MAIN PROPERTIES OF BURGERS TURBULENCE WITH WHITE NOISE INITIAL CONDITIONS

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This paper intends to review the main properties of the solutions of Burgers equation with random initial conditions of white noise type. These properties are closely related to those of the convex hull of a Brownian motion with parabolic drift. A special attention is given to the latter.

1. Introduction

This text aims to survey the main properties of the solutions of the one-dimensional Burgers equation

$$\partial_t u + u\,\partial_x u = 0 \tag{1}$$

with initial condition of white noise[a] type. Burgers introduced in the early 40's this equation in its multidimensional form $\partial_t u + u \cdot \nabla u = 0$ to obtain a simplified model for hydrodynamic turbulence. It is known nowadays that it does not provide an accurate model for hydrodynamic turbulence; see Kraichnan[20] for a discussion on the similarities and the differences with Navier-Stokes equation. Yet, Burgers equation appears in many fields of mathematical physics, such as the formation of the large scale structures of the universe or the dynamic of growing surfaces, see e.g. Woyczynski[26].

The study of the solution of Burgers equation (1) with white noise initial condition takes place in the field of the analysis of solutions of PDE with random initial data. If we think to the phenomenum of turbulence, it seems interesting to exhibit the statistical properties of the solutions of some PDE of fluid mechanics, with random and chaotic initial conditions.

[a]A white noise is the derivative, in the sense of distribution, of a Brownian motion.

Such studies also appear in astrophysics, when one considers the formation of the structures of the universe. Solutions of Burgers equation with random Gaussian initial data seems to be in this case of particular interest, see Vergassolla et *al* [25] for an up-to-date survey. Anyway, the analysis of Burgers turbulence may be viewed as a first step for depicting the solutions of more complicated PDE with random initial data.

The choice of the white noise as initial condition stems from the fact that it appears as a natural model for chaos. Some others initial conditions have yet been also considered. We refer to Bertoin for the analysis of the Brownian case[4] and a survey on the stable noise case[5], and to Leonenko[21] and Woyczynski[26] for other cases. The white noise initial data also arises naturally in statistical physics. Consider a time $t = 0$ particles of mass 1 spread on a regular lattice, say $\mathbb{Z}$, with random initial velocities independent and identically distributed (i.i.d.) with centered law of finite variance. Next, let the system evolves according to the dynamic of free sticky particles: between collisions particles move at constant speed, and when some of them meet, they merge into a new particle, whose mass and momentum are given by the sum of the masses and momenta of the particles involved into the collision. Then, the velocity field of the hydrodynamic limit of such a system of ballistic aggregation is solution to Burgers equation with white noise initial condition; see [11] and also next section for further explanations.

Investigating solutions of Burgers equation with random initial data can lead to interesting problems in probability theory. Indeed, according to the celebrated Hopf-Cole formula the solution $u(\cdot,t)$ of (1) at time t can be expressed in terms of the convex hull of

$$z \mapsto \int_0^z u(x,0)\,dx + \frac{1}{2t}z^2.$$

In the case of a white noise initial condition $u(\cdot,0)$, the analysis of u thus requires a deep analysis of the convex hull of a Brownian motion with parabolic drift, mainly based on the work of Groeneboom[18]. See Section 3 for a survey of this analysis. Some interesting connections with the coalescence and the fragmentation have also to be mentioned, see Bertoin[4].

The rest of the paper intends to review the main properties of the solutions of Burgers equation (1) with initial condition of white noise type. Section 2 recalls necessary background on Burgers equation (with deterministic initial condition). In Section 3, various results on the convex hull of a Brownian motion with parabolic drift are collected. Even if they do not seems to have anything to do with Burgers turbulence, they are the key for the understanding of the proofs of the next sections. In Section 4 the main properties of the solution of (1) with white noise initial condition are depicted. A particular attention is given to its time-evolution. In Section 5, some other types of white noise initial condition are presented. Section 6 concludes with few open problems.

2. Some background on Burgers equation

The purpose of this section is to present some standard features on solutions of Burgers equation (1). We refer to [10,11,19] for proofs.

Even for very smooth initial conditions, solutions of Burgers equation (1) may develop shocks (discontinuities) at finite time. We then loose the existence of strong solutions, as well as the uniqueness of weak solutions. We will focus henceforth on a special (weak) solution of (1), so-called entropy solution, since it is the unique solution of (1) fulfilling some entropy conditions, see [8]. This solution can be obtained in adding a vanishing viscosity term to equation (1). More precisely, when $\varepsilon \to 0+$ the unique strong solution u_ε of

$$\partial_t u + u\,\partial_x u = \varepsilon\,\partial^2_{xx} u$$

converges, excepted maybe on a set of Lebesgue measure 0, to the entropy solution u of (1).

