\circ \otimes w) = \Phi(\circ) \otimes \Phi(w)$;
- $\Phi(\bullet \otimes w) = \Phi(\bullet) \otimes \Phi(w)$.

As a consequence, any element of $S$ can be obtained by taking tensor products of $u_1z$ and $z^{-1}u_1$. In other words, $S = \text{Irr}(O_N^+)$.  

4. Note that because $\pi$ sends $V$ to $W$ coefficient-wise, we have

$$\pi \circ \Delta_{U_N^+} = \left( \Delta_{\overline{O}_N^+} \otimes \Delta_{\overline{O}_N^+} \right) \circ \pi.$$

As a consequence, if $u = (u_{ij})_{1 \leq i, j \leq \dim(u)}$ is a representation of $U_N^+$, then

$$\tilde{\pi}(u) = (\pi(u_{ij}))_{1 \leq i, j \leq \dim(u)}$$

is a representation of $\overline{O}_N^+$. Moreover, we have the following properties:

- $\tilde{\pi}(\overline{w}) = \overline{\tilde{\pi}(w)}$,
Chapter 6. A unitary excursion

• $\hat{\pi}(u \otimes v) = \hat{\pi}(u) \otimes \hat{\pi}(v)$,
• $\hat{\pi}(u^o) = u_1 z$ and $\hat{\pi}(u^\ast) = z^{-1}u_1$.

Because the map $\Phi$ defined before satisfies the same properties when applied to the representations $u^w$ for $w \in M$, we have that for any $w \in M$,

$$\hat{\pi}(u^w) = \Phi(u^w)$$

so that in particular $\hat{\pi}(u^w)$ is non-trivial if $w$ is not the empty word.

5. Let us notice that by the previous point, if $u^w_{ij}$ is a coefficient of the non-trivial irreducible representation $u^w$, then

$\pi(u^w_{ij}) = \hat{\pi}(u^w)_{ij}$

is a coefficient of the non-trivial representation $\hat{\pi}(u^w)$. Thus, composing with the Haar state of $O^+_N$ yields

$$h_{O^+_N} \circ \pi(u^w_{ij}) = 0 = h_{U^+_N}(u^w_{ij})$$

and because the coefficients of irreducible representations form a linear basis of $O(U^+_N)$, this implies that $h_{U^+_N} = h_{O^+_N} \circ \pi$. We can now conclude : if $x \in \ker(\pi)$, then

$$h_{U^+_N}(x^x) = h_{O^+_N} \circ \pi(x^x) = h_{O^+_N}(\pi(x)^* \pi(x)) = 0$$

and by faithfulness of the Haar state, $x = 0$. Thus, $\pi$ is injective, concluding the proof.

6.3.3 Tensor complexification

We mentioned in the beginning of this section that $U_N$ could be classically thought of as a complex version of $O^+_N$. For the sake of completeness, we will now give a precise sense to this. The free complexification is of course not adapted to such a situation, since $O(O_N)$ is a non-commutative commutative algebra, hence does not correspond to a classical group. To deal with classical groups, we need to stay in the commutative realm. This can be done by replacing the free product by a tensor product.

Let $\left( G, u \right)$ and $\left( H, v \right)$ be unitary compact matrix quantum groups and form the tensor product $A = O(G) \otimes O(H)$. We can define the tensor product of the coproducts

$\Delta_G \otimes \Delta_H : A \to O(G) \otimes O(G) \otimes O(H) \otimes O(H)$

but it does not have the correct target algebra. This can easily be remedied by setting

$$\Delta = (id \otimes \Sigma \otimes id) \circ \Delta_G \otimes \Delta_H : A \to A \otimes A,$$

where $\Sigma : O(G) \otimes O(H) \to O(H) \otimes O(G)$ is the flip map sending $x \otimes y$ to $y \otimes x$. One can then check directly that this coproduct is compatible with the representation

$$w = \begin{pmatrix} u \otimes 1 & 0 \\ 0 & 1 \otimes v \end{pmatrix}$$

so that $\left( A, w \right)$ is a unitary compact matrix quantum group.

Definition 6.3.9. The unitary compact matrix quantum group $\left( O(G) \otimes O(H), w \right)$ is called the direct product of $G$ and $H$ and denoted by $G \times H$.

Even though this is not needed in the sequel, let us work out the representation theory of direct products as computed by S. Wang in [Wan95b].
**Theorem 6.3.10** (Wang) Let \((G, u)\) and \((H, v)\) by unitary compact matrix quantum groups. Then, the representations
\[
u^\alpha \otimes \nu^\beta
\]
for \((\alpha, \beta)\) in \((\text{Irr}(G) \setminus \{e_G\}) \times (\text{Irr}(H) \setminus \{e_H\})\) form a complete set of equivalence classes of irreducible and pairwise inequivalent representations of \(G \times H\).

**Proof.** If \(\beta \neq \beta'\), then
\[
\text{Mor}_{G \times H} \left( \nu^\alpha \otimes \nu^\beta, \nu^{\alpha'} \otimes \nu^{\beta'} \right) \cong \text{Mor}_{G \times H} \left( \nu^\alpha \otimes \nu^\beta \otimes \nu^{\beta'}, \nu^{\alpha'} \right) \\
\cong \bigoplus_{\gamma \in [\beta \otimes \beta']} \text{Mor}_{G \times H} \left( \nu^\alpha \otimes \nu^\gamma, \nu^{\alpha'} \right).
\]
If there is a non-zero element in the latter space then, by irreducibility, there exists \(\gamma \subset \beta \otimes \beta'\) such that \(\nu^{\alpha'}\) is a subrepresentation of \(\nu^\alpha \otimes \nu^\gamma\). Thus, the coefficients \(u_{ij}^{\alpha'} \otimes 1\) are linear combinations of coefficients \(u_{ij}^{\alpha} \otimes u_{cd}^{\gamma}\). But since \(\gamma \neq e\), 1 is not in the linear span of the coefficients \(u_{ij}^{\alpha'}\), hence a contradiction by linear independence.

The same argument applies to the case \(\alpha \neq \alpha'\), so that the representations in the statement are pairwise inequivalent. Moreover, by the same argument,
\[
\text{Mor}_{G \times H} \left( \nu^\alpha \otimes \nu^\beta, \nu^{\alpha} \otimes \nu^\beta \right) \cong \text{Mor}_{G \times H} \left( \nu^\alpha \otimes \nu^\beta \otimes \nu^{\beta'}, \nu^\alpha \right) \\
\cong \text{Mor}_{G \times H} \left( \nu^\alpha, \nu^\alpha \right) + \bigoplus_{\gamma \in [\beta \otimes \beta'], \gamma \neq e} \text{Mor}_{G \times H} \left( \nu^\alpha \otimes \nu^\gamma, \nu^\alpha \right) \\
= \text{Mor}_{G \times H} \left( \nu^\alpha, \nu^\alpha \right)
\]
and the latter space is one-dimensional, proving the irreducibility of \(\nu^\alpha \otimes \nu^\beta\).

Eventually, because the coefficients of all these representations generate \(O(G \otimes H)\), we have found all irreducible representations. \(\blacksquare\)

Noticing that \(O(G) \otimes O(H)\) is commutative as soon as both \(O(G)\) and \(O(H)\) are, the recipe to complexify a classical group is now clear. For convenience, let us simply write \(ab\) for \(a \otimes b\) when the context is unambiguous.

**Proposition 6.3.11.** Let \((G, u)\) be an orthogonal compact matrix quantum group. Then, the subalgebra \(A\) of \(O(G \times \mathbb{T})\) generated by \(u_{ij} z\) yields a unitary compact matrix quantum group when endowed with the fundamental representation \(uz = (u_{ij} \otimes z)\).

**Proof.** First observe that \((uz)^* uz = u^t u = \text{Id} = uz(uz)^*\) and that the same holds for \(\nu z = u^t z\).

Moreover, by definition of the coproduct on the tensor product,
\[
\Delta(u_{ij} z) = \Delta(u_{ij}) \Delta(z) = \left( \sum_{k=1}^N u_{ik} \otimes u_{kj} \right) (z \otimes z) = \sum_{k=1}^N u_{ik} z \otimes u_{kj} z.
\]
Since by construction the coefficients of \(uz\) generate \(A\), the proof is complete. \(\blacksquare\)

We will write \(A = O(\mathbb{T})^4\) and call this unitary matrix quantum groups the tensor complexification of \(G\). As an illustration of this notion, let us conclude this section by proving that \(U_N\) is indeed the tensor complexification of \(O_N\).

**Proposition 6.3.12.** There is an isomorphism
\[
U_N^+ \simeq O_N.
\]
\(\text{4. This notation is not standard in any respect, but we simply needed something different from the tilda used for the free complexification.}\)
Proof. Let us consider the natural quotient map
\[ \Psi : \mathcal{O}(U_N^+) \simeq \mathcal{O}(\hat{O}_N) \to \mathcal{O}(\hat{O}_N) \]
given by quotienting by the ideal generated by the commutators \([u_{ij}, u_{k\ell}]\) and \([u_{ij}, z]\) for all \(1 \leq i, j, k, \ell \leq N\). Because this is a quotient by commutators, it factors through the abelianization map
\[ \pi_{ab} : \mathcal{O}(U_N^+) \to \mathcal{O}(U_N), \]
i.e. there exists a surjective \(*\)-homomorphism
\[ \pi : \mathcal{O}(\hat{O}_N) \to \mathcal{O}(U_N) \]
such that \(\pi_{ab} = \pi \circ \Psi\). On the other hand, because \(\hat{O}_N\) is a unitary compact matrix group, there exists \(\pi' : \mathcal{O}(U_N) \to \mathcal{O}(\hat{O}_N)\) which sends the fundamental representation to the fundamental representation coefficient-wise, i.e. such that
\[ \Psi = \pi' \circ \pi_{ab} = \pi' \circ \pi \circ \Psi. \]
By surjectivity of \(\Psi\), \(\pi' \circ \pi = \text{id}_{\mathcal{O}(\hat{O}_N)}\), so that \(\pi\) is injective. Since it is also surjective by definition, it is an isomorphism and the proof is complete. \(\blacksquare\)
CHAPTER 7

FURTHER EXAMPLES

Now that we have adapted our combinatorial tools to the unitary setting, we can use them to study an interesting and important family of examples, called quantum reflection groups. To understand the origin of the definition, let us first discuss classical reflection groups. We have already encountered two examples of classical finite groups which can be generated by elements of order two, which can be thought of as “reflections” in the general sense: the permutation group $S_N$ and the hyperoctaedral group $H_N$. Recall that the second one was made of permutation matrices, but with coefficients allowed to be ±1. This can be generalized in the following way:

**Definition 7.0.1.** Let $N$ and $s$ be integers. The reflection group $H_N^s$ is the group of permutation matrices in $M_N(\mathbb{C})$ with coefficients being $s$-th roots of unity.

### 7.1 Quantum Reflection Groups

#### 7.1.1 Liberation of Reflection Groups

Our purpose in this section is to find a definition of a quantum version of reflection groups. Finding the correct defining relations of the algebra $\mathcal{O}(H_N^s)$ which would also make sense in a non-commutative context is not easy. This problem was in fact one of the main motivations for the introduction of partition quantum groups in [BS09]. We will follow this path by using the following strategy: first describe the intertwiners of reflection groups with coloured partitions, then restrict to non-crossing partitions to produce a quantum object.

It turns out that for $H_N^s$, a partition description of the intertwiner spaces was given by K. Tanabe in [Tan97], though in a different language. We will restate this in terms of coloured partitions using the following two ones. First, we define $q_s \in NC^\bullet(2,2)$ to be the one-block partition with all points coloured in white:

![Partition 1](image1)

Second, we define $p_s \in NC^\bullet(s,0)$ to be the one-block partition with all blocks coloured in white:

![Partition 2](image2)

Let us prove that this is sufficient to recover the reflection groups $H_N^s$ as partition quantum groups.
Chapter 7. Further examples

**Theorem 7.1.1** (Tanabe) Let \( s, N \in \mathbb{N} \) and let \( \mathcal{C}_{s,\text{classical}} \) be the symmetric category of coloured partitions generated by \( q \) and \( p_s \). Then, there is an isomorphism
\[
\mathbb{G}_N(\mathcal{C}_{s,\text{classical}}) = H_N^s
\]

**Proof.** By definition \( \mathbb{G}_N(\mathcal{C}_{s,\text{classical}}) \) is a group of unitary matrices in \( M_N(\mathbb{C}) \) whose coefficients satisfy some extra relations given by the partitions \( q \) and \( p_s \). Let us start with \( q \) and compute:
\[
\left( \text{id} \otimes T_{q_s} \right) \circ \rho_{u \otimes u} (e_{i_1} \otimes e_{i_2}) = \sum_{j_1, j_2=1}^{N} u_{i_1 j_1} u_{i_2 j_2} \otimes T_{q_s} (e_{j_1} \otimes e_{j_2})
\]
\[
= \sum_{j=1}^{N} u_{i_1 j} u_{i_2 j} \otimes e_j \otimes e_j
\]
while
\[
\rho_{u \otimes \pi} \circ T_{q_s} (e_{i_1} \otimes e_{i_2}) = \delta_{i_1 i_2} \sum_{k=1}^{N} \rho_{u \otimes u} (e_{i_1} \otimes e_{i_1})
\]
\[
= \delta_{i_1 i_2} \sum_{j_1, j_2=1}^{N} u_{i_1 j_1} u_{i_2 j_2} \otimes e_{j_1} \otimes e_{j_2}.
\]

These two expressions are equal if and only if
\[
u_{ij} u_{ij'} = 0 = u_{ij} u_{ij'}
\]
for any \( 1 \leq i, i', j, j' \leq N \) such that \( i \neq i' \) and \( j \neq j' \). This precisely means that the matrices in \( \mathbb{G}_N(\mathcal{C}_{s,\text{classical}}) \) have only one non-zero coefficient on each row. Because the columns of a unitary matrix form an orthonormal basis, there is also only one non-zero coefficient on each column, and these coefficients have modulus one.

