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Amaury Freslon

Propriétés d'approximation pour les groupes quantiques discrets

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dirigée par Etienne Blanchard

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M. Saad Baaj	Université de Clermont-Ferrand	Rapporteur
M. Etienne Blanchard	CNRS	Directeur
M. Georges Skandalis	Université Paris VII	Examinateur
M. Stefaan Vaes	K.U. Leuven	Examinateur
M. Leonid Vainerman	Université de Caen	Président

Rapporteur absent lors de la soutenance : M. Narutaka Ozawa (RIMS)

Institut de Mathématiques de Jussieu – Paris Rive Gauche
Bâtiment Sophie Germain
8 place FM/13
75 013 Paris

École doctorale Sciences Mathématiques de Paris centre
Case 188
4 place Jussieu
75 252 Paris cedex 05

Ce sera comme un cercle
Qui se réveille droite,
Une équation montée
Dans l'ordre des degrés,
D'autres géométries
Pour vivre la lumière.
Guillevic, *Carnac*

Propriétés d'approximation pour les groupes quantiques discrets

Résumé

Cette thèse porte sur les *propriétés d'approximation* pour les groupes quantiques discrets et particulièrement sur la *moyennabilité faible*. Notre but est d'appliquer des techniques de théorie géométrique des groupes à l'étude des groupes quantiques.

Nous définissons d'abord la moyennabilité faible dans le cadre des groupes quantiques discrets et nous développons une théorie générale en nous inspirant du cas classique. Nous nous attachons particulièrement à la notion de *constante de Cowling-Haagerup*. Nous définissons aussi une notion de *moyennabilité relative* qui nous permet de démontrer un résultat de stabilité supplémentaire. Un travail similaire est effectué pour la *propriété de Haagerup*. Enfin, nous abordons la question des produits libres de groupes quantiques faiblement moyennables. En nous inspirant des travaux de E. Ricard et X. Qu sur les inégalités de Kintchine, nous démontrons que si deux groupes quantiques discrets ont une constante de Cowling-Haagerup égale à 1, leur produit libre amalgamé sur un sous-groupe quantique fini a également une constante de Cowling-Haagerup égale à 1.

Ensuite, nous donnons des exemples de groupes quantiques discrets faiblement moyennables. Nous utilisons les travaux de M. Brannan sur la propriété de Haagerup ainsi que des idées liées aux *inégalités de Haagerup*. Nous donnons une borne polynomiale pour la norme complètement bornée de certains projecteurs qui nous permet ensuite de "découper" les fonctions de M. Brannan pour prouver la moyennabilité faible. Enfin, nous appliquons des techniques d'équivalence monoïdale pour étendre ces résultats à d'autres classes de groupes quantiques, dont certains ne sont pas unimodulaires.

Mots-clefs

Algèbres d'opérateurs, groupes quantiques, moyennabilité faible, propriété de Haagerup, équivalence monoidale.

Approximation properties for discrete quantum groups

Abstract

This dissertation is concerned with the notion of *approximation property* for discrete quantum groups and in particular *weak amenability*. Our goal is to apply techniques from geometric group theory to the study of quantum groups.

We first give a definition of weak amenability in the setting of discrete quantum groups and we develop some aspects of the general theory, inspired by the classical case. We particularly focus on the notion of *Cowling-Haagerup constant*. We also define a notion of *relative amenability* in this context which allows us to prove an additional stability result. Similar results are worked out for the *Haagerup property*. Eventually, we adress the question of free products of weakly amenable discrete quantum groups. Using the work of E. Ricard and X. Qu on Kintchine inequalities for free products, we prove that if two discrete quantum groups have Cowling-Haagerup constant equal to 1, their free product again has Cowling-Haagerup equal to 1.

Secondly, we give examples of weakly amenable discrete quantum groups. To do this, we use the recent work of M. Brannan on the Haagerup property for free quantum groups together with ideas from various works on *Haagerup inequalities*. More precisely, we give a polynomial bound for the norm of projections on coefficients of an irreducible representation of a free orthogonal quantum groups which allows us to "cut off" M. Brannan's functions and compute the Cowling-Haagerup constant. Finally, we apply techniques of *monoidal equivalence* to extend these results to other classes of discrete quantum groups, some of which are not *unimodular*.

Keywords

Operator algebras, quantum groups, weak amenability, Haagerup property, monoidal equivalence.

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Introduction

Le but de cette thèse est d'étendre au cadre des groupes quantiques discrets certaines des *propriétés d'approximation* utilisées en théorie géométrique des groupes puis d'en faire une étude systématique. Le cas de la moyennabilité a été amplement étudié ces dernières années, par exemple dans [Rua96] et [Tom06], même si certains problèmes fondamentaux restent ouverts dans le cas localement compact. En ce qui concerne les affaiblissements de la moyennabilité, plusieurs articles récents établissent des résultats importants mais aucun traitement général n'a été donné au-delà de l'article [KR99].

Notre attention se portera plus spécifiquement sur la *moyennabilité faible*, une propriété qui n'était jusqu'à maintenant connue que dans des cas triviaux (soit des groupes classiques, soit des groupes quantiques moyennables). Nous aborderons cependant également la *propriété de Haagerup*, pour laquelle nous donnons de nouveaux exemples, ainsi que des questions liées à ces propriétés, notamment la propriété de décroissance rapide.

Aperçu historique et contexte

Les propriétés d'approximation

On peut faire remonter la théorie des propriétés d'approximation aux premiers travaux d'A. Grothendieck [Gro51, Gro55] (voir également les synthèses [Gro52] et [Gro56]) sur les produits tensoriels topologiques d'espaces vectoriels topologiques [6]. Notre intérêt ici est surtout historique, aussi nous renvoyons à [Yos95, Chap I et X] pour les définitions et résultats fondamentaux concernant les espaces vectoriels topologiques et leurs produits tensoriels.

Le problème initial est le suivant : étant donnés deux espaces vectoriels topologiques localement convexes V et W, il existe *a priori* plusieurs façons de munir le produit tensoriel algébrique $V \odot W$ d'une structure d'espace vectoriel topologique localement convexe. Il existe en particulier une complétion "maximale", dite *projective* et notée $V\widehat{\otimes}_{proj}W$, et une complétion "minimale", dite *injective* et notée $V\widehat{\otimes}_{inj}W$. De la définition découle l'existence d'une application linéaire continue

$$V\widehat{\otimes}_{proj}W \to V\widehat{\otimes}_{inj}W.$$

Une question naturelle est de savoir si cette application peut être injective. Elle mène à la notion de *nucléarité*.

Définition. Un espace vectoriel topologique localement convexe V est dit *nucléaire* si pour tout espace vectoriel topologique W, l'application canonique

$$V\widehat{\otimes}_{proj}W \to V\widehat{\otimes}_{inj}W$$

6. La motivation d'A. Grothendieck et la portée de ses travaux dépassent largement le cadre des produits tensoriels topologiques. Il y a ici un choix de présentation de notre part afin de motiver l'introduction des propriétés d'approximations.

du produit tensoriel projectif sur le produit tensoriel injectif est un isomorphisme.

Autrement dit, il n'y a qu'une seule "bonne" façon de définir une topologie localement convexe sur le produit tensoriel algébrique $V \odot W$. Une partie des travaux de A. Grothendieck consiste à donner une caractérisation intrinsèque des espaces nucléaires. Pour ce faire, il introduit d'abord une notion généralisant au cadre des espaces de Banach le concept d'opérateur à trace sur un espace de Hilbert.

Définition. Soient V_1 et V_2 deux espaces de Banach. Une application linéaire continue

$$T : V_1 \to V_2$$

est dite *nucléaire* s'il existe une suite (g_n) d'éléments de V_2 telle que $\|g_n\| \leqslant 1$ pour tout n, une suite (f_n^*) d'éléments du dual topologique V_1^* de V_1 telle que $\|f_n^*\| \leqslant 1$ pour tout n et une suite ρ_n de scalaires dont la série est absolument convergente telles que pour tout $x \in V_1$,

$$T(x) = \sum_n \rho_n f_n^*(x) g_n$$

L'un des résultats majeurs de la théorie des espaces nucléaires, prouvé par A. Grothendieck dans [Gro55, Chap 2 Par 2.1], est une caractérisation de la nucléarité en termes de factorisation de l'application identité. Si V est un espace vectoriel topologique localement convexe et p une semi-norme sur V, on note V_p l'espace de Banach obtenu par séparation et complétion de V pour la semi-norme p.

Théorème (Grothendieck)**.** *Soit V un espace vectoriel topologique. Alors, V est nucléaire si et seulement si pour toute semi-norme p de V il existe une semi-norme $q \geqslant p$ telle que l'application canonique*

$$V_q \to V_p$$

induite par l'application identité de V soit nucléaire.

Soit V un espace vectoriel normé. Le théorème de Banach-Steinhaus implique que V est nucléaire si et seulement s'il est de dimension finie. Toutefois, il peut être intéressant de restreindre la notion de nucléarité en ne considérant que des produits tensoriels avec des espaces de Banach afin d'obtenir une classe plus large d'espaces.

Définition. Un espace de Banach V est dit *nucléaire* (au sens des espaces de Banach) si pour tout espace de Banach W, l'application canonique

$$V \widehat{\otimes}_{proj} W \to V \widehat{\otimes}_{inj} W$$

du produit tensoriel projectif sur le produit tensoriel injectif est un isomorphisme.

A nouveau, la nucléarité peut être caractérisée intrinsèquement à l'aide d'une propriété de factorisation de l'identité. Toutefois, ce ne sont plus des applications nucléaires qui entrent en jeu (puisque nous avons vu qu'elles sont trop restrictives) mais des applications de rang fini.

Définition. Un espace de Banach V possède la *propriété d'approximation* si pour toute partie compact $K \subset V$ et pour tout $\epsilon > 0$ il existe une application linéaire de rang fini

$$T : V \to V$$

telle que

$$\|T(x) - x\| \leqslant \epsilon$$

pour tout $x \in K$.

Autrement dit, il existe une suite d'applications linéaires de rang fini qui converge vers l'application identité V uniformément sur tout compact de V. On a alors le résultat suivant [Gro55, Chap I Prop 35].

Théorème (Grothendieck)**.** *Soit V un espace de Banach. Alors, V est nucléaire si et seulement s'il a la propriété d'approximation.*

Il est alors naturel de chercher à développer des résultats similaires pour d'autres classes d'espaces de Banach, et notamment pour les C*-algèbres. Dans ce cadre, la théorie des représentations donne plus d'informations et donc de restrictions sur les produits tensoriels possibles. Elle permet notamment de mieux comprendre le produit tensoriel maximal (i.e. projectif) et le produit tensoriel minimal (i.e. injectif).

Définition. Une C*-algèbre A est dite *nucléaire* (au sens des C*-algèbres) si pour toute C*-algèbre B, la surjection canonique

$$A \otimes_{max} B \to A \otimes_{min} B$$

du produit tensoriel maximal sur le produit tensoriel minimal est un isomorphisme.

Contrairement au cas des espaces vectoriels topologiques localement convexes et des espaces de Banach, il a fallu une vingtaine d'années pour obtenir une caractérisation complète de la nucléarité d'une C*-algèbre en termes d'approximation de l'application identité. Cette caractérisation est la combinaison de résultats de C. Lance [Lan73] et de M.D. Choi et E. Effros [CE78] et utilise un nouvel ingrédient : la notion de positivité complète.

Définition (Définition 1.3.1)**.** Soit A une C*-algèbre et $T : A \to A$ une application linéaire. L'application T est dite *complètement positive* si les applications linéaires

$$T^{(n)} : \left\{ \begin{array}{ccc} M_n(A) & \to & M_n(A) \\ (a_{i,j}) & \mapsto & (T(a_{i,j})) \end{array} \right.$$

sont toutes positives.

Définition. Une C*-algèbre A possède la *propriété d'approximation complètement positive* s'il existe une suite (T_t) d'applications linéaires de A dans A unifères et complètement positives de rang fini telle que

$$\|T_t(x) - x\| \to 0$$

pour tout x dans A.

Théorème (Lance, Choi-Effros)**.** *Soit A une C*-algèbre A. Alors, A est nucléaire si et seulement si elle a la propriété d'approximation complètement positive.*

Il est relativement facile de donner des exemples de C*-algèbres ne satisfaisant pas la propriété d'approximation complètement positive grâce au lien qui existe entre cette propriété et la notion de *moyennabilité* d'un groupe localement compact définie par J. von Neumann dans [vN30].

Définition. Un groupe localement compact G est dit *moyennable* s'il existe une mesure positive de masse totale 1 sur $L^\infty(G)$ qui est invariante par translation à gauche.

A tout groupe localement compact G est associée une *C*-algèbre réduite* $C_r^*(G)$ définie de la façon suivante : le produit de convolution donne une représentation

$$\lambda : C_c(G) \to \mathcal{B}(L^2(G))$$

appelée *représentation régulière gauche*. Ici, $C_c(G)$ désigne l'$*$-algèbre des fonctions continues à support compact sur G. La fermeture de $\lambda(C_c(G))$ dans $\mathcal{B}(L^2(G))$ est alors une C*-algèbre dite C*-algèbre réduite de G.

Bien que la définition que nous avons donnée de la moyennabilité soit assez différente de celle donnée par la Définition 2.1.6, elles sont équivalentes (voir par exemple [BO08, Thm 2.6.8] pour une démonstration). En utilisant cette équivalence, on peut démontrer le théorème suivant (qui est inclus dans [BO08, Thm 2.6.8]).

Théorème. *Soit* Γ *un groupe discret, alors* $C_r^*(\Gamma)$ *a la propriété d'approximation complètement positive si et seulement si* Γ *est moyennable.*

J. von Neumann a montré dans [vN30] que le groupe libre à deux générateurs $\mathbb{F}_2$ n'est pas moyennable. Par conséquent, la C*-algèbre $C_r^*(\mathbb{F}_2)$ ne satisfait pas la propriété d'approximation complètement positive. Elle satisfait cependant des propriétés d'approximation plus faibles, notamment la *propriété d'approximation métrique* (dont la définition pour les espaces de Banach remonte à [Gro55]).

Définition. Un espace de Banach V possède la *propriété d'approximation métrique* s'il existe une suite (T_t) d'applications linéaires de V dans V de rang fini telle que

$$\|T_t(x) - x\| \to 0$$

pour tout x dans A et

$$\limsup_t \|T_t\| = 1$$

Notons que cette propriété est plus forte que la propriété d'approximation pour les espaces de Banach. Ainsi, si $C_r^*(\mathbb{F}_2)$ n'est pas nucléaire au sens des C*-algèbres, elle l'est au sens des espaces de Banach. Cette situation contraste fortement avec celle des espaces vectoriels topologiques.

C'est U. Haagerup qui a montré dans [Haa78] que la C*-algèbre $C_r^*(\mathbb{F}_N)$ possède la propriété d'approximation métrique pour tout $2 \leqslant N \leqslant \infty$. Il a développé pour cela des méthodes qui ont ensuite été généralisées et étudiées de façon systématique. Il démontre entre autres que les groupes libres possèdent la *propriété de décroissance rapide*, qui est présentée au début de la Section 3.1. Il démontre également que les groupes libres possèdent une autre propriété d'approximation, appelée *propriété de Haagerup* (voir la Définition 2.1.14). Cette propriété s'est avérée être équivalente à l'*a-T-menabilité* de M. Gromov (voir [CCJ$^+$01] pour un traitement détaillé de ces deux notions et de leurs liens) une propriété géométrique possédant des conséquences topologiques et analytiques importantes (elle implique par exemple la *conjecture de Baum-Connes* [HK01]).

Définition. Un groupe Γ est *a-T-menable* s'il existe un espace de Hilbert H sur lequel Γ agit de façon affine, isométrique et propre.

La propriété d'approximation métrique pour la C*-algèbres $C_r^*(\mathbb{F}_2)$ peut en fait être obtenue comme conséquence d'une propriété plus forte, appelée *propriété d'approximation complètement métrique*, qui correspond à une propriété d'approximation pour les groupes appelée *moyennabilité faible* (voir Définition 2.1.7). Comme la propriété de Haagerup, elle peut être considérée comme un affaiblissement de la notion de moyennabilité présentée plus haut (voir la Section 2.1 pour plus de détails sur ce sujet).

Définition. Un espace de Banach V possède la *propriété d'approximation complètement métrique* s'il existe une suite (T_t) d'applications linéaires de V dans V de rang fini telle que

$$\|T_t(x) - x\| \to 0$$

pour tout x dans A et

$$\limsup_t \|T_t\|_{cb} = 1$$

La moyennabilité faible pour les groupes localement compacts a été définie par M. Cowling et U. Haagerup dans [CH89]. Le cas des groupes de Lie simples a été étudié de manière exhaustive par J. de Cannière, M. Cowling et U. Haagerup dans [CH89] et [dCH85]. Cette étude leur a par exemple permis de montrer que les algèbres de von Neumann associées aux réseaux de certains groupes symplectiques ne sont pas isomorphes (voir Exemple 2.1.13). Dans le cas des groupes discrets, de nombreux exemples sont fournis par un résultat de N. Ozawa [Oza08].

Théorème (Ozawa)**.** *Tout groupe discret hyperbolique au sens de Gromov (voir [Gro87]) est faiblement moyennable.*

Remarque. Récemment, la moyennabilité faible s'est révélée être un outil important pour utiliser les techniques de déformation/rigidité initiées par S. Popa, par exemple dans [OP10a], [OP10b] et plus récemment dans [PV13] (on peut également consulter la synthèse [Ioa12]).

Les liens entre la propriété de Haagerup et la moyennabilité faible sont encore mal compris à ce jour. Mentionnons cependant les deux faits suivants :

- Par un résultat de Y. de Cornulier, Y. Stalder et A. Valette [CSV12], la propriété de Haagerup est stable par *produit en couronne* tandis que N. Ozawa a montré dans [Oza12] que les produits en couronne ne sont généralement pas faiblement moyennables.
- La propriété (T) de Kazhdan est incompatible avec la propriété de Haagerup. Toutefois, certains groupes hyperboliques, qui sont faiblement moyennables, possèdent la propriété (T) de Kazhdan (par exemple les réseaux dans $Sp(n, 1)$).

Aucun exemple de groupe faiblement moyennable *avec une constante de Cowling-Haagerup égale à* 1 ne possédant pas la propriété de Haagerup n'est connu. L'existence d'un tel groupe est l'un des grands problèmes ouverts de la théorie.

Pour conclure, mentionnons également l'existence d'une propriété plus faible que la propriété de Haagerup et que la moyennabilité faible, appelée *propriété d'approximation* et introduite par U. Haagerup et J. Kraus dans [HK94]. Il est particulièrement difficile de trouver des groupes ne satisfaisant pas cette propriété. Les premiers exemples (et les seuls à ce jour) ont été fournis par M. De la Salle et V. Lafforgue dans [LDlS11] puis par U. Haagerup et T. de Laat dans [HDL13].

La notion de groupe quantique

Écrire une histoire des groupes quantiques est un exercice difficile et périlleux qui dépasse largement l'objectif de cette introduction. C'est pourquoi nous nous contenterons de citer quelques aspects du développement de cette théorie afin de motiver nos travaux, sans prétendre à l'exhaustivité.

Soit G un groupe abélien localement compact, on pose

$$\widehat{G} = \{\chi : G \to S^1 \text{ morphisme de groupes continu }\}$$

l'ensemble des *caractères* de G, où S^1 désigne le groupe des nombres complexes de module 1. Si on le munit de la multiplication ponctuelle et de la topologie compacte-ouverte, l'ensemble $\widehat{G}$ devient un groupe abélien localement compact, appelé *dual de Pontryagin* de G. Cette dualité dans les groupes abéliens localement compacts possède une remarquable propriété de bidualité, prouvée par S.L. Pontryagin dans [Pon36].

Théorème (Pontryagin)**.** *Soit G un groupe abélien localement compact. Alors, le morphisme de groupes continu*

$$\begin{cases} G & \to & \widehat{\widehat{G}} \\ g & \mapsto & (\chi \mapsto \chi(g)) \end{cases}$$

est un isomorphisme de groupes topologiques.

La dualité de Pontryagin permet d'étendre les techniques d'analyse harmonique comme la transformée de Fourier à tous les groupes abéliens localement compacts. La transformée de Fourier classique sur $\mathbb{R}$ se retrouve d'ailleurs grâce à l'isomorphisme

$$\begin{array}{ccc} \mathbb{R} & \to & \widehat{\mathbb{R}} \\ x & \mapsto & (\chi_x : t \mapsto e^{itx}) \end{array}$$

Faisons dès maintenant une remarque importante : le dual de Pontryagin d'un groupe abélien compact est un groupe abélien discret et, réciproquement, le dual d'un groupe abélien discret est un groupe abélien compact. En particulier, le dual de $\mathbb{Z}$ est le groupe S^1 des nombres complexes de module 1. De façon similaire au cas de $\mathbb{R}$, cette dualité peut s'exprimer à l'aide des séries de Fourier.

Si G est un groupe localement compact mais n'est pas abélien, la théorie précédente ne s'applique plus. En effet, tout caractère de G, étant à valeurs dans un groupe abélien, s'annule sur les commutateurs. Ainsi, $\widehat{G}$ se réduit au dual de l'abélianisé G_{ab} de G. L'histoire des groupes quantiques, en tout cas du point de vue qui nous intéresse, peut être considérée comme une tentative d'étendre la dualité de Pontryagin au-delà du cas abélien.

L'une des premières avancées significatives est due à T. Tannaka. En remarquant que les caractères d'un groupe abélien sont exactement ses représentations irréductibles, ou plus précisément que les représentations irréductibles d'un groupe sont toutes de dimension 1 si et seulement s'il est abélien, on peut envisager de remplacer le dual de Pontryagin par l'ensemble des représentations irréductibles muni d'une structure provenant de celle du groupe. Dans [Tan39], T. Tannaka définit une telle structure duale dans le cas d'un groupe compact en utilisant le produit tensoriel des représentations et la théorie de Peter-Weyl développée dans [PW27]. Comme le produit tensoriel de deux représentations irréductibles n'est pas nécessairement irréductible, il faut considérer toutes les représentations de dimension finie. La théorie des catégories est alors un langage particulièrement adapté pour déduire ce "dual". Plus précisément, on associe à tout groupe compact G une catégorie $\mathrm{Rep}(G)$ dont les objets sont les représentations unitaires de dimension finie de G et dont les morphismes sont les opérateurs d'entrelacement des représentations. On a alors un "dual" défini par

$$\widehat{G} = (\mathrm{Rep}(G), \oplus, \otimes, .^c)$$

où $.^c$ est la conjugaison des représentations. Cette construction fut ensuite raffinée par M.G. Krein dans [Kre63a] et [Kre63b] (qui sont en fait les traductions de [Kre49] et [Kre50]) en caractérisant précisément les catégories pouvant apparaître comme dual d'un groupe compact. L'ensemble de ces résultats est aujourd'hui connu sous le nom de *dualité de Tannaka-Krein*.

Cette dualité a ensuite été généralisée à tous les groupes localement compacts *unimodulaires* par W.F. Stinespring dans [Sti59]. Il utilise notamment l'*unitaire fondamental* V, qui est aujourd'hui un élément essentiel de la théorie et que nous introduirons dans la sous-section 1.1.2 sous le nom de *représentation régulière gauche*. Notons également que W.F. Stinespring utilise pour la première fois des algèbres d'opérateurs, et plus précisément des algèbres de von Neumann, pour décrire le dual d'un groupe localement compact. Parmi les extensions de ces travaux au cas non-unimodulaire, mentionnons l'approche de P. Eymard dans [Eym64] basée sur l'*algèbre de Fourier* et celle proposée par N. Tatsuuma dans [Tat66], inspirée de la dualité de Tannaka-Krein.

Dans tous ces travaux, le groupe et son dual sont de nature différente. Il faut attendre G.I. Kac pour obtenir la première théorie dans laquelle le groupe et son dual sont de même nature. Il s'agit de la théorie des *ring groups* développée dans [Kac61], [Kac65a] et [Kac65b]. Cette structure contient les groupes localement compacts unimodulaires et possède une dualité qui se restreint à la dualité de Pontryagin dans le cas abélien et qui satisfait une généralisation du théorème de bidualité. L'unitaire fondamental de W.F. Stinespring est généralisé à ce cadre et il est prouvé qu'il vérifie l'*équation pentagonale*

$$V_{12}V_{13}V_{23} = V_{23}V_{12}.$$

Cette équation, qui rappelle l'*équation de Yang-Baxter* $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$, a servi de base à de nombreux travaux ultérieurs. Parmi les exemples de *ring groups* construits par V.I. Kac et V.G. Paljutkin, on trouve notamment le plus petit groupe quantique fini (il est d'ordre 8) qui n'est ni commutatif ni cocommutatif, souvent appelée *groupe quantique de Kac-Paljutkin* et construit dans [KP65].

Il y eut ensuite de nombreuses tentatives pour étendre cette dualité aux groupes localement compacts non-unimodulaires, mais la plupart d'entre elles rompent la symétrie des *ring groups* et utilisent un objet dual qui n'est pas de même nature que l'objet de départ. Citons simplement l'important travail de M. Takesaki [Tak72b] qui unifie ces différentes approches.

Une théorie complète fut donnée simultanément par V.I. Kac et L.I. Vainerman dans [KV73] et [KV74] et par M. Enock et J-M. Schwartz dans [ES73] et [ES75]. Elle repose sur la théorie des algèbres de von Neumann, via des objets appelés *algèbres de Hopf-von Neumann*. Dans la suite de leurs travaux, M. Enock et J-M. Schwartz choisirent le nom *algèbres de Kac* pour désigner la structure complète, en hommage au travaux pionniers de ce dernier. Le livre [ES92] donne un exposé complet de la théorie des algèbres de Kac. Bien que la théorie fournisse une dualité parfaite incluant tous les groupes localement compacts, elle présente plusieurs inconvénients. Premièrement, la définition est relativement complexe comparée à celle d'un groupe localement compact.

Définition. Une *algèbre de Kac* est un quadruplet (M, Δ, S, φ), où M est une algèbre de von Neumann, $\Delta : M \to M\overline{\otimes}M$ est un $*$-homomorphisme normal injectif, $S : M \to M$ est un $*$-antihomomorphisme normal involutif et φ est un poids normal semi-fini fidèle sur M tels que :

1. $(\Delta \otimes \imath) \circ \Delta = (\imath \otimes \Delta) \circ \Delta$.
2. $\sigma \circ \Delta \circ S = (S \otimes S) \circ \Delta$.
3. $(\imath \otimes \varphi) \circ \Delta(x) = \varphi(x).1$ pour tout $x \in M_+$.
4. $(\imath \otimes \varphi)[(1 \otimes y^*)\Delta(x)] = S((\imath \otimes \varphi)[\Delta(y^*)(1 \otimes x)])$ pour tous $x, y \in \mathcal{N}_\varphi$.
5. $S \circ \sigma_t^\varphi = \sigma_{-t}^\varphi \circ S$ pour tout $t \in \mathbb{R}$.

Ici, $\mathcal{N}_\varphi = \{x \in M, \varphi(x^*x) < \infty\}$ et $(\sigma_t^\varphi)_{t\in\mathbb{R}}$ désigne le groupe à un paramètre d'automorphismes de φ.

Deuxièmement, on a longtemps manqué d'exemples concrets à étudier en détails. Comme nous allons le voir, ces exemples ont en fait montré l'insuffisance de l'axiomatique des algèbres de Kac. Il existe également une version de la théorie utilisant les C*-algèbres au lieu des algèbres de von Neumann, introduite par J-M. Vallin dans [Val85] (voir également [EV93]).

Le terme *groupe quantique* a été utilisé pour la première fois par V.G. Drinfel'd dans le célèbre article [Dri87] pour désigner certaines algèbres de Hopf obtenues par déformation d'algèbres de Lie. D'autres exemples de déformations ont ensuite été construits par M. Jimbo dans [Jim85]. Parallèlement, S.L. Woronowicz développait une autre approche des groupes quantiques dans [Wor87a], sous le nom de *pseudo-groupe compact matriciel.*

Définition. Un *pseudo-groupe compact matriciel* est une paire (A, u), où A est une C*-algèbre et $u = [u_{i,j}]_{1\leqslant i,j\leqslant n}$ est une matrice à coefficients dans A tels que :

1. La sous-$*$-algèbre $\mathcal{A}$ de A engendrée par les $u_{i,j}$ est dense dans A.
2. Il existe un $*$-homomorphisme $\Delta : A \to A \otimes A$ tel que $\Delta(u_{i,j}) = \displaystyle\sum_{k=1}^{n} u_{i,k} \otimes u_{k,j}$.
3. Il existe une application linéaire antimultiplicative $S : \mathcal{A} \to \mathcal{A}$, non nécessairement bornée, telle que $S(S(a)^*)^* = a$ pour tout a dans $\mathcal{A}$ et
$$\sum_k S(u_{i,k})u_{k,j} = \delta_{i,j}.1 = \sum_k u_{i,k}S(u_{k,j})$$

Cette approche, basée sur la théorie des C*-algèbres, s'inspire des idées de la *géométrie non-commutative* popularisée au même moment par A. Connes (notamment dans le livre [Con90]). L'exemple fondamental de S.L. Woronowicz est le groupe quantique $SU_q(2)$ construit dans [Wor87b], qui s'avère être un analogue de l'exemple $U_q(\mathfrak{sl}(2))$ de M. Jimbo. Les liens précis entre les deux notions de groupes quantiques ont été explorés par M. Rosso dans [Ros90].

Un résultat important de ces travaux est que le groupe quantique $SU_q(2)$ ne peut être décrit par une algèbre de Kac. En effet, l'application S n'est pas un opérateur borné dans ce cas. Une théorie plus large, englobant les algèbres de Kac et les groupes quantiques $SU_q(2)$, était donc nécessaire. Cette théorie a été développée par S.L. Woronowicz dans [Wor88] et [Wor89] puis généralisée dans [Wor98] sous le nom de *groupes quantiques compacts* (voir Définition 1.1.1). C'est dans le cadre de cette théorie que se place cette thèse. Notons qu'en particulier, S.L. Woronowicz généralise dans [Wor88] la dualité de Tannaka-Krein évoquée plus haut, unifiant le point de vue des algèbres d'opérateurs et le point de vue des catégories.

Toutefois, la théorie de S.L. Woronowicz est limitée au cas compact, et ne généralise donc que partiellement la théorie des algèbres de Kac. De plus, S.L. Wornowicz a également construit des exemples non-compacts, tels que les déformations $E_\mu(2)$ du groupe des déplacements du plan construites dans [Wor91], qui ne sont pas non plus des algèbres de Kac. Il manquait alors un cadre précis pour ces groupes quantiques. La plus vaste généralisation proposée est celle des *unitaires multiplicatifs* de S. Baaj et G. Skandalis introduite dans [BS93] et [Baa95].

Définition. Soit H un espace de Hilbert, un unitaire multiplicatif sur H est un opérateur unitaire $V \in \mathcal{B}(H \otimes H)$ tel que

$$V_{12}V_{13}V_{23} = V_{23}V_{12}$$

Il s'agit en particulier du cadre le plus général pour l'étude de la dualité des actions et des produits croisés. La simplicité de la définition des unitaires multiplicatifs permet de les adapter à différents contextes, par exemple à celui des champs continus de C*-algèbres. C'est ainsi qu' É. Blanchard a pu décrire dans [Bla96] de nombreuses constructions "par déformation" comme des déformations continues de C*-algèbres de Hopf.

Cela dit, cette approche est en un sens trop générale et il est difficile de donner des critères satisfaisants sur un unitaire multiplicatif pour assurer que les objets associés auront suffisamment de structure. S.L. Woronowicz a proposé des critères tels que la *maniabilité* [Wor96] ou la *modularité* [SW01] mais la définition restait extrêmement lourde. La *régularité* et même la *semi-régularité* introduite dans [Baa95] se sont, elles, avérées insuffisantes [BSV03].

Une autre tentative, due à A. van Daele, consiste à utiliser un cadre purement algébrique à partir d'une généralisation non unifère des algèbres de Hopf. C'est la notion d'*algèbre de Hopf à multiplicateurs* introduite dans [VD94].

Définition. Une *algèbre de Hopf à multiplicateurs* est une paire (A, Δ), où A est une algèbre associative non-dégénérée et

$$\Delta : A \to M(A \otimes A)$$

est un morphisme d'algèbres tel que

- $\Delta(a)(1 \otimes b)$ et $(a \otimes 1)\Delta(b)$ sont dans $A \otimes A$ pour tous $a, b \in A$.
- $(a \otimes 1 \otimes 1)(\Delta \otimes \imath)(\Delta(b)(1 \otimes c)) = (\imath \otimes \Delta)((a \otimes 1)\Delta(b))(1 \otimes 1 \otimes c)$

et que les applications

$$\begin{cases} T_1 : a \otimes b & \mapsto & \Delta(a)(1 \otimes b) \\ T_2 : a \otimes b & \mapsto & (a \otimes 1)\Delta(b) \end{cases}$$

soient bijectives. Si A est munie d'une involution compatible avec Δ, (A, Δ) est appelée *$*$-algèbre de Hopf à multiplicateurs.*

A. van Daele et ses collaborateurs ont mené une étude systématique de ces objets et construit de nombreux exemples. Ils permettent en particulier de donner une définition intrinsèque des groupes quantiques discrets [VD96] et d'en retrouver toute la structure [VDZ99].

Définition. Un *groupe quantique discret* est une $*$-algèbre de Hopf à multiplicateurs (A, Δ) tel que l'$*$-algèbre A soit une somme d'algèbres de matrices.

Toutefois, certains exemples de déformations de groupes localement compacts ne peuvent être traités par cette théorie (ce problème est discuté en détails dans [KVD97] et [DCVD10]). Des définitions plus élaborées permettant une structure modulaire plus complexe ont été proposées par exemple par T. Masuda et Y. Nakagami dans [MN94] en utilisant des algèbres de von Neumann, travaux qui furent ensuite étendus avec S.L. Woronowicz au cas des C*-algèbres dans [MNW03].

Finalement, une définition des *groupes quantiques localement compacts* a été donnée par J. Kustermans et S. Vaes dans [KV99] et [KV00]. Elle est à ce jour communément acceptée comme la définition la plus appropriée, bien qu'elle suppose l'existence de poids de Haar à gauche et à droite, ce qui n'est pas nécessaire dans le cas des groupes localement compacts. Nous donnons ici la définition dans le cadre des algèbres de von Neumann issue de [KV03], mais il existe aussi une version utilisant les C*-algèbres.

Définition. Un groupe quantique localement compact est une paire (M, Δ), où M est une algèbre de von Neumann et $\Delta : M \to M \overline{\otimes} M$ est un $*$-homomorphisme normal tel qu'il existe deux poids normaux semi-finis φ et ψ sur M satisfaisant :

1. $(\Delta \otimes \imath) \circ \Delta = (\imath \otimes \Delta) \circ \Delta$.
2. $\varphi[(\omega \otimes \imath) \circ \Delta(x)] = \varphi(x)\omega(1)$ pour tout $x \in \mathcal{M}_{\varphi}^{+}$ et tout $\omega \in M_{*}^{+}$.
3. $\psi[(\imath \otimes \omega) \circ \Delta(x)] = \varphi(x)\omega(1)$ pour tout $x \in \mathcal{M}_{\varphi}^{+}$ et tout $\omega \in M_{*}^{+}$.

Ici, $\mathcal{M}_{\varphi}^{+}$ désigne l'ensemble des éléments positifs $x \in M_{+}$ tels que $\varphi(x) < +\infty$.

Contenu de cette thèse

Nous allons maintenant donner une description détaillée par chapitre du plan de cette thèse, ainsi que les énoncés des principaux résultats présentés.

Préliminaires

Ce premier chapitre débute par un exposé détaillé de la théorie des *groupes quantiques compacts* de S.L. Woronowicz telle qu'elle est présentée dans [Wor98]. Nous avons tenté de rendre cet exposé le plus accessible possible, notamment en donnant souvent en exemple le cas des groupes compacts et celui des groupes discrets. Nous avons également mis l'accent sur une description explicite de l'action de la C*-algèbre $C_{\text{red}}(\mathbb{G})$ dans la construction GNS de l'état de Haar qui sera utilisée au Chapitre 3. Nous donnons un exposé similaire de la théorie des *groupes quantiques discrets*, la structure duale de la précédente. Les constructions utilisées ici étaient déjà implicites dans [Wor98] et peuvent être trouvées dans de nombreux articles sur ce sujet, par exemple dans [VV07].

Nous donnons ensuite, sous le titre *Compléments sur les groupes quantiques discrets*, une série de constructions classiques. La construction du produit libre est particulièrement importante puisque certaines de ses propriétés seront étudiées au Chapitre 2. Nous présentons également les *groupes quantiques universels* dont l'étude détaillée fera l'objet du Chapitre 3.

Enfin, nous rappelons quelques éléments de la théorie des applications complètement positives et complètement bornées qui seront fondamentales dans la définition des propriétés étudiées dans cette thèse.

Moyennabilité faible pour les groupes quantiques discrets

Dans le second chapitre de cette thèse, nous développons une théorie générale de la moyennabilité faible dans le cadre des groupes quantiques discrets. Pour ce faire, nous commençons par donner une brève description de la théorie de la moyennabilité faible dans le cadre des groupes discrets afin de pouvoir dresser des parallèles avec nos définitions et nos résultats.

Le point de départ est la notion de *multiplicateur de Herz-Schur* qui a déjà été développée dans le cadre des groupes quantiques localement compacts par M. Junge, M. Neufang et Z.J. Ruan in [JNR09]. Elle permet de donner une définition naturelle de la moyennabilité faible.

Définition (Def 2.2.8)**.** Un groupe quantique discret $\widehat{\mathbb{G}}$ est *faiblement moyennable* s'il existe une suite (a_t) d'éléments de $\ell^{\infty}(\widehat{\mathbb{G}})$ telle que

- a_t est à support fini pour tout t.
- (a_t) converge ponctuellement vers 1.
- $K := \limsup_t \|m_{a_t}\|_{cb}$ est fini.

La borne inférieure des constantes K pour toutes les suites satisfaisant ces propriétés est notée $\Lambda_{cb}(\widehat{\mathbb{G}})$ et appelée la *constante de Cowling-Haagerup* de $\widehat{\mathbb{G}}$. Par convention, $\Lambda_{cb}(\widehat{\mathbb{G}}) = \infty$ si $\widehat{\mathbb{G}}$ n'est pas faiblement moyennable.

Le lien entre cette notion et la propriété d'approximation complètement bornée pour les algèbres d'opérateurs associées n'est pas connu en général. Cette question semble liée à des problèmes de théorie modulaire des algèbres de von Neumann que nous détaillons. Muni de la définition, nous sommes en mesure de prouver certaines propriétés de permanence.

Théorème (Cor 2.2.16, 2.2.17 et 2.2.19)**.** *Soit $\widehat{\mathbb{G}}$ un groupe quantique discret, alors*

1. *Si $\widehat{\mathbb{G}}$ est faiblement moyennable et si $\widehat{\mathbb{H}}$ est un sous-groupe quantique discret de $\widehat{\mathbb{G}}$, alors $\widehat{\mathbb{H}}$ est faiblement moyennable et $\Lambda_{cb}(\widehat{\mathbb{H}}) \leqslant \Lambda_{cb}(\widehat{\mathbb{G}})$.*
2. *Si $\widehat{\mathbb{G}}$ est faiblement moyennable et si $\widehat{\mathbb{H}}$ est un autre groupe quantique discret faiblement moyennable, alors $\widehat{\mathbb{G}} \times \widehat{\mathbb{H}}$ est faiblement moyennable et*
$$\Lambda_{cb}(\widehat{\mathbb{G}} \times \widehat{\mathbb{H}}) \leqslant \Lambda_{cb}(\widehat{\mathbb{G}})\Lambda_{cb}(\widehat{\mathbb{H}}).$$
3. *Si $(\widehat{\mathbb{G}}_i, \pi_{i,j})$ est un système inductif de groupes quantiques discrets dont toutes les applications connectantes sont injectives et dont la limite inductive est $\widehat{\mathbb{G}}$, alors $\sup_i \Lambda_{cb}(\widehat{\mathbb{G}}_i) = \Lambda_{cb}(\widehat{\mathbb{G}})$.*

Nous abordons ensuite une réciproque partielle du premier point du théorème précédent. Pour cela, nous définissons une notion de moyennabilité relative pour les groupes quantiques discrets généralisant celle de P. Eymard dans [Eym72].

Définition (Def 2.2.23)**.** Soit $\widehat{\mathbb{G}}$ un groupe quantique discret. Un sous-groupe quantique $\widehat{\mathbb{H}}$ est *moyennable relativement à* $\widehat{\mathbb{G}}$ s'il existe un état m sur $\ell^{\infty}(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$ tel que pour tout $x \in \ell^{\infty}(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$,
$$(\imath \otimes m) \circ \tau(x) = m(x).1$$
où τ désigne l'action par translation de $\widehat{\mathbb{G}}$ sur le quotient.

Nous généralisons alors un théorème de C. Anatharaman-Delaroche, en utilisant notamment une notion d'*action produit* définie au chapitre 1.

Théorème (Cor 2.2.37)**.** *Soit $\widehat{\mathbb{G}}$ un groupe quantique discret et $\widehat{\mathbb{H}}$ un sous-groupe quantique moyennable relativement à $\widehat{\mathbb{G}}$, alors $\Lambda_{cb}(L^{\infty}(\mathbb{H})) = \Lambda_{cb}(L^{\infty}(\mathbb{G}))$.*

Les idées et les techniques employées jusqu'ici peuvent également être appliquées à l'étude de la propriété de Haagerup, ce que nous faisons dans la Section 2.2.4. La définition est évidente.

Définition (Def 2.2.40)**.** Un groupe quantique discret $\widehat{\mathbb{G}}$ possède la *propriété de Haagerup* s'il existe une suite (a_t) d'éléments de $\ell^{\infty}(\widehat{\mathbb{G}})$ telle que

- $a_t \in C_0(\widehat{\mathbb{G}})$ pour tout t.
- (a_t) converge ponctuellement vers 1.
- m_{a_t} est unifère et complètement positif pour tout t.

La preuve des propriétés de stabilité suivantes est quant à elle relativement simple.

Théorème (Prop 2.2.48)**.** *Soit $\widehat{\mathbb{G}}$ un groupe quantique discret, alors*

1. *Si $\widehat{\mathbb{G}}$ a la propriété de Haagerup et si $\widehat{\mathbb{H}}$ est un sous-groupe quantique discret de $\widehat{\mathbb{G}}$, alors $\widehat{\mathbb{H}}$ a la propriété de Haagerup.*
2. *Si $\widehat{\mathbb{G}}$ a la propriété de Haagaerup et si $\widehat{\mathbb{H}}$ est un autre groupe quantique discret ayant la propriété de Haagerup, alors $\widehat{\mathbb{G}} \times \widehat{\mathbb{H}}$ a la propriété de Haagerup.*
3. *Si $\widehat{\mathbb{G}}$ a la propriété de Haagaerup et si $\widehat{\mathbb{H}}$ est un autre groupe quantique discret ayant la propriété de Haagerup, alors $\widehat{\mathbb{G}} * \widehat{\mathbb{H}}$ a la propriété de Haagerup.*

4. *Si* $(\widehat{\mathbb{G}}_i, \pi_{i,j})$ *est un système inductif de groupes quantiques discrets dont toutes les applications connectantes sont injectives et dont la limite inductive est* $\widehat{\mathbb{G}}$, *alors* $\widehat{\mathbb{G}}$ *a la propriété de Haagerup si et seulement si tous les* $\widehat{\mathbb{G}}_i$ *ont la propriété de Haagerup.*
5. *Soit* $\widehat{\mathbb{G}}$ *un groupe quantique discret unimodulaire et* $\widehat{\mathbb{H}}$ *un sous-groupe quantique moyennable relativement à* $\widehat{\mathbb{G}}$, *alors* $\widehat{\mathbb{H}}$ *a la propriété de Haagerup si et seulement si* $\widehat{\mathbb{G}}$ *a la propriété de Haagerup.*

Enfin, nous abordons le problème de la stabilité de la moyennabilité faible par passage au produit libre. Le problème général est encore ouvert à ce jour, mais des résultats sont connus quand la constante de Cowling-Haagerup des groupes est égale à 1. Nous donnons tout d'abord une généralisation partielle, publiée aux *Comptes rendus de l'Académie de sciences* [Fre12],

Théorème (Thm 2.3.6). *Soit* $(\widehat{\mathbb{G}}_i)_{i\in I}$ *une famille de groupes quantiques discrets dont la constante de Cowling-Haagerup est égale à* 1. *Alors,*

$$\Lambda_{cb}(*_{i\in I}\widehat{\mathbb{G}}_i) = 1.$$

Nous donnons ensuite quelques résultats sur les moyennes de multiplicateurs relativement à un sous-groupe quantique fini. Ces résultats nous permettent de donner une extension du théorème précédent que nous n'étions pas parvenu à prouver lors de la rédaction de [Fre12].

Théorème (Thm 2.3.22). *Soit* $(\widehat{\mathbb{G}}_i)_{i\in I}$ *une famille de groupes quantiques discrets dont la constante de Cowling-Haagerup est égale à* 1 *et soit* $\widehat{\mathbb{H}}$ *un sous-groupe quantique fini commun. Alors,*

$$\Lambda_{cb}((*_{\widehat{\mathbb{H}}}\widehat{\mathbb{G}}_i)) = 1.$$

Les démonstrations de ces deux derniers théorèmes sont plus simples si les groupes quantiques discrets sont supposés moyennables plutôt que faiblement moyennables. Afin de clarifier la présentation et de mettre en lumière les spécificités techniques du cas général, nous donnons séparément les énoncés dans le cas moyennable (ce sont le Corollaire 2.3.3 et le Théorème 2.3.17). Les techniques employées s'appliquent également à la propriété de Haagerup.

Théorème (Thm 2.3.19). *Soit* $(\widehat{\mathbb{G}}_i)_{i\in I}$ *une famille de groupes quantiques discrets ayant la propriété de Haagerup et soit* $\widehat{\mathbb{H}}$ *un sous-groupe quantique fini commun. Alors,* $*_{\widehat{\mathbb{H}}}\widehat{\mathbb{G}}_i$ *a la propriété de Haagerup.*

Exemples de groupes quantiques discrets faiblement moyennables

Le Chapitre 3 constitue le cœur de cette thèse. Nous y donnons des exemples non-triviaux (qui ne sont pas construits à partir de groupes classiques et/ou de groupes quantiques moyennables) de groupes quantiques discrets faiblement moyennables. La preuve repose sur une étude fine et relativement technique de la structure des *groupes quantiques libres orthogonaux* O_N^+. Plus précisément, nous étudions la norme complètement bornée de certains multiplicateurs sur l'algèbre du groupe quantique.

Définition. Soit d un entier, l'application

$$m_{p_d} : \left\{ \begin{array}{ccc} \mathrm{Pol}(O_N^+) & \to & \mathrm{Pol}(O_N^+) \\ u_{i,j}^k & \mapsto & \delta_{k,d}.u_{i,j}^d \end{array} \right.$$

s'étend en une application complètement bornée sur $C_{\mathrm{red}}(O_N^+)$, encore notée m_{p_d}.

La première partie de cette étude est l'établissement d'une *inégalité de Haagerup à valeur opérateurs* dans l'esprit de [Pis03, Thm 9.7.4]. Si $X \in \mathcal{B}(H) \odot \mathrm{Pol}(O_N^+)$ et

$$X^d = (\imath \otimes m_{p_d})(X),$$

une décomposition en blocs matriciels orthogonaux

$$B_{d-j,j}(X^d) = (\imath \otimes p_{d-j})X^d(\imath \otimes p_j)$$

de l'opérateur X^d permet de montrer le résultat suivant.

Théorème (Thm 3.1.10)**.** *Soit N un entier. Il existe une constante K, ne dépendant que de N, telle que pour tout espace de Hilbert H et tout $X \in \mathcal{B}(H) \otimes \mathrm{Pol}(O_N^+)$,*

$$\max_{0\leqslant j\leqslant d}\{\|B_{d-j,j}(X^d)\|\} \leqslant \|X^d\| \leqslant K(d+1)\max_{0\leqslant j\leqslant d}\{\|B_{d-j,j}(X^d)\|\}$$

Cette inégalité est également valable pour tous les groupes quantiques $O^+(F)$, qui sont des versions "déformées" de O_N^+. Un résultat similaire est prouvé pour les groupes quantiques d'automorphismes de C*-algèbres de dimension finie (c'est le Théorème 3.1.16).

La deuxième partie est la preuve d'une majoration polynomiale pour la norme complètement bornée des opérateurs m_{p_d}.

Théorème. *Il existe un polynôme P de degré 2 tel que pour tout entier N,*

$$\|m_{p_d}\|_{cb} \leqslant P(d)$$

dans O_N^+.

Ce résultat technique est utilisé dans la dernière section pour donner de nombreux exemples de groupes quantiques discrets faiblement moyennables. La première méthode pour obtenir ces exemples est de combiner notre estimation de la croissance des normes complètement bornées des opérateurs m_{p_d} avec les applications construites par M. Brannan dans [Bra12a] pour obtenir des applications "tronquées" qui satisfont toutes les hypothèses de la définition de la moyennabilité faible. On traite ainsi tous les groupes quantiques libres unimodulaires.