Provided that the so-called initial potential $W(z) := \int_0^z u(x,0)\,dx$ fulfilled the condition

$$W(z) = o(z^2) \quad \text{as } |z| \to \infty, \tag{2}$$

it is remarkable that the (entropy) solution $u(\cdot,t)$ of Burgers equation (1) at time t can be expressed in terms of the convex hull $\mathcal{H}_t$ of

$$z \mapsto W(z) + \frac{1}{2t} z^2.$$

Indeed, write $a(x,t)$ for the right-most location of the minimum of

$$z \mapsto W(z) + \frac{1}{2t}(z-x)^2.$$

Then, on the one hand $a(x,t)$ coincides with the right-continuous inverse of t times the derivative of the convex hull $\mathcal{H}_t$. On the other hand, a version[b]

[b]A weak solution is only defined up to a set of Lebesgue measure 0, we can thus only speak of a version of it.

of the entropy solution u of (1) is given by the Hopf-Cole formula

$$u(x,t) = \frac{x - a(x,t)}{t},$$

see [10,19]. Notice already that the discontinuities of $x \mapsto u(x,t)$ comes from the discontinuities of $x \mapsto a(x,t)$. Since $x \mapsto a(x,t)$ is right-continuous and increasing, they are only negative and of the first kind.

As mentioned before, we can interpret the entropy solution of (1) in terms of a system of ballistic aggregation. Consider at time $t = 0$, infinitesimal particles spread on the real line according to the uniform density $\rho(dx, 0) = dx$, with velocities given by the velocity field $u(\cdot, 0)$. Then, let the system evolves according to the dynamic of free sticky particles described in the introduction. At time t, the velocity field of the system fit with (a version of) the entropy solution $u(\cdot, t)$ of (1) with initial condition $u(\cdot, 0)$. Moreover, the function $a(x,t)$ defined above represents the rightmost initial location of the particles lying in $]-\infty, x]$ at time t. In other words, the density of mass in the system is given at time t by the Stieljes measure

$$\rho(\,]x,y], t) = a(y,t) - a(x,t).$$

Therefore, the jumps of $x \mapsto a(x,t)$, which correspond to the shocks of $x \mapsto u(x,t)$, also correspond to the macroscopic clusters of particles (clusters of *positive* mass) present in the system at time t. Actually, a jump of $a(\cdot, t)$ at x exactely corresponds to a macroscopic cluster located in x, whose mass is given by $a(x,t) - a(x-, t)$; the notation $a(x-, t)$ refers to the left limit of $a(\cdot, t)$ at x. The velocity V of this cluster is enforced by the conservation of momentum

$$V = \frac{1}{a(x,t) - a(x-,t)} \int_{a(x-,t)}^{a(x,t)} u(z,0)\, dz = \frac{2x - a(x,t) - a(x-,t)}{2t}.$$

In the special case where $x \mapsto a(x,t)$ is a step function, we say that the shock structure is discrete at time t. The path $x \mapsto u(x,t)$ is then shaped as a toothpath made of pieces of line of slope $1/t$ separated by negative jumps (shocks). In terms of ballistic aggregation, a discrete shock structure corresponds to a state of the system where all particles have clumped into macroscopics clusters, whose locations form a discrete sequence of the real line. In a geometrical point of view, the shock structure is discrete if and only if the convex hull $\mathcal{H}_t$ of $z \mapsto W(z) + \frac{1}{2t} z^2$ is piecewise linear. It is convenient in this case to introduce the so-called $\frac{1}{2t}$*-parabolic hull* $\mathcal{P}_t$ of the initial potential W, defined by

$$\mathcal{P}_t(z) = \mathcal{H}_t(z) - \frac{1}{2t}z^2, \quad z \in \mathbb{R}.$$

When the convex hull $\mathcal{H}_t$ is piecewise linear, the parabolic hull $\mathcal{P}_t$ is made of pieces of parabola. Indeed, to a linear piece of $\mathcal{H}_t$ with slope X/t, say $\left(z \mapsto \frac{1}{t}X\,z + k;\, a \le z \le b\right)$, corresponds a piece of parabola of $\mathcal{P}_t$

$$\left(z \mapsto -\frac{1}{2t}z^2 + \frac{1}{t}Xz + k;\, a \le z \le b\right)$$
$$= \left(z \mapsto -\frac{1}{2t}(z-X)^2 + k';\, a \le z \le b\right)$$

with leading coefficient $-\frac{1}{2t}$ and vertex of abscissa X. A moment of thought then shows that there is a one-to-one correspondance between the (pieces of) parabolaes of $\mathcal{P}_t$ and the macroscopic clusters present in the system of ballistic aggregation at time t. Indeed, to a parabola of $\mathcal{P}_t$ corresponds a cluster whose location X is given by the abscissa of the vertex of the parabola. Consider the two extremal contact points between this parabola and the initial potential W. Then, the difference between the abscissaes of these contact points gives the mass of the cluster, whereas the slope of the line going through these two points fits with its velocity, see Figure 1. The state of the system is thus completely determined by $\mathcal{P}_t$.