We now turn to \( p_s \) and compute:
\[
\left( \text{id} \otimes T_{p_s} \right) \circ \rho_{u \otimes s} (e_{i_1} \otimes \cdots \otimes e_{i_s}) = \sum_{j_1, \cdots, j_s=1}^{N} u_{i_1 j_1} \cdots u_{i_s j_s} \otimes T_{p_s} (e_{j_1} \otimes \cdots \otimes e_{j_s})
\]
\[
= \sum_{j=1}^{N} u_{i_1 j} \cdots u_{i_s j} \otimes e_j \otimes \cdots \otimes e_j
\]
while
\[
\rho_s \circ T_{p_s} (e_{i_1} \otimes \cdots \otimes e_{i_s}) = \delta_{i_1 \cdots i_s}.
\]

These two quantities are equal if and only if
\[
\sum_{j=1}^{N} u_{ij}^s = 1.
\]
Given that only one coefficient in the sum can be non-zero, this is equivalent to the fact that for each matrix in \( \mathbb{G}_N(\mathcal{C}_{s,\text{classical}}) \), the non-zero coefficient of each row is an \( s \)-th root of unity, hence the result. \( \blacksquare \)

**Remark 7.1.2.** In his work [Tan97], K. Tanabe deals with the more general family of complex reflection groups \( G(m, p, n) \), of which \( H_N^s \) is the particular case \((s, s, N)\). Defining quantum analogues of the full family \( G(m, p, n) \) is, to this date, still an open problem. However, the complete classification of non-crossing unitary partition quantum groups shows that it cannot be solved in this setting.
We can now illustrate one of the interesting feature of the partition approach to groups and quantum groups. Indeed, as a direct byproduct of Theorem 7.1.1, we obtain a definition of a quantum version of the reflection groups.

**Definition 7.1.3.** Let \( s, N \) be integers and set \( C_s = \langle q_0, p_s \rangle \). The partition quantum group \( \mathbb{G}_N(C_s) \) is called a quantum reflection group and denoted by \( H_N^ {s+} \).

By definition, the ablinization map is a surjection from \( \mathcal{O}(H_N^ {s+}) \) to \( \mathcal{O}(H_N^ {s}) \) mapping the fundamental representation to the fundamental representation, hence \( H_N^ {s+} \) is indeed the classical version of \( H_N^ {s} \). As a first attempt to get some grasp on these objects, let us work out some very basic examples.

**Example 7.1.4.** Let \( s = 1 \), so that \( p_s \) is the white singleton partition \( s_o \). Then,

\[
D_{oo} = (p_1 \otimes p_1) q_o \in C_1
\]
and rotating this yields the identity partition with different colours on the rows. Thus, by Proposition 6.1.7 we can forget colours in \( C_1 \). Since it contains a four-block and a singleton, it contains a three block by Proposition 3.4.5. Thus, \( C_1 = NC^\circ \) and it follows that

\[
H_N^ {1+} = S_N^+.
\]

**Example 7.1.5.** Let \( s = 2 \), so that \( p_2 = D_{oo} \) with both points coloured in white. Rotating it yields the identity partition with different colours, hence by Proposition 6.1.7 we can forget the colours. Since \( C_2 \) contains the four-block and only partitions with even blocks, \( C_2 = NC^\circ_{even} \) by Exercise 18, so that

\[
H_N^ {2+} = H_N^ +.
\]

### 7.1.2 Representation theory

Our goal is now to compute the representation theory of quantum reflection groups. However, our definition of the categories of partitions \( C_s \) is inconvenient for that, because it only gives us generators. It would be better to have an intrinsic characterization of the partitions in \( C_s \). We will present such a characterization now, which requires some vocabulary.

**Definition 7.1.6.** For a coloured partition \( p \in P^\circ(k, 0) \) let \( c_u(p) \) (respectively \( c_l(p) \)) be the difference between the number of white and black points in its upper row (respectively its lower row). The colour sum of \( p \) is then defined as

\[
c(p) = c_u(p) - c_l(p).
\]

We will characterize the partitions in \( C_s \) by the colour sum of their blocks, but for practical reasons it is simpler to define another set of partitions, and then to prove equality.

**Definition 7.1.7.** We define \( D_s \) to be the set of all non-crossing partitions \( p \in NC^\circ \) such that for each block \( b \subset p \),

\[
c(b) = 0 \mod s.
\]

**Remark 7.1.8.** We only defined \( D_s \) as a set, but the reader can check that it is indeed a category of partitions. This will however be proven as a consequence of the equality \( D_s = C_s \). Note that it is at least clear that the defining property of \( D_s \) is preserved under horizontal concatenation, and this is all we will need in the sequel.

Because \( c(q_o) = 0 \) and \( c(p_s) = s \), we have \( C_s \subset D_s \) by definition. We claim that this inclusion is an equality. To keep things clear, we will first work with \( q_o \) alone.

**Proposition 7.1.9.** Let \( D_{oo} \) be the set of all non-crossing partitions \( p \in NC^\circ \) such that for each block \( b \subset p \), \( c(b) = 0 \). Then,

\[
\langle q_o \rangle = D_{oo}
\]

**Proof.** Because \( q_o \subset D_{oo} \), we have an inclusion of the left-hand side into the right-hand side. As for the converse, let us first show that it holds for one-block partitions by induction on the number of points.
For partitions with two points the result is clear.

Let us now assume that the result holds for one-block partitions on at most \(2n\) points (note that for the colour sum to vanish, there must be an even number of points) and consider a one-block partition \(p\) on \(2(n + 1)\) points. Up to rotation, we may assume that \(p\) lies on one line and that its first two points (from the left) are coloured by \(\bullet\). Let \(p'\) be the partition formed by the last \(2n\) points of \(p\). By induction, since

\[
c(p') = c(p) = 0,
\]

\(p' \in \mathcal{D}_\infty\), hence \(p' \in \langle q_0 \rangle\) by the induction hypothesis. But then,

\[
p = (q \otimes p')(id_0 \otimes q' \otimes \mathbb{I}^{\otimes n}),
\]

where \(q'\) is one of the following partitions

\[
\begin{array}{c}
\begin{array}{c}
\bullet \\
\circ
\end{array} \\
\begin{array}{c}
\bullet \\
\circ
\end{array}
\end{array}
\quad \text{or} \quad
\begin{array}{c}
\begin{array}{c}
\bullet \\
\circ
\end{array} \\
\begin{array}{c}
\bullet \\
\circ
\end{array}
\end{array}
\]

If we prove that these two partitions are in \(\langle q_0 \rangle\), then we will be done. The second one is just a rotation of \(q_0\). As for the first one, we have

\[
\tilde{q} = (id_0 \otimes D_{\bullet \circ} \otimes id_\bullet)(q_0 \otimes q_\bullet)(id_0 \otimes D_{\circ \bullet} \otimes id_\circ) \in \langle q_0 \rangle
\]

hence the result.

We can now prove the equality by induction on the number of points. For partitions on at most four points, the result is clear. If it is true for partitions on at most \(2n\) points, let \(p \in \mathcal{D}_\infty\) be a partition on \(2(n + 1)\) points. If it has only one block, then it is in \(\langle q_0 \rangle\) by the first part of the proof. Otherwise, rotate \(p\) so that it has the form \(p = q \otimes p'\). It follows from the definition of \(\mathcal{D}_\infty\) that both \(q\) and \(p'\) are in it, hence the same holds for \(p\), concluding the proof.

The proof of the equality \(\mathcal{C}_s = \mathcal{D}_s\) now follows easily by arguments similar to those of Proposition 7.1.9.

**Proposition 7.1.10.** For any \(1 \leq s < +\infty\), \(\mathcal{C}_s = \mathcal{D}_s\).

**Proof.** Let us first prove that any one-block partition with only white points of \(\mathcal{D}_s\) is in \(\mathcal{C}_s\). Such a partition must be, after a suitable rotation, of the form \(p_{ns}\) for some integer \(n\). But

\[
p_{ns} = (p_s^{\otimes n}) \left( id_{\circ}^{\otimes s-1} \otimes q_0 \otimes id_{\circ}^{\otimes s-2} \otimes q_0 \otimes \cdots \otimes q_0 \otimes id_{\circ}^{\otimes s-1} \right) \in \mathcal{C}_s.
\]

If now \(p\) is a one-block partition with both black and white points, we can rotate it on one line so that the colour of the first two points are \(\bullet\). If \(p'\) is the partition formed by the remaining points, \(c(p') = 0 \mod s\). From this observation, one can prove by induction the \(p \in \mathcal{C}_s\) using the formula

\[
p = (\tilde{q} \otimes p')(id_0 \otimes q' \otimes \mathbb{I}^{\otimes s})
\]

where once again \(q' \in \{\tilde{q}, q_\bullet\}\).

Using this, one can conclude by induction exactly as in the proof of Proposition 7.1.9.

We will now use this result to compute the representation theory of \(H_N^{s+}\). As for \(U_N^{+}\), we first define the abstract object indexing the equivalence classes of irreducible representations. Let \(M_s\) be the free monoid on \(\mathbb{Z}_s\), that is to say the set of all words on \(\{0, \ldots, s - 1\}\). The monoid structure is a priori given by the concatenation of words. However, the fact that \(\mathbb{Z}_s\) is a group
and all points white, i.e.

Eventually, the conjugation is given by

\[ w \ast w' = w_1 \cdots w_{k-1}(w_k + w'_1)w'_2 \cdots w'_k. \]

We define, for \( k \leq 1 \),

\[ \pi w \rightarrow \pi w^{-1}. \]

Moreover, for any \( w, w' \in M_s \),

\[ u^w \otimes u^{w'} = \bigoplus_{w = \alpha z, w' = \beta b} \left( u^{ab} \oplus u^{a \ast b} \right). \]

**Theorem 7.1.11** (Banica-Vergnioux) For \( N \geq 4 \), the irreducible representations of \( H_N^{++} \) can be indexed by the elements of \( M_s \) in such a way that \( u^0 = \varepsilon \), \( u^0 = u \) and for any \( w \in M_s \),

\[ \pi w = u^w. \]

Moreover, for any \( w, w' \in M_s \),

\[ u^w \otimes u^{w'} = \bigoplus_{w = \alpha z, w' = \beta b} \left( u^{ab} \oplus u^{a \ast b} \right). \]

**Proof.** Let us define, for \( 1 \leq i \leq s \), \( \pi_i \) to be the projective partition in \( C_s(i, i) \) with one block and all points white, i.e.

\[ \pi_i = \begin{array}{cccc}
0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 \\
\end{array} \]

Let us also set for convenience \( \pi_0 = q \). We will consider the map \( \Phi : M_s \rightarrow \text{Irr}(H_N^{++}) \) given by

\[ \Phi(w_1 \cdots w_k) = \left[ u_{\pi w_1} \otimes \cdots \otimes u_{\pi w_k} \right]. \]

1. We first prove that \( \Phi \) is surjective. Let \( p \in C_s \) be a projective partition. As usual, we decompose it as

\[ p = (b_0^0 b_0) \otimes p_1 \otimes (b_1^* b_1) \otimes p_2 \otimes \cdots \otimes p_k \otimes (b_k^* b_k), \]

where \( p_i \) is projective with \( r(p_i) = 1 \) and \( b_i \) lies on one line. Each block of \( b_i \) being a block of \( p \), we have \( b_i \in D_s = C_s \), hence (denoting by \( n(p_i) \) the number of points of \( p_i \) composing with

\[ r = b_0^0 \otimes |s_{n(p_1)/2} \otimes b_1^* b_1 \otimes |s_{n(p_2)/2} \otimes \cdots \otimes |s_{n(p_k)/2} \otimes b_k^* b_k \in C_s \]

yields an equivalence between \( p \) and \( p_1 \otimes \cdots \otimes p_k \). For a fixed \( 1 \leq i \leq k \) let \( r_i \) be defined as follows:

- If \( c_u(p_i) \neq 0 \) mod \( s \) and \( x \) is the unique representative of this class in \( \{1, \cdots, s - 1\} \), then \( r_i \) is the unique one-block partition in \( C_s(x, n(p_i)/2) \) with all upper points coloured in white and lower points coloured to match the upper points of \( p_i \).
- If \( c_u(p_i) = 0 \) mod \( s \), then \( r_i \) is the unique one-block partition in \( C_s(2, n(p)/2) \) with upper points coloured with \( \circ \bullet \) and lower points coloured to match the upper points of \( p_i \).

Then,

\[ r' = (p_1 \otimes \cdots \otimes p_k)(r_1 \otimes \cdots \otimes r_k) \in C_s \]

is an equivalence between \( p_1 \otimes \cdots \otimes p_k \) and \( \pi_{s_1} \otimes \cdots \otimes \pi_{s_k} \). By transitivity of the equivalence relation, this proves that \( \Phi \) is surjective by Theorem 4.2.3.
2. As for injectivity, let \( w = w_1 \cdots w_k \) and \( w' = w'_1 \cdots w'_k \) be such that \( \Phi(w) = \Phi(w') \), which is equivalent by Proposition 4.2.5 to

\[
\pi_{w_1} \otimes \cdots \otimes \pi_{w_k} \sim \pi_{w'_1} \otimes \cdots \otimes \pi_{w'_k}
\]

Because two equivalent partitions have the same number of through-blocks, we must have \( k = n \). Moreover, the equivalence is then implemented by the partition

\[
r_{\pi_{w_1}} \otimes \cdots \otimes r_{\pi_{w_k}} \in C_s
\]

so that for any \( 1 \leq i \leq k \),

\[
c_a(w_i) - c_d(w'_i) = c_a(w_i) - c_d(w'_i) = 0 \mod (s).
\]

As a consequence, \( w_i = w'_i \) for all \( 1 \leq i \leq k \) and \( w = w' \).

