RUELLE-TAYLOR RESONANCES OF ANOSOV ACTIONS

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Abstract. We define for $\mathbb{R}^\kappa$-Anosov actions a notion of joint Ruelle resonance spectrum by using the techniques of anisotropic Sobolev spaces in the cohomological setting of joint Taylor spectra. We prove that these Ruelle-Taylor resonances are intrinsic and form a discrete subset of $\mathbb{C}^\kappa$ and that 0 is a always a leading resonance. The joint resonant states at 0 give rise to measures of SRB type and the mixing properties of these measures are related to the existence of purely imaginary resonances. The spectral theory developed in this article applies in particular to the case of Weyl chamber flows and provides a new way to study such flows.

1. Introduction

If P is a differential operator on a manifold M that has purely discrete spectrum as an unbounded operator acting on $L^2(M)$ (e.g. an elliptic operator on a closed Riemannian manifold M), then the eigenvalues and eigenfunctions carry a huge amount of information about the dynamics generated by P. Furthermore, if P is a geometric differential operator (e.g. Laplace-Beltrami operator, Hodge-Laplacian or Dirac operators) the discrete spectrum encodes important topological and geometric invariants of the manifold M.

Unfortunately, in many cases (e.g. if the manifold M is not compact anymore or if P is non-elliptic) the L^2-spectrum of P is not discrete anymore but consists mainly of essential spectrum. Still, there are certain cases where the essential spectrum of P is non-empty, but where there is a hidden intrinsic discrete spectrum attached to P, called the *resonance spectrum*. To be more concrete, let us give a couple of examples of this theory:

- Quantum resonances of Schrödinger operators $P = \Delta + V$ with $V \in C_c^\infty(\mathbb{R}^n)$ on $M = \mathbb{R}^n$ with n odd (see for example [DZ19, Chapter 3] for a textbook account to this classical theory).
- Quantum resonances for the Laplacian on non-compact geometrically finite hyperbolic manifolds $M = \Gamma\backslash\mathbb{H}^{n+1}$: here $P = \Delta_M$ is the Laplace-Beltrami operator on M [MM87, GZ97, GM12].
- Ruelle resonances for Anosov flows [BL07, FS11, DZ16]: here $P = iX$ with X being the vector field generating the Anosov flow.

The definition of the resonances can be stated in different ways (using meromorphically continued resolvents, scattering operators or discrete spectra on auxiliary function spaces), and also the mathematical techniques used to establish the resonances in the above examples are quite diverse (ranging from asymptotics of special functions to microlocal analysis).

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Nevertheless, all three examples above share the common point that the existence of a discrete resonance spectrum can be proven via a parametrix construction, i.e. one constructs a meromorphic family of operators $Q(\lambda)$ (with $\lambda \in \mathbb{C}$) such that

$$(P-\lambda)Q(\lambda) = \mathrm{Id} + K(\lambda),$$

where $K(\lambda)$ is a meromorphic family of compact operators on some suitable Banach or Hilbert space. Once such a parametrix is established, the resonances are the λ where $\mathrm{Id} + K(\lambda)$ is not invertible and the discreteness of the resonance spectrum follows directly from analytic Fredholm theory.

In general, being able to construct such a parametrix and define a theory of resonances involve non-trivial analysis and pretty strong assumptions, but they lead to powerful results on the long time dynamics of the propagator e^{itP}, for example in the study of dynamical systems [Liv04, NZ13, FT17b] or on evolution equations in relativity [HV18]. Furthermore, resonances form an important spectral invariant that can be related to a large variety of other mathematical quantities such as geometric invariants [GZ97, SZ07], topological invariants [DR17, DZ17, DGRS20, KW19] or arithmetic quantities [BGS11]. They also appear in trace formulas and are the divisors of dynamical, Ruelle and Selberg zeta functions [BO99, PP01, GLP13, DZ16, FT17b].

The purpose of this work is to construct a theory of joint resonances spectrum for the generating vector fields of $\mathbb{R}^\kappa$-Anosov actions. As far as we know, this is the first case where a joint spectrum of resonances associated to a family of commuting differential operators has been constructed. The classical example of such $\mathbb{R}^\kappa$-Anosov actions is the *Weyl chamber flow* for compact locally symmetric spaces of rank $\kappa \geq 2$, and even for that case where harmonic analysis and representation theory for Lie groups are powerful tools, our results are new. An interesting consequence of our approach for general Anosov actions is that the "leading joint resonances" allow a natural construction of invariant measures that are similar to the Sinai-Ruelle-Bowen (SRB) measures for Anosov flows. We believe that these measures should be useful in the rigidity conjecture on the classification of such actions.

