Fundamental limits for learning hidden Markov model parameters

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Abstract

We study the frontier between learnable and unlearnable hidden Markov models (HMMs). HMMs are flexible tools for clustering dependent data coming from unknown populations. The model parameters are known to be fully identifiable (up to label-switching) without any modeling assumption on the distributions of the populations as soon as the clusters are distinct and the hidden chain is ergodic with a full rank transition matrix. In the limit as any one of these conditions fails, it becomes impossible in general to identify parameters. For a chain with two hidden states we prove nonasymptotic minimax upper and lower bounds, matching up to constants, which exhibit thresholds at which the parameters become learnable. We also provide an upper bound on the relative entropy rate for parameters in a neighbourhood of the unlearnable region which may have interest in itself.

1 Introduction

1.1 Context and motivation

Finite state space hidden Markov models (HMMs) are widely used in applications to model observations coming from different populations. HMMs can be viewed as particular mixture models. In the latter, given a latent sequence of cluster labels $(X_n)_{n\in\mathbb{N}}$ taking values in a finite set, the observed data $(Y_n)_{n\in\mathbb{N}}$ is a sequence of independent random variables with for each n the distribution of Y_n depending only on X_n. When the X_n are independent, a mixture model is not identifiable: various convex combinations of population probability distributions can lead to the same distribution for the observations. This is true even for observations taking values in a finite alphabet: one cannot recover two different multinomial distributions from a convex combination of them.

For a HMM, one adds the extra structure that $(X_n)_{n\in\mathbb{N}}$ forms a Markov chain. In sharp contrast to the independent setting, with hidden Markov structure one can recover the distribution of data for each population absent virtually any constraint on these distributions (known in this context as the emission distributions). This fact had been observed in applied papers, and a theoretical proof that parameters can be identified with minimal assumptions is recent, given for HMMs taking values in a finite set in [5, 24, 6] and extended to allow for emission distributions modelled nonparametrically (but still with the underlying Markov chain having finite state space) in [18, 4]. HMMs therefore form a tractable class of model nevertheless rich enough to model many practical clustering settings well: see for instance [11, 27, 25, 37, 42, 41]. In this context note that given good estimates of the model parameters one can almost match the optimal clustering and testing behaviour of the Bayes classifier (e.g. see [39], [2]); let us emphasise once more that this is possible essentially absent any constraint on the emission distributions, in contrast to typical clustering algorithms which may require parametric modelling or separation of clusters.

In drawing a contrast between the independent and dependent cases, we have so far omitted to mention that of course an independent model is a degenerate subcase of a Markov model. There are three ways in which the data $(Y_n)_{n\in\mathbb{N}}$ can fail to exhibit dependence: when the population labels themselves are in reality independently distributed; when the emission distributions are identical; or when only one population is observed. It is thus of theoretical and practical importance to understand quantitatively what happens when these limiting situations are approached.

The present work initiates an exploration of the limits of learnability of the hidden Markov parameters as the independent subcase is approached. We focus on the setting of two hidden states and multinomial data, and exhibit principles which should generalise to much wider settings.

1.2 Contribution

Our main result, Theorem 1, gives upper and lower bounds showing the minimax estimation rate for the model parameters, exhibiting that these parameters can be learned if and only if the sample size n is large enough compared to a suitable measure of the closeness of the data to the independent subcase.

Important steps to get the main result are as follows. We introduce a reparametrisation of the model leading to a statistical distance which appears to be a key tool for the understanding of the fundamental limits of learning the HMM parameters near the independent subcase. This statistical distance is proved in Proposition 1 to be equivalent to the distance between the distribution of three consecutive observations, and leads to an explicit upper bound of the relative entropy rate for a specific part of parameters domain, see Proposition 2, which we believe could have interest in itself. Upper bounds for the learning of the new parameters are proved in Theorem 2 while (almost) matching lower bounds are proved in Theorem 3.

1.3 Related work

Theoretical justification of a range of learning methods for HMMs with emission distributions modelled parametrically or nonparametrically have been developed in recent years: moment and tensor methods in [6, 13], and model selection using penalized least squares estimation in [12, 29], using penalized likelihood methods in [30], or using other techniques in [28]. These works all give both asymptotic and nonasymptotic upper bounds controlling the distance between estimators and the unknown parameters. All require the data to truly be dependent, but none quantify explicitly how their sample complexity results depend on the "distance" to independence. Indeed, quantifying this dependence requires a sharp understanding of how the distances between distributions evolve with respect to the distances between parameters, as done for particular parametric finite mixture models in [21, 23, 16].

Results in [13] control the propagation of errors from parameter estimation to the posterior probabilities when calculating the latter via plug-in, implying that good control on the risk of the estimators will ensure the performance of the empirical Bayes classifier is close to that of the true Bayes classifier (whose optimality for clustering is a standard result in decision theory [14]).

A topic closely related to binary classification/clustering is multiple testing, in which one aims to identify within some large data set a collection of data points which come from a "discovery" hypothesis, rather than from the conservative null hypothesis. In this setting control of the false discovery rate has been obtained recently for a knockoffs-based method in [36] and for an empirical Bayes method in [39, 1]; in each case estimation of the HMM parameters is an essential first step. Modelling the proportion of non-null signals as vanishingly small, as our results permit, would allow for further links to the setting of *sparse* multiple testing, considered for example (with independent data) in [3, 10].

Relative entropy rate, or equivalently Kullback-Leibler rate, between HMMs can be expressed using Blackwell's invariant measure [9], but no explicit formulation exists [38]. Providing useful or meaningful upper and lower bounds is a subject of ongoing research [15, 31, 17]. In Proposition 2 we obtain a new bound on the Kullback-Leibler rate between HMMs which compared to the aforementioned works does a better job at capturing the effects of the underlying Markov dependency structure, at the expense of holding only for a restricted subspace of parameters.

To the best of our knowledge no prior theoretical result exists addressing the learning of parameters of a HMM when approaching the independent case. By experimentally studying the EM algorithm when the multinomial emission distributions approach each other, the authors in [35] find a range of parameters for which the EM algorithm behaves badly. We believe such behavior is primarily a result of the investigated region approaching the limit where the parameters become unlearnable, not of a limitation of the EM algorithm specifically.

Finally, let us mention that departure from the independence assumption has been noted to allow for better learning also in HMM settings free from the assumption that the Markov chain has a finite state space [20, 7] (at the expense of stricter assumptions on the emission distributions), and also in other problems including dynamic networks [32, 8], image denoising [33], and deconvolution [19].

1.4 Organisation of the paper

We describe the setting in Section 2 and state our main result in Section 3. The key reparametrisation is given in Section 4 where we state the basic propositions involving the statistical distance we define. Intermediate upper bound results are given in Section 5 while lower bounds are in Section 6. In Section 7 we discuss our results and possible further work. All proofs are deferred to Section 8.

2 Setting

Consider a two-state HMM with multinomial emissions, in which we observe the first n entries of a sequence $\boldsymbol{Y} = (Y_1, Y_2, \dots) \in \{1, \dots, K\}^{\mathbb{N}}$ which, under a parameter $\theta = (p, q, f_0, f_1)$, satisfies

$$\begin{aligned} \mathbb{P}_\theta(Y_n = k \mid \boldsymbol{X}) &= f_{X_i}(k), \\ \boldsymbol{X} = (X_n)_{n\in\mathbb{N}} &\sim \mathrm{Markov}(\pi, Q), \end{aligned} \tag{1}$$

with the $Y_j,\ j \in \mathbb{N}$ conditionally independent given $\boldsymbol{X}$. The vector $\boldsymbol{X}$ of 'hidden states' takes values in $\{0,1\}^{\mathbb{N}}$ and the transition matrix of the chain is given by

$$Q := \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}, \tag{2}$$

with the convention that for $j \geq 1$, $\mathbb{P}_\theta(X_{j+1} = 0 \mid X_j = 0) = 1 - p < 1$ and $\mathbb{P}_\theta(X_{j+1} = 0 \mid X_j = 1) = q > 0$. The densities f_0, f_1 are the 'emission densities' with respect to counting measure on $\{1, \dots, K\}$. Grant also that X_1 is drawn from the stationary distribution of the chain, i.e. $\mathbb{P}_\theta(X_1 = 1) = p/(p+q)$. We throughout use $\mathbb{P}_\theta$ to denote the law of $(\boldsymbol{X}, \boldsymbol{Y})$, and all induced marginal and conditional laws.

In the limit where the sequence $\boldsymbol{Y}$ becomes independent and identically distributed (i.i.d.), learning the parameters becomes impossible due to standard identifiability issues for mixture models : the distribution of Y_1 may be decomposed in many ways as a convex combination of multinomials. This i.i.d. limit can be approached in three ways:

1. $p \approx 0$ or $q \approx 0$, and thus the chain $\boldsymbol{X}$ passes long periods of time in one of the two states;
2. the transition matrix Q is nearly singular, so that $\boldsymbol{X}$ itself is almost i.i.d; this is the case if $|1-p-q| \approx 0$;
3. the emission distributions are close to each other: $\|f_0 - f_1\| \approx 0$, where $\|\cdot\|$ denotes the usual Euclidean norm, $\|f\|^2 = \sum |f(k)|^2$.

We adopt a minimax point of view and encapsulate all the above scenarios within the class of parameters defined, for some $\delta, \epsilon \in (0,1)$ and some $\zeta > 0$, by

$$\Theta = \Theta(\delta, \epsilon, \zeta) = \{\theta \ : p, q \geq \delta, \ |1-p-q| \geq \epsilon, \ \|f_0 - f_1\| \geq \zeta\}.$$

Introduce also the subset

$$\Theta_L = \Theta_L(\delta, \epsilon, \zeta) = \Theta \cap \{1 - |1-p-q| \geq L\}.$$

Remark 1. *Note that $1 - |1-p-q|$ is the absolute spectral gap of the chain $\boldsymbol{X}$, and hence the mixing time of the chain can be upper bounded uniformly in Θ_L since the state space has size 2 (so the chain is automatically reversible). Here L may be arbitrarily small but we think of it as* fixed, *in contrast to δ, ϵ and ζ which are allowed to depend on n. With the introduction of this lower bound we still allow* one *of p, q to be vanishingly small (or arbitrarily close – even equal – to 1), but not both.*

Remark 2. *If ζ is too large compared to $1/K$, $\Theta(\delta, \epsilon, \zeta)$ may be too small to be an interesting parameter space. To avoid this and ensure that $\Theta(\delta, \epsilon, \zeta)$ contains near uniform density pairs, we assume a mild compatibility condition: that*

$$\zeta \leq \frac{\sqrt{2\lfloor K/2 \rfloor}}{4K}. \tag{3}$$

3 Main results

To avoid a label-switching issue discussed in the next section we assume that $f_0 - f_1$ lies in some specified half-plane. Our main result is the following.

Theorem 1. *There exist an estimator $\hat{\theta} = (\hat{p}, \hat{q}, \hat{f}_0, \hat{f}_1)$ and a constant $C = C(K, L) > 0$ such that for all $1 \leq x^2 \leq n\delta^2\epsilon^4\zeta^6$,*

$$\sup_{\theta \in \Theta_L} \mathbb{P}_\theta\Big(\max(|\hat{p} - p|, |\hat{q} - q|) > \frac{Cx}{\sqrt{n\delta^2\epsilon^4\zeta^6}} \max(\delta, \epsilon\zeta)\Big) \leq e^{-x^2},$$

$$\sup_{\theta \in \Theta_L} \mathbb{P}_\theta\Big(\max(\|\hat{f}_0 - f_0\|, \|\hat{f}_1 - f_1\|) > \frac{Cx}{\sqrt{n\delta^2\epsilon^4\zeta^4}}\Big) \leq e^{-x^2}.$$

Grant condition (3). There exist constants $c = c(K) > 0$ and $\epsilon_1 > 0$ such that for $\delta \leq 1/6$, $\epsilon \leq \epsilon_1$, $L \leq 1/3$ and $n\delta^2\epsilon^4\zeta^6 \geq 1$,

$$\inf_{\check{\theta}} \sup_{\theta \in \Theta_L} \mathbb{P}_\theta\Big(\max(|\check{p} - p|, |\check{q} - q|) > \frac{c}{\sqrt{n\delta^2\epsilon^4\zeta^6}} \max(\delta, \epsilon\zeta)\Big) \geq 1/4,$$

$$\inf_{\check{\theta}} \sup_{\theta \in \Theta_L} \mathbb{P}_\theta\Big(\max(\|\check{f}_0 - f_0\|, \|\check{f}_1 - f_1\|) > \frac{c}{\sqrt{n\delta^2\epsilon^4\zeta^4}}\Big) \geq 1/4,$$

where the infima are over all estimators $\check{\theta} = (\check{p}, \check{q}, \check{f}_0, \check{f}_1)$.

The estimator $\hat{\theta}$ is built via plug-in from those constructed later in Theorem 2. Note that the maxima are genuinely required in the lower bounds: in the extreme case where p is close to zero and q is close to 1, one has many samples with $X_i = 0$ and few with $X_i = 1$, so that p and f_0 are easier to estimate accurately than q and f_1.

We deduce immediately the sample complexity for learning the parameters. We do not seek sharp dependence on K in the bounds because we believe our results can be extended to the nonparametric setting, which we leave for further work.

Corollary 1. *Fix a target error magnitude $E > 0$ and a probability level $\alpha > 0$. For the same estimators as in Theorem 1, there exists a constant $C = C(K, L)$ such that for any $\theta \in \Theta_L$ we have*

$$n \geq \frac{\log(1/\alpha)}{\delta^2\epsilon^4\zeta^6} \max\Big(\frac{C\delta^2}{E^2}, \frac{C\epsilon^2\zeta^2}{E^2}, 1\Big) \quad \Longrightarrow \quad \mathbb{P}_\theta(\max(|\hat{p} - p|, |\hat{q} - q|) > E) \leq \alpha,$$
$$n \geq \frac{\log(1/\alpha)}{\delta^2\epsilon^4\zeta^4} \max\Big(\frac{C}{E^2}, \frac{1}{\zeta^2}\Big) \quad \Longrightarrow \quad \mathbb{P}_\theta(\max(\|\hat{f}_0 - f_0\|, \|\hat{f}_1 - f_1\|) > E) \leq \alpha.$$

Conversely there exists a constant $c = c(K) > 0$ such that for all $0 < E \leq c(K)$ and for any estimator $\check{\theta} = (\check{p}, \check{q}, \check{f}_0, \check{f}_1)$ there exists $\theta \in \Theta_L$ such that

$$n \leq \frac{c^2 \max(\delta^2, \epsilon^2\zeta^2)}{E^2\delta^2\epsilon^4\zeta^6} \quad \Longrightarrow \quad \mathbb{P}_\theta(\max(|\check{p} - p|, |\check{q} - q|) > E) \geq 1/4,$$
$$n \leq \frac{c^2}{E^2\delta^2\epsilon^4\zeta^4} \quad \Longrightarrow \quad \mathbb{P}_\theta(\max(\|\check{f}_0 - f_0\|, \|\check{f}_1 - f_1\|) > E) \geq 1/4.$$

Note that to apply Theorem 1 for the lower bounds we would initially also need $n \geq (\delta^2\epsilon^4\zeta^6)^{-1}$ but by monotonicity — i.e. the fact that any measurable function of $(Y_1, \ldots, Y_n)$ is also a measurable function of $(Y_1, \ldots, Y_N)$ for $N \geq n$ — the restriction can be removed.

Let us sketch the main ideas behind the proof of Theorem 1. The full proof is deferred to Section 8, along with all other proofs for this article.

The minimax upper bounds are obtained by producing an estimator that attains the bounds. Building on the work of [6, 18] we know that θ is identifiable from $p_\theta^{(3)}$, and we propose a reparametrisation of the model to simplify the analysis. Indeed, motivated by a desire to simplify the expression for $p_\theta^{(3)}$ (see equations (9) and (10) in Section 4), we introduce new parameters ϕ, ψ and we show in Proposition 1 that $\|p^{(3)}_{\theta(\phi,\psi)} - p^{(3)}_{\theta(\tilde{\phi},\tilde{\psi})}\|$ is equivalent to $\rho(\phi, \psi; \tilde{\phi}, \tilde{\psi})$, where ρ is defined in the proposition and can be seen as the "natural" statistical distance of the problem (see below). Then, we leverage that $p_\theta^{(3)}$ can be estimated in Euclidean distance at the parametric rate $n^{-1/2}$ by the empirical estimator $\hat{p}_n^{(3)}$ defined in Section 5, Lemma 1. This suggests that solving for $(\hat{\phi}, \hat{\psi}) \in \arg\min_{\phi,\psi} \|p^{(3)}_{\theta(\phi,\psi)} - \hat{p}_n^{(3)}\|$ will give a good estimator $(\hat{\phi}, \hat{\psi})$ for (ϕ, ψ). By standard calculations and using the equivalence between $\|p^{(3)}_{\theta(\phi,\psi)} - p^{(3)}_{\theta(\tilde{\phi},\tilde{\psi})}\|$ and $\rho(\phi, \psi; \tilde{\phi}, \tilde{\psi})$ derived in Proposition 1, we obtain bounds on maximum risk of such $(\hat{\phi}, \hat{\psi})$ for estimating (ϕ, ψ) in Theorem 2. Finally, the upper bounds for the original parameters in Theorem 1 are obtained by taking $\hat{\theta} = \theta(\hat{\phi}, \hat{\psi})$.

Incidentally, we remark that the parameterization (ϕ, ψ) turns out to be of special interest: the components of ϕ determine how close the sequence $\boldsymbol{Y}$ is to being i.i.d in an interpretable way (see Section 4), and the parameter ψ is related to the stationary distribution of the sequence $\boldsymbol{Y}$. For this reason, we also establish minimax bounds for the estimation of ϕ and ψ themselves in Theorems 2 and 3.

The minimax lower bounds are obtained by an argument *à la* Le Cam. In particular, it is a famous result of Le Cam [26, 40] that the minimax rate (under quadratic loss) of estimating a functional $g:\Theta\to\mathbb{R}$ is always greater than the maximum value that $|g(\theta)-g(\tilde\theta)|^2$ can take for $\theta,\tilde\theta\in\Theta$ under the constraint that $K(p_\theta^{(n)};p_{\tilde\theta}^{(n)})\le c$, where $K(p_\theta^{(n)};p_{\tilde\theta}^{(n)})$ denotes the *Kullback-Leibler* (KL) divergence between the laws of $(Y_1,\dots,Y_n)$ under parameters θ and $\tilde\theta$, and $0<c<1$ is a small positive constant (see Lemma 2 for the precise formulation we use). Understanding bounds on $|g(\theta)-g(\tilde\theta)|$ in terms of bounds on $K(p_\theta^{(n)};p_{\tilde\theta}^{(n)})$ is also sufficient for obtaining an upper bound on the minimax estimation rate. Since we have dependent observations, the main difficulty of the proof is to relate $K(p_\theta^{(n)};p_{\tilde\theta}^{(n)})$ to a suitable notion of distance between θ and $\tilde\theta$. A key result is Proposition 2 showing that under mild assumptions $K(p_{\theta(\phi,\psi)}^{(n)};p_{\theta(\tilde\phi,\tilde\psi)}^{(n)})$ is upper bounded by a constant times $n\rho^2(\phi,\psi;\tilde\phi,\tilde\psi)$. Then the lower bounds for ϕ (respectively ψ) in Theorem 3 are obtained by lower bounding the value of the optimization problems $\max|\phi_j-\tilde\phi_j|^2$ (respectively $\max|\psi_j-\tilde\psi_j|^2$) subject to $n\rho^2(\phi,\psi;\tilde\phi,\tilde\psi)\le c$ and $\theta(\phi,\psi),\theta(\tilde\phi,\tilde\psi)\in\Theta$ for a small enough constant $c>0$. Finally, the lower bounds for the original parameters in the Theorem 1 are essentially deduced from the bounds for (ϕ,ψ) and inversion of the parameterization.