3. We now have to compute the fusion rules and the reasoning is similar to the case of \( U_N^+ \) in Theorem 6.2.2. Let \( w = w_1 \cdots w_k \) and \( w' = w'_1 \cdots w'_l \) be two words in \( M_n \) and consider \( \Phi(w)^{\square^\ell} \Phi(w') \) for some \( \ell \leq \min(k, n) \). There is an interval in the middle obtained by gluing the upper blocks of \( \pi_{w_k} \) and \( \pi_{w'_l} \), which has colour number \( w_k + w'_l \). But this number must be a multiple of \( s \) by definition of \( D_s = C_s \). Hence, \( w_k = w'_l - 1 \) and applying the same argument we see that there is a decomposition \( w = az \), \( w' = zb \) such that

\[
\Phi(w)^{\square^\ell} \Phi(w') = \Phi(ab).
\]

As for \( \Phi(w)^{\square^\ell} \Phi(w') \), we have similarly that \( w = az \) and \( w' = zb \) with \( z \) of length \( \ell - 1 \). Moreover, the through-block obtained by gluing \( \pi_{w_{k-\ell+1}} \) and \( \pi_{w'_\ell} \) has upper colour number \( w_{k-\ell+1} + w'_\ell \), hence the result.

4. Note eventually that for \( u^w \otimes u^{w'} \) to contain the trivial representation \( u^0 \), one must have \( w' = \overline{w} \). Hence, \( \overline{w} = u \overline{w} \).

Before going to the next section, let us comment on some other examples which discretely appeared. On can consider for instance the category of non-crossing partitions \( C_\infty = \langle q_0 \rangle = D_\infty \). The corresponding unitary compact matrix quantum group is again a quantum reflection group denoted by \( H_\infty^+ \). Its representation theory is the same as \( H_N^+ \) except that the group \( Z_s \) is replaced by the group \( Z \). Of course, the classical version is the reflection group \( H_\infty \) consisting in all permutation matrices with coefficients replaced by arbitrary complex numbers of modulus one.

One may also consider \( C_{\text{alt}} = \langle q \rangle \). It can be proven that this is the category of all non-crossing coloured partitions such that each block, once rotated on one line, has a strictly alternating colouring. With a little effort, one can prove that this is the free complexification of the hyperoctaedral group \( H_N^+ \) and use the same strategy as in the case of \( U_N^+ \) to compute its representation theory. Interestingly, its classical version is again \( H_N^+ \).

### 7.2 Quantum Automorphism Groups of Graphs

We have started our journey in the world of compact quantum groups by a problem from graph theory. Even though the connection explained in Chapter 1 is recent, the idea of using compact matrix quantum groups to study finite graphs had already been exploited before. In particular, it can shed a new light on quantum reflection groups, which strengthens the idea that they are the “correct” generalization of classical reflection groups. As usual, we will first review the basic features of classical automorphism groups of graphs, and this starts by making the notion of a graph precise.

**Definition 7.2.1.** A **finite oriented graph** \( X \) is the data of
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- A finite set $V(X)$ called the set of vertices,
- A set of ordered pairs $E(X) \subset V(X) \times V(X)$ called the set of edges.

If we fix a numbering of the vertices so that $V(X) = \{1, \cdots, N\}$, the adjacency matrix of $X$ is the matrix $A_X$ with $(i,j)$-th coefficient 1 if $(i,j) \in E(X)$ and 0 otherwise.

**Remark 7.2.2.** In Chapter 1, we did not consider orientations on the edges of the graph so that our setting was more restricted. Indeed, unoriented graph can be seen as a particular case of oriented graphs where $(i,j) \in E(X)$ if and only if $(j,i) \in E(X)$.

Let $X$ be a finite oriented graph with $N$ vertices, and let $A_X$ be its adjacency matrix. An automorphism of $X$ is a bijection of its vertices which respects the edges. Seeing permutations of the vertices as matrices in $M_N(\mathbb{C})$, this means that we are considering those permutation matrices which commute with $A_X$. The key point is that this still makes sense if we start with quantum permutation matrices.

**Proposition 7.2.3.** Let $X$ be a finite oriented graph on $N$ vertices with adjacency matrix $A_X$. Then, the quotient $O(S_N^+)$ by the ideal $I$ generated by the coefficients of

$$A_X P - PA_X,$$

**Proof.** We have to check that the ideal in the statement is stable under the involution and that the coproduct of $S_N^+$ factors through the quotient. But this was already done once and for all in the proof of Theorem 3.3.3! Let us write things down nevertheless to convince the reader. Using the notations of Theorem 3.3.3, the ideal $I$ is generated by the elements

$$P_{A_X,i,j} = (A_X P - PA_X)_{i,j} = \sum_{i \rightarrow k} P_{kj} - \sum_{k \rightarrow j} P_{ik},$$

where we write $i \rightarrow j$ if $(i,j) \in E(X)$. It was shown in Point (3) of the proof of Theorem 3.3.3 that

$$\Delta(P_{A_X,i,j}) = \sum_{k=1}^N P_{A_X,i,k} \otimes P_{kj} + P_{ik} \otimes P_{A_X,k,j}.$$

This readily implies

$$\Delta(I) \subset I \otimes O(S_N^+) + O(S_N^+) \otimes I,$$

hence the coproduct of $S_N^+$ factors through the quotient. Because moreover $P_{A_X,i,j}^* = P_{A_X,i,j}$, we have $I^* = I$ and the proof is complete.

**Definition 7.2.4.** The orthogonal compact matrix quantum group defined by Proposition 7.2.3 is called the quantum automorphism group of $X$ and denoted by $QAut(X)$.

Our goal will be to prove that $H_N^+$ is in fact the quantum automorphism group of a graph. This may at first seem surprising, since it is define as a genuinely unitary compact matrix quantum group. But the trick is that it will not be the quantum automorphism group of a graph on $N$ vertices, but on $sN$ vertices. The precise graph can be found by looking at the classical reflection groups $H_N^s$. These are well-known to be the automorphism groups of a disjoint union of $N$ copies of an $s$-cycle oriented cyclically (see Corollary 7.2.11 for an a posteriori proof). More precisely, let $C_s$ be the $s$-cycle graph oriented cyclically, which consists in $s$ points with point $i$ connected to point $i+1$ modulo $s$. Then,

**Theorem 7.2.5 (Banica-Vergnioux)** Let $s \geq 3$ and let $C_{s,N}$ be the disjoint union of $N$ copies of the cyclically oriented $s$-cycle $C_s$. Then,

$$H_N^{s,N} \simeq QAut(C_{s,N}).$$

Moreover, let $C_{2,N}'$ be the disjoint union of $N$ copies of the unoriented graph with two
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vertices connected by an edge. Then,

\[ H^+_N \simeq \text{QAut}(C^\ell_{2,N}). \]

**Remark 7.2.6.** Even classically, it is known that the automorphism group of \( C_{2,N} \) is not \( H^+_N \) but \( S_N \). This comes from the fact that the 2-cycle graph has no non-trivial oriented automorphism, while it has a non-trivial one which reverses the orientation.

The proof of Theorem 7.2.5, which is due to T. Banica and R. Vergnioux in [BV09, Thm 3.2] is not straightforward. It will require an alternate description of \( O(H^+_N) \).

### 7.2.1 Sudoku matrices

We need to understand what the quantum automorphism group of \( C_{s,N} \) can be. Let us number the vertices of \( C_{s,N} \) as follows:

- For \( 1 \leq i \leq N - 1 \), vertex number \( i + \ell N \) is connected to vertex number \( i + (\ell + 1)N \),
- Vertex \( i + sN \) is connected to vertex number \( i \).

With these conventions, the adjacency matrix of \( C_{s,N} \) is

\[
A_{C_{s,N}} = \begin{pmatrix}
0 & I_N & 0 & \cdots & 0 \\
0 & 0 & I_N & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & I_N \\
I_N & 0 & 0 & \cdots & 0
\end{pmatrix} \in M_{sN}(\mathbb{C}).
\]

Let \( M \in M_{sN}(\mathbb{C}) \) be a matrix which commutes with \( A_{C_{s,N}} \). Then, \( M \) must be *block-circulant* in the sense that if we decompose the first \( N \) rows as \( s \) square matrices, then \( M \) is obtained by cyclically permuting these matrices. That observation naturally leads to the following definition:

**Definition 7.2.7.** An \((s,N)\)-sudoku matrix is a quantum permutation matrix \( M \in M_{sN}(\mathcal{B}(H)) \) of the form

\[
M = \begin{pmatrix}
M^{(0)} & M^{(1)} & \cdots & M^{(s-1)} \\
M^{(s-1)} & M^{(0)} & \cdots & M^{(s-2)} \\
\vdots & \vdots & \ddots & \vdots \\
M^{(1)} & M^{(2)} & \cdots & M^{(0)}
\end{pmatrix}
\]

for some matrices \( M^{(0)}, \ldots, M^{(s-1)} \in M_N(\mathcal{B}(H)) \).

We will prove that \( O(H^+_N) \) can be characterized in terms of sudoku matrices, but this first requires establishing some properties of the coefficients of the fundamental representation of \( H^+_N \).

**Lemma 7.2.8.** Let \( u \) be the fundamental representation of \( H^+_N \). Then, for any \( 1 \leq i, j, \ell \leq N \),

1. \( u^{s+\ell}_{ij} = u^{\ell}_{ij} \),
2. \( u^{s\ell}_{ij} = u^{s-\ell}_{ij} \).

**Proof.**

1. Using the fact that the coefficients are pair-wise orthogonal on rows and columns yields

\[
u^{\ell}_{ij} = \left( \sum_{k=1}^{N} u^{s}_{ik} \right) u^{\ell}_{ij} = u^{s}_{ij} u^{\ell}_{ij} = u^{s+\ell}_{ij}.
\]

2. First notice that

\[
u_{ij} = u_{ij} \left( \sum_{k=1}^{N} u_{ik} u^{*}_{ik} \right) = u^{2}_{ij} u^{*}_{ij}
\]
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which yields for any $1 \leq \ell < s$ by a straightforward induction

$$u_{ij}^\ell = u_{ij}^s u_{ij}^{(s-\ell)}.$$  

Another consequence of the first equality is that for any $k \neq j$,

$$u_{ij} u_{ij}^* = u_{ij}^2 u_{ij}^* u_{ik} = u_{ij}^2 (u_{ik} u_{ij})^* = 0.$$  

so that

$$u_{ij}^* = \left( \sum_{k=1}^N u_{ik}^* \right) u_{ij} = u_{ij} u_{ij}^*.$$  

We can now conclude:

$$u_{ij}^{s-\ell} = \left( u_{ij}^{s(s-\ell)} \right)^* = \left( u_{ij}^* (s-\ell) \right)^* = u_{ij}^{s-\ell}.$$  

\[\Box\]

**Remark 7.2.9.** The relations of the statement can also been recovered as the relations implemented by certain specific partitions of $C_s$. Indeed, the first one corresponds to the one-block partition of $C_s(s + \ell, \ell)$ with all points coloured in white while the second one corresponds to the rotation of $p_s$ which belongs to $C_s(s - \ell, \ell)$. We leave it as an exercise to the reader to check those claims.

We can now establish the link between quantum reflection groups and sudoku matrices.

**Proposition 7.2.10.** Let $A$ be the universal $*$-algebra generated by the entries of a $(s,N)$-sudoku matrix and let $\omega$ be a primitive $s$-th root of unity. Then, there exists a pair of mutually inverse $*$-homomorphisms $\Phi : \mathcal{O}(H_N^{s+}) \rightarrow A$ and $\Psi : A \rightarrow \mathcal{O}(H_N^{s+})$ such that

$$\Phi(u_{ij}) = \sum_{p=0}^{s-1} \omega^{-p} M^{(p)}_{ij}$$

and

$$\Psi\left( M^{(p)}_{ij} \right) = \frac{1}{s} \sum_{r=0}^{s-1} \omega^{rp} u_{ij}^r$$

with the convention that $u_{ij}^0 = u_{ij}^*$.  

**Proof.** To show the existence of the $*$-homomorphism, we will use the universal properties of the $*$-algebras involved.

1. Let us start by checking that setting $\Phi(u_{ij})$ satisfies the defining relations of $\mathcal{O}(H_N^{s+})$.  

   Indeed,

   $$\Phi(u_{ij}) \Phi(u_{ij})^* = \sum_{p,p'=0}^{s-1} \omega^{-p+p'} M^{(p)}_{ij} M^{(p')*}_{ij}$$

   $$= \sum_{p,p'=0}^{s-1} \omega^{-p+p'} \delta_{p,p'} \delta_{j,j'} M^{(p)}_{ij}$$

   $$= \delta_{j,j'} \sum_{p=0}^{s-1} M^{(p)}_{ij}$$

   so that the product vanishes if $j \neq j'$, giving the relation corresponding to $q$. Note that we used here the fact that $M$ is a quantum permutation matrix. Moreover, this computation also shows that

   $$\sum_{k=1}^N \Phi(u_{ik}) \Phi(u_{jk}^*) = \sum_{k=1}^N \sum_{p,p'=0}^{s-1} \omega^{-p+p'} M^{(p)}_{ik} M^{(p')*}_{jk}$$

   $$= \delta_{ij} \sum_{p=0}^{s-1} M^{(p)}_{ik}$$

   $$= \delta_{ij}.$$
so that $\Phi(u)\Phi(u)^* = \text{Id}$. Similar computations work for $\Phi(u)^*\Phi(u)$ and $\Phi(u)$. Eventually, because $M_{ij}^{(p)}$ and $M_{ij}^{(p')}$ are orthogonal to one another if $p \neq p'$,

$$
\sum_{j=1}^{N} \Phi(u_{ij})^s = \sum_{j=1}^{N} \left( \sum_{p=0}^{s-1} \omega^p M_{ij}^{(p)} \right)^s
= \sum_{j=1}^{N} \sum_{p=0}^{s-1} \omega^{ps} M_{ij}^{(p)}
= \sum_{j=1}^{N} \sum_{p=0}^{s-1} M_{ij}^{(p)}
= 1.
$$

This shows the existence of the $*$-homomorphism $\Phi$.