1.1. **Statement of the main results.** Let us now introduce the setting and state the main results. Let $\mathcal{M}$ be a closed manifold, $\mathbb{A} \simeq \mathbb{R}^\kappa$ be an abelian group and let $\tau : \mathbb{A} \to \mathrm{Diffeo}(\mathcal{M})$ be a smooth locally free group action. If $\mathfrak{a} := \mathrm{Lie}(\mathbb{A}) \cong \mathbb{R}^\kappa$, we can define a *generating map*

$$X : \begin{cases} \mathfrak{a} & \to C^\infty(\mathcal{M}; T\mathcal{M}) \\ A & \mapsto X_A := \frac{d}{dt}_{|t=0} \tau(\exp(tA)), \end{cases}$$

so that for each basis $A_1, \dots, A_\kappa$ of $\mathfrak{a}$, $[X_{A_j}, X_{A_k}] = 0$ for all j, k. For $A \in \mathfrak{a}$ we denote by $\varphi_t^{X_A}$ the flow of the vector field X_A. It is customary to call the action *Anosov* if there is an $A \in \mathfrak{a}$ such that there is a continuous $d\varphi_t^{X_A}$-invariant splitting

$$T\mathcal{M} = E_0 \oplus E_u \oplus E_s, \tag{1.1}$$

where $E_0 = \mathrm{span}(X_{A_1}, \dots, X_{A_\kappa})$, and there exists a $C > 0, \nu$ such that for each $x \in \mathcal{M}$

$$\forall w \in E_s(x), \forall t \geq 0, \ \|d\varphi_t^{X_A}(x)w\| \leq C^{-\nu t}\|w\|,$$
$$\forall w \in E_u(x), \forall t \leq 0, \ \|d\varphi_t^{X_A}(x)w\| \leq C^{-\nu|t|}\|w\|.$$

Here the norm on $T\mathcal{M}$ is fixed by choosing any smooth Riemannian metric g on $\mathcal{M}$. We say that such an A is *transversely hyperbolic*. It can be easily proved that the splitting is invariant by the whole action. Moreover, there is a maximal open convex cone $\mathcal{W} \subset \mathfrak{a}$ containing A such that for all $A' \in \mathcal{W}$, $X_{A'}$ is also transversely hyperbolic with the same splitting as A (see Lemma 2.2); $\mathcal{W}$ is called a *positive Weyl chamber*. This name is motivated by the classical examples of such Anosov actions that are the Weyl chamber flows for locally symmetric spaces of rank κ (see Example 2.3). There are also several other classes of examples (see e.g. [KS94, SV19]).

Since we have now a family of commuting vector fields, it is natural to consider a joint spectrum for the family $X_{A_1}, \dots, X_{A_\kappa}$ of first order operators if the A_j's are chosen transversely hyperbolic with the same splitting. Guided by the case of a single Anosov flow (done in [BL07, FS11, DZ16]), we define $E_u^* \subset T^*\mathcal{M}$ to be the subbundle such that $E_u^*(E_u \oplus E_0) = 0$. We shall say that $\lambda = (\lambda_1 \dots, \lambda_\kappa) \in \mathbb{C}^\kappa$ is a *joint Ruelle resonance* for the Anosov action if there is a non-zero distribution $u \in C^{-\infty}(\mathcal{M})$ with wavefront set $\mathrm{WF}(u) \subset E_u^*$ such that

$$\forall j = 1, \dots, \kappa, \quad (X_{A_j} + \lambda_j)u = 0. \tag{1.2}$$

The distribution u is called a *joint Ruelle resonant state* (from now on we will denote $C^{-\infty}_{E_u^*}(\mathcal{M})$ the space of distributions u with $\mathrm{WF}(u) \subset E_u^*$). In an equivalent but more invariant way (i.e. independently of the choice of basis $(A_j)_j$ of $\mathfrak{a}$), we can define a *joint Ruelle resonance* as an element $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ of the complexified dual Lie algebra such that there is a non-zero $u \in C^{-\infty}_{E_u^*}(\mathcal{M})$ with

$$\forall A \in \mathfrak{a}, \quad (X_A + \lambda(A))u = 0.$$

It is a priori not clear that this set is discrete nor that the dimension of joint resonant states is finite, but this is a consequence of our work:

Theorem 1. *Let τ be a smooth abelian Anosov action on a closed manifold $\mathcal{M}$ with positive Weyl chamber $\mathcal{W}$. Then the set of joint Ruelle resonances $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ is a discrete set contained in*

$$\bigcap_{A \in \mathcal{W}} \{\lambda \in \mathfrak{a}^*_{\mathbb{C}} \mid \mathrm{Re}(\lambda(A)) \leq 0\}. \tag{1.3}$$

Moreover, for each joint Ruelle resonance $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ the space of joint Ruelle resonant states is finite dimensional.

We emphasize that this theorem is definitely not a straightforward extension of the case of a single Anosov flow. It relies on a deeper result based on the theory of joint spectrum and joint functional calculus developed by Taylor [Tay70b, Tay70a]. This theory allows us to set up a good Fredholm problem on certain functional spaces by using Koszul complexes, as we now explain.