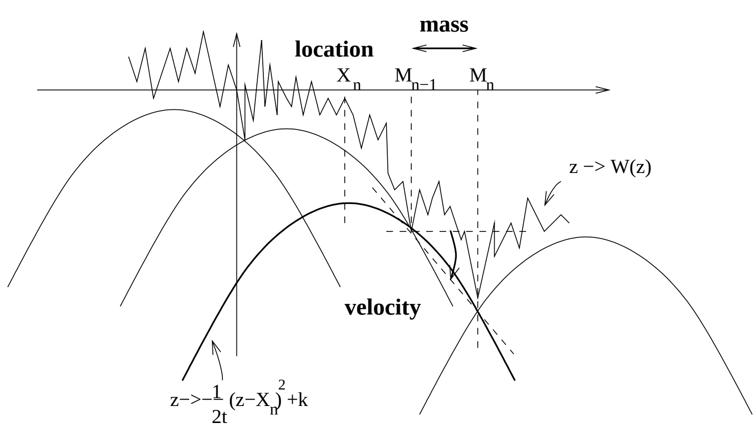

Figure 1. Geometrical interpretation of a shock.

Finally, we emphasize that the above analysis still makes sense when the initial condition $u(\cdot, 0)$ is not a real function, but is defined as the derivative in the sense of Schwartz of an initial potential W fulfilling condition (2). The solution $u(\cdot, t)$ is then a real function at any time $t > 0$ and when $t \to 0+$, it converges in the sense of Schwartz to $u(\cdot, 0)$, which is still said to be the initial condition. The white noise initial condition is to be understood in this sense.

3. Parabolic hull of a Brownian motion

According to the work of Groeneboom[17] (see also Pitman[22]), it is well known that the convex hull of a Brownian motion W is a.s. piecewise linear. A standard application of Girsanov Theorem shows that this property still holds for the convex hull of a Brownian motion with parabolic drift, see Groeneboom[18] and also Avellaneda & E[3].

Theorem 3.1. *The convex hull $\mathcal{H}_t$ of a (two-sided) Brownian motion with parabolic drift $\left(W_z + \frac{1}{2t}z^2;\, z \in \mathbb{R}\right)$ is piecewise linear with probability one.*

Recall from the previous section that when the convex hull $\mathcal{H}_t$ is piecewise linear, the $\frac{1}{2t}$-parabolic hull of W is made of pieces of parabola. We can index these pieces of parabola on $\mathbb{Z}$, with indices increasing from left to right and the convention that the parabola number 1 is the first parabola whose vertex is located at the right of 0. We write X_n for the abscissa of the vertex of the piece of parabola number n and also M_{n-1} and M_n for the abscissaes of its end-points; see Figure 1. One may notice that, in the notation of the previous section, $M_n = a(X_n, t)$.

The parabolic hull $\mathcal{P}_t$ is fully determined by the sequence $(X_n, M_n)_{n\in\mathbb{Z}}$. A characterization of this sequence can be easily derived from the work of Groeneboom[18] on Brownian motions with parabolic drift. It involves the Laplace transform $C(\lambda)$ of the integral of a Brownian excursion e of duration 1. According to Groeneboom's formula (see [18] Lemma 4.2.(iii))

$$\begin{aligned} C(\lambda) &:= \mathbb{E}\left(\exp\left(-\lambda \int_0^1 e_s\, ds\right)\right) \\ &= \lambda\sqrt{2\pi}\sum_{n=1}^{\infty} \exp\left(-2^{-1/3} w_n \lambda^{2/3}\right), \text{ for } \lambda > 0, \end{aligned} \tag{3}$$

where $0 > -w_1 > -w_2 > \cdots$ denotes the zeros of the Airy function $\mathtt{Ai}$ (see [1] on p 446). We also introduce, following Groeneboom's notations, the function $g : \mathbb{R} \to \mathbb{R}^+$ defined by its Fourier transform

$$\hat{g}(\lambda) \;=\; \int_{\mathbb{R}} e^{\mathtt{i}\lambda s}\, g(s)\, ds = \frac{2^{1/3}}{\mathtt{Ai}\left(\mathtt{i}2^{-1/3}\lambda\right)}\,. \tag{4}$$

Theorem 3.2.
The sequences $\{(0,M_0),(X_n,M_n)_{n\geq 1}\}$ *and* $\{(0,M_0),(X_{-n+1},M_{-n})_{n\geq 1}\}$ *are two Markov chains, independent conditionally on* M_0*, with transitions given by*

$$\begin{aligned}\mathbb{P}\left(X_n \in dx_n, M_n - M_{n-1} \in dm_n \,|\, X_{n-1} = x_{n-1}, M_{n-1} = a_{n-1}\right) = \\ \frac{1}{t\sqrt{2\pi m_n}} C\left(m_n^{3/2}/t\right) \frac{g\left((2t)^{-2/3}(m_n + a_{n-1} - x_n)\right)}{g\left((2t)^{-2/3}(a_{n-1} - x_{n-1})\right)} \\ \times \exp\left(\frac{(a_{n-1}-x_n)^3 - (a_{n-1}-x_{n-1})^3}{6t^2}\right) dx_n\, dm_n. \quad (5)\end{aligned}$$

Moreover, the law of M_0 *is given by*

$$\mathbb{P}(M_0 \in da) = \frac{1}{2^{5/3}t^{2/3}} g\left(-(2t)^{-2/3}a\right) g\left((2t)^{-2/3}a\right) da. \qquad \blacksquare$$

This result has been recently recovered by Frachebourg and Martin[13].