2. We now turn to $\Psi$. Let us first prove that the image of the coefficients of $M$ are indeed orthogonal projections:

$$
\Psi \left( M_{ij}^{(p)} \right)^2 = \frac{1}{s^2} \sum_{r,r'=0}^{s-1} \omega^{(r+r')p} u_{ij}^{r+r'}
= \frac{1}{s} \sum_{\ell=0}^{s-1} \omega^{\ell p} u_{ij}^{\ell}
= \Psi \left( M_{ij}^{(p)} \right)
$$

and

$$
\Psi \left( M_{ij}^{(p)} \right)^* = \frac{1}{s} \sum_{r=0}^{s-1} \omega^{-rp} u_{ij}^{r*}
= \frac{1}{s} \sum_{r=0}^{s-1} \omega^{(s-r)p} u_{ij}^{s-r}
= \Psi \left( M_{ij}^{(p)} \right).
$$

The last condition is the sum on rows and columns. We have

$$
\sum_{p=0}^{s-1} \sum_{j=1}^{N} \Psi \left( M_{ij}^{(p)} \right) = \frac{1}{s} \sum_{j=1}^{N} \sum_{p,r=0}^{s-1} \omega^{rp} u_{ij}^{r}
= \frac{1}{s} \sum_{j=1}^{N} \sum_{r=0}^{s-1} u_{ij}^{r} \sum_{p=0}^{s-1} \omega^{rp}
= \frac{1}{s} \sum_{j=1}^{N} \sum_{r=0}^{s-1} u_{ij}^{0}
= \sum_{j=1}^{N} u_{ij}^{s}
= 1
$$

and the same holds for the sum over rows by taking the transpose matrices.

3. To conclude, the only thing left is the check that $\Phi$ and $\Psi$ are mutually inverse. Let us do
it:
\[
\Psi \circ \Phi(u_{ij}) = \sum_{p=0}^{s-1} \omega^{-p} \sum_{r=0}^{s-1} \omega^r u^r_{ij} \frac{1}{s} = \frac{1}{s} \sum_{r=0}^{s-1} u^r_{ij} \sum_{p=0}^{s-1} \omega^{(r-1)p} = \frac{1}{s} \sum_{r=0}^{s-1} u^r_{ij} \delta_{r,1} = u_{ij}
\]

and
\[
\Phi \circ \Psi(M^{(p)}_{ij}) = \frac{1}{s} \sum_{r=0}^{s-1} \omega^{r p} \left( \sum_{\ell=0}^{s-1} \omega^{-(\ell)} M^{(\ell)}_{ij} \right)^r = \frac{1}{s} \sum_{r=0}^{s-1} \sum_{\ell=0}^{s-1} \omega^{(p-\ell) r} M^{(\ell)}_{ij} = \frac{1}{s} \sum_{\ell=0}^{s-1} M^{(\ell)}_{ij} \sum_{r=0}^{s-1} \sum_{\ell=0}^{s-1} \omega^{(p-\ell) r} = M^{(p)}_{ij}.
\]

We are now ready to prove that quantum reflection groups are quantum automorphism groups of graphs.

**Proof of Theorem 7.2.5.** Let us recall our numbering of the vertices of \(C_{s,N}\):

- For \(1 \leq i \leq N - 1\), vertex number \(i + \ell N\) is connected to vertex number \(i + (\ell + 1)N\),
- Vertex \(i + sN\) is connected to vertex number \(i\).

With these conventions, the adjacency matrix is
\[
A_{C_{s,N}} = \begin{pmatrix}
0 & I_N & 0 & \cdots & 0 \\
0 & 0 & I_N & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & I_N \\
I_N & 0 & 0 & \cdots & 0
\end{pmatrix} \in M_{sN}(\mathbb{C}).
\]

Note that for \(s = 2\), this is not the adjacency matrix of \(C_{2,N}\) but of \(C'_{2,N}\), hence the proof below works for all cases. We will proceed in two steps.

1. We first prove that a quantum permutation matrix \(P \in M_{sN}(B(H))\) commutes with \(A_{C_{s,N}}\) if and only if it is an \((s,N)\)-sudoku matrix. Let us write \(P\) in block form as
\[
P = \begin{pmatrix}
P^{(0,0)} & P^{(0,1)} & \cdots & P^{(0,s-1)} \\
P^{(1,0)} & P^{(1,1)} & \cdots & P^{(1,s-1)} \\
\vdots & \vdots & \ddots & \vdots \\
P^{(s-1,0)} & M^{(s-1,1)} & \cdots & M^{(s-1,s-1)}
\end{pmatrix}.
\]

Then, the block columns of \(PB\) are those of \(P\) shifted by one (modulo \(s\)) on the right. Thus, the \((i,j)\)-th block of \(PAC_{s,N}\) is \(P^{(i,j-1)}\). Similarly, the block rows are shifted by one on the left (modulo \(s\)) in \(A_{C_{s,N}}P\) so that the \((i,j)\)-th block is \(P^{(i+1,j)}\). Thus, if \(P\) commutes with \(A_{C_{s,N}}\), then \(P^{(i,j-1)} = P^{(i+1,j)}\) and it follows by induction that \(P^{(i,j)}\) only depends on the class of \(j - i\) modulo \(s\), i.e. \(P\) is an \((s,N)\)-sudoku matrix. Conversely, it is clear that an \((s,N)\)-sudoku matrix commutes with \(A_{C_{s,N}}\).
2. By definition, \( \mathcal{O}(\text{QAut}(C_{s,N})) \) is the quotient of \( \mathcal{O}(S_N^+) \) by the relations

\[
P_{AC_{s,N}} = A_{C_{s,N}} P.
\]

By Point (1), this is the same as the universal \( \ast \)-algebra generated by the coefficients of an \((s,N)\)-sudoku matrix, which we now know to be isomorphic to \( \mathcal{O}(H_N^{s+}) \). All that is left is to prove that the isomorphism \( \Phi \) of Proposition 7.2.10 behaves well with respect to the coproducts.

This first requires identifying the coproduct on \( \mathcal{O}(\text{QAut}(C_{s,N})) \) in the sudoku picture. Notice that if \( i = i^* + pN \) and \( j = j^* + p'N \) with \( i^*, j^* \in \{1, \cdots, N\} \), then

\[
M_{i,j} = M_{i^*+pN, j^*+p'N}^{(p-p')}.
\]

Using this equality, we see that for any \( 1 \leq i,j \leq N \) and \( 0 \leq p \leq s-1 \),

\[
\Delta (M_{ij}^{(p)}) = \Delta (M_{i+pN, j})
\]

\[
= \sum_{k=1}^{sN} M_{i+pN, k} \otimes M_{k, j}
\]

\[
= \sum_{p'=0}^{s-1} \sum_{k=1}^{N} M_{i+pN+k+p'N} \otimes M_{k+p'N, j}
\]

\[
= \sum_{p'=0}^{s-1} \sum_{k=1}^{N} M_{k, k}^{(p-p')} \otimes M_{k, j}^{(p')},
\]

Note that all the exponents are taken modulo \( s \). We are now left with an elementary computation:

\[
\Delta \circ \Phi(u_{ij}) = \sum_{p=0}^{s-1} \omega^{-p} \Delta (M_{ij}^{(p)})
\]

\[
= \sum_{p=0}^{s-1} \omega^{-p} \sum_{p'=0}^{s-1} \sum_{k=1}^{N} M_{ik}^{(p-p')} \otimes M_{kj}^{(p')},
\]

where \( i^* \) and \( j^* \) are the unique representatives of \( i \) and \( j \) modulo \( N \) in \( \{1, \cdots, N\} \), and

\[
(\Phi \otimes \Phi) \circ \Delta(u_{ij}) = \sum_{k=1}^{N} \Phi(u_{ik}) \otimes \Phi(u_{kj})
\]

\[
= \sum_{k=1}^{N} \sum_{p=1, p'=0}^{s-1} \omega^{-p-p'} M_{ik}^{(p)} \otimes M_{kj}^{(p')}
\]

\[
= \sum_{k=1}^{N} \sum_{p'=0}^{s-1} \sum_{p''=0}^{s-1} \omega^{-p''} M_{ik}^{(p''-p')} \otimes M_{kj}^{(p')} = \Delta \circ \Phi(u_{ij}).
\]

Let us conclude by the \textit{a posteriori} proof that \( H_N^s \) was indeed the automorphism group of \( C_{s,N} \).

**Corollary 7.2.11.** There is an isomorphism

\[
H_N^s \simeq \text{Aut}(C_{s,N}).
\]

**Proof.** Simply consider the isomorphism induced by \( \Phi \) between the maximal abelian quotients of the \( \ast \)-algebras of \( \mathcal{O}(H_N^{s+}) \) and \( \mathcal{O}(\text{QAut}(C_{s,N})) \).
7.2.2 Free wreath products

To conclude this chapter, we will give another description of quantum reflection groups, based on the classical notion of wreath product. Let us recall what this is:

**Definition 7.2.12.** Let $G$ be a group and let $N$ be an integer. The (permutational) wreath product of $G$ by $S_N$ is the semi-direct product

\[ G^N \rtimes S_N, \]

where the action is given by

\[ \sigma.(g_1, \cdots, g_N) = (g_{\sigma^{-1}(1)}, \cdots, g_{\sigma^{-1}(N)}). \]

It is denoted by $G \wr S_N$.

The reason why we give this definition is that there is a well-known isomorphism

\[ H_N^+ = \mathbb{Z}_s \wr S_N. \]

Our purpose is to prove a similar isomorphism involving $H_N^{++}$ on the left-hand side, and a new construction on the right-hand side, called the free wreath product. To explain the basic idea underlying the definition of that generalization of the wreath product, note that the set underlying the classical wreath product is just the direct product $G^N \rtimes S_N$. To make this free, it is natural to consider the $*$-algebra

\[ \mathcal{O}(\mathbb{G})^{*N} \ast \mathcal{O}(S_N^+). \]

Now, the whole point of the construction of a semi-direct product is that the group law is not the product law. Similarly, we must here devise a coproduct which is not the canonical one for the free product. The issue is that the notion of semi-direct product is rather tricky at the level of compact quantum groups and does not work very well in this context. However, in our setting we know that the whole structure of a unitary compact matrix quantum group is encoded in its fundamental representation, so that we can try first to find a nice fundamental representation of $G \wr S_N$, and then to transport it to the free product.

Let us assume therefore that $G$ is a unitary compact matrix group and let $\rho : G \to M_n(\mathbb{C})$ be a fundamental representation. Our first goal is to use $\rho$ to realize $G \rtimes S_N$ as a group of matrices. For each $1 \leq i \leq N$ we define a representation $\rho_i$ of $G$ on

\[ V = (\mathbb{C}^n)^{\otimes N} = \mathbb{C}^n \oplus \cdots \oplus \mathbb{C}^n \]

by the formula

\[ \rho_i(g) = \text{Id}_{\mathbb{C}^n}^{\otimes i-1} \oplus \rho(g) \oplus \text{Id}_{\mathbb{C}^n}^{N-i}. \]

We also need a representation of $S_N$ on $V$. Let us denote the canonical basis of the $i$-th copy of $\mathbb{C}^n$ by $(e_x^{(i)})_{1 \leq p \leq n}$. Then we set, for $\sigma \in S_N$,

\[ \pi(\sigma) \left( e_x^{(i)} \right) = e_x^{\sigma(i)}. \]

We claim that these representations are enough to recover the wreath product.

**Proposition 7.2.13.** The subgroup $H$ of $\text{GL}(V)$ generated by $\rho_i(G)$ and $\pi(S_N)$ is isomorphic to $G \wr S_N$.

**Proof.** First note that $\rho_i(G)$ and $\rho_j(G)$ commute for $i \neq j$, so that $H$ is generated by $N$ commuting copies of $G$ and a copy of $S_N$. Moreover, for any $1 \leq i \leq N$ and $1 \leq p \leq n$,

\[
\pi(\sigma) \rho_i(g) \pi(\sigma^{-1}) \left( e_p^{(\sigma(i))} \right) = \pi(\sigma) \rho_i(g) \left( e_p^{(i)} \right) \\
= \pi(\sigma) \left( \sum_{q=1}^n \rho(h)_{pq} e_q^{(i)} \right) \\
= \sum_{q=1}^n \rho(h)_{pq} e_q^{\sigma(i)} \\
= \rho_{\sigma(i)}(h) \left( e_p^{\sigma(i)} \right)
\]
while
\[ \pi(\sigma) \rho_i(g) \pi(\sigma^{-1}) \left( e_p^{(j)} \right) = e_p^{(j)}, \]
if \( j \neq \sigma(i) \) so that in the end
\[ \pi(\sigma) \rho_i(g) \pi(\sigma^{-1}) = \rho_{\sigma(i)}(g). \]
In other words, the action of \( \sigma \) on \( G^N \) is implemented by conjugation in \( H \), i.e. \( H \) is the semi-direct product by that action.

The previous proposition can be restated by saying that the representation
\[ \tilde{\rho} : (g_1, \cdots, g_N, \sigma) \mapsto \pi(\sigma) \left( \prod_{i=1}^{N} \rho_i(g_i) \right) \]
is fundamental, so that the unitary compact matrix group
\[ (\mathcal{O}(G^N \times S_N), \tilde{\rho}) \]
is exactly \( G \wr S_N \). Moreover, the coefficients of this representation are given by
\[ \tilde{\rho}_{ip,jq}(g_1, \cdots, g_N, \sigma) = \delta_{\sigma(i),j} \rho(g_i)_{pq} = \rho(g_i)_{pq} \delta_{ij}(\sigma) \]
where \( c_{ij} \) are the coefficient functions on \( S_N \).

We now know how to define a unitary compact matrix quantum group structure on
\[ H = \mathcal{O}(\mathbb{G})^* N \ast \mathcal{O}(S_N^+), \]
for a unitary compact matrix quantum group \((G, u)\). Indeed, we should simply consider the fundamental representation \( w \) with coefficients
\[ w_{ip,jq} = \nu_i(u_{pq}) P_{ij}, \]
where \( \nu_i : \mathcal{O}(\mathbb{G}) \to H \) is the canonical inclusion into the \( i \)-th copy of \( \mathcal{O}(\mathbb{G}) \). Let us check for sanity that this defines a unitary matrix.