Théorème (Thm 3.3.2)**.** *Soit N un entier et $F \in GL_N(\mathbb{C})$ une matrice unitaire à un scalaire près telle que $F\overline{F} \in \mathbb{R}.\mathrm{Id}$. Alors, les groupes quantiques discrets $O^+(F)$ et $U^+(F)$ sont faiblement moyennables et ont une constante de Cowling-Haagerup égale à 1.*

Il est naturel de chercher à exploiter la moyennabilité faible pour obtenir des résultats de structure sur l'algèbre de von Neumann associée au groupe quantique. Le premier résultat dans cette direction est dû à Y. Isono, qui a montré dans [Iso12] que les algèbres de von Neumann des groupes quantiques libres O_N^+ et U_N^+ sont *fortement solides*.

Il semble difficile d'adapter la preuve de la majoration polynomiale à d'autres groupes quantiques discrets, pour des raisons techniques qui sont expliquées dans la Remarque 3.2.7. Pour étendre nos résultat, nous avons donc utilisé une autre stratégie, basée sur la notion d'*équivalence monoïdale*. Les liens entre cette notion et les propriétés d'approximation n'ont été que très peu étudiés (à l'exception notable de l'exactitude dans [VV07]). Dans le cas le plus simple (celui des propriétés d'approximations *centrales* introduites dans la Définition 3.3.25), l'équivalence monoïdale transporte certaines propriétés d'équivalence en vertu de la proposition suivante due à S. Vaes :

Proposition. *Soit φ une équivalence monoïdale entre deux groupes quantiques compacts $\mathbb{G}_1$ et $\mathbb{G}_2$. Alors, pour tout $\alpha \in \mathrm{Irr}(\mathbb{G}_1)$, on a*

$$\|m_{p_\alpha}\|_{cb} = \|m_{p_{\varphi(\alpha)}}\|_{cb}.$$

Grâce à ce résultat, nous pouvons d'une part étendre la classe d'exemples de groupes quantiques discrets faiblement moyennables et d'autre part étendre également la classe d'exemples de groupes quantiques discrets possédant la propriété de Haagerup.

Théorème (Cor 3.3.20)**.** *Soit $\mathbb{G}$ un groupe quantique compact monoïdalement équivalent à O_N^+ ou U_N^+ pour un certain $N \geqslant 3$ au groupe quantique d'automorphismes de (B, ψ) pour une C^*-algèbre de dimension finie B de dimension au moins 6 munie de sa δ-trace ψ, alors $\widehat{\mathbb{G}}$ a la propriété de Haagerup et est faiblement moyennable avec une constante de Cowling-Haagerup égale à 1.*

Nous avons choisi de présenter le résultat précédent comme simple corollaire et de réserver le titre de théorème à une formulation alternative. Cette formulation donne une caractérisation intrinsèque des groupes quantiques discrets auxquels les techniques précédentes s'appliquent.

Théorème (Thm 3.3.23)**.** *Considèrons les objets suivants :*

- *Une matrice $F_1 \in GL_N(\mathbb{C})$ telle que $F_1\overline{F}_1 \in \mathbb{R}.\mathrm{Id}$ et $\mathrm{Tr}(F_1^*F_1) \in \mathbb{N}$.*
- *Une matrice $F_2 \in GL_N(\mathbb{C})$ telle que $\mathrm{Tr}(F_2^*F_2) \in \mathbb{N}$.*
- *Une C^*-algèbre de dimension finie B munie d'un état ψ tel que $\mathrm{Sp}(mm^*) \in \mathbb{N} \setminus \{5\}$.*

Alors, les groupes quantiques discrets $\mathbb{F}O^+(F_1)$, $\mathbb{F}U^+(F_2)$ et $\widehat{\mathbb{G}}(B, \psi)$ ont la propriété de Haagerup et sont faiblement moyennables avec une constante de Cowling-Haagerup égale à 1.

Le théorème précédent est remarquable en ce qu'il donne des exemples qui ne sont pas unimodulaires. De tels exemples semblent inaccessibles avec les méthodes directes employées précédemment, notamment par M. Brannan dans [Bra12a] et [Bra13]. Dans le cas non-unimodulaire, on ne peut espérer prouver une généralisation de la solidité forte. Toutefois, Y. Isono donne dans [Iso12] des résultats d'absence de sous-algèbre de Cartan pour l'algèbre de von Neumann.

Notre étude du lien entre équivalence monoïdale et propriétés d'approximations reste très sommaire et soulève plusieurs questions. Nous tentons dans le dernier paragraphe de formaliser ces futurs axes de recherche.

Conventions and notations

Unless explicitely stated otherwise, inner products on vector spaces will be chosen *right-linear*. Let us detail some notations which will be used in this dissertation.

- For two Hilbert spaces H and K, $\mathcal{B}(H, K)$ will denote the set of bounded linear maps from H to K and $\mathcal{B}(H) := \mathcal{B}(H, H)$. In the same way we will use the notations $\mathcal{K}(H, K)$ and $\mathcal{K}(H)$ for compact linear maps. We will denote by $\mathcal{B}(H)_*$ the predual of $\mathcal{B}(H)$, i.e. the Banach space of all normal linear forms on $\mathcal{B}(H)$. For any two vectors $\xi, \eta \in H$, we define a linear form $\omega_{\eta,\xi}$ on $\mathcal{B}(H)$ in the following way :

$$\omega_{\eta,\xi}(T) = \langle \eta, T.\xi \rangle.$$

- If B is a subset of a topological vector space C, $[B]$ will denote the *closed linear span* of B in C.
- The symbol $\otimes$ will denote the *minimal* (or spatial) tensor product of C*-algebras or the topological tensor product of Hilbert spaces. The spatial tensor product of von Neumann algebras will be denoted $\overline{\otimes}$ and the algebraic tensor product (over $\mathbb{C}$) will be denoted $\odot$.
- On any tensor product $H \otimes H'$ of Hilbert spaces, we define the *flip*

$$\Sigma : \left\{ \begin{array}{ccc} H \otimes H' & \rightarrow & H' \otimes H \\ x \otimes y & \mapsto & y \otimes x \end{array} \right.$$

  and the *operator flip*

$$\sigma : \left\{ \begin{array}{ccc} \mathcal{B}(H \otimes H') & \rightarrow & \mathcal{B}(H' \otimes H) \\ T & \mapsto & \Sigma \circ T \circ \Sigma \end{array} \right.$$

  We will use the usual leg-numbering notations : for an operator X acting on a tensor product, we set $X_{12} := X \otimes 1$, $X_{23} := 1 \otimes X$ and $X_{13} := (\Sigma \otimes 1)(1 \otimes X)(\Sigma \otimes 1)$.
- The identity map of an algebra A will be denoted $\imath_A$ or simply $\imath$ if there is no possible confusion. The *multiplier algebra* of a C*-algebra A will be denoted $M(A)$.
- If A is a $*$-algebra together with a state ψ, we will denote by $L^2(A, \psi)$ the GNS construction using the right-linear inner product $\langle a, b \rangle = \psi(a^*b)$ and by $L^2(A, \psi)^{op}$ the GNS construction using the left-linear inner product $\langle a, b \rangle = \varphi(ab^*)$. Note that in the latter case, we do not get a representation of the algebra A but a representation of the opposite algebra A^{op} (acting on the right).
- If A is a C*-algebra and if $u = [u_{i,j}]$ is a matrix with coefficients in A, then $\overline{u}$ will denote the matrix $[u_{i,j}^*]$.

Chapter 1

Preliminaries

This first chapter is devoted to introducing the main objects and results which will be used in this dissertation. We intend to make this work as self-contained as possible and that is the reason why we will sometimes go into some details and sometimes only overview the different topics in the sequel, depending on their relevance to us.

The chapter is organized as follows :

- In Section 1.1 we introduce compact quantum groups using the formalism of S.L. Woronowicz. We then make explicit the structure of discrete quantum groups which was implicit already in [Wor98]. Nothing is new in this section but we pay particular attention to the description of the GNS representation of the Haar state using the irreducible representations which will be essential in Chapter 3. All the computations come from [Wor98] but we had not found it explicitely presented in this way in the litterature.
- In Section 1.2 we gather several results and constructions using discrete quantum groups which will be needed at various stages of this dissertation. We also give a description of the free orthogonal and free unitary quantum groups and of the quantum automorphism groups of finite-dimensional C*-algebras. These are central objects in the theory of discrete quantum groups and will be our main source of examples.
- In Section 1.3 we give a few technical elements concerning completely positive and completely bounded maps. These are the fundamental notions involved in the theory of approximation properties for operator algebras.

1.1 Quantum groups

1.1.1 Compact quantum groups

Compact quantum groups were first introduced by S.L. Woronowicz through the slightly less general notion of *compact matrix pseudogroup* (see Definition 1.2.37) in [Wor87a]. He then gave a comprehensive account of the general theory in [Wor98]. Another survey, encompassing the non-separable case, can be found in [MVD98]. Let us also mention the book [Tim08], which gives a very detailed treatment of several topics of quantum groups theory and in particular of compact quantum groups in both the algebraic and analytical setting.

Definition and first properties

Building on some ideas of non-commutative topology, one can think of compact quantum groups as non-commutative compact topological spaces (i.e. unital C*-algebras) together with a compatible "group" structure. The appropriate definition of this "group" structure is given by the following definition.

Definition 1.1.1. A *compact quantum group* $\mathbb{G}$ is a pair $(C(\mathbb{G}), \Delta)$ where $C(\mathbb{G})$ is a unital C*-algebra and

$$\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$$

is a unital $*$-homomorphism such that

$$(\Delta \otimes \imath) \circ \Delta = (\imath \otimes \Delta) \circ \Delta$$

and

$$[\Delta(C(\mathbb{G}))(1 \otimes C(\mathbb{G}))] = C(\mathbb{G}) \otimes C(\mathbb{G}) = [\Delta(C(\mathbb{G}))(C(\mathbb{G}) \otimes 1)].$$

The first condition is called *coassociativity* since it reflects the same phenomenon as associativity but at the dual level. The second condition is sometimes called *(quantum) cancellation property* because of Example 1.1.2.

Example 1.1.2. Let G be compact semigroup and define a $*$-homomorphism Δ_G from the C*-algebra $C(G)$ of continuous functions on G to $C(G \times G)$ by the formula

$$\Delta_G(f)(g, h) = f(gh).$$

Using the fact that $C(G \times G)$ is isomorphic to $C(G) \otimes C(G)$, we get a coproduct on $C(G)$. Its coassociativity comes from the associativity of the group law on G. According to [Wor98, Rmk 3], the quantum cancellation property is equivalent to the assertion that every element in G is bisimplifiable. Thus, if G is a compact group, we have constructed a compact quantum group $(C(G), \Delta_G)$.

Example 1.1.3. Let Γ be a discrete group and recall that its *reduced C*-algebra* is the norm closure in $\mathcal{B}(L^2(\Gamma))$ of the linear span of the convolution operators $\lambda(\gamma)$ (which is simply γ acting by left translation of the variable). Define a coproduct $\Delta_{\widehat{\Gamma}}$ on the reduced C*-algebra $C_r^*(\Gamma)$ of Γ by the formula

$$\Delta_{\widehat{\Gamma}}(\lambda(g)) = \lambda(g) \otimes \lambda(g).$$

This turns $(C_r^*(\Gamma), \Delta_{\widehat{\Gamma}})$ into a compact quantum group. Note that the C*-algebra $C_r^*(\Gamma)$ has long been considered as a suitable replacement for the Pontryagin dual when the group is not abelian (see for instance [Con94, Chap 2 Sec 4] and references therein). This is made more precise by the theory of quantum groups since $(C_r^*(\Gamma), \Delta_{\widehat{\Gamma}})$ is exactly the compact quantum dual of Γ. We will consequently denote it by $\widehat{\Gamma}$.

In fact, Definition 1.1.1 has been designed in order to make Example 1.1.2 "canonical" in the sense of the following Proposition [Wor87a, Thm 1.5].

Proposition 1.1.4. *Let $\mathbb{G}$ be a compact quantum group such that $C(\mathbb{G})$ is a commutative C*-algebra. Then, there exists a compact group G and a $*$-isomorphism $\varphi : C(\mathbb{G}) \to C(G)$ such that $(\varphi \otimes \varphi) \circ \Delta = \Delta_G \circ \varphi$.*

Remark 1.1.5. Example 1.1.3 is also "canonical" but in a subtler way. In fact, if $\mathbb{G}$ is a compact quantum group which is *cocommutative*, i.e. such that $\sigma \circ \Delta = \Delta$, then there exists a discrete group Γ such that $\mathbb{G}$ is isomorphic to some C*-completion of the group algebra $\mathbb{C}[\Gamma]$ with the obvious generalization of the coproduct $\Delta_{\widehat{\Gamma}}$. The fact that this is not necessarily isomorphic to the construction of Example 1.1.3 is a typical non-amenability phenomenon.

The main feature of compact quantum groups is the existence of a generalization of the Haar measure, which happens to be both left and right invariant (see [Wor98, Thm 1.3] or [MVD98, Thm 4.4]).

Theorem 1.1.6 (Woronowicz)**.** *Let $\mathbb{G}$ be a compact quantum group. Then, there is a unique* Haar state *on $\mathbb{G}$, that is to say a state h on $C(\mathbb{G})$ such that for all $a \in C(\mathbb{G})$,*

$$(\imath \otimes h) \circ \Delta(a) = h(a).1$$
$$(h \otimes \imath) \circ \Delta(a) = h(a).1$$

Example 1.1.7. Let G be a compact group, let μ be its normalized Haar measure and set

$$h(f) = \int_G f(x)d\mu(x).$$

The invariance of μ means that for any $f \in C(G)$ and any $g \in G$,

$$\int_G f(xy)d\mu(y) = \int_G f(y)d\mu(y) = \int_G f(yx)d\mu(y).$$

This precisely means that the state h is the Haar state of $(C(G), \Delta_G)$.

Example 1.1.8. Let Γ be a discrete group. A direct computation shows that the Haar state h of $\widehat{\Gamma}$ is defined on the generators $\lambda(g)$ by $h(\lambda(g)) = \delta_{g,e}$, where e denotes the neutral element of Γ.

Unlike the case of compact and discrete groups, the Haar state fails to be tracial in general. This purely quantum phenomenon gives rise to the following definition.

Definition 1.1.9. A compact quantum group $\mathbb{G}$ is said to be *of Kac type* if its Haar state is a trace.

One may think of compact quantum groups of Kac type as "closer" to classical compact groups. This will become more precise in Chapter 2 where some crucial proofs fail for non-Kac type quantum groups.

Representation theory

Thanks to the Haar state, one can recover a full analogue of the Peter-Weyl theory of representations of compact groups in the quantum setting. Before describing it, let us give a few definitions concerning representations of compact quantum groups.

Definition 1.1.10. Let $\mathbb{G}$ be a compact quantum group. A *(left) representation* of $\mathbb{G}$ on a Hilbert space H is an element u of the multiplier algebra of $C(\mathbb{G}) \otimes \mathcal{K}(H)$ which satisfies

$$(\Delta \otimes \imath)(u) = u_{13}u_{23}. \tag{1.1}$$

The representation is said to be *unitary* if it is unitary as an element of the C*-algebra $M(C(\mathbb{G}) \otimes \mathcal{K}(H))$.

Let us look at the particular case of finite-dimensional representations. An element $u \in C(\mathbb{G}) \otimes M_n(\mathbb{C})$ can be seen, through the isomorphism

$$C(\mathbb{G}) \otimes M_n(\mathbb{C}) \simeq M_n(C(\mathbb{G})),$$

as an $n \times n$ matrix $[u_{i,j}]_{i,j}$ with coefficients in $C(\mathbb{G})$. Under this identification, Equation (1.1) becomes

$$\Delta(u_{i,j}) = \sum_{k=0}^{n} u_{i,k} \otimes u_{k,j}. \tag{1.2}$$

Remark 1.1.11. Apart from the regular representation, all the representations that we are going to deal with will be finite dimensional, hence we will not have to worry about technicalities linked to the multiplier algebras.

Example 1.1.12. Let G be a compact group and let $\pi : G \to M_n(\mathbb{C})$ be a representation. Let $u_{i,j}$ be the composition of the (i,j)-th coordinate function of $M_n(\mathbb{C})$ with π. Then we get n^2 continuous functions $u_{i,j}$ on G and the equation $\pi(g.h) = \pi(g).\pi(h)$ reads

$$u_{i,j}(g.h) = \sum_{k=0}^{n} u_{i,k}(g)u_{k,j}(h).$$

Composing with the isomorphism $C(G\times G) \simeq C(G)\otimes C(G)$ we get exactly Equation (1.2). It is not very difficult to see that any finite dimensional representation of $(C(G), \Delta_G)$ arises in this way from a representation of G.

Example 1.1.13. Let Γ be a discrete group. Because of the definition of the coproduct $\Delta_{\widehat{\Gamma}}$, $\lambda(g)$ can be seen as a one-dimensional representation of $\widehat{\Gamma}$ for any $g \in \Gamma$.

We now proceed to describe the structure of the category of finite-dimensional unitary representations of a compact quantum group $\mathbb{G}$. We will first describe the morphisms of this category and then the additional structure with which it can be endowed.

Definition 1.1.14. Let $\mathbb{G}$ be a compact quantum group and let u and v be two representations of $\mathbb{G}$ on Hilbert spaces H_u and H_v respectively. An *intertwiner* (or *morphism*) between u and v is a map $T \in \mathcal{B}(H_u, H_v)$ such that

$$v(1 \otimes T) = (1 \otimes T)u.$$

The set of intertwiners between u and v will be denoted $\mathrm{Mor}(u, v)$.

A representation u will be said to be *irreducible* if $\mathrm{Mor}(u, u) = \mathbb{C}.Id$ and it will be said to be *contained* in v if there is an isometric intertwiner between u and v. We will say that two representations are *equivalent* (resp. *unitarily equivalent*) if there is an intertwiner between them which is an isomorphism (resp. a unitary). Let us define two fundamental operations on representations.

Definition 1.1.15. Let $\mathbb{G}$ be a compact quantum group and let u and v be two representations of $\mathbb{G}$ on Hilbert spaces H_u and H_v respectively. The *sum* of u and v is the diagonal sum of the operators u and v seen as an element of $M(C(\mathbb{G}) \otimes \mathcal{K}(H_u \oplus H_v))$. It is a representation denoted $u \oplus v$. The *tensor product* of u and v is the element

$$u_{12}v_{13} \in M(C(\mathbb{G}) \otimes \mathcal{K}(H_u \otimes H_v)).$$

It is a representation denoted $u \otimes v$.

There is also an additional operation available in this category. First, let us denote by $1_{\mathbb{G}}$ the trivial representation of $\mathbb{G}$ which is simply the element $1 \otimes 1 \in C(\mathbb{G}) \otimes \mathbb{C}$. For any representation u of a compact quantum group $\mathbb{G}$, there is, according to [Wor98, Sec 6], a unique (up to isomorphism) representation $\overline{u}$ such that

$$\mathrm{Mor}(1_{\mathbb{G}}, u \otimes \overline{u}) \neq \{0\} \neq \mathrm{Mor}(1_{\mathbb{G}}, \overline{u} \otimes u), \tag{1.3}$$

The representation $\overline{u}$ is called the *contragredient* of u. Note that containing the trivial representation is equivalent to having a fixed point.

Remark 1.1.16. If u is a unitary representation, $\overline{u}$ need not be unitary itself, but is always equivalent to a unitary representation. This is another manifestation of the non-traciality of the Haar state in the following sense : a compact quantum group $\mathbb{G}$ is of Kac type if and only if for any unitary representation u, the representation $\overline{u}$ is again unitary.

The key fact for the study of representations of compact quantum groups is that we only have to focus on finite-dimensional representations, thanks to [Wor98, Sec 6].

Theorem 1.1.17 (Woronowicz)**.** *Let $\mathbb{G}$ be a compact quantum group. Then, any representation of $\mathbb{G}$ is equivalent to a unitary one and is a sum of irreducible representations. Moreover, any irreducible representation is finite-dimensional.*

Example 1.1.18. If Γ is a discrete group, it is not difficult to prove that the irreducible representations of the compact quantum group $\widehat{\Gamma}$ are exactly the representations $\lambda(g)$ for $g \in \Gamma$. Moreover, the tensor product of $\lambda(g_1)$ and $\lambda(g_2)$ is $\lambda(g_1.g_2)$ and the contragredient of $\lambda(g)$ is $\lambda(g^{-1})$.

Endowed with the direct sum, tensor product and contragredient operations, the category of finite-dimensional unitary representations of a compact quantum group is a *complete concrete monoidal W*-tensor category* [Wor88, Thm 1.2]. S.L. Woronowicz's generalization of the Tannaka-Krein duality theorem [Wor88, Thm 1.3], which is a converse of this fact, gives a way of constructing compact quantum groups out of specific categories, for example analogues of deformations of Lie groups [Ros90] or the so-called *easy quantum groups* (see e.g. [BS09]).

Theorem 1.1.19 (Woronowicz)**.** *Let $\mathfrak{C}$ be a concrete complete monoidal W*-tensor category. Then, there exists a compact quantum group $\mathbb{G}$ such that $\mathfrak{C}$ is isomorphic to the category of finite-dimensional unitary representations of $\mathbb{G}$.*

If u is a representation of a compact quantum group $\mathbb{G}$ on a Hilbert space H_u, a *coefficient* of u is any element of $C(\mathbb{G})$ of the form $(\imath \otimes \omega_{\eta,\xi})(u)$. We will denote by $\mathrm{Pol}(\mathbb{G})$ the set of all coefficients of finite-dimensional representations of $\mathbb{G}$. It is easy to see that the structure of the category of finite-dimensional unitary representation translates into a $*$-algebra structure on $\mathrm{Pol}(\mathbb{G})$. The following results [Wor98, Thm 1.2] show the relevance of this object to the study of compact quantum groups.

Theorem 1.1.20 (Woronowicz)**.** *The $*$-algebra $\mathrm{Pol}(\mathbb{G})$ can be turned into a Hopf $*$-algebra where the coproduct is given by the restriction of the coproduct on $C(\mathbb{G})$. Morover, this algebra is norm-dense in $C(\mathbb{G})$.*

Let us detail the Hopf algebra structure on $\mathrm{Pol}(\mathbb{G})$ (we refer the reader to [Abe77] for the necessary material on Hopf algebras). We know from Theorem 1.1.17 that we can restrict ourselves to irreducible unitary representations. Let u be such a representation and fix an orthonormal basis (e_i) of the Hilbert space H_u on which it acts. If we set $u_{i,j} = (\imath \otimes \omega_{e_i,e_j})(u)$ (equivalently, $u_{i,j}$ is the (i,j)-th coefficient of the matrix $u \in M_{\dim(H_u)}(C(\mathbb{G}))$ if the identification with the matrix algebra has been done using the basis (e_i)), we get a linear basis of the space of coefficients of u. Consequently, we only have to describe the structure maps on the $u_{i,j}$'s. If we denote the antipode by S and the counit by ε, we have

$$\begin{aligned}
\Delta(u_{i,j}) &= \sum_{k=1}^{\dim(H_u)} u_{i,k} \otimes u_{k,j} \\
S(u_{i,j}) &= u_{j,i}^* \\
\varepsilon(u_{i,j}) &= \delta_{i,j}.
\end{aligned}$$

Note moreover that the Haar state h satisfies $h(u_{i,j}) = 0$ for any finite-dimensional representation u, apart from the trivial one.

Example 1.1.21. If Γ is a discrete group, it is a consequence of Example 1.1.18 that the Hopf-$*$-algebra $\mathrm{Pol}(\widehat{\Gamma})$ is isomorphic to the group algebra $\mathbb{C}[\Gamma]$. This explains the notation $\mathrm{Pol}(\mathbb{G})$ for this algebra, which should be thought of as the algebra of polynomial functions on the discrete quantum group $\widehat{\mathbb{G}}$.

Remark 1.1.22. Note that the Haar state of $\mathbb{G}$ is always faithfull on $\mathrm{Pol}(\mathbb{G})$, whereas it may not be faithful on $C(\mathbb{G})$.

The structure maps S and ε are not bounded in general with respect to the norm of $C(\mathbb{G})$ and hence do not extend to the whole C*-algebra. Using the point of view of duals of discrete groups, the problem of the boundedness of the counit is linked to the question whether the trivial representation extends to the reduced C*-algebra, which is known to be equivalent to amenability of the discrete group. Anticipating on the generalization of Pontryagin duality, we can give the following definition.

Definition 1.1.23. A compact quantum group $\mathbb{G}$ is said to have an *amenable dual* if the counit of $\mathrm{Pol}(\mathbb{G})$ extends to $C(\mathbb{G})$ and the Haar state h is faithful on $C(\mathbb{G})$.

The problems concerning the antipode are purely quantum. However, we know precisely the cases when the antipode is well behaved thanks to [Wor98, Thm 1.5].

Theorem 1.1.24 (Woronowicz)**.** *Let $\mathbb{G}$ be a compact quantum group. Then, the following are equivalent :*

- *$\mathbb{G}$ is of Kac type.*
- *The antipode S of $\mathrm{Pol}(\mathbb{G})$ satisfies $S^2 = \mathrm{Id}$.*
- *The antipode S of $\mathrm{Pol}(\mathbb{G})$ satisfies $S(x^*) = S(x)^*$ for all $x \in \mathrm{Pol}(\mathbb{G})$.*
- *The antipode S of $\mathrm{Pol}(\mathbb{G})$ extends to a bounded map on any C*-completion of $\mathrm{Pol}(\mathbb{G})$.*

The GNS construction of the Haar state

We will now give an explicit description of the GNS construction of the Haar state of a compact quantum group $\mathbb{G}$ using the irreducible representations of $\mathbb{G}$. We do not prove any new result but it will be convenient in the sequel to have some formulas explicitely written down. According to our conventions, we will here chose the scalar product $\langle a, b\rangle = h(a^*b)$ to make the GNS construction.

Definition 1.1.25. Let us denote by $L^2(\mathbb{G})$ the Hilbert space of the GNS construction for h, by ξ_h the cyclic vector and by π_h the associated representation of $C(\mathbb{G})$. The image of π_h will be denoted $C_{\mathrm{red}}(\mathbb{G})$ and called the *reduced form* of $\mathbb{G}$.

Remark 1.1.26. We can restate Definition 1.1.23 in the following more natural way : a compact quantum group $\mathbb{G}$ has an amenable dual if the counit ε of $\mathrm{Pol}(\mathbb{G})$ extends to $C_{\mathrm{red}}(\mathbb{G})$.

Let $\mathrm{Irr}(\mathbb{G})$ be the set of unitary equivalence classes of irreducible unitary representations of $\mathbb{G}$. If $\alpha \in \mathrm{Irr}(\mathbb{G})$, we will denote by u^α a representative of the class α and by H_α the finite dimensional Hilbert space on which u^α acts. For $\alpha \in \mathrm{Irr}(\mathbb{G})$, the characterization of the contragredient representation $\overline{\alpha}$ of α given by Equation (1.3) implies the existence of an antilinear isomorphism

$$j_\alpha : H_\alpha \to H_{\overline{\alpha}}.$$

The matrix $j_\alpha^* j_\alpha \in \mathcal{B}(H_\alpha)$ is unique up to multiplication by a real number. We will say that j_α is *normalized* if

$$\mathrm{Tr}(j_\alpha^* j_\alpha) = \mathrm{Tr}((j_\alpha^* j_\alpha)^{-1}).$$

Remark 1.1.27. The above condition only determines j_α up to some complex number of modulus one. However, this will be of no consequence in the sequel.

If j_α is normailzed, we set $Q_\alpha = j_\alpha^* j_\alpha$, $\dim_q(\alpha) = \mathrm{Tr}(Q_\alpha)$ and

$$t_\alpha(1) = \sum j_\alpha(e_i) \otimes e_i,$$

where (e_i) is some fixed orthonormal basis of H_α. Note that by construction,

$$t_\alpha : \mathbb{C} \to H_{\overline{\alpha}} \otimes H_\alpha$$

is an intertwiner. Let us define a map

$$\Psi_\alpha : \left\{ \begin{array}{ccc} H_{\overline{\alpha}} \otimes H_\alpha & \to & C_{\mathrm{red}}(\mathbb{G}) \\ \eta \otimes \xi & \mapsto & \pi_h[(1 \otimes j_{\overline{\alpha}}(\eta)^*) u^\alpha (1 \otimes \xi)] \end{array} \right.$$

According to [Wor98, Eq 6.8] we have, for any $z, z' \in H_{\overline{\alpha}} \otimes H_\alpha$,

$$h(\Psi_\alpha(z)^* \Psi_\alpha(z')) = \frac{1}{\dim_q(\alpha)} \langle z, z' \rangle.$$

and

$$\Psi = \bigoplus_\alpha \sqrt{\dim_q(\alpha)} \Psi_\alpha . \xi_h : \bigoplus_\alpha (H_{\overline{\alpha}} \otimes H_\alpha) \to L^2(\mathbb{G})$$

is an isometric isomorphism of Hilbert spaces. If we let $E_{i,j}$ denote the operator on H_α sending e_i to e_j and the other vectors of the basis to 0, we can define another map

$$\Phi_\alpha : \left\{ \begin{array}{ccc} H_{\overline{\alpha}} \otimes H_\alpha & \longrightarrow & \mathcal{B}(H_\alpha) \\ j_\alpha(e_i) \otimes e_j & \mapsto & E_{i,j} \end{array} \right.$$

Now, we observe that $\Theta_\alpha = \Psi_\alpha \circ \Phi_\alpha^{-1} : \mathcal{B}(H_\alpha) \to C_{\mathrm{red}}(\mathbb{G})$ sends $E_{i,j}$ to $\pi_h(u^\alpha_{i,j})$ and that

$$\begin{array}{rcl} h(\Theta_\alpha(E_{i,j})^* \Theta_\alpha(E_{k,l})) & = & \dfrac{1}{\dim_q(\alpha)} \langle \Phi_\alpha^{-1}(E_{i,j}), \Phi_\alpha^{-1}(E_{k,l}) \rangle \\ & = & \dfrac{1}{\dim_q(\alpha)} \langle j_\alpha(e_i) \otimes e_j, j_\alpha(e_k) \otimes e_l \rangle \\ & = & \dfrac{\delta_{j,l}}{\dim_q(\alpha)} \langle e_k, Q_\alpha e_i \rangle \\ & = & \dfrac{1}{\dim_q(\alpha)} \mathrm{Tr}(Q_\alpha E_{i,j}^* E_{k,l}). \end{array}$$

Thus, if we endow $\mathcal{B}(H_\alpha)$ with the scalar product

$$\langle A, B\rangle_\alpha = \dim_q(\alpha)^{-1}\operatorname{Tr}(Q_\alpha A^*B),$$

we get an isometric isomorphism of Hilbert spaces

$$\Theta = \bigoplus_\alpha \Theta_\alpha.\xi_h : \bigoplus_\alpha \mathcal{B}(H_\alpha) \to L^2(\mathbb{G}).$$

Note that since the duality map $S_\alpha : A \mapsto \langle A, .\rangle_\alpha$ is bijective on the finite-dimensional space $\mathcal{B}(H_\alpha)$, one can endow $\oplus_\alpha \mathcal{B}(H_\alpha)_*$ with a Hilbert space structure making it isomorphic to $L^2(\mathbb{G})$ via $\Theta\circ(\oplus_\alpha S_\alpha^{-1})$. This is the "natural" isomorphism since it sends $\omega \in \mathcal{B}(H_\alpha)_*$ to $\pi_h[(\imath\otimes\omega)(u^\alpha)].\xi_h$, as one easily checks on the elements of the form $\omega_{\eta,\xi}$.

Let u^α and u^β be two irreducible representations of $\mathbb{G}$ and assume for the sake of simplicity that *every irreducible subrepresentation of $u^\alpha \otimes u^\beta$ appears with multiplicity* 1. This will always be satisfied for the quantum groups we consider in Chapter 3. Let $v_\gamma^{\alpha,\beta} : H_\gamma \to H_\alpha \otimes H_\beta$ be an isometric intertwiner and note that

$$v_\gamma^{\alpha,\beta} Q_\gamma = (Q_\alpha \otimes Q_\beta)v_\gamma^{\alpha,\beta}.$$

We have,

$$\begin{aligned}
(\imath\otimes\omega_{\eta,\xi})(u^\alpha)(\imath\otimes\omega_{\eta',\xi'})(u^\beta) &= (\imath\otimes\omega_{\eta,\xi}\otimes\omega_{\eta',\xi'})(u^\alpha_{12}u^\beta_{13})\\
&= (\imath\otimes\omega_{\eta,\xi}\otimes\omega_{\eta',\xi'})(u^\alpha \otimes u^\beta)\\
&= (\imath\otimes\omega_{\eta,\xi}\otimes\omega_{\eta',\xi'})\left(\sum_{\gamma\subset\alpha\otimes\beta}(\imath\otimes v_\gamma^{\alpha,\beta})u^\gamma(\imath\otimes v_\gamma^{\alpha,\beta})^*\right)\\
&= \sum_{\gamma\subset\alpha\otimes\beta}(\imath\otimes[\omega_{\eta,\xi}\otimes\omega_{\eta',\xi'}]^\gamma)(u^\gamma)
\end{aligned}$$

where $\omega^\gamma(x) = \omega(v_\gamma^{\alpha,\beta}\circ x\circ(v_\gamma^{\alpha,\beta})^*)$ for $\omega \in \mathcal{B}(H_\alpha\otimes H_\beta)_*$. Using the duality map S_α^{-1}, we can write the map induced on $C_{\text{red}}(\mathbb{G})$ by the product under our identification : for $A \in \mathcal{B}(H_\alpha)$ and $B \in \mathcal{B}(H_\beta)$,

$$\Theta_\alpha(A).\Theta_\beta(B) = \sum_{\gamma\subset\alpha\otimes\beta}\Theta_\gamma((v_\gamma^{\alpha,\beta})^*(A\otimes B)v_\gamma^{\alpha,\beta}). \tag{1.4}$$

We can now give an explicit formula for the GNS representation π_h. Let x be a coefficient of u^α and let $\xi \in p_\beta L^2(\mathbb{G}) \simeq \mathcal{B}(H_\beta)$. Identify x with $\pi_h(x)\xi_h$, which is an element of $p_\alpha L^2(\mathbb{G}) \simeq \mathcal{B}(H_\alpha)$. Making the identification by Θ implicit, we have :

$$x.\xi = \sum_{\gamma\subset\alpha\otimes\beta}(v_\gamma^{\alpha,\beta})^*(x\otimes\xi)v_\gamma^{\alpha,\beta} = \sum_{\gamma\subset\alpha\otimes\beta}\operatorname{Ad}(v_\gamma^{\alpha,\beta})(x\otimes\xi).$$

Let us mention that the equations of [Wor98, Sec 6] can be given a more common form which is the generalization of *Schur's orthogonality relations*. Let us first state them in full generality as in [Wor98, Eq 6.18 and Eq 6.19]

Proposition 1.1.28. *Let $\mathbb{G}$ be a compact quantum group and let α and β be two irreducible representations of $\mathbb{G}$. Then, for any $\eta, \xi \in H_\alpha$ and $\eta', \xi' \in H_\beta$, we have*

$$\begin{aligned}
h[(\imath\otimes\omega_{\eta',\xi'})(u^\alpha)(\imath\otimes\omega_{\eta,\xi})(u^\beta)^*] &= \frac{\delta_{\alpha,\beta}}{\dim_q(\alpha)}\langle\eta',\eta\rangle\langle\xi, Q_\alpha\xi'\rangle\\
h[(\imath\otimes\omega_{\eta',\xi'})(u^\alpha)^*(\imath\otimes\omega_{\eta,\xi})(u^\beta)] &= \frac{\delta_{\alpha,\beta}}{\dim_q(\alpha)}\langle\eta', Q_\alpha^{-1}\eta\rangle\langle\xi,\xi'\rangle
\end{aligned}$$

If an orthonormal basis has been chosen for the carrier space of every irreducible representations, these equations can be rewritten

$$\begin{aligned} h(u^{\alpha}_{i,j}(u^{\beta}_{l,m})^*) &= \delta_{\alpha,\beta}\delta_{i,l}\frac{(Q_\alpha)_{m,j}}{\dim_q(\alpha)} \\ h((u^{\alpha}_{i,j})^* u^{\beta}_{l,m}) &= \delta_{\alpha,\beta}\delta_{j,m}\frac{(Q^{-1}_\alpha)_{l,i}}{\dim_q(\alpha)} \end{aligned}$$

Actions of compact quantum groups on C*-algebras

Since we will define several compact quantum groups by the property of being universal with respect to actions on a given C*-algebra, we briefly recall the definition.

Definition 1.1.29. Let $\mathbb{G}$ be a compact quantum group and let A be a C*-algebra. A *(left) action* α of $\mathbb{G}$ on A is a $*$-homomorphism $\rho : A \to C(\mathbb{G}) \otimes A$ such that

$$(\Delta \otimes \imath) \circ \rho = (\imath \otimes \rho) \circ \rho$$

and $\rho(A)(C(\mathbb{G}) \otimes 1)$ is dense in $C(\mathbb{G}) \otimes A$.

Example 1.1.30. Let G be a compact group together with an action ρ by homeomorphisms on a locally compact space X. Then the map $\widetilde{\rho}$ sending a function $f \in C(X)$ to

$$(g, x) \in G \times X \mapsto f(\rho_g(x))$$

is an action of $(C(G), \Delta_G)$ on $C(X)$. It is not difficult to see that any action of $(C(G), \Delta_G)$ on a commutative C*-algebra comes in this way from an action of G.

We will also need to consider actions of compact quantum groups preserving a "measured structure" on the C*-algebra.

Definition 1.1.31. Let $\mathbb{G}$ be a compact quantum group and let A be a C*-algebra together with a distinguished state φ. A (left) action ρ of $\mathbb{G}$ on (A, φ) is an action ρ of $\mathbb{G}$ on A such that for all $x \in A$,

$$(\imath \otimes \varphi) \circ \rho(x) = \varphi(x).1$$

Actions of compact quantum groups will also be used in Subsection 3.3.2 when dealing with monoidal equivalence. However, the (few) necessary material will be introduced there in order to keep this preliminary chapter concise.

1.1.2 Discrete quantum groups and Pontryagin duality

We now proceed to define and describe discrete quantum groups. The main point of quantum groups, and the reason why S.L. Woronowicz first developped his theory, is to give a general setting in which the classical Pontryagin duality of locally compact abelian groups holds in full generality. The right context to present duality between compact and discrete quantum groups would thus be the theory of locally compact quantum groups of J. Kustermans and S. Vaes as introduced in [KV00]. However, since it is quite technical and quite unnecessary for our purpose, we will avoid this theory. Instead, we will use the theory of *multiplicative unitaries* as introduced by S. Baaj and G. Skandalis in [BS93].

Regular representation and Pontryagin duality

We first define the regular representation of a compact quantum group $\mathbb{G}$ and explain how it implements both the group and its dual. Recall that $(L^2(\mathbb{G}), \pi_h, \xi_h)$ denotes the GNS construction of the Haar state and that $C_{\mathrm{red}}(\mathbb{G})$ is the image of $C(\mathbb{G})$ under the map π_h.

Remark 1.1.32. Note that the coproduct on $\mathbb{G}$ induces a coproduct on $C_{\mathrm{red}}(\mathbb{G})$ turning $\mathbb{G}_{\mathrm{red}} = (C_{\mathrm{red}}(\mathbb{G}), \Delta)$ into a compact quantum group. This new quantum group has the same representation theory as $\mathbb{G}$ and in particular $\mathrm{Pol}(\mathbb{G}) = \mathrm{Pol}(\mathbb{G}_{\mathrm{red}})$, hence we should not distinguish between them. For that reason, the subscript "red" in $\mathbb{G}_{\mathrm{red}}$ will be omitted in the sequel.

It is proved in [Wor98, Thm 4.1] that there is a unique unitary operator W on the Hilbert space $L^2(\mathbb{G}) \otimes L^2(\mathbb{G})$ such that

$$W^*(\xi \otimes \pi_h(a)\xi_h) = (\pi_h \circ \pi_h) \circ \Delta(a)(\xi \otimes \xi_h)$$

for any $\xi \in L^2(\mathbb{G})$ and $a \in C(\mathbb{G})$. The operator W is called the *(left) regular representation* of $\mathbb{G}$ and it is proved in [Wor98, Thm 4.1] that it is indeed a representation of $\mathbb{G}$ on $L^2(\mathbb{G})$. This means in particular that W is a multiplier of $C_{\mathrm{red}}(\mathbb{G}) \otimes \mathcal{K}(L^2(\mathbb{G}))$. This also implies that W satisfies the following *pentagonal equation* :

$$W_{12}W_{13}W_{23} = W_{23}W_{12}$$

and hence is a *multiplicative unitary* in the sense of [BS93, Def 1.1]. It is known that such a multiplicative unitary encodes the whole data of the compact quantum group, which can be recovered using the following formulæ :

$$C_{\mathrm{red}}(\mathbb{G}) = [(\imath \otimes \mathcal{B}(H)_*)(W)] \tag{1.5}$$

$$\Delta(x) = W^*(1 \otimes x)W \tag{1.6}$$

Example 1.1.33. If G is a compact group, a direct computation shows that for any function $f \in L^2(G \times G)$,

$$(Wf)(x, y) = f(x, x^{-1}y).$$

This justifies the name "regular representation" attributed to W.

Given any multiplicative unitary with sufficiently nice properties (for instance *regularity* or *manageability*), one can build two quantum groups in duality using the general construction of [BS93] based on formulæ (1.5) and (1.6). That construction works in particular for the regular representation of a compact quantum group.

Definition 1.1.34. Let $\mathbb{G}$ be a compact quantum group. The *dual discrete quantum group* $\widehat{\mathbb{G}}$ is the pair $(C_0(\widehat{\mathbb{G}}), \widehat{\Delta})$, where

$$\begin{aligned} C_0(\widehat{\mathbb{G}}) &= [(\mathcal{B}(H)_* \otimes \imath)(W)] \\ \widehat{\Delta}(x) &= \Sigma W(x \otimes 1)W^*\Sigma \end{aligned}$$

Setting $\widehat{W} = \Sigma W^*\Sigma$, one obtains exact analogues of Equations (1.5) and (1.6) :

$$\begin{aligned} C_0(\widehat{\mathbb{G}}) &= [(\imath \otimes \mathcal{B}(H)_*)(\widehat{W})] \\ \widehat{\Delta}(x) &= \widehat{W}^*(1 \otimes x)\widehat{W} \end{aligned}$$

Remark 1.1.35. Note that unlike the compact case, the image of the coproduct $\widehat{\Delta}$ is not contained in $C_0(\widehat{\mathbb{G}}) \otimes C_0(\widehat{\mathbb{G}})$. However, the image of the coproduct is contained in the multiplier algebra $M(C_0(\widehat{\mathbb{G}}) \otimes C_0(\widehat{\mathbb{G}}))$ and one can prove that discrete quantum groups fit into the setting of A. van Daele's *algebraic quantum groups* (see [VD98]).

Since $C_0(\widehat{\mathbb{G}})$ is defined as a subalgebra of $\mathcal{B}(L^2(\mathbb{G}))$ we can take its bicommutant to define a von Neumann algebra $\ell^\infty(\widehat{\mathbb{G}})$ to which the coproduct extends naturally to a normal map. Similarly, the coproduct on $C_{\text{red}}(\mathbb{G})$ extends to its bicommutant in $\mathcal{B}(L^2(\mathbb{G}))$, which will be denoted $L^\infty(\mathbb{G})$. These two objects fit into the framework of locally compact quantum groups in the von Neumann algebraic setting of [KV03].

Example 1.1.36. Let us detail how discrete groups enter this setting. Mimicking the case of compact groups, we can define a coproduct Δ_Γ on the C*-algebra $C_0(\Gamma)$ of continuous functions on Γ vanishing at infinity by the formula

$$\Delta_\Gamma(f)(g,h) = f(gh).$$

The image of this map is included in the algebra $C_b(\Gamma \times \Gamma)$ of bounded functions on $\Gamma \times \Gamma$, which is isomorphic to the multiplier algebra of $C_0(\Gamma) \otimes C_0(\Gamma)$. Its coassociativity comes from the associativity of the group law on Γ. If we perform the duality construction with the compact quantum group $\widehat{\Gamma}$, we get precisely the pair $(C_0(\Gamma), \Delta_\Gamma)$ standardly represented on $\ell^2(\Gamma)$. This justifies the use of the symbol $\widehat{\Gamma}$ to denote this compact quantum group. Note moreover that $L^\infty(\widehat{\Gamma})$ is nothing but the von Neumann algebra $\mathcal{L}\Gamma = C^*_r(\Gamma)''$ of Γ.

The structure of discrete quantum groups

Given a compact quantum group $\mathbb{G}$, it is possible to give a complete description of the dual discrete quantum group $\widehat{\mathbb{G}}$ using the representation theory of $\mathbb{G}$. This can be thought of as a far-reaching generalization of Pontryagin duality, which identifies the dual group of an abelian compact group with its characters, which are exactly the irreducible representations in that case. The description we give here will be constantly use in this dissertation.

Proposition 1.1.37. *Let $\mathbb{G}$ be a compact quantum group. Then, there are isomorphisms*

$$\begin{aligned} C_0(\widehat{\mathbb{G}}) &\simeq \bigoplus_{\alpha \in \mathrm{Irr}(\mathbb{G})} \mathcal{B}(H_\alpha) \\ \ell^\infty(\widehat{\mathbb{G}}) &\simeq \prod_{\alpha \in \mathrm{Irr}(\mathbb{G})} \mathcal{B}(H_\alpha) \end{aligned}$$

Under this identification, we will denote by p_α the minimal central projection in $\ell^\infty(\widehat{\mathbb{G}})$ corresponding to $\mathcal{B}(H_\alpha)$. Let us explain how to recover the coproduct in that context. Let $\alpha \in \mathrm{Irr}(\mathbb{G})$ and $x \in \mathcal{B}(H_\alpha)$. Then for any $\beta, \gamma \in \mathrm{Irr}(\mathbb{G})$ and $T \in \mathrm{Mor}(\alpha, \beta \otimes \gamma)$,

$$\widehat{\Delta}(x) \circ T = T \circ x$$

and this caracterizes $\widehat{\Delta}(x)$. Note that if

$$\mathbb{V} = \bigoplus_{\alpha \in \mathrm{Irr}(\mathbb{G})} u^\alpha$$

is the diagonal sum of all the irreducible representations, then $\widehat{\Delta}$ is also characterized by the equation

$$(\imath \otimes \widehat{\Delta})(\mathbb{V}) = \mathbb{V}_{12}\mathbb{V}_{13}.$$

In fact, $\mathbb{V}$ is simply the image of the regular representation W under the previous isomorpisms.

There is of course no Haar state on a discrete quantum group $\widehat{\mathbb{G}}$ (the C*-algebra $C_0(\widehat{\mathbb{G}})$ is not even unital, unless it is finite-dimensional), but there are two *Haar weights* which are respectively left and right invariant. They can be constructed from the representations of $\mathbb{G}$ in the following way. Define, for any $\alpha \in \mathrm{Irr}(\mathbb{G})$ two states φ_α and ψ_α on $\mathcal{B}(H_\alpha)$ by

$$\begin{cases} \varphi_\alpha(x) & = & \dfrac{\mathrm{Tr}(Q_\alpha x)}{\dim_q(\alpha)} \\ \psi_\alpha(x) & = & \dfrac{\mathrm{Tr}(Q_\alpha^{-1} x)}{\dim_q(\alpha)} \end{cases}$$

Then the weights

$$h_L : x \mapsto \sum_{\alpha \in \mathrm{Irr}(\mathbb{G})} \dim_q(\alpha)^2 \varphi_\alpha(p_\alpha x) \tag{1.7}$$

$$h_R : x \mapsto \sum_{\alpha \in \mathrm{Irr}(\mathbb{G})} \dim_q(\alpha)^2 \psi_\alpha(p_\alpha x) \tag{1.8}$$

are respectively left and right invariant under $\widehat{\Delta}$, i.e. for every positive element $x \in C_0(\widehat{\mathbb{G}})$ such that $h_L(x) < +\infty$ (respectively $h_R(x) < +\infty$),

$$\begin{cases} (\imath \otimes h_L) \circ \widehat{\Delta}(x) & = & h_L(x).1 \\ (h_R \otimes \imath) \circ \widehat{\Delta}(x) & = & h_R(x).1 \end{cases}$$

Remark 1.1.38. It is a general fact that on a locally compact quantum group, Haar weights (the existence of which is assumed) are unique up to multiplication by a scalar, thus the above formulæ describe all the Haar weights of $\widehat{\mathbb{G}}$.