It is known that the "excursions" of the Brownian motion above its convex hull are distributed, conditionally on the convex hull, as independent Brownian excursions, see Groeneboom[17] and Pitman[22]. The next theorem states a similar path decomposition of the Brownian motion conditionally on its parabolic hull, see [14] for proof. We write $e^{[m]}$ for a Brownian excursion of duration m and

$$\begin{cases} \sigma(m) = \min\left\{\frac{2}{x(m-x)} e_x^{[m]};\, x \in]0,m[\right\} \\ \eta(m) = \text{right-most location of this minimum.}\end{cases}$$

Theorem 3.3. *The "excursions" of the Brownian motion above its parabolic hull* $\mathcal{P}_t$

$$\mathcal{E}^{(n)} = \left(W(M_{n-1}+x) - \mathcal{P}_t(M_{n-1}+x);\, 0 \leq x \leq M_n - M_{n-1}\right)$$

are independent conditionally on $\mathcal{P}_t$*, with as conditional law, the law* $\nu(m_n,t)$ *of*

$$\left(x \mapsto e_x^{[m_n]} - \frac{1}{2t} x(m_n - x) \,|\, \sigma(m_n) \geq 1/t\right)$$

where $m_n = M_n - M_{n-1}$. $\blacksquare$

Remark: A straighforward application of Girsanov Theorem shows that the law $\nu(m,t)$ is absolutely continuous with respect to the law $\mathbb{P}^{[m]}$ of $e^{[m]}$. Actually,

$$d\nu(m,t) = \frac{\exp\left(-\frac{1}{t}\int_0^m e_x^{[m]}\, dx\right)}{\mathbb{E}\left(\exp\left(-\frac{1}{t}\int_0^m e_x^{[m]}\, dx\right)\right)} d\mathbb{P}^{[m]}.$$

The law of the variables $\sigma(m)$ and $\eta(m)$ plays a key role in the analysis of Burgers turbulence with white noise initial data. It is specified in the next theorem, in terms of the function C defined above. See [15] for proof.

Theorem 3.4. *The scaling property of Brownian excursions enforces the identity in law*

$$(\sigma(m), \eta(m)) \stackrel{law}{=} \left(m^{-3/2}\sigma(1), m\eta(1)\right).$$

For any $a > 0$ and $0 < x < 1$, the probability density function of $(\sigma(1), \eta(1))$ is given by

$$\mathbb{P}(\sigma(1) \in da, \eta(1) \in dx) = \frac{e^{-a^2/24}}{\sqrt{8\pi x(1-x)}} C\left(ax^{3/2}\right) C\left(a(1-x)^{3/2}\right) da\, dx\,.$$

Moreover, $\mathbb{P}(\sigma(1) \geq a) = e^{-a^2/24}C(a)$, for $a \geq 0$. ■

4. Burgers turbulence with white noise initial velocity

In this section, we turn our attention to the solutions of Burgers equation (1) with initial condition $u(\cdot, 0)$ distributed as a white noise. In other words, we consider an initial potential $(W_x;\, x \in \mathbb{R})$ distributed as a two-sided Brownian motion. We first describe the solution at a fixed time $t > 0$, and then focus on its time-evolution.

4.1. *State at a fixed time $t > 0$*

According to Theorem 1 (Section 3), when W is distributed as a Brownian motion, the convex hull of the path $x \mapsto W_x + \frac{1}{2t}x^2$ is piecewise linear with probability one. As a consequence (see Section 2), when $u(\cdot, 0)$ is distributed as a white noise, the shock structure is discrete a.s. We recall that in this case, the solution $x \mapsto u(x, t)$ is a toothpath, fully determined by the sequence $((X_n, M_n);\, n \in \mathbb{Z})$ described in Theorem 2. Indeed, X_n gives the location of its n^{th} shock at the right of the origine, and $(M_n - M_{n-1})/t$ the strength of this shock. In terms of ballistic aggregation, the state of the system is the following. All particles have a.s. clumped into macroscopic clusters located in $(X_n;\, n \in \mathbb{Z})$, with mass and velocity given by $(m_n = M_n - M_{n-1};\, n \in \mathbb{Z})$ and

$$\left(V_n = \frac{2X_n - M_n - M_{n-1}}{2t};\, n \in \mathbb{Z}\right).$$

Besides, it is to be mentioned that the scaling property of the white noise propagates to the turbulence and induces the identity in law, see e.g. [3],

$$(u(x,t);\, x \in \mathbb{R}) \stackrel{\text{law}}{=} \left(t^{-1/3} u\left(x t^{-2/3}, 1\right);\, x \in \mathbb{R}\right).$$

4.2. *Time evolution of the turbulence*

The previous section gives a complete description of the state of the turbulence at a fixed time $t > 0$. The natural question now is to understand its time evolution. It will convenient in this view to use the ballistic interpretation of the turbulence.

As time runs, the clusters present in the system aggregate according to the dynamic of sticky particles. This clustering is deterministic, because so is the dynamic of sticky particles. But it induces a loss of information in the sense that we cannot recover the state of the system at a time t_1 from the state of the system at a time $t_2 > t_1$. Suppose now that time runs *backwards*. Then, clusters dislocate and due to the loss of information, this dislocation goes *randomly*. If we do understand how a cluster breaks into pieces in backward times, then we will understand how it did aggregate in forward times. Roughly, in this subsection we will answer to the question: what does the genealogical tree of a given cluster look like?