**Lemma 7.2.14.** The matrix \( w \in M_{nN}(H) \) is unitary, as well as the matrix \( \overline{w} \).

**Proof.** We simply compute
\[
\begin{align*}
\sum_{k=1}^{N} \sum_{r=1}^{n} w_{ip,kr} w_{jq,kr}^* &= \sum_{k=1}^{N} \sum_{r=1}^{n} \nu_i(u_{pr}) P_{ik} P_{jk} \nu_j^*(u_{qr}) \\
&= \sum_{r=1}^{n} \nu_i(u_{pr}) \left( \sum_{k=1}^{N} P_{ik} P_{jk} \right) \nu_j^*(u_{qr}) \\
&= \delta_{ij} \sum_{r=1}^{n} \nu_i(u_{pr}) u_{qr}^* \\
&= \delta_{ij} \delta_{pq}
\end{align*}
\]
so that \( ww^* = \text{id} \). Similar computations work for \( w^*w \), as well as for \( \overline{w} \overline{w}^* \). \( \blacksquare \)

The problem is to check that there is a coproduct on the \( \ast \)-algebra \( H \) which turns this into a representation, and this is were things break down. Indeed, assume that there exists a \( \ast \)-homomorphism \( \delta : H \to H \otimes H \) such that
\[ \delta(\nu_i(u_{pq}) P_{ij}) = \sum_{k=1}^{N} \sum_{z=1}^{n} \nu_i(u_{pr}) P_{ik} \otimes \nu_k(u_{rq}) P_{kj}. \]
Then, summing over $j$ in both sides of the equation yields

$$\delta(\nu_i(u_{pq})) = \sum_{k=1}^{N} \sum_{r=1}^{n} \nu_i(u_{pr}) P_{ik} \otimes \nu_k(u_{rq}).$$

Assume for instance that $u$ is orthogonal, i.e. that all its coefficients are self-adjoint. Then,

$$\sum_{k=1}^{N} \sum_{r=1}^{n} P_{ik} \nu_i(u_{pr}) \otimes \nu_k(u_{rq}) = (\nu_i(u_{pq}))^*$$

$$= \delta(\nu_i(u_{pq})^*)$$

$$= \sum_{k=1}^{N} \sum_{r=1}^{n} \nu_i(u_{pr}) P_{ik} \otimes \nu_k(u_{rq})$$

so that

$$\sum_{k=1}^{N} \sum_{r=1}^{n} (P_{ik} \nu_i(u_{pr}) - \nu_i(u_{pr}) P_{ik}) \otimes \nu_k(u_{rq}) = 0.$$ 

Such a relation does not hold in the free product so that if we want to have a chance of turning $w$ into a representation, we should add some relations to $\mathcal{H}$ making the above equation true.

**Definition 7.2.15.** Let $\mathbb{G}$ be a unitary compact matrix quantum group and let $N$ be an integer. The free wreath product algebra is the quotient $A$ of $\mathcal{O}(\mathbb{G})^* N * \mathcal{O}(S_N^+)$ by the relations

$$\nu_i(x) P_{ij} = P_{ij} \nu_i(x)$$

for all $1 \leq i, j \leq N$ and all $x \in \mathcal{O}(\mathbb{G})$.

The fundamental observation of J. Bichon in [Bic04] is that these relations are enough to define a unitary compact matrix quantum group structure.

**Proposition 7.2.16.** There exists a unique $\ast$-homomorphism $\Delta : A \to A \otimes A$ such that for all $1 \leq p, q \leq n$ and $1 \leq i, j \leq N$,

$$\Delta(w_{ip,jq}) = \sum_{k=1}^{N} \sum_{r=1}^{n} \nu_i(u_{pr}) P_{ik} \otimes \nu_k(u_{rq}) P_{kj}.$$

**Proof.** We will proceed in two steps, first proving the existence of a suitable map from $\mathcal{H}$ to $A \otimes A$ and then checking that it passes to the quotient.

1. We start by defining the coproduct on $\nu_i(\mathcal{O}(\mathbb{G}))$. Let us first consider the $\ast$-homomorphism

$$\Phi_{ik} = (\nu_i \otimes \nu_k) \circ \Delta : \nu_i(\mathbb{G}) \to \mathcal{H} \otimes \mathcal{H} \to A \otimes A.$$ 

whose range is contained in $\nu_i(\mathcal{O}(\mathbb{G})) \otimes \nu_k(\mathcal{O}(\mathbb{G})) \subset A \otimes A$. On the latter subalgebra, the map

$$\Psi_{ik} : (x \otimes y) \mapsto x P_{ik} \otimes y$$

is a $\ast$-homomorphism. Indeed, because of the commutation relations defining $A$,

$$\left(x P_{ij} \otimes y (x' P_{ik} \otimes y')\right) = \left(x P_{ik} x' P_{ik} \otimes yy'\right) = x x' P_{ik} \otimes yy'.$$

Let us now consider the linear map

$$\Phi_i = \sum_{k=1}^{N} \Psi_{ik} \circ \Phi_{ik}.$$

It is clear that $\Phi_i(x^*) = \Phi_i(x)^*$, and observing that for $k \neq k'$,

$$(\Psi_{ik} \circ \Phi_{ik}(x)) (\Psi_{ik'} \circ \Phi_{ik'}(y)) = 0,$$

we see that $\Phi_i$ is multiplicative, hence defines a $\ast$-homomorphism from $\nu_i(\mathcal{O}(\mathbb{G}))$ to $A \otimes A$. 

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2. We still have to define the coproduct on $O(S_N^+)$, but it suffices to consider the usual coproduct of $S_N^+$, seen as a $*$-homomorphism

$$\Delta : O(S_N^+) \to H \otimes H \to A \otimes A.$$ 

3. By the universal property of the free product, there exists a unique $*$-homomorphism $\Phi : H \to A \otimes A$ restricting to $\nu_i(O(G))$ and to the usual coproduct on $O(S_N^+)$. Let us check that it yields the correct formula for $w_{ip,jq}$:

$$\Phi(w_{ip,jq}) = \left( \sum_{k=1}^{N} \sum_{r=1}^{n} \nu_i(u_{pr}) P_{ik} \otimes \nu_k(u_{rq}) \right) \left( \sum_{k'=1}^{N} P_{ik'} \otimes P_{k'j} \right)$$

$$= \sum_{k,k'=1}^{N} \sum_{r=1}^{n} \nu_i(u_{pr}) P_{ik} P_{ik'} \otimes \nu_k(u_{rq}) P_{k'j}$$

$$= \sum_{k=1}^{N} \sum_{r=1}^{n} \nu_i(u_{pr}) P_{ik} \otimes \nu_k(u_{rq}) P_{kj}$$

$$= \sum_{k=1}^{N} \sum_{r=1}^{n} w_{ip,kr} \otimes w_{kr,jq}.$$

4. To conclude, we must prove that $\Phi$ vanishes on the commutators defining $A$. This is a simple computation:

$$\Phi(\nu_i(u_{pq}) P_{ij}) = \sum_{k=1}^{N} \sum_{r=1}^{n} \nu_i(u_{pr}) P_{ik} \otimes \nu_k(u_{rq}) P_{kj}$$

$$= \sum_{k=1}^{N} \sum_{r=1}^{n} P_{ik} \nu_i(u_{pr}) \otimes P_{kj} \nu_k(u_{rq})$$

$$= \sum_{k=1}^{N} \sum_{r=1}^{n} P_{ik} \nu_i(u_{pr}) P_{ik} \otimes P_{kj} \nu_k(u_{rq})$$

$$= \Phi(P_{ij}) \Phi(\nu_i(u_{pq}))$$

$$= \Phi(P_{ij} \nu_i(u_{pq}))$$

\[\blacksquare\]

**Corollary 7.2.17.** The pair $(A, w)$ is a unitary compact matrix quantum group. It is called the (permutational) free wreath product of $G$ by $S_N^+$ and denoted by $G \wr_\star S_N^+$.

**Proof.** We already proved that $w$ and $\overline{w}$ are unitary in Lemma 7.2.14 and Proposition 7.2.16 shows that the coproduct is well-defined. The only thing left to check is that the coefficients of $w$ generate $A$. For this simply observe that

$$\sum_{j=1}^{N} w_{ip,jq} = \nu_i(u_{pq})$$

and

$$\sum_{q=1}^{n} w_{ip,jq} w_{ip,jq}^* = P_{ij}$$

hence the $*$-algebra generated by the coefficients of $w$ contains both $O(S_N^+)$ and $O(\nu_i(G))$ for all $1 \leq i \leq N$. In other words, it coincides with $A$. \[\blacksquare\]
Remark 7.2.18. Recall that if $G$ is a classical unitary compact matrix group, then the maximal abelian quotient of $\mathcal{O}(G)^\times N \ast \mathcal{O}(S_N^+)$ is

$$\mathcal{O}(G)^\times N \times \mathcal{O}(S_N) = \mathcal{O}(G^N \times S_N) = \mathcal{O}(G \wr S_N)$$

and the same is of course true starting with $\mathcal{A}$. Moreover, the previous discussion shows that $w$ is mapped to a fundamental representation of $G \wr S_N$ which was defined above, so that we recover the full group structure of the classical wreath product from the free wreath product.

Now that we have a quantum version of the wreath product construction, we may imagine that there is an isomorphism $H_N^+ \simeq \mathbb{Z}_s \wr \ast S_N^+$. This is in fact true, but proving it first requires a suitable description of the unitary compact matrix quantum group $\mathbb{Z}_s$.

**Lemma 7.2.19.** Let $B_s$ be the quotient of $\mathcal{O}(S_N^+)$ by the relations

$$P_{xy} = P_{x'y'}$$

for all $1 \leq x, y, x', y' \leq s$ such that $y - x = y' - x' \mod s$. Then, if $u_{ij}$ denotes the image of $P_{ij}$ in the quotient, $(B_s, u)$ is a unitary compact matrix quantum group isomorphic to $\mathbb{Z}_s$.

**Proof.** Let us consider the following matrix:

$$B_s = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & 0.
\end{pmatrix}$$

It generates a subgroup of $S_N$ isomorphic to $\mathbb{Z}_s$ and its coefficients satisfy the condition in the statement. As a consequence, if $Q_{ij}$ is the $(i, j)$-th coefficient function on $M_s(\mathbb{C})$, there is a surjective $\ast$-homomorphism

$$\Phi : B_s \to \mathcal{O}(\mathbb{Z}_s)$$

sending $u_{ij}$ to $Q_{ij}$. Moreover, the condition in the statement implies that $B_s$ is spanned as a vector space by the elements $(P_{ij})_{0 \leq j \leq s-1}$ so that it has dimension at most $s$. Because $\mathcal{O}(\mathbb{Z}_s)$ has dimension $s$, this forces $\Phi$ to be an isomorphism, proving both assertions. $\square$

We are now ready for the last important result of this chapter, which is originally due to T. Banica and R. Vergnioux in [BB09].

**Theorem 7.2.20** (Banica-Vergnioux) There is an isomorphism

$$H_N^+ \simeq \mathbb{Z}_s \wr \ast S_N^+.$$

**Proof.** We must build an isomorphism $\Phi : \mathcal{O}(H_N^+) \to \mathcal{O}(\mathbb{Z}_s \wr \ast S_N^+)$ sending a fundamental representation of the first unitary compact matrix quantum group to a fundamental representation of the second one. The fundamental representation $w$ of the right-hand side has size $sN$, so that it will be more practical to work with the sudoku characterization of $\mathcal{O}(H_N^+)$ which has the same size. Then, the natural guess is

$$\Phi (M_{ij}^{(p)}) = w_{ij0,jp}.$$

To see that this is well-defined, simply notice that if we set

$$A_{ij}^{(p,q)} = w_{ijp,jq},$$

then the matrix $A = A_{ij}^{(p,q)}$ is a quantum permutation matrix such that $A_{ij}^{(p,q)} = A_{ij}^{(0,q-s-p)}$, hence an $(s, N)$ sudoku matrix. As a consequence, the formula above defines a surjective $\ast$-homomorphism $\Phi : \mathcal{O}(H_N^+) \to \mathcal{O}(\mathbb{Z}_s \wr \ast S_N^+)$.
Conversely, we have to prove that there exists a $*$-homomorphism sending $w_{ip,jq}$ to $A^{(p,q)}_{ij}$. We will do this using the universal property of the free product, hence we must first define the $*$-homomorphism at the level of $\nu_i(O(G))$ and $O(S_N^+^+)$. But we have already seen in Corollary 7.2.17 how we can recover all the factors from the fundamental representation $w$. Let us set for a fixed $1 \leq i \leq N$

$$N^{(p,q)}_i = \sum_{j=1}^{N} M^{(q-p)}_{ij}.$$ 

These are orthogonal projections satisfying

- $\sum_{p=0}^{s-1} N^{(p,q)}_i = \sum_{j=1}^{N} \sum_{p=0}^{s-1} M^{(q-p)}_{ij} = 1$ and similarly for the sum over $q$,

- $N^{(p,q)}_i = N^{(p',q')}_i$ if $q - p = q' - p' \mod s$.

By Lemma 7.2.19, we therefore have a surjective $*$-homomorphism $\Psi_i : O(Z_s) \to O(H_N^+)$ sending $u_{pq}$ to $N^{(p,q)}_i$.

In a similar fashion, we set for $1 \leq i, j \leq N$ and $0 \leq p \leq s - 1$

$$Q_{ij} = \sum_{q=0}^{s-1} A^{(p,q)}_{ij} A^{(p,q)*}_{ij} = \sum_{q=0}^{s-1} M^{(q)}_{ij}.$$ 

These are orthogonal projections, and

$$\sum_{i=1}^{N} Q_{ij} = \sum_{i=1}^{N} \sum_{q=0}^{s-1} M^{(q)}_{ij} = 1 = \sum_{j=1}^{N} Q_{ij}$$

so that they form a quantum permutation matrix. As a consequence, there exists a unique $*$-homomorphism $\Psi' : O(S_N^+) \to O(H_N^+)$ sending $P_{ij}$ to $Q_{ij}$.

