Variants of the Haagerup property relative to non-commutative L_p-spaces

Baptiste Olivier

TECHNION - Haifa

09th June 2014

Introduction : the Haagerup property (H)

The **Haagerup property** (H) (also called **a-T-menability**) appeared in 1979 in a seminal result of U. Haagerup.

It is a **non-rigidity property for topological groups**.

Property (H) is known to be a **strong negation of Kazhdan 's property** (T).

Examples of groups with property (H) :

- amenable groups;
- groups acting properly on trees, spaces with walls (free groups...);
- $SO(n,1)$, $SU(n,1)$.

Applications of property (H) :

- rigidity for von Neumann algebras;
- weak amenability;
- Baum-Connes conjecture (Higson, Kasparov).

Definitions

G l.c.s.c. group, B Banach space.
$\pi : G \to O(B)$ an orthogonal representation.

- π is said to have almost invariant vectors (a.i.v.) if there exists a sequence of unit vectors $v_n \in B$ such that

$$\lim_n \sup_{g \in K} ||\pi(g)v_n - v_n|| = 0 \text{ for all } K \subset G \text{ compact.}$$

- π is said to have vanishing coefficients (or is C_0) if

$$\lim_{g \to \infty} < \pi(g)v, w > \ = 0 \text{ for all } v \in B, w \in B^*.$$

(1) G is said to have property (H_B) if there exists an orthogonal representation $\pi : G \to O(B)$ which is C_0 and has a.i.v. .
(2) G is said to be a-F_B-menable if there exists a proper action by affine isometries of G on the space B.

Known results for $B = L_p$

- When $B = \mathcal{H}$ is a Hilbert space, (1) $\Leftrightarrow$ (2) give equivalent definitions of property (H).
- Property (H_B) is a strong negation of property (T_B), and the a-F_B-menability is a strong negation of property (F_B).

Theorem (Nowak/Chatterji, Drutu, Haglund)

- G has $(H) \Rightarrow G$ is a-$F_{L_p(0,1)}$-menable for all $p \geq 1$.
- G has $(H) \Leftrightarrow G$ is a-$F_{L_p(0,1)}$-menable for all $1 \leq p \leq 2$.

There exist groups G which are a-$F_{L_p(0,1)}$-menable for p large and have property (T) :

Theorem (Yu, Nica)

For $p > 2$ large enough, hyperbolic groups admit a proper action by affine isometries on ℓ_p, as well as on $L_p(0,1)$.

Non-commutative L_p-spaces

For $1 \leq p < \infty$, $\mathcal{M}$ a von Neumann algebra, and τ a normal faithful semi-finite trace on $\mathcal{M}$, we have

$$L_p(\mathcal{M}) = \overline{\{x \in \mathcal{M} \mid \tau(|x|^p) < \infty\}}^{||.||_p}$$

where $||x||_p = \tau(|x|^p)^{\frac{1}{p}}$.

Basic properties :

- $L_p(\mathcal{M})$ is a u.c.u.s. Banach space if $p > 1$, and $L_p(\mathcal{M})^* \simeq L_{p'}(\mathcal{M})$ where $\frac{1}{p} + \frac{1}{p'} = 1$.
- $L_p(\mathcal{M}) \simeq L_p(\mathcal{N})$ isometrically if and only if the algebras $\mathcal{M}$ and $\mathcal{N}$ are Jordan-isomorphic (Sherman).

Examples :

- $\mathcal{M} = L^\infty(X, \mu)$ with $\tau(f) = \int f \; d\mu$:
  $L_p(\mathcal{M}) = L_p(X, \mu)$ is a classical (commutative) L_p-space.
- $\mathcal{M} = \mathcal{B}(\mathcal{H})$ with $\tau = \mathrm{Tr}$ the usual trace :
  $L_p(\mathcal{M}) = \{x \in \mathcal{B}(\mathcal{H}) \mid \mathrm{Tr}(|x|^p) < \infty \}$ is the Schatten p-ideal, denoted by S_p.

$O(L_p(\mathcal{M}))$ and the Mazur map

Let $1 \leq p < \infty$, $p \neq 2$.

Theorem (Yeadon)

Let $U \in O(L_p(\mathcal{M}))$. Then there exist $u \in \mathcal{U}(\mathcal{M})$, B a positive operator affiliated with $\mathcal{M}$ with spectral projections commuting with $\mathcal{M}$, and $J : \mathcal{M} \to \mathcal{M}$ a Jordan-isomorphism such that

$$\mathbf{Ux{=}uBJ(x)} \text{ and } \tau(B^p J(x)) = \tau(x) \text{ for all } x \in \mathcal{M}_+ \cap L_p(\mathcal{M}).$$

$$O(L_p(\mathcal{M})) \textit{ big enough} \leftrightarrow (\ (H_{L_p(\mathcal{M})}) \Leftrightarrow (H)\ )$$

$$O(L_p(\mathcal{M})) \ \textit{ not big enough} \leftrightarrow (\ (H_{L_p(\mathcal{M})}) \not\Leftrightarrow (H)\ )$$

Conjugation by the Mazur map :

$$M_{p,q}(\alpha|x|) = \alpha|x|^{p/q} \text{ where } \alpha|x| \text{ is the polar decomposition of } x \in L_p.$$

Consider $V = M_{q,p} \circ U \circ M_{q,p} : L_q \to L_q$. Then :

$$\mathbf{V = uB^{p/q}J} \ \in \mathbf{O(L_q}(\mathcal{M})).$$

General facts about property $(H_{L_p(\mathcal{M})})$

- (G, G') has $(T_{L_p(\mathcal{M})})$, and G has $(H_{L_p(\mathcal{M})}) \Rightarrow G'$ is compact.
- Property $(H_{L_p(\mathcal{M})})$ is inherited by closed subgroups.
- Property $(H_{L_p(\mathcal{M})})$ only depends on the $||.||_p$-isometric class of $L_p(\mathcal{M})$.
- If $\mathcal{M}$ is a factor (i.e. $\mathcal{M} \cap \mathcal{M}' = \mathbb{C}$), then $(H_{L_p(\mathcal{M})}) \Rightarrow (H)$.