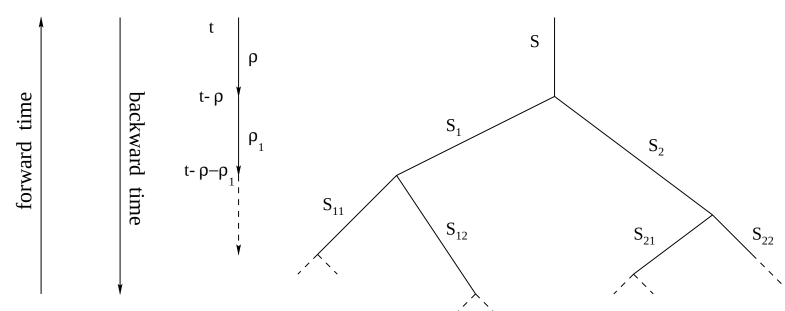

Figure 2. Genealogical tree of a Cluster

Henceforth, we focus on the fragmentation of the clusters in backwards time. The next theorem specified on which parameters the fragmentation of a cluster does depend on.

Theorem 4.1. *Conditionally on the state of the system at time t, each cluster present at time t breaks into pieces independently of the others, and according to a conditional law only depending on its mass and time t.*

Physically, the independence of the fragmentation of a cluster from its location and velocity may be viewed as a consequence of the invariance of the system under translation and Galilean transformations. The fact that it does not depend on the fragmentation of the other clusters may be understood as follow. Consider at time 0 two (infinitesimal) particles, which belong at time t to two different clusters. These two particles cannot interact up to time t, else they would stick and belong to the same cluster. Therefore, the particles which made up a cluster at time t cannot interact before time t with the other particles. Since in addition the initial velocities of the particles are uncorrelated, the aggregation processes of the clusters are expected to be independent.

Proof: We only sketch the proof of Theorem 4, and refer to [14] for details. The main point is to traduce the fragmentation of the clusters in terms of the parabolic hull of the initial potential W. Recall there is a one-to-one correspondance between the clusters present at time t in the system and the (pieces of) parabolaes of the $\frac{1}{2t}-$parabolic hull of the initial potential. Consider a given cluster at time t and its corresponding parabola with leading coefficient $-\frac{1}{2t}$. At time $s < t$, its corresponding parabola of the $\frac{1}{2s}-$parabolic hull of W is stretched in the vertical direction, since its leading coefficient is $-\frac{1}{2s}$. Let time s decreases from t to 0. The parabola corresponding to the cluster get more and more stretched, up to a time $t^* < t$ where it enters into contact with the initial potential W. This time t^* corresponds to the time at which the cluster splits into two clusters. Let time s decreases further. We now have two parabolaes corresponding to the two clusters. They are stretched in the vertical direction, up to the moment where one of them touches W in a new point, and also splits into two new parabolaes, giving at all three parabolaes/clusters. And so on.

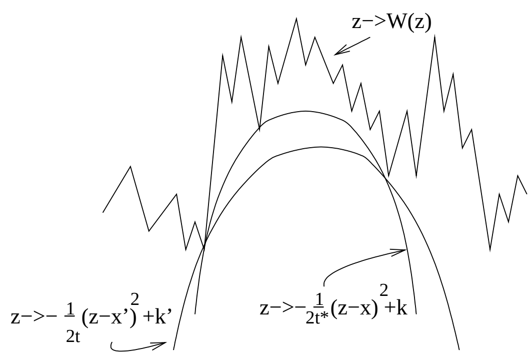

Figure 3. Time t^* of splitting.

A moment of thought thus shows that the fragmentation of a given cluster at time t only depends on the "excursion " $\mathcal{E}$ of the initial potential W above the parabola corresponding to the cluster. When W is distributed as a Brownian motion, it follows from Theorem 3 that conditionally on the state of the system at time t, each cluster breaks into pieces independently of the others. Moreover, since the conditional law of $\mathcal{E}$ given $\mathcal{P}_t$ only depends on time t and the mass m of the cluster, the fragmentation of the cluster only depends on m and t, and *not* on its velocity or location. ■

According to the previous theorem, we can focus on a single cluster of mass m at time t. We now turn our attention to its first splitting.

Theorem 4.2. *With probability one a cluster splits into exactely two clusters at its first splitting. The law of the time* t^* *of the splitting of a cluster of mass* m *at time* t *and of the mass* m^* *of the left-most cluster arising during this splitting is given by*

$$\mathbb{P}(t^* \in ds, m^* \in dm_1) =$$

$$\frac{m^{3/2}}{s^2\sqrt{8\pi m_1 m_2}} \exp\left(\frac{m^3}{24}\left(\frac{1}{t^2}-\frac{1}{s^2}\right)\right) \frac{C\left(m_1^{3/2}/s\right) C\left(m_2^{3/2}/s\right)}{C\left(m^{3/2}/t\right)} \qquad (6)$$

for $(s, m_1) \in]0,t[\times]0,m[$*, with the notation* $m_2 = m - m_1$ *and* C *defined by (3).*

Moreover, we have for $0 < s < t$

$$\mathbb{P}(t^* \leq s) = \exp\left(\frac{m^3}{24}\left(\frac{1}{t^2}-\frac{1}{s^2}\right)\right) \frac{C\left(m^{3/2}/s\right)}{C\left(m^{3/2}/t\right)}. \qquad (7)$$

We refer to [14] for numerical illustrations of these laws.