Note that unlike the classical case, the left and right Haar weights on a discrete quantum group need not be equal (or even proportional) in general. In other words, discrete quantum groups may not be *unimodular*. This quantum phenomenon is only another aspect of the non-traciality of the Haar state on a compact quantum group. More precisely, we have the following result, which is part of [Wor98, Thm 1.5] :

Proposition 1.1.39. *A discrete quantum group $\widehat{\mathbb{G}}$ is unimodular if and only if its dual compact quantum group $\mathbb{G}$ is of Kac type.*

This means that $\mathbb{G}$ is of Kac type if and only all the asociated matrices Q_α are equal to Id_{H_α}or equivalently if and only if $\dim_q(\alpha) = \dim(\alpha)$ for every irreducible representation α.

1.2 Complements on discrete quantum groups

This section is a kind of survey of the theory of discrete quantum groups. We will define several notions, give a few results, explain how to construct new discrete quantum groups from old ones and eventually describe the main examples of discrete quantum groups which we will be interested in.

1.2.1 Actions of discrete quantum groups

Actions of discrete quantum groups appear in two different flavors, depending whether one considers "topological actions" (i.e. on a C*-algebra) or "measurable actions" (i.e. on a von Neumann algebra).

The C*-algebra case

We will not be interested in actions on C*-algebras in this dissertation, except as far as the notion of exactness is concerned. Consequently, we just give a brief series of definitions.

Definition 1.2.1. A *(left) action* of a discrete quantum group $\widehat{\mathbb{G}}$ on a C*-algebra A is a $*$-homomorphism $\rho : A \mapsto M(C_0(\widehat{\mathbb{G}}) \otimes A)$ such that

$$(\widehat{\Delta} \otimes \imath) \circ \rho = (\imath \otimes \rho) \circ \rho$$

and $(\widehat{\varepsilon} \otimes \imath) \circ \alpha = \imath_A$.

Example 1.2.2. Let Γ be a discrete group together with an action ρ by homeomorphisms on a locally compact space X. Then, the map $\widetilde{\rho}$ sending a function $f \in C_0(X)$ to

$$(g, x) \in \Gamma \times X \mapsto f(\rho_g(x))$$

is an action of $(C_0(\Gamma), \Delta_\Gamma)$ on $C_0(X)$.

Associated to an action of a discrete quantum group on a C*-algebra are a full and a reduced crossed-product. As we do not need any details on this in the sequel, we will just explain how to construct the reduced one. We start with a faithfull representation π of A on a Hilbert space H. Then, the closed linear space

$$\widehat{\mathbb{G}} \ltimes_r A = [(\imath \otimes \pi)(A)(C_{\text{red}}(\mathbb{G}) \otimes 1)]$$

is in fact a C*-algebra called the *reduced crossed-product* of A by ρ. It is a non-obvious fact that this construction does not depend on the choice of a faithful representation of A we chose.

Definition 1.2.3. A discrete quantum group $\widehat{\mathbb{G}}$ is said to be *exact* if the functor $\widehat{\mathbb{G}} \ltimes_r$ turns $\widehat{\mathbb{G}}$-equivariant short exact sequences into short exact sequences.

Recall the following definition of exactness for C*-algebras.

Definition 1.2.4. A C*-algebra A is said to be *exact* if the functor $A \otimes$ turns short exact sequences of C*-algebras into short exact sequences.

As in the classical case, the two notions are linked by the following theorem (see [Bla01, Prop 4.1] for a proof).

Proposition 1.2.5. *A discrete quantum group $\widehat{\mathbb{G}}$ is exact if and only if the C*-algebra $C_{red}(\mathbb{G})$ is exact.*

The von Neumann algebra case

Actions of quantum groups on von Neumann algebras are in some sense more tractable than actions on C*-algebras. In particular, though we lose track of the possible non-exactness of $\widehat{\mathbb{G}}$, we get a unitary implementation which will be useful when dealing with relative amenability. The main reference on this subject is [Vae01].

Definition 1.2.6. A *(left) action* of $\widehat{\mathbb{G}}$ on a von Neumann algebra M is a unital normal $*$-homomorphism $\rho : M \to \ell^{\infty}(\widehat{\mathbb{G}})\overline{\otimes}M$ such that

$$(\widehat{\Delta} \otimes \imath) \circ \rho = (\imath \otimes \rho) \circ \rho.$$

Remark 1.2.7. By a straightforward adaptation of Example 1.2.2, there is a correpondance between actions of Γ on measure spaces and actions of $\widehat{\Gamma}$ on commutative von Neuman algebras for any discrete group Γ.

The *fixed point algebra* of the action ρ is the subalgebra $M^{\rho} = \{x \in M, \rho(x) = 1 \otimes x\}$ of M. A subalgebra N of M is said to be *stable* under the action if $\rho(N) \subset \ell^{\infty}(\widehat{\mathbb{G}})\overline{\otimes}N$. In that case there is a restricted action of $\widehat{\mathbb{G}}$ on N which will still be denoted ρ.

There is of course only one crossed-product construction in the von Neumann algebraic setting, which straightforwardly generalizes the classical definition.

Definition 1.2.8. Let $\widehat{\mathbb{G}}$ be a discrete quantum group acting on a von Neumann algebra M. The *crossed-product* $\widehat{\mathbb{G}} \ltimes_{\rho} M$ is the von Neumann subalgebra of $\mathcal{B}(L^2(\mathbb{G}))\overline{\otimes}M$ generated by $\rho(M)$ and $L^{\infty}(\mathbb{G}) \otimes 1$.

The crossed-product is endowed with a *dual action* $\widehat{\rho}$ of $\mathbb{G}^{\mathrm{op}}$ (i.e. with respect to the flipped coproduct) defined by $\widehat{\rho}(\rho(m)) = 1 \otimes \rho(m)$ and $\widehat{\rho}(a \otimes 1) = [(\sigma \circ \Delta)(a)] \otimes 1$ for all $m \in M$ and $a \in L^{\infty}(\mathbb{G})$. S. Baaj and G. Skandalis proved in [BS93, Thm 7.5] that this dual action satisfies a quantum version of the Takesaki-Takai duality (the proof is done in the C*-algebra setting but [BS93, Rmq 7.7] explains how one can extend it to the von Neumann algebra setting).

Theorem 1.2.9 (Baaj, Skandalis)**.** *Let $\widehat{\mathbb{G}}$ be a discrete quantum group. Then, the crossed-product $\mathbb{G}^{op} \ltimes_{\widehat{\rho}} (\widehat{\mathbb{G}} \ltimes_{\rho} M)$ is Morita equivalent to M.*

Let $\widehat{\mathbb{G}}$ be a discrete quantum group and let M be a von Neumann algebra together with a fixed normal semi-finite faithful (in short n.s.f.) weight θ with GNS construction $(K, \imath, \Lambda_{\theta})$. It is proved in [Vae01] that any action of $\widehat{\mathbb{G}}$ on M is unitarily implementable, i.e. there is a unitary

$$U^{\rho} \in \ell^{\infty}(\mathbb{G})\overline{\otimes}\mathcal{B}(K)$$

(which happens to be the adjoint of a representation of $\widehat{\mathbb{G}}$) such that

$$\rho(x) = U^{\rho}(1 \otimes x)(U^{\rho})^*$$

for any $x \in M$.

1.2.2 Subgroups and quotients

The notion of quantum subgroup is not clearly defined in general and its perusal requires some care (see for example [DKSS12]). However, quantum subgroups turn out to be a quite tractable notion as soon as one restricts to the discrete, or equivalently compact, case. Let us describe two different ways of defining a quantum subgroup of a given discrete quantum group.

1. Let $\mathbb{G}$ be a compact quantum group, let $\mathfrak{C}$ be its category of finite-dimensional unitary representations and let $\mathfrak{D}$ be a full subcategory of $\mathfrak{C}$ containing the trivial representation and stable under the direct sum, tensor product and contragredient operations. Then, according to Theorem 1.1.19, there exists a compact quantum group $\mathbb{H}$ such that $\mathfrak{D}$ is its category of finite-dimensional unitary representations. The algebra $\mathrm{Pol}(\mathbb{H})$ naturally embeds as a Hopf $*$-subalgebra into $\mathrm{Pol}(\mathbb{G})$, thus giving by completion a subalgebra $C(\mathbb{H})$ of $C_{\mathrm{red}}(\mathbb{G})$ which is stable under the coproduct. The restriction of the Haar state on $C_{\mathrm{red}}(\mathbb{G})$, which is faithful, yields a faithful Haar state on $C(\mathbb{H})$, which is consequently the reduced C*-algebra $C_{\mathrm{red}}(\mathbb{H})$ of $\mathbb{H}$.
2. Reciprocally, let A be a C*-subalgebra of $C_{\mathrm{red}}(\mathbb{G})$ such that $\Delta(A) \subset A \otimes A$ and $[\Delta(A)(1 \otimes A)] = A \otimes A = [\Delta(A)(A \otimes 1)]$. This implies that $\mathbb{H} = (A, \Delta)$ is a *bona fide* compact quantum group and even (because the Haar state is obviously faithful) that $A = C_{\mathrm{red}}(\mathbb{H})$. Now, the inclusion $M_n(C(\mathbb{H})) \subset M_n(C(\mathbb{G}))$ implies that any representation of $\mathbb{H}$ induces a representation of $\mathbb{G}$. One can consequently identify the category of finite-dimensional unitary representations of $\mathbb{H}$ with a full subcategory of the category of finite-dimensional unitary representations of $\mathbb{G}$ which is stable under all the operations.

In any of these two equivalent (according to [Ver04, Lem 2.1]) situations, we will say that $\widehat{\mathbb{H}}$ is a *discrete quantum subgroup* of $\widehat{\mathbb{G}}$.

Example 1.2.10. Let Γ be a discrete group and let Λ be a discrete subgroup of Γ. By virtue of the canonical inclusion $C_{\mathrm{red}}(\Lambda) \subset C_{\mathrm{red}}(\Gamma)$, Λ can be seen as a discrete quantum subgroup of Γ. It is easy to prove that reciprocally any discrete quantum subgroup of Γ is cocommutative, hence comes from a discrete subgroup of Γ.

Example 1.2.11. Let G be a compact group and let H be a compact group which is a quotient of G. The quotient map being continuous and surjective, it induces an injective $*$-homomorphism from $C(H)$ into $C(G)$. Hence, $\widehat{H}$ is a discrete quantum subgroup of $\widehat{G}$. Noticing that any discrete quantum subgroup of $\widehat{G}$ has a commutative C*-algebra, we see that they all come from quotients of G.

Assume $\mathbb{G}$ to be of Kac type, then $L^\infty(\mathbb{H}) \subset L^\infty(\mathbb{G})$ is an inclusion of finite von Neumann algebras and there is consequently a faithful normal conditional expectation $\mathbb{E}_{\mathbb{H}} : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{H})$. This map sends $\mathrm{Pol}(\mathbb{G})$ to $\mathrm{Pol}(\mathbb{H})$ and thus gives a conditional expectation on the level of the reduced C*-algebras. Though it is not straightforward to prove, such a conditional expectation also exists without the traciality assumption (see [Ver04, Prop 2.2] for a proof).

Proposition 1.2.12. *Let $\widehat{\mathbb{G}}$ be a discrete quantum group, let $\widehat{\mathbb{H}}$ be a discrete quantum subgroup of $\widehat{\mathbb{G}}$ and denote the respective Haar states of $\mathbb{G}$ and $\mathbb{H}$ by $h_{\mathbb{G}}$ and $h_{\mathbb{H}}$. Then, there exists a faithful condition expectation $\mathbb{E}_{\mathbb{H}} : C_{red}(\mathbb{G}) \to C_{red}(\mathbb{H})$ such that $h_{\mathbb{H}} \circ \mathbb{E}_{\mathbb{H}} = h_{\mathbb{G}}$. Moreover, $\mathbb{E}_{\mathbb{H}}$ extends to a faithful normal condition expectation from $L^\infty(\mathbb{G})$ to $L^\infty(\mathbb{H})$, still denoted $\mathbb{E}_{\mathbb{H}}$.*

Let $p_{\mathbb{H}}$ denote the central projection $\sum_{\alpha \in \mathrm{Irr}(\mathbb{H})} p_\alpha$, which is an element of $\ell^\infty(\widehat{\mathbb{G}})$. We can use $p_{\mathbb{H}}$ to describe the structure of $\widehat{\mathbb{H}}$ using the structure of $\widehat{\mathbb{G}}$ (see [Fim10, Prop 2.3] for details).

Proposition 1.2.13. *With the notations above, we have*

1. $\widehat{\Delta}(p_{\mathbb{H}})(p_{\mathbb{H}} \otimes 1) = p_{\mathbb{H}} \otimes p_{\mathbb{H}}$

2. $\ell^\infty(\widehat{\mathbb{H}}) = p_{\mathbb{H}}\ell^\infty(\widehat{\mathbb{G}})$
3. $\widehat{\Delta}_{\mathbb{H}}(a) = \widehat{\Delta}(a)(p_{\mathbb{H}} \otimes p_{\mathbb{H}})$
4. *If h_L is a left Haar weight for $\widehat{\mathbb{G}}$, then $h_{L,\mathbb{H}} : x \mapsto h_L(p_{\mathbb{H}}x)$ is a left Haar weight for $\widehat{\mathbb{H}}$.*

We end by detailing the construction of the quotient and quasi-regular representation associated to a discrete quantum subgroup. It is easy to see, using the above statements, that the map $a \mapsto \widehat{\Delta}(a)(1 \otimes p_{\mathbb{H}})$ defines a right action (right actions of discrete quantum groups are defined in the same way as left actions with the obvious modifications) of $\widehat{\mathbb{H}}$ on $\ell^\infty(\widehat{\mathbb{G}})$. Let $\ell^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$ denote the fixed point subalgebra of this action. Using Proposition 1.2.13 again, we see that the restriction of the coproduct $\widehat{\Delta}$ to $\ell^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$ yields a left action of $\widehat{\mathbb{G}}$ on this von Neumann algebra which will be denoted τ and called the *translation action*.

Let h_L be the left-invariant weight of $\widehat{\mathbb{G}}$ defined by Equation (1.7). We know from [Fim10, Prop 2.4] that the map

$$T : x \mapsto (\imath \otimes h_{L,\mathbb{H}})[\widehat{\Delta}(x)(1 \otimes p_{\mathbb{H}})]$$

is a normal faithful operator-valued weight from $\ell^\infty(\widehat{\mathbb{G}})$ onto $\ell^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$ and that there consequently exists a n.s.f. weight θ on $\ell^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$ such that $h_L = \theta \circ T$. Let U be the unitary implementation of the action τ with respect to the weight θ. Then, $\mathcal{R} = U^*$ will be called the *quasi-regular representation* of $\widehat{\mathbb{G}}$ modulo the quantum subgroup $\widehat{\mathbb{H}}$ (representations of discrete quantum groups, or more generally locally compact quantum groups are defined in the same way as representations of compact quantum groups, see [KV00]).

1.2.3 Building new quantum groups

We will now detail several ways of combining discrete quantum groups to build new ones. All these constructions are generalizations of classical ones. Let us first make the notion of "maximal C*-algebra of a discrete quantum group" more precise, since we will use it several times in the sequel.

Definition 1.2.14. Let $\mathbb{G}$ be a compact quantum group and let $C_{\max}(\mathbb{G})$ be the enveloping C*-algebra of the $*$-algebra $\mathrm{Pol}(\mathbb{G})$. The coproduct extends by universality to a coproduct $\Delta_{\max}$ on $C_{\max}(\mathbb{G})$ and yields another version $\mathbb{G}_{\max} = (C_{\max}(\mathbb{G}), \Delta)$ of $\mathbb{G}$ called the *maximal form* of $\mathbb{G}$.

Here by "another version" we mean that the Hopf $*$-algebra $\mathrm{Pol}(\mathbb{G}_{\max})$ is canonically isomorphic to $\mathrm{Pol}(\mathbb{G})$, and that the restriction of the Haar states coincide on these algebras. Thus, one should really think of $\mathbb{G}$, $\mathbb{G}_{\mathrm{red}}$ and $\mathbb{G}_{\max}$ as being the same object, or at least that their dual discrete quantum groups are the same. Note that by universality, there is always a surjective map

$$\lambda_{\mathbb{G}} : C_{\max}(\mathbb{G}) \to C_{\mathrm{red}}(\mathbb{G})$$

which intertwines the coproducts and that the usual characterization of amenability still holds (see [BMT01, Thm. 3.6] for a proof).

Proposition 1.2.15. *A discrete quantum group $\widehat{\mathbb{G}}$ is amenable if and only if the map $\lambda_{\mathbb{G}}$ is injective.*

Direct products

Let us first recall the definition of the maximal tensor product of C*-algebra.

Definition 1.2.16. Let A and B be two C*-algebras. The *maximal tensor product* of A and B is the envelopping C*-algebra of their algebraic tensor product (over $\mathbb{C}$) $A \odot B$. It will be denoted $A \otimes^{max} B$.

Let $\mathbb{G}$ and $\mathbb{H}$ be two compact quantum groups with Haar states $h_{\mathbb{G}}$ and $h_{\mathbb{H}}$ respectively. The maximal tensor product $C_{\max}(\mathbb{G}) \otimes^{max} C_{\max}(\mathbb{H})$ of $C_{\max}(\mathbb{G})$ and $C_{\max}(\mathbb{H})$ can be turned into a compact quantum group in the following way : let $\Delta_{\mathbb{G}}$ and $\Delta_{\mathbb{H}}$ denote the coproducts of $\mathbb{G}$ and $\mathbb{H}$ respectively, then the map

$$\Delta = (\imath \otimes \sigma \otimes \imath) \circ (\Delta_{\mathbb{G}} \otimes \Delta_{\mathbb{H}})$$

defines by universality a coproduct on $C_{\max}(\mathbb{G}) \otimes^{max} C_{\max}(\mathbb{H})$. The following theorem summarizes the main results of [Wan95b].

Theorem 1.2.17 (Wang)**.** *The pair $\mathbb{G} \otimes \mathbb{H} = (C_{max}(\mathbb{G}) \otimes^{max} C_{max}(\mathbb{H}), \Delta)$ is a compact quantum group. Moreover, its Haar state is the tensor product of the Haar states and $C_{red}(\mathbb{G} \otimes \mathbb{H})$ is isomorphic to the minimal tensor product $C_{red}(\mathbb{G}) \otimes C_{red}(\mathbb{H})$.*

Example 1.2.18. Let Γ_1 and Γ_2 be two discrete groups and perform this construction with the compact quantum groups $\widehat{\Gamma}_1 = (C_r^*(\Gamma_1), \Delta_{\widehat{\Gamma}_1})$ and $\widehat{\Gamma}_2 = (C_r^*(\Gamma_2), \Delta_{\widehat{\Gamma}_2})$. It is well known that the C*-algebra $C_r^*(\Gamma_1) \otimes C_r^*(\Gamma_2)$ is isomorphic to the reduced group C*-algebra $C_r^*(\Gamma_1 \times \Gamma_2)$ of $\Gamma_1 \times \Gamma_2$, and one can prove that there is an isomorphism of compact quantum groups

$$\widehat{\Gamma}_1 \otimes \widehat{\Gamma}_2 \simeq \widehat{\Gamma_1 \times \Gamma_2}.$$

By analogy with this example, the dual of the tensor product of two compact quantum groups $\mathbb{G}$ and $\mathbb{H}$ will be denoted $\widehat{\mathbb{G}} \times \widehat{\mathbb{H}}$ and called the *direct product* of $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$. The representation theory of $\mathbb{G} \otimes \mathbb{H}$ can be explicitly described using the representations of $\mathbb{G}$ and $\mathbb{H}$. This is done in [Wan95b, Thm 2.11].

Theorem 1.2.19 (Wang)**.** *Let $\mathbb{G}$ and $\mathbb{H}$ be compact quantum groups. Then, the irreducible representations of $\mathbb{G} \otimes \mathbb{H}$ are the trivial one and the representations of the form*

$$u \otimes_{ext} v = u_{13} v_{24},$$

where u is an irreducible representation of $\mathbb{G}$ and v is an irreducible representation of $\mathbb{H}$.

Inductive limits

Let us now turn to inductive limits. Let $(\widehat{\mathbb{G}}_i)$ be a family of discrete quantum groups together with $*$-homomorphisms

$$\pi_{i,j} : C(\mathbb{G}_i) \to C(\mathbb{G}_j)$$

intertwining the coproducts and satisfying $\pi_{j,k} \circ \pi_{i,j} = \pi_{i,k}$. We will call the data $(\widehat{\mathbb{G}}_i, \pi_{i,j})$ an *inductive system of discrete quantum groups.*

Let $C(\mathbb{G})$ be the inductive limit C*-algebra of this system. Because the connecting maps intertwine the coproducts, one can define a coproduct Δ on $C(\mathbb{G})$ by universality using the coproducts of the $\mathbb{G}_i$'s. Then, the pair $\mathbb{G} = (C(\mathbb{G}), \Delta)$ is a compact quantum group (see [Wan95a, Prop 3.2] for a proof). We will say that the dual discrete quantum group $\widehat{\mathbb{G}}$ is the *inductive limit* of the system $(\widehat{\mathbb{G}}_i, \pi_{i,j})$. One can easily check that the Haar state on $\mathbb{G}$ is exactly equal to the inductive limit of the Haar states and that the reduced C*-algebra of in the inductive limit is the inductive limit of the reduced C*-algebras. The following proposition describes the representation theory of $\mathbb{G}$ under an injectivity assumption. It is certainly well known, but since we did not succeed in finding a reference, we include a proof for the sake of completeness.

Proposition 1.2.20. *Let $(\widehat{\mathbb{G}}_i, \pi_{i,j})$ be an inductive system of discrete quantum groups with inductive limit $\widehat{\mathbb{G}}$ and assume that all the maps $\pi_{i,j}$ are injective. Then, there is a one-to-one correspondance between the irreducible representations of $\mathbb{G}$ and the increasing union of the sets of irreducible representations of the $\mathbb{G}_i$'s.*

Proof. The maps $\pi_{i,j}$ being injective, we can identify each $\widehat{\mathbb{G}}_i$ with a discrete quantum subgroup of the $\widehat{\mathbb{G}}_j$'s for $j \geqslant i$. This gives inclusions of the sets of irreducible representations and we denote by $\mathcal{S}$ the increasing union of these sets. We can also identify each $C(\mathbb{G}_i)$ with a C*-subalgebra of $C(\mathbb{G})$ in such a way that

$$\overline{\bigcup C(\mathbb{G}_i)} = C(\mathbb{G}).$$

Under this identification, the discrete quantum groups $\widehat{\mathbb{G}}_i$ are quantum subgroups of $\widehat{\mathbb{G}}$, hence any irreducible representation of some $\mathbb{G}_i$ yields an irreducible representation of $\mathbb{G}$ and we have proved that $\mathcal{S} \subset \mathrm{Irr}(\mathbb{G})$. Moreover, the algebra

$$\mathcal{A} := \bigcup_i \mathrm{Pol}(\mathbb{G}_i)$$

is a dense Hopf-$*$-subalgebra of $C(\mathbb{G})$ spanned by coefficients of irreducible representations. Because of Shur's orthogonality relations, the density implies that the coefficients of all irreducible representations of $\mathbb{G}$ are in $\mathcal{A}$, i.e. $\mathcal{A} = \mathrm{Pol}(\mathbb{G})$. This means that any irreducible representation of $\mathbb{G}$ comes from an element of $\mathcal{S}$ and $\mathrm{Irr}(\mathbb{G}) = \mathcal{S}$. □

Example 1.2.21. Let $(\Gamma_i, \pi_{i,j})$ be an inductive system of discrete groups. Each map $\pi_{i,j}$ induces covariantly a map between the reduced C*-algebras of the corresponding groups, yielding an inductive system of discrete quantum groups. It is well known that the inductive limit C*-algebra $\lim C^*_r(\Gamma_i)$ is isomorphic to the reduced group C*-algebra $C^*_r(\lim \Gamma_i)$ of the inductive limit of the system $(\Gamma_i, \pi_{i,j})$, and one can prove that this map is an isomorphism of discrete quantum groups.

Free products – the group case

The free product construction will be of particular importance in this dissertation, both from the point of view of operator algebras and from the point of view of quantum groups. We will therefore define these notions in details. For the sake of simplicity, we only explain the construction of the free product of two objects. The generalization to any family of objects is always quite straightforward and left to the reader. Let us start with the case of classical groups, which is both the easiest and the model for the other definitions.

Let Γ_1 and Γ_2 be (discrete) groups and let Λ be another group together with injective group homomorphisms $i_k : \Lambda \to \Gamma_k$ for $k = 1, 2$. Otherwise said, Λ is a *common subgroup* of Γ_1 and Γ_2. Then, there is a unique group $\Gamma_1 *_\Lambda \Gamma_2$ together with group homomorphisms $j_k : \Gamma_k \to \Gamma_1 *_\Lambda \Gamma_2$ for $k = 1, 2$ satisfying the following universal property : for any group

Ξ and any two group homomorphisms $\phi_k : \Gamma_k \to \Xi$, $k = 1, 2$, such that $\phi_1 \circ i_1 = \phi_2 \circ i_2$, there is a unique group homomorphism $\phi : \Gamma_1 *_\Lambda \Gamma_2 \to \Xi$ such that the diagram

$$\begin{tikzcd}
\Lambda \arrow[r, "i_1"] \arrow[d, "i_2"] & \Gamma_1 \arrow[d, "j_1"] \arrow[ddr, bend left, "\phi_1"] & \\
\Gamma_2 \arrow[r, "j_2"] \arrow[drr, bend right, "\phi_2"'] & \Gamma_1 *_\Lambda \Gamma_2 \arrow[dr, "\phi"] & \\
& & \Xi
\end{tikzcd}$$

commutes. The group $\Gamma_1 *_\Lambda \Gamma_2$ is called the *free product of* Γ_1 *and* Γ_2 *amalgamated over* Λ. One can give a complete description of the elements of these groups (we refer the reader to [Ser77] for a comprehensive treatment of this construction).

Proposition 1.2.22. *Let Γ_1, Γ_2 be discrete groups and let Λ be a common subgroup. We identify the groups with their images in $\Gamma_1 *_\Lambda \Gamma_2$ under the maps j_1 and j_2. For $k = 1, 2$, let us fix systems S_k of representatives of the right classes of Γ_k modulo Λ with the neutral element e_k of Γ_k as a representative of the class Λ. Then, any element $g \in \Gamma_1 *_\Lambda \Gamma_2$ can be written in a unique way as*

$$g = g_1 g_2 \dots g_n h,$$

where $g_i \in S_{l_i} \setminus \{1\}$, $l_i \neq l_{i+1}$ and $h \in \Lambda$.

Free products – the operator algebra case

There is a similar construction, starting with two algebras A_1 and A_2 and a common subalgebra B, yielding the *algebraic* free product amalgamated over B, denoted $A_1 \circledast_B A_2$. We will not explain this construction but only use it to define free products of C*-algebras. Let A_1 and A_2 be C*-algebras and let B be a common C*-subalgebra of A_1 and A_2, i.e. we have injective $*$-homomorphisms $i_k : B \to A_k$ for $k = 1, 2$. In this situation, the algebraic free product of A_1 and A_2 amalgamated over B, $A_1 \circledast_B A_2$, is a $*$-algebra. Its envelopping C*-algebra (whose existence was first proved in [Bla78, Sec 3]) is called the *maximal* free product of A_1 and A_2 amalgamated over B and is denoted $A *_B^{max} A_2$. One can prove (see [VDN, Sec 1.4 and Sec 1.5] for details) that the natural maps $j_k : A_k \to A_1 *_B^{max} A_2$ for $k = 1, 2$ are in fact injective $*$-homomorphisms and we can thus identify A_1 and A_2 with their images. The definition of the maximal amalgamated free product implies that it satisfies a universal property.

Proposition 1.2.23. *Let C be a C*-algebra together with $*$-homomorphisms $\phi_k : A_k \to C$, for $k = 1, 2$, such that $\phi_1 \circ i_1 = \phi_2 \circ i_2$. Then, there is a unique $*$-homomorphism*

$\phi : A_1 *_B^{max} A_2 \to C$ such that the following diagram commutes :

$$\begin{tikzcd}
B \arrow[r, "i_1"] \arrow[d, "i_2"] & A_1 \arrow[d, "j_1"] \arrow[ddr, bend left, "\phi_1"] & \\
A_2 \arrow[r, "j_2"] \arrow[drr, bend right, "\phi_2"'] & A_1 *_B^{max} A_2 \arrow[dr, "\phi"] & \\
& & C
\end{tikzcd}$$

Remark 1.2.24. Let us give an alternative way of constructing the maximal amalgamated free product. We start by forming the maximal free product $A_1 *^{max} A_2$ of A_1 and A_2 without amalgamation. Then we form the (closed two-sided) ideal I of $A_1 *^{max} A_2$ generated by all the elements of the form $j_1 \circ i_1(x) - j_2 \circ i_2(x)$ for $x \in B$. Then, the universal property implies that there is an isomorphism $(A_2 *^{max} A_2)/I \simeq A_1 *_B^{max} A_2$.

Remark 1.2.25. Free products can also be used to recover tensor products. In fact, if $A_1 *^{max} A_2$ is the maximal free product of two C*-algebras A_1 and A_2 and if J is the (closed two-sided) ideal in $A_1 *^{max} A_2$ generated by all the commutators $[j_1(a_1), j_2(a_2)]$ for $a_1 \in A_1$ and $a_2 \in A_2$, then there is an isomorphism (again given by the universal property) $(A_1 *^{max} A_2)/J \simeq A_1 \otimes^{max} A_2$.

The construction of the reduced free product is more involved and does not work in general. It was first done by D. Avitzour in [Avi82] and independantly (in greater generality) by D. Voiculescu in [Voi85]. Using the same notations as before, we will assume the existence of conditional expectations $\mathbb{E}_k : A_k \to B$ for $k = 1, 2$ (this is no real restriction as far as discrete quantum groups are concerned). We will also assume the C*-algebras A_1, A_2 and B to be unital for the sake of simplicity. Our aim is to build a representation of the algebraic free product and then take the completion with respect to the operator norm. However, it will be more convenient to build this representation on a *Hilbert C*-module* rather than on a Hilbert space.

For $k = 1, 2$, denote by H_k the Hilbert B-module obtained by separating and completing A_k with respect to the B-valued inner product $\langle a, b\rangle = \mathbb{E}_k(a^*b)$. The action of A_k on itself by left multiplication induces a representation π_k on the algebra of bounded adjointable B-linear operators $\mathcal{B}(H_k)$ of H_k. We will denote by ξ_k the image of the unit of A_k in H_k. The triple (H_k, π_k, ξ_k) is the *GNS construction* associated to the conditional expectation $\mathbb{E}_k$. Note that

$$H_k^\circ = [\widehat{a}, a \in A_k, \mathbb{E}_k(a) = 0]$$

is the orthogonal complement of $B.\xi_k$ and is naturally endowed with an action of B. The image of an element $a \in A_k$ in H_k will be denoted $\widehat{a}$ (note that $\widehat{a}$ may be zero for some non-zero a). We can now build a representation of the algebraic amalgamated free product on the so-called Fock space. Let

$$\mathcal{F} = \mathbb{C}.\Omega \bigoplus \bigoplus_{n \geqslant 1, l_1 \neq l_2 \neq \cdots \neq l_n} H_{l_1}^\circ \otimes_B \cdots \otimes_B H_{l_n}^\circ$$

where $\otimes_B$ is the interior tensor product of Hilbert B-modules (with respect to the right action of B on one side and the restriction to B of the left action of A_k on the other side) and Ω is a norm one vector called the *vacuum vector*. Then, for $k = 1, 2$, if A_k° denotes the kernel of $\mathbb{E}_k$, an element $a_k \in A_k$ will act on an element $h_{l_1} \otimes \cdots \otimes h_{l_n}$ as

$$\widehat{a} \otimes h_{l_1} \otimes \cdots \otimes h_{l_n}$$

if $k \neq l_1$ and as

$$(\pi_k(a_k).h_{l_1} - \langle \xi_k, \pi_k(a_k).h_{l_1} \rangle \xi_k) \otimes h_{l_2} \cdots \otimes h_{l_n} + \langle \xi_k, \pi_k(a_k).h_{l_1} \rangle h_{l_2} \otimes \cdots \otimes h_{l_n}$$

if $k = l_1$.

This yields a representation of A_1 and A_2 coinciding on B, hence a representation of the algebraic amalgamated free product $A_1 \circledast_B A_2$ on $\mathcal{B}(\mathcal{F})$. Note that this representation is faithful if the conditional expectations $\mathbb{E}_1$ and $\mathbb{E}_2$ are GNS-faithful. Taking the norm-closure in $\mathcal{B}(\mathcal{F})$ yields a C*-algebra called the *reduced* free product of A_1 and A_2 amalgamated over B and denoted $A_1 *_B A_2$. Note that using the universal property of Proposition 1.2.23, we see that there is a canonical surjective map from the maximal to the reduced amalgamated free product.

Remark 1.2.26. There is also a notion of amalgamated free product for von Neumann algebras, that we now briefly describe. Start with two von Neumann algebras M_1 and M_2 together with a common von Neumann subalgebra N which is the image of conditional expectations. Considering them as C*-algebras, we can still perform the previous construction to obtain a representation of the algebraic amalgamated free product. Then, taking the bicommutant of the image of the algebraic free product yields a von Neumann algebra called the reduced amalgamated free product of the von Neumann algebras M_1 and M_2 over N and denoted $M_1 *_N M_2$.

Free products – the quantum group case

We will now use what precedes to define a notion of free products for discrete quantum groups. Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be compact quantum groups and let $\mathbb{H}$ be a compact quantum group such that there are injective $*$-homomorphisms $i_k : C_{\max}(\mathbb{H}) \to C_{\max}(\mathbb{G}_k)$, for $k = 1, 2$, intertwining the coproducts (i.e. $\widehat{\mathbb{H}}$ is a *common quantum subgroup* of $\widehat{\mathbb{G}}_1$ and $\widehat{\mathbb{G}}_2$). Then, the maximal amalgamated free product of $C_{\max}(\mathbb{G}_1)$ and $C_{\max}(\mathbb{G}_2)$ over $C_{\max}(\mathbb{H})$ can be endowed with a compact quantum group structure such that the injective $*$-homomorphisms j_k intertwine the coproducts. In fact, if we denote, for $k = 1, 2$ the coproduct of $\mathbb{G}_k$ by Δ_k, then setting

$$\Delta \circ j_k(x) = (j_k \otimes j_k) \circ \Delta_k(x)$$

for any $x \in C_{\max}(\mathbb{G}_k)$ defines a coproduct on the algebraic amalgamated free product. This map can then be extended by universality to the maximal amalgamated free product. It is easy to check that $\mathbb{K} = (C_{\max}(\mathbb{G}_1) *^{max}_{C_{\max}(\mathbb{H})} C_{\max}(\mathbb{G}_2), \Delta)$ is a compact quantum group. It was proven in [Wan95a, Thm 3.8] that its Haar state is the free product of the Haar states on $\mathbb{G}_1$ and $\mathbb{G}_2$. From this, one can prove that the reduced free product of $C_{\mathrm{red}}(\mathbb{G}_1)$ and $C_{\mathrm{red}}(\mathbb{G}_2)$ amalgamated over $C_{\mathrm{red}}(\mathbb{H})$ with respect to the conditional expectations coming from Proposition 1.2.12 is exactly the C*-algebra of the reduced form of the compact quantum group obtained by the universal free product.

Remark 1.2.27. One can also prove that the von Neumann algebra $L^\infty(\mathbb{K})$ is isomorphic to the amalgamated free product $L^\infty(\mathbb{G}_1) *_{L^\infty(\mathbb{H})} L^\infty(\mathbb{G}_2)$.

Example 1.2.28. Let Γ_1 and Γ_2 be discrete groups, let Λ be a common discrete subgroup and perform this construction with the compact quantum groups $\widehat{\Gamma}_1$, $\widehat{\Gamma}_2$ and $\widehat{\Lambda}$. It is well-known that the C*-algebra $C_r^*(\Gamma_1) *_{C_r^*(\Lambda)} C_r^*(\Gamma_2)$ is isomorphic to the reduced group algebra $C_r^*(\Gamma_1 *_\Lambda \Gamma_2)$, and one can even prove that there is an isomorphism of compact quantum groups

$$\widehat{\Gamma}_1 *_{\widehat{\Lambda}} \widehat{\Gamma}_2 \simeq \widehat{\Gamma_1 *_\Lambda \Gamma_2}.$$

By analogy with this example, the discrete quantum group $\widehat{\mathbb{K}}$ will be called the *amalgamated free product* of $\widehat{\mathbb{G}}_1$ and $\widehat{\mathbb{G}}_2$ over $\widehat{\mathbb{H}}$ and denoted $\widehat{\mathbb{G}}_1 *_{\widehat{\mathbb{H}}} \widehat{\mathbb{G}}_2$. There is no easy way to describe the representation theory of an amalgamated free product of quantum groups in terms of the representation theories of the quantum groups involved (see e.g. the example in page 129 of [Ver04]). Nevertheless, Proposition 1.2.22 has an analogue [Wan95a, Thm 3.10] if one restricts to the case of free products without amalgamation.

Theorem 1.2.29 (Wang). *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be compact quantum groups and choose a system of representatives $(u_k^\alpha)_\alpha$ of the irreducible representations of $\mathbb{G}_k$ for $k = 1, 2$. Then, the irreducible representations of the free product are exactly the trivial representation and the representations of the form*

$$v_{l_1}^{\alpha_1} \otimes v_{l_2}^{\alpha_2} \otimes \cdots \otimes v_{l_n}^{\alpha_n},$$

where $l_i \neq l_{i+1}$.

Remark 1.2.30. Note that in the general case, the representations described above are still representations of the amalgamated free product and that one can easily prove that they *contain* all irreducible representations. In other words, $\mathrm{Irr}(\mathbb{G}_1) \cup \mathrm{Irr}(\mathbb{G}_2)$ *generates* the category of finite-dimensional representations.

1.2.4 Universal quantum groups

We end this section by introducing two important families of compact (or discrete) quantum groups which will be studied in Chapter 3. Both family could be called "universal quantum groups", though, as we will see, the reasons for this differ. We want to emphasize the fact that the terminology "universal quantum groups" is non-standard and may have various meaning depending on authors. As far as we are concerned, a universal quantum group will be one of the quantum groups that we are going to define and study hereafter.

Free quantum groups

Free unitary and free orthogonal quantum groups were first defined by A. Van Daele and S. Wang in [VDW96, Wan95a] and the definition was later slightly modified by T. Banica in [Ban96]. We will recall the definition and main properties of these free quantum groups. Note that the term "free" or "free orthogonal" is a little ambiguous. It is for instance used in the context of *easy quantum groups* (see e.g. [BS09]) to denote several different types of quantum groups. In this dissertation, we will adopt the following restricted sense : a free quantum group is one the quantum groups defined in Definition 1.2.34.

Definition 1.2.31. Let $N \in \mathbb{N}$ and let $F \in GL_N(\mathbb{C})$. We denote by $A_u(F)$ the universal unital C*-algebra generated by N^2 elements $(u_{i,j})$ such that the matrices $u = [u_{i,j}]$ and $F\overline{u}F^{-1}$ are *unitary*.

Remark 1.2.32. In the previous definition, we implicitely identified the matrix F with the element $1 \otimes F \in C(\mathbb{G}) \otimes M_N(\mathbb{C})$. This identification will always be made in this dissertation.

Definition 1.2.33. Let $N \in \mathbb{N}$ and let $F \in GL_N(\mathbb{C})$ such that $F\overline{F} \in \mathbb{R}.\mathrm{Id}$. We denote by $A_o(F)$ the universal unital C*-algebra generated by N^2 elements $(v_{i,j})$ such that the matrix $v = (v_{i,j})$ is *unitary* and $v = F\overline{v}F^{-1}$.

One can easily check, using the universality of the constructions, that there is a unique coproduct Δ_u (resp. Δ_o) on $A_u(F)$ (resp. $A_o(F)$) such that for all i, j,

$$\Delta_u(u_{i,j}) = \sum_{k=1}^{N} u_{i,k} \otimes u_{k,j}$$

$$\Delta_o(v_{i,j}) = \sum_{k=1}^{N} v_{i,k} \otimes v_{k,j}$$

Endowing these C*-algebras with the respective coproduct yields compact quantum groups.

Definition 1.2.34. A pair $(A_u(F), \Delta_u)$ is called a *free unitary quantum group* and will be denoted $U^+(F)$. A pair $(A_o(F), \Delta_o)$ is called a *free orthogonal quantum group* and will be denoted $O^+(F)$. Their discrete duals will be denoted respectively $\mathbb{F}U^+(F)$ and $\mathbb{F}O^+(F)$.

Remark 1.2.35. The restriction on the matrix F in the definition of $O^+(F)$ is equivalent to requiring the fundamental representation v to be irreducible. That assumption is necessary in order to get a nice description of the representation theory of $O^+(F)$.

Remark 1.2.36. The case $F = I_N$ is peculiar and has been studied in more details than the others. The associated compact quantum groups are often denoted U_N^+ and O_N^+ in the litterature. We will use these notations and denote their discrete duals by $\mathbb{F}U_N^+$ and $\mathbb{F}O_N^+$.

These quantum groups are "universal" in a strong sense that we are going to explain now. First recall the following definition [Wor87a, Def 1.1].

Definition 1.2.37. A *compact matrix pseudogroup* is a pair $(\mathbb{G}, w)$ where $\mathbb{G}$ is a compact quantum group and w is a finite dimensional representation of $\mathbb{G}$, called the *fundamental representation*, such that the coefficients of w generate $C(\mathbb{G})$ as a C*-algebra.

Every compact matrix pseudogroup is a "compact quantum subgroup" of a free unitary quantum group. In fact, if $(\mathbb{G}, w)$ is a compact matrix pseudogroup, then there exists a matrix F such that $F\overline{w}F^{-1}$ is unitary (because the contragredient of a unitary representation is always unitarizable). Thus, by universality, there is a surjective map

$$A_u(F) \longrightarrow C(\mathbb{G})$$

sending $u_{i,j}$ to $w_{i,j}$. Such a map obviously intertwines the coproducts. Assume moreover that the fundamental representation w is equivalent to its contragredient representation $\overline{w}$. Then, if F is a matrix such that $F\overline{w}F^{-1} = w$, there is a similar surjective map

$$A_o(F) \longrightarrow C(\mathbb{G}).$$

This fact is somehow an analogue of the following classical fact : any compact subgroup of $GL(V)$, with V a finite-dimensional complex vector space, embeds into some $U(N)$ and any compact subgroup of $GL(W)$, with W a finite-dimensional real vector space, embeds into some $O(N)$. In this sense, we can see $U^+(F)$ and $O^+(F)$ as quantum generalizations of the usual unitary and orthogonal groups. Let us give another intuition for that from [BG10].

Definition 1.2.38. Let $N \in \mathbb{N}$. The *free sphere* of dimension N is the universal C*-algebra $C(\mathbb{S}^N)$ generated by N self-adjoint elements $x_1, \dots, x_N$ such that

$$\sum_i x_i^2 = 1.$$

Remark 1.2.39. According to the philosophy of non-commutative geometry, one should think of the C*-algebra $C(\mathbb{S}^N)$ as the "algebra of continuous functions" on the "free sphere" $\mathbb{S}^N$. In fact, the abelianization of $C(\mathbb{S}^N)$ is exactly the algebra of continuous functions on the classical sphere S^N. We have simply "liberated" the generators from the commutativity assumption, hence the name.

There is a natural action α of O_N^+ on $C(\mathbb{S}^N)$ defined by

$$\alpha(x_i) = \sum_{j=1}^N u_{i,j} \otimes x_j.$$

Moreover, for any compact quantum group $\mathbb{G}$ and any "isometric" action β of $\mathbb{G}$ on $C(\mathbb{S}^N)$, there is a unique $*$-homomorphism

$$\Phi : C(O_N^+) \to \mathbb{G}$$

intertwining the coproducts and the actions (i.e. such that $(\Phi \otimes \imath) \circ \alpha = \beta$). In other words, in exactly the same way as $O(N)$ is the isometry group of the classical sphere S^N, O_N^+ is the *quantum isometry group* of the free sphere. Here "isometric" refers to the standard compact spectral triple with which free spheres can be endowed and we refer the reader [BG10] for details concerning this construction and the proofs.

The representation theory of free orthogonal quantum groups was computed by T. Banica in [Ban96], and appeared to be the same as the representation theory of $SU(2)$.

Theorem 1.2.40 (Banica)**.** *One can index the equivalence classes of irreducible representations of $O^+(F)$ by the set $\mathbb{N}$ of integers (u^0 being the trivial representation and $u^1 = u$ the fundamental one). Each irreducible representation is isomorphic to its contragredient and the tensor product is given (inductively) by*

$$u^1 \otimes u^n = u^{n-1} \oplus u^{n+1}.$$

Moreover,

$$\dim_q(u^n) = \frac{q^{n+1} - q^{-n-1}}{q - q^{-1}},$$

where $q + q^{-1} = \mathrm{Tr}(F^*F)$ and $0 < q \leqslant 1$. We will use the shorthand notation D_n for $\dim_q(u^n)$ in the sequel.

Remark 1.2.41. One always has $q + q^{-1} \geqslant N$.

Remark 1.2.42. The analogy of $O(F)^+$ with $SU(2)$ is even stronger. Let $F \in GL_2(\mathbb{C})$ such that $F\overline{F} = c.\mathrm{Id}$, let q be the real number defined in Theorem 1.2.40. Then, $O^+(F)$ is isomorphic to S.L. Woronowicz's quantum $SU(2)$ group $SU_{-\operatorname{sign}(c)q}(2)$ (see [Wor87b] for the definition of $SU_q(2)$).

The representation theory of $U^+(F)$ was also explicitely computed by T. Banica in [Ban97]. However, we will only need the following result [Ban97, Thm 1].

Theorem 1.2.43 (Banica)**.** *Let $F \in GL_N(\mathbb{C})$ be such that $F\overline{F} \in \mathbb{R}.\mathrm{Id}$. Then, the discrete quantum group $\mathbb{F}U^+(F)$ is the* free complexification *of $\mathbb{F}O^+(F)$, and in particular a quantum subgroup of $\mathbb{Z} * \mathbb{F}O^+(F)$.*

It is natural to ask for a classification of the quantum groups $O^+(F)$ and $U^+(F)$ depending on the matrix F. This question was investigated in [Wan02] to which we refer the reader for details.

Theorem 1.2.44 (Wang)**.** *Let $N \in \mathbb{N}$ and $F, F' \in GL_N(\mathbb{C})$ be such that $F\overline{F} \in \mathbb{R}.\mathrm{Id}$ and $F'\overline{F}' \in \mathbb{R}.\mathrm{Id}$. Then, the compact quantum groups $O^+(F)$ and $O^+(F')$ are isomorphic if and only if there is a unitary $w \in GL_N(\mathbb{C})$ such that $F' = w^tFw$.*

S. Wang also gives in [Wan02] a complete system of representatives of the isomorphism classes. We will rather use the equivalent description of J. Bichon, A. De Rijdt and S. Vaes given in [BDRV06, Rmk 5.7]. Let $F \in GL_N(\mathbb{C})$ be such that $F\overline{F} = \pm\,\mathrm{Id}$, then there are real numbers $0 < \lambda_1 \leqslant \cdots \leqslant \lambda_k < 1$ and a unitary w such that, denoting the diagonal matrix with coefficients $\lambda_1, \dots, \lambda_k$ by $D(\lambda_1, \dots, \lambda_k)$,

$$w^tFw = \begin{pmatrix} 0 & D(\lambda_1, \dots, \lambda_k) & 0 \\ D(\lambda_1, \dots, \lambda_k)^{-1} & 0 & \\ 0 & 0 & \mathrm{Id}_{N-2k} \end{pmatrix}$$

if $F\overline{F} = +\,\mathrm{Id}$ and

$$w^tFw = \begin{pmatrix} 0 & D(\lambda_1, \dots, \lambda_k) \\ -D(\lambda_1, \dots, \lambda_k)^{-1} & 0 \end{pmatrix}$$

if $F\overline{F} = -\,\mathrm{Id}$.

These free unitary and free orthogonal quantum groups have attracted much attention in the past years since they provide a toy-model for the use of techniques from geometric group theory in the context of discrete quantum groups. In fact, they can be thought of as analogues of free groups in the quantum setting. To see this, let I be the (closed two-sided) ideal in $A_o(I_N)$ generated by the elements $u_{i,j}$ with $i \neq j$. Then, the quotient C*-algebra is exactly the universal C*-algebra of the group $(\mathbb{Z}/2\mathbb{Z})^{*N}$ (i.e. the N-fold free product of $\mathbb{Z}/2\mathbb{Z}$ by itself) and the quotient map intertwines the coproducts. This means that $(\mathbb{Z}/2\mathbb{Z})^{*N}$ is a quotient of $\mathbb{F}O_N^+$. Similarly, the free group $\mathbb{F}_N$ is a quotient of U_N^+. Let us end this section by summarizing the known structure results concerning the quantum groups O_N^+, which will illustrate the previous analogy with free groups.