By the universal property of unital free products, there exists a unique $*$-homomorphism $\Psi : O(G)^* \star O(S_N^+)$ which coincides with $\Psi_i$ on $\nu_i(O(G))$ and with $\Psi'$ on $O(S_N^+)$. Moreover, because

$$N^{(p,q)}_i Q_{ij} = \left( \sum_{k=1}^{N} M^{(q-p)}_{ik} \right) \left( \sum_{r=0}^{s-1} M^{(r)}_{ij} \right)$$

$$= \sum_{k=1}^{N} \sum_{r=0}^{s-1} M^{(q-p)}_{ik} M^{(r)}_{ij}$$

$$= M^{(q-p)}_{ij}$$

$$= \sum_{k=1}^{N} \sum_{r=0}^{s-1} M^{(r)}_{ij} M^{(q-p)}_{ik}$$

$$= Q_{ij} N^{(p,q)}_i,$$

the $*$-homomorphism $\Psi$ factors through $A$. By construction, it is inverse to $\Phi$, hence the result.

As a consequence, we can prove the classical isomorphism given as a motivation for this section:

**Corollary 7.2.21.** There is an isomorphism

$$H_N^+ \sim Z_s \wr S_N.$$
Proof. This follows from the fact that taking the maximal abelian quotient of $O(H_N^{s+})$ yields $O(H_N^s)$ with its usual group structure because of their descriptions through categories of partitions, while we have already mentioned in Remark 7.2.18 that the maximal abelian quotient of $O(G \wr \ast S_N^+)$ is $O(G \wr S_N)$ with its wreath product structure.

Remark 7.2.22. The reader is invited to check that the proofs still work for $s = \infty$, yielding isomorphisms

\[ H_N^{\infty+} \simeq \mathbb{Z} \wr S_N^+ \text{ and } H_N^{\infty} \simeq \mathbb{Z} \wr S_N. \]

Let us conclude with a more general comment. A classical result asserts that if $X$ is a connected oriented graph and $X_N$ is the disjoint union of $N$ copies of $X$, then

\[ \text{Aut}(X_N) = \text{Aut}(X) \wr S_N. \]

This was in fact the original motivation behind the definition of the free wreath product by J. Bichon in [Bic04], where he proved the following quantum analogue:

**Theorem 7.2.23** (Bichon) Let $X$ be a connected oriented graph and let $X_N$ be the disjoint union of $N$ copies of $X$. Then,

\[ Q\text{Aut}(X_N) = Q\text{Aut}(X) \wr S_N^+. \]
In this final section, we will explain a very recent result concerning the structure of the quantum permutation groups $S_N^+$ called residual finite-dimensionality. This result has several motivations. One of them is the theory of finite-dimensional approximation of operator algebras. This is not the subject of this text, but we cannot help but suggest reading the book [BO08] for a comprehensive treatment of that fascinating subject. In particular, Theorem 8.1.7 implies that the von Neumann algebras $L^\infty(S_N^+)$ satisfies the so-called Connes embedding conjecture, a major open problem in von Neumann algebra theory.

Another motivation comes from the connection with quantum information theory through graph isomorphism games, and this is the path that we will take.

8.1 Finite-dimensional strategies

Let us get back to the graph isomorphism game described in Section 1.1. Recall that perfect quantum strategies are given by specific quantum permutation matrices in the sense of Definition 1.1.5, and that these are nothing but representations of the $*$-algebra $O(S_N^+)$ on a Hilbert space $H$. From the physical point of view, of peculiar interest are the finite-dimensional ones, i.e. those for which $H$ can be chosen to be finite-dimensional. Indeed, for practical purposes it is important to know whether the strategy can be implemented with a finite number of degrees of freedom. Formalizing this leads to the following general problem:

**Question 8.1.1.** Given two finite graphs $X$ and $Y$ such that there exists a perfect quantum strategy for the corresponding isomorphism game, does there exist a perfect finite-dimensional quantum strategy?

This turns out to be a deep problem with connections to important conjectures in quantum information theory, quantum computation and operator algebras. We will not dig into this here but simply consider the following related question at the level of quantum permutation groups.

**Question 8.1.2.** Does the $*$-algebra $O(S_N^+)$ have many finite-dimensional representations?

The question is purposely vague because part of it is precisely to find a suitable meaning to the word “many”. One reasonable notion of having many finite-dimensional representations is the possibility of separating the points of the $*$-algebra by them. This leads to the following notion:

**Definition 8.1.3.** A $*$-algebra $A$ is said to be residually finite-dimensional if for any $x \neq 0$ in $A$, there exists a finite-dimensional $*$-algebra $B$ and a $*$-homomorphism $\pi : A \to B$ such that $\pi(x) \neq 0$.

The above definition can be restated in several ways. For instance, any finite-dimensional $*$-algebra $B$ embeds through left multiplication into $\mathcal{L}(B) \cong M_{\dim(B)}(\mathbb{C})$, so that we can equivalently assume that the points of $A$ are separated by finite-dimensional representations. Moreover,
gathering all these maps shows the existence of a family of integers \((n_i)_{i \in I}\) such that there is an embedding of \(*\)-algebras\(^1\)

\[ A \hookrightarrow \prod_{i \in I} M_{n_i}(\mathbb{C}). \]

There is no general recipe for proving such a property for a general \(*\)-algebra. However, when the algebra comes from a discrete group, then one may translate this into a property of the group. Here is the natural analogue:

**Definition 8.1.4.** A discrete group \(\Gamma\) is said to be **residually finite** if and only if there exists finite groups \((\Lambda_i)_{i \in I}\) such that there is a group embedding

\[ \Gamma \hookrightarrow \prod_{i \in I} \Lambda_i. \]

In other words, the points of \(\Gamma\) are separated by its finite quotients. If \(\Gamma\) is residually finite, then let

\[ x = \sum_{g \in F} \lambda_g a_g \in \mathbb{C}[\Gamma]. \]

be an arbitrary element, where \(F\) is some finite subset of \(\Gamma\). If \(\Lambda_g\) is a finite quotient of \(\Gamma\) in which \(g\) is not trivial, then the image of \(x\) in \(\mathbb{C}[\Lambda]\) is non-zero, where

\[ \Lambda = \prod_{g \in F} \Lambda_g. \]

Since \(\Lambda\) is finite, \(\mathbb{C}[\Lambda]\) is finite-dimensional and we have proven that \(\mathbb{C}[\Gamma]\) is residually finite-dimensional.

Upon seeing this, it is natural to wonder whether the converse also holds. This is false in general, but true if the group \(\Gamma\) is moreover finitely generated. The complete proof of this fact is beyond our scope, but we can at least give a precise idea of why it holds.

**Theorem 8.1.5** Let \(\Gamma\) be a finitely generated discrete group. If \(\mathbb{C}[\Gamma]\) is residually finite-dimensional, then \(\Gamma\) is residually finite.

**Sketch of proof.** The proof proceeds in two steps. The first one can be done using our setting of unitary compact matrix group. We first prove that \(\Gamma\) embeds into a unitary compact matrix group \(G\). To do this, let us consider the \(*\)-algebra \(A\) of all coefficient functions of finite-dimensional unitary representations of \(\Gamma\). For such a coefficient function \(\rho_{ij}\), define a function \(\Delta(\rho_{ij})\) on \(\Gamma \times \Gamma\) by

\[ \Delta(\rho_{ij}) : (g, h) \mapsto \rho_{ij}(gh) = \sum_{k=1}^{\dim(\rho)} \rho_{ik}(g) \otimes \rho_{kj}(h). \]

This uniquely defines a \(*\)-homomorphism

\[ \Delta : A \to A \otimes A \]

so that if we can construct a fundamental representation, we will have a unitary compact matrix group. Let \(S = \{g_1, \ldots, g_N\}\) be a generating set for \(\Gamma\) and fix for each \(1 \leq \ell \leq N\) a finite-dimensional representation \(\pi_\ell\) of \(\mathbb{C}[\Gamma]\) such that

\[ \pi_\ell(a_{g_\ell} - a_\epsilon) \neq 0. \]

By definition, \(\pi_\ell\) restricts to a unitary representation \(\rho_\ell\) of \(\Gamma\) such that \(\rho_\ell(g_\ell) \neq \text{Id.} \) As a consequence, setting

\[ \rho = \bigoplus_{\ell=1}^{N} \rho_\ell, \]

\(^1\) No assumption is made concerning the cardinality of the set \(I\).
the pair \((A, \rho)\) satisfies the axiom of a unitary compact matrix quantum group\(^2\). Since \(A\) is moreover commutative, there exists a unitary compact matrix group \(G\) such that \(A = \mathcal{O}(G)\). Recall that \(G\) can be recovered through the characters of \(A\). For each \(g \in \Gamma\), the evaluation map \(ev_g\) yields such a character. Moreover, two such characters are distinct because, as the points of \(C[\Gamma]\) are separated by finite-dimensional representation, the points of \(\Gamma\) are separated by finite-dimensional unitary representations. As a consequence, \(\Gamma\) embeds into \(G\).

The second step is a theorem by Mal’cev from [Mal40], stating that any finitely generated linear\(^3\) group is residually finite. The interested reader can refer for instance to [BO08, Thm 6.4.13] for an elementary proof of this fact. Since we just proved that \(\Gamma\) is linear, the proof is complete. ■

Based on this result, we can give the following definition:

**Definition 8.1.6.** A unitary compact matrix quantum group \(G\) is said to be *residually finite* if \(\mathcal{O}(G)\) is a residually finite-dimensional \(*\)-algebra.

The main point of this restatement is that we have turned our algebraic problem into a quantum group problem, opening the door to the use of representation theory techniques, and in particular the combinatorics of partitions. We will illustrate this by answering Question 8.1.2:

**Theorem 8.1.7** (Brannan-Chirvasitu-F.) The quantum permutation groups \(S_N^+\) are residually finite for all \(N\).

### 8.2 The proof

#### 8.2.1 Topological generation

The strategy for the proof of Theorem 8.1.7 for \(S_N^+\) is induction on \(N\). This may seem strange, but it is rather natural if one thinks in terms of discrete groups. Indeed, let \(\Gamma\) be a finitely generated discrete group and let \(\Gamma_1\) and \(\Gamma_2\) be quotients of \(\Gamma\). Assume that the two following properties hold:

- Both \(\Gamma_1\) and \(\Gamma_2\) are residually finite,
- Any element of \(\Gamma\) has a non-trivial image either in \(\Gamma_1\) or in \(\Gamma_2\).

Then, \(\Gamma\) is obviously residually finite\(^4\).

This observation suggests to reduce the problem to quotients which are easier to handle while being large enough to recover the original group. To express it at the level of compact matrix quantum groups, we should try to write the second condition in terms of the \(*\)-algebras \(C[\Gamma_1]\), \(C[\Gamma_2]\) and \(C[\Gamma]\). First note that the quotient map \(\pi_i : \Gamma \to \Gamma_i\) for \(i = 1, 2\) induces a surjective \(*\)-homomorphism \(C[\Gamma] \to C[\Gamma_i]\) mapping the fundamental representation to the fundamental representation coefficient-wise. Moreover, if \(\Lambda = \ker(\pi_1) \cap \ker(\pi_2)\), then both \(\pi_1\) and \(\pi_2\) factor through \(\Gamma \to \Gamma/\Lambda\), so that the \(*\)-algebra maps also factor through \(C[\Gamma] \to C[\Gamma/\Lambda]\).

In other words, the second condition is equivalent to the impossibility of factoring both \(\pi_1\) and \(\pi_2\) at the same time.

The last step to obtain a tractable quantum version of the above strategy is to translate our observations in a “compact” perspective. This will be done thanks to the key notion of topological generation. This idea was first introduced by A. Chirvasitu in [Chi15] (though not under that name). To explain it, let us write \(\mathbb{H} < \mathfrak{G}\) if \(\mathfrak{G} = (\mathcal{O}(\mathfrak{G}), u)\) and \(\mathbb{H} = (\mathcal{O}(\mathbb{H}), v)\) are unitary compact matrix quantum groups with a surjective \(*\)-homomorphism \(\pi : \mathcal{O}(\mathfrak{G}) \to \mathcal{O}(\mathbb{H})\)

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2. This compact group is called the Bohr compactification of \(\Gamma\).
3. A discrete group is said to be linear if it is a subgroup of \(GL_N(K)\) for some field \(K\) and some integer \(N \in \mathbb{N}\).
4. We are saying here that residually residually finite groups are residually finite, which is almost tautological.
such that $\pi(u_{ij}) = v_{ij}$. We already mentionned in the beginning of Section 1.3.3 that this is a natural analogue of the inclusion of compact groups. Note that in such a situation, if both $\mathbb{G}$ and $\mathbb{H}$ are classical compact groups, then any character on $\mathcal{O}(\mathbb{H})$ yields a character on $\mathcal{O}(\mathbb{G})$ through composition with $\pi$, hence $\mathbb{H}$ embeds as a closed subgroup of $\mathbb{G}$, explaining the notation.

**Definition 8.2.1.** Consider $\mathbb{G}_1, \mathbb{G}_2 < \mathbb{G}$ given by surjections $\pi_1$ and $\pi_2$. We say that $\mathbb{G}$ is *topologically generated* by $\mathbb{G}_1$ and $\mathbb{G}_2$ if there is no $\mathbb{H} < \mathbb{G}$ (except for $\mathbb{G}$ itself), given by a surjection $\pi$ such that both $\pi_1$ and $\pi_2$ factor through $\pi$.

**Remark 8.2.2.** Assume that $G_1, G_2 < G$ are two closed compact subgroups, and let $K$ be the closure of the subgroup of $G$ generated by $G_1$ and $G_2$. Then, $G_1, G_2 < K < G$ so that $\pi$ factors through the restriction map $\pi_K : \mathcal{O}(G) \to \mathcal{O}(K)$.

In other words, $G$ is topologically generated by $G_1$ and $G_2$ if and only if it is generated by them as a topological group, hence the name.

**Remark 8.2.3.** For the sake of simplicity, we will work exclusively with orthogonal compact matrix quantum groups in this section. However, all the general arguments carry to the setting of unitary compact matrix quantum groups (and even general compact quantum groups). We leave it to the interested reader to find the natural analogues of the statements and proofs.

The core result we need is the following:

**Proposition 8.2.4.** If $\mathbb{G}$ is topologically generated by two residually finite orthogonal compact matrix quantum subgroups $\mathbb{G}_1$ and $\mathbb{G}_2$, then $\mathbb{G}$ is residually finite.