4 Change of parameterisation

We reparametrise the model in such a way that the i.i.d. limiting cases are highlighted, by changing variables to $\phi=(\phi_1,\phi_2,\phi_3)$ and $\psi=(\psi_1,\psi_2)$ defined as

$$\phi(\theta)=\begin{pmatrix}\frac{q-p}{p+q} & 1-p-q & \|f_0-f_1\|\end{pmatrix},\quad \psi(\theta)=\begin{pmatrix}\frac{qf_0+pf_1}{p+q} & \frac{f_0-f_1}{\|f_0-f_1\|}\end{pmatrix}.$$

Here we have separated the scalar parameters ϕ from the vector parameters ψ. Defining

$$r(\phi)=\tfrac14(1-\phi_1^2)\phi_2\phi_3^2, \tag{4}$$

it follows from the discussion in Section 2 that the data $\boldsymbol{Y}$ is close to i.i.d. exactly when $r(\phi)\approx0$. [This is of course true also of other combinations of the components of ϕ, but as equation (10) will show, $r(\phi)$ is the appropriate combination measuring the "distance" to the i.i.d. case.]

Define

$$\begin{aligned}\Phi=\Phi(\delta,\epsilon,\zeta)&=\{(\phi(\theta),\psi(\theta)):\theta\in\Theta(\delta,\epsilon,\zeta)\},\\ \Phi_L=\Phi_L(\delta,\epsilon,\zeta)&=\{(\phi(\theta),\psi(\theta)):\theta\in\Theta_L(\delta,\epsilon,\zeta)\},\end{aligned}$$

and note that for $(\phi,\psi)\in\Phi$ we have

$$-\frac{1-\delta}{1+\delta}\le\phi_1\le\frac{1-\delta}{1+\delta},\quad \epsilon\le|\phi_2|\le1-2\delta,\quad \zeta\le\phi_3\le\sqrt2,\quad |r(\phi)|\ge\delta\epsilon\zeta^2/4, \tag{5}$$

while for $(\phi,\psi)\in\Phi_L$ we additionally have

$$|\phi_2|\le1-L. \tag{6}$$

Remark 3. *When $K = 2$, in view of identifiability issues discussed in the next subsection, ψ_2 is not needed in the parametrisation, since we may universally make the choice*

$$\psi_2 = \Big(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\Big).$$

Remark 4. *The parametrisation $\theta \mapsto (\phi, \psi)$ is invertible: we calculate*

$$\begin{aligned} p &= \tfrac{1}{2}(1-\phi_2)(1-\phi_1), \\ q &= \tfrac{1}{2}(1-\phi_2)(1+\phi_1), \\ f_0 &= \psi_1 - \tfrac{1}{2}\phi_1\phi_3\psi_2 + \tfrac{1}{2}\phi_3\psi_2, \\ f_1 &= \psi_1 - \tfrac{1}{2}\phi_1\phi_3\psi_2 - \tfrac{1}{2}\phi_3\psi_2. \end{aligned}$$

Remark 5. *Suppose ψ_1 is a probability density function with respect to counting measure on $\{1, \dots, K\}$, ψ_2 is a function satisfying $\|\psi_2\| = 1$ and $\sum_k \psi_2(k) = 0$, and ϕ satisfies $|\phi_1| \le 1$, $|\phi_2| \le 1$ and $\phi_3 \ge 0$. Then (ϕ, ψ) lies in $\Phi(\delta, \epsilon, \zeta)$ if and only if*

$$\tfrac{1}{2}(1-\phi_2)(1-|\phi_1|) \ge \delta, \quad \tfrac{1}{2}(1-\phi_2)(1+|\phi_1|) \le 1, \quad |\phi_2| \ge \epsilon, \quad \phi_3 \ge \zeta, \tag{7}$$
$$\psi_1(k) - \tfrac{1}{2}\phi_1\phi_3\psi_2(k) - \tfrac{1}{2}\phi_3|\psi_2(k)| \ge 0, \quad \forall k \le K. \tag{8}$$

The model (1) is identifiable for the parameter set Θ only up to 'label-switching', since $\boldsymbol{Y}$ has the same distribution under the parameters (p, q, f_0, f_1) and (q, p, f_1, f_0); in the parametrisation (ϕ, ψ), the distribution of $\boldsymbol{Y}$ is the same under $(\phi_1, \phi_2, \phi_3, \psi_1, \psi_2)$ and under $(-\phi_1, \phi_2, \phi_3, \psi_1, -\psi_2)$. However, it was proved in [6] that aside from this label-switching, the model parameters can be identified from the law of just three consecutive observations. To that end, for any integer m denoting by $P_\theta^{(m)}$ the law of $(Y_1, \dots, Y_m)$ under parameter $\theta \in \Theta$, and by $p_\theta^{(m)}$ the corresponding density with respect to counting measure on $\{1, \dots, K\}^m$, we calculate

$$p_\theta^{(3)} = \left(\frac{q}{p+q}\right) g \otimes f_0 \otimes g + \left(\frac{p}{p+q}\right) h \otimes f_1 \otimes h, \tag{9}$$

where $g = (1-p)f_0 + pf_1$ and $h = qf_0 + (1-q)f_1$, and where $\otimes$ denotes the tensor product so that

$$(f \otimes g \otimes h)(a, b, c) = f(a)g(b)h(c), \quad (a, b, c) \in \{1, \dots, K\}^3.$$

In the (ϕ, ψ) parametrisation, writing just $p_{\phi,\psi}^{(3)}$ for $p_{\theta(\phi,\psi)}^{(3)}$ in a slight abuse of notation, we have

$$\begin{aligned} p_{\phi,\psi}^{(3)} = \psi_1 \otimes \psi_1 \otimes \psi_1 &+ r(\phi)\big(\psi_2 \otimes \psi_2 \otimes \psi_1 + \psi_1 \otimes \psi_2 \otimes \psi_2\big) \\ &+ \phi_2 r(\phi) \psi_2 \otimes \psi_1 \otimes \psi_2 - \phi_1\phi_2\phi_3 r(\phi) \psi_2 \otimes \psi_2 \otimes \psi_2, \end{aligned} \tag{10}$$

where we recall the notation $r(\phi) = \frac{1}{4}(1-\phi_1^2)\phi_2\phi_3^2$.

We define a statistical distance ρ directly on the parameter space Φ which is equivalent to the Euclidean distance between the densities $p_{\phi,\psi}^{(3)}$ and $p_{\tilde{\phi},\tilde{\psi}}^{(3)}$. The function ρ is not a true metric because it may not satisfy the triangle inequality and because, due to the identifiability issues reflected by the appearance of factors of $\operatorname{sgn}(\langle \psi_2, \tilde{\psi}_2 \rangle)$ in its definition, we may have $\rho(\phi, \psi; \tilde{\phi}, \tilde{\psi}) = 0$ with $(\phi, \psi) \neq (\tilde{\phi}, \tilde{\psi})$. Here $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on $\mathbb{R}^K$, $\langle f, g \rangle = \sum_{i=1}^K f(k)g(k)$.

Proposition 1. *For r as in equation (4) define m by*

$$m(\phi) = (r(\phi), \phi_2 r(\phi), \phi_1\phi_2\phi_3 r(\phi)), \tag{11}$$

and define

$$\begin{aligned}\rho(\phi,\psi;\tilde{\phi},\tilde{\psi}) = \max\big\{&|m_1(\phi)-m_1(\tilde{\phi})|,\ |m_2(\phi)-m_2(\tilde{\phi})|,\ |m_3(\phi)-\mathrm{sgn}(\langle\psi_2,\tilde{\psi}_2\rangle)\cdot m_3(\tilde{\phi})|,\\ &\|\psi_1-\tilde{\psi}_1\|,\ \max\{|m_1(\phi)|,|m_1(\tilde{\phi})|\}\cdot\|\psi_2-\mathrm{sgn}(\langle\psi_2,\tilde{\psi}_2\rangle)\cdot\tilde{\psi}_2\|\big\}.\end{aligned} \tag{12}$$

There exist constants $c_1, c_2 > 0$ (which depend on K) such that for all $(\phi,\psi), (\tilde{\phi},\tilde{\psi}) \in \bigcup_{\delta,\epsilon,\zeta}\Phi(\delta,\epsilon,\zeta)$ we have

$$c_1\rho(\phi,\psi;\tilde{\phi},\tilde{\psi}) \le \|p^{(3)}_{\phi,\psi} - p^{(3)}_{\tilde{\phi},\tilde{\psi}}\| \le c_2\rho(\phi,\psi;\tilde{\phi},\tilde{\psi}).$$

Optimal estimation rates can be obtained if we adequately understand the Kullback–Leibler divergence between distributions with different parameters. The Kullback–Leibler divergence between $P^{(n)}_{\theta(\phi,\psi)}$ and $P^{(n)}_{\theta(\tilde{\phi},\tilde{\psi})}$ can be related to the statistical distance $\rho(\phi,\psi;\tilde{\phi},\tilde{\psi})$ in a neighbourhood of the independent subcase.

Proposition 2. *Assume there exists $c \in (0,1)$ such that $\min(f_0, f_1, \tilde{f}_0, \tilde{f}_1) \ge c$. There exist constants $C, \epsilon_0 > 0$ depending only on c such that if $\max(|\phi_2|, |\tilde{\phi}_2|) \le \epsilon_0$, then with ρ as in equation (12),*

$$K(P^{(n)}_{\theta(\phi,\psi)}, P^{(n)}_{\theta(\phi,\psi)}) \le Cn\rho(\phi,\psi;\tilde{\phi},\tilde{\psi})^2.$$

We note that only the lower bound on $\|p^{(3)}_{\phi,\psi} - p^{(3)}_{\tilde{\phi},\tilde{\psi}}\|$ in Proposition 1 is used in our paper (it is used in proving Theorem 2). The upper bound on $\|p^{(3)}_{\phi,\psi} - p^{(3)}_{\tilde{\phi},\tilde{\psi}}\|$ is still of interest as it establishes the tightness (up to constants) of the corresponding lower bound, thereby proving the equivalence between $\|p^{(3)}_{\phi,\psi} - p^{(3)}_{\tilde{\phi},\tilde{\psi}}\|$ and $\rho(\phi,\psi;\tilde{\phi},\tilde{\psi})$ and showing that ρ is the natural and adequate statistical metric for this problem. Furthermore, in combination with Proposition 2, Pinsker's inequality, and the fact that all norms on the set $\{1,\dots,K\}^3$ are equivalent, it shows that whenever $\max(|\phi_2|, |\tilde{\phi}_2|)$ is small enough,

$$K(P^{(n)}_{\theta(\phi,\psi)}, P^{(n)}_{\theta(\phi,\psi)}) \le C'n\|p^{(3)}_{\phi,\psi} - p^{(3)}_{\tilde{\phi},\tilde{\psi}}\|^2 \le C''nK(P^{(3)}_{\theta(\phi,\psi)}, P^{(3)}_{\theta(\phi,\psi)}),$$

for constants $C', C'' > 0$, once again highlighting the prominent role of the law of 3 consecutive observations in HMM modeling, and illustrating that optimal estimators (up to numerical constants) can be built solely on the basis of the empirical distribution of blocks of 3 consecutive observations. This shows that as long as the chain $\boldsymbol{Y}$ is not "too dependent", it behaves almost as if we had observed i.i.d. blocks of 3 consecutive observations (in which case we would have that $K(P^{(n)}_{\theta(\phi,\psi)}, P^{(n)}_{\theta(\phi,\psi)}) = (n/3)K(P^{(3)}_{\theta(\phi,\psi)}, P^{(3)}_{\theta(\phi,\psi)})$, for all n divisible by 3).

5 Upper bounds

We obtain the following upper bounds for estimating ϕ and ψ. Since we are studying limits as the quantities of interest become small, the relative risk may be of as much interest as the absolute risk, and we provide bounds for both quantities. The bounds demonstrate that learning model parameters is possible in the regime where n is large enough in relation to δ, ϵ and ζ. Observe firstly that estimation of $p^{(3)}$ is possible at a parametric rate.

Lemma 1. *Define the empirical estimator* $\hat{p}_n^{(3)} : \{1, \dots, K\}^n \to [0,1]$ *by*

$$\hat{p}_n^{(3)}(a,b,c) = \frac{1}{n} \sum_{i=1}^{n-2} \mathbb{1}\{Y_i = a,\ Y_{i+1} = b,\ Y_{i+2} = c\}. \tag{13}$$

Then for some constant $C = C(K, L)$ *and any* $x \geq 1$

$$\sup_{(\phi,\psi)\in\Phi_L(\delta,\epsilon,\zeta)} \mathbb{P}_{(\phi,\psi)}(\|\hat{p}^{(3)} - p^{(3)}\| \geq Cx/\sqrt{n}) \leq e^{-x^2}.$$

Theorem 2. *Assume* Φ_L *is non-empty and let* $\hat{\phi}, \hat{\psi}$ *be any measurable functions satisfying, for* $\hat{p}_n^{(3)}$ *as in equation* (13)*,*

$$\|p^{(3)}_{\hat{\phi},\hat{\psi}} - \hat{p}_n^{(3)}\| \leq 2 \inf_{(\tilde{\phi},\tilde{\psi})\in\Phi_L} \|p^{(3)}_{\tilde{\phi},\tilde{\psi}} - \hat{p}_n^{(3)}\|.$$

There exists a constant $C = C(K, L) > 0$ *such that the following hold.*

1. *Assume* $1 \leq x^2 \leq n\delta^2\epsilon^4\zeta^6$. *Then*

$$\sup_{(\phi,\psi)\in\Phi_L(\delta,\epsilon,\zeta)} \mathbb{P}_{\phi,\psi}\Big(\Big|\frac{1-\hat{\phi}_1^2}{1-\phi_1^2} - 1\Big|^2 \geq \frac{2Cx^2}{n\delta^2\epsilon^4\zeta^6}\Big)$$
$$\leq \sup_{(\phi,\psi)\in\Phi_L(\delta,\epsilon,\zeta)} \mathbb{P}_{\phi,\psi}\Big(\min(|\hat{\phi}_1 - \phi_1|, |\hat{\phi}_1 + \phi_1|)^2 \geq \frac{Cx^2}{n\epsilon^4\zeta^6}\Big) \leq e^{-x^2}.$$

2. *Assume* $1 \leq x^2 \leq n\delta^2\epsilon^2\zeta^4$. *Then*

$$\sup_{(\phi,\psi)\in\Phi_L(\delta,\epsilon,\zeta)} \mathbb{P}_{\phi,\psi}\Big(\Big|\frac{\hat{\phi}_2}{\phi_2} - 1\Big|^2 \geq C\frac{x^2}{n\delta^2\epsilon^4\zeta^4}\Big)$$
$$\leq \sup_{(\phi,\psi)\in\Phi_L(\delta,\epsilon,\zeta)} \mathbb{P}_{\phi,\psi}\Big(|\hat{\phi}_2 - \phi_2|^2 \geq C\frac{x^2}{n\delta^2\epsilon^2\zeta^4}\Big) \leq e^{-x^2}.$$

3. *Assume* $1 \leq x^2 \leq n\delta^2\epsilon^4\zeta^6$. *Then*

$$\sup_{(\phi,\psi)\in\Phi_L(\delta,\epsilon,\zeta)} \mathbb{P}_{\phi,\psi}\Big(\Big|\frac{\hat{\phi}_3}{\phi_3} - 1\Big|^2 \geq C\frac{x^2}{n\delta^2\epsilon^4\zeta^6}\Big)$$
$$\leq \sup_{(\phi,\psi)\in\Phi_L(\delta,\epsilon,\zeta)} \mathbb{P}_{\phi,\psi}\Big(|\hat{\phi}_3 - \phi_3|^2 \geq C\frac{x^2}{n\delta^2\epsilon^4\zeta^4}\Big) \leq e^{-x^2}.$$

4. *Assume* $1 \leq x^2 \leq n$. *Then*

$$\sup_{(\phi,\psi)\in\Phi_L(\delta,\epsilon,\zeta)} \mathbb{P}_{\phi,\psi}\Big(\|\hat{\psi}_1 - \psi_1\|^2 \geq \frac{Cx^2}{n}\Big) \leq e^{-x^2}.$$

5. *Assume* $1 \leq x^2 \leq n\delta^2\epsilon^2\zeta^4$ *and* $K > 2$. *Then*

$$\sup_{(\phi,\psi)\in\Phi_L(\delta,\epsilon,\zeta)} \mathbb{P}_{\phi,\psi}\Big(\min\big(\|\hat{\psi}_2 - \psi_2\|^2, \|\hat{\psi}_2 + \psi_2\|^2\big) \geq \frac{Cx^2}{n\delta^2\epsilon^2\zeta^4}\Big) \leq e^{-x^2}.$$

Recall that estimating ψ_2 is unnecessary when $K = 2$ (see Remark 3). Note that the absolute loss in each case is bounded, and one can deduce that the bounds for ϕ_2 and for ψ hold without an upper bound on x, with e^{-x^2} on the right replaced by zero (for C large enough).

6 Lower bounds

We prove lower bounds, matching the previous upper bounds in a suitable regime and demonstrating the impossibility of learning model parameters when n is not large enough in relation to δ, ϵ and ζ. The particular value $1/4$ on the right sides in the following is not essential: what is important is that the probabilities are bounded away from zero. The lower bounds over Φ_L remain true over the larger set Φ.