Proof: We write as before $\mathcal{E}$ for the "excursion" of the initial potential W above the parabola corresponding to the cluster at time t. Recall from the proof of the previous theorem that the time t^* corresponds to the time at which the parabola enters into contact with the initial potential in a new point. When the initial potential is distributed as a Brownian motion, the cluster splits a.s. into *two* clusters, because the parabola enters a.s. into contact with the Brownian motion in a *single* new point, see [14] for proof. The location of this contact point gives the distribution of mass between the two new clusters. Indeed, it should be plain from the mechanism described above that $1/t^*$ and m^* correspond to the maximum and the location of the maximum of

$$x \mapsto \frac{2}{x(m-x)}\,\mathcal{E}(x) + \frac{1}{t}\,.$$

When W is distributed as a Brownian motion, the conditional law of $\mathcal{E}$ given $\mathcal{P}_t$ is $\nu(m,t)$. Therefore, $1/t^*$ and m^* are distributed as the variables $\sigma(m)$ and $\eta(m)$ conditioned by $\{\sigma(m) \geq 1/t\}$. Formulaes (6) and (7) follow thus from Theorem 3. ∎

The previous result depicts the first splitting. Combined with a Markov property at the time of fragmentation (see [14]), it yields a complete description of the fragmentation of a cluster. This description can be formulated as follow. We denote by $m_1, \dots, m_k$ the masses of the clusters resulting at time $s = t - r$ of the fragmentation of a cluster of mass m at time t. The mass m_1 refers to the mass of the left-most cluster, the mass m_k to the one of the right-most cluster. We write also

$$\mathcal{M}^{(m,t)}(r) := (m_1, \dots, m_k).$$

Theorem 4.3. *The process* $\left(r \mapsto \mathcal{M}^{(m,t)}(r);\, 0 < r < t\right)$ *is a pure-jump (inhomogeneous) strong Markov process, with rate of jump at time* r

$$\mathbb{P}\left(\begin{matrix}\mathcal{M}^{(m,t)}(r+h) = (m_1, \dots, m_{i,1}, m_{i,2}, \dots, m_k) \\ m_{i,1} \in d\lambda_1\end{matrix} \,\middle|\, \begin{matrix}\mathcal{M}^{(m,t)}(r) = \\ (m_1, \dots, m_i, \dots, m_k)\end{matrix}\right)$$
$$\overset{h\to 0+}{\sim} h\, \frac{m_i^{3/2}\, d\lambda_1}{\sqrt{2\pi\lambda_1\lambda_2}} \times \frac{C\left(\lambda_1^{3/2}/t-r\right) C\left(\lambda_2^{3/2}/t-r\right)}{C\left(m_i^{3/2}/t-r\right)}$$

with the function C *defined by (3) and* $\lambda_2 = m_i - \lambda_1$.

We refer to [14] for the proof of the Markov property and [15] for the computation of the rate of jump.

We end this section with a remark about the dynamic of fragmentation. The property stated in Theorem 4 bears the same flavor as the so-called *fragmentation property* considered by Pitman[23] and Bertoin[6]. Nevertheless, the fragmentation process $r \mapsto \mathcal{M}^{(m,t)}(r)$ we study here is *not* homogeneous in time and therefore differs from those considered by Pitman and Bertoin. Besides, a cluster of mass m at time t statistically breaks into pieces in the same way as a cluster of mass $mt^{-2/3}$ at time 1. This permits to associate a *time homogeneous* Markov process to $r \mapsto \mathcal{M}^{(m,t)}(r)$. Indeed, the process

$$\hat{\mathcal{M}}^{(m,t)}(s) = t^{-2/3} e^{2s/3} \mathcal{M}^{(m,t)}(te^{-s}),\ s \in \mathbb{R}^+$$

is a time homogeneous strong Markov process, whose dynamic can be depicted as follow. Each cluster making up $\hat{\mathcal{M}}^{(m,t)}$ grows deterministically as $s \mapsto e^{2s/3}$ and also splits randomly, independently of the others, according to the fragmentation rate

$$F(\lambda_1, \lambda - \lambda_1) = \frac{\lambda^{3/2}}{\sqrt{8\pi\lambda_1(\lambda - \lambda_1)}} \times \frac{C\left(\lambda_1^{3/2}\right) C\left((\lambda - \lambda_1)^{3/2}\right)}{C\left(\lambda^{3/2}\right)}.$$

5. Burgers turbulence with some other initial velocities of white noise type

In this section, we consider other initial conditions of white noise type for equation (1). We outline in Section 5.1 the main properties of the solution of Burgers equation (1) with as initial condition $u(\cdot, 0)$, a white noise on $\mathbb{R}^+$ and 0 on $\mathbb{R}^-$. In Section 5.2, we depict the case where $u(\cdot, 0)$ is a periodic white noise. We omit the proof.