**Proof.** Let $\mathcal{I}$ be the intersection of the kernels of all finite-dimensional $*$-representations of $\mathcal{O}(\mathbb{G})$, let $\mathcal{A} = \mathcal{O}(\mathbb{G})/\mathcal{I}$ be the corresponding quotient with canonical surjection $\pi : \mathcal{O}(\mathbb{G}) \to \mathcal{A}$

and set $v_{ij} = \pi(u_{ij})$. Let $x \in \mathcal{I}$ and assume that $\pi_i(x) \neq 0$ for some $i \in \{1, 2\}$. Then, because $\mathbb{G}_i$ is residually finite, there exists a finite-dimensional representation $\rho$ of $\mathcal{O}(\mathbb{G}_i)$ such that $\rho \circ \pi_i(x) \neq 0$. But $\rho \circ \pi_i$ is also a finite-dimensional representation of $\mathcal{O}(\mathbb{G})$, hence its kernel contains $\mathcal{I}$, a contradiction. As a consequence, both $\pi_1$ and $\pi_2$ factor through $\pi$. To conclude we therefore simply have to show that $(\mathcal{A}, v)$ is an orthogonal compact matrix quantum group.

To do so, first note that $\mathcal{I}$ is an intersection of $*$-ideals, hence $\mathcal{A}$ is a $*$-algebra. Moreover, if $x \in \mathcal{I}$, and if $\rho_1, \rho_2$ are finite-dimensional representations of $\mathcal{O}(\mathbb{G})$, then

$$\rho_1 \otimes \rho_2 \circ \Delta$$

is also a finite-dimensional representation, hence $(\rho \otimes \rho) \circ \Delta(x) = 0$. By Lemma 1.3.3, this means that

$$\Delta(x) \subseteq \ker(\rho_1) \otimes \mathcal{O}(\mathbb{G}) + \mathcal{O}(\mathbb{G}) \otimes \ker(\rho_2).$$

In other words, writing

$$\Delta(x) = \sum_{i=1}^{n} x_i \otimes y_i$$

we have that for any $1 \leq i \leq n$, either $\rho_1(x_i) = 0$ or $\rho_2(y_i) = 0$. We claim that this implies

$$\Delta(x) \in \mathcal{I} \otimes \mathcal{O}(\mathbb{G}) + \mathcal{O}(\mathbb{G}) \otimes \mathcal{I}.$$ 

Indeed, if this was not the case, then there would exist $1 \leq i \leq n$ such that $x_i, y_i \notin \mathcal{I}$. By definition of $\mathcal{I}$, this means that there exists finite-dimensional representations $\rho_1, \rho_2$ such that $\rho_1(x_i) \neq 0$ and $\rho_2(y_i) \neq 0$, a contradiction.

It follows that there exists a $*$-homomorphism $\tilde{\Delta} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ uniquely determined by

$$\tilde{\Delta} \circ \pi = (\pi \otimes \pi) \circ \Delta.$$
In particular,
\[ \tilde{\Delta}(v_{ij}) = \sum_{k=1}^{N} v_{ik} \otimes v_{kj} \]
and \((\mathcal{A}, \nu)\) is an orthogonal compact matrix quantum group. By the definition of topological generation, we must therefore have \(\mathcal{A} = O(\mathbb{G})\) and \(\nu = u\), i.e. \(\mathcal{I} = \{0\}\).

In the sequel, we will need a criterion to prove topological generation, which relies on the following restatement of Definition 8.2.1.

**Lemma 8.2.5.** Consider \(G_1, G_2 \subset G\) given by surjections \(\pi_1\) and \(\pi_2\). Then, \(G\) is topologically generated by \(G_1\) and \(G_2\) if there is no \(H \subset G\) (except for \(G\) itself), given by a surjection \(\pi\) such that the \(\ast\)-homomorphism
\[ \rho = (\pi_1 \otimes \pi_2) \circ \Delta : O(G) \to O(G_1) \otimes O(G_2) \]
factors through \(\pi\).

**Proof.** It is clear that if both \(\pi_1\) and \(\pi_2\) factor through \(\pi\), then so does \(\rho\). Conversely, assume that \(\ker(\pi) \subset \ker(\rho)\). Then, for \(x \in \ker(\pi)\),
\[
\pi_1(x) = \pi_1((\text{id} \otimes \varepsilon) \circ \Delta(x)) = \pi_1((\text{id} \otimes \varepsilon \circ \pi_2) \circ \Delta(x)) = (\text{id} \otimes \varepsilon) \circ ((\pi_1 \otimes \pi_2) \circ \Delta(x)) = 0
\]
and similarly for \(x \in \ker(\pi_2)\).

**Proposition 8.2.6.** Let \(G\) be an orthogonal compact matrix quantum group and let \(G_1, G_2 \subset G\) be quantum subgroups. If for any \(k \in \mathbb{N}\),
\[
\text{Mor}_G\left(u \otimes^k, \varepsilon\right) = \text{Mor}_{G_1}\left(u_1 \otimes^k, \varepsilon\right) \cap \text{Mor}_{G_2}\left(u_2 \otimes^k, \varepsilon\right),
\]
then \(G\) is topologically generated by \(G_1\) and \(G_2\).

**Proof.** First note that if \(f : (\mathbb{C}^N) \otimes^k \to \mathbb{C}\) is invariant under \(\rho_{u \otimes^k}\), then it is also invariant under
\[
(id \otimes \pi_i) \circ \rho_{u \otimes^k} = \rho_{u_i \otimes^k}
\]
so that the left-hand side is always contained in the right-hand side.

Let us introduce the following inductive notation: \(\Delta^{(1)} = \Delta\) and for \(k \geq 1\),
\[
\Delta^{(k+1)} = (\Delta^{(k)} \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta^{(k)}) \circ \Delta,
\]
where the last equality comes from the coassociativity of the coproduct (see Equation (1.2)).

With the notations of Lemma 8.2.5, we consider the \(\ast\)-ideal
\[
\mathcal{I} = \bigcap_{k \in \mathbb{N}} \ker\left(\rho^{\otimes k} \circ \Delta^{(k)}\right).
\]

Using the fact that
\[
(\rho^{\otimes k} \circ \Delta^{(k)} \otimes \rho^{\otimes k'} \circ \Delta^{(k')}) \circ \Delta = \rho^{\otimes (k+k')} \circ \Delta^{(k+k')},
\]
and the same reasoning as in the proof of Proposition 8.2.4, we infer that
\[
\Delta(\mathcal{I}) \subset O(\mathbb{G}) \otimes \mathcal{I} + \mathcal{I} \otimes O(\mathbb{G}),
\]
hence the quotient \(\mathcal{A} = O(\mathbb{G})/\mathcal{I}\) has an orthogonal compact matrix quantum group structure coming from the images \(v_{ij}\) of \(u_{ij}\).
Let now $\mathbb{H}' < \mathbb{G}$ such that $\rho$ factors through $\pi' : \mathcal{O}(\mathbb{G}) \to \mathcal{O}(\mathbb{H}')$. Then, for any $k \in \mathbb{N}$,
\[
\ker(\pi') \subset \ker \left( \Delta^{(k)} \circ \pi' \right) = \ker \left( \pi'^{\otimes k} \circ \Delta^{(k)} \right) \subset \ker \left( \rho^{\otimes k} \circ \Delta^{(k)} \right)
\]
so that $\ker(\pi') \subset \ker(\pi)$ and $\pi$ factors through $\pi'$. In other words, if we prove that $\mathbb{H} = \mathbb{G}$, then the proof will be complete.

By Lemma 8.2.5, $\mathcal{I} \subset \ker(\pi_i)$ for $i = 1, 2$ so that $\mathbb{G}_i < \mathbb{H}$. The comment at the beginning of the proof therefore yields
\[
\text{Mor}_{\mathbb{H}}(u^{\otimes k}, \varepsilon) \subset \text{Mor}_{\mathbb{G}_1}(u_1^{\otimes k}, \varepsilon) \cap \text{Mor}_{\mathbb{G}_2}(u_2^{\otimes k}, \varepsilon).
\]
Similarly, because $\mathbb{H} < \mathbb{G}$,
\[
\text{Mor}_{\mathbb{G}}(u^{\otimes k}, \varepsilon) \subset \text{Mor}_{\mathbb{H}}(u^{\otimes k}, \varepsilon)
\]
Using now our assumption, we deduce that $\text{Mor}_{\mathbb{G}}(u^{\otimes k}, \varepsilon) = \text{Mor}_{\mathbb{H}}(u^{\otimes k}, \varepsilon)$ for all $k \in \mathbb{N}$. The same equality then holds for any tensor powers of $u$ and $v$, hence $\mathbb{G} = \mathbb{H}$ by Theorem 3.3.3, and this finishes the proof.

8.2.2 At least six points

The following result [BCF20, Thm 3.3 and Thm 3.12] is the key tool to prove residual finite-dimensionality. In this statement, we see the inclusion $S^+_{N-1} < S^+_N$ through the surjection
\[
\pi_1 : \mathcal{O}(S^+_N) \to \mathcal{O}(S^+_N)
\]
sending $u_{11}$ to 1.

**Theorem 8.2.7** (Brannam-Chirvasitu-F.) For any $N \geq 6$, the quantum permutation group $S^+_N$ is topologically generated by $S^+_{N-1}$ and $S_N$.

**Proof of Theorem 8.2.7.** Set $V = \mathbb{C}^N$ and let $P_N, P_{N-1}$ and $P^\text{class}_N$ denote the fundamental representations of $S^+_N, S^+_{N-1}$ and $S_N$ respectively. We will say that a map is $S^+_N$ (respectively $S^+_{N-1}$, respectively $S_N$) invariant if it commutes with appropriate tensor powers of $P_N$ (respectively $P_{N-1}$, respectively $P^\text{class}_N$). By Proposition 8.2.6, it is enough to prove that if
\[
f : V^{\otimes k} \to \mathbb{C}
\]
is a linear map which is both $S^+_{N-1}$-invariant and $S_N$-invariant, then $f$ is $S^+_N$-invariant.

Let us therefore consider such a map $f$ and, for $1 \leq i \leq N$, let $V_i = e_i^+$. Because $f$ is $S^+_{N-1}$ invariant, its restriction to $V_i^{\otimes k}$ is a linear combination of partitions maps $: \text{there exist complex numbers } (\lambda_p)_{p \in NC(k)} \text{ such that}$
\[
f|_{V_i^{\otimes k}} = \sum_{p \in NC(k)} \lambda_p f_p.
\]
Let us set
\[
\tilde{f} = f - \sum_{p \in NC(k)} \lambda_p f_p.
\]
This is still invariant under $S^+_{N-1}$ and $S_N$ and vanishes on $V_i^{\otimes k}$. Our task is to show that it vanishes on the whole of $V^{\otimes k}$.

For this purpose, let us set $V'_i = \mathbb{C} e_i$, so that
\[
V^{\otimes k} = \bigoplus_{\ell_1, \ldots, \ell_k} V_1^{\ell_1} \otimes \cdots \otimes V_1^{\ell_k}
\]
where $\epsilon$ is either prime or nothing. Let us consider one of these summands where $V_i$ appears $\ell$ times and denote it by $W$. Since $S^+_{N-1}$ acts trivially on $V'_i$, there exists a linear $S^+_{N-1}$-equivariant isomorphism
\[
\Phi : W \to V_1^{\otimes \ell}.
\]
As a consequence, there exist complex numbers $(\mu_p)_{p \in NC(\ell)}$ such that

$$\tilde{f} \circ \Phi^{-1} = \sum_{p \in NC(\ell)} \mu_p f_p.$$  

The idea now is to use the linear independence of the partition maps to conclude that $\mu_p = 0$ for all $p$, hence that $\tilde{f} \circ \Phi^{-1} = 0$.

To do this, set $V_{1,N} = V_1 \cap V_N$ and observe that

$$\Phi^{-1}(V_{1,N}^{\otimes \ell}) \subset V_N^{\otimes k}.$$  

Now, by $S_N$-invariance, we can exchange $e_1$ and $e_N$ without changing the value of $\tilde{f}$, hence it vanishes on $V_N^{\otimes k}$. Thus, $\tilde{f} \circ \Phi^{-1}$ vanishes on $V_{1,N}^{\otimes \ell}$. Since $N \geq 6$, $\dim(V_{1,N}) \geq 4$ so that noncrossing partition maps on $V_{1,N}^{\otimes \ell}$ are linearly independent by Theorem 4.1.1. This forces $\mu_p = 0$ for all $p$, hence $\tilde{f} = 0$. ■

8.2.3 At most five points

So far, Theorem 8.2.7 is useless for an inductive proof since we do not know the base case: is $\mathcal{O}(S_5^+)$ residually finite-dimensional? Before addressing this question, let us comment on the case $N = 4$. It was shown by B. Collins and T. Banica in [BC08, Thm 4.1] that there is an embedding

$$\pi : C(S_4^+) \hookrightarrow C(SU(2), M_4(\mathbb{C})).$$

As a consequence, the $*$-representations $\pi_g : x \mapsto \pi(x)(g)$ for all $g \in SU(2)$ separate the points, i.e. $S_4^+$ is residually finite-dimensional.