Theorem 3. *Grant the compatibility condition (3). There exist constants $c = c(K) > 0$ and $\epsilon_0 > 0$ such that whenever $\epsilon \le \epsilon_0$, $\delta \le 1/6$ and $L \le 1/3$ the following hold. [The infima are over all estimators, i.e. all measurable functions of the data $(Y_1, \dots, Y_n)$.]*

1. *Assume $n\delta^2\epsilon^4\zeta^6 \ge 1$. Then*

$$\inf_{\hat{\phi}_1} \sup_{(\phi,\psi)\in\Phi_L(\delta,\epsilon,\zeta)} \mathbb{P}_{\phi,\psi}\Big(\min(|\hat{\phi}_1 - \phi_1|^2, |\hat{\phi}_1 + \phi_1|^2) \ge \frac{c}{n\epsilon^4\zeta^6}\Big)$$
$$\ge \inf_{\hat{\phi}_1} \sup_{(\phi,\psi)\in\Phi_L(\delta,\epsilon,\zeta)} \mathbb{P}_{\phi,\psi}\Big(\Big|\frac{1-\tilde{\phi}_1^2}{1-\phi_1^2} - 1\Big|^2 \ge \frac{2c}{n\delta^2\epsilon^4\zeta^6}\Big) \ge 1/4.$$

2. *Assume $n\delta^2\epsilon^4\zeta^4 \ge 1$. Then*

$$\inf_{\hat{\phi}_2} \sup_{(\phi,\psi)\in\Phi_L(\delta,\epsilon,\zeta)} \mathbb{P}_{\phi,\psi}\Big(|\hat{\phi}_2 - \phi_2|^2 \ge \frac{c}{n\delta^2\epsilon^2\zeta^4}\Big)$$
$$\ge \inf_{\hat{\phi}_2} \sup_{(\phi,\psi)\in\Phi_L(\delta,\epsilon,\zeta)} \mathbb{P}_{\phi,\psi}\Big(\Big|\frac{\hat{\phi}_2}{\phi_2} - 1\Big|^2 \ge \frac{c}{n\delta^2\epsilon^4\zeta^4}\Big) \ge 1/4.$$

3. *Assume $n\delta^2\epsilon^4\zeta^6 \ge 1$. Then*

$$\inf_{\hat{\phi}_3} \sup_{(\phi,\psi)\in\Phi_L(\delta,\epsilon,\zeta)} \mathbb{P}_{\phi,\psi}\Big(|\hat{\phi}_3 - \phi_3|^2 \ge \frac{c}{n\delta^2\epsilon^4\zeta^4}\Big)$$
$$\ge \inf_{\hat{\phi}_3} \sup_{(\phi,\psi)\in\Phi_L(\delta,\epsilon,\zeta)} \mathbb{P}_{\phi,\psi}\Big(\Big|\frac{\hat{\phi}_3}{\phi_3} - 1\Big|^2 \ge \frac{c}{n\delta^2\epsilon^4\zeta^6}\Big) \ge 1/4.$$

4. *For any n, δ, ϵ and ζ,*

$$\inf_{\hat{\psi}_1} \sup_{(\phi,\psi)\in\Phi_L(\delta,\epsilon,\zeta)} \mathbb{P}_{\phi,\psi}\Big(\|\hat{\psi}_1 - \psi_1\|^2 \ge \frac{c}{n}\Big) \ge 1/4.$$

5. *Assume $n\delta^2\epsilon^2\zeta^4 \ge 1$ and $K > 2$. Then*

$$\inf_{\hat{\psi}_2} \sup_{(\phi,\psi)\in\Phi_L(\delta,\epsilon,\zeta)} \mathbb{P}_{\phi,\psi}\Big(\min(\|\hat{\psi}_2 - \psi_2\|, \|\hat{\psi}_2 + \psi_2\|)^2 \ge \frac{c}{n\delta^2\epsilon^2\zeta^4}\Big) \ge 1/4.$$

7 Conclusions and future directions

In this work we have quantified the impact on learnability of approaching the i.i.d. boundary within the set of parameters of a hidden Markov model. The limiting cases occur when one hidden state is absorbing, when the underlying Markov chain becomes a sequence of independent variables, or when the emission distributions are equal. We have proved both upper and lower bounds for the estimation rates of the parameters in a hidden Markov models with two hidden states and finitely many possible outcomes. Our results characterize the frontier in the parameter space between learnable and unlearnable parameters and quantify how large the sample has to be in order to get estimators with prescribed error with high probability.

Some tricky regions of the parameter space are not fully captured in the upper and lower bounds. Specifically, the condition on n in the lower bound for estimating ϕ_2 differs by a factor of ϵ^2 from the corresponding condition in the upper bound. Also, in the upper bound for ϕ_1, we do not describe the precise estimation behaviour in the region $n\delta^2\epsilon^4\zeta^6 < x^2 < n\epsilon^4\zeta^6$: in this range we can obtain something by applying the bound with $y^2 = \min(x^2, n\delta^2\epsilon^4\zeta^6)$ but we cannot expect that this gives the correct dependence on x. [There is no issue in the region $x^2 \geq n\epsilon^4\zeta^6$ since we may replace the bound e^{-x^2} with zero, similarly to the comment after the theorem regarding ϕ_2 and ψ.] A similar gap exists for estimating ϕ_3. Our results already work for a wide range of parameters, and extending to the few remaining cases is an interesting issue for future research.

Regarding the upper bounds, we analysed a minimum distance estimator for theoretical convenience, and we think the same upper bounds should hold for more practical estimators (for example empirical least squares estimators and tensor-based methods). Our proof method relies on the fact that the two steps of estimating $p^{(3)}$ and of estimating, given $p^{(3)}$, the HMM parameters themselves, decouple. This is because, with good mixing properties for the Markov chain, estimation of $p^{(3)}$ can be done uniformly at a rate not depending on the HMM parameters (Lemma 1). When the spectral gap is small the underlying Markov chain mixes slowly, spending long periods remaining in whichever of the two states it is in, so that estimation of $p^{(3)}$ becomes hard for parameters for which there is small spectral gap. These are not the same parameters for which recovering the HMM parameters given $p^{(3)}$ is most difficult, and so to obtain accurate rates without a spectral gap requires carefully addressing the two steps simultaneously, which is beyond the scope of the paper (we could obtain a suboptimal rate using the current methods just with careful tracking of the spectral gap, since it is lower bounded by $1 - 2\delta$, but upper and lower bounds obtained in this way mismatch by a factor of δ). Note the above arguments explain the requirement for a spectral gap, not an absolute spectral gap; we believe our results will in fact hold in the near-periodic case when the spectral gap is close to 2 and the absolute spectral gap is close to zero, but this would require some extra technical calculations in the proof of Lemma 1.

We believe similar results hold with more than two hidden states and with arbitrary nonparametric emission distributions. Investigation of the fundamental limits for learning more general HMMs and misspecified modelling will be the object of further work. Developments of our findings for clustering, multiple testing and sparse settings will also be the object of further work, and all will depend fundamentally on the results obtained here.

On the practical side, usual estimation algorithms can be expected to exhibit bad computational behaviour when the unknown true parameters lie near the learning frontier. We have not tackled this issue here and we believe it merits substantive investigation, both in building robust algorithms and in detecting the poor performance in the problematic region. This last question is interesting both from a practical and a theoretical point of view.

8 Proofs

8.1 Proof of Proposition 1

Recall the definition (11) of m as

$$m(\phi) = (r(\phi), \phi_2 r(\phi), \phi_1\phi_2\phi_3 r(\phi)), \quad r(\phi) = \tfrac{1}{4}(1-\phi_1^2)\phi_2\phi_3^2.$$

We write $\tilde{m} = m(\tilde{\phi})$, and we write ψ_{ijk} for $\psi_i \otimes \psi_j \otimes \psi_k$ and $\tilde{\psi}_{ijk}$ for $\tilde{\psi}_i \otimes \tilde{\psi}_j \otimes \tilde{\psi}_k$. Then from equation (10) we have

$$\begin{aligned} p^{(3)}_{\phi,\psi} - p^{(3)}_{\tilde{\phi},\tilde{\psi}} = (\psi_{111} - \tilde{\psi}_{111}) + \{m_1(\psi_{221} + \psi_{122}) - \tilde{m}_1(\tilde{\psi}_{221} + \tilde{\psi}_{122})\} \\ + \{m_2\psi_{212} - \tilde{m}_2\tilde{\psi}_{212}\} - \{m_3\psi_{222} - \tilde{m}_3\tilde{\psi}_{222}\}. \end{aligned} \tag{14}$$

Recalling that $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on $\mathbb{R}^K$, we have $\langle \psi_1, 1\rangle = 1$, $\langle \psi_2, 1\rangle = 0$, $\|\psi_2\| = 1$ and $\|1\| = K^{1/2}$. Let $\langle \cdot, \cdot \rangle$ also denote the Euclidean inner product on $\mathbb{R}^{K\times K\times K}$, wherein for functions $f_i, \tilde{f}_i : \{1, \dots, K\} \to \mathbb{R}, \ i \leq 3$ we have

$$\langle f_1 \otimes f_2 \otimes f_3, \tilde{f}_1 \otimes \tilde{f}_2 \otimes \tilde{f}_3\rangle = \langle f_1, \tilde{f}_1\rangle\langle f_2, \tilde{f}_2\rangle\langle f_3, \tilde{f}_3\rangle.$$

Lower bounding $\|p^{(3)}_{\phi,\psi} - p^{(3)}_{\tilde{\phi},\tilde{\psi}}\|$ For any function $f : \{1, \dots, K\} \to \mathbb{R}$, we have

$$\langle p^{(3)}_{\phi,\psi} - p^{(3)}_{\tilde{\phi},\tilde{\psi}}, f \otimes 1 \otimes 1\rangle = \langle \psi_{111} - \tilde{\psi}_{111}, f \otimes 1 \otimes 1\rangle = \langle \psi_1 - \tilde{\psi}_1, f\rangle.$$

Then

$$\begin{aligned} \|\psi_1 - \tilde{\psi}_1\| &= \sup_{\|f\|=1} |\langle \psi_1 - \tilde{\psi}_1, f\rangle| \\ &= \sup_{\|f\|=1} |\langle p^{(3)}_{\phi,\psi} - p^{(3)}_{\tilde{\phi},\tilde{\psi}}, f \otimes 1 \otimes 1\rangle| \\ &\leq \|p^{(3)}_{\phi,\psi} - p^{(3)}_{\tilde{\phi},\tilde{\psi}}\| \sup_{\|f\|=1} \|f \otimes 1 \otimes 1\| = K\|p^{(3)}_{\phi,\psi} - p^{(3)}_{\tilde{\phi},\tilde{\psi}}\|, \end{aligned} \tag{15}$$

and similarly,

$$\begin{aligned} \langle p^{(3)}_{\phi,\psi} - p^{(3)}_{\tilde{\phi},\tilde{\psi}}, 1 \otimes f \otimes f\rangle &= \langle \psi_{111} - \tilde{\psi}_{111}, 1 \otimes f \otimes f\rangle + \langle m_1\psi_{122} - \tilde{m}_1\tilde{\psi}_{122}, 1 \otimes f \otimes f\rangle \\ &= \langle \psi_1 - \tilde{\psi}_1, f\rangle^2 + m_1\langle \psi_2, f\rangle^2 - \tilde{m}_1\langle \tilde{\psi}_2, f\rangle^2. \end{aligned} \tag{16}$$

Choosing $f = \psi_2 + \operatorname{sgn}(\langle \psi_2, \tilde{\psi}_2\rangle) \cdot \tilde{\psi}_2$ (with the convention that $\operatorname{sgn}(0) = +1$), we observe that

$$\langle \psi_2, f\rangle = 1 + |\langle \psi_2, \tilde{\psi}_2\rangle| = \operatorname{sgn}(\langle \psi_2, \tilde{\psi}_2\rangle) \cdot \langle \tilde{\psi}_2, f\rangle.$$

In particular we note that $\langle \psi_2, f\rangle^2 = \langle \tilde{\psi}_2, f\rangle^2 = (1+|\langle \psi_2, \tilde{\psi}_2\rangle|)^2 \geq 1$. Since also $\|f\|^2 = 2+2|\langle \psi_2, \tilde{\psi}_2\rangle| \leq 4$, returning to (16) we observe that

$$\begin{aligned} |m_1 - \tilde{m}_1| &\leq \|f\|^2\|\psi_1 - \tilde{\psi}_1\|^2 + \|1 \otimes f \otimes f\|\|p^{(3)}_{\phi,\psi} - p^{(3)}_{\tilde{\phi},\tilde{\psi}}\| \\ &\leq 4\|\psi_1 - \tilde{\psi}_1\|^2 + 4K^{1/2}\|p^{(3)}_{\phi,\psi} - p^{(3)}_{\tilde{\phi},\tilde{\psi}}\| \\ &\leq 4(K^{7/2} + K^{1/2})\|p^{(3)}_{\phi,\psi} - p^{(3)}_{\tilde{\phi},\tilde{\psi}}\|, \end{aligned} \tag{17}$$

where for the last line we have used equation (15) and the fact that $\|p^{(3)}_{\phi,\psi} - p^{(3)}_{\tilde{\phi},\tilde{\psi}}\|^2 \le K^3$. We continue by considering the expression $f \otimes 1 \otimes f$, for which we have

$$\begin{aligned}\langle p^{(3)}_{\phi,\psi} - p^{(3)}_{\tilde{\phi},\tilde{\psi}}, f \otimes 1 \otimes f\rangle &= \langle \psi_{111} - \tilde{\psi}_{111}, f \otimes 1 \otimes f\rangle + \langle m_2\psi_{212} - \tilde{m}_2\tilde{\psi}_{212}, f \otimes 1 \otimes f\rangle \\ &= \langle \psi_1 - \tilde{\psi}_1, f\rangle^2 + m_2\langle \psi_2, f\rangle^2 - \tilde{m}_2\langle \tilde{\psi}_2, f\rangle^2\end{aligned}$$

Recognising symmetry with equation (16), we again choose $f = \psi_2 + \mathrm{sgn}(\langle \psi_2, \tilde{\psi}_2\rangle) \cdot \tilde{\psi}_2$ to obtain

$$|m_2 - \tilde{m}_2| \le 4(K^{7/2} + K^{1/2})\|p^{(3)}_{\phi,\psi} - p^{(3)}_{\tilde{\phi},\tilde{\psi}}\|. \tag{18}$$

Finally, considering the expression $f \otimes f \otimes f$, we observe that

$$\begin{aligned}\langle p^{(3)}_{\phi,\psi} - p^{(3)}_{\tilde{\phi},\tilde{\psi}}, f \otimes f \otimes f\rangle &= \langle \psi_{111} - \tilde{\psi}_{111}, f \otimes f \otimes f\rangle + \langle m_1(\psi_{221} + \psi_{122}) - \tilde{m}_1(\tilde{\psi}_{221} + \tilde{\psi}_{122}), f \otimes f \otimes f\rangle \\ &\quad + \langle m_2\psi_{212} - \tilde{m}_2\tilde{\psi}_{212}, f \otimes f \otimes f\rangle - \langle m_3\psi_{222} - \tilde{m}_3\tilde{\psi}_{222}, f \otimes f \otimes f\rangle.\end{aligned}$$

In other words,

$$\begin{aligned}\langle p^{(3)}_{\phi,\psi} - p^{(3)}_{\tilde{\phi},\tilde{\psi}}, f \otimes f \otimes f\rangle &= \langle \psi_1 - \tilde{\psi}_1, f\rangle^3 + 2\big(m_1\langle \psi_2, f\rangle^2\langle \psi_1, f\rangle - \tilde{m}_1\langle \tilde{\psi}_2, f\rangle^2\langle \tilde{\psi}_1, f\rangle\big) \\ &\quad + \big(m_2\langle \psi_2, f\rangle^2\langle \psi_1, f\rangle - \tilde{m}_2\langle \tilde{\psi}_2, f\rangle^2\langle \tilde{\psi}_1, f\rangle\big) - \big(m_3\langle \psi_2, f\rangle^3 - \tilde{m}_3\langle \tilde{\psi}_2, f\rangle^3\big).\end{aligned}$$

Once more choosing $f = \psi_2 + \mathrm{sgn}(\langle \psi_2, \tilde{\psi}_2\rangle) \cdot \tilde{\psi}_2$, we obtain (recall that by construction $\|f\| \le 2$, $1 \le \langle \psi_2, f\rangle^2 = \langle \tilde{\psi}_2, f\rangle^2 \le 4$, and also $\mathrm{sgn}(\langle \psi_2, \tilde{\psi}_2\rangle)\langle \psi_2, f\rangle = \langle \tilde{\psi}_2, f\rangle$)

$$\begin{aligned}|m_3 - \mathrm{sgn}(\langle \psi_2, \tilde{\psi}_2\rangle) \cdot \tilde{m}_3| &\le 8\|\psi_1 - \tilde{\psi}_1\|^3 + 8\|p^{(3)}_{\phi,\psi} - p^{(3)}_{\phi,\psi}\| \\ &\quad + 8\big|m_1\langle \psi_1, f\rangle - \tilde{m}_1\langle \tilde{\psi}_1, f\rangle\big| + 4\big|m_2\langle \psi_1, f\rangle - \tilde{m}_2\langle \tilde{\psi}_1, f\rangle\big|.\end{aligned}$$

For some constant $C = C(K)$ we have

$$\begin{aligned}\big|m_1\langle \psi_1, f\rangle - \tilde{m}_1\langle \tilde{\psi}_1, f\rangle\big| &\le |\langle \psi_1, f\rangle|\big|m_1 - \tilde{m}_1\big| + |\tilde{m}_1||\langle \psi_1 - \tilde{\psi}_1, f\rangle| \\ &\le 2\|\psi_1\|\big|m_1 - \tilde{m}_1\big| + 2|\tilde{m}_1|\|\psi_1 - \tilde{\psi}_1\| \\ &\le C\|p^{(3)}_{\phi,\psi} - p^{(3)}_{\tilde{\phi},\tilde{\psi}}\|,\end{aligned}$$

where for the last line we have used equations (15) and (17) and that $\|\psi_1\| \le K^{1/2}$ and $|\tilde{m}_1| \le \tilde{\phi}_3^2/4 \le \|\tilde{f}_0 - \tilde{f}_1\|^2/4 \le K/4$.