- The von Neumann algebras $L^\infty(O_N^+)$ are full prime solid non-amenable type II_1-factors with the Akemann-Ostrand property [VV07].
- The von Neumann algebras $L^\infty(O_N^+)$ have the Haagerup property [Bra12a] (and thus do not have property (T), which had already been proven in [Fim10]).
- The C*-algebras $C_{\mathrm{red}}(O_N^+)$ are exact and simple (for $N \geqslant 3$) [VV07].
- The C*-algebras $C_{\mathrm{red}}(O_N^+)$ are projectionless [Voi11].
- The discrete quantum groups $\mathbb{F}O_N^+$ have the property of rapid decay [Ver07].
- The discrete quantum groups $\mathbb{F}O_N^+$ are K-amenable and satisfy the strong Baum-Connes property [Voi11].
- $K_0(C_{\mathrm{red}}(O_N^+)) = \mathbb{Z} = K_0(C_r^*(\mathbb{F}_N))$
  $K_1(C_{\mathrm{red}}(O_N^+)) = \mathbb{Z} \neq \mathbb{Z}^N = K_1(C_r^*(\mathbb{F}_N))$ [Voi11].
- $\beta_1^{(2)}(\mathbb{F}O_N^+) = 0 \neq N - 1 = \beta_1^{(2)}(\mathbb{F}_N)$ [Ver12].

Quantum automorphism groups of finite spaces

Compact quantum automorphism groups of finite spaces were first introduced by S. Wang in [Wan98]. Here by "finite space" we mean "non-commutative finite space" i.e. any finite dimensional C*-algebra. Before introducing them in full generality, let us give one of the most important examples.

Definition 1.2.45. Let $N \in \mathbb{N}$ and let $A_s(N)$ be the universal C*-algebra generated by N^2 orthogonal projections $(u_{i,j})$ such that for all i_0 and j_0,

$$\sum_{j=1}^{N} u_{i_0,j} = 1 = \sum_{i=1}^{N} u_{i,j_0}.$$

Remark 1.2.46. The above description can be made closer to Definition 1.2.34. Let us say that a matrix $[u_{i,j}]$ is *magic orthogonal* if all its coefficients are orthogonal projections summing up to 1 on each row and each column. Then, $A_s(N)$ is the universal C*-algebra generated by N^2 elements $u_{i,j}$ such that the matrix $[u_{i,j}]$ is magic orthogonal.

Like in the case of free quantum groups, the universality allows us to endow the C*-algebra $A_s(N)$ with a coproduct Δ_s by the formula

$$\Delta_s(u_{i,j}) = \sum_{k=1}^{N} u_{i,k} \otimes u_{k,j}$$

The pair $S_N^+ = (A_s(N), \Delta_s)$ is called the *quantum permutation group* on N elements. S. Wang noticed that for $1 \leqslant N \leqslant 3$, the C*-algebra $A_s(N)$ is in fact equal to $C(S_N)$, the algebra of continuous functions on the classical permutation group. However, he also proved that for $N \geqslant 4$, the C*-algebra $A_s(N)$ is infinite-dimensional, giving rise to some new, purely quantum phenomena. Note that $C(S_N)$ can always be recovered as the maximal abelian quotient of $A_s(N)$.

This construction leads to several further topics. One possible direction, which will not be discussed in this dissertation, is the study of more general quantum permutation groups, for example quantum symmetry groups of graphs, as studied in [Bic03] and [BB07] (see also the survey [BBC07]).

Another direction is based on [Wan98, Thm 3.1]. First notice that S_N^+ has a natural action ρ on $\mathbb{C}^N = \sum_i \mathbb{C}.e_i$, defined by

$$\rho(e_i) = \sum_{j=1}^{N} u_{i,j} \otimes e_j.$$

Moreover, this action is universal in a strong sense.

Theorem 1.2.47 (Wang). *The compact quantum group S_N^+ is the compact quantum automorphism group of the C*-algebra $\mathbb{C}^N$, i.e. for any compact quantum group $\mathbb{G}$ together with an action ρ' on $\mathbb{C}^N$, there is a unique $*$-homomorphism $\Phi : C(S_N^+) \to C(\mathbb{G})$ intertwining the coproducts and such that $(\Phi \otimes \imath) \circ \rho = \rho'$.*

This suggests to construct compact quantum automorphism groups for other finite-dimensional C*-algebras. However, S. Wang made the fundamental remark that there is no universal quantum group acting on such a C*-algebra in general, but that there is one as soon as the preservation of a state is imposed. In other words, any *finite non-commutative measure space* has a compact quantum automorphism group. The definition was later generalized by Banica in [Ban99c] and [Ban02]. It relies on the following fundamental result.

Theorem 1.2.48 (Wang, Banica)**.** *Let B be a finite dimensional C*-algebra and let ψ be a state on B. Then, there is a unique compact quantum group $\mathbb{G}(B, \psi)$, together with an action ρ on (B, ψ), such that the following universal property holds : if $\mathbb{H}$ is a compact quantum group and if ρ' is an action of $\mathbb{H}$ on (B, ψ), then there exists a unique $*$-homomorphism $\Phi : C(\mathbb{G}(B, \psi)) \to C(\mathbb{H})$ intertwining the coproducts and such that $\rho' = (\imath \otimes \Phi) \circ \rho$.*

The dual discrete quantum group of $\mathbb{G}(B, \psi)$ will be denoted $\widehat{\mathbb{G}}(B, \psi)$.

Remark 1.2.49. It was proven by A. De Rijdt in [DR07, Thm 3.6.3] that $\mathbb{G}(B_1, \psi_1)$ is isomorphic to $\mathbb{G}(B_2, \psi_2)$ as compact quantum groups if and only if there exists an isomorphism $\varphi : B_1 \to B_2$ such that $\psi_2 \circ \varphi = \psi_1$.

The compact quantum group $\mathbb{G}(B, \psi)$ will be called the *compact quantum automorphism group* of (B, ψ). The term "compact" will be omitted in the sequel since we only work with compact quantum groups. Let us detail two particular examples of this construction.

- Let N be an integer and $B = \mathbb{C}^N$. If ψ is the integration with respect to the uniform measure on $\mathbb{C}^N$, then the compact quantum group $\mathbb{G}(B, \psi)$ is isomorphic to the quantum permutation group S_N^+.
- Let N be an integer and equip $B = M_N(\mathbb{C})$ with the standard (non-normalized) trace τ. Then, it is proved in [Ban99c, Cor 4.1] that the C*-algebra $C_{\max}(\mathbb{G}(B, \tau))$ is isomorphic to the C*-subalgebra of $C_{\max}(O_N^+)$ generated by elements of the form $u_{i,j}u_{k,l}^*$, where u is the fundamental representation of O_N^+. Moreover, this isomorphism intertwines the coproducts.

Note that if $\dim(B) \leqslant 3$, then B is commutative and we get the usual permutation group S_N. For greater dimensions, the C*-algebra $C(\mathbb{G}(B, \psi))$ is infinite dimensional and the compact quantum group $\mathbb{G}(B, \psi)$ has an amenable dual if and only if $\dim(B) \leqslant 4$. This is a direct consequence of the computation of the fusion rules and the quantum Kesten criterion (see [Ban99b, Thm 6.1] and [Ban99a, Thm 5.1], or [Kye08, Thm 4.4] for the most general statement).

Of course, the compact quantum group $\mathbb{G}(B, \psi)$ depends on the state ψ, but it was proven by T. Banica in [Ban99c] and [Ban02] that under some technical assumption on ψ, the fusion rules can be nicely described and in fact do not even depend on the C*-algebra B. Let us first define the family of states we are interested in.

Definition 1.2.50. A state ψ on a C*-algebra B is said to be a δ*-form* if the multiplication map $m : L^2(B, \psi) \otimes L^2(B, \psi) \to L^2(B, \psi)$ satisfies $mm^* = \delta^2.\mathrm{Id}$.

Remark 1.2.51. If we decompose the finite-dimensional C*-algebra B as a sum of matrix algebras $\oplus_{i=1}^p M_{n_i}(\mathbb{C})$, the state ψ is given by

$$\psi(x_1 + \cdots + x_p) = \sum_{i=1}^{p} \mathrm{Tr}(F_i x_i)$$

if every x_i belongs to $M_{n_i}(\mathbb{C})$. Note that the matrices F_i are positive definite. Then, ψ is a δ-form if and only if $\mathrm{Tr}(F_i^{-1}) = \delta^2$ for every i.

Example 1.2.52. Let ψ be a *tracial* δ-form. Then, $F_i = \lambda_i \mathrm{Id}$ for every i. The δ-form condition gives $n_i\lambda_i^{-1} = \delta^2$ and the state condition yields

$$\sum_{i=1}^{p} n_i \lambda_i = 1.$$

From this, we easily deduce that $\lambda_i = n_i/\dim(B)$ and $\delta^2 = \dim(B)$. Thus, there is exactly one δ-trace on a finite-dimensional C*-algebra, called the *canonical δ-trace*. Note that this trace can be recovered as the usual non-normalized trace on the matrix algebra $\mathcal{B}(B)$ restricted to B which is embedded through the natural action by left multiplication.

Theorem 1.2.53 (Banica). *Let B be a finite dimensional C*-algebra and let ψ be a δ-form on B. Then, one can index the equivalence classes of irreducible representations of $\mathbb{G}(B,\psi)$ by the set $\mathbb{N}$ of integers (u^0 being the trivial representation and $u^1 \oplus u^0 = u$ the fundamental one), each one is isomorphic to its contragredient and the tensor product is given (inductively) by*

$$u^1 \otimes u^n = u^{n-1} \oplus u^n \oplus u^{n+1}.$$

Moreover, the quantum dimensions of the irreducible representations are given by

$$\dim_q(u^n) = \frac{q^{2n+1} - q^{-2n-1}}{q - q^{-1}} = D_{2n},$$

where $q + q^{-1} = \delta$ and $0 < q \leqslant 1$.

Remark 1.2.54. According to the comment after Proposition 1.1.39, the compact quantum group $\mathbb{G}(B,\psi)$ is of Kac type if and only if the quantum dimension of its fundamental representation is equal to its classical dimension, i.e.

$$(q + q^{-1})^2 = \dim_q(u) = \dim(u) = \dim(B),$$

i.e. $\delta = \sqrt{\dim(B)}$. Using the notations of Remark 1.2.51, we have by the Cauchy-Schwarz inequality

$$N_i^2 \leqslant \operatorname{Tr}(F_i)\operatorname{Tr}(F_i^{-1})$$

for every $1 \leqslant i \leqslant p$. Summing up these inequalities, we get

$$\sum_{i=1}^{p} N_i^2 \leqslant \sum_{i=1}^{p} \operatorname{Tr}(F_i)\operatorname{Tr}(F_i^{-1}) = \delta^2 \sum_{i=1}^{p} \operatorname{Tr}(F_i) = \dim(B) = \sum_{i=1}^{p} N_i^2.$$

Hence, the Cauchy-Schwarz inequality is an equality for every i, which means that all the matrices F_i are scalar multiples of the identity. Otherwise said, the compact quantum group $\mathbb{G}(B,\psi)$ is of Kac type if and only if ψ is the canonical δ-trace on B.

The fusion rules of $\mathbb{G}(B,\psi)$ are in fact the fusion rules of $SO(3)$. This is in some sense the next step of difficulty after the fusion rules of $SU(2)$. This is the reason why it is natural to try to study in this context the properties already proved for free quantum groups. Less is known, but restricting to C*-algebras B endowed with the canonical δ-trace τ, we can list several properties, all of which were proved in [Bra13].

- The von Neumann algebras $L^\infty(\mathbb{G}(B,\tau))$ are full prime non-amenable type II_1 factors with the Akemann-Ostrand property (for $N \geqslant 8$).
- The von Neumann algebras $L^\infty(\mathbb{G}(B,\tau))$ have the Haagerup property (and thus cannot have property (T)).
- The C*-algebras $C_{\mathrm{red}}(\mathbb{G}(B,\tau))$ are exact and simple (for $N \geqslant 8$).
- The discrete quantum groups $\widehat{\mathbb{G}}(B,\tau)$ have the property of rapid decay.

Let us also mention that it was proven by C. Voigt in [Voi13, Thm 5.2] that the discrete dual of $\mathbb{G}(M_N(\mathbb{C}),\tau)$ is K-amenable and that

$$K_0(C_{\mathrm{red}}(\mathbb{G}(M_N(\mathbb{C}),\tau))) = \mathbb{Z} \oplus \mathbb{Z}_n \text{ and } K_1(C_{\mathrm{red}}(\mathbb{G}(M_N(\mathbb{C}),\tau))) = \mathbb{Z}.$$

1.3 Complete positivity and complete boundedness

We conlude this preliminary chapter with a few technical elements from the theory of completely bounded maps which will be central in this dissertation. Completely bounded maps are one of the fundamental notion of the theory of operator spaces and the book [Pis03] is probably the best reference on the subject. However, we will need very few results of the theory of completely bounded maps, all of which are contained in [BO08, Sec 1.5 and Appendix B].

Though we are mainly interested in complete boundedness, we start by studying the stronger notion of complete positivity which will also be needed.

Definition 1.3.1. Let A and B be C*-algebras. A linear map $T : A \to B$ is said to *completely positive* (in short c.p.) if for any integer $n \in \mathbb{N}$, the map

$$T \otimes \mathrm{Id}_n : A \otimes M_n(\mathbb{C}) \to B \otimes M_n(\mathbb{C})$$

is positive. If moreover T is unital, it is said to be *unital completely positive*, in short u.c.p.

Remark 1.3.2. Using the isomorphism between $A \otimes M_n(\mathbb{C})$ and $M_n(A)$ (and similarly for B), we see that T is completely positive if and only if for any positive matrix $[a_{i,j}]$ with coefficients in A, the matrix $[T(a_{i,j})]$ is again positive.

Example 1.3.3. Any $*$-homomorphism is completely positive since its extensions to matrix algebras are again $*$-homomorphisms and hence preserve positivity.

Example 1.3.4. Let $\varphi : A \to \mathbb{C}$ be a positive linear form, then φ is completely positive. In fact, for any integer n, let $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$ and let $[a_{i,j}] \in M_n(A)$ be positive. Then,

$$\langle (\varphi \otimes \mathrm{Id}_n)([a_{i,j}])\xi, \xi\rangle = \sum_{i,j=1}^{n} \overline{\xi_i}\xi_j \varphi(a_{i,j}) = \varphi\left(\sum_{i,j=1}^{n} \overline{\xi_i}\xi_j a_{i,j}\right) = \varphi(\xi^*[a_{i,j}]\xi) \geqslant 0.$$

Example 1.3.5. Let H be a Hilbert space and let $\pi : A \to \mathcal{B}(H)$ be a (non-necessarily faithful) representation. Then, for any other Hilbert space K and any bounded linear map $V : H \to K$, the map

$$T : x \mapsto V^*\pi(x)V$$

is completely positive.

The last example is in fact very general, according to the celebrated *Stinespring's factorization theorem* (see e.g. [BO08, Thm 1.5.3] for a proof).

Theorem 1.3.6 (Stinespring)**.** *Let A be a unital C*-algebra, let H be a Hilbet space and let $T : A \to \mathcal{B}(H)$ be a completely positive map. Then, there exists another Hilbert space K, a representation $\pi : A \to \mathcal{B}(K)$ and a bounded linear map $V : H \to K$ such that for all $x \in A$,*

$$T(x) = V^*\pi(x)V.$$

Let us illustrate this theorem with an easy but useful consequence, the so-called *Kadison's inequality.*

Proposition 1.3.7. *Let A be a unital C*-algebra, let H a Hilbert space and let $T : A \to \mathcal{B}(H)$ a* unital *completely positive map. Then, for any $x \in A$,*

$$T(x^*x) \geqslant T(x^*)T(x).$$

We now turn to completely bounded maps, parralleling what we have just done for completely positive maps.

Definition 1.3.8. Let A and B be two C*-algebras. A linear map $T : A \to B$ is said to *completely bounded* (in short c.b.) if there exists a constant $C > 0$ such that for any integer $n \in \mathbb{N}$, the map

$$T \otimes \mathrm{Id}_n : A \otimes M_n(\mathbb{C}) \to B \otimes M_n(\mathbb{C})$$

has norm less than C. The infimum of the constants C satisfying this property is called the *completely bounded norm* of T and denoted $\|T\|_{cb}$.

Example 1.3.9. Any $*$-homomorphism is completely bounded since all its extensions to matrix algebras are again $*$-homomorphisms, hence contractive.

Example 1.3.10. Let $\varphi : A \to \mathbb{C}$ be a bounded linear form. Then, φ is completely bounded. In fact, for any integer $n \in \mathbb{N}$ and $[a_{i,j}] \in M_n(A)$, we have

$$\begin{aligned}
\|[(\varphi(a_{i,j})]\| &= \sup\{|\langle[\varphi(a_{i,j})]\eta, \xi\rangle|, \xi, \eta \in \mathbb{C}^n, \|\xi\| = \|\eta\| = 1\} \\
&= \sup\left\{\left|\sum_{i,j=1}^{n} \overline{\xi_i}\eta_j\varphi(a_{i,j})\right|, \sum_{i=1}^{n}|\xi_i|^2 = \sum_{j=1}^{n}|\eta_j|^2 = 1\right\} \\
&= \sup\left\{\left|\varphi\left(\sum_{i,j=1}^{n} \overline{\xi_i}\eta_j a_{i,j}\right)\right|, \sum_{i=1}^{n}|\xi_i|^2 = \sum_{j=1}^{n}|\eta_j|^2 = 1\right\} \\
&\leqslant \|\varphi\| \sup\left\{\left|\sum_{i,j=1}^{n} \overline{\xi_i}\eta_j a_{i,j}\right|, \sum_{i=1}^{n}|\xi_i|^2 = \sum_{j=1}^{n}|\eta_j|^2 = 1\right\} \\
&\leqslant \|\varphi\|\|[a_{i,j}]\|.
\end{aligned}$$

Example 1.3.11. Let H be a Hilbert space and let $\pi : A \to \mathcal{B}(H)$ be a (non-necessarily faithful) representation. Then, for any other Hilbert space K and for any bounded linear maps $P, Q : H \to K$, the map

$$T : x \mapsto Q^*\pi(x)P$$

is completely bounded and $\|T\|_{cb} \leqslant \|P\|\|Q\|$.

Remark 1.3.12. Combining this remark with Stinespring's extension theorem, we see that any completely positive map is completely bounded.

Again this example is very general. The analogue of Stinespring's factorization is called *Wittstock's factorization theorem* (see e.g. [BO08, Thm B.7] for a proof).

Theorem 1.3.13 (Wittstock)**.** *Let A be a unital C*-algebra, let H be a Hilbert space and let $T : A \to \mathcal{B}(H)$ be a completely bounded map. Then, there exists a Hilbert space K, a representation $\pi : A \to \mathcal{B}(K)$ and two bounded linear maps $P, Q : H \to K$ such that for all $x \in A$,*

$$T(x) = Q^*\pi(x)P$$

Let us end with two criteria concerning these properties. The first one is concerned with automatic complete boundedness and complete positivity. The proof in the positive case is very well-known and can be found e.g. in [Con00, Prop 34.6]. The argument also works in the bounded case though we did not suceed in finding a reference. That is the reason why we will sketch the proof.

Proposition 1.3.14. *Let A be a C*-algebra, let X be a compact topological space and let $T : A \to C(X)$ be a linear map. Then, T is completely bounded if and only if it is bounded (and $\|T\|_{cb} = \|T\|$). Moreover, T is completely positive if and only if it is positive.*

Proof. Let x be any point in X and let φ_x be the evaluation map at x. Then, for any $a \in A \otimes M_n(\mathbb{C})$,

$$(\varphi_x \otimes \mathrm{Id}_n) \circ (T \otimes \mathrm{Id}_n)(a) = [(\varphi_x \circ T) \otimes \mathrm{Id}_n](a)$$

If T is bounded, we know from Example 1.3.10 that the left hand side has norm less than $\|\varphi_x \circ T\|\|a\| \leqslant \|T\|\|a\|$. This means that the matrix-valued function $(T \otimes \mathrm{Id}_n)(a)$ is bounded with norm less than $\|T\|\|a\|$, hence the result. The proof of complete positivity is similar. $\square$

The second one allows us to derive complete positivity from complete boundedness.

Proposition 1.3.15. *Let A be a unital C*-algebra and let $T : A \to A$ be a linear map such that $T(1) = 1$ and $\|T\|_{cb} = 1$. Then, T is unital completely positive.*

Proof. It is well-known (see e.g. [Con00, Prop 33.9]) that a linear map S on a unital C*-algebra satisfying both $S(1) = 1$ and $\|S\| = 1$ is positive. Applying this to the map $T \otimes \mathrm{Id}_n$ for any integer n proves that they are all positive, hence the result. $\square$

Chapter 2

Weak amenability for discrete quantum groups

This chapter deals with the general theory of weak amenability. We first give a definition of a weakly amenable discrete quantum group, building on earlier works on multipliers for quantum groups. We then develop several aspects of the theory inspired from the classical setting. Although we are able to generalize several classical results to this setting, the picture remains quite incomplete and we comment in some details the defects of the quantum theory in several places.

The chapter is organized as follows :

- Section 2.1 is a brief review of the classical theory of weak amenability. Along the way, we also recall the corresponding notions for operator algebras which will be used in the sequel.
- Section 2.2 starts with some reformulation of the theory of multipliers in the context of discrete quantum groups. Though it has already appeared in the litterature for locally compact quantum groups, we give here a treatment which emphasizes the link with the classical theory of Herz-Schur multipliers on discrete groups. This leads us to the definition of weak amenability for discrete quantum groups. From this definition we draw several quite elementary consequences. We also discuss the problem of linking the Cowling-Haagerup constant of a discrete quantum group with the one of its associated operator algebras, although we were not able to provide new results in this regard. We then define relative amenability for discrete quantum groups and investigate its link with weak amenability. Finally, we give a similar, though shorter, treatment of the Haagerup property.
- Eventually, we turn in Section 2.3 to a more difficult problem, namely the link between weak amenability and free products. In fact, the basic strategy of the proof follows the classical one but the context of quantum groups forces us to rely on more operator algebraic techniques, thus involving careful estimates of norms of certain operators. To conclude, we explain how these results can be extended when amalgamation is allowed over some finite quantum subgroups. This section is mostly an extended version of our paper [Fre12].

2.1 The classical setting

We quickly review the fundamental results of the theory of weak amenability for classical discrete groups, to serve as a guideline for our treatment of the quantum theory. We refer the reader to [BO08, Chap 12] or to the survey [BN06] for a more comprehensive treatment with proofs.

Weak amenability, and more generally approximation properties, are based on the notion of Herz-Schur multipliers on groups. Let Γ be a discrete group and denote by

$$\lambda : \Gamma \to \mathcal{B}(\ell^2(\Gamma))$$

its left regular representation. From any bounded function

$$\varphi : \Gamma \to \mathbb{C},$$

one can construct a map m_φ on the linear span of $\{\lambda(g), g \in \Gamma\}$ by the formula

$$m_\varphi(\lambda(g)) = \varphi(g)\lambda(g).$$

The map m_φ is the *multiplier* associated to φ. It is then very natural to look for some criterion on φ ensuring that the map m_φ extends to a (completely) bounded or (completely) positive map on $C^*_r(\Gamma)$. The issue of complete positivity is a little easier to handle thanks to the following definition.

Definition 2.1.1. Let Γ be a discrete group and let $\varphi : \Gamma \to \mathbb{C}$ be bounded. The map φ is said to be *positive definite* if for any integer $n \in \mathbb{N}$ and for any $a_1, \dots, a_n \in \mathbb{C}$,

$$\sum_{k=0}^{n} \varphi(g_i g_j^{-1}) a_i \overline{a_j} \geqslant 0$$

A proof of the next proposition can be found e.g. in [BO08, Thm D.3].

Proposition 2.1.2. *Let Γ be a discrete group and let $\varphi : \Gamma \to \mathbb{C}$ be bounded. Then, m_φ extends to a completely positive map on $C^*_r(\Gamma)$ if and only if φ is positive definite.*

For complete boundedness, the answer is a bit more complicated but we still have a complete characterization, generally attributed to J. Gilbert (see [Jol92] for a proof).

Proposition 2.1.3. *Let Γ be a discrete group and let $\varphi : \Gamma \to \mathbb{C}$ be bounded. Then, m_φ extends to a completely bounded map on $C^*_r(\Gamma)$ if and only if there exists a Hilbert space K and two families $(\xi_s)_{s\in\Gamma}$ and $(\eta_t)_{t\in\Gamma}$ of vectors in K such that for all $s, t \in \Gamma$,*

$$\varphi(s) = \langle \eta_t, \xi_{st} \rangle.$$

If m_φ is completely bounded, we will say that it is a *Herz-Schur multiplier.*

Remark 2.1.4. The condition $\varphi(s) = \langle \eta_t, \xi_{st} \rangle$ is usually written $\varphi(st^{-1}) = \langle \eta_t, \xi_s \rangle$. However, the first formula will be more tractable when looking for a quantum analogue.

Example 2.1.5. It is easy to see that if φ has finite support, then m_φ is completely bounded. Using Proposition 2.1.3, one can even prove that any ℓ^2-function gives rise to a completely bounded multiplier. In fact, $\xi_s = g \mapsto \varphi(sg^{-1})\delta_g$ and $\eta_t = \delta_t$ are two suitable families of vectors in $\ell^2(\Gamma)$.

Before defining weak amenability, we would like to justify the word "weak" by giving one of the (many) equivalent characterizations of amenability.

Definition 2.1.6. A discrete group Γ is *amenable* if there exists a net of functions $\varphi_t : \Gamma \to \mathbb{C}$ such that

- φ_t as finite support for all t.
- $\varphi_t(x) \to 1$ for all $x \in \Gamma$.
- m_{φ_t} is completely positive for all t (or equivalently, φ_t is positive definite for all t).

Definition 2.1.7. A discrete group Γ is *weakly amenable* if there exists a net of functions $\varphi_t : \Gamma \to \mathbb{C}$ such that

- φ_t as finite support for all t.
- $\varphi_t(x) \to 1$ for all $x \in \Gamma$.
- $K := \limsup_t \|m_{\varphi_t}\|_{cb}$ is finite.

The lower bound of the constants K for all nets satisfying these properties is denoted $\Lambda_{cb}(\Gamma)$ and called the *Cowling-Haagerup constant* of Γ. By convention, $\Lambda_{cb}(\Gamma) = \infty$ if Γ is not weakly amenable.

Example 2.1.8. N. Ozawa proved in [Oza08] that all Gromov hyperbolic groups are weakly amenable.

Example 2.1.9. Let G be a simple Lie group and let Γ be a lattice (i.e. a discrete subgroup with finite covolume) in G. Then Γ is weakly amenable if and only if G is of real rank one, and its Cowling-Haagerup constant depends only on the group G :

- If $G = SO(n,1)$ or $G = SU(n,1)$, then $\Lambda_{cb}(\Gamma) = 1$.
- If $G = Sp(n,1)$, then $\Lambda_{cb}(\Gamma) = 2n - 1$.
- If $G = F_{4(-20)}$, then $\Lambda_{cb}(\Gamma) = 21$.

The reader is referred to [dCH85, CH89] for the proofs of these (highly non-trivial) facts.

Recall the following notions of weak amenability for operator algebras.

Definition 2.1.10. A C*-algebra A is said to be *weakly amenable* if there exists a net (T_t) of linear maps from A to itself such that

- T_t has finite rank for all t.
- $\|T_t(x) - x\| \to 0$ for all $x \in A$.
- $K := \limsup_t \|T_t\|_{cb}$ is finite.

The lower bound of the constants K for all nets satisfying these properties is denoted $\Lambda_{cb}(A)$ and called the *Cowling-Haagerup constant* of A. By convention, $\Lambda_{cb}(A) = \infty$ if the C*-algebra A is not weakly amenable.

A von Neumann algebra N is said to be *weakly amenable* if there exists a net (T_t) of normal linear maps from N to itself such that

- T_t has finite rank for all t.
- $(T_t(x) - x) \to 0$ in the weak-$*$ topology for all $x \in N$.
- $K := \limsup_t \|T_t\|_{cb}$ is finite.

The lower bound of the constants K for all nets satisfying these properties is denoted $\Lambda_{cb}(N)$ and called the *Cowling-Haagerup constant* of N. By convention, $\Lambda_{cb}(N) = \infty$ if the von Neumann algebra N is not weakly amenable.

Remark 2.1.11. Note that given a von Neumann algebra N, its Cowling-Haagerup constant as a C*-algebra need not be equal to its Cowling-Haagerup constant as a von Neumann algebra. For instance, $\mathcal{B}(\ell^2(\mathbb{N}))$ is weakly amenable as a von Neumann algebra (it is even amenable) but not as a C*-algebra (it is not even exact). Except otherwise stated, $\Lambda_{cb}(N)$ will always denote the Cowling-Haagerup constant of N as a von Neumann algebra.

These three notions of weak amenability are linked by the following fundamental result (recall that $\mathcal{L}\Gamma$ denotes the bicommutant of $\lambda(\Gamma)$ in $\mathcal{B}(\ell^2(\Gamma))$).

Theorem 2.1.12 (Haagerup). *Let Γ be a discrete group. Then,*

$$\Lambda_{cb}(\Gamma) = \Lambda_{cb}(C_r^*(\Gamma)) = \Lambda_{cb}(\mathcal{L}\Gamma).$$

Example 2.1.13. Let $n, m \in \mathbb{N}$ and consider lattices $\Gamma_1 \subset Sp(n, 1)$ and $\Gamma_2 \subset Sp(m, 1)$. Then, Example 2.1.9 together with Theorem 2.1.12 imply that the von Neumann algebras $\mathcal{L}\Gamma_1$ and $\mathcal{L}\Gamma_2$ cannot be ismorphic as soon as $n \neq m$. This isomorphism problem had been an open question for some time and one of the motivations for introducting weak amenability.

Another way of weakening the amenability property is to remove the finite support condition. This yields the so-called *Haagerup property* first introduced in [Haa78].

Definition 2.1.14. A discrete group Γ has the *Haagerup property* if there exists a net of maps $\varphi_t : \Gamma \to \mathbb{C}$ such that
- φ_t vanishes at infinity for all t.
- $\varphi_t(x) \to 1$ for all $x \in \Gamma$.
- m_{φ_t} is completely positive for all t (or equivalently, φ_t is positive definite for all t).

Again, there is a corresponding notion at the level of operator algebras. However, it requires the choice of a (GNS-faithful) state. The definition for von Neumann algebras can be traced back to [Haa78] (see also [Cho83]) and the definition for C*-algebras comes from [Don11]. Note that the original definitions were restricted to the tracial case but can be written in full generality.

Definition 2.1.15. A C*-algebra A is said to have the *Haagerup property with respect to a (GNS-faithful) state ψ* if there exists a net (T_t) of linear maps from A to itself such that
- $\psi \circ T_t \leqslant \psi$ and T_t extends to a compact map on $L^2(A, \psi)$ for all t.
- $\|T_t(x) - x\| \to 0$ for all $x \in A$.
- T_t is completely positive for all t.

A von Neumann algebra N is said to have the *Haagerup property with respect to a (GNS-faithful) state ψ* if there exists a net (T_t) of normal linear maps from N to itself such that
- $\psi \circ T_t \leqslant \psi$ and T_t extends to a compact map on $L^2(N, \psi)$ for all t.
- $(T_t(x) - x) \to 0$ in the weak-$*$ topology for all $x \in N$.
- T_t is completely positive for all t.

Remark 2.1.16. A finite von Neumann algebra N is said to have the Haagerup property if it has the Haagerup property with respect to a faithful tracial state and it was proved in [Jol02, Prop 2.4] that this notion is independant of the choice of the faithful tracial state.

The following result was proved by M. Choda in [Cho83, Thm 3]. Note that the statement implicitely uses the fact that the von Neumann algebra of a discrete group is always finite. This fails for general discrete quantum groups and that is the reason why we will need Definition 2.1.15 later on.

Theorem 2.1.17 (Choda). *Let Γ be a discrete group. Then, Γ has the Haagerup property if and only if $\mathcal{L}\Gamma$ has the Haagerup property.*

Note that there is also a notion of Haagerup property for general von Neumann algebras which does not depend on the choice of a particular state, called the *compact approximation property.*

Definition 2.1.18. A von Neumann algebra N is said to have the compact approximation property if there exists a net (T_t) of normal linear maps from N to itself such that

- The map $x \mapsto T_t(x).\xi$ is compact as a map from N to $L^2(N)$ for all $\xi \in L^2(N)$ and all t.
- $(T_t(x) - x) \to 0$ in the weak-$*$ topology for all $x \in N$.
- T_t is completely positive for all t.

This definition is obviously independant of the choice of a standard form $L^2(N)$ for N. However, this notion is *a priori* not very interesting in the context of group algebras because of the following statement, proved in [AD95, Thm 4.16].

Theorem 2.1.19 (Anantharaman-Delaroche)**.** *Let Γ be a discrete group. Then, $\mathcal{L}\Gamma$ has the compact approximation property if and only if it has the Haagerup property.*

2.2 General theory

2.2.1 Multipliers

We now study the notion of weak amenability for discrete quantum groups. The first step is to make sense of the notion of multiplier, and more precisely of Herz-Schur multiplier. Completely bounded multipliers for locally compact quantum groups have been widely studied in recent years, for instance in [HNR11], [JNR09], [KR97], [KR99], [Daw12] and [Daw13]. However, many aspects of the theory become simpler when one restricts to discrete quantum groups. That is the reason why we give a brief survey of the main results in this setting. This simplified presentation also allows us to prove some new results.

Instead of using complex-valued functions on the group, we will use elements of the von Neumann algebra $\ell^\infty(\widehat{\mathbb{G}})$. Thinking of the irreducible representations of $\mathbb{G}$ as the "points" of $\widehat{\mathbb{G}}$, we get the following definition.

Definition 2.2.1. Let $\widehat{\mathbb{G}}$ be a discrete quantum group and $a \in \ell^\infty(\widehat{\mathbb{G}})$. The *left multiplier* associated to a is the map $m_a : \mathrm{Pol}(\mathbb{G}) \to \mathrm{Pol}(\mathbb{G})$ defined by

$$(m_a \otimes \imath)(u^\alpha) = (1 \otimes ap_\alpha)u^\alpha,$$

for every irreducible representation α of $\mathbb{G}$.

Remark 2.2.2. Recall that W denotes the left regular representation of $\mathbb{G}$. Using the fact that $\mathbb{V}$ is the image of W under the isomorphism of Proposition1.1.37 (see the comments after that proposition), the definition can be rephrased simply as

$$(m_a \otimes \imath)(W) = (1 \otimes a)W.$$

This means that for any $\omega \in \mathcal{B}(L^2(\mathbb{G}))_*$, one has

$$m_a((\imath \otimes \omega)(W)) = (\imath \otimes \omega)((1 \otimes a)W) = (\imath \otimes \omega.a)(W),$$

where $\omega.a(x) = \omega(ax)$. This definition makes sense in a more general setting and corresponds to the definition of J. Kraus and Z-J. Ruan in [KR99] for Kac algebras and to the definition of M. Junge, M. Neufang and Z-J. Ruan in [JNR09] (see also [Daw12]) for locally compact quantum groups.

Remark 2.2.3. Let us assume that there exists a linear form $\omega_a \in \mathcal{B}(L^2(\mathbb{G}))^*$ such that $(\omega_a \otimes \imath)(W) = a$. Then, $m_a = (\omega_a \otimes \imath) \circ \Delta$. Indeed,

$$\begin{aligned}(m_a \otimes \imath)(W) &= (1 \otimes a)W \\ &= (\omega_a \otimes \imath \otimes \imath)(W_{13})W \\ &= (\omega_a \otimes \imath \otimes \imath)(W_{13}W_{23}) \\ &= (\omega_a \otimes \imath \otimes \imath) \circ (\Delta \otimes \imath)(W) \\ &= ([(\omega_a \otimes \imath) \circ \Delta] \otimes \imath)(W).\end{aligned}$$

This links Definition 2.2.1 with the "convolution operators" used by M. Brannan in [Bra12a] and [Bra13] to study the Haagerup property for discrete quantum groups.

From this definition, there is an obvious notion of support and pointwise convergence in $\ell^\infty(\widehat{\mathbb{G}})$.

Definition 2.2.4. A net (a_t) of elements of $\ell^\infty(\widehat{\mathbb{G}})$ is said to *converge pointwise* to an element $a \in \ell^\infty(\widehat{\mathbb{G}})$ if

$$a_t p_\alpha \to a p_\alpha$$

for any irreducible representation α of $\mathbb{G}$. An element $a \in \ell^\infty(\widehat{\mathbb{G}})$ is said to have *finite support* if ap_α is non-zero only for finitely many irreducible representations α.

Remark 2.2.5. If a_t is a sequence in $\ell^\infty(\widehat{\mathbb{G}})$ converging pointwise to a, then the equality $xp_\varepsilon = \widehat{\varepsilon}(x)p_\varepsilon$ implies that $\widehat{\varepsilon}(a_t) \to \widehat{\varepsilon}(a)$.

As in the classical case, we need an intrinsic characterization of those bounded functions giving rise to completely bounded multipliers. An analogue of Proposition 2.1.3 has been given in the setting of locally compact quantum groups by M. Daws [Daw12, Prop 4.1 and Thm 4.2].

Theorem 2.2.6 (Daws)**.** *Let $\widehat{\mathbb{G}}$ be a discrete quantum group and let $a \in \ell^\infty(\widehat{\mathbb{G}})$. Then, m_a extends to a competely bounded multiplier on $\mathcal{B}(L^2(\mathbb{G}))$ if and only if there exists a Hilbert space K and two maps $\xi, \eta \in \mathcal{B}(L^2(\mathbb{G}), L^2(\mathbb{G}) \otimes K)$ such that $\|\xi\|\|\eta\| = \|m_a\|_{cb}$ and*

$$(1 \otimes \eta)^* \widehat{W}_{12}^* (1 \otimes \xi) \widehat{W} = a \otimes 1. \tag{2.1}$$

Moreover, we then have $m_a(x) = \eta^*(x \otimes 1)\xi$.

Proof. We only give the explicit forms of the operators ξ and η since it will be needed later on. We can restrict to the case when m_a extends to a completely contractive map on $\mathcal{B}(L^2(\mathbb{G}))$, still denoted m_a. By Theorem 1.3.13 (Wittstock's factorization theorem), there is a representation $\pi : \mathcal{B}(L^2(\mathbb{G})) \to \mathcal{B}(K)$ and two isometries $P, Q \in \mathcal{B}(L^2(\mathbb{G}), K)$ such that for all $x \in \mathcal{B}(L^2(\mathbb{G}))$, $m_a(x) = Q^*\pi(x)P$. Define two maps ξ and η by

$$\left\{\begin{array}{lcl} \xi &=& (\imath \otimes \pi)(\widehat{W}^*)(1 \otimes P)\widehat{W}(1 \otimes \xi_h) \\ \eta &=& (\imath \otimes \pi)(\widehat{W}^*)(1 \otimes Q)\widehat{W}(1 \otimes \xi_h) \end{array}\right.$$

The remainder of the proof is pure computation. □

This theorem means that there is an intrinsic norm on $\ell^\infty(\widehat{\mathbb{G}})$ which does not depend on the space on which one considers the multipliers. Corollary 2.2.7 makes this idea more explicit.

Corollary 2.2.7. *Let $\widehat{\mathbb{G}}$ be a discrete quantum group and let $a \in \ell^\infty(\widehat{\mathbb{G}})$. The following are equivalent*

1. *m_a extends to a completely bounded map on $\mathcal{B}(L^2(\mathbb{G}))$.*
2. *m_a extends to a completely bounded map on $C_{red}(\mathbb{G})$.*
3. *m_a extends to a completely bounded map on $L^\infty(\mathbb{G})$.*

Moreover, the completely bounded norms of these maps are all equal.

Proof. We only prove the equivalence between (1) and (2), the same argument applies for the equivalence between (1) and (3).

Assume (1), then as m_a maps $\mathrm{Pol}(\mathbb{G})$ into itself, its completely bounded extension restricts to a completely bounded map on $C_{\mathrm{red}}(\mathbb{G})$ with smaller norm.

Assume (2), then ξ and η can still be defined if m_a is a completely bounded map on $C_{\mathrm{red}}(\mathbb{G})$ (π being then a representation of $C_{\mathrm{red}}(\mathbb{G})$) and the formula

$$x \mapsto \eta^*(x \otimes 1)\xi$$

defines a completely bounded extension of m_a to all of $\mathcal{B}(L^2(\mathbb{G}))$ with norm less than $\|\xi\|\|\eta\|$. □

2.2.2 Definition and first consequences

Building on the previous section, we are now able to give a definition of weak amenablity for discrete quantum groups.

Definition 2.2.8. A discrete quantum group $\widehat{\mathbb{G}}$ is said to be *weakly amenable* if there exists a net (a_t) of elements of $\ell^\infty(\widehat{\mathbb{G}})$ such that

- a_t has finite support for all t.
- (a_t) converges pointwise to 1.
- $K := \limsup_t \|m_{a_t}\|_{cb}$ is finite.

The lower bound of the constants K for all nets satisfying these properties is denoted $\Lambda_{cb}(\widehat{\mathbb{G}})$ and called the *Cowling-Haagerup constant* of $\widehat{\mathbb{G}}$. By convention, $\Lambda_{cb}(\widehat{\mathbb{G}}) = \infty$ if $\widehat{\mathbb{G}}$ is not weakly amenable.

Example 2.2.9. It is clear from [Tom06] that amenable discrete quantum groups are weakly amenable with Cowling-Haagerup constant equal to 1.

Remark 2.2.10. It is clear that a discrete group Γ is weakly amenable in the sense of Definition 2.1.7 if and only if the discrete quantum group $(C_0(\Gamma), \Delta_\Gamma)$ is weakly amenable in the sense of Definition 2.2.8 and that the Cowling-Haagerup constants are the same.

It is natural to ask for a counterpart to Theorem 2.1.12 in the context of discrete quantum groups. Let us give it in two steps.

Proposition 2.2.11. *Let $\widehat{\mathbb{G}}$ be a weakly amenable discrete quantum group. Then, $C_{red}(\mathbb{G})$ and $L^\infty(\mathbb{G})$ are weakly amenable and $\Lambda_{cb}(\widehat{\mathbb{G}}) \geqslant \Lambda_{cb}(C_{red}(\mathbb{G}))$ and $\Lambda_{cb}(\widehat{\mathbb{G}}) \geqslant \Lambda_{cb}(L^\infty(\mathbb{G}))$.*

Proof. This is straightforward since the maps m_{a_t} obviously satisfy the conditions of Definition 2.1.10. □

The proof of the converse inequalities is more tricky. When the discrete quantum group is unimodular, the following theorem was proved by J. Kraus and Z-J. Ruan in [KR99, Thm 5.14].

Theorem 2.2.12 (Kraus-Ruan)**.** *Let* $\widehat{\mathbb{G}}$ *be a* unimodular *discrete quantum group, then*

$$\Lambda_{cb}(\widehat{\mathbb{G}}) = \Lambda_{cb}(C_{red}(\mathbb{G})) = \Lambda_{cb}(L^{\infty}(\mathbb{G})).$$

Remark 2.2.13. By analogy with the classical case, one can define the *Fourier algebra* of a discrete quantum group $\widehat{\mathbb{G}}$ to be the predual $L^1(\mathbb{G})$ of $L^{\infty}(\mathbb{G})$ and the *reduced Fourier-Stieltjes algebra* to be the dual $B_{\lambda}(\widehat{\mathbb{G}})$ of $C_{\mathrm{red}}(\mathbb{G})$. Using these definitions, the result can be extended (see again [KR99, Thm 5.14]).

Corollary 2.2.14. *Let* $\widehat{\mathbb{G}}$ *be a* unimodular *discrete quantum group, then*

$$\Lambda_{cb}(\widehat{\mathbb{G}}) = \Lambda_{cb}(C_{red}(\mathbb{G})) = \Lambda_{cb}(B_{\lambda}(\widehat{\mathbb{G}})) = \Lambda_{cb}(L^1(\mathbb{G})) = \Lambda_{cb}(L^{\infty}(\mathbb{G})).$$

Whether the above theorem is still true in the general case is, up to now, an open question. In fact, if $T : L^{\infty}(\mathbb{G}) \to L^{\infty}(\mathbb{G})$ is a completely bounded map, the natural way of associating a multiplier a to T is to set, for any $\alpha \in \mathrm{Irr}(\mathbb{G})$,

$$ap_{\alpha} = (h \otimes \imath)((T \otimes \imath)(u^{\alpha})(u^{\alpha})^*).$$

The problem then boils down to controlling $\|m_a\|_{cb}$ with $\|T\|_{cb}$. Looking at the proof of Theorem 2.1.12 (e.g. in [BO08, Thm 12.3.10]) suggests to look at the isometry, say $\mathfrak{U}$, induced by the coproduct on the GNS constructions of h and $h \otimes h$ respectively. The following formula then holds :

$$m_a = \mathfrak{U}^*[(T \otimes \imath) \circ \Delta]\mathfrak{U}.$$

However, the map $\theta : x \mapsto \mathfrak{U}^* x \mathfrak{U}$ becomes very complicated in the case of a general quantum group and the image of $L^{\infty}(\mathbb{G})\overline{\otimes}L^{\infty}(\mathbb{G})$ is not even contained in $L^{\infty}(\mathbb{G})$.

In the case of unimodular quantum groups, one can use the canonical conditional expectation (see [KR99, Lem 5.9])

$$\mathbb{E}_{\Delta} : L^{\infty}(\mathbb{G})\overline{\otimes}L^{\infty}(\mathbb{G}) \to \Delta(L^{\infty}(\mathbb{G}))$$

to rewrite the equality as

$$m_a = \Delta^{-1} \circ \mathbb{E}_{\Delta} \circ [(T \otimes \imath) \circ \Delta].$$

As soon as the discrete quantum group is not unimodular, an easy computation using the explicit formulas of [Wor98, Thm 1.4] for the modular theory of the Haar state proves that the image of the coproduct is not invariant under the one-parameter automorphism group. This, by virtue of a theorem of M. Takesaki [Tak72a], implies that there can be no conditional expectation onto the image of the coproduct.

Noticing that $\Delta^{-1} \circ \mathbb{E}_{\Delta}$ is nothing but the map induced at the level of von Neumann algebras by the adjoint of the coproduct (i.e. by the map $\mathfrak{U}^*$), one could also try to replace it with the "skew-dual" of [AC82] which is defined by (see [CNT87, Prop VIII.3] for the formula as we give it below)

$$h(\Delta^{\sharp}(x)y) = (h \otimes h)((\sigma_{i/2} \otimes \sigma_{i/2})(x)\Delta(y))$$

and is known to be completely positive by [AC82, Thm 3.5]. However, an easy computation yields that for any $\alpha, \beta \in \mathrm{Irr}(\mathbb{G})$,

$$\Delta^{\sharp}(u^{\alpha}_{i,j} \otimes u^{\beta}_{k,l}) = \frac{\delta_{\alpha,\beta}\delta_{j,k}}{\dim_q(\alpha)} u^{\alpha}_{i,l}.$$

This means that $\Delta^\sharp \circ (T_t \otimes \imath) \circ \Delta$ will not converge pointwise to the identity, hence this strategy also fails.

Let us now turn to the consequences of Definition 2.2.8. The main advantage of Theorem 2.2.12 is that it enables us to use C*-algebra techniques and results to study weakly amenable discrete quantum groups. The following link with exactness is the easiest example of this.

Corollary 2.2.15. *Let $\widehat{\mathbb{G}}$ be a weakly amenable discrete quantum group, then $\widehat{\mathbb{G}}$ is exact.*

Proof. Any weakly amenable C*-algebra is exact according to [BN06, Thm 3.9]. Combining this with Proposition 2.2.11 and Proposition 1.2.5 yields the result. □

However, since Theorem 2.2.12 only works in the unimodular case, we will rather use our intrinsic definition. We give here the main permanence properties that one can easily deduce in that way.