5.1. *The one-sided white noise case*

In this subsection, we deals with the initial condition

$$u(\cdot, 0) = \begin{cases} 0 & \text{on }]-\infty, 0] \\ \text{white noise} & \text{on }]0, \infty[\,. \end{cases}$$

In terms of ballistic aggregation, such an initial condition arises at the hydrodynamic limit of the following system. At time $t = 0$ the sticky particles are spread uniformely on $\mathbb{Z}$ and those on the right of the origine receive random i.i.d. velocities (with finite variance), where as those on the left of the origine stay at rest.

The phenomenum of main interest here is the propagation to the left of the chaos initially located on the right of 0. The solution $x \mapsto u(x, t)$ has a shock front, which travels to the left as time t runs. At the left of this shock front $u(\cdot, t)$ equals 0, where as at its right $x \mapsto u(x, t)$ is a.s. a toothpath, made of pieces of line of slope $1/t$ separated by a discrete sequence of shocks, see Figure 4. The location X_n and $M_n = t \times$the strength of the n^{th} shock at the right of the shock front form a Markov chain, with transitions given by (5). We write henceforth x_t and M_t for the location and t times the strength of the shock front.

It is convenient to use the ballistic description of $u(\cdot, t)$. There exists a so-called front cluster, travelling to the left, on the left of which there are infinitesimal particles at rest. On its right, all particles have clumped into macroscopic clusters, whose locations and masses are given by $(X_n, M_n)_{n \in \mathbb{N}}$. The location and the mass of the front cluster correspond to x_t and M_t.

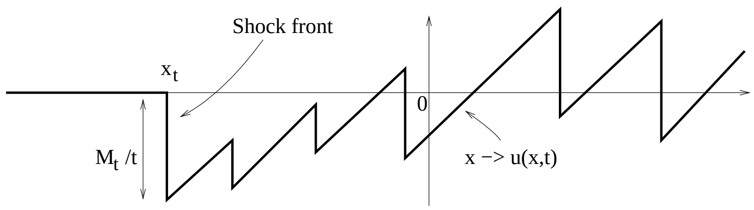

Figure 4. Shape of $x \mapsto u(x,t)$.

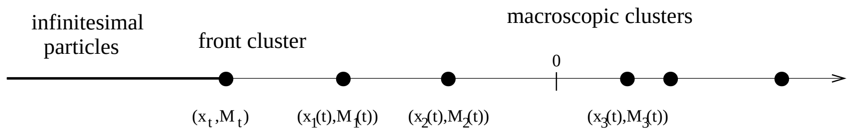

Figure 5. Shape of the system of sticky particles.

The first property to mention about the shock front is the time-scaling identity in law

$$(x_t, M_t) \overset{\text{law}}{=} \left(t^{2/3}x_1, t^{2/3}M_1\right). \tag{8}$$

This property originates from the scaling property of the white noise and permits to focus on time $t = 1$. The second property to be noticed, is that the shock front is completely described at time $t = 1$ by the variables x_1 and M_1. Indeed, according to the conservation of mass and momentum the velocity V_1 of the shock front is given by $V_1 = -\frac{1}{2}M_1$. This equality can be extended at any time $t > 0$ by

$$V_t = -\frac{1}{2t}M_t\,. \tag{9}$$

It is an easy task to derive from the work of Groeneboom[18] the law of (x_1, M_1), in terms of the function g defined by (4) and the function $h(m,\cdot) : \mathbb{R}^+ \to R^+$ defined by the series

$$h(m,x) = 2^{1/3}\sum_{n=1}^{\infty}\frac{\mathtt{Ai}\left(2^{2/3}m - w_n\right)}{\mathtt{Ai}'\left(-w_n\right)}\exp\left(-2^{1/3}xw_n\right),$$

where as before, $0 > -w_1 > -w_2 > \cdots$ represents the zeros of the Airy function $\mathtt{Ai}$ ranked in decreasing order. See [16] for proof and also the law of x_1 alone.

Theorem 5.1. *In the above notation, the law of* (x_1, M_1) *is given by*

$$\mathbb{P}(-x_1 \in dx, M_1 \in dM) = \frac{e^{-x^3/3}M}{2}\, g\left(2^{-2/3}M\right) h\left(2^{-4/3}x^2, 2^{-2/3}(M-x)\right)$$

for $M, x > 0$.

We now turn our attention to the time-evolution of the shock front. It is conspicious from the ballistic description of the system, that the dynamic of the shock front is governed by two phenomena. First its movment to the left is continuously braked by the infinitesimal particles at rest on its left. Second, macroscopic clusters on its right sometimes catch it and then increase sharply its velocity. We are mainly interested by the evolution of the location x_t of the shock front. The identity (8) suggets that x_t behave roughly as $t \mapsto -t^{2/3}$. But we stress that the identity (8) is only true for a *fixed* time $t > 0$ and therefore does not give the time-evolution of $t \mapsto x_t$. The identity (9) enforces the equality

$$x_t = -\int_0^t M_s \frac{ds}{2s},$$

so that the evolution of the shock front can be fully expressed in terms of the process $t \mapsto M_t$, which is characterized in the following theorem.