Similarly, using equation (18) and the fact that $|\tilde{m}_2|$ is suitably bounded, we have for some $C = C(K)$

$$\big|m_2\langle \psi_1, f\rangle - \tilde{m}_2\langle \tilde{\psi}_1, f\rangle\big| \le C\|p^{(3)}_{\phi,\psi} - p^{(3)}_{\tilde{\phi},\tilde{\psi}}\|.$$

We deduce for some different constant $C = C(K)$ that

$$|m_3 - \mathrm{sgn}(\langle \psi_2, \tilde{\psi}_2\rangle) \cdot \tilde{m}_3| \le C\|p^{(3)}_{\phi,\psi} - p^{(3)}_{\tilde{\phi},\tilde{\psi}}\|. \tag{19}$$

Finally, for ψ_2 we show that for some C we have

$$\max(|m_1|, |\tilde{m}_1|)\|\psi_2 - \mathrm{sgn}(\langle \psi_2, \tilde{\psi}_2\rangle) \cdot \tilde{\psi}_2\| \le C\|p^{(3)}_{\phi,\psi} - p^{(3)}_{\tilde{\phi},\tilde{\psi}}\|. \tag{20}$$

If $\psi_2 = \tilde{\psi}_2$ there is nothing to prove, so we assume without loss of generality that $\psi_2 \neq \tilde{\psi}_2$. Also assume that $|m_1| \geq |\tilde{m}_1|$, the final bound then following by symmetry. Returning to equation (16) with f to be chosen, we see that

$$m_1\big(\langle \psi_2, f\rangle^2 - \langle \tilde{\psi}_2, f\rangle^2\big) = \langle \psi_1 - \tilde{\psi}_1, f\rangle^2 - \langle p^{(3)}_{\phi,\psi} - p^{(3)}_{\tilde{\phi},\tilde{\psi}}, 1 \otimes f \otimes f\rangle - \langle \tilde{\psi}_2, f\rangle^2 (m_1 - \tilde{m}_1).$$

Since $\langle \psi_2, f\rangle^2 - \langle \tilde{\psi}_2, f\rangle^2 = \langle \psi_2 - \tilde{\psi}_2, f\rangle \langle \psi_2 + \tilde{\psi}_2, f\rangle$ we obtain

$$|m_1 \langle \psi_2 - \tilde{\psi}_2, f\rangle \langle \psi_2 + \tilde{\psi}_2, f\rangle| \leq \|f\|^2 \|\psi_1 - \tilde{\psi}_1\|^2 + K\|f\|^2 \|p^{(3)}_{\phi,\psi} - p^{(3)}_{\tilde{\phi},\tilde{\psi}}\| + |m_1 - \tilde{m}_1| \langle \tilde{\psi}_2, f\rangle^2. \quad (21)$$

Observe that $\psi_2 + \tilde{\psi}_2$ is orthogonal to $\psi_2 - \tilde{\psi}_2$ (this arises from the fact that ψ_2 and $\tilde{\psi}_2$ have unit norms) and choose

$$f = \frac{\psi_2 + \tilde{\psi}_2}{\|\psi_2 + \tilde{\psi}_2\|} + \frac{\psi_2 - \tilde{\psi}_2}{\|\psi_2 - \tilde{\psi}_2\|};$$

note that

$$\langle \psi_2 - \tilde{\psi}_2, f\rangle \langle \psi_2 + \tilde{\psi}_2, f\rangle = \|\psi_2 - \tilde{\psi}_2\| \|\psi_2 + \tilde{\psi}_2\|.$$

Since also $\|f\| \leq 2$ and $|\langle \tilde{\psi}_2, f\rangle| \leq 2$, continuing from equation (21) and using equations (15) and (17) we see that for a constant $C = C(K)$

$$\begin{aligned} |m_1| \|\psi_2 - \tilde{\psi}_2\| \|\psi_2 + \tilde{\psi}_2\| &\leq 4\|\psi_1 - \tilde{\psi}_1\|^2 + 4K\|p^{(3)}_{\phi,\psi} - p^{(3)}_{\tilde{\phi},\tilde{\psi}}\| + 4|m_1 - \tilde{m}_1| \\ &\leq 2C\|p^{(3)}_{\phi,\psi} - p^{(3)}_{\tilde{\phi},\tilde{\psi}}\|. \end{aligned}$$

Observing that

$$\begin{aligned} \|\psi_2 - \tilde{\psi}_2\|^2 \|\psi_2 + \tilde{\psi}_2\|^2 &= \|\psi_2 - \operatorname{sgn}(\langle \psi_2, \tilde{\psi}_2\rangle) \cdot \tilde{\psi}_2\|^2 \|\psi_2 + \operatorname{sgn}(\langle \psi_2, \tilde{\psi}_2\rangle) \cdot \tilde{\psi}_2\|^2 \\ &= \|\psi_2 - \operatorname{sgn}(\langle \psi_2, \tilde{\psi}_2\rangle) \cdot \tilde{\psi}_2\|^2 \big(2 + 2|\langle \psi_2, \tilde{\psi}_2\rangle|\big) \\ &\geq 2\|\psi_2 - \operatorname{sgn}(\langle \psi_2, \tilde{\psi}_2\rangle) \cdot \tilde{\psi}_2\|^2, \end{aligned}$$

and recalling we assumed that $|m_1| \geq |\tilde{m}_1|$, equation (20) follows.

The proof that $\|p^{(3)}_{\phi,\psi} - p^{(3)}_{\tilde{\phi},\tilde{\psi}}\|$ is lower bounded up to a constant by $\rho(\phi, \psi; \tilde{\phi}, \tilde{\psi})$ follows by combining equations (15) and (17)–(20)

Upper bounding $\|p^{(3)}_{\phi,\psi} - p^{(3)}_{\tilde{\phi},\tilde{\psi}}\|$ From equation (14),

$$\begin{aligned} \|p^{(3)}_{\phi,\psi} - p^{(3)}_{\tilde{\phi},\tilde{\psi}}\| \leq \|\psi_{111} - \tilde{\psi}_{111}\| &+ |m_1 - \tilde{m}_1| \|\psi_{221} + \psi_{122}\| + |\tilde{m}_1| \|\psi_{221} - \tilde{\psi}_{221}\| \\ &+ |\tilde{m}_1| \|\psi_{122} - \tilde{\psi}_{122}\| + |m_2 - \tilde{m}_2| \|\psi_{212}\| + |\tilde{m}_2| \|\psi_{212} - \tilde{\psi}_{212}\| \\ &+ |m_3 - \tilde{m}_3| \|\psi_{222}\| + |\tilde{m}_3| \|\psi_{222} - \tilde{\psi}_{222}\|. \end{aligned} \quad (22)$$

Note that the bound remains valid if we replace the final two terms by

$$|m_3 + \tilde{m}_3| \|\psi_{222}\| + |\tilde{m}_3| \|\psi_{222} + \tilde{\psi}_{222}\|;$$

we focus on the case where $\operatorname{sgn}(\langle \psi_2, \tilde{\psi}_2\rangle) = +1$ for which the original decomposition yields suitable bounds, but the proof in the other case is similar using the alternative decomposition.

As used already in proving the lower bound on $\|p^{(3)}_{\phi,\psi} - p^{(3)}_{\tilde\phi,\tilde\psi}\|$, we note that

$$\max(\|\psi_{221}\|, \|\psi_{122}\|, |\tilde m_1|, \|\psi_{212}\|, |\tilde m_2|, \|\psi_{222}\|, |\tilde m_3|) \le C,$$

for some $C = C(K)$. To conclude the proof it thus suffices to bound the tensor product terms $\|\psi_{ijk} - \tilde\psi_{ijk}\|$ in terms of the differences $\|\psi_1 - \tilde\psi_1\|, \|\psi_2 - \tilde\psi_2\|$. First we decompose

$$\begin{aligned}\|\psi_1 \otimes \psi_1 \otimes \psi_1 - \tilde\psi_1 \otimes \tilde\psi_1 \otimes \tilde\psi_1\| &\le \|\psi_1 \otimes \psi_1 \otimes \psi_1 - \tilde\psi_1 \otimes \psi_1 \otimes \psi_1\| \\ &\quad + \|\tilde\psi_1 \otimes \psi_1 \otimes \psi_1 - \tilde\psi_1 \otimes \tilde\psi_1 \otimes \psi_1\| + \|\tilde\psi_1 \otimes \tilde\psi_1 \otimes \psi_1 - \tilde\psi_1 \otimes \tilde\psi_1 \otimes \tilde\psi_1\|,\end{aligned}$$

so that

$$\|\psi_{111} - \tilde\psi_{111}\| \le \|\psi_1 - \tilde\psi_1\|(\|\psi_1\|^2 + \|\psi_1\|\|\tilde\psi_1\| + \|\tilde\psi_1\|^2) \le 3K\|\psi_1 - \tilde\psi_1\|. \tag{23}$$

We also note, recalling that ψ_2 and $\tilde\psi_2$ have unit norms, that

$$\begin{aligned}\|\psi_{221} - \tilde\psi_{221}\|^2 &= \|\psi_{221}\|^2 + \|\tilde\psi_{221}\|^2 - 2\langle\psi_{221}, \tilde\psi_{221}\rangle \\ &= \|\psi_1\|^2 + \|\tilde\psi_1\|^2 - 2\langle\psi_2, \tilde\psi_2\rangle^2\langle\psi_1, \tilde\psi_1\rangle \\ &= \|\psi_1 - \tilde\psi_1\|^2 + 2\langle\psi_1, \tilde\psi_1\rangle\big(1 - \langle\psi_2, \tilde\psi_2\rangle^2\big) \\ &\le \|\psi_1 - \tilde\psi_1\|^2 + 2\|\psi_1\|\|\tilde\psi_1\||1 - \langle\psi_2, \tilde\psi_2\rangle^2|.\end{aligned}$$

Observe that

$$\|\psi_2 - \tilde\psi_2\|^2 = 2(1 - \langle\psi_2, \tilde\psi_2\rangle), \tag{24}$$

and hence

$$\big|1 - \langle\psi_2, \tilde\psi_2\rangle^2\big| = \big|1 + \langle\psi_2, \tilde\psi_2\rangle\big|\big|1 - \langle\psi_2, \tilde\psi_2\rangle\big| \le 2|1 - \langle\psi_2, \tilde\psi_2\rangle| = \|\psi_2 - \tilde\psi_2\|^2.$$

We deduce that

$$\|\psi_{221} - \tilde\psi_{221}\|^2 \le \|\psi_1 - \tilde\psi_1\|^2 + 2\|\psi_1\|\|\tilde\psi_1\|\|\psi_2 - \tilde\psi_2\|^2 \le \|\psi_1 - \tilde\psi_1\|^2 + 2K\|\psi_2 - \tilde\psi_2\|^2. \tag{25}$$

By symmetry, the same bound holds for $\|\psi_{122} - \tilde\psi_{122}\|$ and for $\|\psi_{212} - \tilde\psi_{212}\|$. Furthermore $\|\psi_{222}\| = 1$, and using (24),

$$\begin{aligned}\|\psi_{222} - \tilde\psi_{222}\|^2 &= \|\psi_{222}\|^2 + \|\tilde\psi_{222}\|^2 - 2\langle\psi_{222}, \tilde\psi_{222}\rangle \\ &= 2 - 2\langle\psi_2, \tilde\psi_2\rangle^3 \\ &= 2\big(1 - \langle\psi_2, \tilde\psi_2\rangle\big)\big(1 + \langle\psi_2, \tilde\psi_2\rangle + \langle\psi_2, \tilde\psi_2\rangle^2\big) \\ &\le 3\|\psi_2 - \tilde\psi_2\|^2.\end{aligned} \tag{26}$$

The claim follows from inserting equations (23), (25) and (26) into equation (22).

8.2 Proof of Proposition 2

Write $X_{1:k}$ and $Y_{1:k}$ for the vectors $(X_1, \dots, X_k)$ and $(Y_1, \dots, Y_k)$ respectively, and recall that $P^{(n)}_\theta$ denotes the law of $Y_{1:n}$ for parameter θ. Without loss of generality we may assume that $\operatorname{sgn}(\langle\psi_2, \tilde\psi_2\rangle) = +1$, since one may substitute $\tilde\phi' = (-\tilde\phi_1, \tilde\phi_2, \tilde\phi_3)$ and $\tilde\psi' = (\tilde\psi_1, -\tilde\psi_2)$ for $\tilde\phi$ and $\tilde\psi$ and obtain $P^{(n)}_{\tilde\theta} = P^{(n)}_{\tilde\theta'}$, hence

$K(P_\theta^{(n)}; P_{\tilde\theta}^{(n)}) = K(P_\theta^{(n)}; P_{\tilde\theta'}^{(n)})$, but $\mathrm{sgn}(\langle \psi_2, \tilde\psi_2' \rangle) = -\mathrm{sgn}(\langle \psi_2, \tilde\psi_2 \rangle)$. Recall that $K(P;Q)$ is upper bounded by the chi-square distance $\chi^2(P,Q) = \mathbb{E}_Q[(\mathrm{d}P/\mathrm{d}Q - 1)^2]$ (e.g. [40, Lemma 2.7]). Then using that $\mathbb{P}_\theta(Y_1 = \cdot) = \psi_1(\cdot)$ and $\mathbb{P}_{\tilde\theta}(Y_1 = \cdot) = \tilde\psi_1(\cdot) \geq c$, we have

$$K(P_\theta^{(1)}; P_{\tilde\theta}^{(1)}) \leq \sum_{y \in \mathcal{Y}} \frac{[\mathbb{P}_\theta(Y_1 = y) - \mathbb{P}_{\tilde\theta}(Y_1 = y)]^2}{\mathbb{P}_{\tilde\theta}(Y_1 = y)} \leq \frac{\|\psi_1 - \tilde\psi_1\|^2}{c}. \tag{27}$$

This yields the case $n = 1$ since the definition (12) implies that $\rho(\phi, \psi; \tilde\phi, \tilde\psi) \geq \|\psi_1 - \tilde\psi_1\|^2$.

Now assume that $n \geq 2$. By the chain rule for relative divergence (used inductively), we have

$$K(P_\theta^{(n)}; P_{\tilde\theta}^{(n)}) = K(P_\theta^{(1)}; P_{\tilde\theta}^{(1)}) + \sum_{k=1}^{n-1} \mathbb{E}_\theta[K(\mathbb{P}_\theta(Y_{k+1} \in \cdot \mid Y_{1:k}); \mathbb{P}_{\tilde\theta}(Y_{k+1} \in \cdot \mid Y_{1:k}))]. \tag{28}$$

The first term was addressed above, and we now consider the remaining terms. Again bounding the KL divergence by the chi-square distance, we have

$$\begin{aligned} K(\mathbb{P}_\theta(Y_{k+1} \in \cdot \mid Y_{1:k}); \mathbb{P}_{\tilde\theta}(Y_{k+1} \in \cdot \mid Y_{1:k})) &\leq \sum_{y \in \mathcal{Y}} \frac{[\mathbb{P}_\theta(Y_{k+1} = y \mid Y_{1:k}) - \mathbb{P}_{\tilde\theta}(Y_{k+1} = y \mid Y_{1:k})]^2}{\mathbb{P}_{\tilde\theta}(Y_{k+1} = y \mid Y_{1:k})} \\ &\leq \frac{\|\mathbb{P}_\theta(Y_{k+1} = \cdot \mid Y_{1:k}) - \mathbb{P}_{\tilde\theta}(Y_{k+1} = \cdot \mid Y_{1:k})\|^2}{\min_{y \in \mathcal{Y}} \mathbb{P}_{\tilde\theta}(Y_{k+1} = y \mid Y_{1:k})}. \end{aligned}$$

But, for any $k \geq 1$,

$$\begin{aligned} \mathbb{P}_\theta(Y_{k+1} = y \mid Y_{1:k}) &= \sum_{x \in \{0,1\}^{k+1}} \mathbb{P}_\theta(Y_{k+1} = y \mid Y_{1:k}, X_{1:k+1} = x) \mathbb{P}_\theta(X_{1:k+1} = x \mid Y_{1:k}) \\ &= \sum_{x \in \{0,1\}^{k+1}} f_{x_{k+1}}(y) \mathbb{P}_\theta(X_{1:k+1} = x \mid Y_{1:k}) \\ &= \sum_{x \in \{0,1\}} f_x(y) \mathbb{P}_\theta(X_{k+1} = x \mid Y_{1:k}), \end{aligned} \tag{29}$$

where we have used that $Y_{k+1} \mid (Y_{1:k}, X_{1:k+1})$ has the same law as $Y_{k+1} \mid X_{k+1}$. Therefore when $\min(\tilde f_0, \tilde f_1) \geq c$ we must have for all $Y_{1:k}$ and all $k \geq 1$

$$\mathbb{P}_{\tilde\theta}(Y_{k+1} = y \mid Y_{1:k}) \geq c \sum_{x \in \{0,1\}} \mathbb{P}_{\tilde\theta}(X_{k+1} = x \mid Y_{1:k}) = c.$$

Hence we have established that for all $Y_{1:k}$ and all $k \geq 1$

$$K(\mathbb{P}_\theta(Y_{k+1} \in \cdot \mid Y_{1:k}); \mathbb{P}_{\tilde\theta}(Y_{k+1} \in \cdot \mid Y_{1:k})) \leq \frac{\|\mathbb{P}_\theta(Y_{k+1} = \cdot \mid Y_{1:k}) - \mathbb{P}_{\tilde\theta}(Y_{k+1} = \cdot \mid Y_{1:k})\|^2}{c}. \tag{30}$$

Let us now rewrite $\mathbb{P}_\theta(Y_{k+1} = y \mid Y_{1:k})$ in the parametrisation (ϕ, ψ). For convenience we introduce the notation $P_k(x) := \mathbb{P}_\theta(X_{k+1} = x \mid Y_{1:k})$ for the prediction filters, and we similarly write $\tilde P_k(x) :=$

$\mathbb{P}_{\tilde\theta}(X_{k+1}=x \mid Y_{1:k})$. By equation (29) and Remark 4,

$$\begin{aligned}\mathbb{P}_\theta(Y_{k+1}=y \mid Y_{1:k}) &= f_0(y)P_k(0)+f_1(y)P_k(1)\\ &= \Big(\psi_1(y)-\tfrac12\phi_1\phi_3\psi_2(y)+\tfrac12\phi_3\psi_2(y)\Big)P_k(0)\\ &\quad+\Big(\psi_1(y)-\tfrac12\phi_1\phi_3\psi_2(y)-\tfrac12\phi_3\psi_2(y)\Big)P_k(1)\\ &= \psi_1(y)+\tfrac12\big(P_k(0)-P_k(1)-\phi_1\big)\phi_3\psi_2(y).\end{aligned}$$

Define

$$V_k \coloneqq \phi_3(P_k(0)-P_k(1)-\phi_1)=\phi_3(1-2P_k(1)-\phi_1), \qquad \tilde V_k \coloneqq \tilde\phi_3(1-2\tilde P_k(1)-\tilde\phi_1), \qquad k\ge 1.$$

Then combining equations (27), (28) and (30), we obtain

$$\begin{aligned}K(P_\theta^{(n)};P_{\tilde\theta}^{(n)}) &\le \frac{\|\psi_1-\tilde\psi_1\|^2}{c}+\frac1c\sum_{k=1}^{n-1}\mathbb{E}_{\tilde\theta}\Big[\Big\|\psi_1-\tilde\psi_1+\frac12 V_k\psi_2-\frac12\tilde V_k\tilde\psi_2\Big\|^2\Big]\\ &\le \frac{2n-1}{c}\|\psi_1-\tilde\psi_1\|^2+\frac{1}{2c}\sum_{k=1}^{n-1}\mathbb{E}_{\tilde\theta}[\|V_k\psi_2-\tilde V_k\tilde\psi_2\|^2]\\ &\le \frac{2n-1}{c}\|\psi_1-\tilde\psi_1\|^2+\frac{\|\psi_2-\tilde\psi_2\|^2}{c}\sum_{k=1}^{n-1}\mathbb{E}_{\tilde\theta}[V_k^2]+\frac1c\sum_{k=1}^{n-1}\mathbb{E}_{\tilde\theta}[(V_k-\tilde V_k)^2],\end{aligned} \tag{31}$$

where in the last line we have used that $\|\tilde\psi_2\|^2=1$.