Corollary 2.2.16. *Let $\widehat{\mathbb{G}}$ be a weakly amenable discrete quantum group and let $\widehat{\mathbb{H}}$ be a discrete quantum subgroup of $\widehat{\mathbb{G}}$. Then, $\widehat{\mathbb{H}}$ is weakly amenable and $\Lambda_{cb}(\widehat{\mathbb{H}}) \leqslant \Lambda_{cb}(\widehat{\mathbb{G}})$.*

Proof. Let $\epsilon > 0$ and let (a_t) be a net of finitely supported elements in $\ell^\infty(\widehat{\mathbb{G}})$ converging pointwise to 1 and such that $\limsup \|m_{a_t}\|_{cb} \leqslant \Lambda_{cb}(\widehat{\mathbb{G}}) + \epsilon$. Then, using the notations of Proposition 1.2.13, $(a_t p_{\mathbb{H}})$ is a net of finitely supported elements in $\ell^\infty(\widehat{\mathbb{H}})$ which converges pointwise to 1. Using the conditional expecation of Proposition 1.2.12, we see that $\|m_{a_t p_{\mathbb{H}}}\|_{cb} = \|\mathbb{E}_{\mathbb{H}} \circ m_{a_t}\|_{cb} \leqslant \|m_{a_t}\|_{cb}$. Thus, $\Lambda_{cb}(\widehat{\mathbb{H}}) \leqslant \Lambda_{cb}(\widehat{\mathbb{G}}) + \epsilon$. □

Corollary 2.2.17. *Let $\widehat{\mathbb{G}}$ and $\widehat{\mathbb{H}}$ be two discrete quantum groups. Then,*

$$\Lambda_{cb}(\widehat{\mathbb{G}} \times \widehat{\mathbb{H}}) \leqslant \Lambda_{cb}(\widehat{\mathbb{G}})\Lambda_{cb}(\widehat{\mathbb{H}}).$$

Proof. Let $\epsilon > 0$ and let (a_t) and (b_s) be nets of finitely supported elements respectively in $\ell^\infty(\widehat{\mathbb{G}})$ and in $\ell^\infty(\widehat{\mathbb{H}})$ converging pointwise to 1 and such that $\limsup \|m_{a_t}\|_{cb} \leqslant \Lambda_{cb}(\widehat{\mathbb{G}}) + \epsilon$ and $\limsup \|m_{b_s}\|_{cb} \leqslant \Lambda_{cb}(\widehat{\mathbb{H}}) + \epsilon$. Set

$$c_{(t,s)} = a_t \otimes b_s \in \ell^\infty(\widehat{\mathbb{G}} \times \widehat{\mathbb{H}}).$$

From the description of the representation theory of tensor products given by Theorem 1.2.19, we see that $(c_{(t,s)})$ is a net of finitely supported elements converging pointwise to 1. Moreover, since $m_{c_{(t,s)}} = m_{a_t} \otimes m_{b_s}$, we have $\Lambda_{cb}(\widehat{\mathbb{G}} \times \widehat{\mathbb{H}}) \leqslant (\Lambda_{cb}(\widehat{\mathbb{G}}) + \epsilon)(\Lambda_{cb}(\widehat{\mathbb{H}}) + \epsilon)$, which concludes the proof. □

Remark 2.2.18. It is a general fact that for any two C*-algebras A and B, we have $\Lambda_{cb}(A \otimes B) = \Lambda_{cb}(A)\Lambda_{cb}(B)$ (see e.g. [BO08, Thm 12.3.13]). Hence, we always have

$$\Lambda_{cb}(C_{\text{red}}(\mathbb{G} \otimes \mathbb{H})) = \Lambda_{cb}(C_{\text{red}}(\mathbb{G}))\Lambda_{cb}(C_{\text{red}}(\mathbb{H})).$$

Moreover, Theorem 2.2.12 implies that $\Lambda_{cb}(\widehat{\mathbb{G}} \times \widehat{\mathbb{H}}) = \Lambda_{cb}(\widehat{\mathbb{G}})\Lambda_{cb}(\widehat{\mathbb{H}})$ as soon as the discrete quantum groups are unimodular. It is very likely that this result is true in general but we were not able to prove it. Note that we can summarize this by the inequality

$$\Lambda_{cb}(C_{\text{red}}(\mathbb{G}))\Lambda_{cb}(C_{\text{red}}(\mathbb{H})) \leqslant \Lambda_{cb}(\widehat{\mathbb{G}} \times \widehat{\mathbb{H}}) \leqslant \Lambda_{cb}(\widehat{\mathbb{G}})\Lambda_{cb}(\widehat{\mathbb{H}}).$$

Corollary 2.2.19. *Let $(\widehat{\mathbb{G}}_i, \pi_{i,j})$ be an inductive system of discrete quantum groups with inductive limit $\widehat{\mathbb{G}}$ and limit maps $\pi_i : C(\mathbb{G}_i) \to C(\mathbb{G})$. Then, if all the maps π_i are injective,*

$$\sup_i \Lambda_{cb}(\widehat{\mathbb{G}}_i) = \Lambda_{cb}(\widehat{\mathbb{G}}).$$

In particular, the inductive limit is weakly amenable if and only if the quantum groups are all weakly amenable with uniformly bounded Cowling-Haagerup constant.

Proof. The injectity of the limit maps ensures that each $\widehat{\mathbb{G}}_i$ can be seen as a discrete quantum subgroup of $\mathbb{G}$. Hence, Corollary 2.2.16 gives the inequality

$$\sup_i \Lambda_{cb}(\widehat{\mathbb{G}}_i) \leqslant \Lambda_{cb}(\widehat{\mathbb{G}}).$$

To prove the converse inequality, fix an $\epsilon > 0$ and let $(a_t^i)_t$ be a net of finitely supported elements in $\ell^\infty(\widehat{\mathbb{G}}_i)$ converging pointwise to 1 and such that $\limsup \|m_{a_t^i}\|_{cb} \leqslant \Lambda_{cb}(\widehat{\mathbb{G}}_i) + \epsilon$. Using the description of the representation theory of inductive limits given by Proposition 1.2.20, we can see $(a_t^i)_{(i,t)}$ as a net of finitely supported elements of $\ell^\infty(\widehat{\mathbb{G}})$ converging pointwise to 1 by setting $a_t^i p_\alpha = 0$ for any $\alpha \notin \mathrm{Irr}(\mathbb{G}_i)$. It is clear that this does not change the completely bounded norm of the operator $m_{a_t^i}$. The conditions on the completely bounded norms then gives $\Lambda_{cb}(\widehat{\mathbb{G}}) \leqslant \sup_i \Lambda_{cb}(\widehat{\mathbb{G}}_i) + \epsilon$. □

Let us end this subsection with another (weakened) version of the last result in the case when the connecting maps are not necessarily injective.

Proposition 2.2.20. *Let $(\widehat{\mathbb{G}}_i, \pi_{i,j})$ be an inductive system of discrete quantum groups with inductive limit $\widehat{\mathbb{G}}$ and limit maps $\pi_i : C(\mathbb{G}_i) \to C(\mathbb{G})$. Then,*

$$\sup_i \Lambda_{cb}(\pi_i(C_{red}(\mathbb{G}_i)) = \Lambda_{cb}(C_{red}(\widehat{\mathbb{G}})).$$

Proof. The inequality $\sup_i \Lambda_{cb}(\pi_i(C_{\mathrm{red}}(\mathbb{G}_i)) \leqslant \Lambda_{cb}(C_{\mathrm{red}}(\mathbb{G}))$ is straightforward from the fact that $(\pi_i(C_{\mathrm{red}}(\mathbb{G}_i)), \Delta)$ is the dual of a discrete quantum subgroup of $\widehat{\mathbb{G}}$. The converse one can be seen as a consequence of the following more general result. □

Proposition 2.2.21. *Let $(A_i, \pi_{i,j})$ be a direct system of C*-algebras such that for each i, there is a conditional expectation $\mathbb{E}_i$ from the inductive limit A onto the C*-subalgebra $\pi_i(A_i)$. Then $\Lambda_{cb}(A) \leqslant \sup_i \Lambda_{cb}(\pi_i(A_i))$.*

Proof. The proof is certainly well-known (see [DS05, Prop 4] for the case where the connecting maps are injective), but we give it for completeness. Let $\epsilon > 0$ and $\mathcal{F} \subset A$ be a finite subset, set $\Lambda = \sup_i(\Lambda_{cb}(\pi_i(A_i)))$ and

$$\eta = \frac{\sqrt{(2+\Lambda)^2 + 4\epsilon} - (2+\Lambda)}{2}.$$

We can see A as the closure of the union of the $\pi_i(A_i)$ (see for example [RLL00, Prop 6.2.4]), thus there is an index i_0 and a finite subset $\mathcal{G}$ of A_{i_0} such that $d(\mathcal{F}, \mathcal{G}) \leqslant \eta$, i.e. for any $x \in \mathcal{F}$ there is a $y \in \mathcal{G}$ such that $\|x - y\| \leqslant \eta$. Let T be a finite-rank linear map from $\pi_{i_0}(A_{i_0})$ to itself approximating the identity up to η on $\mathcal{G}$ and with

$$\|T\|_{cb} \leqslant \Lambda + \eta.$$

Set $T_{\mathcal{F},\epsilon} = T \circ \mathbb{E}_{i_0}$. This is a finite-rank linear map from A to itself with $\|T_{\mathcal{F},\epsilon}\|_{cb} \leqslant \Lambda + \epsilon$. Moreover, for any $x \in \mathcal{F}$, if y is an element of $\mathcal{G}$ such that $\|x - y\| \leqslant \eta$, one has

$$\begin{aligned}
\|T_{\mathcal{F},\epsilon}(x) - x\| &= \|T \circ \mathbb{E}_{i_0}(x) - x\| \\
&\leqslant \|T \circ \mathbb{E}_{i_0}(y) - y\| + \|T \circ \mathbb{E}_{i_0}(x - y) - (x - y)\| \\
&\leqslant \eta + \|T \circ \mathbb{E}_{i_0}\|\|x - y\| + \|x - y\| \\
&\leqslant \eta + (\Lambda + \eta)\eta + \eta \\
&= \eta(2 + \Lambda + \eta) \\
&= \epsilon.
\end{aligned}$$

We have proven that for any finite subset $\mathcal{F}$ of A and any $\epsilon > 0$ there is a linear map $T_{\mathcal{F},\epsilon}$ with completely bounded norm less than $\Lambda + \epsilon$ which approximates the identity uniformly up to ϵ on $\mathcal{F}$. This implies that $\Lambda_{cb}(A) \leqslant \Lambda$. □

2.2.3 Quantum subgroups and relative amenability

Let us now come back to the issue of quantum subgroups and consider it in some detail. There is obviously no general converse to Corollary 2.2.16. However, it is known that under some conditions, weak amenability may pass to overgroups. For instance, weak amenability of any lattice in a Lie group implies weak amenability of the group (there is an obvious generalization of Definition 2.1.7 for general locally compact groups). Even though this example comes from the locally compact world, it suggests that some "finite covolume" assumption may be enough to have a converse to Corollary 2.2.16. A possible substitute for this is the notion of co-amenability.

P. Eymard defined in [Eym72] a subgroup $H \subset G$ to be *co-amenable* if there is a mean on the homogeneous space G/H (i.e. a state on $\ell^\infty(G/H)$) which is invariant with respect to the translation action of G. He investigated this property as a weakening of the notion of amenability, since a group is amenable if and only if its trivial subgroup is co-amenable. It was proved in [AD95, Thm 4.9] that if H is co-amenable in G and is weakly amenable, then G is weakly amenable and $\Lambda_{cb}(G) = \Lambda_{cb}(H)$. Since this proof is based on von Neumann algebraic techniques, it can be partially extended to the setting of quantum groups.

Remark 2.2.22. A more direct (and C*-algebraic) proof of [AD95, Thm 4.9] is given in [BO08, Prop 12.3.11]. However, it is quite ill-suited to the setting of quantum groups since it is based on the use of a section of the quotient which fails to exist in any reasonable sense in the quantum case, at least as soon as the subgroup we consider is not *divisible* in the sense of [VV13, Def 4.1].

We first have to define a notion of co-amenable quantum subgroup. Because the word "co-amenable" is already widely used in quantum group theory (meaning roughly amenability of the dual quantum group), we will rather use the alternative terminology *relatively amenable*.

Definition 2.2.23. Let $\widehat{\mathbb{G}}$ be discrete quantum group and let $\widehat{\mathbb{H}}$ be a discrete quantum subgroup of $\widehat{\mathbb{G}}$. We say that $\widehat{\mathbb{H}}$ is *amenable relative to* $\widehat{\mathbb{G}}$ if the quotient space has an invariant mean for the translation action, i.e. if there exists a state m on $\ell^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$ such that for all $x \in \ell^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$,

$$(\imath \otimes m) \circ \tau(x) = m(x).1$$

Remark 2.2.24. One could define an action of a locally compact quantum group on a von Neumann algebra to be *amenable* if such an invariant mean exists. This notion would be in some sense "dual" to the notion of amenable action defined in [AD79] for locally compact groups and in [JP90] for Kac algebras. In fact, [JP90, Thm 3.5] implies that, at least in the unimodular case, if $\widehat{\mathbb{H}}$ is amenable relative to $\widehat{\mathbb{G}}$ and if the translation action on $\ell^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$ is amenable in the sense of [JP90, Def 3.1], then $\widehat{\mathbb{G}}$ must be amenable. However, we are not interested in the study of such a general property in the present work.

This definition gives an additional characterization of amenability. Recall that a representation U of a discrete quantum group $\widehat{\mathbb{G}}$ on a Hilbert space H_U is said to have a *left-invariant mean* is there is a state $m \in \mathcal{B}(H_U)^*$ such that for every $x \in C_0(\widehat{\mathbb{G}})$,

$$(\imath \otimes m)[U^*(1 \otimes x)U] = m(x).1$$

Proposition 2.2.25. *A discrete quantum group $\widehat{\mathbb{G}}$ is amenable if and only if the trivial subgroup is amenable relative to $\widehat{\mathbb{G}}$.*

Proof. First assume $\widehat{\mathbb{G}}$ to be amenable. Then, any representation of $\widehat{\mathbb{G}}$ is amenable by [BCT05, Thm 6.3], i.e. has a left-invariant mean (see [BCT05, Def 6.1]). This implies that for any discrete quantum subgroup $\widehat{\mathbb{H}}$ of $\widehat{\mathbb{G}}$, the quasi-regular representation of $\widehat{\mathbb{G}}$ modulo $\widehat{\mathbb{H}}$ has a left-invariant mean. Otherwise said, any discrete quantum subgroup of an amenable discrete quantum group is relatively amenable, in particular the trivial subgroup.

Assume now that the trivial subgroup is amenable relative to $\widehat{\mathbb{G}}$. Since the quotient by the trivial subgroup yields the von Neumann algebra $\ell^\infty(\mathbb{G})$ with the translation action being the coproduct, we can deduce by [Tom06, Thm 3.8] that $\widehat{\mathbb{G}}$ is amenable. □

It is not easy to find examples of relatively amenable discrete quantum groups, but we at least have one source of examples, which illustrates the "finite covolume" aspect : *cofinite* quantum subgroups.

Definition 2.2.26. A quantum subgroup $\widehat{\mathbb{H}}$ of a discrete quantum group $\widehat{\mathbb{G}}$ is said to be cofinite if the quotient von Neumann algebra $\ell^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$ is finite-dimensional.

Proposition 2.2.27. *Let $\widehat{\mathbb{H}}$ be a cofinite quantum subgroup of a* unimodular *discrete quantum group $\widehat{\mathbb{G}}$. Then, $\widehat{\mathbb{H}}$ is amenable relative to $\widehat{\mathbb{G}}$.*

Proof. If $\widehat{\mathbb{H}}$ is a cofinite quantum subgroup of $\widehat{\mathbb{G}}$, the quasi-regular representation $\mathcal{R}$ of $\widehat{\mathbb{G}}$ modulo $\widehat{\mathbb{H}}$ is finite dimensional. Since unimodular discrete quantum groups are examples of unimodular algebraic quantum groups, it follows from [BCT05, Cor. 7.11] that any finite-dimensional representation of such a quantum group is amenable. Then, according to [BCT05, Thm. 7.8], there is an invariant mean m on $\mathcal{B}(\ell^2(\widehat{\mathbb{G}}/\widehat{\mathbb{H}}))$ for $\mathcal{R}$ and $\widehat{\mathbb{H}}$ is amenable relative to $\widehat{\mathbb{G}}$.

□

Example 2.2.28. Let $N > 3$ be an integer and let $\mathbb{G}$ be the free orthogonal quantum group O_N^+ defined in Definition 1.2.34. The subalgebra of $L^\infty(O_N^+)$ generated by the elements $u_{i,j}u_{k,l}$ is stable under the coproduct and thus defines a discrete quantum subgroup $\widehat{\mathbb{H}}$ of $\widehat{\mathbb{G}}$ called its *even (or projective) part*. It is clear that under the usual (set-theoretic) identification $\mathrm{Irr}(\mathbb{G}) = \mathbb{N}$ given by Theorem 1.2.40, $\mathrm{Irr}(\mathbb{H})$ corresponds to the even integers and it is not very difficult to see that $\ell^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}}) = \mathbb{C} \oplus \mathbb{C}$. Thus $\widehat{\mathbb{H}}$ is amenable relative to $\widehat{\mathbb{G}}$. As already mentionned in Subsection 1.2.4, this quantum subgroup is isomorphic to the quantum automorphism group of $M_N(\mathbb{C})$ with respect to its canonical δ-trace defined by Theorem 1.2.48.

Remark 2.2.29. Relative amenability does not pass to subgroups, even in the case of classical groups. Examples of triples of discrete groups $K < H < G$ with K amenable relative to G but not to H were constructed in [MP03] and [Pes03].

We will now use ideas from the works of C. Anantharaman-Delaroche on actions of groups on von Neumann algebras [AD79] and follow the path of [AD95] to prove Theorem 2.2.37. This means that we are going to prove a more general result, involving the notion of *amenable equivalence* of von Neumann algebras which was introduced in [AD95, Def 4.1].

Definition 2.2.30. Let M and N be two von Neumann algebras. We say that M is *amenably dominated* by N if there exists a von Neumann algebra N_1 which is Morita equivalent to N and contains M in such a way that there is a norm-one projection from N_1 onto M. We then write $M \prec_a N$. We say that M and N are *amenably equivalent* if $M \prec_a N$ and $N \prec_a M$.

Our aim is to prove the following generalization of a classical result (see Paragraphe 4.10 in [AD95]).

Theorem 2.2.31. *Let $\widehat{\mathbb{G}}$ be a discrete quantum group and let $\widehat{\mathbb{H}}$ be a discrete quantum subgroup which is amenable relative to $\widehat{\mathbb{G}}$. Then, $L^\infty(\mathbb{H})$ is amenably equivalent to $L^\infty(\mathbb{G})$.*

In order to prove this theorem, we have to build a norm one projection from a well-chosen von Neumann algebra onto $L^\infty(\mathbb{G})$. We will do this by adapting some ideas of [AD79] to the setting of discrete quantum groups.

Lemma 2.2.32. *Let M_1, M_2 be von Neumann algebras and let ρ be an action of a discrete quantum group $\widehat{\mathbb{G}}$ on M_2. Assume that we have a von Neumann subalgebra N_2 of M_2 which is stable under the action ρ and a norm-one (non-necessarily normal) equivariant projection $P : M_2 \to N_2$ (i.e. $\rho \circ P = (1 \otimes P) \circ \rho$). Then, there exists a norm one projection $Q : M_1 \overline{\otimes} M_2 \to M_1 \overline{\otimes} N_2$ such that*

$$Q(x_1 \otimes x_2) = x_1 \otimes P(x_2)$$

for all $x_1 \in M_1$ and $x_2 \in M_2$. Moreover, P is equivariant with respect to $\mu = (\sigma \otimes \imath) \circ (\imath \otimes \rho)$.

Proof. The existence of the projection Q was proved in [Tom69, Thm 4]. The fact that the construction preserves equivariance is a simple generalization of [AD79, Lem 2.1] but we give it for completeness. Note that if the projection P was normal, we could simply extend its tensor product with Id_{M_1} to the tensor product von Neumann algebra and conclude the proof. Thus, the subtelty of the statement lies in dealing with the non-normality of the projection. Let the von Neumann algebras M_1 and M_2 act on Hilbert spaces H_1 and H_2 respectively and fix a Hilbert basis $(e_i)_{i\in I}$ of H_1, the elements of which will be identified with the corresponding one-dimensional projections. For any finite subset J of I, set

$$e_J = \sum_{i \in J} e_i \text{ and } P_J = \mathrm{Id}_{e_J H_1} \otimes P.$$

We can define a net $(P'_J)_J$ of maps from $\mathcal{B}(H_1)\overline{\otimes} M_2$ to $\mathcal{B}(H_1)\overline{\otimes} N_2$ by setting

$$P'_J(x) = P_J((e_J \otimes 1)x(e_J \otimes 1)).$$

Let Lim be any Banach limit on the set of sequences indexed by finite subsets of I and set, for any $\xi, \eta \in H_1 \otimes H_2$,

$$\langle Q(x)\xi, \eta\rangle = \mathrm{Lim}\langle P'_J(x)\xi, \eta\rangle.$$

This defines a norm one projection such that $Q(x_1 \otimes x_2) = x_1 \otimes P(x_2)$ for every $x_1 \in \mathcal{B}(H_1)$ and $x_2 \in M_2$. Moreover, the obvious equivariance of the projections P'_J with respect to ρ yields the equality

$$(\sigma \otimes \imath) \circ (\imath \otimes \rho) \circ P'_J = (\imath \otimes P'_J) \circ (\sigma \otimes \imath) \circ (\imath \otimes \rho).$$

Passing to the Banach limit Lim gives the corresponding equivariance property for Q. It is now easy to check that the restriction of Q to $M_1 \overline{\otimes} M_2$ is the desired projection (see the proof of [AD79, Lem 2.1] for details). □

Proposition 2.2.33. *Let $\widehat{\mathbb{G}}$ be a discrete quantum group and let (M, N, P) be a triple consisting in a von Neumann algebra M endowed with an action ρ of $\widehat{\mathbb{G}}$, a von Neumann subalgebra N of M which is stable under the action ρ and a norm-one equivariant projection $P : M \to N$. Then, there exists a norm-one projection $\widetilde{Q} : \widehat{\mathbb{G}} \ltimes M \to \widehat{\mathbb{G}} \ltimes N$.*

Proof. By Lemma 2.2.32, there exists a norm one projection

$$Q : \mathcal{B}(L^2(\mathbb{G}))\overline{\otimes} M \to \mathcal{B}(L^2(\mathbb{G}))\overline{\otimes} N$$

which is equivariant with respect to $\mu = (\sigma \otimes \imath) \circ (\imath \otimes \rho)$ and such that

$$Q(x_1 \otimes x_2) = x_1 \otimes P(x_2).$$

Let us consider the explicit $*$-isomorphisms of [Vae01, Thm 2.6]

$$\begin{array}{rcccl} \Phi_M & : & \mathcal{B}(L^2(\mathbb{G}))\overline{\otimes} M & \to & \mathbb{G}^{op} \ltimes (\widehat{\mathbb{G}} \ltimes M) \\ \Phi_N & : & \mathcal{B}(L^2(\mathbb{G}))\overline{\otimes} N & \to & \mathbb{G}^{op} \ltimes (\widehat{\mathbb{G}} \ltimes N) \end{array}$$

Since any von Neumann algebra can be recovered in a crossed-product as the fixed points algebra under the dual action by [Vae01, Thm 2.7], we only have to prove that the norm-one projection

$$\widetilde{Q} = \Phi_N \circ Q \circ \Phi_M^{-1} : \mathbb{G}^{op} \ltimes (\widehat{\mathbb{G}} \ltimes M) \to \mathbb{G}^{op} \ltimes (\widehat{\mathbb{G}} \ltimes N)$$

is equivariant with respect to the the bidual action to conclude.

Let us use the notations of [Vae01]. The bidual action on the double crossed product can be transported to $\mathcal{B}(L^2(\mathbb{G}))\overline{\otimes} M$ in the following way : there is an operator $\mathcal{J} : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G}'^{\mathrm{op}})$ and a map γ such that

$$\widehat{\widehat{\rho}} \circ \Phi = (\mathcal{J} \otimes \Phi) \circ \gamma.$$

It is clear that the $\widehat{\widehat{\rho}}$-equivariance of $\widetilde{Q}$ is equivalent to the γ-equivariance of Q.

To prove the latter, recall that γ decomposes as $\mathrm{Ad}_{\Sigma V^* \Sigma \otimes 1} \circ \mu$. We already know by Lemma 2.2.32 that

$$\mu \circ Q = (\imath \otimes Q) \circ \mu$$

and, using the approximating projections P'_J, we see that $1 \otimes Q$ also commutes to $\mathrm{Ad}_{\Sigma V^* \Sigma \otimes 1}$. Hence,

$$\gamma \circ Q = (\imath \otimes Q) \circ \gamma.$$

□

We can now prove our main result.

Proof of Theorem 2.2.31. The fact that the mean is invariant precisely means that m is an equivariant norm-one projection onto the von Neumann subalgebra $\mathbb{C}.1$ of M. Consequently, proposition 2.2.33 applied to the triple $(\ell^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}}), \mathbb{C}, m)$ yields a norm-one projection from $\widehat{\mathbb{G}} \ltimes \ell^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$ onto $\widehat{\mathbb{G}} \ltimes \mathbb{C} = L^\infty(\mathbb{G})$. Since according e.g. to [Vae05, Rmk 4.3], $\widehat{\mathbb{G}} \ltimes \ell^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$ is Morita equivalent to $L^\infty(\mathbb{H})$, we have proven that $L^\infty(\mathbb{G}) \prec_a L^\infty(\mathbb{H})$. The conditional expectation of 1.2.12 then gives $L^\infty(\mathbb{H}) \prec_a L^\infty(\mathbb{G})$, concluding the proof. □

Remark 2.2.34. Notice that since the basic construction $\langle L^\infty(\mathbb{G}), L^\infty(\mathbb{H})\rangle$ is isomorphic to $\widehat{\mathbb{G}} \ltimes \ell^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$ (see e.g. the beginning of Section 4 of [Vae05]), Proposition 2.2.33 proves that $L^\infty(\mathbb{H})$ is *amenable relative* to $L^\infty(\mathbb{G})$ in the sense of [MP03, Def 4] (or equivalently that $L^\infty(\mathbb{H}) \subset L^\infty(\mathbb{G})$ is an amenable inclusion).

As a first consequence, we mention the link between relative amenablity and amenability.

Corollary 2.2.35. *Let $\widehat{\mathbb{G}}$ be a* unimodular *discrete quantum group and let $\widehat{\mathbb{H}}$ be a discrete quantum subgroup such that $\widehat{\mathbb{G}}$ is amenable relative to $\widehat{\mathbb{H}}$. Then, $\widehat{\mathbb{H}}$ is amenable if and only if $\widehat{\mathbb{G}}$ is amenable.*

Proof. If $\widehat{\mathbb{G}}$ is amenable, then $\widehat{\mathbb{H}}$ is a amenable because it is a quantum subgroup. If $\widehat{\mathbb{H}}$ is amenable, $L^\infty(\mathbb{H})$ is injective by [Rua96, Thm 4.5]. Because they are equivalent by Theorem 2.2.31, $L^\infty(\mathbb{G})$ is alos injective, implying again by [Rua96, Thm 4.5] that $\widehat{\mathbb{G}}$ is amenable. □

Example 2.2.36. Let us give an example related to this result, using the notion of *half-liberated easy quantum group* defined in [BCS10]. The argument is inspired from an unpublished work of M. Brannan. These are compact matrix pseudogroups $(\mathbb{G}, u)$ satisfying the two following properties :

- The map $u_{i,j} \mapsto -u_{i,j}$ extends to $*$-homomorphism $\mathcal{J} : C_{\max}(\mathbb{G}) \to C_{\max}(\mathbb{G})$.
- The projective part $P(\mathbb{G})$ of $\mathbb{G}$ is a classical compact group (more precisely $abc = cba$ for every $a, b, c \in C(\mathbb{G})$).

Here by *projective part* we mean the compact quantum group $P\mathbb{G}$ formed by the C*-subalgebra of $C_{\max}(\mathbb{G})$ generated by coefficients of $u \otimes \overline{u}$ together with the restriction of the coproduct. Since $\mathcal{J}$ obviously preserves the Haar state, it yields a well-defined map at the level of von Neumann algebras. Moreover, $\mathcal{J}$ fixes $L^\infty(P\mathbb{G})$ and thus,

$$\mathbb{E} : x \mapsto \frac{x + \mathcal{J}(x)}{2}$$

is the conditional expectation from $L^\infty(\mathbb{G})$ onto $L^\infty(P\mathbb{G})$. Since $\mathcal{J}$ is a $*$-homomorphism, it is positive and we have, for all $x \geqslant 0$,

$$\mathbb{E}(x) \geqslant \frac{x}{2}.$$

This, by virtue of the *Pimsner-Popa inequality* [JS97, Thm 5.1.3], implies that the index $[L^\infty(\mathbb{G}) : L^\infty(P\mathbb{G})]$ is at most 2. The inclusion being proper, the index is exactly 2. It is of course likely that the discrete quantum subgroup $\widehat{P\mathbb{G}}$ of $\widehat{\mathbb{G}}$ is cofinite with index 2, but we do not have a proof of this fact.

However, finite index von Neumann subalgebras are relatively amenable, and $L^\infty(P\mathbb{G})$ is amenable because it is commutative. This proves that $L^\infty(\mathbb{G})$ is amenable. Since $\mathbb{G}$ is of Kac type, it is also amenable by [Rua96, Thm 4.5].

We can now derive our result on weak amenability. Again, we have to restrict to the unimodular case if we want to be able to extract informations on $\Lambda_{cb}(\widehat{\mathbb{G}})$ from $\Lambda_{cb}(L^{\infty}(\mathbb{G}))$.

Corollary 2.2.37. *Let $\widehat{\mathbb{G}}$ be a discrete quantum group and let $\widehat{\mathbb{H}}$ be a discrete quantum subgroup which is amenable relative to $\widehat{\mathbb{G}}$, then $\Lambda_{cb}(L^{\infty}(\mathbb{H})) = \Lambda_{cb}(L^{\infty}(\mathbb{G}))$. If moreover $\widehat{\mathbb{G}}$ (and consequently $\widehat{\mathbb{H}}$) is* unimodular*, then $\Lambda_{cb}(\widehat{\mathbb{G}}) = \Lambda_{cb}(\widehat{\mathbb{H}})$.*

Proof. It was proved in [AD95, Thm 4.9] that amenably equivalent von Neumann algebras have equal Cowling-Haagerup constant. □

Remark 2.2.38. Combining Theorem 2.2.31 with [BF11, Thm 5.1] yields the following result : let $\widehat{\mathbb{G}}$ be a *unimodular* discrete quantum group and let $\widehat{\mathbb{H}}$ be a discrete quantum subgroup which is amenable relative to $\widehat{\mathbb{G}}$. Then, if $L^{\infty}(\mathbb{H})$ has the Haagerup property, $L^{\infty}(\mathbb{G})$ also has the Haagerup property.

Remark 2.2.39. Assume that $\widehat{\mathbb{G}}$ has Kazhdan's property (T) as defined in [Fim10, Def 3.1] and let $\widehat{\mathbb{H}}$ be a discrete quantum subgroup which is amenable relative to $\widehat{\mathbb{G}}$. Denote by H the Hilbert space $L^2(\mathbb{G})$ seen as the standard correspondence between $L^{\infty}(\mathbb{G})$ and $L^{\infty}(\mathbb{H})$. By Theorem 2.2.31, H is a *left injective correspondance* in the sense of [AD95, Def 3.1]. By [AD95, Prop 3.6], it also a *left amenable correspondance* in the sense of [AD95, Def 2.1], i.e. the correspondence $H \otimes \overline{H}$ weakly contains the identity correspondence of $L^{\infty}(\mathbb{G})$. Moreover, we know that $H \otimes \overline{H}$ is precisly the Hilbert space $\ell^2(\widehat{\mathbb{G}} \ltimes \ell^{\infty}(\widehat{\mathbb{G}}/\widehat{\mathbb{H}}))$ associated to the GNS construction for the dual weight $\widetilde{\theta}$ on the crossed-product. Thanks to [Vae01], the GNS construction for the dual weight may be explicitely described. In fact, the map

$$\mathcal{I} : (a \otimes 1)\alpha(x) \mapsto a \otimes x$$

for $a \in L^{\infty}(\mathbb{G})$ and $x \in \ell^{\infty}(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$ extends to an isomorphism between $\ell^2(\widehat{\mathbb{G}} \ltimes \ell^{\infty}(\widehat{\mathbb{G}}/\widehat{\mathbb{H}}))$ and $L^2(\mathbb{G}) \otimes \ell^2(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$. Let us endow the latter Hilbert space with the structure of a correspondance from $L^{\infty}(\mathbb{G})$ to itself induced by the quasi-regular representation, i.e. the left action π_l and the right action π_r are given for every $a \in L^{\infty}(\mathbb{G})$ by

$$\pi_l(a) = \mathcal{R}^*(a \otimes 1)\mathcal{R} \text{ and } \pi_r(a) = (Ja^*J) \otimes 1.$$

Then, the previous isomorphism intertwines these actions with the natural left and right actions of $L^{\infty}(\mathbb{G})$ on $\ell^2(\widehat{\mathbb{G}} \ltimes \ell^{\infty}(\widehat{\mathbb{G}}/\widehat{\mathbb{H}}))$ (inherited from its canonical identification with $H \otimes \overline{H}$). Thus, the correspondence associated with the quasi-regular representation weakly contains the identity correspondance. Since $L^{\infty}(\widehat{\mathbb{G}})$ has property (T) in the sense of [CJ85] by [Fim10, Thm 3.1], the correspondance associated with the quasi-regular representation actually contains the identity correspondence. Since any property (T) discrete quantum group is unimodular by [Fim10, Prop 3.2], we can apply [JP92, Lem 7.1] to conclude that $\mathcal{R}$ contains the trivial representation, i.e. has a fixed vector. This implies by [Fim10, Lem 2.3] that the quotient $\ell^{\infty}(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$ is finite-dimensional. In short, any relatively amenable quantum subgroup of a property (T) discrete quantum group is cofinite.

We end this section by a brief discussion of the case of quotients and extensions. We will in fact only need classical group theory here, in order to make two points.

- The discrete groups $\mathbb{Z}^2$ and $SL(2, \mathbb{Z})$ are weakly amenable (and have Cowling-Haagerup constant equal to 1). However, the semi-direct product $\mathbb{Z}^2 \rtimes SL(2, \mathbb{Z})$ is not weakly amenable (this was originally proved by in [Haa88], another proof can be found in [OP10a, Cor 2.12]). Thus, weak amenability does not pass to extensions of discrete groups.

- The discrete group $\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$ is finitely generated, and thus is a quotient of the free group $\mathbb{F}_N$ for a suitable integer N (for instance $N = 4$). But all free groups are weakly amenable with Cowling-Haagerup constant equal to 1, thus weak amenability does not pass to quotients neither.

2.2.4 The Haagerup property

Along the lines of what has been done so far for weak amenability, one can also study the Haagerup property for discrete quantum groups. Such a study has been recently carried out in the setting of locally compact quantum groups in [DFSW14], which is more general than ours. However, our approach is focused on the approximation point of view, which makes some proofs different. The definition, inspired by Definitions 2.1.14 and 2.2.8, is straightforward.

Definition 2.2.40. A discrete quantum group $\widehat{\mathbb{G}}$ is said to have the *Haagerup property* if there exists a net (a_t) of elements of $\ell^\infty(\widehat{\mathbb{G}})$ such that
- $a_t \in C_0(\widehat{\mathbb{G}})$ for all t.
- (a_t) converges pointwise to 1.
- m_{a_t} is completely positive for all t.

Example 2.2.41. If Γ is a discrete group. Then, it has the Haagerup property in the sense of Definition 2.1.14 if and only if the discrete quantum group $(C_0(\Gamma), \Delta_\Gamma)$ has the Haagerup property.

Example 2.2.42. It is clear from [Tom06] that amenable discrete quantum groups have the Haagerup property.

Remark 2.2.43. Obviously, Definition 2.2.40 coincides with the definition of the Haagerup property given in [DFSW14, Def 5.1].

Note that the multiplier associated to an element in $\ell^\infty(\widehat{\mathbb{G}})$ is always h-invariant by definition. With this in mind, the following fact is obvious.

Proposition 2.2.44. *Let $\widehat{\mathbb{G}}$ be a discrete quantum group with the Haagerup property. Then,*
- *$C_{red}(\mathbb{G})$ has the Haagerup property relative to the Haar state h.*
- *$L^\infty(\mathbb{G})$ has the Haagerup property relative to the Haar state h.*
- *$L^\infty(\mathbb{G})$ has the compact approximation property.*

Remark 2.2.45. If $\widehat{\mathbb{G}}$ has the Haagerup property and Kazhdan's property (T) in the sense of [Fim10, Def 3.1], then the von Neumann algebra $L^\infty(\mathbb{G})$ also has both the Haagerup property and property (T) and is thus finite-dimensional. This in turn means that $\widehat{\mathbb{G}}$ is a finite quantum group.

Restricting to the unimodular case, we can generalize Theorem 2.1.17. This restriction of course comes from the same reason as in Theorem 2.2.12. We refer to the end of Section 2.1 for the definitions of the operator-algebraic properties.

Theorem 2.2.46. *Let $\widehat{\mathbb{G}}$ be a* unimodular *discrete quantum group. Then, the following are equivalent :*
- *$\widehat{\mathbb{G}}$ has the Haagerup property.*
- *$C_{red}(\mathbb{G})$ has the Haagerup property relative to the Haar state h.*
- *$L^\infty(\mathbb{G})$ has the Haagerup property.*

- $L^\infty(\mathbb{G})$ *has the compact approximation property.*

Proof. We first focus on the assumptions concerning the von Neumann algebra. Note that in the finite case, the Haagerup property implies the compact approximation property (see e.g. [AD95, Rem. 4.14 and Prop 4.16]). Thus, we will assume that $L^\infty(\mathbb{G})$ has the compact approximation property and apply the same ideas as in Theorem 2.2.12 to prove that $\widehat{\mathbb{G}}$ has the Haagerup property. Let (T_t) be a net of normal unital completely positive maps giving the compact approximation property for $L^\infty(\mathbb{G})$ and let a_t be the unique element in $\ell^\infty(\widehat{\mathbb{G}})$ such that for all $\rho \in \mathrm{Irr}(\mathbb{G})$,

$$a_t p_\alpha = (h \otimes \imath)((T_t \otimes \imath)(u^\alpha)(u^\alpha)^*).$$

Then, we know from [KR99, Thm 5.5] that

$$m_{a_t} = \Delta^{-1} \circ \mathbb{E}_\Delta \circ (T_t \otimes \imath) \circ \Delta,$$

which is unital and completely positive. Thus, we only have to check that the elements a_t are in $C_0(\widehat{\mathbb{G}})$. Denote by $\|.\|_{\infty,2}$ the norm of operators from $L^\infty(\mathbb{G})$ to $L^2(\mathbb{G})$ and let $(Q_i^t)_i$ be a net of finite-rank maps approximating $x \mapsto T_t(x).\xi_h$ in $\|.\|_{\infty,2}$. Then, by density, one can find maps P_t^i with range included in the linear span of finitely many coefficients of irreducible representations and such that, for every t,

$$\|Q_t^i - P_t^i\|_{\infty,2} \underset{i}{\to} 0.$$

This implies that $x \mapsto T_t(x).\xi_h$ is also the limit of the net $(P_t^i)_i$. Let $\mathfrak{U}$ denote the isometry induced by the coproduct on $L^2(\mathbb{G})$. Noticing that $\mathfrak{U}^*(u_{i,j}^\alpha \otimes u_{k,l}^\beta) = 0$ as soon as $\alpha \neq \beta$, we see that

$$x \mapsto \mathfrak{U}^* \circ (P_t^i \otimes \imath)[\Delta(x)(1 \otimes \xi_h)]$$

has range included in the linear span of the coefficients of finitely many irreduible representations. Now, the map $x \mapsto m_{a_t}(x).\xi_h$ is by construction the norm limit of the above maps. This is easily seen to imply that a_t is in $C_0(\widehat{\mathbb{G}})$.

The same arguments can be used to prove the equivalence between the Haagerup property for $\widehat{\mathbb{G}}$ and the Haagerup property for $C_{\mathrm{red}}(\mathbb{G})$ relative to h. □

Remark 2.2.47. Theorem 2.2.46 shows that our definition of the Haagerup property coincides in the unimodular case with the definition given by M. Brannan in [Bra12a, Sec 3].

We conclude with a list of permanence properties, most of which are straightforward consequences of Definition 2.1.15 or Theorem 2.2.46.

Proposition 2.2.48. *The following hold*

1. *A discrete quantum subgroup of a discrete quantum group with the Haagerup property also has the Haagerup property.*
2. *A direct product of discrete quantum groups with the Haagerup property also has the Haagerup property.*
3. *A free product of discrete quantum groups with the Haagerup property also has the Haagerup property.*
4. *A direct limit of discrete quantum groups with the Haagerup property with injective connecting maps also has the Haagerup property.*

5. *If $\widehat{\mathbb{G}}$ is a* unimodular *discrete quantum group and if $\widehat{\mathbb{H}}$ is a discrete quantum subgroup which is amenable relative to $\widehat{\mathbb{G}}$, then the Haagerup property for $\widehat{\mathbb{H}}$ implies the Haagerup property for $\widehat{\mathbb{G}}$.*

Proof. Assertion (1) is proved like Corollary 2.2.16.

Assertion (2) is proved like Corollary 2.2.17, noticing that the tensor product of two elements in $C_0(\widehat{\mathbb{G}})$ and $C_0(\widehat{\mathbb{H}})$ respectively lies in $C_0(\widehat{\mathbb{G}}\times\widehat{\mathbb{H}})$ and that a tensor product of unital completely positive maps is again unital and completely positive.

The proof of assertion (3), is a little subtler. First note that according to [Daw13, Thm 5.9], the complete positivity of a multiplier m_a implies that a can in fact be chosen to be of the form $(\omega_a\otimes\imath)(W)$ for some state ω_a on $C_{\max}(\mathbb{G})$. This means that the free product $m_a * m_b$ of completely positive maps coresponds to the multiplier m_c with

$$c = (\omega_c\otimes\imath)(W),$$

where ω_c is the free product of the states ω_a and ω_b. So let us take nets $(\omega_{a_t}\otimes\imath)(W_1)$ and $(\omega_{b_t}\otimes\imath)(W_2)$ implementing the Haagerup property for two discrete quantum groups $\widehat{\mathbb{G}}_1$ and $\widehat{\mathbb{G}}_2$. Setting $\omega_{c_t}=\omega_{a_t}*\omega_{b_t}$, we get a net of elements $c_t=(\omega_{c_t}\otimes\imath)(W)$ converging pointwise to the identity and yielding completely positive multipliers (because ω_{c_t} is again a state). Now, [Boc93, Thm 3.9] asserts that $m_{c_t}=m_{a_t}*m_{b_t}$ is compact as soon m_{a_t} and m_{b_t} are. Let us prove that this implies that $c_t\in C_0(\widehat{\mathbb{G}}_1*\widehat{\mathbb{G}}_2)$. If $\widehat{\mathbb{K}}=\widehat{\mathbb{G}}_1*\widehat{\mathbb{G}}_2$, we have for any $\alpha\in\mathrm{Irr}(\mathbb{K})$,

$$\begin{aligned}(h\otimes\imath)((m_{c_t}\otimes\imath)(u^\alpha)(u^\alpha)^*) &= (h\otimes\imath)((\omega_{c_t}\otimes\imath\otimes\imath)(u^\alpha_{13}u^\alpha_{23})(u^\alpha)^*)\\ &= (h\otimes\imath)((\omega_{c_t}\otimes\imath\otimes\imath)[u^\alpha_{13}u^\alpha_{23}(u^\alpha_{23})^*])\\ &= (h\otimes\imath)((\omega_{c_t}\otimes\imath\otimes\imath)(u^\alpha_{13}))\\ &= c_t p_\alpha.\end{aligned}$$

Now, using the same argument as in the proof of Theorem 2.2.46, we can conclude that $c_t\in C_0(\widehat{\mathbb{G}}_1*\widehat{\mathbb{G}}_2)$.

Assertion (4) is proved like Corollary 2.2.19.

Assertion (5) was already mentioned in Remark 2.2.38. Note that it can also be recovered using the fact, proved in [AD95, Prop. 4.17], that if N is a von Neumann algebra which is amenably dominated by another von Neumann algebra with the compact approximation property, then N also has the compact approximation property. □

The finitely generated group $\mathbb{Z}^2\rtimes SL(2,\mathbb{Z})$ does not have the Haagerup property because the infinite subgroup $\mathbb{Z}^2$ has relative property (T) (see [Mar82] or [Sha99] for a quantitative statement). Hence, the Haagerup property does not pass to extensions, nor to quotients since free groups have the Haagerup property by [Haa78]. Finally, noticing that

$$\mathbb{Z}^2\rtimes SL(2,\mathbb{Z}) = (\mathbb{Z}^2\rtimes\mathbb{Z}/4\mathbb{Z}) *_{\mathbb{Z}^2\rtimes\mathbb{Z}/2\mathbb{Z}} (\mathbb{Z}^2\rtimes\mathbb{Z}/6\mathbb{Z})$$

proves that the Haagerup property does not pass to amalgamated free products. However, we will see at the end of section 2.3 that it passes to amalgamated free products *over a finite quantum subgroup.*

2.3 Free products

Free products appear to be more complicated to handle as far as weak amenability is concerned. In fact, it is still not known whether every free product of weakly amenable groups is weakly amenable. Up to now, there has been two positive results in this direction, the second one generalizing the first one. For the sake of clarity, we will give generalizations of both results separately. We will then adress the case of amalgamated free products and prove that our results extend on condition that the subgroup over which the amalgamation is done is finite.

2.3.1 The amenable case

The fact that a free product of amenable groups has Cowling-Haagerup constant equal to 1 (though it is not amenable) was first proved by M. Bożejko and M.A. Picardello in [BP93] (even allowing amalgamation over a finite subgroup). But it can also be recovered as an easy consequence of the following theorem [RX06, Thm 4.3].

Theorem 2.3.1 (Ricard, Xu). *Let $(A_i, \varphi_i)_{i\in I}$ be C^*-algebras with distinguished states (φ_i) having faithful GNS construction. Assume that for each i, there is a net of finite-rank unital completely positive maps $(V_{i,j})$ on A_i converging to the identity pointwise and preserving the state (i.e. φ_i is* CP-approximable *in the sense of [Eck13, Def 1.1]). Then, the reduced free product of the family (A_i, φ_i) has Cowling-Haagerup constant equal to* 1.

All we have to do is to find such a net of unital completely positive maps which leaves the Haar states invariant when the discrete quantum groups are amenable. This is given by the following characterization of amenability [Tom06, Thm 3.8].

Theorem 2.3.2 (Tomatsu). *A discrete quantum group $\widehat{\mathbb{G}}$ is amenable if and only if there exists a net (ω_t) of normal states on $C_{red}(\mathbb{G})$ such that the nets of completely positive maps*

$$((\omega_t \otimes \imath) \circ \Delta) \text{ and } ((\imath \otimes \omega_t) \circ \Delta)$$

converge pointwise to the identity.

The h-invariance of these maps is given by the left and right invariance of the Haar state on compact quantum groups. Thus, we have the following :

Corollary 2.3.3. *Let $(\widehat{\mathbb{G}}_i)_{i\in I}$ be a family of amenable discrete quantum groups, then*

$$\Lambda_{cb}(*_{i\in I} C_{red}(\mathbb{G}_i)) = 1.$$

If the discrete quantum groups involved are all unimodular, their free product is also unimodular and we have, by Theorem 2.2.12,

$$\Lambda_{cb}(*_{i\in I} \widehat{\mathbb{G}}_i) = 1.$$

This already yields many new examples of weakly amenable discrete qantum groups.

Example 2.3.4. Let $(G_i)_{i\in I}$ be a family of (non-trivial) compact groups, then their duals $\widehat{G}_i$ (in the sense of quantum groups) are amenable. Thus $*_{i\in I}\widehat{G}_i$ is a non-commutative and non-cocommutative discrete quantum group with Cowling-Haagerup constant equal to 1.

2.3.2 The weakly amenable case

We will now prove that a free product of weakly amenable discrete quantum groups with Cowling-Haagerup constant equal to 1 has Cowling-Haagerup constant equal to 1. This result has been proven in the classical case by E. Ricard and Q. Xu [RX06, Thm 4.3] using the following key result [RX06, Prop 4.11].

Theorem 2.3.5 (Ricard, Xu). *Let $(B_i, \psi_i)_{i\in I}$ be unital C^*-algebras with distinguished states (ψ_i) having faithful GNS constructions. Let $A_i \subset B_i$ be unital C^*-subalgebras such that the states $\varphi_i = \psi_{i|A_i}$ also have faithful GNS construction. Assume that for each i, there is a net of finite-rank maps $(V_{i,j})$ on A_i converging to the identity pointwise, preserving the state and such that $\limsup_j \|V_{i,j}\|_{cb} = 1$. Assume moreover that for each pair (i, j), there is a completely positive unital map $U_{i,j} : A_i \to B_i$ preserving the state and such that*

$$\|V_{i,j} - U_{i,j}\|_{cb} + \|V_{i,j} - U_{i,j}\|_{\mathcal{B}(L^2(A_i,\varphi_i),L^2(B_i,\psi_i))} + \|V_{i,j} - U_{i,j}\|_{\mathcal{B}(L^2(A_i,\varphi_i)^{op},L^2(B_i,\psi_i)^{op})} \underset{j}{\to} 0.$$

Then, the reduced free product of the family (A_i, φ_i) has Cowling-Haagerup constant equal to 1.

Having a characterization of weak amenability in terms of approximations of 1 on the group is the main ingredient to apply this theorem. Thus, Theorem 2.2.6 will prove crucial in the proof of the quantum version.

Theorem 2.3.6. *Let $(\widehat{\mathbb{G}}_i)_{i\in I}$ be a family of discrete quantum groups with Cowling-Haagerup constant equal to 1, then $\Lambda_{cb}(*_{i\in I}\widehat{\mathbb{G}}_i) = 1$.*

The proof of this theorem is quite involved. In order to make it more clear, we will divide it into several lemmata, most of which are rather technical. We first introduce some general notations.