Theorem 5.2. *The process* $t \mapsto \hat{M}_t := \frac{1}{\sqrt{t}} M_t$ *is a pure-jump inhomogeneous and increasing Markov process, with rate of jump*

$$\mathbb{P}\left(\hat{M}_{t+h} - \hat{M}_t \in dm \,|\, \hat{M}_t = M\right) \overset{h\to 0+}{\sim} h\, \frac{(M+m)\,dm}{t^{5/4}\sqrt{8\pi m}} C\left(m^{3/2}/t^{1/4}\right) \frac{g\left(2^{-2/3}t^{-1/6}(M+m)\right)}{g\left(2^{-2/3}t^{-1/6}M\right)}$$

for any $M, m, t > 0$.

We can also gives the asymptotic behaviour of $t \mapsto x_t$ for small and large time t

Proposition 5.1. *When time t tends to 0 or* ∞, *we have with probability one the asymptotics*

$$\limsup_{t\to 0/\infty} \frac{-x_t}{(t^2 \log|\log t|)^{1/3}} = \frac{2^{5/3}}{3},$$

$$\text{and} \quad \lim_{t\to 0/\infty} \frac{-x_t}{t^{2/3}} |\log t|^{2/3+\delta} = \infty, \quad \forall\, \delta > 0.$$

Some other aspects of the solution $u(\cdot, t)$ have also been investigated. The main contributions are perhaps the description of the flux of particles crossing a given point and the study of the different scaling regimes of the solution by Frachebourg, Jacquemet and Martin[12], see also [7]. Besides, it can be noticed that the genealogy of a macroscopic cluster present at the right of the shock front, is statistically the same as the genealogy considered in Section 4. Finally, we mention the work of Tribe & Zaboronski[24] and also of Frachebourg et *al.*[12] on the case where the initial condition is given by a white noise on a finite interval, and 0 elsewhere.

5.2. *The periodic white noise case*

We focus henceforth on the solution of Burgers equation (1) with initial condition $u(\cdot, 0)$ distributed as a periodic white noise. In other words, we consider the case where the initial potential W is 1-periodic and is distributed on $[0, 1]$ as a Brownian bridge of duration 1. Since the solution $x \mapsto u(x, t)$ is also 1-periodic at any time $t > 0$, we can focus on a period.

It is convenient for investigating such a solution to use the ballistic description of $x \mapsto u(x, t)$. The system of sticky particles associated to $u(., t)$ is 1-periodic and can therefore be thought as a circular system, corresponding to the hydrodynamic limit of the following system. Consider at time $t = 0$, N particles uniformely spread on a circle, with random angular velocities $(\omega_i)_{1,N}$ i.i.d., of finite variance and fulfilling $\sum_{i=1}^{N} \omega_i$. Then, let the system evolves according to the next dynamic. Between collisions the particles evolve on the circle with constant angular velocities and when some particles meet, they merge into a new particle with conservation of mass and momenta.

As before, the shock structure of $u(\cdot, t)$ is discrete a.s. at any time $t > 0$. In a circular point of view it means that all particles have clumped into a finite number of macroscopic clusters. Moreover, it can be shown that when time t tends to ∞ there remains a.s. a single cluster of mass 1 and velocity 0. Its location follows the uniform law on the circle. The genealogy of this final cluster is distributed as the limit law of the genealogy of a cluster of mass 1 at time t in Section 4, when time $t \to \infty$. This permits to compute the probability density of a given state in terms of the function C defined by (3). Indeed, the probability density to have exactely N clusters of mass

$m_1, \ldots, m_N$ $(m_1 + \ldots + m_N = 1)$ located at $\theta_1 < \cdots < \theta_N$ equals

$$\frac{\exp\left(-\phi\left((m_i,\theta_i)_{1,N}\right)/t^2\right)}{\left(t\sqrt{2\pi}\right)^{N-1}} \prod_{j=1}^{N} \exp\left(-\frac{m_j^3}{24\,t^2}\right) \frac{C\left(m_j^{3/2}/t\right)}{\sqrt{m_j}},$$

where ϕ is a completely determined "polynomial-like" function of $(m_i, \theta_i)_{1,N}$, see [15] Section 4 Proposition 1. Since the formula of ϕ is somewhat complicated, we refer to [15] for its very definition.

6. Some open problems

To conclude we evoke some open problems. Many question on the one-dimensional Burgers turbulence remains open. For example, concerning the periodic case, it would be worth to obtain a simple formula for the law of the number N of clusters present at time t. For more general initial conditions, we may wonder if it is possible to extend some of the above results (see [5] for a discussion in the stable noise case)? Yet, going in higher dimensions appears now as the most challenging problems in Burgers turbulence, see Vergassola et *al.*[25] for motivations and simulations.

Besides, for a best understanding of the phenomenum of turbulence, it would be worth to exhibit some statistical properties of the solutions of some PDE of fluid mechanics (especially of Navier-Stoke equation), with adapted random initial conditions.

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