Let us now find an inductive formula for V_k. First we observe that, for any $k\ge 2$

$$\begin{aligned}P_k(x) &\coloneqq \mathbb{P}_\theta(X_{k+1}=x \mid Y_{1:k})\\ &= \sum_{x'\in\{0,1\}}\mathbb{P}_\theta(X_{k+1}=x \mid Y_{1:k},X_k=x')\mathbb{P}_\theta(X_k=x' \mid Y_{1:k})\\ &= \sum_{x'\in\{0,1\}}\mathbb{P}_\theta(X_{k+1}=x \mid X_k=x')\mathbb{P}_\theta(X_k=x' \mid Y_{1:k})\\ &= \sum_{x'\in\{0,1\}}Q_{x',x}\mathbb{P}_\theta(X_k=x' \mid Y_{1:k-1},Y_k),\end{aligned}$$

and we further calculate

$$\begin{aligned}\mathbb{P}_\theta(X_k=x' \mid Y_{1:k-1},Y_k=y_k) &= \frac{\mathbb{P}_\theta(X_k=x',Y_k=y_k \mid Y_{1:k-1})}{\mathbb{P}_\theta(Y_k \mid Y_{1:k-1})}\\ &= \frac{f_{x'}(y_k)\mathbb{P}_\theta(X_k=x' \mid Y_{1:k-1})}{\sum_{x''\in\{0,1\}}\mathbb{P}_\theta(y_k \mid Y_{1:k-1},X_k=x'')\mathbb{P}_\theta(X_k=x'' \mid Y_{1:k-1})}\\ &= \frac{f_{x'}(y_k)P_{k-1}(x')}{\sum_{x''\in\{0,1\}}f_{x''}(y_k)P_{k-1}(x'')}.\end{aligned}$$

Similarly, for $k = 1$,

$$\begin{aligned}\mathbb{P}_\theta(X_2 = x \mid Y_1 = y_1) &= \frac{\mathbb{P}_\theta(X_2 = x, Y_1 = y_1)}{\mathbb{P}_\theta(y_1)} \\ &= \frac{\sum_{x' \in \{0,1\}} \mathbb{P}_\theta(X_2 = x, Y_1 = y_1 \mid X_1 = x')\mathbb{P}_\theta(X_1 = x')}{\sum_{x' \in \{0,1\}} f_{x'}(y_1)\mathbb{P}_\theta(X_1 = x')} \\ &= \frac{\sum_{x' \in \{0,1\}} f_{x'}(y_1) Q_{x',x}\mathbb{P}_\theta(X_1 = x')}{\sum_{x' \in \{0,1\}} f_{x'}(y_1)\mathbb{P}_\theta(X_1 = x')}\end{aligned}$$

To summarise, we have proved the recursive formula

$$P_k(x) = \begin{cases} \frac{\sum_{x' \in \{0,1\}} Q_{x',x} f_{x'}(Y_k) P_{k-1}(x')}{\sum_{x' \in \{0,1\}} f_{x'}(Y_k) P_{k-1}(x')} & \text{if } k \geq 2, \\ \frac{\sum_{x' \in \{0,1\}} f_{x'}(Y_1) Q_{x',x}\mathbb{P}_\theta(X_1 = x')}{\sum_{x' \in \{0,1\}} f_{x'}(Y_1)\mathbb{P}_\theta(X_1 = x')} & \text{if } k = 1. \end{cases}$$

Therefore when $k \geq 2$,

$$\begin{aligned}V_k &= \phi_3\Big(1 - 2\frac{Q_{0,1} f_0(Y_k)P_{k-1}(0) + Q_{1,1} f_1(Y_k)P_{k-1}(1)}{f_0(Y_k)P_{k-1}(0) + f_1(Y_k)P_{k-1}(1)} - \phi_1\Big) \\ &= \phi_3\Big(1 - \frac{2pf_0(Y_k)P_{k-1}(0) + 2(1-q)f_1(Y_k)P_{k-1}(1)}{f_0(Y_k)P_{k-1}(0) + f_1(Y_k)P_{k-1}(1)} - \phi_1\Big) \\ &= \phi_3\Big(1 - \frac{2pf_0(Y_k) + 2P_{k-1}(1)[(1-q)f_1(Y_k) - pf_0(Y_k)]}{f_0(Y_k) + P_{k-1}(1)[f_1(Y_k) - f_0(Y_k)]} - \phi_1\Big).\end{aligned}$$

We write for convenience

$$\begin{aligned}D_k &= f_0(Y_k) + P_{k-1}(1)[f_1(Y_k) - f_0(Y_k)], \\ N_k &= (1-\phi_1)\phi_3 D_k - 2\phi_3 p f_0(Y_k) - 2\phi_3 P_{k-1}(1)[(1-q)f_1(Y_k) - pf_0(Y_k)],\end{aligned}$$

so that $V_k = N_k/D_k$. We rewrite the previous expressions solely in terms of the parameters (ϕ, ψ) [recall the inversion formulae in Remark 4]. First,

$$\begin{aligned}D_k &= \psi_1(Y_k) - \frac{1}{2}\phi_1\phi_3\psi_2(Y_k) + \frac{1}{2}\phi_3\psi_2(Y_k) - P_{k-1}(1)\phi_3\psi_2(Y_k) \\ &= \psi_1(Y_k) + \frac{1}{2}\phi_3\psi_2(Y_k)[1 - 2P_{k-1}(1) - \phi_1] \\ &= \psi_1(Y_k) + \frac{V_{k-1}}{2}\psi_2(Y_k).\end{aligned}$$

Also,

$$2pf_0 = (1-\phi_1)(1-\phi_2)\Big[\psi_1 - \frac{1}{2}\phi_1\phi_3\psi_2 + \frac{1}{2}\phi_3\psi_2\Big],$$

and

$$
\begin{aligned}
(1-q)f_1 - pf_0 &= (1-q-p)f_1 - p(f_0 - f_1) \\
&= \phi_2\Big(\psi_1 - \frac{1}{2}\phi_1\phi_3\psi_2 - \frac{1}{2}\phi_3\psi_2\Big) - \frac{1}{2}(1-\phi_2)(1-\phi_1)\phi_3\psi_2 \\
&= \phi_2\psi_1 - \frac{1}{2}\phi_3\psi_2\Big(\phi_2 + \phi_1\phi_2 + (1-\phi_2)(1-\phi_1)\Big) \\
&= \phi_2\psi_1 - \frac{1}{2}\phi_3\psi_2\Big(1 - \phi_1 + 2\phi_1\phi_2\Big).
\end{aligned}
$$

Using the last three displays and the fact that $2\phi_3 P_{k-1}(1) = -V_{k-1} + \phi_3 - \phi_1\phi_3$, we obtain that

$$
\begin{aligned}
N_k = {}& (1-\phi_1)\phi_3\Big[\psi_1(Y_k) + \frac{V_{k-1}}{2}\psi_2(Y_k)\Big] \\
& - \phi_3(1-\phi_1)(1-\phi_2)\Big[\psi_1(Y_k) - \frac{1}{2}\phi_1\phi_3\psi_2(Y_k) + \frac{1}{2}\phi_3\psi_2(Y_k)\Big] \\
& + (V_{k-1} - \phi_3 + \phi_1\phi_3)\Big[\phi_2\psi_1(Y_k) - \frac{1}{2}\phi_3\psi_2(Y_k)\Big(1 - \phi_1 + 2\phi_1\phi_2\Big)\Big].
\end{aligned}
$$

Grouping together the terms proportional to V_{k-1} and the others,

$$
\begin{aligned}
N_k = {}& V_{k-1}\Big[\phi_2\psi_1(Y_k) + \frac{1}{2}\Big(-\phi_3 + \phi_1\phi_3 - 2\phi_1\phi_2\phi_3 + (1-\phi_1)\phi_3\Big)\psi_2(Y_k)\Big] \\
& + \psi_1(Y_k)\Big[(1-\phi_1)\phi_3 - \phi_3(1-\phi_1)(1-\phi_2) - \phi_3(1-\phi_3)\phi_2\Big] \\
& + \frac{1}{2}\psi_2(Y_k)\Big[-\phi_3^2(1-\phi_1)^2(1-\phi_2) + \phi_3^2(1-\phi_1)(1-\phi_1+2\phi_1\phi_2)\Big].
\end{aligned}
$$

We remark that,

$$
-\phi_3 + \phi_1\phi_3 - 2\phi_1\phi_2\phi_3 + (1-\phi_1)\phi_3 = -2\phi_1\phi_2\phi_3,
$$

and

$$
(1-\phi_1)\phi_3 - \phi_3(1-\phi_1)(1-\phi_2) - \phi_3(1-\phi_3)\phi_2 = 0,
$$

and

$$
\begin{aligned}
& -\phi_3^2(1-\phi_1)^2(1-\phi_2) + \phi_3^2(1-\phi_1)(1-\phi_1+2\phi_1\phi_2) \\
&\quad = -\phi_3^2(1-\phi_1)^2(1-\phi_2) + \phi_3^2(1-\phi_1)^2 + 2\phi_1\phi_2\phi_3^2(1-\phi_1) \\
&\quad = \phi_2\phi_3^2(1-\phi_1)^2 + 2\phi_1\phi_2\phi_3^2(1-\phi_1) \\
&\quad = \phi_2\phi_3^2[1 - 2\phi_1 + \phi_1^2 + 2\phi_1 - 2\phi_1^2] \\
&\quad = \phi_2\phi_3^2(1-\phi_1^2).
\end{aligned}
$$

That is,

$$
N_k = \phi_2 V_{k-1}\big[\psi_1(Y_k) - \phi_1\phi_3\psi_2(Y_k)\big] + \frac{\phi_2\phi_3^2(1-\phi_1^2)}{2}\psi_2(Y_k)
$$

which means that for $k \geq 2$,

$$V_k = \frac{\phi_2[\psi_1(Y_k) - \phi_1\phi_3\psi_2(Y_k)]V_{k-1} + 2r(\phi)\psi_2(Y_k)}{\psi_1(Y_k) + \frac{1}{2}\psi_2(Y_k)V_{k-1}}.$$

For $k = 1$, recalling that $Y_1 \sim \psi_1$ and $\mathbb{P}_\theta(X_1 = 1) = p/(p+q)$, we have

$$\begin{aligned}
V_1 &= \phi_3(1 - 2P_1(1) - \phi_1) \\
&= \phi_3\Big(1 - \phi_1 - 2\frac{f_0(Y_1)\frac{pq}{p+q} + f_1(Y_1)\frac{(1-q)p}{p+q}}{\psi_1(Y_1)}\Big) \\
&= \phi_3\Big(1 - \phi_1 - 2\frac{f_1(Y_1)\frac{p}{p+q} + \phi_3\psi_2(Y_1)\frac{pq}{p+q}}{\psi_1(Y_1)}\Big) \\
&= \phi_3\Big(1 - \phi_1 - 2\frac{[\psi_1(Y_1) - \frac{1}{2}\phi_1\phi_3\psi_2(Y_1) - \frac{1}{2}\phi_3\psi_2(Y_1)]\frac{1-\phi_1}{2} + \phi_3\psi_2(Y_1)\frac{(1-\phi_2)(1-\phi_1^2)}{4}}{\psi_1(Y_1)}\Big) \\
&= -\phi_3^2\psi_2(Y_1)\frac{-\frac{(1-\phi_1)(1+\phi_1)}{2} + \frac{(1-\phi_2)(1-\phi_1^2)}{2}}{\psi_1(Y_1)} \\
&= \frac{\frac{1}{2}(1-\phi_1^2)\phi_2\phi_3^2\psi_2(Y_1)}{\psi_1(Y_1)}
\end{aligned}$$

Letting $m_1 = r(\phi)$, $m_2 = r(\phi)\phi_2$, and $m_3 = r(\phi)\phi_1\phi_2\phi_3$, we have obtained the inductive formula

$$V_k = \begin{cases} \frac{[m_2\psi_1(Y_k) - m_3\psi_2(Y_k)]\frac{V_{k-1}}{m_1} + 2m_1\psi_2(Y_k)}{\psi_1(Y_k) + \frac{1}{2}\psi_2(Y_k)V_{k-1}} & \text{if } k \geq 2, \\ \frac{2m_1\psi_2(Y_1)}{\psi_1(Y_1)} & \text{if } k = 1. \end{cases}$$

The strategy is now to bound $V_k - \tilde{V}_k$ for $k \geq 2$ in term of $V_1 - \tilde{V}_1$ using the above inductive formula. To do so, we will need an upper bound for V_k (respectively $\tilde{V}_k$) which we establish now. We claim that $|V_k| \leq 4|m_1|/c$ for all $k \geq 1$ provided ϵ_0 is taken small enough. Indeed, $|\psi_2(Y_1)| \leq \|\psi_2\| = 1$ and $c \leq \psi_1(Y_1) \leq 1$, hence $|V_1| \leq 2|m_1|/c \leq 4|m_1|/c$. Now suppose that $|V_{k-1}| \leq 4|m_1|/c$; then, under the assumptions of the proposition with for $\epsilon_0 = \epsilon_0(c)$ small enough, using equation (5) to see that $|m_1| \leq |\phi_2| \leq \epsilon_0$, $|\phi_1\phi_2\phi_3| \leq \sqrt{2}|\phi_2| \leq \sqrt{2}\epsilon_0$, we have

$$\begin{aligned}
|V_k| &\leq \frac{(|m_2| + |m_3|)\frac{4}{c} + 2|m_1|}{c - \frac{1}{2}\frac{4|m_1|}{c}} \\
&\leq |m_1|\frac{(|\phi_2| + |\phi_1\phi_2\phi_3|)\frac{4}{c} + 2}{c - 4|m_1|/c} \\
&\leq \frac{4|m_1|}{c}.
\end{aligned} \tag{32}$$

Similarly $|\tilde{V}_k| \leq 4|\tilde{m}_1|/c$ for all $k \geq 1$. We are now in position to bound $V_k - \tilde{V}_k$ for $k \geq 2$. Recall $V_k = N_k/D_k$ and similarly write $\tilde{V}_k = \tilde{N}_k/\tilde{D}_k$. Then

$$V_k - \tilde{V}_k = \frac{N_k}{D_k} - \frac{\tilde{N}_k}{\tilde{D}_k} = \frac{\tilde{D}_k N_k - D_k\tilde{N}_k}{D_k\tilde{D}_k} = \frac{(\tilde{D}_k - D_k)N_k}{D_k\tilde{D}_k} + \frac{N_k - \tilde{N}_k}{\tilde{D}_k}.$$

As when bounding $|V_k|$, we can assume that ϵ_0 is small enough to have $D_k \geq c/2$ and $\tilde{D}_k \geq c/2$, and

$$|N_k| \leq (|m_2| + |m_3|)\frac{4}{c} + 2|m_1| \leq 4|m_1|.$$

Therefore,

$$|V_k - \tilde{V}_k| \leq \frac{16|m_1|}{c^2}|D_k - \tilde{D}_k| + \frac{2}{c}|N_k - \tilde{N}_k|. \tag{33}$$

But, recalling the definition (12) of ρ, we have

$$\begin{aligned}
|D_k - \tilde{D}_k| &= \Big|\psi_1(Y_k) - \tilde{\psi}_1(Y_k) + \frac{1}{2}\Big(\psi_2(Y_k)V_{k-1} - \tilde{\psi}_2(Y_k)\tilde{V}_{k-1}\Big)\Big| \\
&\leq |\psi_1(Y_k) - \tilde{\psi}_1(Y_k)| + \frac{|\psi_2(Y_k)|}{2}|V_{k-1} - \tilde{V}_{k-1}| + \frac{|\tilde{V}_{k-1}|}{2}|\psi_2(Y_k) - \tilde{\psi}_2(Y_k)| \\
&\leq \|\psi_1 - \tilde{\psi}_1\| + \frac{|V_{k-1} - \tilde{V}_{k-1}|}{2} + \frac{2|m_1|}{c}\|\psi_2 - \tilde{\psi}_2\| \\
&\leq \Big(1 + \frac{2}{c}\Big)\rho(\phi, \psi; \tilde{\phi}, \tilde{\psi}) + \frac{|V_{k-1} - \tilde{V}_{k-1}|}{2},
\end{aligned}$$

and

$$\begin{aligned}
N_k - \tilde{N}_k &= [m_2\psi_1(Y_k) - \tilde{m}_2\tilde{\psi}_1(Y_k) - m_3\psi_2(Y_k) + \tilde{m}_3\tilde{\psi}_2(Y_k)]\frac{V_{k-1}}{m_1} \\
&\quad + \Big(\frac{V_{k-1}}{m_1} - \frac{\tilde{V}_{k-1}}{\tilde{m}_1}\Big)\Big(\tilde{m}_2\tilde{\psi}_1(Y_k) - \tilde{m}_3\tilde{\psi}_2(Y_k)\Big) + 2m_1\psi_2(Y_k) - 2\tilde{m}_1\tilde{\psi}_2(Y_k),
\end{aligned}$$

from which we deduce that

$$\begin{aligned}
|N_k - \tilde{N}_k| &\leq \Big(|m_2|\|\psi_1 - \tilde{\psi}_1\| + |m_2 - \tilde{m}_2| + |m_3|\|\psi_2 - \tilde{\psi}_2\| + |m_3 - \tilde{m}_3|\Big)\Big|\frac{V_{k-1}}{m_1}\Big| \\
&\quad + \Big(|V_{k-1} - \tilde{V}_{k-1}| + |m_1 - \tilde{m}_1|\Big|\frac{V_{k-1}}{m_1}\Big|\Big)\frac{|\tilde{m}_2| + |\tilde{m}_3|}{|\tilde{m}_1|} \\
&\quad + 2|m_1|\|\psi_2 - \tilde{\psi}_2\| + 2|m_1 - \tilde{m}_1| \\
&\leq 4|m_1 - \tilde{m}_1| + \frac{4|m_2 - \tilde{m}_2|}{c} + \frac{4|m_3 - \tilde{m}_3|}{c} \\
&\quad + \frac{4|m_2|\|\psi_1 - \tilde{\psi}_1\|}{c} + \Big(\frac{4|m_3|}{c|m_1|} + 2\Big)|m_1|\|\psi_2 - \tilde{\psi}_2\| + \frac{c|V_{k-1} - \tilde{V}_{k-1}|}{8},
\end{aligned}$$

where the last line holds when ϵ_0 is small enough. Inserting these bounds into equation (33), we find that there is a constant B depending solely on c such that for all $k \geq 2$

$$\begin{aligned}
|V_k - \tilde{V}_k| &\leq B\rho(\phi, \psi; \tilde{\phi}, \tilde{\psi}) + \Big(\frac{1}{4} + \frac{8|m_1|}{c^2}\Big)|V_{k-1} - \tilde{V}_{k-1}| \\
&\leq B\rho(\phi, \psi; \tilde{\phi}, \tilde{\psi}) + \frac{1}{2}|V_{k-1} - \tilde{V}_{k-1}|
\end{aligned}$$

again when ϵ_0 is small enough. Hence for $k \geq 2$,

$$\begin{aligned}|V_k - \tilde{V}_k| &\leq 2(1-2^{-k})B\rho(\phi,\psi;\tilde{\phi},\tilde{\psi}) + 2^{1-k}|V_1 - \tilde{V}_1| \\ &\leq \frac{3B}{2}\rho(\phi,\psi;\tilde{\phi},\tilde{\psi}) + \frac{|V_1 - \tilde{V}_1|}{2}.\end{aligned}$$

To finish the proof, it is enough to show that $|V_1 - \tilde{V}_1|$ is bounded by a multiple constant of $\rho(\phi,\psi;\tilde{\phi},\tilde{\psi})$, which follows from its definition and the same arguments as above. Thus for some constant $B' > 0$ depending only on c

$$\max_{k=1,\dots,n} |V_k - \tilde{V}_k| \leq B'\rho(\phi,\psi;\tilde{\phi},\tilde{\psi}). \tag{34}$$

The conclusion follows by combining equations (31), (32) and (34).