Let $\widehat{\mathbb{G}}$ be a discrete quantum group, let $0 < \epsilon < 1$ and let $a \in \ell^\infty(\widehat{\mathbb{G}})$ be such that $\|m_a\|_{cb} \leqslant 1 + \epsilon$. Let

$$\xi, \eta : L^2(\mathbb{G}) \to L^2(\mathbb{G}) \otimes K$$

denote the two maps given by Theorem 2.2.6 with $\|\xi\| = \|\eta\| \leqslant \sqrt{1+\epsilon}$ and set $\gamma = (\xi+\eta)/2$ and $\delta = (\xi - \eta)/2$. Obtaining an approximation by a unital completely positive map is the first step.

Lemma 2.3.7. *Assume that $a = \widehat{S}(a)^*$ and that $m_a(1) = 1$. Then, there exists a u.c.p. map on $\mathcal{B}(L^2(\mathbb{G}))$ approximating m_a up to 6ϵ in completely bounded norm.*

Proof. We know from [KR99, Prop 2.6] that for any $x \in C(\mathbb{G})$,

$$\xi^*(x \otimes 1)\eta = m_a(x^*)^* = m_{\widehat{S}(a)^*}(x).$$

Thus,

$$m_a(x) = \frac{1}{2}(m_a(x) + m_{\widehat{S}(a)^*}(x)) = \frac{1}{2}((\eta^*(x\otimes 1)\xi + \xi^*(x \otimes 1)\eta)) = M_\gamma(x) - M_\delta(x),$$

where

$$M_\gamma(x) = \gamma^*(x \otimes 1)\gamma \text{ and } M_\delta(x) = \delta^*(x \otimes 1)\delta.$$

The maps M_γ and M_δ are completely positive, thus $\|M_\gamma\|_{cb} = \|\gamma\|^2 \leqslant 1+\epsilon$ and evaluating at 1 gives $\|1 + \delta^*\delta\| \leqslant 1 + \epsilon$, i.e. $\|M_\delta\|_{cb} = \|\delta^*\delta\| \leqslant \epsilon$.

We now want to perturb M_γ into a *unital* completely positive map. To do this, first note that

$$\|1 - \gamma^*\gamma\| = \|\delta^*\delta\| \leqslant \epsilon < 1,$$

which implies that $\gamma^*\gamma$ is invertible, and set $\widetilde{\gamma} = \gamma|\gamma|^{-1}$ where $|\gamma| = (\gamma^*\gamma)^{1/2}$. Note that $\|\widetilde{\gamma} - \gamma\| \leqslant \epsilon$. Thus, $M_{\widetilde{\gamma}}$ is a unital completely positive map and

$$\begin{aligned}
\|M_{\widetilde{\gamma}} - M_\gamma\|_{cb} &= \|M_{\gamma+(\widetilde{\gamma}-\gamma)} - M_\gamma\|_{cb} \\
&\leqslant \|\widetilde{\gamma} - \gamma\|\|\gamma\| + \|\widetilde{\gamma} - \gamma\|\|\gamma\| + \|\widetilde{\gamma} - \gamma\|\|\widetilde{\gamma} - \gamma\| \\
&\leqslant \epsilon(2 + 3\epsilon) \leqslant 5\epsilon.
\end{aligned}$$

This proves that $M_{\widetilde{\gamma}}$ is a unital completely positive map approximating m_a on $C(\mathbb{G})$ up to 6ϵ in completely bounded norm. $\square$

Set $D = \mathcal{B}(L^2(\mathbb{G}))$. We now want to prove that the previous approximation also works when the maps are seen as operators on $L^2(D,\tau)$ and $L^2(D,\tau)^{op}$, where $\tau(x) = \langle \xi_h, x(\xi_h)\rangle$. Let us start with a purely computational lemma.

Lemma 2.3.8. *For any $\zeta \in K$, $(\imath \otimes \pi)(\widehat{W})(\xi_h \otimes \zeta) = \xi_h \otimes \zeta$.*

Proof. Note that since by definition $W(\xi \otimes \xi_h) = \xi \otimes \xi_h$ for any $\xi \in H$, we also have $\widehat{W}(\xi_h \otimes \xi) = \xi_h \otimes \xi$. For any $\theta_1, \theta_2, \xi \in L^2(\mathbb{G})$,

$$\begin{aligned}\langle \xi, (\imath \otimes \omega_{\theta_1,\theta_2})(\widehat{W})\xi_h\rangle &= \langle \xi \otimes \theta_1, \widehat{W}(\xi_h \otimes \theta_2)\rangle \\ &= \langle \xi, \xi_h\rangle\langle \theta_1, \theta_2\rangle \\ &= \omega_{\theta_1,\theta_2}(1)\langle \xi, \xi_h\rangle.\end{aligned}$$

Thus by density, we have $\langle \xi, (\imath\otimes\omega)(\widehat{W})\xi_h\rangle = \omega(1)\langle \xi, \xi_h\rangle$ for any $\omega \in \mathcal{B}(L^2(\mathbb{G}))_*$. Now, let $\zeta_1, \zeta_2 \in K$ and $\xi \in L^2(\mathbb{G})$, then

$$\begin{aligned}\langle \xi \otimes \zeta_1, (\imath \otimes \pi)(\widehat{W})(\xi_h \otimes \zeta_2)\rangle &= \langle \xi, (\imath \otimes \omega_{\zeta_1,\zeta_2} \circ \pi)(\widehat{W})\xi_h\rangle \\ &= \omega_{\zeta_1,\zeta_2}(\pi(1))\langle \xi, \xi_h\rangle \\ &= \langle \xi \otimes \zeta_1, \xi_h \otimes \zeta_2\rangle.\end{aligned}$$

$\square$

This gives us a systematic way to investigate the L^2-norm of some specific operators.

Lemma 2.3.9. *Let T be any bounded linear operator from $L^2(\mathbb{G})$ to K and set*

$$A(T) = (\imath \otimes \pi)(\widehat{W})^*(1 \otimes T)\widehat{W}(\imath \otimes \xi_h) \in \mathcal{B}(L^2(\mathbb{G}), L^2(\mathbb{G}) \otimes K).$$

and $M_{A(T)}(x) = A(T)^*(x \otimes 1)A(T)$. Then, $\tau(M_{A(T)}(x^*x)) \leqslant \|T\|^2\tau(x^*x)$ and $M_{A(T)}$ is a bounded operator on $L^2(D,\tau)$ of norm less than $\|T\|^2$. If moreover $A(T)^*A(T)$ is invertible, then $M_{A(T)|A(T)|^{-1}}$ is τ-invariant.

Proof. Let us compute

$$\begin{aligned}A(T)\xi_h &= (\imath \otimes \pi)(\widehat{W})^*(1 \otimes T)\widehat{W}(\xi_h \otimes \xi_h) \\ &= (\imath \otimes \pi)(\widehat{W})^*(1 \otimes T)(\xi_h \otimes \xi_h) \\ &= (\imath \otimes \pi)(\widehat{W})^*(\xi_h \otimes T(\xi_h)) \\ &= \xi_h \otimes T(\xi_h).\end{aligned}$$

From this, we get

$$\langle \xi_h, A(T)^*(x \otimes 1)A(T)\xi_h\rangle = \langle A(T)\xi_h, (x \otimes 1)A(T)(\xi_h)\rangle = \langle \xi_h, x(\xi_h)\rangle\|T(\xi_h)\|^2$$

and Proposition 1.3.7 (Kadison's inequality) yields

$$\begin{aligned}\tau(M_{A(T)}(x)^*M_{A(T)}(x)) &\leqslant \|A(T)\|^2\tau(M_{A(T)}(x^*x)) \\ &\leqslant \|T\|^2\|A(T)\|^2\tau(x^*x) \\ &\leqslant \|T\|^4\tau(x^*x).\end{aligned}$$

Let us now turn to $A(T)^*A(T)$. First,

$$\begin{aligned}
A(T)^*A(T)\xi_h &= (\imath\otimes\xi_h^*)\widehat{W}^*(1\otimes T^*)(\imath\otimes\pi)(\widehat{W})(\xi_h\otimes T(\xi_h)) \\
&= (\imath\otimes\xi_h^*)\widehat{W}^*(1\otimes T^*)(\xi_h\otimes T(\xi_h)) \\
&= (\imath\otimes\xi_h^*)\widehat{W}^*(\xi_h\otimes T^*T(\xi_h)) \\
&= (\imath\otimes\xi_h^*)(\xi_h\otimes T^*T(\xi_h)) \\
&= \langle\xi_h, T^*T(\xi_h)\rangle\xi_h \\
&= \|T(\xi_h)\|^2\xi_h
\end{aligned}$$

and ξ_h is an eigenvector for $A(T)^*A(T)$. If $A(T)^*A(T)$ is invertible, then

$$(A(T)^*A(T))^{-1/2}\xi_h = \|T(\xi_h)\|^{-1}\xi_h.$$

Thus, $A(T)|A(T)|^{-1}\xi_h = \xi_h\otimes\|T(\xi_h)\|^{-1}T(\xi_h)$ and

$$\begin{aligned}
\tau(M_{A(T)|A(T)|^{-1}}(x)) &= \langle A(T)|A(T)|^{-1}\xi_h, (x\otimes 1)A(T)|A(T)|^{-1}\xi_h\rangle \\
&= \langle\xi_h, x(\xi_h)\rangle \\
&= \tau(x).
\end{aligned}$$

□

Applying Lemma 2.3.9 to $M_\delta = A([P-Q]/2)$, and setting $\|x\|_2 = \tau(x^*x)^{1/2}$ for $x\in D$, we can compute

$$\|(m_a - M_\gamma)(x)\|_2^2 = \|M_\delta(x)\|_2^2 = \tau(M_\delta(x)^*M_\delta(x)) \leqslant \|\delta\|^4\|x\|_2^2 \leqslant \epsilon^4\|x\|_2^2$$

i.e. $\|(m_a - M_\gamma)(x)\|_2 \leqslant \epsilon^2\|x\|_2$ and M_γ approximates m_a up to ϵ^2 in $\mathcal{B}(L^2(D,\tau))$. We now only have to control $\|M_{\widetilde{\gamma}} - M_\gamma\|_{\mathcal{B}(L^2(D,\tau))}$.

Lemma 2.3.10. $\tau((M_\gamma(x) - M_{\widetilde{\gamma}}(x))^*(M_\gamma(x) - M_{\widetilde{\gamma}}(x)))^{1/2} \leqslant 5\epsilon\tau(x^*x)^{1/2}$.

Proof. We have $\gamma = A((P+Q)/2)$. Thus, setting $T = ((P+Q)/2)$ and observing that

$$\left\|\left(T\xi_h - \frac{1}{\|T\xi_h\|}T\xi_h\right)\right\| = \left\|\xi_h\otimes\left(T\xi_h - \frac{1}{\|T\xi_h\|}T\xi_h\right)\right\| = \|(\gamma-\widetilde{\gamma})\xi_h\| \leqslant \|\gamma-\widetilde{\gamma}\| \leqslant \epsilon,$$

we can compute, again with Lemma 2.3.9,

$$\begin{aligned}
\tau(M_{\gamma-\widetilde{\gamma}}(x^*x)) &\leqslant \langle(x^*x\otimes 1)(\gamma-\widetilde{\gamma})\xi_h, (\gamma-\widetilde{\gamma})\xi_h\rangle \\
&= \left\langle(x^*x\otimes 1)\left(\xi_h\otimes\left(T\xi_h - \frac{1}{\|T\xi_h\|}T\xi_h\right)\right), \xi_h\otimes\left(T\xi_h - \frac{1}{\|T\xi_h\|}T\xi_h\right)\right\rangle \\
&= \langle(x^*x)\xi_h,\xi_h\rangle\left\|T\xi_h - \frac{1}{\|T\xi_h\|}T\xi_h\right\|^2 \\
&\leqslant \epsilon^2\tau(x^*x) \\
&\leqslant \epsilon^2\|x\|_2^2
\end{aligned}$$

and $\|M_{\gamma-\widetilde{\gamma}}(x)\|_2^2 = \tau(M_{\gamma-\widetilde{\gamma}}(x)^*M_{\gamma-\widetilde{\gamma}}(x)) \leqslant \|\gamma-\widetilde{\gamma}\|^2\tau(M_{\gamma-\widetilde{\gamma}}(x^*x)) \leqslant \epsilon^4\|x\|_2^2$. Now, for all $x \in D$, we have

$$\begin{aligned}
\|M_\gamma(x) - M_{\widetilde{\gamma}}(x)\|_2 &= \|M_\gamma(x) - M_{\gamma+(\widetilde{\gamma}-\gamma)}(x)\|_2 \\
&\leqslant \|M_{\widetilde{\gamma}-\gamma}(x)\|_2 + \tau((\widetilde{\gamma}-\gamma)^*(x^*\otimes 1)\gamma\gamma^*(x\otimes 1)(\widetilde{\gamma}-\gamma))^{1/2} \\
&\quad + \tau(\gamma^*(x^*\otimes 1)(\widetilde{\gamma}-\gamma)(\widetilde{\gamma}-\gamma)^*(x\otimes 1)\gamma)^{1/2} \\
&\leqslant \epsilon^2\|x\|_2 + (1+\epsilon)\tau((\widetilde{\gamma}-\gamma)^*(x^*\otimes 1)(x\otimes 1)(\widetilde{\gamma}-\gamma))^{1/2} \\
&\quad + \epsilon\tau(\gamma^*(x^*\otimes 1)(x\otimes 1)\gamma)^{1/2} \\
&\leqslant \epsilon^2\|x\|_2 + (1+\epsilon)\tau(M_{\widetilde{\gamma}-\gamma}(x^*x))^{1/2} + \epsilon\tau(M_\gamma(x^*x))^{1/2} \\
&\leqslant \epsilon^2\|x\|_2 + (1+\epsilon)\epsilon\|x\|_2 + \epsilon(1+\epsilon)\|x\|_2 \\
&\leqslant 5\epsilon\|x\|_2.
\end{aligned}$$

Finally, $\|(m_a - M_{\widetilde{\gamma}})(x)\|_2 \leqslant 6\epsilon\|x\|_2$. □

We have to chek that this approximation also works in $\mathcal{B}(L^2(D,\tau)^{op})$, but this is straightforward.

Lemma 2.3.11. *$M_{\widetilde{\gamma}}$ also approximates m_a up to 6ϵ in $\mathcal{B}(L^2(D,\tau)^{op})$.*

Proof. To estimate the opposite L^2-norm, one only needs to do all the previous computations exchanging P and Q. Since they play symmetric rôles, we get the same result. □

Thus, we are able to approximate m_a by a unital completely positive map in all the required norms. There remains only to check the τ-invariance condition.

Lemma 2.3.12. *The maps $M_{\widetilde{\gamma}}$ and m_a are τ-preserving.*

Proof. For $M_{\widetilde{\gamma}}$, this comes from Lemma 2.3.9. For m_a, this comes from the following computation

$$\begin{aligned}
\tau(m_a(x)) &= \langle \eta^*(x\otimes 1)\xi(\xi_h), \xi_h\rangle \\
&= \langle (x\otimes 1)\xi(\xi_h), \eta(\xi_h)\rangle \\
&= \langle (x\otimes 1)A(P)(\xi_h), A(Q)(\xi_h)\rangle \\
&= \langle (x\otimes 1)(\xi_h\otimes P(\xi_h), \xi_h\otimes Q(\xi_h)\rangle \\
&= \langle x(\xi_h), \xi_h\rangle\langle P(\xi_h), Q(\xi_h)\rangle \\
&= \tau(x)\tau(m_a(1))
\end{aligned}$$

and the fact that we have assumed m_a to be unital. □

We are now ready to prove the theorem.

Proof of Theorem 2.3.6. For each i, set $A_i = C_{\text{red}}(\mathbb{G}_i)$ and $B_i = \mathcal{B}(L^2(\mathbb{G}_i))$. Consider a net $(a_{i,t})_t$ of finitely supported elements in $\ell^\infty(\widehat{\mathbb{G}}_i)$ converging pointwise to the identity and such that $\limsup_t \|m_{a_{i,t}}\|_{cb} = 1$ and note that since $\widehat{\varepsilon}(a_{i,t}) \to 1$ (because of the pointwise convergence assumption, see Remark 2.2.5), we can, up to extracting a suitable subsequence, assume it to be non-zero and divide by it so that $m_{a_{i,t}}$ becomes unital. For any $0 < \epsilon < 1$, there is a $t(\epsilon)$ such that $\|m_{a_{i,t(\epsilon)}}\|_{cb} \leqslant 1+\epsilon$ (the same being automatically true for $m_{\widehat{S}(a_{i,t(\epsilon)})}$). Since

$$\widehat{S}\circ * \circ \widehat{S}\circ * = \imath,$$

we can replace $a_{i,t}$ by $(a_{i,t}+\widehat{S}(a_{i,t}))/2$ so that all the hypothesis of Lemma 2.3.7 are satisfied. Then, by lemmata 2.3.7, 2.3.9, 2.3.10, 2.3.11 and 2.3.12 we get a unital completely positive approximation in completely bounded norm and in both L^2-norms. Applying Theorem 2.3.5 proves that $\Lambda_{cb}(*_i A_i) = 1$. Since the original maps were all multipliers, the resulting finite-dimensional approximation is also implemented by multipliers and $\Lambda_{cb}(*_{i\in I}\widehat{\mathbb{G}}_i) = 1$. □

2.3.3 Amalgamation over a finite quantum subgroup

Though it is not explicitely written in the paper [RX06], Theorem 2.3.5 extends to the amalgamated case in the following form :

Theorem 2.3.13 (Ricard, Xu). *Let C be a C^*-algebra, let $(B_i)_{i\in I}$ be unital C^*-algebras together with GNS-faithful conditional expectations $\mathbb{E}_{B_i} : B_i \to C$. Let $A_i \subset B_i$ be unital C^*-subalgebras with GNS-faithful conditional expectations $\mathbb{E}_{A_i} : A_i \to C$ which are the restrictions of $\mathbb{E}_{B_i}$. Assume that for each i, there is a net of finite-rank maps $(V_{i,j})_j$ on A_i converging to the identity pointwise, satisfying $\mathbb{E}_{A_i}\circ V_{i,j} = \mathbb{E}_{A_i}$ and such that $\limsup_j \|V_{i,j}\|_{cb} = 1$. Assume moreover that for each pair (i,j), there is a completely positive unital map $U_{i,j} : A_i \to B_i$ satisfying $\mathbb{E}_{B_i}\circ V_{i,j} = \mathbb{E}_{A_i}$ and such that*

$$\|V_{i,j}-U_{i,j}\|_{cb}+\|V_{i,j}-U_{i,j}\|_{\mathcal{B}(L^2(A_i,\mathbb{E}_{A_i}),L^2(B_i,\mathbb{E}_{B_i}))}+\|V_{i,j}-U_{i,j}\|_{\mathcal{B}(L^2(A_i,\mathbb{E}_{A_i})^{op},L^2(B_i,\mathbb{E}_{B_i})^{op})} \underset{j}{\to} 0.$$

Assume moreover that the maps $V_{i,j}$ and $U_{i,j}$ are the identity on C for all i, j. Then, the reduced amalgamated free product $*_C(A_i,\mathbb{E}_{A_i})$ has Cowling-Haagerup constant equal to 1.

It is easy to see that these extra conditions are satisfied for a free product of groups amalgamated over a *finite* subgroup, thus recovering and improving the whole result of M. Bożejko and M.A. Picardello [BP93]. The same strategy works for discrete quantum groups, but again with some additional technicalities. Let us first detail a natural way of averaging multipliers over a finite quantum subgroup, which is the first step to the proof. We will use the notations of Subsection 1.2.2.

Definition 2.3.14. Let $\widehat{\mathbb{G}}$ be a discrete quantum group, $\widehat{\mathbb{H}}$ a *finite* quantum subgroup of $\widehat{\mathbb{G}}$ and let $a\in\ell^\infty(\widehat{\mathbb{G}})$. The *averaging of a over* $\widehat{\mathbb{H}}$ is the element $c\in\ell^\infty(\widehat{\mathbb{G}})$ defined by

$$c = (\widehat{h}\otimes\imath)[(p_{\mathbb{H}}\otimes\imath)\widehat{\Delta}(a)].$$

We now prove that this averaging process is well-behaved with respect to completely bounded norms and that it yields the identity on $\widehat{\mathbb{H}}$.

Lemma 2.3.15. *The averaging of a over $\widehat{\mathbb{H}}$ satisfies $\|m_c\|_{cb} \leqslant |\widehat{h}(p_{\mathbb{H}})|\|m_a\|_{cb}$. Moreover, m_c is a multiple of the identity on $C_{red}(\mathbb{H})\subset C_{red}(\mathbb{G})$.*

Proof. If $\eta,\xi : L^2(\mathbb{G})\to L^2(\mathbb{G})\otimes K$ are the maps coming from Theorem 2.2.6, we set

$$\xi' = (\widehat{h}\otimes\imath\otimes\imath)[\widehat{W}_{12}^*(p_{\mathbb{H}}\otimes\xi)\widehat{W}]. \tag{2.2}$$

Let us make the meaning of this definition clear : if $x\in\ell^2(\mathbb{G})\otimes L^2(\mathbb{G})$, then $\widehat{W}x$ belongs to the same space so that we can apply ξ to its second leg, yielding an element $y\in\ell^2(\widehat{\mathbb{G}})\otimes L^2(\mathbb{G})\otimes K$ to which we can apply $\widehat{W}_{12}^*$. Now, taking $p_{\mathbb{H}}$ into account we see that the term inside the brackets is a linear map from $\ell^2(\mathbb{H})\otimes L^2(\mathbb{G})$ to $\ell^2(\mathbb{H})\otimes L^2(\mathbb{G})\otimes K$. Eventually, applying $\widehat{h}$ to the first leg (note that it is defined on all $\mathcal{B}(\ell^2(\widehat{\mathbb{H}}))$ because it is finite-dimensional) gives a map from $L^2(\mathbb{G})$ to $L^2(\mathbb{G})\otimes K$.

We are going to prove that $c\otimes 1 = (1\otimes\eta^*)\widehat{W}_{12}^*(1\otimes\xi')\widehat{W}$, which will imply our claim on the completely bounded norm since $\|\xi'\| \leqslant |\widehat{h}(p_{\mathbb{H}})|\|\xi\|$. First note that since

$$(\widehat{\Delta}\otimes\imath)(x) = \widehat{W}_{12}^*(1\otimes x)\widehat{W}_{12},$$

we have

$$\begin{array}{rcl}\widehat{\Delta}(a)\otimes 1 & = & \widehat{W}_{12}^*(1\otimes 1\otimes\eta^*)(1\otimes\widehat{W}_{12}^*)(1\otimes 1\otimes\xi)(1\otimes\widehat{W})\widehat{W}_{12} \\ & = & (1\otimes 1\otimes\eta^*)(\widehat{W}_{12}^*\otimes 1)(1\otimes\widehat{W}_{12}^*)(\widehat{W}_{12}\otimes 1)(1\otimes 1\otimes\xi)\widehat{W}_{12}^*\widehat{W}_{23}\widehat{W}_{12} \\ & = & (1\otimes 1\otimes\eta^*)(\widehat{W}_{23}^*\widehat{W}_{13}^*\otimes 1)(1\otimes 1\otimes\xi)\widehat{W}_{13}\widehat{W}_{23}\end{array}$$

where we used twice the pentagonal equation for $\widehat{W}$. Applying $\widehat{h}(p_{\mathbb{H}}\,.)$ to the first leg yields the result. Now, if $\alpha\in\mathrm{Irr}(\mathbb{H})$, we get using the invariance of $\widehat{h}$,

$$\begin{array}{rcl} cp_\alpha & = & (\widehat{h}\otimes\imath)[(p_{\mathbb{H}}\otimes\imath)\widehat{\Delta}(a)]p_\alpha \\ & = & (\widehat{h}\otimes\imath)[(p_{\mathbb{H}}\otimes p_{\mathbb{H}})\widehat{\Delta}(a)]p_\alpha \\ & = & (\widehat{h}\otimes\imath)[\widehat{\Delta}(p_{\mathbb{H}}a)]p_\alpha \\ & = & \widehat{h}(p_{\mathbb{H}}a)p_\alpha\end{array}$$

and $m_c = \widehat{h}(p_{\mathbb{H}}a)\,\mathrm{Id}$ on $C_{\mathrm{red}}(\mathbb{H})$. □

Using this averaging technique, we have an $\widehat{\mathbb{H}}$-invariant version of weak amenability.

Proposition 2.3.16. *Let $\widehat{\mathbb{G}}$ be a discrete quantum group, $\widehat{\mathbb{H}}$ a* finite *quantum subgroup of $\widehat{\mathbb{G}}$ and let (a_t) be a net of finitely supported elements in $\ell^\infty(\widehat{\mathbb{G}})$ converging pointwise to* 1. *Then, there is a net (b_t) of finite-rank elements in $\ell^\infty(\widehat{\mathbb{G}})$ converging pointwise to* 1*, such that*

$$\limsup_t \|m_{b_t}\|_{cb} \leqslant \limsup_t \|m_{a_t}\|_{cb}$$

and m_{b_t} is the identity on $C_{red}(\mathbb{H})\subset C_{red}(\mathbb{G})$.

Proof. Let c_t be the averaging of a_t over $\widehat{\mathbb{H}}$ by the previous construction and let $\mathrm{Supp}(a_t)$ be the set of equivalence classes of irreducible representations α such that $a_tp_\alpha\neq 0$. Then,

$$c_tp_\alpha = (\widehat{h}\otimes\imath)\left[(p_{\mathbb{H}}\otimes p_\alpha)\widehat{\Delta}(a)\right] = (\widehat{h}\otimes\imath)\left[\sum_{\beta\in\mathrm{Irr}(\mathbb{H})}\sum_{\gamma\in\mathrm{Supp}(a)}\widehat{\Delta}(p_\gamma)(p_\beta\otimes p_\alpha)\widehat{\Delta}(a)\right]$$

By definition, $\widehat{\Delta}(p_\gamma)(p_\beta\otimes p_\alpha)\neq 0$ if and only if $\gamma\subset\beta\otimes\alpha$, which by Frobenius reciprocity (see e.g. [Tim08, Prop 3.1.11]) is equivalent to $\overline{\alpha}\subset\overline{\gamma}\otimes\beta$. Hence, c_tp_α is non-zero only if α belongs to the finite set

$$\bigcup_{\beta\in\mathrm{Irr}(\mathbb{H})}\bigcup_{\gamma\in\mathrm{Supp}(a)}\{\alpha\in\mathrm{Irr}(\mathbb{G}),\overline{\alpha}\subset\overline{\gamma}\otimes\beta\}$$

and c_t has finite support. The same holds for

$$b_t = \frac{1}{\widehat{h}(p_{\mathbb{H}}a_t)}c_t,$$

which induces multipliers m_{b_t} which are the identity on $C_{\mathrm{red}}(\mathbb{H})$.

Assume now that a_t converges pointwise to 1, and note that the inequality of the completely bounded norms is obvious from Lemma 2.3.15. Fix $\beta\in\mathrm{Irr}(\mathbb{G})$, $\epsilon>0$ and let D be the (finite) set of $\gamma\in\mathrm{Irr}(\mathbb{G})$ which are contained in $\alpha\otimes\beta$ for some $\alpha\in\mathrm{Irr}(\mathbb{H})$. We denote by p_D the sum of the projections p_γ for $\gamma\in D$. Let t be such that :

- $\|a_t p_D - p_D\| \leqslant \dfrac{\epsilon}{4}$
- $\|a_t p_{\mathbb{H}} - p_{\mathbb{H}}\| \leqslant \dfrac{\epsilon}{4}$
- $|\widehat{h}(a_t p_{\mathbb{H}})| > \dfrac{1}{2}$

Then,

$$\begin{aligned}
\|b_t p_\beta - p_\beta\| &= \||\widehat{h}(a_t p_{\mathbb{H}})|^{-1}(\widehat{h}\otimes\imath)[(p_{\mathbb{H}}\otimes p_\beta)\widehat{\Delta}(a_t)] - p_\beta\| \\
&= |\widehat{h}(a_t p_{\mathbb{H}})|^{-1}\|(\widehat{h}\otimes\imath)[(p_{\mathbb{H}}\otimes p_\beta)\widehat{\Delta}(a_t) - (a_t p_{\mathbb{H}}\otimes p_\beta)]\| \\
&= |\widehat{h}(a_t p_{\mathbb{H}})|^{-1}\|(\widehat{h}\otimes\imath)[(p_{\mathbb{H}}\otimes p_\beta)\widehat{\Delta}(a_t p_D) - (a_t p_{\mathbb{H}}\otimes p_\beta)]\| \\
&\leqslant |\widehat{h}(a_t p_{\mathbb{H}})|^{-1}\|(\widehat{h}\otimes\imath)[(p_{\mathbb{H}}\otimes p_\beta)\widehat{\Delta}(a_t p_D - p_D)]\| \\
&+ |\widehat{h}(a_t p_{\mathbb{H}})|^{-1}\|(\widehat{h}\otimes\imath)[(p_{\mathbb{H}}\otimes p_\beta)\widehat{\Delta}(p_D) - p_{\mathbb{H}}\otimes p_\beta)]\| \\
&+ |\widehat{h}(a_t p_{\mathbb{H}})|^{-1}\|(\widehat{h}\otimes\imath)[(p_{\mathbb{H}} - a_t p_{\mathbb{H}})\otimes p_\beta)]\| \\
&\leqslant |\widehat{h}(a_t p_{\mathbb{H}})|^{-1}(\|a_t p_D - p_D\| + \|a_t p_{\mathbb{H}} - p_{\mathbb{H}})\\
&\leqslant 2\left(\frac{\epsilon}{4}+\frac{\epsilon}{4}\right) = \epsilon
\end{aligned}$$

using the fact that $(p_{\mathbb{H}}\otimes p_\beta)\widehat{\Delta}(p_D) = p_{\mathbb{H}}\otimes p_\beta$. □

We can now generalize the result of M. Bożejko and M.A. Picardello.

Theorem 2.3.17. *Let $(\widehat{\mathbb{G}}_i)_{i\in I}$ be a family of amenable discrete quantum groups and let $\widehat{\mathbb{H}}$ be a common* finite *quantum subgroup, then*

$$\Lambda_{cb}(*_{\widehat{\mathbb{H}}}\widehat{\mathbb{G}}_i) = 1.$$

Proof. Let (a_t) be a net of finitely supported elements converging pointwise to 1 and such that m_{a_t} is unital and completely positive. Apply the previous proposition to a_t to obtain an element b_t. Apply it again to $\widehat{S}(b_t)^*$ to produce a third element b'_t. Then, (b'_t) is a net of finitely supported elements converging pointwise to 1 and such that $m_{b'_t}$ is the identity on $C_{\mathrm{red}}(\mathbb{H})$. In the case of a unital completely positive multiplier, the proof of Theorem 2.2.6 yields a map $\xi : L^2(\mathbb{G}_i) \to L^2(\mathbb{G}_i)\otimes K$ such that $m_{a_t}(x) = \xi^*(x\otimes\imath)\xi$. This implies that $m_{b'_t}(x) = (\xi')^*(x\otimes 1)\xi'$ is unital and completely positive. Moreover, the conditional expectation $\mathbb{E}_{\mathbb{H}}\circ m_{b'_t}$ is invariant with respect to the Haar state, because $m_{b'_t}$ is invariant. Since, by [Ver04, Prop 2.2], $\mathbb{E}_{\mathbb{H}}$ is the unique conditional expectation satisfying this invariance, we have $\mathbb{E}_{\mathbb{H}}\circ m_{b'_t} = \mathbb{E}_{\mathbb{H}}$.

Now that all the multipliers are the identity on $C_{\mathrm{red}}(\mathbb{H})$ and preserve the conditional expectation $\mathbb{E}_{\mathbb{H}}$, their amalgamated free product makes sense (see for example [BO08, Thm 4.8.5] for details on the construction of the free product of u.c.p. maps). We can then follow the proof of Proposition 2.2.48 to conclude. □

Remark 2.3.18. Recall that since an extension of amenable groups is again amenable, the groups $\mathbb{Z}^2\rtimes\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}^2\rtimes\mathbb{Z}/6\mathbb{Z}$ are amenable. However, as noticed in the end of Section 2.2.3, the group

$$\mathbb{Z}^2\rtimes SL(2,\mathbb{Z}) = (\mathbb{Z}^2\rtimes\mathbb{Z}/4\mathbb{Z}) *_{\mathbb{Z}^2\rtimes\mathbb{Z}/2\mathbb{Z}} (\mathbb{Z}^2\rtimes\mathbb{Z}/6\mathbb{Z})$$

is not weakly amenable. This proves that the finiteness condition in Theorem 2.3.17 cannot be removed.

Let us briefly mention that this averaging technique can also be used to improve the statement of Proposition 2.2.48 concerning the Haagerup property.

Theorem 2.3.19. *Let $\widehat{\mathbb{G}}_i$ be a family of discrete quantum groups with the Haagerup property and let $\widehat{\mathbb{H}}$ be a common* finite *quantum subgroup. Then, $*_{\widehat{\mathbb{H}}}\widehat{\mathbb{G}}_i$ has the Haagerup property.*

Proof. We first prove that completely positive multipliers on $C_{\text{red}}(\mathbb{G}_i)$ can be averaged so that they become the identity on $C_{\text{red}}(\mathbb{H})$. Let us first recall that by [KR99, Prop 2.6], if $c \in \ell^\infty(\widehat{\mathbb{G}}_i)$ satisfies $m_c(x) = \eta^*(x \otimes 1)\xi$ for some maps $\xi, \eta \in \mathcal{B}(L^2(\mathbb{G}_i), L^2(\mathbb{G}_i) \otimes K)$, then

$$m_{\widehat{S}(c)^*}(x) = \xi^*(x \otimes 1)\eta,$$

where $\widehat{S}$ denotes the antipode of $\widehat{\mathbb{G}}_i$. Let (a_t) be a net of elements in $C_0(\widehat{\mathbb{G}}_i)$ converging pointwise to 1 and such that m_{a_t} is unital and completely positive. Let b_t be the averaging of a_t over $\widehat{\mathbb{H}}$. Average again $\widehat{S}(b_t)^*$ to produce a third element b'_t. Then, by Lemma 2.3.15, (b'_t) is a net of elements in $C_0(\widehat{\mathbb{G}}_i)$ converging pointwise to 1 and such that $m_{b'_t}$ is the identity on $C_{\text{red}}(\mathbb{H})$. In the case of a unital completely positive multiplier, the proof of Theorem 2.2.6 yields a map $\xi : L^2(\mathbb{G}_i) \to L^2(\mathbb{G}_i) \otimes K$ such that $m_{a_t}(x) = \xi^*(x \otimes \imath)\xi$. This implies that $m_{b'_t}(x) = (\xi')^*(x \otimes 1)\xi'$ is unital and completely positive. Moreover, the conditional expectation $\mathbb{E}_{\mathbb{H}} \circ m_{b'_t}$ is invariant with respect to the Haar state, because $m_{b'_t}$ is invariant. Since, by [Ver04, Prop 2.2], $\mathbb{E}_{\mathbb{H}}$ is the unique conditional expectation satisfying this invariance, we have $\mathbb{E}_{\mathbb{H}} \circ m_{b'_t} = \mathbb{E}_{\mathbb{H}}$.

Now that all the multipliers are the identity on $C_{\text{red}}(\mathbb{H})$ and preserve the conditional expectation $\mathbb{E}_{\mathbb{H}}$, their amalgamated free product makes sense (see for example [BO08, Thm 4.8.5] for details on the construction of the free product of u.c.p. maps). We can then follow the proof of Proposition 2.2.48 to conclude. □

When the quantum groups are not amenable but are weakly amenable with Cowling-Haagerup constant equal to 1, one can try the same strategy. Averaging first a_t and then $S(a_t)^*$ on $\widehat{\mathbb{H}}$ gives an element b'_t in $\ell^\infty(\widehat{\mathbb{G}})$ and two maps η' and ξ' in $\mathcal{B}(L^2(\mathbb{G}), L^2(\mathbb{G}) \otimes K)$ satisfying

$$m_{b'_t}(x) = (\eta')^*(x \otimes 1)\xi'$$

and such that the multiplier $m_{b'_t}$ is the identity on $C_{\text{red}}(\mathbb{H})$. Setting $\gamma' = (\eta' + \xi')/2$ and $\widetilde{\gamma}' = \gamma'|\gamma'|^{-1}$, the unital completely positive approximation we are interested in is

$$M_{\widetilde{\gamma}'}(x) = (\widetilde{\gamma}')^*(x \otimes 1)\widetilde{\gamma}'.$$

The problem would then be to prove that this operator is the identity on $C_{\text{red}}(\mathbb{H})$. We do not know whether this fact holds or not. However, we can use again an averaging trick, but at the level of C*-algebras, to build a new unital completely positive approximation which will be the identity on $C_{\text{red}}(\mathbb{H})$. If $T : C_{\text{red}}(\mathbb{G}) \to \mathcal{B}(L^2(\mathbb{G}))$ is a linear map, we define a linear map $R_{\widehat{\mathbb{H}}}(T)$ by

$$R_{\widehat{\mathbb{H}}}(T) : x \mapsto \int_{\mathcal{U}(C_{\text{red}}(\mathbb{H}))} T(xv^*)v dv,$$

where the integration is done with respect to the normalized Haar measure of the compact group $\mathcal{U}(C_{\text{red}}(\mathbb{H}))$ (recall that $C_{\text{red}}(\mathbb{H})$ is finite-dimensional). Similarly, we define a linear map $L_{\widehat{\mathbb{H}}}(T)$ by

$$L_{\widehat{\mathbb{H}}}(T) : x \mapsto \int_{\mathcal{U}(C_{\text{red}}(\mathbb{H}))} u^*T(ux) du.$$

Let us give some elementary properties of these two operations.

Lemma 2.3.20. *If T is completely bounded, then $R_{\widehat{\mathbb{H}}}(T)$ (resp. $L_{\widehat{\mathbb{H}}}(T)$) is also completely bounded with $\|R_{\widehat{\mathbb{H}}}(T)\|_{cb} \leqslant \|T\|_{cb}$ (resp. $\|L_{\widehat{\mathbb{H}}}(T)\|_{cb} \leqslant \|T\|_{cb}$). Moreover, for any $a, b \in C_{red}(\mathbb{H})$,*

$$R_{\widehat{\mathbb{H}}} \circ L_{\widehat{\mathbb{H}}}(T)(axb) = a[R_{\widehat{\mathbb{H}}} \circ L_{\widehat{\mathbb{H}}}(T)(x)]b. \tag{2.3}$$

Proof. Set $S = R_{\widehat{\mathbb{H}}}(T)$ (the computation is similar for $L_{\widehat{\mathbb{H}}}(T)$). For any integer n and any $x \in C_{\mathrm{red}}(\mathbb{G}) \otimes M_n(\mathbb{C})$, we have

$$\begin{aligned}
\|(S \otimes \mathrm{Id}_{M_n(\mathbb{C})})(x)\| &\leqslant \int_{\mathcal{U}(C_{\mathrm{red}}(\mathbb{H}))} \|(T \otimes \mathrm{Id}_{M_n(\mathbb{C})})(x(v^* \otimes 1))(v \otimes 1)\| dv \\
&\leqslant \int_{\mathcal{U}(C_{\mathrm{red}}(\mathbb{H}))} \|T\|_{cb}\|x\| dv \\
&= \|T\|_{cb}\|x\|.
\end{aligned}$$

Writing any element in $C_{\mathrm{red}}(\mathbb{H})$ as a linear combination of four unitaries, we can restrict ourselves to prove Equation (2.3) when a and b are unitaries. In that case, the changes of variables $u = u'a$ and $v = v'b$ yield

$$\begin{aligned}
R_{\widehat{\mathbb{H}}} \circ L_{\widehat{\mathbb{H}}}(T)(a^*xb) &= \iint_{\mathcal{U}(C_{\mathrm{red}}(\mathbb{H})) \times \mathcal{U}(C_{\mathrm{red}}(\mathbb{H}))} u^*T(ua^*xbv^*)vdudv \\
&= \iint_{\mathcal{U}(C_{\mathrm{red}}(\mathbb{H})) \times \mathcal{U}(C_{\mathrm{red}}(\mathbb{H}))} a^*(u')^*T(u'x(v')^*)v'bdu'dv' \\
&= a^*[R_{\widehat{\mathbb{H}}} \circ L_{\widehat{\mathbb{H}}}(T)(x)]b.
\end{aligned}$$

□

With this in hand, we will be able to average the completely positive maps approximating the multipliers. Let us check that this averaging behaves nicely on the multipliers $m_{b'(t)}$.

Lemma 2.3.21. *The maps $A_t = R_{\widehat{\mathbb{H}}} \circ L_{\widehat{\mathbb{H}}}(m_{b'(t)})$ have finite rank, converge pointwise to the identity, are equal to the identity on $C_{red}(\mathbb{H})$ and satisfy $\limsup_t \|A_t\|_{cb} = \limsup_t \|m_{b'(t)}\|_{cb}$.*

Proof. The pointwise convergence, the identity property and the assertion on the completely bounded norms follow from the construction. To prove that the rank is finite, first note that if $\alpha \in \mathrm{Irr}(\mathbb{G})$ and $u, v \in C_{\mathrm{red}}(\mathbb{H})$, then $u(u^{\alpha}_{i,j})v$ belongs to the linear span of coefficients of irreducible subrepresentations γ of $\beta_1 \otimes \alpha \otimes \beta_2$ for $\beta_1, \beta_2 \in \mathrm{Irr}(\mathbb{H})$. Thus, by Frobenius reciprocity,

$$A_t(u^{\alpha}_{i,j}) = \iint_{\mathcal{U}(C_{\mathrm{red}}(\mathbb{H})) \times \mathcal{U}(C_{\mathrm{red}}(\mathbb{H}))} u^* m_{b'(t)}(u(u^{\alpha}_{i,j})v^*)vdudv$$

is equal to 0 as soon as α is not in finite the set

$$\bigcup_{\beta_1, \beta_2 \in \mathrm{Irr}(\mathbb{H})} \bigcup_{\gamma \in \mathrm{Supp}(b'(t))} \{\alpha \in \mathrm{Irr}(\mathbb{G}), \overline{\alpha} \in \beta_2 \otimes \overline{\gamma} \otimes \beta_1\}.$$

□

The following theorem is the quantum version of the best known statement on stability of weak amenability with respect to free products for classical discrete groups.

Theorem 2.3.22. *Let $(\widehat{\mathbb{G}}_i)_{i \in I}$ be a family of weakly amenable discrete quantum groups such that $\Lambda_{cb}(\widehat{\mathbb{G}}_i) = 1$ for every $i \in I$ and let $\widehat{\mathbb{H}}$ be a common* finite *quantum subgroup. Then,*

$$\Lambda_{cb}(*_{\widehat{\mathbb{H}}} \widehat{\mathbb{G}}_i) = 1.$$

Proof. Using the notations of Theorem 2.3.5, we set :

- $A_i = C_{\mathrm{red}}(\mathbb{G}_i)$
- $B_i = \mathcal{B}(L^2(\mathbb{G}_i))$
- $C = C_{\mathrm{red}}(\mathbb{H})$
- $\mathbb{E}_{A_i} = \mathbb{E}_{\mathbb{H}}$

To define the conditional expectations $\mathbb{E}_{B_i}$, first consider the orthogonal projection

$$P^i_{\mathbb{H}} : L^2(\mathbb{G}_i) \to L^2(\mathbb{H}).$$

Then, $\mathbb{E}'_i : x \mapsto P^i_{\mathbb{H}} x P^i_{\mathbb{H}}$ is a conditional expectation from $\mathcal{B}(L^2(\mathbb{G}_i))$ to $\mathcal{B}(L^2(\mathbb{H}))$ with the property that for any coefficient x of an irreducible representation in $\mathrm{Irr}(\mathbb{G}_i) \setminus \mathrm{Irr}(\mathbb{H})$, $\mathbb{E}'_i(x) = 0$. In fact, the restriction of $\mathbb{E}'_i$ to $C_{\mathrm{red}}(\mathbb{G}_i)$ is precisely the conditional expectation $\mathbb{E}_{\mathbb{H}}$ of Proposition 1.2.12. Because $C_{\mathrm{red}}(\mathbb{H})$ is finite-dimensional, there is also a conditional expectation

$$\mathbb{E}''_i : \mathcal{B}(L^2(\mathbb{H})) \to C_{\mathrm{red}}(\mathbb{H}).$$

We set $\mathbb{E}_{B_i} = \mathbb{E}''_i \circ \mathbb{E}'_i$. Since $\mathbb{E}_{A_i}$ is the restriction of $\mathbb{E}'_i$to A_i it is also the restriction of $\mathbb{E}_{B_i}$.

Let us fix an index i, let $(a_j)_j$ be a net of elements in $\ell^\infty(\mathbb{G}_i)$ implementing weak amenability. Averaging a_j over $\widehat{\mathbb{H}}$ we get an element b_j. Averaging $S(b_j)^*$ yields another element b'_j and we can assume that $b'_j = S(b'_j)^*$ as in the proof of Theorem 2.3.6. We therefore set $V_{i,j} = R_{\widehat{\mathbb{H}}} \circ L_{\widehat{\mathbb{H}}}(m_{b'_j})$. By Lemma 2.3.21, these maps satisfy all the required properties. Recall that $\gamma'_j = (\xi'_j + \eta'_j)/2$ and $\widetilde{\gamma}'_j = \gamma'_j|\gamma'_j|^{-1}$. Set

$$\zeta_j = \int_{\mathcal{U}(C_{\mathrm{red}}(\mathbb{H}))} (1 \otimes u^*)\widetilde{\gamma}_j u du$$

and observe that

$$R_{\widehat{\mathbb{H}}} \circ L_{\widehat{\mathbb{H}}}(M_{\widetilde{\gamma}_j}) = M_{\zeta_j} : x \mapsto \zeta_j^*(1 \otimes x)\zeta_j$$

is a completely positive map which is the identity on $C_{\mathrm{red}}(\mathbb{H})$ by Equation (2.3). We therefore set $U_{i,j} = M_{\zeta_j}$. Then, Lemma 2.3.21 yields

$$\|V_{i,j} - U_{i,j}\|_{cb} = \|R_{\widehat{\mathbb{H}}} \circ L_{\widehat{\mathbb{H}}}(m_{b'_j} - M_{\widetilde{\gamma}_j})\|_{cb} \leqslant \|m_{b'_j} - M_{\widetilde{\gamma}_j}\|_{cb},$$

so that, by [Fre12, Lem 4.3] the convergence in completely bounded norm holds. We still have to check the compatibility of the maps with the conditional expectations and the L^2-norm convergence.

Let us consider the conditional expectation

$$\mathbb{E} = \mathbb{E}_{B_i} \circ U_{i,j} : C_{\mathrm{red}}(\mathbb{G}_i) \to C_{\mathrm{red}}(\mathbb{H}).$$

We claim that $\mathbb{E}(x) = 0$ whenever x is a coefficient of a representation in $\mathrm{Irr}(\mathbb{G})\setminus\mathrm{Irr}(\mathbb{H})$. In fact, it follows directly from the explicit expression of ξ'_j given in Lemma 2.3.15 that there is a vector $v_{\xi'_j} \in K$ such that for any $y \in C_{\mathrm{red}}(\mathbb{H})$, $\xi'_j(y) = y \otimes v_{\xi'_j}$. The same holds for η'_j and γ'_j (with different vectors), hence also for $\widetilde{\gamma}'_j$. From this, a straightforward calculation yields

$$P_{\mathbb{H}} M_{\widetilde{\gamma}'_j}(x) P_{\mathbb{H}} = 0. \tag{2.4}$$

Since elements in $\mathcal{U}(C_{\mathrm{red}}(\mathbb{H}))$ respect the decomposition $L^2(\mathbb{G}) = L^2(\mathbb{H}) \oplus L^2(\mathbb{H})^\perp$, Equation (2.4) also holds for M_{ζ_j}, proving the claim. Since $\mathbb{E}(x) = x$ for any $x \in C_{\mathrm{red}}(\mathbb{H})$, we see that $\mathbb{E} = \mathbb{E}_{\mathbb{H}}$. The same argument shows that $\mathbb{E}_{A_i} \circ V_{i,j} = \mathbb{E}_{A_i}$.

Using this compatibility, the same argument as in 2.3.12 shows that $M_{\widetilde{\gamma}'_j}$ also approximates $m_{b'_j}$ in both L^2-norms (here L^2-norms means norms as operator between the Hilbert modules associated to the conditional expectations). The proof of Lemma 2.3.20 also works for the L^2-norms so that we can conclude that $U_{i,j}$ approximates $V_{i,j}$ in both L^2-norms, concluding the proof. $\square$

This result cannot be extended to free products amalgamated over arbitrary quantum subgroups, as noticed in Remark 2.3.18.