Only $N = 5$ remains, and this is indeed a non-trivial matter which can be solved using the classification of subfactors. It was showed by T. Banica in [Ban18, Thm 7.10] that $S_5^+$ enjoys a much stronger property: the canonical map

$$\pi_{ab} : \mathcal{O}(S_5^+) \rightarrow \mathcal{O}(S_5)$$

does not factor through any compact matrix quantum group. The idea of the proof is that by [Ban99a] and [TW18], any quantum subgroup of $S_5^+$ yields a subfactor planar algebra at index 5, with extra properties if it contains $S_5$. Moreover, this correspondence is injective. Now one has to look at the complete list of subfactor planar algebras at index 5 satisfying the extra properties (see for instance the survey [JMS14]) and check that none of the corresponding quantum groups contains $S_5$, which is not very difficult. We can now complete our proof:

**Proof of Theorem 8.1.7 for $S_5^+$.** First, we can extend the statement of Theorem 8.2.7 to $N = 5$. Indeed, if $S_5^+$ was not topologically generated by $S_5$ and $S_4^+$, then there would be an $S_5 < \hat{H} < S_5^+$ which is not classical because $S_4^+ < \hat{H}$, contradicting the aforementioned result. Thus, $S_4^+$ and $S_5$ topologically generate $S_5^+$. We can now conclude by induction starting from the fact that $S_4^+$ is residually finite. ■

**Remark 8.2.8.** Let us mention the following interesting problem: is there a compact matrix quantum group through which the quotient map

$$\pi_{ab} : \mathcal{O}(S_N^+) \rightarrow \mathcal{O}(S_N)$$

does not factor through any compact matrix quantum group? If not, then the inclusion $S_N < S_N^+$ is said to be *maximal*. For $N \leq 3$, maximality trivially follows from the equality of the two quantum groups. For $N = 4$, it can be proven by checking the list of all quantum subgroups of $S_4^+$ given in [BB09] and we already mentioned the proof of T. Banica in [Ban18, Thm 7.10] for $N = 5$. This is all that is known to this day.
8.3 Further examples

To conclude this chapter, we will prove that our other travelling companions $O_N^+$ and $H_N^{s+}$ also have the Connes embedding property are residually finite. We will start with $H_N^{s+}$, using a trick relying on the free wreath product decomposition of Section 7.2.2.

**Theorem 8.3.1** Let $s, N$ be integers. Then, the quantum group $H_N^{s+}$ is residually finite.

*Proof.* It was proven in Theorem 7.2.20 that $H_N^+ = \mathbb{Z}_s \ltimes S_N^+$ and this can be restated in the following way: consider the sequence of $*$-algebras $A_k$ where $A_0 = \mathcal{O}(S_N^+)$ and for any $k \geq 0,$

$$A_{k+1} = C[\mathbb{Z}_s] \ast \frac{A_k}{\langle [C[\mathbb{Z}_s], u_{k+1}] | 1 \leq j \leq N \rangle}.$$  

Then, $A_N = \mathcal{O}(H_N^{s+}).$ All that we need is therefore a stability result for residual finite-dimensionality. Let us first notice that $\langle u_{k+1} | 1 \leq j \leq N \rangle \simeq C^N$ so that

$$A_{k+1} \simeq \left( C[\mathbb{Z}_s] \otimes C^N \right) \ast_{C^N} A_k \hookrightarrow \left( C[\mathbb{Z}_s] \otimes A_k \right) \ast_{C^N} \left( C[\mathbb{Z}_s] \otimes A_k \right).$$

It therefore suffices to show that $A \ast_E A$ is residually finite-dimensional as soon as $A$ is and $B$ is finite-dimensional. We will not give the proof but simply explain the two steps:

1. Using a free product decomposition of a Hilbert space on which $A$ acts faithfully, it is easy to see that any element has non-zero image into a similar free product with $A$ replaced by a finite-dimensional quotient. It is therefore sufficient to do it for finite-dimensional $A.$

2. The result is then a particular case of [ADEL04, Thm 4.2], where the strategy is to embed $A$ in a matrix algebra in a trace-preserving way. One can then use [BD04, Thm 2.3] to conclude. 

**Remark 8.3.2.** The same strategy works to prove that if $\Gamma$ is a finite group, then $\Gamma \ltimes S_N^+$ is residually finite-dimensional. Observing with the same argument as in [BCF20, Lem 2.13] that if $\Gamma$ is residually finite, then $\Gamma \ltimes S_N^+$ is topologically generated by $\Lambda \ltimes S_N^+$ for all finite quotients $\Lambda$ of $\Gamma,$ we see that the free wreath product is residually finite as soon as $\Gamma$ is.

For $O_N^+$, there are two available proofs of residual finiteness. One uses topological generation in the spirit of Theorem 8.2.7 (this is [BCV17, Thm 4.1]) and the other one first proves the result for the free unitary quantum groups $U_N^+$ and then uses the fact that $O_N^+$ contains a “finite-index” quantum subgroup which is also a quantum subgroup of $U_N^+$ (this is [Chi15, Thm 3.1]). Here, we will indicate how the result for $H_N^{s+}$ can be used to simplify the topological generation approach.

**Theorem 8.3.3** The quantum group $O_N^+$ is residually finite for all $N.$

*Proof.* We will use a topological generation result to prove this by induction. First note that by definition, $H_N^- < O_N^+.$ Let now $A$ be the quotient of $\mathcal{O}(O_N^+)$ by the ideal generated by $U_{11} - 1.$ We claim that $A \simeq \mathcal{O}(O_{N-1}^+).$ Indeed, if we consider any matrix $(v_{ij})_{1 \leq i,j \leq (N-1)}$ of operators which is orthogonal, then there exists by the universal property a map such that

- $U_{11} \mapsto 1,$
- $U_{1i}, U_{j1} \mapsto 0,$
- $U_{ij} \mapsto v_{(i-1)(j-1)}$ for $i,j > 1.$

This factors by definition through a surjection from $A$ to the $*$-algebra generated by the $v_{ij}$’s so that the claim is proven. We will write $B_N^+ = (A,v)^5,$ where $v$ is the image of $U$ in the quotient.

---

5. This is called the quantum bistochastic group by analogy with the classical group of all bistochastic matrices.
Let us now show that $A$ in fact comes from a partition quantum group. Let $p \in NC(1,0)$ be the singleton partition, whose corresponding linear map $T_p : C^N \to C$ sends all basis vectors to 1. Then,

$$(\text{id} \otimes T) \circ \rho_U(e_i) = \sum_{j=1}^{N} U_{ij} \otimes 1$$

while $\varepsilon \circ T(e_i) = 1$, so that if we add $p$ to the category of partitions $NC_2$ of $O_N^+$, we get the extra relation

$$\sum_{j=1}^{N} U_{ij} = 1$$

for all $1 \leq i \leq N$. This is in fact the defining relation of $B_N^+$ in disguise. Indeed, set

$$\xi_1 = \sum_{i=1}^{N} e_i$$

and $\xi_j = e_1 - e_j$ for $j \neq 1$. This is an orthonormal basis of $C^N$, and if we denote by $V$ the image of $U$ in this basis, then the relation given by $p$ becomes $V_{11} = 1$.

As a consequence, $B_N^+$ is isomorphic to the partition quantum group associated to the category of partitions $NC_{1,2} = \langle p, NC_2 \rangle$.

A straightforward induction shows that $NC_{1,2}$ is the category of all partitions with blocks of size at most two. It therefore follows from Theorem 4.1.1 that for $N \geq 4$,

$$\text{Mor}_{B_N^+}(u^\otimes k, \varepsilon) \cap \text{Mor}_{H_N^+}(u^\otimes k, \varepsilon) = \text{Vect} \{ f_p \mid p \in NC_{2,1}(k) \} \cap \text{Vect} \{ f_p \mid p \in NC_{\text{even}}(k) \}
= \text{Vect} \{ f_p \mid p \in NC_{2,1}(k) \cap NC_{\text{even}}(k) \}
= \text{Vect} \{ f_p \mid p \in NC_2(k) \}
= \text{Mor}_{O_N^+}(U^{\otimes k}, \varepsilon),$$

i.e. $O_N^+$ is topologically generated by $H_N^+$ and $B_N^+ \simeq O_N^+$.

To conclude we would need the result for $O_3^+$. Unfortunately, this is the only case which is still open at the time these notes are written. It is nevertheless possible to “jump over it” thanks to [BCV17, Thm 4.2] : $O_4^+$ is topologically generated by the free product $O_2^+ \ast O_2^+$ and the permutation group $S_4$. As a consequence, it is residually finite, and we can now conclude. ■
APPENDIX A : TWO THEOREMS ON COMPLEX MATRIX ALGEBRAS

For the sake of completeness, we give a detailed proof of two results from the representation theory of algebras which were used in these lectures. Since we only need them for algebras of matrices over the field of complex number, we will only give the proofs in that case, allowing us to simplify some of the arguments.

Let us recall a few basic facts for convenience: a subalgebra $A \subset M_n(\mathbb{C})$ is said to be irreducible if there is no vector in $\mathbb{C}^n$ fixed by all the elements of $A$. The nature of irreducible matrix algebras is elucidated by the following celebrated result of Burnside:

**Theorem** A (Burnside’s Theorem) Let $A \subset M_n(\mathbb{C})$ be an irreducible subalgebra. Then,

$$ A = M_n(\mathbb{C}). $$

**Proof.** The proof proceeds in two steps. First, we will prove that $A$ contains a rank 1 matrix. Then, we will deduce from this that it contains all rank 1 matrices, hence all matrices. We will do this following [HR80]. Before we start, note that by irreducibility, for any non-zero $x \in \mathbb{C}^n$,

$$ A.x := \{ T(x) \mid T \in A \} = \mathbb{C}^n. $$

For the first part, we will proceed by contradiction. Let $T \in A$ be a matrix with minimal rank and assume that $d = \text{rk}(T) \geq 2$. Then, there exists $x_1, x_2 \in \mathbb{C}^n$ such that the vectors $T(x_1)$ and $T(x_2)$ are linearly independent. Let us choose $S \in A$ such that $ST(x_1) = x_2$. Then, $T(x_1)$ and

$$ TST(x_1) = T(x_2) $$

are linearly independent, so that the operator $TST - \lambda T$ is non-zero for all $\lambda \in \mathbb{C}$. However, notice that $TS - \mu \text{Id}$ is a linear operator on the range of $T$, hence it has an eigenvalue: there exists $\mu \in \mathbb{C}$ such that $TS - \mu \text{Id}$ is not invertible. Then,

$$ TST - \mu T = (TS - \mu \text{Id})T $$

has rank strictly between 0 and $d$, contradicting minimality. As a conclusion, there is a rank-one matrix in $A$.

The rank-one matrix obtained in the previous paragraph can be written as the operator

$$ T_{\phi,y} : x \mapsto \phi(x)y $$

for some linear form $\phi \in (\mathbb{C}^n)^*$ and a vector $y \in \mathbb{C}^n$. Because $Ay = \mathbb{C}^n$, we also have $T_{\phi,z} \in A$ for all $z \in \mathbb{C}^n$. Similarly, $A$ acts on $(\mathbb{C}^n)^*$ by

$$ (S, \psi) \mapsto \psi \circ S. $$

Assume that there is a fixed linear form $\eta$ and pick vectors $y_1 \notin \ker(\eta)$ and $y_2 \in \ker(\eta)$. By irreducibility there exists $S \in A$ such that $S(y_1) = y_2$, but then

$$ \eta \circ S(x_1) = 0 \neq \eta(x_1), $$
contradicting invariance. Therefore, $A$ acts irreducibly on $(\mathbb{C}^n)^*$, hence \( \{ \phi \circ S \mid S \in A \} = (\mathbb{C}^n)^* \). Putting things together, for any \( \psi \in (\mathbb{C}^n)^* \) and \( z \in \mathbb{C}^n \), there exists \( S_1, S_2 \in A \) such that
\[
S_1 \circ T \circ S_2 : x \mapsto \psi(x)z.
\]
As a consequence, $A$ contains all rank-one matrices, hence equals $M_n(\mathbb{C})$. ■

Note that the argument only requires the existence of an eigenvalue for the matrix $ST$, hence works for any algebraically closed field.

Our second result concerns the double commutant of a matrix $*$-algebra. Recall that given a subalgebra $A \subset M_n(\mathbb{C})$, its commutant $A'$ is by definition
\[
A' = \{ T \in M_n(\mathbb{C}) \mid AT = TA \}.
\]

**Theorem B** (Double Commutant Theorem) Let $A \subset M_n(\mathbb{C})$ be a subalgebra which is stable under taking adjoints. Then, $A'' = A$.

**Proof.** By definition $A \subset A''$. Moreover, if $V \subset \mathbb{C}^n$ is stable under the action of $A$, then by stability under taking adjoints, its orthogonal complement is also stable. As a consequence, there is an orthogonal decomposition
\[
\mathbb{C}^n = V_1 \oplus \cdots \oplus V_m
\]
into irreducible subspaces. Each of these subspaces yields an irreducible representation of $A$, and some of them may be equivalent. Let \( \{i_1, \cdots, i_k\} \) be such that any subspace is equivalent to $V_{i_j}$ for some $j$, and let us denote by $n_j$ the number of such subspaces. By Schur’s Lemma, $A$ is then block diagonal with $k$ blocks corresponding to the restrictions to each $V_{i_j}^{n_j}$.

Let us consider one of those diagonal blocks, which has size $n_j \times \dim(V_{i_j})$, and assume that this is all of $\mathbb{C}^n$. The restriction of $A$ to any of the summands is $M_{\dim(V_{i_j})}(\mathbb{C})$ by Theorem A. Moreover, for a given element $T \in A$, and two summands $V, V'$ of $V_{i_j}$, the fact that they are equivalent means that there is a linear isomorphism $\varphi : V \to V'$ such that
\[
\varphi \circ T|_V = T|_{V'} \circ \varphi.
\]
By Schur’s Lemma, $\varphi$ is a multiple of the identity, hence $T|_V = T|_{V'}$. As a conclusion, $A$ consists in all matrices of the form diag($T, \cdots, T$) for $T \in M_{\dim(V_{i_j})}(\mathbb{C})$. We now investigate the commutant of $A$, which is nothing but the self-intertwiners of $V_{i_j}^{n_j}$. Such an intertwiner is given by a block $n_j \times n_j$ matrix, where the $(a, b)$-th block is an intertwiner from the $a$-th copy of $V_{i_j}$ to the $b$-th one. By Schur’s Lemma again, each block must be a scalar matrix so that $A'$ is $M_{n_j}(\mathbb{C})$ seen as block scalar matrices in $M_{n_j \times \dim(V_{i_j})}(\mathbb{C})$. As a consequence, the bicommutant consists in all $M_{\dim(V_{i_j})}(\mathbb{C})$-block diagonal matrices that is, $A$ again.

Back to the general case, a final application of Schur’s Lemma shows that $A$ is block diagonal, with blocks of the previous form and the result follows. ■

**Corollary A.** Let $A \subset M_N(\mathbb{C})$ be a subalgebra stable under taking adjoints. Then, $A$ is isomorphic to a direct sum of matrix algebras.
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References


References