8.3 Proof of Theorem 2

We start with the proof of Lemma 1, that $p^{(3)}$ can be estimated at a parametric rate.

Proof of Lemma 1. We use a Markov chain concentration result from [34]. Theorem 3.4 therein (but note there is an updated version of the paper on arXiv) tells us that for any stationary Markov chain $\boldsymbol{Z} = (Z^{(1)}, Z^{(2)}, \dots)$ of pseudo-spectral gap γ_{ps} (defined as in [34]) and any function h satisfying $\mathbb{E}[h(Z^{(1)})^2] \leq \sigma^2$ and $\|h\|_\infty \leq b$,

$$\mathbb{P}(|\sum_{i=1}^n h(Z^{(i)}) - Eh(Z^{(1)})| \geq x) \leq 2\exp\Big(-\frac{x^2\gamma_{\text{ps}}}{8(n+1/\gamma_{\text{ps}})\sigma^2 + 20bx}\Big). \tag{35}$$

We apply to the chain $\boldsymbol{Z}$ defined by $Z^{(n)} = (X_n, X_{n+1}, X_{n+2}, Y_n, Y_{n+1}, Y_{n+2})$; we begin by showing the pseudo-spectral gap of this chain is bounded from below. Proposition 3.4 of the same reference shows that the reciprocal of the pseudo-spectral gap of any chain is bounded above by twice the mixing time $t_{\text{mix}}^{\boldsymbol{Z}}$ of the chain, defined as the first time that the law of $\boldsymbol{Z}$, regardless of the starting distribution, is within 1/4 of its invariant distribution in total variation distance. We note that $t_{\text{mix}}^{\boldsymbol{Z}}$ is equal to the mixing time $t_{\text{mix}}^{\boldsymbol{X}^{(3)}}$ of the chain $((X_n, X_{n+1}, X_{n+2})_{n\geq 0})$. This latter quantity is equal to $t_{\text{mix}}^{\boldsymbol{X}} + 2$ where $t_{\text{mix}}^{\boldsymbol{X}}$ denotes the mixing time of the chain $\boldsymbol{X}$ itself. Finally, the matrix Q has eigenvalues 1 and ϕ_2, and an explicit computation yields that $\max_{ij}|Q_{ij}^n - \pi_j| = \max_i(\pi_i)|\phi_2|^n$ so that the mixing time of $\boldsymbol{X}$ is at most

$$\Big\lceil \frac{\log 4}{\log(1/|\phi_2|)} \Big\rceil \leq \Big\lceil \frac{\log 4}{\log(1/(1-L))} \Big\rceil \leq \Big\lceil \frac{\log 4}{L} \Big\rceil,$$

which is a constant since L is fixed. The pseudo-spectral gap of the chain $\boldsymbol{Z}$ is thus lower bounded by some constant $\gamma = \gamma(L)$.

Applying equation (35) with $h(Z) = \mathbb{1}\{Z_4 = a, Z_5 = b, Z_6 = c\}$, which satisfies $\mathbb{E}h^2 \leq 1$ and $\|h\|_\infty \leq 1$, we see that

$$\mathbb{P}_{\phi,\psi}\big(n|\hat{p}^{(3)}(a,b,c) - p_{\phi,\psi}^{(3)}(a,b,c)| \geq x\big) \leq 2\exp\Big(-\frac{\gamma x^2}{8n + 8/\gamma + 20x}\Big),$$

hence for some constant $c' > 0$

$$\mathbb{P}_{\phi,\psi}\big(|\hat{p}^{(3)}(a,b,c) - p_{\phi,\psi}^{(3)}(a,b,c)| \geq x/\sqrt{n}\big) \leq 2\exp\Big(-c'\min\Big(x^2, x^2 n, x\sqrt{n}\Big)\Big).$$

Using that $\|\hat{p}^{(3)} - p^{(3)}\| \le K^3 \max_{a,b,c} |\hat{p}^{(3)}(a,b,c) - p^{(3)}(a,b,c)|$ and a union bound, we deduce for some $C = C(K, L)$ and for $x \le \sqrt{n}$ that

$$\mathbb{P}_{\phi,\psi}(\|\hat{p}^{(3)} - p^{(3)}\| \ge K^3 x/\sqrt{n}) \le 2K^3 \exp(-Cx^2).$$

For $x \ge 1$ we may absorb the factor $2K^3$ into the exponential by changing the constant C, and by replacing x with $C'x$ we can remove this constant, yielding the result in the case where $C'x \le \sqrt{n}$. In the other case, since $\|\hat{p}^{(3)} - p^{(3)}\|$ is bounded (by $K^{3/2}$), by increasing the constant C' if necessary we have $C'x/\sqrt{n} \ge K^{3/2}$ so that the probability in question is equal to $0 \le e^{-x^2}$. □

To prove Theorem 2, observe that by Lemma 1 there exist events $\mathcal{A}_n$ of probability at least e^{-x^2} on which

$$\|\hat{p}_n^{(3)} - p_{\phi,\psi}^{(3)}\| \le Cx/\sqrt{n}.$$

The true parameter (ϕ, ψ) lies in Φ_L so that any estimators constructed in Theorem 2 satisfy

$$\|p_{\hat{\phi},\hat{\psi}}^{(3)} - \hat{p}_n^{(3)}\| \le 2\|p_{\phi,\psi}^{(3)} - \hat{p}_n^{(3)}\|,$$

and hence on the event $\mathcal{A}_n$ further satisfy

$$\|p_{\hat{\phi},\hat{\psi}}^{(3)} - p_{\phi,\psi}^{(3)}\| \le \|p_{\hat{\phi},\hat{\psi}}^{(3)} - \hat{p}_n^{(3)}\| + \|p_{\phi,\psi}^{(3)} - \hat{p}_n^{(3)}\| \le 3\|p_{\phi,\psi}^{(3)} - \hat{p}_n^{(3)}\| \le 3Cx/\sqrt{n},$$

By Proposition 1 we deduce for a constant C' that $\rho(\hat{\phi}, \hat{\psi}; \phi, \psi) \le C'x/\sqrt{n}$ on $\mathcal{A}_n$. For estimating ψ, observe that $\|\hat{\psi}_1 - \psi_1\| \le \rho(\hat{\phi}, \hat{\psi}; \phi, \psi)$ and $|r(\phi)| \min(\|\hat{\psi}_2 - \psi_2\|, \|\hat{\psi}_2 + \psi_2\|) \le \rho(\hat{\phi}, \hat{\psi}; \phi, \psi)$. The upper bound for estimating ψ_1 is immediate and, recalling from equation (5) that $|r(\phi)| \ge \delta\epsilon\zeta^2/4$, we also deduce the bound for ψ_2.

For the bounds on ϕ, observe firstly that it suffices to prove the upper bounds on the absolute risk since, taking ϕ_2 as an example, for $(\phi, \psi) \in \Phi_L(\delta, \epsilon, \zeta)$ we have

$$\begin{aligned}\mathbb{P}_{\phi,\psi}\Big(|\hat{\phi}_2/\phi_2 - 1|^2 \ge \frac{C}{n\delta^2\epsilon^4\zeta^4}\Big) &= \mathbb{P}_{\phi,\psi}\Big(|\hat{\phi}_2 - \phi_2|^2 \ge \frac{C\phi_2^2}{n\delta^2\epsilon^4\zeta^4}\Big) \\ &\le \mathbb{P}_{\phi,\psi}\Big(|\hat{\phi}_2 - \phi_2| \ge \frac{C\epsilon^2}{n\delta^2\epsilon^4\zeta^4}\Big).\end{aligned} \tag{36}$$

(See also after equation (37) for a similar argument with ϕ_1.) Define

$$\omega_1(\phi, \psi; \eta) := \sup\Big\{|\phi_1 - \operatorname{sgn}(\langle\psi_2, \tilde{\psi}_2\rangle) \cdot \tilde{\phi}_1| \ : \ \rho(\phi, \psi; \tilde{\phi}, \tilde{\psi}) \le \eta\Big\},$$

and

$$\omega_j(\phi, \psi; \eta) := \sup\Big\{|\phi_j - \tilde{\phi}_j| \ : \ \rho(\phi, \psi; \tilde{\phi}, \tilde{\psi}) \le \eta\Big\}, \qquad j = 2, 3.$$

We have the following.

Proposition 3. *Let $\eta \in [0, 1]$. There exist constants c, C for which the following hold.*

$$\begin{aligned}\eta < c(1 - \phi_1^2)\phi_2^2\phi_3^3 &\implies \omega_1(\phi, \psi; \eta) \le \frac{C\eta}{\phi_2^2\phi_3^3}, \\ \eta < c(1 - \phi_1^2)|\phi_2|\phi_3^2 &\implies \omega_2(\phi, \psi; \eta) \le \frac{C\eta}{(1 - \phi_1^2)|\phi_2|\phi_3^2}, \\ \eta < c(1 - \phi_1^2)\phi_2^2\phi_3^3 &\implies \omega_3(\phi, \psi; \eta) \le \frac{C\eta}{(1 - \phi_1^2)\phi_2^2\phi_3^2}.\end{aligned}$$

The conditions of Theorem 2 ensure that on the event $\mathcal{A}_n$ we may apply Proposition 3 with $\eta = C'x/\sqrt{n}$. We deduce the upper bounds for estimating the components of ϕ immediately upon replacing $\hat{\phi}_1, \hat{\phi}_2$ and ϕ_3 on the right sides in Proposition 3 by their lower bounds [for ϕ_1 we note that $\min(|\hat{\phi}_1 - \phi_1|, |\hat{\phi}_1 + \phi_1|) \leq |\phi_1 - \mathrm{sgn}(\langle \psi_2, \tilde{\psi}_2 \rangle) \cdot \tilde{\phi}_1|$].

Proof of Proposition 3. Recall that $m(\phi) = (r(\phi), \phi_2 r(\phi), \phi_1\phi_2\phi_3 r(\phi))$ with $r(\phi) = \frac{1}{4}(1-\phi_1^2)\phi_2\phi_3^2$. If $r(\phi) = 0$ then in each case no $\eta \in [0,1]$ satisfies the conditions and so there is nothing to prove. Otherwise, note that m is invertible when restricted to $\{\phi : r(\phi) \neq 0\} \supset \Phi(\delta, \epsilon, \zeta)$ and its inverse is given by $\phi(m)$ defined by

$$\phi_1(m) = m_3/(4m_1^2 m_2 + m_3^2)^{1/2}$$
$$\phi_2(m) = m_2/m_1,$$
$$\phi_3(m) = (4m_1^2 m_2 + m_3^2)^{1/2}/m_2.$$

For arbitrary $(\phi, \psi; \tilde{\phi}, \tilde{\psi})$ satisfying $\rho(\phi, \psi; \tilde{\phi}, \tilde{\psi}) \leq \eta$, we define

$$\Delta_1 \coloneqq m_1(\tilde{\phi}) - m_1(\phi), \quad \Delta_2 \coloneqq m_2(\tilde{\phi}) - m_2(\phi), \quad \Delta_3 \coloneqq \mathrm{sgn}(\langle \psi_2, \tilde{\psi}_2 \rangle) \cdot m_3(\tilde{\phi}) - m_3(\phi).$$

Define also

$$\begin{aligned} g(\phi) &\coloneqq 4m_1(\phi)^2 m_2(\phi) + m_3(\phi)^2 \\ &= \{m_2(\phi)\phi_3\}^2 \\ &= \Big\{\frac{1}{4}(1-\phi_1^2)\phi_2^2\phi_3^3\Big\}^2, \end{aligned}$$

and, for $\Delta = (\Delta_1, \Delta_2, \Delta_3)$,

$$\begin{aligned} h_\phi(\Delta) &\coloneqq g(\tilde{\phi}) - g(\phi) \\ &= 4(m_1(\phi) + \Delta_1)^2(m_2(\phi) + \Delta_2) + (m_3(\phi) + \Delta_3)^2 - \{4m_1(\phi)^2 m_2(\phi) + m_3(\phi)^2\}. \end{aligned}$$

Observe that

$$\begin{aligned} h_\phi(\Delta) = 8m_1(\phi)m_2(\phi)\Delta_1 + 8m_1(\phi)\Delta_1\Delta_2 + 4m_2(\phi)\Delta_1^2 \\ + 4\Delta_1^2\Delta_2 + 4m_1(\phi)^2\Delta_2 + 2m_3(\phi)\Delta_3 + \Delta_3^2. \end{aligned}$$

Bounding ω_1 We decompose,

$$\begin{aligned} \phi_1 - \mathrm{sgn}(\langle \psi_2, \tilde{\psi}_2 \rangle) \cdot \tilde{\phi}_1 &= \frac{m_3(\phi)}{\sqrt{4m_1(\phi)^2 m_2(\phi) + m_3(\phi)^2}} - \frac{\mathrm{sgn}(\langle \psi_2, \tilde{\psi}_2 \rangle) \cdot m_3(\tilde{\phi})}{\sqrt{4m_1(\tilde{\phi})^2 m_2(\tilde{\phi}) + m_3(\tilde{\phi})^2}} \\ &= \frac{m_3(\phi)}{\sqrt{g(\phi)}} - \frac{m_3(\phi) + \Delta_3}{\sqrt{g(\phi) + h_\phi(\Delta)}} \\ &= m_3(\phi)\Big\{\frac{1}{\sqrt{g(\phi)}} - \frac{1}{\sqrt{g(\phi) + h_\phi(\Delta)}}\Big\} - \frac{\Delta_3}{\sqrt{g(\phi) + h_\phi(\Delta)}} \\ &= \frac{m_3(\phi)}{\sqrt{g(\phi)(g(\phi) + h_\phi(\Delta))}}\Big\{\sqrt{g(\phi) + h_\phi(\Delta)} - \sqrt{g(\phi)}\Big\} - \frac{\Delta_3}{\sqrt{g(\phi) + h_\phi(\Delta)}} \\ &= \frac{m_3(\phi)}{\sqrt{g(\phi)(g(\phi) + h_\phi(\Delta))}} \frac{h_\phi(\Delta)}{\sqrt{g(\phi) + h_\phi(\Delta)} + \sqrt{g(\phi)}} - \frac{\Delta_3}{\sqrt{g(\phi) + h_\phi(\Delta)}}. \end{aligned}$$

Now we observe that $m_3(\phi)/\sqrt{g(\phi)}$ is equal to ϕ_1, so indeed

$$\phi_1 - \operatorname{sgn}(\langle \psi_2, \tilde{\psi}_2 \rangle) \cdot \tilde{\phi}_1 = \frac{\phi_1 h_\phi(\Delta) - \Delta_3(\sqrt{g(\phi) + h_\phi(\Delta)} + \sqrt{g(\phi)})}{\sqrt{g(\phi) + h_\phi(\Delta)}(\sqrt{g(\phi) + h_\phi(\Delta)} + \sqrt{g(\phi)})}.$$

Call the numerator of this last fraction N and call its denominator D. Writing $h_\phi(\Delta)$ as $h_\phi(\Delta) = \xi_\phi(\Delta) + \gamma_\phi(\Delta)$, where $\gamma_\phi(\Delta) \coloneqq 2m_3(\phi)\Delta_3 + \Delta_3^2$, we see that

$$N = \phi_1 \xi_\phi(\Delta) + \phi_1 \gamma_\phi(\Delta) - \Delta_3 \big\{ (g(\phi) + h_\phi(\Delta))^{1/2} + g(\phi)^{1/2} \big\}.$$

In order to obtain the optimal upper bound, we need to do a fine analysis of this expression. To this end, we calculate

$$\begin{aligned}
A &\coloneqq \phi_1 \gamma_\phi(\Delta) - \Delta_3 \{(g+h)^{1/2} + g^{1/2}\} \\
&= 2\Delta_3\{\phi_1 m_3(\phi) - g^{1/2}\} + \phi_1 \Delta_3^2 - \Delta_3\{(g+h)^{1/2} - g^{1/2}\} \\
&= -2\Delta_3(1-\phi_1^2)g^{1/2} + \phi_1\Delta_3^2 - \frac{\Delta_3 h}{(g+h)^{1/2} + g^{1/2}} \\
&= -2\Delta_3(1-\phi_1^2)g^{1/2} - \Delta_3 \frac{\gamma_\phi(\Delta) - \phi_1\Delta_3((g+h)^{1/2} + g^{1/2})}{(g+h)^{1/2} + g^{1/2}} - \frac{\Delta_3 \xi_\phi(\Delta)}{(g+h)^{1/2} + g^{1/2}},
\end{aligned}$$

where the last line follows because $\phi_1 m_3(\phi) = \phi_1^2 g(\phi)^{1/2}$. We now focus on the middle term of this last display, which we will express as a function of A.

$$\begin{aligned}
B &\coloneqq \gamma_\phi(\Delta) - \phi_1\Delta_3((g+h)^{1/2} + g^{1/2}) \\
&= 2\Delta_3(m_3(\phi) - \phi_1 g^{1/2}) + \Delta_3^2 - \phi_1\Delta_3\{(g+h)^{1/2} - g^{1/2}\} \\
&= \Delta_3^2 - \frac{\phi_1\Delta_3 h}{(g+h)^{1/2} + g^{1/2}} \\
&= \Delta_3^2 - \frac{\phi_1\Delta_3\gamma_\phi(\Delta)}{(g+h)^{1/2} + g^{1/2}} - \frac{\phi_1\Delta_3\xi_\phi(\Delta)}{(g+h)^{1/2} + g^{1/2}} \\
&= \Delta_3 \frac{\Delta_3\{(g+h)^{1/2} + g^{1/2}\} - \phi_1\gamma_\phi(\Delta)}{(g+h)^{1/2} + g^{1/2}} - \frac{\phi_1\Delta_3\xi_\phi(\Delta)}{(g+h)^{1/2} + g^{1/2}} \\
&= -\frac{\Delta_3 A}{(g+h)^{1/2} + g^{1/2}} - \frac{\phi_1\Delta_3\xi_\phi(\Delta)}{(g+h)^{1/2} + g^{1/2}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
A &= -2\Delta_3(1-\phi_1^2)g^{1/2} - \frac{\Delta_3 B}{(g+h)^{1/2} + g^{1/2}} - \frac{\Delta_3\xi_\phi(\Delta)}{(g+h)^{1/2} + g^{1/2}} \\
&= -2\Delta_3(1-\phi_1^2)g^{1/2} + \frac{\Delta_3^2 A}{\{(g+h)^{1/2} + g^{1/2}\}^2} + \frac{\phi_1\Delta_3^2\xi_\phi(\Delta)}{\{(g+h)^{1/2} + g^{1/2}\}^2} - \frac{\Delta_3\xi_\phi(\Delta)}{(g+h)^{1/2} + g^{1/2}},
\end{aligned}$$

from which we deduce that

$$N = \phi_1\xi_\phi(\Delta) + \frac{-2\Delta_3(1-\phi_1^2)g^{1/2} + \frac{\phi_1\Delta_3^2\xi_\phi(\Delta)}{\{(g+h)^{1/2}+g^{1/2}\}^2} - \frac{\Delta_3\xi_\phi(\Delta)}{(g+h)^{1/2}+g^{1/2}}}{1 - \Delta_3^2/\{(g+h)^{1/2} + g^{1/2}\}^2}.$$