Chapter 3

Examples of weakly amenable discrete quantum groups

In this chapter, we give examples of discrete quantum groups which are weakly amenable. These examples fall into the two families of *universal quantum groups* introduced in Subsection 1.2.4, i.e. free quantum groups and quantum automorphism groups of finite-dimensional C*-algebras. More precisely, we are able to deduce weak amenability in all the cases where the Haagerup property is known, which is precisely when the discrete quantum groups are unimodular. Using techniques from the theory of monoidal equivalence, we can then further extend both weak amenability and the Haagerup property to some other, non-unimodular discrete quantum groups. This chapter is the heart of this dissertation since it contains the major part of our work. Most of the results explained hereafter appeared in the paper [Fre13].

The chapter is organized as follows :

- Section 3.1 is devoted to the study of the *operator-valued Haagerup inequality*. This is an inequality partially generalizing the Rapid Decay property when norms are replaced by completely bounded norms. Though our main aim is Theorem 3.3.2, this result is interesting in its own right and this is the reason why we prove it separately. We prove operator-valued inequalities both for all free quantum groups and all quantum automorphism groups of finite-dimensional C*-algebras. The proof is quite involved even for free quantum groups and the case of quantum automorphism groups, which was not contained in [Fre13], is given with less details since it proceeds along the same lines.
- In Section 3.2 we prove, using a strategy inspired from the proof of a similar result for free groups by U. Haagerup, that the completely bounded norm of projections on coefficients of irreducible representations in $O^+(F)$ grows at most quadratically. This Theorem 3.2.8 can be considered as our main achievement since all the results of Section 3.3 will be easily deduced from it. We do not prove a similar result for quantum automorphism groups for two reasons : the strategy fails in that case (more precisely the induction technique used in the proof of Proposition 3.2.5, see Remark 3.2.7) and the result will be obtained in a very simple way in the last section. We also give a lower bound for the growth of these norms, which is linear.
- Section 3.3 contains the main results of this dissertation. We prove weak amenability for unimodular free quantum groups using the technical results worked out in the previous sections. We then investigate the relationship between approximation properties and monoidal equivalence. Using it, we can broadly extend our weak amenability result. We end with some comments on the problems raised by this

work.

3.1 Operator-valued Haagerup inequalities

Haagerup inequalities were first introduced by U. Haagerup in [Haa78] for free groups in order to prove the metric approximation property. P. Jolissaint then gave a comprehensive treatment of this notion in [Jol90] under the name of *Rapid Decay property*, in short property RD. Among several applications of this property, let us mention that its K-theoretic consequences proved crucial in the works of A. Connes and H. Moscovici [CM90] on the Novikov conjecture and of V. Lafforgue [Laf02] and [Laf12] on Banach KK-theory and the Baum-Connes conjecture with coefficients for property (T) groups.

In the context of free groups, the basic problem is to bound the norm of the convolution operator $\lambda(f)$ associated to a finitely supported function f on $\mathbb{F}_N$ using $\|f\|_2$. This is not possible in general, but if the function is supported in the set W_d of words of length d, then it was proven in [Haa78, Lemma 1.4] that

$$\|f\|_2 \leqslant \|\lambda(f)\|_{C_r^*(\mathbb{F}_N)} \leqslant (d+1)\|f\|_2. \tag{3.1}$$

In particular, the Cauchy-Schwarz inequality applied to (3.1) yields, for any finitely supported function $f : \mathbb{F}_N \to \mathbb{C}$,

$$\|\lambda(f)\| \leqslant 2\sqrt{\sum_{g\in\mathbb{F}_N} (|g|+1)^4|f(g)|^2}.$$

where $|g|$ denotes the length of the word g. Such an inequality is very useful since the so-called *Sobolev norm* on the right-hand side is often much easier to compute than the norm on the left-hand side. Later on, similar inequalities were developped for functions f having values in $\mathcal{B}(H)$ for some Hilbert space H. The general case was settled in two different ways by A. Buchholz in [Buc99, Thm 2.8] and by U. Haagerup (the paper is unpublished but the proof can be found in [Pis03, Thm 9.7.4]). These results, however, are rather analogues of the original one since the norm $\|.\|_2$ on the right-hand side of Equation (3.1) has to be replaced by a non-hilbertian one. A precise statement will be given in Subsection 3.1.1.

Remark 3.1.1. On the side of quantum groups, property RD was defined and studied by R. Vergnioux in [Ver07]. These results were later used by M. Brannan to study the metric approximation property in [Bra12a] and [Bra13], where he also gave new examples. However, we will not need this notion hereafter.

3.1.1 Free quantum groups

We first focus on the free orthogonal quantum groups introduced in Definition 1.2.34. More precisely, this section will be concerned with the study of the discrete quantum groups $\mathbb{F}O^+(F)$. We would like to emphasize the fact that no assumption is made on the matrix F here, except from $F\overline{F} \in \mathbb{R}.\,\mathrm{Id}$.

Some notations

Our strategy to obtain the operator-valued Haagerup inequality is inspired from the proof of U. Haagerup as given in G. Pisier's book [Pis03, Thm 9.7.4]. Throughout the proof, we will make parallels with the case of free groups, hoping that this will give some intuition.

From now on, we fix an integer $N > 2$ and a matrix $F \in GL_N(\mathbb{C})$ satisfying $F\overline{F} \in \mathbb{R}.\mathrm{Id}$. We will index the irreducible representations of $O^+(F)$ by integers according to Theorem 1.2.40 and write $\mathcal{H}$ for the Hilbert space $L^2(O^+(F))$ which is identified with $\oplus_k \mathcal{B}(H_k)$ as explained in Subsection 1.1.1 (H_k being the carrier Hilbert space of the irreducible representation indexed by k). Let H be a fixed Hilbert space and let $X \in \mathcal{B}(H) \odot \mathrm{Pol}(O^+(F))$ (it is enough to control the norm on such elements since they form a dense subalgebra). Choose $d \in \mathbb{N}$ and set

$$X^d = (\imath \otimes m_{p_d})(X).$$

These objects should be thought of as "operator-valued functions with finite support" on $\mathbb{F}O^+(F)$, X^d having "support in the elements of length d". Our aim in this section is to control the norm of X^d using some finite-dimensional "blocks".

Remark 3.1.2. Recall from [Ver07] that there is a natural length function on $\mathrm{Irr}(O^+(F))$ such that the irreducible representation u_d has length d. Using this notion, one could give a rigorous definition of "operator-valued functions with support in the words of length d". This, however, will not be needed afterwards.

The following lemma summarizes some standard calculations which will be used several times in the sequel. Recall that $D_n = (q^n - q^{-n})/(q - q^{-1})$.

Lemma 3.1.3. *Let $a > b$ be integers, then $D_{a-b}^{-1} \leqslant D_b/D_a \leqslant q^{a-b}$. Moreover, for any integer c, $q^c D_c \leqslant (1-q^2)^{-1}$.*

Proof. Let $n \in \mathbb{Z}$ such that $n \geqslant -b$. Decomposing $u^{b+n} \otimes u^{a+n+1}$ and $u^{b+n+1} \otimes u^{a+n}$ into sums of irreducible representations according to Theorem 1.2.40 yields

$$D_{b+n}D_{a+n+1} = D_{a-b+1} + \cdots + D_{a+b+2n+1} \leqslant D_{a-b-1} + \cdots + D_{a+b+2n+1} = D_{b+n+1}D_{a+n}$$

This inequality means that the sequence $(D_{b+n}/D_{a+n})_{n \geqslant -b}$ is increasing. Thus, any term is greater than its first term D_{a-b}^{-1} and less than its limit q^{a-b}. The second part of the lemma is straightforward since $q^c D_c = (1 - q^{2c+2})/(1 - q^2)$. $\square$

Block decomposition

The aim of this section is to reduce the problem of controlling the norm of X^d as much as possible by restricting our attention to small pieces which convey all the information we need. This will serve both to prove the operator-valued Haagerup inequality and to get the polynomial bound for the completely bounded norm of the projections on coefficients of irreducible representations. We start by decomposing the operators into more elementary ones. For any two integers a and b, we set

$$\left\{ \begin{array}{lcl} B_{a,b}(X) & := & (\imath \otimes p_a)X(\imath \otimes p_b) \\ B_{a,b}(X^d) & := & (\imath \otimes p_a)X^d(\imath \otimes p_b) \end{array} \right.$$

This is simply X (resp. X^d) seen as an operator from $\mathcal{B}(H_b)$ to $\mathcal{B}(H_a)$ and obviously has norm less than $\|X\|$ (resp. $\|X^d\|$). Such an operator will be called a *block*. The operator X^d admits a particular decomposition with respect to these blocks.

Lemma 3.1.4. *Set $X_j^d = \displaystyle\sum_{k=0}^{+\infty} B_{d-j+k,j+k}(X^d)$. Then, $X^d = \displaystyle\sum_{j=0}^{d} X_j^d$.*

Proof. Clearly,

$$X^d = \sum_{a,b} B_{a,b}(X^d).$$

If we decompose X^d as $\sum_i T_i \otimes x_i$, with x_i a coefficient of u^d and $T_i \in \mathcal{B}(H)$, we see that X^d sends $H \otimes (p_b\mathcal{H})$ into $\oplus_c(H \otimes (p_c\mathcal{H}))$ where the sum runs over all irreducible subrepresentations u^c of $u^d \otimes u^b$. Thus, we deduce from Theorem 1.2.40 that $B_{a,b}(X^d)$ vanishes as soon as a is not of the form $d+b-2j$ for some $0 \leqslant j \leqslant \min(d,b)$. Consequently,

$$X^d = \sum_{b=0}^{+\infty} \sum_{j=0}^{\min(d,b)} B_{d+b-2j,b}(X^d) = \sum_{j=0}^{d} \sum_{b=j}^{+\infty} B_{d+b-2j,b}(X^d).$$

Writing $b = k + j$, we get the desired result. □

This result should be thought of as a decomposition according to the "number of deleted letters" in the action of X^d. We illsutrate this in the case of free groups. Consider an operator-valued function $f : \mathbb{F}_N \to \mathcal{B}(H)$ which vanishes on words of length different from d. For any finitely supported operator-valued function g, the convolution product evaluated on a fixed word x reads

$$(f * g)(x) = \sum_{y.z=x} f(y).g(z).$$

Now, y must be of length d, so we can split the sum according to the number of letters in y that are deleted when multiplying it by z, namely

$$(f * g)(x) = \sum_{j=0}^{d} \sum_{|x|=|y|+|z|-j} f(y).g(z).$$

Writting $|z| = j + k$ then gives the analogous decomposition. Indeed, denoting by W_n the set of words of length n and by χ_F the characteristic function of a set F, we have

$$f * g = \sum_{j=0}^{d} \left(\sum_{k=0}^{+\infty} f * (g.\chi_{W_{j+k}}) \right) .\chi_{W_{d-j+k}}$$

Let us come back to $\mathbb{F}O^+(F)$. Thanks to the triangle inequality, we can restrict ourselves to the study of $\|X_j^d\|$. Proposition 3.1.7 further reduces the problem to the study of only one specific block in X_j^d. Before stating and proving it, we have to introduce several notations and elementary facts.

Recall from Subsection 1.1.1 that for $u^c \subset u^a \otimes u^b$, $v_c^{a,b} : H_c \mapsto H_a \otimes H_b$ denotes an isometric intertwiner and let $M_k^+ : \mathcal{H} \otimes \mathcal{B}(H_k) \to \mathcal{H}$ be the orthogonal sum of the operators $\mathrm{Ad}(v_{l+k}^{l,k})$. Under our identification of $\mathcal{H}$ with $\oplus\mathcal{B}(H_k)$, the restriction of M_k^+ to $\mathcal{B}(H_l)\otimes\mathcal{B}(H_k)$ is the map induced by the product composed with the orthogonal projection onto $\mathcal{B}(H_{l+k})$.

In the case of free groups, the analogue of the operator M_k^+ is quite easy to describe. In fact, let g be a word in $\mathbb{F}_N$ and let h be another word of length k. If $|g.h| < |g||h|$ (i.e. if there is at least one cancellation in the product), then $M_k^+(g,h) = 0$. Otherwise, $M_k^+(g,h) = g.h$. If this operator does not seem natural, it is because we are in fact more interested in its adjoint : $(M_k^+)^*$ simply takes a word w of length at least k and cuts it into its first $|w| - k$ letters and its last k letters.

If we endow $\mathcal{B}(H_k)$ with the scalar product $\langle .,. \rangle_k$ defined in Subsection 1.1.1, it can be seen as a subspace of $\mathcal{H}$ and we can compute the norm of the restriction of M_k^+ to $\mathcal{B}(H_l) \otimes \mathcal{B}(H_k)$ with respect to the Hilbert space structure on $\mathcal{H} \otimes \mathcal{H}$.

Proposition 3.1.5. *With the notation of Theorem 1.2.40,* $\|M_k^+(p_l \otimes \imath)\|^2 = D_l D_k / D_{l+k}$*. Moreover,*

$$(M_k^+)^* p_{l+k} = \frac{D_l D_k}{D_{l+k}} \operatorname{Ad}((v_{l+k}^{l,k})^*)$$

Proof. Let $x \in \mathcal{B}(H_l) \otimes \mathcal{B}(H_k)$, then

$$\begin{array}{rcl} \|M_k^+(x)\|^2 & = & \dfrac{1}{D_{l+k}} \operatorname{Tr}(Q_{k+l} M_k^+(x)^* M_k^+(x)) \\ & = & \dfrac{1}{D_{l+k}} \operatorname{Tr}(Q_{k+l} (v_{l+k}^{l,k})^* x^* v_{l+k}^{l,k} (v_{l+k}^{l,k})^* x v_{l+k}^{l,k}) \\ & \leqslant & \dfrac{1}{D_{l+k}} \operatorname{Tr}(Q_{l+k} (v_{l+k}^{l,k})^* x^* x v_{l+k}^{l,k}) \\ & = & \dfrac{1}{D_{l+k}} \operatorname{Tr}(v_{l+k}^{l,k} Q_{l+k} (v_{l+k}^{l,k})^* x^* x) \\ & \leqslant & \dfrac{1}{D_{l+k}} (\operatorname{Tr} \otimes \operatorname{Tr})((Q_l \otimes Q_k) x^* x) \\ & = & \dfrac{D_l D_k}{D_{l+k}} \|x\|^2 \end{array}$$

Thus, $\|M_k^+(p_l \otimes \imath)\|^2 \leqslant D_l D_k / D_{l+k}$ and the norm is attained at $x = v_{l+k}^{l,k} (v_{l+k}^{l,k})^*$. The explicit form of the adjoint comes from the same computation. $\square$

Remark 3.1.6. Note that this computation also proves that $\|M_k^+\|^2 = \dfrac{1 - q^{2k+2}}{1 - q^2}$ and, in particular, that $\|M_1^+\|^2 = 1 + q^2 \leqslant 2$.

With this in hand, we can state and prove the main technical result of this section. By convention, a product indexed by the empty set (this is the case $j = 0$ in the statement below) will be equal to 1.

Proposition 3.1.7. *For integers a, b and c, set*

$$N_{a,b}^c = 1 - \frac{D_{(a-b+c)/2} D_{(b-a+c)/2-1}}{D_{a+1} D_b}$$

whenever this expression makes sense. Then, if we set

$$\chi_j^d(k) = \sqrt{\frac{D_{d-j} D_{j+k}}{D_{d-j+k} D_j}} \prod_{i=0}^{j-1} (N_{d-j+i,k+i}^{d-j+k})^{-1},$$

we have, for every k, $\|B_{d-j+k,j+k}(X^d)\| \leqslant \chi_j^d(k) \|B_{d-j,j}(X^d)\|$.

Proof. Let us first focus on the one-dimensional case. Let x be a coefficient of u^d seen as an element of $\mathcal{B}(H_d)$ and choose an integer k. Let us compare the two vectors

$$\left\{ \begin{array}{rcl} A & = & [M_k^+(p_{d-j} x p_j \otimes \imath)(M_k^+)^*](\xi) \\ B & = & (p_{d-j+k} x p_{j+k})(\xi) \end{array} \right.$$

for $\xi \in p_{j+k}\mathcal{H} = \mathcal{B}(H_{j+k})$. Setting $V = (\imath \otimes v_{j+k}^{j,k})^* (v_{d-j}^{d,j} \otimes \imath) v_{d-j+k}^{d-j,k}$, we get an intertwiner between u^{d-j+k} and $u^{d \otimes (j+k)}$. Since that inclusion has multiplicity 1, there is a complex number $\mu_j^d(k)$ such that

$$V = \mu_j^d(k) v_{d-j+k}^{d,j+k}.$$

Now, using Equation (1.4) and Proposition 3.1.5, we have

$$A = V^*(x \otimes \xi)\left(\frac{D_j D_k}{D_{j+k}}\right)V \text{ and } B = (v^{d,j+k}_{d-j+k})^*(x \otimes \xi)v^{d,j+k}_{d-j+k}$$

and consequently $B = \lambda^d_j(k)A$, with $\lambda^d_j(k) = (D_j D_k/D_{j+k})^{-1}|\mu^d_j(k)|^{-2}$. Let us compute $|\mu^d_j(k)|$. If we set $v^{a,b}_{+} = (v^{a,b}_{a+b})^*$ and define two morphisms of representations

$$\left\{\begin{array}{lcl} \mathcal{T}_A & = & (v^{d-j,j}_{+} \otimes v^{j,0}_{+} \otimes \imath_k)(\imath_{d-j} \otimes t_j \otimes \imath_k)v^{d-j,k}_{d-j+k} \\ \mathcal{T}_B & = & (v^{d-j,j}_{+} \otimes v^{j,k}_{+})(\imath_{d-j} \otimes t_j \otimes \imath_k)v^{d-j,k}_{d-j+k} \end{array}\right.$$

we have, up to some complex numbers of modulus 1,

$$\mathcal{T}_A = \|\mathcal{T}_A\|(v^{d,j}_{d-j} \otimes \imath)v^{d-j,k}_{d-j+k} \text{ and } \mathcal{T}_B = \|\mathcal{T}_B\|v^{d,j+k}_{d-j+k}.$$

Since moreover $(\imath \otimes v^{j,k}_{j+k})^*\mathcal{T}_A = \mathcal{T}_B$, we get $|\mu^d_j(k)|^2 = \|\mathcal{T}_B\|^2/\|\mathcal{T}_A\|^2$. The equality we used is illustrated on the following diagram :

$$\begin{tikzcd}
 & H_{d-j+k} \arrow[dl, "\mathcal{T}_B"'] \arrow[dr, "\mathcal{T}_A"] \arrow[drr, bend left, "\mathcal{T}_B"] & & \\
H_d \otimes H_{j+k} \arrow[d, "x\otimes\xi"'] & & H_d \otimes H_j \otimes H_k \arrow[r, "\imath\otimes(v^{j+k}_{j,k})^*"] & H_d \otimes H_{j+k} \arrow[d, "x\otimes\xi"] \\
H_d \otimes H_{j+k} \arrow[dr, "\mathcal{T}_B^*"'] & & H_d \otimes H_j \otimes H_k \arrow[dl, "\mathcal{T}_A^*"] & H_d \otimes H_{j+k} \arrow[l, "\imath\otimes v^{j+k}_{j,k}"] \\
 & H_{d-j+k} & &
\end{tikzcd}$$

Thanks to [Ver05, Prop. 2.3] and [Ver07, Lem. 4.8], we can compute the norms of $\mathcal{T}_A$ and $\mathcal{T}_B$ to obtain

$$|\mu^d_j(k)|^2 = \prod_{i=0}^{j-1}\frac{N^{d-j+k}_{d-j+i,k+i}}{N^{d-j}_{d-j+i,i}} = \prod_{i=0}^{j-1}N^{d-j+k}_{d-j+i,k+i}.$$

Note that for $j = 0$, the above product is not defined. However, $\lambda^d_0(k) = 1$ since $\mathcal{T}_A = \mathcal{T}_B$ in that case. As $\lambda^d_j(k)$ does not depend on ξ, we have indeed proved the following equality in $\mathcal{B}(\mathcal{H})$:

$$p_{d-j+k}xp_{j+k} = \lambda^d_j(k)[M^+_k(p_{d-j}xp_j \otimes \imath)(M^+_k)^*].$$

Now we go back to the operator-valued case. We have $X^d = \sum_i T_i \otimes x_i$, where $x_i \in \mathrm{Pol}(O^+(F))$ is a coefficient of u^d and $T_i \in \mathcal{B}(H)$, hence

$$\lambda^d_j(k)[(\imath \otimes M^+_k)(B_{d-j,j}(X^d) \otimes \imath)(\imath \otimes M^+_k)^*] = B_{d-j+k,j+k}(X^d).$$

Using the norms of the restrictions of M^+_k computed above, we get

$$\|B_{d-j+k,j+k}(X^d)\| \leqslant \lambda^d_j(k)\|(\imath \otimes M^+_k)B_{d-j,j}(X^d)(\imath \otimes M^+_k)^*\| \leqslant \chi^d_j(k)\|B_{d-j,j}(X^d)\|.$$

□

In the case of free groups, [Pis03, Thm 9.7.4] states that the coefficients $\chi^d_j(k)$ can all be chosen equal to 1. We do not know whether this is still true for $\mathbb{F}O^+(F)$, but at least it seems out of reach with our method.

Operator-valued Haagerup inequality

We first draw an easy corollary which will be used several times in the sequel. Recall that q is the unique real number such that $0 < q \leqslant 1$ and $q + q^{-1} = \mathrm{Tr}(F^*F)$.

Corollary 3.1.8. *There is a constant $K(q)$, depending only on q, such that for any $d \in \mathbb{N}$ and $0 \leqslant j \leqslant d$, $\|X_j^d\| \leqslant K(q)\|B_{d-j,j}(X^d)\|$.*

Proof. According to Lemma 3.1.3, we have

$$\frac{D_{d-j}D_{k-1}}{D_{d-j+i+1}D_{k+i}} \leqslant q^{i+1}q^{i+1} = q^{2i+2},$$

thus $(N_{d-j+i,k+i}^{d-j+k})^{-1} \leqslant (1 - q^{2i+2})^{-1}$. Again by Lemma 3.1.3, $D_{d-j}/D_{d-j+k} \leqslant q^k$ and $D_{j+k}/D_j \leqslant D_k$, hence

$$\chi_j^d(k) \leqslant \sqrt{q^k D_k}\prod_{i=0}^{j-1}\frac{1}{1-q^{2i+2}} \leqslant \frac{1}{\sqrt{1-q^2}}\prod_{i=0}^{+\infty}\frac{1}{1-q^{2i+2}} = K(q).$$

□

Combining the results above, we can give our analogue of the operator-valued version of the Haagerup inequality. Let us first recall the precise statement in the case of free groups. Let $N \geqslant 2$ be an integer and let $f : \mathbb{F}_N \to \mathcal{B}(H)$ be a finitely supported operator-valued function. For two integers j and d satisfying $0 \leqslant j \leqslant d$ and two elements $x \in W_{d-j}$ and $y \in W_j$, we set

$$f_{x,y}^{[d,j]} = f(xy^{-1})\delta_{|xy^{-1}|=d}$$

We can see these scalars as matrix coefficients of an operator

$$f^{[d,j]} : \ell^2(W_j, H) \to \ell^2(W_{d-j}, H)$$

the norm of which will be denoted $\|f\|_{[d,j]}$. With this in hand, we can state the classical result [Buc99, Thm 2.8], [Pis03, Thm 9.7.4].

Theorem 3.1.9 (Buchholz, Haagerup)**.** *The following inequalities hold*

$$\max_{0\leqslant j\leqslant d}\{\|f\|_{[d,j]}\} \leqslant \left\|\sum_{z\in W_d} f(z)\otimes\lambda(z)\right\| \leqslant (d+1)\max_{0\leqslant j\leqslant d}\{\|f\|_{[d,j]}\}$$

We can now state and prove our quantum version of this inequality.

Theorem 3.1.10. *Let N be an integer and let $F \in GL_N(\mathbb{C})$ be such that $F\overline{F} \in \mathbb{R}.\mathrm{Id}$. Then, there exists a constant $K(q)$, depending only on q, such that for any Hilbert space H and any $X \in \mathcal{B}(H)\otimes \mathrm{Pol}(O^+(F))$,*

$$\max_{0\leqslant j\leqslant d}\{\|B_{d-j,j}(X^d)\|\} \leqslant \|X^d\| \leqslant K(q)(d+1)\max_{0\leqslant j\leqslant d}\{\|B_{d-j,j}(X^d)\|\}$$

Proof. The first inequality comes from the fact that $\|B_{d-j,j}(X^d)\| \leqslant \|X^d\|$ for all integers d and j and the second one comes from the triangle inequality combined with Proposition 3.1.7. □

Taking $H = \mathbb{C}$, we get the inequality

$$\|x^d\| \leqslant \max_{0\leqslant j\leqslant d} \|p_{d-j}x^d p_j\| \tag{3.2}$$

for any coefficient x^d of u^d. This is weaker than the definition of property RD given in [Ver07, Prop and Def 3.5]. In fact, this is strictly weaker since it is satisfied for any matrix F while property RD does not hold when the discrete quantum group is not unimodular (see [Ver07, Prop 4.7]).

Remark 3.1.11. There is also a notion of *strong Haagerup inequality* which was developped in [Kem05] and [KS07] in the scalar case. The operator-valued case was studied by M. De la Salle in [DLS09]. The above result and the work of M. Brannan [Bra12b] on strong Haagerup inequalities in the quantum setting give evidence for the existence of such a strong operator-valued Haagerup inequality for $\mathbb{F}O_N^+$.

3.1.2 Quantum automorphism groups

We now extend the previous results and prove an operator-valued Haagerup inequality for quantum automorphism groups of finite-dimensional C*-algebras. The proof uses exactly the same strategy as for free quantum groups. Again, irreducible representations are indexed by integers and we will use the same notations as before. However, the fusion rules are now that of $SO(3)$ instead of $SU(2)$. As we will see, this makes the computations more involved, in particular because we have to distinguish the subrepresentations of a given tensor product according to some parity. We will only sketch some of the arguments here since they are straightforward adaptations of the arguments used before. We have to start with some structure results concerning the category of finite-dimensional representations of quantum automorphism groups.

Further preliminaries

We give a very short description of some properties of the category of finite-dimensional representations of quantum automorphism groups. The following facts are more or less explicitely contained in [Ban99c] and [Ban02]. A concise, though far more comprehensive, presentation can be found in [Bra13, Subsec 3.1].

The category of finite-dimensional representations of any quantum automorphism group can be seen to be isomorphic to the so-called 2*-cabled Temperley-Lieb category.* The only concrete data we need on this category is some structure of the morphism spaces. We know that irreducible representations can be labelled by integers. If $u^c \subset u^a \otimes u^b$, one can explicitely construct an intertwiner $\rho_c^{a,b} : H_c \to H_a \otimes H_b$ such that

$$(\rho_c^{a,b})^*\rho_c^{a,b} = (R_c^{a,b})^2.\,\mathrm{Id}_{H_c}$$

where, using the notations of Proposition 3.1.7,

$$(R_c^{a,b})^2 = \frac{D_{2a}}{D_{a+b-c}D_{a-b+c}} \prod_{i=0}^{a+b-c-1} N^{2c}_{a-b+c+i,b-a+c+i}.$$

This implies that $v_c^{a,b} = (R_c^{a,b})^{-1}\rho_c^{a,b}$ is an isometric intertwiner. The operators v and ρ are related by a crucial *absorption property* : if a, b, c and x are integers such that $u^x \subset u^a \otimes u^b$, then

$$(\imath \otimes v_{b+c}^{b,c})(\rho_x^{a,b} \otimes \imath) = \rho_x^{a,b+c}. \tag{3.3}$$

Operator-valued Haagerup inequality

From now on, we fix a finite-dimensional C*-algebra B, a δ-form ψ on B and we denote by $\mathbb{G}$ the compact quantum automorphism group of (B, ψ). Again, we will write $\mathcal{H}$ for the Hilbert space $L^2(\mathbb{G})$ identified with $\oplus_k \mathcal{B}(H_k)$. Let H be a fixed Hilbert space and let $X \in \mathcal{B}(H) \odot \mathrm{Pol}(\mathbb{G})$. Fix $d \in \mathbb{N}$ and set

$$X^d = (\imath \otimes m_{p_d})(X).$$

As in Section 3.1, our proof will rely on a block decomposition of the operators X and X^d. However, the fusion rules force us to decompose according to some "parity" of the subrepresentations in the following analogue of Lemma 3.1.4.

Lemma 3.1.12. *Set, for* $0 \leqslant j \leqslant d$,

$$\left\{\begin{array}{lcl} X^0_{d,j} & = & \displaystyle\sum_{k=0}^{+\infty} B_{d-j+k,j+k}(X^d) \\ X^1_{d,j} & = & \displaystyle\sum_{k=0}^{+\infty} B_{d-j+k,j+k+1}(X^d) \end{array}\right.$$

Then,

$$X^d = \sum_{j=0}^{d} X^0_{d,j} + \sum_{j=0}^{d-1} X^1_{d,j}.$$

Proof. We start from the equality,

$$X^d = \sum_{a,b} B_{a,b}(X^d).$$

From Theorem 1.2.53 we know that $B_{a,b}(X^d)$ vanishes as soon as a is not of the form $d+b-j$ for some $0 \leqslant j \leqslant 2\min(d,b)$. Consequently,

$$\begin{aligned} X^d &= \sum_{b=0}^{+\infty} \sum_{j=0}^{2\min(d,b)} B_{d+b-j,b}(X^d) \\ &= \sum_{j=0}^{d} \sum_{b=j}^{+\infty} B_{d+b-2j,b}(X^d) + \sum_{j=0}^{d-1} \sum_{b=j+1}^{+\infty} B_{d+b-(2j+1),b}(X^d) \end{aligned}$$

Writing $b = j+k$ in the first sum and $b = j+k+1$ in the second one, we get the result. $\square$

Remark 3.1.13. We will sometimes use the notation $X^1_{j,d}$ or $B_{d-j,j+1}(X^d)$ for $j = d$ in the sequel. This is pure notational convenience since $B_{k,d+k+1}(X^d) = 0$ for every k.

As previously, we will reduce the problem to finding a bound on a specific block of $\|X^i_{d,j}\|$ for $i = 0, 1$, using the straightforward generalization of the operator M^+_k. Note that in this context, a computation similar to that of Proposition 3.1.5 yields

$$\|M^+_k(p_l \otimes \imath)\|^2 = D_{2l}D_{2k}/D_{2(l+k)}$$

and, on $\mathcal{B}(H_{l+k})$,

$$(M^+_k)^* p_{l+k} = (D_{2l}D_{2k}/D_{2(l+k)})\,\mathrm{Ad}((v^{l,k}_{l+k})^*).$$

With this in hand, we can state and prove the analogue of Proposition 3.1.7.

Proposition 3.1.14. *If we set*

$$\begin{cases} \chi^0_{j,d}(k) &= \sqrt{\dfrac{D_{2(d-j)}D_{2(j+k)}}{D_{2(d-j+k)}D_{2j}}} \displaystyle\prod_{i=0}^{2j-1} \frac{1}{N^{2(d-j+k)}_{2(d-j)+i,2k+i}} \\ \chi^1_{j,d}(k) &= \sqrt{\dfrac{D_{2(d-j)}D_{2(j+k+1)}}{D_{2(d-j+k)}D_{2(j+1)}}} \displaystyle\prod_{i=0}^{2j} \frac{N^{2(d-j)}_{2(d-j)+i-1,i+1}}{N^{2(d-j+k)}_{2(d-j)+i-1,2k+i+1}} \end{cases}$$

then we have, for every k,

$$\begin{cases} \|B_{d-j+k,j+k}(X^d)\| &\leqslant \chi^0_{j,d}(k)\|B_{d-j,j}(X^d)\| \\ \|B_{d-j+k,j+k+1}(X^d)\| &\leqslant \chi^1_{j,d}(k)\|B_{d-j,j+1}(X^d)\| \end{cases}$$

Proof. Let us first focus on the one-dimensional case. Let x be a coefficient of u^d seen as an element of $\mathcal{B}(H_d)$ and fix an integer k. Let us compare the two vectors

$$\begin{cases} A &= [M_k^+(p_{d-j}xp_j \otimes \imath)(M_k^+)^*](\xi) \\ B &= (p_{d-j+k}xp_{j+k})(\xi) \end{cases}$$

for $\xi \in p_{j+k}\mathcal{H} = \mathcal{B}(H_{j+k})$. Setting $V = (\imath \otimes v^{j,k}_{j+k})^*(v^{d,j}_{d-j} \otimes \imath)v^{d-j,k}_{d-j+k}$, we have an intertwiner between u^{d-j+k} and $u^{d\otimes(j+k)}$. Since that inclusion has multiplicity 1, there is a complex number $\mu^0_{j,d}(k)$ such that

$$V = \mu^0_{j,d}(k)v^{d,j+k}_{d-j+k}.$$

Now, noticing that thanks to Equation (1.4),

$$A = V^*(x \otimes \xi)\left(\frac{D_{2j}D_{2k}}{D_{2(j+k)}}\right)V \text{ and } B = (v^{d,j+k}_{d-j+k})^*(x \otimes \xi)v^{d,j+k}_{d-j+k}$$

we have $B = \lambda^0_{d,j}(k)A$, with $\lambda^0_{d,j}(k) = (D_{2k}D_{2j}/D_{2(j+k)})^{-1}|\mu^0_{d,j}(k)|^{-2}$. Let us compute $|\mu^0_{d,j}(k)|$. We know that

$$(\rho^{d,j}_{d-j} \otimes \imath)v^{d-j,k}_{d-j+k} = R^{d,j}_{d-j}(v^{d,j}_{d-j} \otimes \imath)v^{d-j,k}_{d-j+k} \text{ and } \rho^{d,j+k}_{d-j+k} = R^{d,j+k}_{d-j+k}v^{d,j+k}_{d-j+k}.$$

Using the absorption property of Equation (3.3) we see that

$$(\imath \otimes v^{j,k}_{j+k})^*(\rho^{d,j}_{d-j} \otimes \imath)v^{d-j,k}_{d-j+k} = \rho^{d,j+k}_{d-j+k},$$

hence

$$|\mu^0_{d,j}(k)|^2 = (R^{d,j+k}_{d-j+k})^2/(R^{d,j}_{d-j})^2 = \prod_{i=0}^{2j-1} \frac{N^{2(d-j+k)}_{2(d-j)+i,2k+i}}{N^{2(d-j)}_{2(d-j)+i,i}} = \prod_{i=0}^{j-1} N^{2(d-j+k)}_{2(d-j)+i,2k+i}.$$

Again, the product is not defined for $j = 0$. However, $\lambda^d_0(k) = 1$ since $A = B$ in that case. As $\lambda^0_{d,j}(k)$ does not depend on ξ, we have indeed proved the following equality in $\mathcal{B}(\mathcal{H})$:

$$p_{d-j+k}xp_{j+k} = \lambda^0_{d,j}(k)[M_k^+(p_{d-j}xp_j \otimes \imath)(M_k^+)^*].$$

Now we go back to the operator-valued case. We have $X^d = \sum T_i \otimes x_i$, where $x_i \in \mathrm{Pol}(\mathbb{G})$ is a coefficient of u^d and $T_i \in \mathcal{B}(H)$, hence

$$\lambda^0_{d,j}(k)[(\imath \otimes M_k^+)(B_{d-j,j}(X^d) \otimes \imath)(\imath \otimes M_k^+)^*] = B_{d-j+k,j+k}(X^d).$$

Using the norms of the restrictions of M_k^+ computed above, we get

$$\|B_{d-j+k,j+k}(X^d)\| \leqslant \lambda_{d,j}^0(k)\|(\imath \otimes M_k^+)B_{d-j,j}(X^d)(\imath \otimes M_k^+)^*\| \leqslant \chi_{d,j}^0(k)\|B_{d-j,j}(X^d)\|.$$

The second inequality is obtained exactly in the same way and we only sketch the proof in that case. After reducing the problem to the one-dimensional case, we use the operators

$$\begin{cases} A' &= [M_k^+(p_{d-j}xp_{j+1} \otimes \imath)(M_k^+)^*](\xi) \\ B' &= (p_{d-j+k}xp_{j+k+1})(\xi) \end{cases}$$

and the intertwiner $V' = (\imath \otimes v_{j+k+1}^{j+1,k})^*(v_{d-j}^{d,j+1} \otimes \imath)v_{d-j+k}^{d-j,k} = \mu_{d,j}^1(k)v_{d-j+k}^{d,j+k+1}$. We again have $B' = \lambda_{d,j}^1(k)A'$, with $\lambda_{d,j}^1(k) = (D_{2k}D_{2j+1}/D_{2(j+k+1)})^{-1}|\mu_{d,j}^1(k)|^{-2}$. Using the absorption rule for intertwiners yields

$$|\mu_{d,j}^1(k)|^2 = \frac{(R_{d-j}^{d,j+1})^2}{(R_{d-j+k}^{d,j+k+1})^2} = \prod_{i=0}^{2j} \frac{N_{2(d-j)+i-1,i+1}^{2(d-j)}}{N_{2(d-j)+i-1,2k+i+1}^{2(d-j+k)}}.$$

Hence the formula for $\chi_{d,j}^1(k)$. $\square$

We only need a bound on the coefficients $\chi_{d,j}^i(k)$ to conclude. Recall that q is the unique real number such that $0 < q \leqslant 1$ and $q + q^{-1} = \delta$.

Corollary 3.1.15. *There is a constant $K(q)$, depending only on q, such that for any $d \in \mathbb{N}$ and $0 \leqslant j \leqslant d$,*

$$\|X_{d,j}^0\| \leqslant K(q)\|B_{d-j,j}(X^d)\| \text{ and } \|X_{d,j}^1\| \leqslant K(q)\|B_{d-j,j+1}(X^d)\|$$

Proof. According to Lemma 3.1.3, we have

$$\frac{D_{2(d-j)}D_{2k-1}}{D_{2(d-j)+i+1}D_{2k+i}} \leqslant q^{i+1}q^{i+1} = q^{2i+2},$$

thus $(N_{2(d-j)+i,2k+i}^{2(d-j+k)})^{-1} \leqslant (1-q^{2i+2})^{-1}$. Again by Lemma 3.1.3, we have

$$D_{2(d-j)}/D_{2(d-j+k)} \leqslant q^{2k} \text{ and } D_{2(j+k)}/D_{2j} \leqslant D_{2k},$$

hence

$$\chi_{d,j}^0(k) \leqslant \sqrt{q^{2k}D_{2k}} \prod_{i=0}^{2j-1} \frac{1}{1-q^{2i+2}} \leqslant \frac{1}{\sqrt{1-q^2}} \prod_{i=0}^{+\infty} \frac{1}{1-q^{2i+2}} = K(q)$$

Similarly, we have

$$\frac{D_{2(d-j)-1}D_{2k}}{D_{2(d-j)+i}D_{2k+i+1}} \leqslant q^{i+1}q^{i+1} = q^{2i+2},$$

implying

$$\frac{N_{2(d-j)+i-1,i+1}^{2(d-j)}}{N_{2(d-j)+i-1,2k+i+1}^{2(d-j+k)}} \leqslant \frac{1}{N_{2(d-j)+i-1,2k+i+1}^{2(d-j+k)}} \leqslant \frac{1}{1-q^{2i+2}}$$

and $D_{2(d-j)}D_{2(j+k+1)}/D_{2(d-j+k)}D_{2(j+1)} \leqslant q^{2k}D_{2k}$ by Lemma 3.1.3, yielding

$$\chi_{d,j}^1(k) \leqslant \sqrt{q^{2k}D_{2k}} \prod_{i=0}^{2j} \frac{1}{1-q^{2i+2}} \leqslant \frac{1}{\sqrt{1-q^2}} \prod_{i=0}^{+\infty} \frac{1}{1-q^{2i+2}} = K(q).$$

$\square$

Combining Proposition 3.1.14 and Corollary 3.1.15, we eventually get the operator-valued Haagerup inequality (note that if $j = d$, $\|B_{d-j,j+1}(X^d)\| = \|B_{0,d+1}(X^d)\| = 0$).

Theorem 3.1.16. *Let B be a finite-dimensional C^*-algebra and let ψ be a δ-form on B. Then, there exists a constant $K(q)$, depending only on q, such that for any Hilbert space H and any $X \in \mathcal{B}(H) \otimes \mathrm{Pol}(\mathbb{G}(B, \psi))$,*

$$\max_{0 \leqslant j \leqslant d} \{\|B_{d-j,j}(X^d)\| + \|B_{d-j,j+1}(X^d)\|\} \leqslant \|X^d\|$$

and

$$\|X^d\| \leqslant K(q)(d+1) \max_{0 \leqslant j \leqslant d} \{\|B_{d-j,j}(X^d)\| + \|B_{d-j,j+1}(X^d)\|\}$$

3.2 The completely bounded norm of projections

In this section, we turn back to free quantum groups. Our main result will be the proof of an explicit polynomial bound for the completely bounded norm of the projection onto the linear span of coefficients of an irreducible representation u^d in $C_{\mathrm{red}}(O^+(F))$. This should be considered as the most important achievement of this dissertation, since all the important results are rather simple consequences of it. Before going into the details of the proof, which is quite involved, we would like to give some motivation.

First note that this projection is simply the multiplier m_{p_d} associated to $p_d \in \ell^\infty(\widehat{\mathbb{G}})$. Thus, we can reformulate the proof of the Haagerup property for $\mathbb{F}O_N^+$ by M. Brannan in terms of these multipliers. In fact, if we choose for every integer k and real number t a scalar coefficient $b_k(t)$, we can define a net of elements

$$a(t) = \sum_{k=0}^{+\infty} b_k(t) p_k \in \ell^\infty(\widehat{\mathbb{G}}).$$

The associated multipliers are

$$m_{a(t)} = \sum_{k=0}^{+\infty} b_k(t) m_{p_k}$$

We know from [Bra12a] that we can choose coefficients $b_k(t)$ that decrease exponentially with t and such that m_{a_t} is unital and completely positive. Thus, if the completely bounded norm of the operators m_{p_d} can be bounded polynomially, we can cut off the multipliers by restricting the sum to a finite number of k's while keeping some control on the completely bounded norm. The net we obtain will then satisfy all the hypothesis of Definition 2.2.8 and $\mathbb{F}O^+(F)$ will be weakly amenable. For the sake of completeness, we also prove a lower bound on the completely bounded norm in order to give some information on the possible defect of optimality of the upper bound.

3.2.1 Upper bound

We want to find some polynomial P such that $\|X^d\| \leqslant P(d)\|X\|$. Thanks to Proposition 3.1.7, the problem reduces to finding a polynomial Q such that, for every $0 \leqslant j \leqslant d$,

$$\|B_{d-j,j}(X^d)\| \leqslant Q(d)\|X\|.$$

This will be done using the following recursion formula.

Proposition 3.2.1. *Set*

$$N_1^+ = \bigoplus_l \frac{D_{l+1}}{D_1 D_l} M_1^+(p_l \otimes \imath) : \mathcal{H} \otimes \mathcal{B}(H_1) \to \mathcal{H}.$$

According to Proposition 3.1.5, $(N_1^+)^*$ is the sum of the operators $\mathrm{Ad}((v_{l+1}^{l,1})^*)$. Then, there are coefficients $C_j^d(s)$ such that for $0 \leqslant j \leqslant d$,

$$\begin{aligned} B_{d-j+1,j+1}(X) \quad &- \quad (\imath \otimes M_1^+)(B_{d-j,j}(X) \otimes \imath)(\imath \otimes N_1^+)^* \\ = B_{d-j+1,j+1}(X^{d+2}) \quad &+ \quad \sum_{s=0}^{\min(j,d-j)} C_j^d(s) B_{d-j+1,j+1}(X^{d-2s}) \end{aligned} \tag{3.4}$$

Proof. The idea of the proof is similar to the one used in the proof of Proposition 3.1.7. We first consider the one-dimensional case. Let x be a coefficient of u^l seen as an element of $\mathcal{B}(H_l)$. Fix an element $\xi \in p_{j+1}\mathcal{H}$. Again, the vectors

$$\begin{cases} A &= [M_1^+(p_{d-j} x p_j \otimes \imath)(N_1^+)^*](\xi) \\ B &= (p_{d-j+1} x_l p_{j+1})(\xi) \end{cases}$$

are proportional. Note that if $l > d+2$, $l < |d-2j|$ or $l-d$ is not even, both operators are 0. Note also that if $l = d+2$, $A = 0$. The other values of l can be written $d-2s$ for some positive integer s between 0 and $\min(j, d-j)$. In that case, the existence of a scalar $\nu_j^d(s)$ such that $B = \nu_j^d(s)A$ follows from the same argument as in the proof of Proposition 3.1.7. Let us compute $\nu_j^d(s)$, noticing that thanks to the normalization of N_1^+, the constant $\nu_j^d(s)$ only corresponds to the "μ-part" of the constant λ of Proposition 3.1.7. This time, we have to set

$$\begin{cases} \mathcal{T}_A &= (v_+^{d-s-j,j-s} \otimes v_+^{j-s,s} \otimes \imath_1)(\imath_{d-j-s} \otimes t_{j-s} \otimes \imath_{s+1}) v_{d-j+1}^{d-j-s,s+1} \\ \mathcal{T}_B &= (v_+^{d-s-j,j-s} \otimes v_+^{j-s,s+1})(\imath_{d-j-s} \otimes t_{j-s} \otimes \imath_{s+1}) v_{d-j+1}^{d-j-s,s+1} \end{cases}$$

We can then use the same argument as in the proof of Proposition 3.1.7 , that we illsutrate with the following diagram :

$$\begin{tikzcd}
 & H_{d-j+1} \arrow[dl, "\mathcal{T}_B"'] \arrow[dr, "\mathcal{T}_A"] \arrow[drr, bend left, "\mathcal{T}_B"] & & \\
H_l \otimes H_{j+1} \arrow[d, "x\otimes\xi"'] & & H_l \otimes H_j \otimes H_1 \arrow[r, "\imath\otimes(v_{j,1}^{j+1})^*"] & H_l \otimes H_{j+1} \arrow[d, "x\otimes\xi"] \\
H_l \otimes H_{j+1} \arrow[dr, "\mathcal{T}_B^*"'] & & H_l \otimes H_j \otimes H_1 \arrow[dl, "\mathcal{T}_A^*"] & H_l \otimes H_{j+1} \arrow[l, "\imath\otimes v_{j,1}^{j+1}"] \\
 & H_{d-j+1} & &
\end{tikzcd}$$

Again, applying [Ver05, Prop. 2.3] and [Ver07, Lem. 4.8] yields

$$\nu_j^d(s) = \frac{\|\mathcal{T}_A\|^2}{\|\mathcal{T}_B\|^2} = \prod_{i=0}^{j-s-1} \frac{N_{d-s-j+i,s+i}^{d-j}}{N_{d-j-s+i,s+i+1}^{d-j+1}}.$$

As in the proof of Proposition 3.1.7, we can now go back to the operator-valued case. We have

$$X = \sum_{l} \sum_{i=0}^{k(l)} T_l^{(i)} \otimes x_l^{(i)}$$

where $x_l^{(i)} \in \mathrm{Pol}(O^+(F))$ are coefficients of u^l and $T_l^{(i)} \in \mathcal{B}(H)$. Setting

$$X^l = \sum_{i=0}^{k(l)} T_l^{(i)} \otimes x_l^{(i)},$$

we have

$$B_{d-j+1,j+1}(X^l) = \nu_j^d(s)(\imath \otimes M_1^+)(B_{d-j,j}(X^l) \otimes \imath)(\imath \otimes N_1^+)^*$$

and setting $C_j^d(s) = 1 - \nu_j^d(s)^{-1}$ concludes the proof. □

Remark 3.2.2. One could prove similar formulæ in the case of quantum automorphism groups. However, because of the fusion rules, there would also be a $B_{d-j+1,j+1}(X^{d+1})$ appearing on the right. One could also modify the strategy, using the inclusion $j \subset j \otimes 1$ to write a formula involving $B_{d-j+1,j}(X)$. We will see afterwards why such equalities cannot yield a bound on the completely bounded norm of projections on coefficients of irreducible representations.

In the case of free groups, many terms vanish in the previous computations. As explained before, the adjoint of the operator M_1^+ isolates the last lettre of a word, hence it corresponds to the Schur multiplier with symbol $(\langle x(\beta), x(\gamma)\rangle)_{\beta,\gamma}$ defined in the proof of [Pis03, Thm 9.7.4] (where $x(\beta)$denotes the last letter of the word β). It is then proved that all the coefficients $C_j^d(s)$ vanish. Thus, the proof is over in that case.

In the quantum case, we have to deal with all the coefficients $C_j^d(s)$. This requires some control on them and on the coefficients $\chi_j^d(s)$, which is given by the following lemma.