  Question : $(H_{L_p(\mathcal{M})}) \Rightarrow (H)$ for all von Neumann algebra $\mathcal{M}$?

  Question : Is the property of vanishing coefficients preserved by the conjugation by the Mazur map ?

Known cases : $\rightarrow \pi(g) = u_g J_g$ for all $g \in G$

$\rightarrow \pi(g) = B_g J_g$ for all $g \in G$

$(H_{L_p(\mathcal{M})})$ for $\mathcal{M} = \ell_\infty$ and $\mathcal{M} = \mathcal{B}(\mathcal{H})$

Case $\mathcal{M} = \ell_\infty$:

Let $\pi^p : G \to O(\ell_p)$ be an orthogonal representation, and π^2 its conjugate by $M_{p,2}$. Then there exist unitary characters $\chi_i : H_i \to \mathbb{C}$ on open subgroups $H_i \subset G$, such that π has the form :

$$\pi^2(g) = \oplus_i (\mathrm{Ind}_{\mathrm{H_i}}^{\mathrm{G}} \chi_{\mathrm{i}})(\mathrm{g}) \text{ for all } g \in G.$$

Theorem

- If G is connected, then G has property $(H_{\ell_p}) \Leftrightarrow G$ is compact.
- If G is totally disconnected, then G has property $(H_{\ell_p}) \Leftrightarrow G$ is amenable.

Case $\mathcal{M} = \mathcal{B}(\mathcal{H})$:

Arazy : if $U \in O(S_p)$, then there exist $u, v \in \mathcal{U}(\mathcal{H})$ such that

$$Ux = uxv \text{ or } Ux = u^t xv \text{ for all } x \in S_p.$$

Theorem

G has property $(H_{S_p}) \Leftrightarrow G$ has property (H).

Property $(H_{L_p(0,1)})$

Connected Lie groups with property (H) were determined by Chérix, Cowling and Valette : in particular, non-compact simple connected Lie groups are the ones locally isomorphic to $SO(n,1)$ or $SU(n,1)$.

Theorem

Let $1 \leq p < \infty$. Let G be a connected Lie group with Levi decomposition $G = SR$ such that the semi-simple part S has finite center. Then TFAE :
(i) G has property $(H_{L_p([0,1])})$;
(ii) G has property (H) ;
(iii) G is locally isomorphic to a product $\prod_{i \in I} S_i \times M$ where M is amenable, I is finite, and for all $i \in I$, S_i is a group $SO(n_i,1)$ or $SU(m_i,1)$ with $n_i \geq 2$, $m_i \geq 1$.

About the proof of (iii) $\Rightarrow$ (i) :
(1) deal the case of the groups $SO(n,1)$ and $SU(n,1)$;
(2) use a finite-covering argument.

Question : Can this proof be adapted to the case of $\widetilde{SU(n,1)}$?

Results about a-$F_{L_p(\mathcal{M})}$-menability

Theorem : relation with property $(H_{L_p(\mathcal{M})})$

Let $\mathcal{M}$ be a von Neumann algebra.

- Then : $(H_{L_p(\mathcal{M})}) \Rightarrow$ a-$F_{L_p(\mathcal{M}\otimes\ell_\infty)}$-menability.
- If moreover $\mathcal{M}$ is a I_∞ or II_∞ factor, then we have : $(H_{L_p(\mathcal{M})}) \Rightarrow$ a-$F_{L_p(\mathcal{M})}$-menability.

Remark : The converse is not true. From Yu's construction, one can obtain proper actions on S_p for some Kazhdan's groups.

Theorem

Denote by R the hyperfinite II_1 factor. Then we have

$$(H) \Rightarrow \text{a-}F_{L_p(\mathcal{M})}\text{-menability}$$

for the following von Neumann algebras :

- $\mathcal{M} = R \otimes \ell_\infty$;
- $\mathcal{M} = R \otimes \mathcal{B}(\ell_2)$.

Discussion toward further results

Question : what about results of type a-$F_{L_p(\mathcal{M})}$-menability $\Rightarrow (H)$?

Known method : for (X, μ) a measured space and $1 \leq p \leq 2$, we have

$$L_p(X, \mu) \text{ embeds isometrically in } \mathcal{H}.$$

Remarks :

- The map $(x, y) \mapsto ||x - y||_p^p$ is not a kernel conditionnally of negative type on $L_p(\mathcal{M}) \times L_p(\mathcal{M})$ whenever $\mathcal{M}_2(\mathbb{R}) \subset \mathcal{M}$.
- Isometric embeddings of type $L_p(\mathcal{M}) \subset L_q(\mathcal{N})$ can be used to prove

$$\text{a-}F_{L_p(\mathcal{M})}\text{-menability} \Rightarrow \text{a-}F_{L_q(\mathcal{N})}\text{-menability}.$$

Thank you for your attention!