Since $m_2(\phi) \geq 0$, we see that $\xi_\phi(\Delta)$ has maximal amplitude when $\Delta_1 = \mathrm{sgn}(m_1(\phi))\eta$ and when $\Delta_2 = \eta$, in which case we have

$$\begin{aligned}|\xi_\phi(\Delta)| &= 8|m_1(\phi)|m_2(\phi)\eta + 8|m_1(\phi)|\eta^2 + 4m_2(\phi)\eta^2 + 4\eta^3 + 4m_1(\phi)^2\eta \\ &\leq 12m_1(\phi)^2\eta + 12|m_1(\phi)|\eta^2 + 4\eta^3,\end{aligned}$$

where the last line follows since $m_2(\phi) \leq |m_1(\phi)|$. Now we observe that under the condition of the lemma, we have $\eta \lesssim |m_1(\phi)|$, and so we can find a constant $C > 0$ such that

$$|\xi_\phi(\Delta)| \leq Cm_1(\phi)^2\eta.$$

Also, we have that $|\gamma_\phi(\Delta)| \leq 2|m_3(\phi)|\eta + \eta^2$, and so

$$|h_\phi(\Delta)| \leq Cm_1(\phi)^2\eta + 2|m_3(\phi)|\eta + \eta^2,$$

Noting that $\phi_3 \leq \sqrt{K}$, for $c_0 = c_0(K)$ sufficiently small in the assumption of the proposition we have $|h_\phi(\Delta)| \leq g(\phi)/2$. Consequently, noting also that $|\Delta_3| \leq \eta$ and $\eta \leq 4c_0 g^{1/2}$, we find that

$$\begin{aligned}|N| &\lesssim |\phi_1|m_1(\phi)^2\eta + \eta(1-\phi_1^2)g(\phi)^{1/2} \\ &\lesssim \eta(1-\phi_1^2)^2\phi_2^2\phi_3^3,\end{aligned}$$

and

$$|D| \gtrsim g(\phi) \gtrsim (1-\phi_1^2)^2\phi_2^4\phi_3^6.$$

Hence we have

$$|\phi_1 - \mathrm{sgn}(\langle\psi_2, \tilde{\psi}_2\rangle) \cdot \tilde{\phi}_1| \lesssim \frac{\eta}{\phi_2^2\phi_3^3}.$$

Bounding ω_2 We rewrite,

$$\begin{aligned}\phi_2 - \tilde{\phi}_2 &= \frac{m_2(\phi)}{m_1(\phi)} - \frac{m_2(\phi) + \Delta_2}{m_1(\phi) + \Delta_1} \\ &= \frac{m_2(\phi)(m_1(\phi) + \Delta_1) - (m_2(\phi) + \Delta_2)m_1(\phi)}{m_1(\phi)(m_1(\phi) + \Delta_1)}.\end{aligned}$$

Hence,

$$\phi_2 - \tilde{\phi}_2 = \frac{\Delta_1 m_2(\phi) - \Delta_2 m_1(\phi)}{m_1(\phi)(m_1(\phi) + \Delta_1)}.$$

Under the assumptions of the theorem, we have that $\eta \leq m_1(\phi)/2$, and thus

$$\begin{aligned}|\phi_2 - \tilde{\phi}_2| &\leq \frac{2\eta(m_2(\phi) + |m_1(\phi)|)}{m_1(\phi)^2} \\ &\leq \frac{4\eta}{|m_1(\phi)|} \\ &= \frac{16\eta}{(1-\phi_1^2)|\phi_2|\phi_3^2}.\end{aligned}$$

Bounding ω_3 We rewrite,

$$\begin{aligned}\phi_3 - \tilde{\phi}_3 &= \frac{\sqrt{g(\phi)}}{m_2(\phi)} - \frac{\sqrt{g(\phi)+h_\phi(\Delta)}}{m_2(\phi)+\Delta_2} \\ &= \frac{m_2(\phi)(\sqrt{g(\phi)} - \sqrt{g(\phi)+h_\phi(\Delta)})}{m_2(\phi)(m_2(\phi)+\Delta_2)} + \frac{\Delta_2\sqrt{g(\phi)}}{m_2(\phi)(m_2(\phi)+\Delta_2)} \\ &= \frac{-h_\phi(\Delta)}{(m_2(\phi)+\Delta_2)(\sqrt{g(\phi)+h_\phi(\Delta)}+\sqrt{g(\phi)})} + \frac{\Delta_2\phi_3}{m_2(\phi)+\Delta_2} \\ &= \frac{-h_\phi(\Delta) + \Delta_2\phi_3(\sqrt{g(\phi)+h_\phi(\Delta)}+\sqrt{g(\phi)})}{(m_2(\phi)+\Delta_2)(\sqrt{g(\phi)+h_\phi(\Delta)}+\sqrt{g(\phi)})}\end{aligned}$$

Let us call the numerator of the fraction on the right of the last display N, and the denominator D. We further decompose $h_\phi(\Delta)$ as $h_\phi(\Delta) = \xi_\phi(\Delta) + \gamma_\phi(\Delta)$, where $\gamma_\phi(\Delta) \coloneqq 4m_1(\phi)^2\Delta_2$. We see that

$$\begin{aligned}N &= -\xi_\phi(\phi) - 4m_1(\phi)^2\Delta_2 + \phi_3\Delta_2((g+h)^{1/2} + g^{1/2}) \\ &= -\xi_\phi(\phi) - 4m_1(\phi)^2\Delta_2 + \phi_3\Delta_2\{(g+h)^{1/2} + g^{1/2}\} \\ &= -\xi_\phi(\phi) - 4m_1(\phi)^2\Delta_2 + 2\phi_3\Delta_2 g^{1/2} + \phi_3\Delta_2\{(g+h)^{1/2} - g^{1/2}\} \\ &= -\xi_\phi(\phi) + \Delta_2(1+\phi_1^2)\phi_3 g^{1/2} + \frac{\phi_3\Delta_2 h}{(g+h)^{1/2}+g^{1/2}},\end{aligned}$$

where the last line follows because $m_1(\phi)^2 = \frac{1}{4}(1-\phi_1^2)\phi_3 g^{1/2}$. Since $m_2(\phi) \geq 0$, we see that $\xi_\phi(\Delta)$ has maximal amplitude when $\Delta_1 = \operatorname{sgn}(m_1(\phi))\eta$ and when $\Delta_2 = \eta$, in which case we have

$$\begin{aligned}|\xi_\phi(\Delta)| &= 8|m_1(\phi)|m_2(\phi)\eta + 8|m_1(\phi)|\eta^2 + 4m_2(\phi)\eta^2 + 4\eta^3 + 2|m_3(\phi)|\eta + \eta^2 \\ &\lesssim \{|m_1(\phi)|m_2(\phi) + 2|m_3(\phi)|\}\eta + \eta^2 \\ &\lesssim (1-\phi_1^2)\phi_2^2\phi_3^3\max\{(1-\phi_1^2)|\phi_2|\phi_3,\, |\phi_1|\}\eta + \eta^2,\end{aligned}$$

where the second line follows because under the assumptions of the proposition we have that $m_2(\phi) \lesssim |m_1(\phi)|$ and $\eta \leq m_2(\phi)/2$ (note that $\phi_3 \leq K^{1/2}$). Since $h_\phi(\Delta) = \xi_\phi(\Delta) + 4m_1(\phi)^2\Delta_2$, we also have

$$|h_\phi(\Delta)| \lesssim (1-\phi_1^2)\phi_2^2\phi_3^3\max\{(1-\phi_1^2)\phi_3,\, |\phi_1|\}\eta + \eta^2,$$

Hence,

$$\begin{aligned}|N| &\lesssim (1-\phi_1^2)\phi_2^2\phi_3^3\max\{(1-\phi_1^2)|\phi_2|\phi_3,\, |\phi_1|\}\eta + \eta^2 \\ &\quad + \eta\phi_3 g^{1/2} + \frac{\eta^2\phi_3(1-\phi_1^2)\phi_2^2\phi_3^3\max\{(1-\phi_1^2)\phi_3, |\phi_1|\} + \eta^3\phi_3}{g^{1/2}} \\ &\lesssim (1-\phi_1^2)\phi_2^2\phi_3^3\max\{(1-\phi_1^2)|\phi_2|\phi_3,\, |\phi_1|\}\eta + \eta^2 \\ &\quad + \eta\phi_3 g^{1/2} + \eta^2\phi_3\max\{(1-\phi_1^2)\phi_3, |\phi_1|\} + \frac{\eta^3}{(1-\phi_1^2)\phi_2^2\phi_3^2}\end{aligned}$$

But by assumption $\eta \lesssim (1-\phi_1^2)\phi_2^2\phi_3^3$, and $4g^{1/2} = (1-\phi_1^2)\phi_2^2\phi_3^2$, thus

$$|N| \lesssim (1-\phi_1^2)\phi_2^2\phi_3^3\max\{\phi_3,\, |\phi_1|\}\eta + \eta^2.$$

Note that $\max(\phi_3, |\phi|_1) \le \sqrt{K}$. Moreover, under the assumptions of the proposition and using that $\phi_3 \le \sqrt{K}$, it is the case that $|\Delta_2| \le \eta \lesssim m_2(\phi)$. Therefore $|D| \gtrsim m_2(\phi)\sqrt{g(\phi)}$, and

$$|\phi_3 - \tilde{\phi}_3| \lesssim \frac{\eta}{(1-\phi_1^2)\phi_2^2\phi_3^2} + \frac{\eta^2}{(1-\phi_1^2)^2\phi_2^4\phi_3^5}.$$

Finally, since have assumed that $\eta < \frac{(1-\phi_1^2)\phi_2^2\phi_3^2}{8}$, we see that the second term is at most a constant times the first, so that it can be absorbed by increasing the constant C. $\square$

8.4 Proof of Theorem 3

We give a standard two-point testing lower bound, summarising ideas that can be found for example in Chapter 2 of [40].

Lemma 2. *Given data $X^{(n)} \sim p_u^{(n)}$ for parameter $u \in \mathcal{U}$, the following lower bounds hold for estimating u.*

Suppose $\mathcal{U} \subseteq \mathbb{R}$ and for some $r \le 1/2$ assume that there exist parameters u_0, u_1 satisfying

i. $|u_1/u_0 - 1| \ge 4r$,

ii. $K(p_{u_1}^{(n)}; p_{u_0}^{(n)}) \le 1/100$.

where we recall K denotes the Kullback–Leibler divergence. Then

$$\inf_{\hat{u}} \sup_{u \in \mathcal{U}} \mathbb{P}_u(|\hat{u}/u - 1| \ge r) \ge 1/4,$$

where the infimum is over all estimators $\hat{u}$ based on the data $X^{(n)}$.

If instead $(\mathcal{U}, d)$ is a pseudo-metric space and for some $r \ge 0$ there exist parameters u_0, u_1 satisfying

i. $d(u_0, u_1) \ge 2r$

ii. $K(p_{u_1}^{(n)}, p_{u_0}^{(n)}) \le 1/100$,

then

$$\inf_{\hat{u}} \sup_{u \in \mathcal{U}} \mathbb{P}_u(d(\hat{u}, u) \ge r) \ge 1/4.$$

Proof. Given an estimator $\hat{u}$ we may construct a test T of $u = u_0$ vs $u = u_1$,

$$T = \mathbb{1}\Big\{\Big|\frac{\hat{u}}{u_0} - 1\Big| > \Big|\frac{\hat{u}}{u_1} - 1\Big|\Big\}.$$

Observe that

$$\begin{aligned}\Big|\frac{\hat{u}}{u_0} - 1\Big| &= \Big|\frac{u_1}{u_0} - 1 + \frac{\hat{u} - u_1}{u_1}\frac{u_1}{u_0}\Big| \\ &\ge 4r - \Big|\frac{\hat{u}}{u_1} - 1\Big|(1 + 4r).\end{aligned}$$

Then

$$\begin{aligned}\mathbb{P}_{u_1}(T=0) &= \mathbb{P}_{u_1}\Big(\Big|\frac{\hat{u}}{u_0}-1\Big| \leq \Big|\frac{\hat{u}}{u_1}-1\Big|\Big)\\ &\leq \mathbb{P}_{u_1}\Big(4r - \big|\frac{\hat{u}}{u_1}-1\big|(1+4r) \leq \Big|\frac{\hat{u}}{u_1}-1\Big|\Big)\\ &\leq \mathbb{P}_{u_1}\Big(\Big|\frac{\hat{u}}{u_1}-1\Big| \geq r\Big),\end{aligned}$$

where for the last line we have used that $4r/(2+4r) \geq r$ for $r \leq 1/2$. Also note that on the event $\{T=1\} \cap \{|\hat{u}/u_0 - 1| < r\}$ we have also $|\hat{u}/u_1 - 1| < r$ and hence

$$\begin{aligned}|u_1/u_0 - 1| &= |\hat{u}/u_0 - 1 - (\hat{u}/u_1 - 1) - (\hat{u}/u_1 - 1)(u_1/u_0 - 1)|\\ &< 2r + r|u_1/u_0 - 1|,\end{aligned}$$

so that $|u_1/u_0 - 1| < 2r/(1-r)$ on this event. Having assumed $r \leq 1/2$ and $|u_1/u_0 - 1| \geq 4r$ we deduce that $\{T=1\} \cap \{|\hat{u}/u_0 - 1| < r\} = \emptyset$ so that $\{T=1\} \subseteq \{|\hat{u}/u_0 - 1| \geq r\}$, and hence we have shown

$$\inf_{\hat{u}} \sup_{u} \mathbb{P}_u\Big(\Big|\frac{\hat{u}}{u}-1\Big| \geq r\Big) \geq \inf_{\hat{u}} \max_{i=0,1} \mathbb{P}_{u_i}\Big(\Big|\frac{\hat{u}}{u_i}-1\Big| \geq r\Big) \geq \inf_{T} \max_{i=0,1} \mathbb{P}_{u_i}(T \neq i),$$

where the latter infimum is over all tests T. In the pseudo-metric case a reduction considering the test $T = \mathbb{1}\{d(\hat{u}, u_0) > d(\hat{u}, u_1)\}$ and directly using the triangle inequality likewise yields

$$\inf_{\hat{u}} \sup_{u} \mathbb{P}_u\big(d(\hat{u}, u) \geq r\big) \geq \inf_{T} \max_{i=0,1} \mathbb{P}_{u_i}(T \neq i).$$

It remains to lower bound the maximum probability of testing error by $1/4$. Introducing the event $A = \big\{\frac{p_{u_0}^{(n)}}{p_{u_1}^{(n)}} \geq 1/2\big\}$, we see

$$\mathbb{P}_{u_0}(T \neq 0) \geq \mathbb{E}_{u_1}\big[\tfrac{p_{u_0}^{(n)}}{p_{u_1}^{(n)}} \mathbb{1}_A T\big] \geq \tfrac{1}{2}[\mathbb{P}_{u_1}(T=1) - \mathbb{P}_{u_1}(A^c)]$$

Thus, writing $p_1 = \mathbb{P}_{u_1}(T=1)$, we see

$$\begin{aligned}\max(\mathbb{P}_{u_0}(T \neq 0), \mathbb{P}_{u_1}(T \neq 1)) &\geq \max(\tfrac{1}{2}(p_1 - \mathbb{P}_{u_1}(A^c)), 1 - p_1)\\ &\geq \inf_{p \in [0,1]} \max(\tfrac{1}{2}(p - \mathbb{P}_{u_1}(A^c)), 1 - p).\end{aligned}$$

The infimum is attained when $\frac{1}{2}(p - \mathbb{P}_{u_1}(A^c)) = 1 - p$ and takes the value $\frac{1}{3}\mathbb{P}_{u_1}(A)$ so that

$$\inf_{T} \max_{i=0,1} \mathbb{P}_{u_i}(T \neq i) \geq \tfrac{1}{3}\mathbb{P}_{u_1}(A).$$

Next observe

$$\begin{aligned}\mathbb{P}_{u_1}(A) = \mathbb{P}_{u_1}\big[\tfrac{p_{u_1}^{(n)}}{p_{u_0}^{(n)}} \leq 2\big] &= 1 - \mathbb{P}_{u_1}^n\big[\log\big(\tfrac{p_{u_1}^{(n)}}{p_{u_0}^{(n)}}\big) > \log 2\big] \geq 1 - \mathbb{P}_{\theta_1}^n\big[\big|\log\big(\tfrac{p_{u_1}^{(n)}}{p_{u_0}^{(n)}}\big)\big| > \log 2\big]\\ &\geq 1 - (\log 2)^{-1}\mathbb{E}_{u_1}\big|\log\big(\tfrac{p_{u_1}^{(n)}}{p_{u_0}^{(n)}}\big)\big|,\end{aligned}$$

where we have used Markov's inequality to attain the final expression. By the second Pinsker inequality (e.g. Proposition 6.1.7b in [22]), using the upper bound on the Kullback–Leibler divergence we can continue the chain of inequalities to see

$$\mathbb{P}_{u_1}(A) \geq 1 - (\log 2)^{-1}\big[K(p_{u_1}^{(n)}, p_{u_0}^{(n)}) + \sqrt{2K(p_{u_1}^{(n)}, p_{u_0}^{(n)})}\big] \geq 1 - (\log 2)^{-1}(\mu + \sqrt{2\mu}).$$

For any $c < 1/3$, we may choose $\mu = \mu(c)$ small enough that the testing error satisfies

$$\inf_T \max_{i=0,1} \mathbb{P}_{u_i}(T \neq i) \geq \tfrac{1}{3}\Big(1 - \frac{\mu + \sqrt{2\mu}}{\log 2}\Big) > c,$$

and in particular a numerical calculation shows that $\mu = 1 + \frac{1}{4}\log 2 - \sqrt{1 + \frac{1}{2}\log 2} > 1/100$ works for $c = 1/4$. $\square$

In view of Proposition 2, for any $(\phi, \psi), (\tilde{\phi}, \tilde{\psi}) \in \Phi$ corresponding to strictly positive emission densities, we have for ϕ_2 and $\tilde{\phi}_2$ small enough that

$$K(p_{\phi,\psi}^{(n)}, p_{\tilde{\phi},\tilde{\psi}}^{(n)}) \leq Cn\rho(\phi, \psi; \tilde{\phi}, \tilde{\psi})^2,$$

where $C > 0$ is a constant depending only on K and a lower bound for the emission densities. We remark that for all the hypotheses we will exhibit below, we will have that ϕ_2 and $\tilde{\phi}_2$ are of order ϵ, which is upper bounded by ϵ_1 by assumption, so that choosing the latter small enough the above bound on $K(p_{\phi,\psi}^{(n)}, p_{\tilde{\phi},\tilde{\psi}}^{(n)})$ will apply. Then, to prove Item (1), it suffices to apply Lemma 2 to $u = 1 - \phi_1^2$ and prove the existence of parameters $(\phi, \psi), (\tilde{\phi}, \tilde{\psi}) \in \Phi_L(\delta, \epsilon, \zeta)$ satisfying for small enough $c_1 > 0$ and some $c_2 > 0$

$$\rho(\phi, \psi; \tilde{\phi}, \tilde{\psi}) \leq c_1/\sqrt{n}, \quad \text{and} \quad \Big|\frac{1 - \tilde{\phi}_1^2}{1 - \phi_1^2} - 1\Big| \geq c_2/\sqrt{n\delta^2\epsilon^4\zeta^6} \tag{37}$$

which will give the lower bound on the absolute risk. Regarding the relative risk, we then note that for any $a \geq 0$, since $|\phi_1| \leq 1$ and $1 - \phi_1^2 \geq \delta$, so that we may assume the same of $\hat{\phi}_1$, we have

$$\begin{aligned}\mathbb{P}_{\phi,\psi}(\min(|\hat{\phi}_1 - \phi_1|, |\hat{\phi}_1 + \phi_1|) \geq a) &\geq \mathbb{P}_{\phi,\psi}(|(1 - \hat{\phi}_1^2) - (1 - \phi_1^2)| \geq 2a) \\ &\geq \mathbb{P}_{\phi,\psi}(|(1 - \hat{\phi}_1^2)/(1 - \phi_1^2) - 1| \geq 2a/\delta).\end{aligned}$$

(See also equation (36) for a similar calculation with ϕ_2.)