Lemma 3.2.3. *Assume that* $0 < q \leqslant 1/\sqrt{3}$. *Then, for any* $0 \leqslant j \leqslant d$,

$$\sum_{s=0}^{\min(j,d-j)} |C_j^d(s)|\chi_{j-s}^{d-2s}(s+1) \leqslant 1.$$

Proof. We first give another expression of $|C_j^d(s)|$. Decomposing into sums of irreducible representations yields

$$D_{d-s-j+i+1}D_{s+i+1} - D_{d-s-j}D_s = D_{d-j+2} + \cdots + D_{d-j+2i+2} = D_iD_{d-j+i+2}$$
$$D_{d-s-j+i+1}D_{s+i} - D_{d-s-j}D_{s-1} = D_{d-j+1} + \cdots + D_{d-j+2i+1} = D_iD_{d-j+i+1}$$

which implies that

$$N_{d-j-s+i,s+i+1}^{d-j+1} = \frac{D_iD_{d-j+i+2}}{D_{d-s-j+i+1}D_{s+i+1}} \text{ and } N_{d-s-j+i,s+i}^{d-j} = \frac{D_iD_{d-j+i+1}}{D_{d-s-j+i+1}D_{s+i}}.$$

Hence

$$\nu_j^d(s) = \prod_{i=0}^{j-s-1} \frac{N_{d-s-j+i,s+i}^{d-j}}{N_{d-j-s+i,s+i+1}^{d-j+1}} = \prod_{i=0}^{j-s-1} \frac{D_{d-j+i+1}D_{s+i+1}}{D_{s+i}D_{d-j+i+2}} = \frac{D_jD_{d-j+1}}{D_sD_{d-s+1}}.$$

Again, noticing that $D_j D_{d-j+1} - D_s D_{d-s+1} = D_{d-j-s} D_{j-s-1}$ yields

$$|C_j^d(s)| = |1 - \nu_j^d(s)^{-1}| = \frac{D_{d-j-s} D_{j-s-1}}{D_{d-j+1} D_j}.$$

According to Lemma 3.1.3, we thus have

$$|C_j^d(s)| \leqslant q^{s+1} q^{s+1} = q^{2s+2}$$

Now we turn to $\chi_{j-s}^{d-2s}(s+1)$. In fact, we are going to bound $\chi_j^d(s+1)$ independently of d and j. Decomposing into sums of irreducible representations, we get

$$D_{d-j+i+1} D_{k+i} - D_{d-j} D_{k-1} = D_{d-j+k+1} + \cdots + D_{d-j+k+2i+1} = D_i D_{d-j+k+i+1},$$

which implies that $N_{d-j+i,k+i}^{d-j+k} = D_i D_{d-j+k+i+1} / D_{d-j+i+1} D_{k+i}$. Now we can compute

$$\begin{aligned}
\frac{\chi_j^d(s+1)}{\chi_j^d(s)} &= \sqrt{\frac{D_{j+s+1} D_{d-j+s}}{D_{j+s} D_{d-j+s+1}}} \prod_{i=0}^{j-1} \frac{D_{s+1+i} D_{d-j+s+i+1}}{D_{s+i} D_{d-j+s+i+2}} \\
&= \sqrt{\frac{D_{j+s+1} D_{d-j+s}}{D_{j+s} D_{d-j+s+1}}} \frac{D_{j+s} D_{d-j+s+1}}{D_s D_{d+s+1}} \\
&= \frac{\sqrt{D_{j+s} D_{d-j+s+1} D_{d-j+s} D_{j+s+1}}}{D_s D_{d+s+1}}.
\end{aligned}$$

Using Lemma 3.1.3 again, we get

$$\frac{\chi_j^d(s+1)}{\chi_j^d(s)} \leqslant \sqrt{q^j D_j q^{d-j} D_{d-j}} \leqslant \frac{1}{1-q^2}.$$

Since $\chi_j^d(1) \leqslant (1-q^2)^{-1}$, we have proved that $\chi_j^d(s+1) \leqslant (1-q^2)^{-s-1}$. This bound is independant of d and j, thus it also works for $\chi_{j-s}^{d-2s}(s+1)$. Combining this with our previous estimate, we can compute

$$\sum_{s=0}^{\min(j,d-j)} |C_j^d(s)| \chi_{j-s}^{d-2s}(s+1) \leqslant \sum_{s=0}^{+\infty} \left(\frac{q^2}{1-q^2}\right)^{s+1} = \frac{q^2}{1-2q^2}.$$

The last term is less than 1 as soon as $q \leqslant 1/\sqrt{3}$. □

Remark 3.2.4. Note that $0 < q \leqslant 1/\sqrt{3}$ if and only if $q + q^{-1} \geqslant 4/\sqrt{3}$. Since

$$2 < \frac{4}{\sqrt{3}} < 3,$$

Lemma 3.2.3 holds as soon as the matrix F is of size at least 3.

Gathering all our results will now yield the estimate we need. To make things more clear, we will proceed in two steps. First we bound the norms of the blocks of X^d.

Proposition 3.2.5. *There exists a polynomial Q such that for every $d \in \mathbb{N}$ and $0 \leqslant j \leqslant d$, $\|B_{d-j,j}(X^d)\| \leqslant Q(d)\|X\|$.*

Proof. First note that $B_{d,0}(X^d) = B_{d,0}(X)$ and $B_{0,d}(X^d) = B_{0,d}(X)$, hence we only have to consider the case $1 \leqslant j \leqslant d-1$. Moreover, applying the triangle inequality to the recursion relation of Proposition 3.2.1 yields

$$\begin{aligned}\|B_{d-j+1,j+1}(X^{d+2})\| &\leqslant (1+\|M_1^+\|\|N_1^+\|)\|X\| \\ &+ \sum_{s=0}^{\min(j,d-j)} |C_j^d(s)|\|B_{d-j+1,j+1}(X^{d-2s})\|.\end{aligned}$$

We proceed by induction, with the following induction hypothesis

$H(d)$: "For any integer $l \leqslant d$ and any $0 \leqslant j \leqslant l$, $\|B_{l-j,j}(X^l)\| \leqslant Q(l)\|X\|$ with $Q(X) = 2X+1$."

Because of the remark at the beginning of the proof, $H(0)$ and $H(1)$ are true. Knowing this, we just have to prove that for any d, $H(d)$ implies the inequality for $d+2$. Indeed, this will prove that assuming $H(d)$, both the inequalities for $d+1$ (noticing that $H(d)$ implies $H(d-1)$) and $d+2$ are true, hence $H(d+2)$ will hold.

Assume $H(d)$ to be true for some d and apply the recursion formula above. The blocks in the right-hand side of the inequality are of the form $B_{d-j+1,j+1}(X^{d-2s})$. By Proposition 3.1.7 and $H(d)$,

$$\begin{aligned}\|B_{d-j+1,j+1}(X^{d-2s})\| &= \|B_{(d-2s)-(j-s)+s+1,(j-s)+s+1}(X^{d-2s})\| \\ &\leqslant \chi_{j-s}^{d-2s}(s+1)\|B_{(d-2s)-(j-s),(j-s)}(X^{d-2s})\| \\ &\leqslant \chi_{j-s}^{d-2s}(s+1)Q(d-2s)\|X\|.\end{aligned}$$

Then, bounding $Q(d-2s)$ by $Q(d)$ and using Lemma 3.2.3 yields

$$\|B_{d-j+1,j+1}(X^{d+2})\| \leqslant 3\|X\| + Q(d)\|X\| \leqslant Q(d+2)\|X\|.$$

Since $\|B_{d-j+1,j+1}(X^{d+2})\| = \|B_{(d+2)-(j+1),j+1}(X^{d+2})\|$, the inequality is proved for $1 \leqslant j+1 \leqslant d+1$. In other words, we have $\|B_{d-J,J}(X^{d+2})\| \leqslant Q(d+2)\|X\|$ for any $1 \leqslant J \leqslant d+1$. As noted at the beginning of the proof, this is enough to get $H(d+2)$. $\square$

Remark 3.2.6. Note that bounding each $Q(d-2s)$ by $Q(d)$, which may seem a rough majorization, has in fact very little impact since the polynomial Q has to be of degree at least 1 to make the recursion work.

Remark 3.2.7. The strategy by induction used in this proof fails for quantum automorphism groups. The subtelty comes from the fact, proved in Lemma 3.1.4, that we have to deal not only with blocks of the form $B_{d-j,j}(X^d)$ but also with blocks of the form $B_{d-j,j+1}(X^d)$. Assume that the polynomial bound is known for any block of X^l with $l \leqslant d$. We have in particular to bound the norm of the block

$$B_{(d+1)-j,j+1}(X^{d+1}) = B_{d-j+1,j+1}(X^{d+1}).$$

If such a block appears in a formula of the type of Equation (3.4), then there will also be the block $B_{d-j+1,j+1}(X^{d+2})$ and we will not be able to make the induction work. We can nevertheless make the induction work in the particular case of $\mathbb{G}(M_N(\mathbb{C}),\tau)$ (where τ is the canonical δ-trace) because it is a quantum subgroup of O_N^+, so that we can restrict the induction process of the latter. This suggests that in general the GNS representation of $\mathbb{G}(B,\psi)$ is "too small" and we need some intermediate subspaces (those coming from O_N^+ in the case of $M_N(\mathbb{C})$) giving the missing steps.

Secondly we bound the norm of X^d itself.

Theorem 3.2.8. *Let $F \in GL_N(\mathbb{C})$ be such that $F\overline{F} \in \mathbb{R}.\mathrm{Id}$ and let $0 \leqslant q \leqslant 1$ be the real number defined in Theorem 1.2.40. Then, if $q \leqslant 1/\sqrt{3}$ (in particular if $N \geqslant 3$), there exists a polynomial P such that for every integer d,*

$$\|m_{p_d}\|_{cb} \leqslant P(d).$$

Proof. We use the notations of Proposition 3.2.5. We know from Corollary 3.1.8 that $\|X_j^d\| \leqslant K(q)\|B_{d-j,j}(X^d)\|$, thus $\|X_j^d\| \leqslant K(q)Q(d)\|X\|$. If we set

$$P(X) = K(q)(X+1)Q(X),$$

we get $\|X^d\| \leqslant P(d)\|X\|$ by applying the triangle inequality to the decomposition of Lemma 3.1.4. □

Remark 3.2.9. One could slightly improve this bound by noticing that since we can replace $Q(d)$ by 1 when $j = 0$ or d, $\|X^d\| \leqslant K(q)(2d^2 - d + 1)\|X\|$.

Remark 3.2.10. When $q = 1$, $\mathbb{F}O^+(F)$ is the dual of the classical compact group $SU(2)$ or of the compact quantum group $SU_{-1}(2)$. It was proven in [Ver07, Sec 4.1] that this discrete quantum dual of the former has property RD, and that consequently $\|m_{p_d}\|$ grows at most polynomially. Since any bounded map from a C*-algebra into a *commutative* C*-algebra is completely bounded with completely bounded norm equal to the usual norm by Proposition 1.3.14, Theorem 3.2.8 also works in that case. Note that $SU_{-1}(2)$ also has the Rapid Decay property (see again [Ver07, Sec 4.1]). This and Theorem 3.2.8 of course suggests that the polynomial bound holds for the dual of $SU_q(2)$ for all values of q, though the majorizations of Lemma 3.2.3 are not good enough to provide such a statement.

3.2.2 Lower bound

It is proved in [Pis03, Thm 9.7.4] that in the free group case, the completely bounded norm of the projections on words of fixed length grows at most linearly. Our technique cannot determine whether such a result still holds in the quantum case, but proves the slightly weaker fact that the growth is at most quadratic. However, we can prove that it is also at least linear. Let us first recall that the sequence (μ_k) of (dilated) Chebyshev polynomials of the second kind is defined by $\mu_0(X) = 1$, $\mu_1(X) = X$ and

$$X\mu_k(X) = \mu_{k-1}(X) + \mu_{k+1}(X)$$

Proposition 3.2.11. *Let $F \in GL_N(\mathbb{C})$ be such that $F\overline{F} \in \mathbb{R}.\mathrm{Id}$. Then, there exists a polynomial R of degree one such that, for all integers d,*

$$\|m_{p_d}\|_{cb} \geqslant R(d).$$

Proof. Since $\|m_{p_d}\|_{cb} \geqslant \|m_{p_d}\|$, we will simply prove a lower bound for this second norm. Let $\chi_n \in \mathrm{Pol}(\mathbb{G})$ be the character of the representation u^n, i.e.

$$\chi_n = (\imath \otimes \mathrm{Tr})(u_n).$$

Our aim is to prove that looking at the action of m_{p_d} on $\chi_{d+2} - \chi_d$ is enough to get the lower bound.

It is known (see [Ban96]) that sending χ_n to the restriction to $[-2, 2]$ of μ_n yields an isomorphism between the C*-subalgebra of $C_{\mathrm{red}}(O^+(F))$ generated by the elements χ_n

and $C([-2,2])$. Moreover, the restriction of these polynomials to the interval $[-2,2]$ form a Hilbert basis with respect to the scalar product associated to the semicircular law

$$d\nu = \frac{\sqrt{4-t^2}}{2\pi}dt.$$

Let us denote by $\pi : C([-2,2]) \to \mathcal{B}(L^2([-2,2], d\nu))$ the faithful representation by multiplication operators. What precedes means precisely that we have, for any finitely supported sequence (a_n),

$$\left\|\sum_n a_n\chi_n\right\|_{C_{\mathrm{red}}(O_N^+)} = \left\|\sum_n a_n\mu_{n|[-2,2]}\right\|_\infty = \left\|\sum_n a_n\pi(\mu_n)\right\|_{\mathcal{B}(L^2([-2,2],d\nu))}.$$

Let e_i denote the image of μ_i in $L^2([-2,2], d\nu)$ and denote by T_n the operator sending e_i to e_{i+n} for $n \geqslant 0$. Letting E_j denote the linear span of the vectors e_i for $0 \leqslant i \leqslant j$, we can also define an operator T_{-n} which is 0 on E_{n-1} and sends e_i to e_{i-n} for $i \geqslant n$. The last operator we need, denoted S_n, sends $e_i \in E_n$ to e_{n-i} and is 0 on $E_n^\perp$. These translation operators obviously have norm 1. Moreover, a simple computation using Theorem 1.2.40 (or equivalently the recursion relation of the Chebyshev polynomials) shows that

$$\pi(\mu_{n+2} - \mu_n) = T_{n+2} - S_n - T_{-(n+2)}.$$

Thus, $\|\chi_{n+2} - \chi_n\| = \|\pi(\mu_{n+2} - \mu_n)\| \leqslant 3$. On the other hand, it easily seen that $\mu_n(2) = n + 1$. In fact, this is true for $\mu_1(X) = X$ and $\mu_2(X) = X^2 - 1$ and we have the recursion relation

$$2\mu_n(2) = \mu_{n+1}(2) + \mu_{n-1}(2).$$

This implies that $\|\chi_n\| = \|\mu_n\|_\infty \geqslant n + 1$. Combining these two facts yields

$$\|m_{p_d}\| \geqslant \frac{\|m_{p_d}(\chi_{d+2} - \chi_d)\|}{\|\chi_{d+2} - \chi_d\|} = \frac{\| - \chi_d\|}{\|\chi_{d+2} - \chi_d\|} \geqslant \frac{d+1}{3}$$

and setting $R(X) = (X + 1)/3$ concludes the proof. □

3.3 Monoidal equivalence and approximation properties

3.3.1 Weak amenability for unimodular free quantum groups

As explained in the beginning of Section 3.1, our strategy is now to cut off nice multipliers on the quantum group $\mathbb{F}O^+(F)$. The only available source of such multipliers is the Haagerup property. In fact, we even need to know that the Haagerup property can be implemented by multipliers which "decrease rapidly at infinity". This is the problem studied hereafer. We will then collect all our results to compute the Cowling-Haagerup constant of all unimodular free quantum groups.

Complements on the Haagerup property

All the results proved so far hold in great generality, i.e. for any $\mathbb{F}O^+(F)$ with F of size at least 3 satisfying $F\overline{F} \in \mathbb{R}.\mathrm{Id}$. We now adress the problem of finding suitable multipliers as explained before. Such multipliers were constructed by M. Brannan in [Bra12a] in the case $F = \mathrm{Id}$. We show here that this result can be extended to the case of a scalar multiple of a unitary matrix. In the other cases, i.e. when the discrete quantum group is not unimodular, the problem must be left open for the moment.

Proposition 3.3.1. *Let $F \in GL_N(\mathbb{C})$ be a scalar multiple of a unitary matrix. Then, the discrete quantum groups $\mathbb{F}O^+(F)$ and $\mathbb{F}U^+(F)$ have the Haagerup property.*

Proof. According to Theorem 1.2.44, there is, up to isomorphism of the associated quantum groups, only two matrices F to consider, namely Id_N and (when N is even)

$$J_N = \begin{pmatrix} 0 & I_{N/2} \\ -I_{N/2} & 0 \end{pmatrix}.$$

We claim that in the second case, setting $b_k(t) = \mu_k(t)/\mu_k(N)$ and

$$a(t) = \sum_k b_k(t) p_k$$

yields the Haagerup property. In fact, the arguments of [Bra12a] apply in this context and the only thing we have to prove is that N is not isolated in the spectrum of χ_1 in the C*-algebra $C_{\max}(O^+(J_N))$. Let $\theta \in \mathbb{R}$, note that we can assume $N \geqslant 4$ and consider the matrices

$$R^\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & \mathrm{Id}_{N/2-2} \end{pmatrix} \text{ and } U^\theta = \begin{pmatrix} R^\theta & 0 \\ 0 & R^\theta \end{pmatrix}.$$

Then, the matrix U^θ is orthogonal and commutes to F. Thus, by universality,

$$u_{i,j} \mapsto U^\theta_{i,j}$$

extends to a well defined character on $C_{\max}(O^+(J_N))$ sending χ_1 to

$$\mathrm{Tr}(U^\theta) = N - 4 + 4\cos(\theta).$$

This proves that $[N-4, N]$ is contained in the spectrum of χ_1 (in fact, this trick proves that $[2, N]$ is always contained in the spectrum and that $[0, N]$ is contained in the spectrum as soon as N is a multiple of 4). Thus $\mathbb{F}O^+(F)$ has the Haagerup property, as well as its free complexification $\mathbb{F}U^+(F)$ by Proposition 2.2.48. □

First examples

We are now ready for one of our main theorems. As all the results of this chapter, it heavily relies on the work of the previous sections and the proof is consequently quite short.

Theorem 3.3.2. *Let $N \geqslant 2$ be an integer and let $F \in GL_N(\mathbb{C})$ be a scalar multiple of a unitary matrix such that $F\overline{F} \in \mathbb{R}.\mathrm{Id}$. Then, the discrete quantum groups $\mathbb{F}O^+(F)$ and $\mathbb{F}U^+(F)$ are weakly amenable and their Cowling-Haagerup constant is equal to 1.*

Proof. For $N = 2$, this result is already known by amenability of the discrete quantum group $\mathbb{F}O^+(F) = \widehat{SU_{\pm 1}(2)}$. Thus, we will assume $N > 2$. We are going to use the net $(a(t))$ of elements in $\ell^\infty(\mathbb{F}O^+(F))$ introduced in the proof of Proposition 3.3.1. For $i \geqslant 0$, set

$$a_i(t) = \sum_{k=0}^{i} b_k(t) p_k.$$

Then, $(a_i(t))_{i,t}$ is a net of finitely supported elements converging pointwise to the identity and we have to prove that the completely bounded norms of the associated multipliers satisfy the boundedness condition. If we fix some $2 < t_0 < 3$, then [Bra12a, Prop. 4.4] asserts the existence of a constant K_0, depending only on t_0, such that for any $t_0 \leqslant t < N$, $0 < b_k(t) < K_0(t/N)^k$. According again to [Bra12a, Prop. 4.4], the multipliers associated to the elements $a(t) = \sum_k b_k(t)p_k$ (where the sum runs from 0 to infinity) are unital and completely positive. Moreover, for any $t_0 \leqslant t < N$,

$$\|m_{a(t)} - m_{a_i(t)}\|_{cb} \leqslant \sum_{k\geqslant i} K_0 \left(\frac{t}{N}\right)^k \|m_{p_k}\|_{cb}.$$

This sum tends to 0 as i goes to infinity since Theorem 3.2.8 implies that it is the rest of an absolutely converging series. This implies that $\limsup \|m_{a_i(t)}\|_{cb} = 1$. In other words, $\Lambda_{cb}(\mathbb{F}O^+(F)) = 1$. By Theorem 2.3.6 and Theorem 2.2.12, we also have $\Lambda_{cb}(\mathbb{Z}*\mathbb{F}O^+(F)) = 1$, hence $\Lambda_{cb}(\mathbb{F}U^+(F)) = 1$ by Theorem 1.2.43. □

This result can be extended a little further to some free quantum groups with F not satisfying $F\overline{F} \in \mathbb{R}.\mathrm{Id}$ (Definition 1.2.34 in fact makes sense for any invertible matrix F).

Corollary 3.3.3. *Let $F \in GL_N(\mathbb{C})$ be a scalar multiple of a unitary matrix. Then, the discrete quantum groups $\mathbb{F}O^+(F)$ and $\mathbb{F}U^+(F)$ have the Haagerup property. Moreover, they are weakly amenable with Cowling-Haagerup constant equal to* 1.

Proof. S. Wang proved in [Wan02] that for a general F, $\mathbb{F}U^+(F)$ and $\mathbb{F}O^+(F)$ can be decomposed as a free product (without amalgamation) of free orthogonal and free unitary quantum groups for matrices F' satisfying $F'\overline{F'} \in \mathbb{R}.\mathrm{Id}$. If F is a scalar multiple of a unitary matrix, all the subgroups in the free product are unimodular (i.e. the matrices F' are scalar multiple of unitary matrices). Hence, Theorem 3.3.2 combined with assertion (3) of Proposition 2.2.48 and 2.3.6 yields the result. □

Remark 3.3.4. It is a consequence of Theorem 3.3.2 and Corollary 2.2.15 that when F is a multiple of a unitary matrix, $\mathbb{F}O^+(F)$ and $\mathbb{F}U^+(F)$ are exact because they are weakly amenable. This had previously been proven by S. Vaes and R. Vergnioux in [VV07] using an argument of amenability of the boundary action (and can be extended to free products by general results on the stability of exactness, see e.g. [BO08, Cor 4.8.3]).

Remark 3.3.5. Let us point out that Theorem 3.3.2, Theorem 2.3.6 and the isomorphisms of [Rau12, Thm 4.1] and [Web13, Prop 3.2] imply that the free bistochastic quantum groups B_N^+ and their symmetrized versions $(B_N^+)'$ and $(B_N^+)^\sharp$ have the Haagerup property and are weakly amenable with Cowling-Haagerup constant equal to 1.

We already mentioned that weak amenability is an important tool in the classification of von Neumann algebras. To illustrate this, we give an important consequence of Theorem 3.3.2, proved by Y. Isono in [Iso12, Thm A] (we are in fact giving the corollary of this theorem here). To state it, let us recall a few definitions concerning von Neumann algebras. If M is a von Neumann algebra and A is a von Neumann subalgebra of M, the *normalizer* of A in M is the group

$$\mathcal{N}_M(A) = \{u \in \mathcal{U}(M), u^*Au \subset A\}.$$

A von Neumann algebra M is said to be *strongly solid* if for any diffuse (i.e. without any minimal projection) amenable von Neumann subalgebra A of M, $\mathcal{N}_M(A)''$ is an amenable von Neumann algebra.

Theorem 3.3.6 (Isono)**.** *Let $F \in GL_N(\mathbb{C})$ be a scalar multiple of a unitary matrix. Then, the von Neumann algebras $L^\infty(O^+(F))$ and $L^\infty(U^+(F))$ are strongly solid.*

In particular, strongly solid von Neumann algebras have no *Cartan subalgebra* i.e. no maximal abelian von Neumann subalgebra A such that $\mathcal{N}_M(A)'' = M$. From the point of view of classification, this means for example that the von Neuman algebras $L^\infty(O_N^+)$ and $L^\infty(U_N^+)$ cannot be isomorphic to the crossed-product $\Gamma \ltimes L^\infty(X)$ for any free ergodic action of a group Γ on a standard measure space X. More generally, it cannot be isomorphic to any von Neumann algebra $\mathcal{L}(X, \mathcal{R})$ associated to an ergodic equivalence relation $\mathcal{R}$ on a standard probability space X, as defined in [FM77].

3.3.2 Monoidal equivalence and approximation properties

We will now present the notion of *monoidal equivalence* of compact quantum groups and study its interplay with approximation properties. As we will see, this proves to be the right tool to extend our results to a large collection of universal quantum groups.

Last preliminaries

Roughly speaking, monoidal equivalence consists in looking at a compact quantum group as a monoidal category. By this, we mean that we can forget the data of the Hilbert spaces associated to finite-dimensional representations (the so-called *forgetful functor*) to obtain a monoidal category which may be equivalent to the category of some other, non-isomorphic compact quantum group. Adding the forgetful functor, such an equivalence of categories would force the compact quantum groups to be isomorphic by Theorem 1.1.19. Let us give the definition [BDRV06, Def 3.1] to make this idea more precise.

Definition 3.3.7. Two compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$ are *monoidally equivalent* if there is a bijection $\varphi : \mathrm{Irr}(\mathbb{G}_1) \to \mathrm{Irr}(\mathbb{G}_2)$ such that the image of the trivial representation is the trivial representation, together with linear isomorphisms

$$\varphi : \mathrm{Mor}(\alpha_1 \otimes \cdots \otimes \alpha_n, \beta_1 \otimes \cdots \otimes \beta_m) \to \mathrm{Mor}(\varphi(\alpha_1) \otimes \cdots \otimes \varphi(\alpha_n), \varphi(\beta_1) \otimes \cdots \otimes \varphi(\beta_m))$$

such that $\varphi(1) = 1$ and for any morphisms S and T,

$$\begin{aligned} \varphi(S \circ T) &= \varphi(S) \circ \varphi(T) \\ \varphi(S^*) &= \varphi(S)^* \\ \varphi(S \otimes T) &= \varphi(S) \otimes \varphi(T) \end{aligned}$$

As explained in [BDRV06, Rmk 3.3], two compact quantum groups are monoidally equivalent if and only if they have the same fusion rules and the same *quantum $6j$-symbols* (see [CFS95] for details on this notion). These symbols are very close to the coefficients $N_{a,b}^c$ appearing in the previous sections. This suggests a link between our work and monoidal equivalence. To investigate it, let us first give another characterization of monoidal equivalence. Quite surprisingly, this is a "dynamical" characterization which looks very different from the categorical definition. To state it, we first need to give some results and notations from the theory of ergodic coactions of compact quantum groups as developped by P. Podleś in [Pod95] and by F. Boca and M.B. Landstad in [Boc95] and [Lan95].

Let $\mathbb{G}$ be a compact quantum group together with an action $\rho : B \to C(\mathbb{G}) \otimes B$ on a unital C*-algebra B. The action is said to be *ergodic* if the space of fixed-points

$B^\rho = \{b \in B, \rho(b) = 1 \otimes b\}$ is equal to $\mathbb{C}.1$. In that case, there is a unique invariant state ω on B, defined by

$$\omega(b).1 = (h \otimes \imath) \circ \rho(b).$$

Moreover, for any irreducible representation $\alpha \in \mathrm{Irr}(\mathbb{G})$, the linear form (using the notations of the definition of the Haar weights on the discrete dual in Subsection 1.1.2)

$$h_\alpha(x) = \dim_q(\alpha)^2 (h \otimes \psi_\alpha)((x \otimes 1)u^\alpha)$$

yields a projection $P_\alpha = (h_\alpha \otimes \imath) \circ \rho$ in B called the *spectral projection* associated to the irreducible representation α. The image of this projection is the corresponding *spectral subspace*, denoted B_α. Note that the spectral subspace associated to the trivial representation is exactly the space of fixed points of the action. More generally, one can check that if $C(\mathbb{G})_\alpha$ denotes the linear span of the coefficients of u^α in $C(\mathbb{G})$, then

$$B_\alpha = \{b \in B, \rho(b) \in C(\mathbb{G})_\alpha \otimes B\}.$$

This decomposition of the C*-algebra B is very similar to the decomposition of $C(\mathbb{G})$ in subspaces of coefficients of irreducible representations (which is the case of $\mathbb{G}$ acting on itself by the coproduct Δ). The $*$-algebra

$$\mathcal{B} = \bigoplus_{\alpha \in \mathrm{Irr}(\mathbb{G})} B_\alpha$$

is dense in B and the state ω is faithful on $\mathcal{B}$. The action is said to be reduced (resp. universal) if B is the completion of $\mathcal{B}$ in the GNS construction of ω (resp. the envelopping C*-algebra of $\mathcal{B}$).

Remark 3.3.8. The restriction of ρ to $\mathcal{B}$ induces a coaction of the Hopf $*$-algebra $\mathrm{Pol}(\mathbb{G})$.

In order to define the notion of multiplicity, we have to look at the *intertwiner spaces*

$$\mathrm{Mor}(H_\alpha, B) = \{T : H_\alpha \to B, \alpha(T(\xi)) = (\imath \otimes T)(u^\alpha(1 \otimes \xi)), \forall \xi \in H_\alpha\},$$

where H_α is the carrier Hilbert space of the representation u^α. The dimension of the (finite-dimensional) vector space $\mathrm{Mor}(H_\alpha, B)$ is called the *multiplicity* of α in ρ and denoted $\mathrm{mult}(\alpha)$.

Remark 3.3.9. The fact that the intertwiner space is finite-dimensional implies that B_α is also finite-dimensional and $\dim(B_\alpha) = \mathrm{mult}(\alpha). \dim(H_\alpha) = \mathrm{mult}(\alpha). \dim(\alpha)$. As suggested by this equality, spectral subspaces can also be interpreted in another way : to the *ergodic* action ρ, one can associate a representation X^ρ of $\mathbb{G}$ on the GNS space of the invariant state ω by the formula

$$X^\rho(b \otimes x) = \rho(b)(1 \otimes x).$$

Then, the intertwiner space $\mathrm{Mor}(H_\alpha, B)$ corresponds to intertwiners between the irreducible representation u^α and X^ρ and $\mathrm{mult}(\alpha)$ is the multiplicity of this inclusion. The spectral subspaces are recovered thanks to the the equality $B_\alpha = \{T(\xi), \xi \in H_\alpha\}$.

The antilinear normalized map $j_\alpha^* : H_\alpha \to H_{\overline{\alpha}}$ induces a map J_α from $\mathrm{Mor}(H_\alpha, B)$ to $\mathrm{Mor}(H_{\overline{\alpha}}, B)$ and the trace of the positive definite matrix $J_\alpha^* J_\alpha \in \mathrm{Mor}(H_\alpha, B)$ is called the *quantum multiplicity* of α in ρ, denoted $\mathrm{mult}_q(\alpha)$. By [Lan95, Thm A] (see also [BDRV06, Thm 2.9]), we always have

$$\mathrm{mult}(\alpha) \leqslant \mathrm{mult}_q(\alpha) \leqslant \dim_q(\alpha).$$

The action ρ is said to be of *full quantum multiplicity* if $\mathrm{mult}_q(\alpha) = \dim_q(\alpha)$ for all α. A proof of the following theorem can be found in [BDRV06, Thm 3.9 and Prop 3.13].

Theorem 3.3.10 (Bichon – De Rijdt – Vaes)**.** *Two compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$ are monoidally equivalent if and only if there exists a C^*-algebra B, called the* linking algebra*, endowed with two commuting reduced ergodic actions of full quantum multiplicity of $\mathbb{G}_1$ and $\mathbb{G}_2$ respectively.*

As far as free orthogonal quantum groups are concerned, it is proved in [BDRV06, Thm 5.3] that $O^+(F)$ and $O^+(F')$ are monoidally equivalent if and only if the sign of $F\overline{F}$ and the associated number q are the same. In particular, any free orhtogonal quantum group is monoidally equivalent to exactly one of the quantum groups $SU_q(2)$.

Applications of monoidal equivalence

We now investigate the link between monoidal equivalence and the multipliers associated to projections on coefficients of a fixed irreducible representation. The following proposition was communicated to us by S. Vaes.

Proposition 3.3.11. *Let φ be a monoidal equivalence between two compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$. Then, for every $\alpha \in \mathrm{Irr}(\mathbb{G}_1)$, we have*

$$\|m_{p_\alpha}\|_{cb} = \|m_{p_{\varphi(\alpha)}}\|_{cb}.$$

Proof. Let B be the linking algebra given by Theorem 3.3.10 and fix $\alpha \in \mathrm{Irr}(\mathbb{G}_1)$. The explicit description of the action $\rho : B_{\mathrm{red}} \to C(\mathbb{G}_1) \otimes B_{\mathrm{red}}$ implies that

$$(m_{p_\alpha} \otimes \imath) \circ \rho = \rho \circ P_\alpha.$$

The injective $*$-homomorphism ρ being completely isometric, we deduce

$$\|P_\alpha\|_{cb} \leqslant \|m_{p_\alpha}\|_{cb}.$$

Now, S. Vaes and R. Vergnioux constructed in the proof of [VV07, Thm 6.1] an injective $*$-homomorphism $\theta : C(\mathbb{G}_2) \to B_{\mathrm{red}} \otimes B_{\mathrm{red}}^{op}$ such that

$$(P_\alpha \otimes \imath) \circ \theta = \theta \circ m_{p_{\varphi(\alpha)}},$$

yielding

$$\|m_{p_{\varphi(\alpha)}}\|_{cb} \leqslant \|P_\alpha\|_{cb}.$$

Thus, $\|m_{p_{\varphi(\alpha)}}\|_{cb} \leqslant \|m_{p_\alpha}\|_{cb}$ and by symmetry (using the inverse monoidal equivalence), the completely bounded norms are equal. □

Proposition 3.3.11 means in particular that proving the polynomial bound of Theorem 3.2.8 in the case of $SU_q(2)$ (say at least for $0 < |q| \leqslant 3^{-1/2}$) would give an alternative proof of Theorem 3.2.8. However, it is not clear to us that such a computation would be easier. We can now give a second class of examples of weakly amenable discrete quantum groups. To do this, we first extend Theorem 3.2.8 to quantum automorphism groups of finite spaces.

Theorem 3.3.12. *Let B be a finite-dimensional C^*-algebra and let ψ be a δ-form on B with*

$$\delta \geqslant \frac{4}{\sqrt{3}}.$$

Then, there exists a polynomial $\widetilde{P}$ such that for every integer d,

$$\|m_{p_d}\|_{cb} \leqslant \widetilde{P}(d).$$

Proof. Consider in $C(SU_q(2))$ the C*-subalgebra $C(SO_q(3))$ generated by the coefficients of $u^{\otimes 2}$, where u denotes the fundamental representation of $SU_q(2)$. The restriction of the coproduct turns this C*-algebra into a compact quantum group, denoted $SO_q(3)$, which can be identified with the compact quantum automorphism group of $M_2(\mathbb{C})$ with respect to a $|q+q^{-1}|$-form (see [Soł10]). By definition, its irreducible representations can be identified with the even irreducible representations of $SU_q(2)$, and re-indexing them by $\mathbb{N}$ gives the $SO(3)$-fusion rules $u^1 \otimes u^n = u^{n-1} \oplus u^n \oplus u^{n+1}$. Consequently, we have for every integer d,

$$\|m_{p_d}\|_{cb} \leqslant P(2d)$$

as soon as $0 < |q| \leqslant 3^{-1/2}$. We know from [DRVV10, Thm 4.7] that if ψ is any δ-form on B, the compact quantum automorphism group of (B, ψ) is monoidally equivalent to $SO_q(3)$ if and only if $|q + q^{-1}| = \delta$. Thus, by Proposition 3.3.11, we have, with $\widetilde{P}(X) = P(2X)$,

$$\|m_{p_d}\|_{cb} \leqslant \widetilde{P}(d)$$

for any integer d as soon as δ is big enough. As noted in Remark 3.2.4, the lower bound is $\delta \geqslant 4/\sqrt{3}$. $\square$

Again, the proof of weak amenability is now straightforward.

Theorem 3.3.13. *Let B be a finite-dimensional C*-algebra with* $\dim(B) \geqslant 6$ *and let ψ be the δ-trace on B. Then, the discrete quantum group $\widehat{\mathbb{G}}(B, \psi)$ is weakly amenable with Cowling-Haagerup constant equal to* 1.

Proof. If ψ is the δ-trace, $\delta = \sqrt{\dim(B)}$ and it was proven in [Bra13, Thm 4.2] that the elements

$$a(t) = \sum_{k=0}^{+\infty} \frac{\mu_{2k}(\sqrt{t})}{\mu_{2k}(\sqrt{\dim(B)})} p_k$$

implement the Haagerup property. Since $\sqrt{5} \leqslant 4/\sqrt{3} \leqslant \sqrt{6}$, we can apply the same proof as in Theorem 3.3.2 as soon as the dimension of B is at least 6, yielding weak amenability with Cowling-Haagerup constant equal to 1. $\square$

Remark 3.3.14. As explained in Remark 1.2.54, Theorem 3.3.13 holds exactly for unimodular quantum automorphism groups.

Remark 3.3.15. A particular case of the previous theorem is the *quantum permutation groups* S_N^+ (for $N \geqslant 6$) defined by S. Wang in [Wan98]. We can also deduce the Haagerup property and weak amenability with Cowling-Haagerup constant equal to 1 for its symetrized version $(S_N^+)'$ (see [BS09] for the definition).

Remark 3.3.16. If $\dim(B) \leqslant 4$, the discrete quantum group $\widehat{\mathbb{G}}(B, \psi)$ is amenable and is therefore weakly amenable with Cowling-Haagerup constant equal to 1. Hence, the only case which is not covered by the previous theorem is the case of quantum automorphism groups of five-dimensional C*-algebras. There are two such C*-algebras, namely $\mathbb{C}^5$ and $M_2(\mathbb{C}) \oplus \mathbb{C}$. The quantum automorphism groups of these spaces are known to have the Haagerup property and we of course believe that they are weakly amenable with Cowling-Haagerup constant equal to 1, although we do not have a proof of this fact.

As in the case of free quantum groups, one can consider a finite-dimensional C*-algebra B together with a state which is not necessarily a δ-form. Then, there is a free product decomposition where each factor is the quantum automorphism group of some ideal in B together with a δ-form, with δ ranging through the spectrum of the operator mm^*. Using this unpublished result of M. Brannan, we get an easy corollary of Theorem 3.3.13.

Corollary 3.3.17. *Let B be a finite-dimensional C*-algebra with* $\dim(B) \geqslant 6$ *and let* ψ *be a trace on* B *such that* $\mathrm{Sp}(mm^*) \subset \mathbb{N}\setminus\{5\}$*. Then, the discrete quantum group* $\widehat{\mathbb{G}}(B, \psi)$ *has the Haagerup property and is weakly amenable with Cowling-Haagerup constant equal to* 1.

M. Brannan proved in [Bra13, Thm 5.13] that quantum automorphism groups of finite-dimensional C*-algebras are *bi-exact* in the sense that their von Neumann algebras satisfy Y. Isono's condition $(AO)^+$ (see [Iso12, Def 3.1.1]). This, together with Y. Isono's general result [Iso12, Thm A] immediately yields the following.

Theorem 3.3.18. *Let B be a finite-dimensional C*-algebra with* $\dim(B) \geqslant 6$ *and let* ψ *be a trace on* B *such that* $\mathrm{Sp}(mm^*) \subset \mathbb{N}\setminus\{5\}$*. Then, the von Neumann algebra* $L^\infty(\mathbb{G}(B, \psi))$ *is strongly solid.*

Non-unimodular issues

We end this dissertation with some further consequences of Proposition 3.3.11 and some questions it naturally raises. Let us say that an element $a \in \ell^\infty(\widehat{\mathbb{G}})$ is *central* if it is of the form

$$a = \sum_{\alpha\in\mathrm{Irr}(\mathbb{G})} b_\alpha p_\alpha$$

where $b_\alpha \in \mathbb{C}$ (i.e. a belongs to the centre of $\ell^\infty(\widehat{\mathbb{G}})$). We then have the following.

Corollary 3.3.19. *Let* φ *be a monoidal equivalence between two compact quantum groups* $\mathbb{G}_1$ *and* $\mathbb{G}_2$ *and let* $a = \sum b_\alpha p_\alpha \in \mathcal{Z}(\ell^\infty(\widehat{\mathbb{G}}_1))$ *be a central element. Then, the central element* $\varphi(a) = \sum b_\alpha p_{\varphi(\alpha)}$ *satisfies*

$$\|m_a\|_{cb} = \|m_{\varphi(a)}\|_{cb}.$$

Moreover, m_a *is unital and completely positive if and only if* $m_{\varphi(a)}$ *is unital and completely positive.*

Proof. Making linear combinations in the proof of Proposition 3.3.11, we see (using the same notations) that

$$(m_a \otimes \imath) \circ \rho = \rho \circ \left(\sum b_\alpha P_\alpha\right) \text{ and } \left(\sum b_\alpha P_\alpha \otimes \imath\right) \circ \theta = \theta \circ m_{\varphi(a)}, \tag{3.5}$$

giving the equality of the completely bounded norms. Assuming that m_a is unital and completely positive, we see that $m_{\varphi(a)}$ is unital and has completely bounded norm equal to 1. Thus, by Proposition 1.3.15, $m_{\varphi(a)}$ is completely positive. □

This means that weak amenability, as well as the Haagerup property, transfer through monoidal equivalence as soon as it can be implemented by central elements. This gives us examples of non-amenable, non-unimodular discrete quantum groups having approximation properties.

Corollary 3.3.20. *Let* $\mathbb{G}$ *be compact quantum group which is monoidally equivalent to* O_N^+ *or* U_N^+ *for some* N *or to the compact quantum automorphism group of* (B, ψ) *for some finite-dimensional C*-algebra* B *of dimension at least* 6 *endowed with its* δ*-trace* ψ*. Then,* $\widehat{\mathbb{G}}$ *has the Haagerup property and is weakly amenable with Cowling-Haagerup constant equal to* 1.

Remark 3.3.21. Using these arguments we can in fact recover [Bra13, Thm 4.2] directly from [Bra12a, Thm 4.5].

Remark 3.3.22. Note that under the conditions of Corollary 3.3.20, the linking algebra B giving the monoidal equivalence also has the Haagerup property relative to the unique invariant state and is weakly amenable with Cowling-Haagerup constant equal to 1. This means in particular that the C*-algebras $A_0(F_1, F_2)$ have the Haagerup property and are weakly amenable with Cowling-Haagerup constant equal to 1 as soon as F_1 or F_2 is unitary up to a scalar.

Let us give a alternative statement which is slightly more general, focusing on the intrinsic characterization of the discrete quantum groups.

Theorem 3.3.23. *Consider the following objects :*

- *A matrix* $F_1 \in GL_N(\mathbb{C})$ *such that* $F_1\overline{F}_1 \in \mathbb{R}.\mathrm{Id}$ *and* $\mathrm{Tr}(F_1^*F_1) \in \mathbb{N}$.
- *A matrix* $F_2 \in GL_N(\mathbb{C})$ *such that* $\mathrm{Tr}(F_2^*F_2) \in \mathbb{N}$.
- *A finite-dimensional C*-algebra* B *together with a state* ψ *such that* $\mathrm{Sp}(mm^*) \in \mathbb{N} \setminus \{5\}$.

Then, the discrete quantum groups $\mathbb{F}O^+(F_1)$, $\mathbb{F}U^+(F_2)$ *and* $\widehat{\mathbb{G}}(B, \psi)$ *have the Haagerup property and are weakly amenable with Cowling-Haagerup constant equal to* 1.

Proof. First note that $\mathbb{F}O^+(F_1)$ is monoidally equivalent to a unimodular free orthogonal quantum group. Thus, the first point comes from Corollary 3.3.20. For the second point, set

$$F_3 = \begin{pmatrix} 0 & 1 \\ -q^{-2} & 0 \end{pmatrix}$$

and observe that $U^+(F_2)$ is monoidally equivalent to $U^+(F_3)$ by [BDRV06, Thm 6.2], which is in turn monoidally equivalent to $U^+_{q+q^{-1}}$ and we again conclude with Corollary 3.3.19. Finally, using the free product decomposition alluded to before , Corollary 3.3.19 and the fact that $\delta \leqslant 4$ implies that $\dim(B) \leqslant 4$, we get the last point. $\square$

When the discrete quantum group $\mathbb{F}O^+(F)$ is not unimodular, (generalized) strong solidity has not yet been proved, but we can resort to the weaker notion of *semisolidity* : a von Neumann algebra M is said to be *semisolid* if for any finite projection $p \in M$ and any type II von Neumann subalgebra A of pMp, the *relative commutant* $N' \cap (pMp)$ of N in pMp is amenable. With this in hand, we have the following corollary of [Iso12, Thm C].

Theorem 3.3.24 (Isono)**.** *Let* $\widehat{\mathbb{G}}$ *be one of the discrete quantum groups considered in Theorem 3.3.23. Then,*

- *If* $L^\infty(\mathbb{G})$ *is a type* II_1 *factor, then it is strongly solid.*
- *If* $L^\infty(\mathbb{G})$ *is a type* III_1 *factor, then it has no Cartan subalgebra.*

These results raise the following question :

Question. *When is it possible to implement an approximation property by multipliers associated to central elements on a discrete quantum group ?*

To tackle this problem, we first give a very natural definition.

Definition 3.3.25. Let A be an approximation property (i.e. amenability, weak amenability or the Haagerup property). We say that a discrete quantum group $\widehat{\mathbb{G}}$ has *central A* if there are central multipliers implementing the property A on $\widehat{\mathbb{G}}$.

We can now express some positive partial answers to our question.

Proposition 3.3.26. *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be two compact quantum groups. Then,*

1. *If $\widehat{\mathbb{G}}_1$ has central A, then it has A (the converse is true for any discrete group since every element is central).*
2. *If $\widehat{\mathbb{G}}_1$ has central A and if $\mathbb{G}_1$ is monoidally equivalent to $\mathbb{G}_2$, then $\widehat{\mathbb{G}}_2$ has central A.*
3. *In the previous case, the linking algebra also has central A (with respect to the projections on spectral subspaces). Moreover, if the linking algebra has central A then both quantum groups have central A.*
4. *If $\widehat{\mathbb{G}}_1$ has the Haagerup property and is* unimodular*, then it has the central Haagerup property.*
5. *If $\widehat{\mathbb{G}}_1$ is amenable and* unimodular*, then it is centrally amenable.*

Proof. (1) is obvious and (2) and (3) are direct consequences of Corollary 3.3.19. The proof of (4) is a consequence of M. Brannan's averaging technique used in the proof of [Bra12a, Thm 3.7]. In fact, let $(a_t) \in \ell^\infty(\widehat{\mathbb{G}}_1)$ be a net of elements implementing the Haagerup property. For each t, there is a state ω_t on $C_{\max}(\mathbb{G}_1)$ such that m_{a_t} is the convolution operator C_{ω_t}. Set

$$T_{\omega_t} = \Delta^{-1} \circ \mathbb{E} \circ (C_{\omega_t} \otimes \imath) \circ \Delta.$$

It is proved in [Bra12a, Thm 3.7] that this operator is the multiplier associated to the central element

$$\sum_\alpha \frac{\omega_t(\chi_\alpha^*)}{\dim_q(\alpha)} p_\alpha.$$

It is still unital completely positive and L^2-compact. From the fact that (a_t) converges pointwise to 1, we see that ω_t converges pointwise to the counit and that consequently, T_{ω_t} converges pointwise to the identity. Hence, $\widehat{\mathbb{G}}_1$ has the central Haagerup property. The proof of (5) is similar. □

Remark 3.3.27. Let $\widehat{\mathbb{G}}$ be a discrete quantum group with the Haagerup property and let Ω be a 2-cocycle on $\widehat{\mathbb{G}}$ (see [BDRV06, Def 4.1] for the definition). If $\widehat{\mathbb{G}}$ is unimodular, then the twisted discrete quantum group $\widehat{\mathbb{G}}_\Omega$ also has the Haagerup property. In fact, the compact quantum groups $\mathbb{G}$ and $\mathbb{G}_\Omega$ are monoidally equivalent by [BDRV06, Rem 4.4]. Note moreover that since this monoidal equivalence preserves the dimensions, the discrete quantum group $\widehat{\mathbb{G}}_\Omega$ is also unimodular.

Remark 3.3.28. One can define the *central Cowling-Haagerup constant* $\Lambda_{cb}^{central}(\widehat{\mathbb{G}})$ of $\widehat{\mathbb{G}}$ to be the constant obtained by considering only central multipliers. Then, monoidal equivalence implies the equality of these central Cowling-Haagerup constants but does not *a priori* give any information on the usual Cowling-Haagerup constants, apart from the obvious inequality $\Lambda_{cb}(\widehat{\mathbb{G}}) \leqslant \Lambda_{cb}^{central}(\widehat{\mathbb{G}})$.

Note that the averaging trick used in the proof of (4) and (5) is not good enough to provide a similar result for weak amenability. Moreover, the unimodularity assumption cannot be dropped in (5). Indeed, the discrete dual of $SU_q(2)$ is amenable for any q and any $O^+(F)$ is monoidally equivalent to some $SU_q(2)$, though its dual is not amenable. Thus, $SU_q(2)$ is centrally amenable if and only if $q = 1$. However, if $q + q^{-1}$ is an integer, it follows from Theorem 3.3.23 that the discrete dual of $SU_q(2)$ has the central Haagerup property and is centrally weakly amenable with Cowling-Haagerup constant equal to 1. The same holds for the discrete dual of $SO_q(3)$.

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