Similar conditions to (37) suffice for proving the other parts of Theorem 3 and we proceed now to verifying the existence of suitable parameters (ϕ, ψ) and $(\tilde{\phi}, \tilde{\psi})$, with the help of the following lemma.

Lemma 3. *For a given ϕ, assume conditions (6) and (7) and assume that $\phi_3 \leq \sqrt{2\lfloor K/2 \rfloor}/(2K)$. Then there exists ψ such that (ϕ, ψ) lies in Φ_L and the corresponding emission densities f_0, f_1 are bounded below by some constant $c = c(K) > 0$.*

In particular, for $|\phi_1| \leq 1 - 3\delta$, $\epsilon \leq \phi_2 \leq \min(1/3, 1 - L)$, $\zeta \leq \phi_3 \leq 2\zeta$, then such a ψ exists under the compatibility condition (3).

Proof. Take

$$\psi_1(k) = 1/K, \quad \psi_2(k) = (2\lfloor K/2 \rfloor)^{-1/2}(\mathbb{1}\{k \text{ odd}, k < K\} - \mathbb{1}\{k \text{ even}\}), \quad k \leq K.$$

Under the assumed condition on ϕ_3 and recalling that $|\phi_1| \leq 1$ by assumption, we observe from the expressions for f_0, f_1 given in Remark 4 that these are lower bounded by $1/(2K)$. In the particular case, one simply notes that all the conditions hold for such ϕ. $\square$

Proof of Items (1) and (3) We prove the lower bounds for estimating ϕ_1 and ϕ_3 together. For some small constant $c > 0$, set $R = c\epsilon^{-2}\zeta^{-3}n^{-1/2}$ and, writing $S = (2 - 6\delta - R)R/(6\delta - 9\delta^2)$, set

$$\phi = (1 - 3\delta, \epsilon, \zeta\sqrt{1+S}),$$
$$\tilde{\phi} = (1 - 3\delta - R, \epsilon, \zeta).$$

Recalling the definition $r(\phi) = (1 - \phi_1^2)\phi_2\phi_3^2/4$, the choice of ϕ_3 ensures that $r(\phi) = r(\tilde{\phi})$, and we note that under the assumptions of the theorem we have $R \leq \delta \leq 1/6$ so that $S \leq R/\delta \leq 1$ and $\zeta \leq \phi_3 \leq 2\zeta$. By Lemma 3 there exists ψ such that $(\phi, \psi), (\tilde{\phi}, \psi) \in \Phi_L$ and for this ψ we see that

$$\rho(\phi, \psi; \tilde{\phi}, \psi) = |\phi_1\phi_2\phi_3 r(\phi) - \tilde{\phi}_1\tilde{\phi}_2\tilde{\phi}_3 r(\tilde{\phi})| = \phi_2 r(\phi)|\phi_1\phi_3 - \tilde{\phi}_1\tilde{\phi}_3|.$$

Using that $\sqrt{1+t} \leq 1 + t$ for $t \geq 0$ we have

$$|\phi_1\phi_3 - \tilde{\phi}_1\tilde{\phi}_3| = (1 - 3\delta)\zeta(\sqrt{1+S} - 1) + R\zeta \leq (S + R)\zeta \leq 2R\zeta/\delta,$$

hence since $r(\phi) = (6\delta - 9\delta^2)\epsilon\zeta^2(1 + S)/4 \leq 3\delta\epsilon\zeta^2$, we obtain

$$\rho(\phi, \psi; \tilde{\phi}, \tilde{\psi}) \leq 6\epsilon^2\zeta^3 R \leq 6cn^{-1/2}.$$

Recalling that $R \leq \delta \leq 1/6$ and that $r(\phi) = r(\tilde{\phi})$ one calculates

$$\frac{1 - \tilde{\phi}_1^2}{1 - \phi_1^2} - 1 = S \geq R/(12\delta).$$

For c small enough we see that the conditions in equation (37) are satisfied, yielding the claimed bound for estimating ϕ_1.

To prove the lower bound for estimating ϕ_3 it suffices to lower bound $|\phi_3/\tilde{\phi}_3 - 1|$. Here we use the bound $\sqrt{1+x} - 1 \geq x/(2\sqrt{1+x}) \geq x/(2\sqrt{2})$ for $0 \leq x \leq 1$ to see for a constant $c' > 0$ that

$$|\phi_3/\tilde{\phi}_3 - 1| \geq c'R/\delta.$$

The bound for ϕ_3 follows from applying Lemma 2.

Proof of Item (2) For a constant $c > 0$, define $R = c\delta^{-1}\epsilon^{-1}\zeta^{-2}n^{-1/2}$, define $\phi, \tilde{\phi}$ by

$$\phi = (1 - 3\delta, \epsilon, \zeta(1 + R/\epsilon)^{1/2})$$
$$\tilde{\phi} = (1 - 3\delta, \epsilon + R, \zeta),$$

and observe that by construction $r(\phi) = r(\tilde{\phi})$. Noting that $\phi_2 \leq 2\epsilon \leq 1 - L$ and $\phi_3 \leq 2\zeta$ because the assumptions of Theorem 3 ensure that $R \leq \epsilon \leq 1/3$, we deduce using Lemma 3 that there exists some $\psi = \tilde{\psi}$ such that $(\phi, \psi), (\tilde{\phi}, \tilde{\psi}) \in \Phi_L(\delta, \epsilon, \zeta)$.

Next observe, using that $(1 + x)^{1/2} \leq 1 + x$,

$$\phi_1|\phi_2\phi_3 - \tilde{\phi}_2\tilde{\phi}_3| \leq |\phi_2||\phi_3 - \tilde{\phi}_3| + |\tilde{\phi}_3||\phi_2 - \tilde{\phi}_2| = \epsilon\zeta(\sqrt{1 + R/\epsilon} - 1) + \zeta R \leq 2\zeta R \leq R,$$

the last inequality following from the fact that under the compatibility condition (3) we have $\zeta \leq 1/2$. We deduce

$$\rho(\phi, \psi; \tilde{\phi}, \tilde{\psi}) = r(\phi)\max(|\phi_2 - \tilde{\phi}_2|, |\phi_1\phi_2\phi_3 - \tilde{\phi}_1\tilde{\phi}_2\tilde{\phi}_3|) = Rr(\phi).$$

Again using that $\phi_3 \leq 2\zeta$ and noting also that $(1-\phi_1^2) = 6\delta - 9\delta^2 \leq 6\delta$, we see that for some $C' > 0$ we have

$$\rho(\phi, \psi; \tilde{\phi}, \tilde{\psi}) \leq C'\delta\epsilon\zeta^2 R \leq cC' n^{-1/2}.$$

As with Items (1) and (3), for c small enough in the definition of R we may apply Lemma 2 to deduce the claimed lower bound since $|\tilde{\phi}_2/\phi_2 - 1| = R/\epsilon$.

Proof of Item (4) Set $\phi = \tilde{\phi} = (0, \epsilon, \zeta)$, set, as in Lemma 3,

$$\psi_1(k) = 1/K, \quad \psi_2(k) = (2\lfloor K/2 \rfloor)^{-1/2}(\mathbb{1}\{k \text{ odd}, k < K\} - \mathbb{1}\{k \text{ even}\}), \quad k \leq K,$$

and define $\tilde{\psi}_1 = \psi_1 + cn^{-1/2}\psi_2$. Note that under the compatibility condition (3) we have $(\phi, \psi), (\tilde{\phi}, \tilde{\psi}) \in \Phi_L$ for n larger than some $C = C(K, c)$, or for all $n \geq 1$ if c is small enough. Then

$$\rho(\phi, \psi; \tilde{\phi}, \tilde{\psi}) = cn^{-1/2}$$

and we apply Lemma 2 to deduce the result.

Proof of Item (5) Set $\phi = \tilde{\phi} = (1-3\delta, \epsilon, \zeta)$, choose $\psi_1 = \tilde{\psi}_1$ to be the uniform density on $\{1, \dots, K\}$. As with the previous parts, an application of Lemma 2 will yield the theorem if we can exhibit $\psi_2, \tilde{\psi}_2$ such that the induced emission densities are bounded below by some $c' = c'(K) > 0$, $\|\psi_2 - \tilde{\psi}_2\| = R := c(n\delta^2\epsilon^2\zeta^4)^{-1/2}$ for some $c > 0$, $\operatorname{sgn}(\langle \psi_2, \tilde{\psi}_2 \rangle) = +1$, and $\rho(\phi, \psi; \tilde{\phi}, \tilde{\psi}) \leq c_1 n^{-1/2}$ for a small constant c_1. Such a choice is possible, for a small enough constant c, under the compatibility condition (3) and for $n\delta^2\epsilon^2\zeta^4 \geq 1$; for example, define ψ_2 as in Lemma 3 by

$$\psi_2(k) = (2\lfloor K/2 \rfloor)^{-1/2}(\mathbb{1}\{k \text{ odd}, k < K\} - \mathbb{1}\{k \text{ even}\}), \quad k \leq K,$$

and, for h defined by $h(1) = 2^{-1/2}$, $h(3) = -2^{-1/2}$ and $h(k) = 0$ for all other k, define

$$\tilde{\psi}_2 = (\psi_2 + \alpha h)/(1+\alpha), \quad \alpha = R/(2-R).$$

This satisfies $\|\tilde{\psi}_2 - \psi_2\| = R$, $\|\tilde{\psi}_2\| = 1$, $\langle \tilde{\psi}_2, 1 \rangle = 0$ and $\langle \tilde{\psi}_2, \psi_2 \rangle \geq 0$. For $k \notin \{1, 3\}$ the condition (8) of Remark 5 holds with $1/(2K)$ in place of 0 on the right, and for $k \in \{1, 3\}$ a direct calculation shows that the condition with $1/(4K)$ in on the right if R is upper bounded by some $c' = c'(K)$,which is the case for $c = c(K)$ sufficiently small. Then

$$\rho(\phi, \psi; \tilde{\phi}, \tilde{\psi}) = |r(\phi)|\|\psi_2 - \tilde{\psi}_2\| \leq \delta\epsilon\zeta^2 R \leq cn^{-1/2}.$$

8.5 Proofs for Section 3

Proof of Theorem 1. We begin with the upper bounds. From the inversion formulae in Remark 4 we have

$$\max(\|\hat{f}_0 - f_0\|, \|\hat{f}_1 - f_1\|) \leq \|\hat{\psi}_1 - \psi_1\| + \tfrac{1}{2}\|\hat{\phi}_1\hat{\phi}_3\hat{\psi}_2 - \phi_1\phi_3\psi_2\| + \tfrac{1}{2}\|\hat{\phi}_3\hat{\psi}_2 - \phi_3\psi_2\|$$

Recalling that $|\hat{\phi}_1| \leq 1$, that $0 \leq \phi_3 \leq K^{1/2}$ and that $\|\psi_2\| = \|\hat{\psi}_2\| = 1$, we decompose the second term on the right, with an implicit decomposition of the third term included:

$$\begin{aligned}
\|\hat{\phi}_1\hat{\phi}_3\hat{\psi}_2 - \phi_1\phi_3\psi_2\| &\leq |\hat{\phi}_1|\|\hat{\phi}_3\hat{\psi}_2 - \phi_3\psi_2\| + |\phi_3||\hat{\phi}_1 - \phi_1| \\
&\leq |\hat{\phi}_3 - \phi_3| + K^{1/2}\|\hat{\psi}_2 - \psi_2\| + \phi_3|\hat{\phi}_1 - \phi_1|.
\end{aligned}$$

It follows that for some constant C we have

$$\max(\|\hat{f}_0 - f_0\|, \|\hat{f}_1 - f_1\|) \leq C \max(\|\hat{\psi}_1 - \psi_1\|, \|\hat{\psi}_2 - \psi_2\|, |\hat{\phi}_3 - \phi_3|, \phi_3|\hat{\phi}_1 - \phi_1|).$$

Applying Proposition 3 as in the proof of Theorem 2, one can show that for some $C > 0$

$$\mathbb{P}_{\phi,\psi}\Big(\phi_3^2|\hat{\phi}_1 - \phi_1|^2 \geq \frac{Cx^2}{n\epsilon^4\zeta^4}\Big) \leq e^{-x^2}.$$

The upper bounds for estimating f_0 and f_1 then follow from Theorem 2.

Similarly, Remark 4 and the fact that $|\phi_2| \leq 1$ give

$$\max(|\hat{p} - p|, |\hat{q} - q|) \leq \tfrac{1}{2}(1 + |\hat{\phi}_1|)|\hat{\phi}_2 - \phi_2| + \tfrac{1}{2}|\hat{\phi}_1 - \phi_1||1 - \phi_2| \leq 2\max(|\hat{\phi}_1 - \phi_1|, |\hat{\phi}_2 - \phi_2|).$$

The upper bounds then again follow from Theorem 2.

For the lower bounds, writing $\theta(\phi,\psi) = (p, q, f_0, f_1)$ and $\theta(\tilde{\phi},\tilde{\psi}) = (\tilde{p}, \tilde{q}, \tilde{f}_0, \tilde{f}_1)$, observe by Lemma 2 that it suffices to lower bound $\max(|p - \tilde{p}|, |q - \tilde{q}|)$ and $\max(\|f_0 - \tilde{f}_0\|, \|f_1 - \tilde{f}_1\|)$ corresponding to choices of $(\phi,\psi), (\tilde{\phi},\tilde{\psi})$ made in the proof of Theorem 3.

From the inversion formulae in Remark 4 we calculate, for any $\phi, \tilde{\phi}$,

$$\begin{aligned} &2\max(|p - \tilde{p}|, |q - \tilde{q}|) \\ &\quad \geq \max((1 + |\phi_1|)|\phi_2 - \tilde{\phi}_2| - |1 - \tilde{\phi}_2||\phi_1 - \tilde{\phi}_1|, |\phi_1 - \tilde{\phi}_1||1 - \tilde{\phi}_2| - (1 - |\phi_1|)|\phi_2 - \tilde{\phi}_2|). \end{aligned} \tag{38}$$

If $\delta > \epsilon\zeta$ set $\phi = (1 - 3\delta, \epsilon, \zeta(1 + S)^{1/2})$ and $\tilde{\phi} = (1 - 3\delta - R, \epsilon, \zeta)$, where $R = c(n\epsilon^4\zeta^6)^{-1/2}$ for some $c > 0$ and where $S \in [R/(12\delta), R/\delta]$ is, as in the proof of Theorem 3 Item (1), such that $r(\phi) = r(\tilde{\phi})$. If $\delta \leq \epsilon\zeta$ instead set $\phi = (1 - 3\delta, \epsilon, \zeta(1 + R/\epsilon)^{1/2})$, $\tilde{\phi} = (1 - 3\delta, \epsilon + R, \zeta)$ with $R = c(n\epsilon^2\delta^2\zeta^4)^{-1/2}$. In either case the proof of Theorem 3 demonstrates that for suitable $\psi = \tilde{\psi}$ we have $K(p^{(n)}_{\phi,\psi}, p^{(n)}_{\tilde{\phi},\tilde{\psi}}) \leq 1/100$ for c small enough hence by Lemma 2

$$\inf_{\check{\theta}} \sup_{\theta\in\Theta} \mathbb{P}_\theta\Big(\max(|\check{p} - p|, |\check{q} - q|) > c' \max(|p - \tilde{p}|, |q - \tilde{q}|)\Big) \geq 1/4.$$

Inserting from equation (38) we conclude the bound in either case.

For (f_0, f_1), again set $\phi = (1 - 3\delta, \epsilon, \zeta(1 + S)^{1/2})$, $\tilde{\phi} = (1 - 3\delta - R, \epsilon, \zeta)$ where $R = c\epsilon^{-2}\zeta^{-3}n^{-1/2}$, and choose $\psi = \tilde{\psi}$ by Lemma 3. As with p and q we deduce that for some $c' > 0$ we have

$$\inf_{\check{\theta}} \sup_{\theta\in\Theta} \mathbb{P}_\theta\Big(\max(\|\check{f}_0 - f_0\|, \|\check{f}_1 - f_1\|) > c' \max(\|f_0 - \tilde{f}_0\|, \|f_1 - \tilde{f}_1\|)\Big) \geq 1/4.$$

Using the inversion formulae in Remark 4 and the fact that $\psi = \tilde{\psi}$ and $\|\psi_2\| = 1$, one calculates

$$2\max(\|f_0 - \tilde{f}_0\|, \|f_1 - \tilde{f}_1\|) = |\phi_1\phi_3 - \tilde{\phi}_1\tilde{\phi}_3| + |\phi_3 - \tilde{\phi}_3| \geq |\phi_3 - \tilde{\phi}_3|$$

For the current choice of $\phi, \tilde{\phi}$, calculating as in proving Theorem 3 Item (3), we have $|\phi_3 - \tilde{\phi}_3| \geq C\zeta R/\delta$ for some $C > 0$ and we deduce the lower bound. □

Proof of Corollary 1. It suffices to substitute $\alpha = e^{-x^2}$ into Theorem 1 and solve for error equal to E, while ensuring that $x^2 = \log(1/\alpha)$ is suitably bounded. □

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