The generating map of the Anosov action τ can be viewed as an operator

$$X : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}; \mathfrak{a}^*_{\mathbb{C}}), \quad (Xu)(A) := X_A u,$$

and similarly for $X + \lambda$ if $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. We can then define for each $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ the differential operators $d_{(X+\lambda)} : C^\infty(\mathcal{M}; \Lambda^j \mathfrak{a}^*_{\mathbb{C}}) \to C^\infty(\mathcal{M}; \Lambda^{j+1}\mathfrak{a}^*_{\mathbb{C}})$ by setting

$$d_{(X+\lambda)}(u \otimes \omega) := ((X + \lambda)u) \wedge \omega \text{ for } u \in C^\infty(\mathcal{M}),\ \omega \in \Lambda^j \mathfrak{a}^*_{\mathbb{C}}.$$

Due to the commutativity of the family of vector fields X_A for $A \in \mathfrak{a}$, it can be easily checked that $d_{(X+\lambda)} \circ d_{(X+\lambda)} = 0$ (see Lemma 3.2). Moreover, as a differential operator, it extends to a continuous map

$$d_{(X+\lambda)} : C^{-\infty}_{E_u^*}(\mathcal{M}; \Lambda^j \mathfrak{a}^*_{\mathbb{C}}) \to C^{-\infty}_{E_u^*}(\mathcal{M}; \Lambda^{j+1}\mathfrak{a}^*_{\mathbb{C}})$$

and defines an *associated Koszul complex*

$$0 \longrightarrow C^{-\infty}_{E_u^*}(\mathcal{M}) \overset{d_{(X+\lambda)}}{\longrightarrow} C^{-\infty}_{E_u^*} \otimes \Lambda^1 \mathfrak{a}^*_{\mathbb{C}} \dots \overset{d_{(X+\lambda)}}{\longrightarrow} C^{-\infty}_{E_u^*}(\mathcal{M}) \otimes \Lambda^\kappa \mathfrak{a}^*_{\mathbb{C}} \longrightarrow 0, \tag{1.4}$$

We prove the following results on the cohomologies of this complex:

Theorem 2. *Let τ be a smooth abelian Anosov action*[1] *on a closed manifold $\mathcal{M}$ with generating map X. Then for each $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and $j = 0, \dots, \kappa$, the cohomology*

$$\ker d_{(X+\lambda)}|_{C^{-\infty}_{E_u^*}(\mathcal{M})\otimes\Lambda^j\mathfrak{a}^*_{\mathbb{C}}} / \operatorname{ran} d_{(X+\lambda)}|_{C^{-\infty}_{E_u^*}(\mathcal{M})\otimes\Lambda^{j-1}\mathfrak{a}^*_{\mathbb{C}}}$$

is finite dimensional. It is non-trivial only at a discrete subset of $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$.

We want to remark that the statement about the cohomologies in Theorem 2 is not only a stronger statement than Theorem 1, but that the cohomological setting is in fact a fundamental ingredient in proving the discreteness of the resonance spectrum and its finite multiplicity. Our proof relies on the theory of joint *Taylor spectrum* (developed by J. Taylor in [Tay70b, Tay70a]), defined using such Koszul complexes carrying a suitable notion of Fredholmness. In our proof of Theorem 2 we show that the Koszul complex furthermore provides a good framework for a parametrix construction via microlocal methods. More precisely, the parametrix construction is not done on the topological vector spaces $C^{-\infty}_{E_u^*}(\mathcal{M})$ but on a scale of Hilbert spaces $\mathcal{H}_{NG}$, depending on the choice of an escape function $G \in C^\infty(T^*\mathcal{M})$ and a parameter $N \in \mathbb{R}^+$, by which one can in some sense approximate $C^{-\infty}_{E_u^*}(\mathcal{M})$. The spaces $\mathcal{H}_{NG}$ are *anisotropic Sobolev spaces* which roughly speaking allow $H^N(\mathcal{M})$ Sobolev regularity in all directions except in E_u^* where we allow for $H^{-N}(\mathcal{M})$ Sobolev regularity. They can be rigorously defined using microlocal analysis, following the techniques of Faure-Sjöstrand [FS11]. By further use of pseudodifferential and Fourier integral operator theory we can then construct a parametrix $Q(\lambda)$, which is a family of bounded operators on $\mathcal{H}_{NG} \otimes \Lambda\mathfrak{a}^*_{\mathbb{C}}$ depending holomorphically on $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and fulfilling

$$d_{(X+\lambda)}Q(\lambda) + Q(\lambda)d_{(X+\lambda)} = \mathrm{Id} + K(\lambda). \tag{1.5}$$

[1] We actually prove Theorem 1 and Theorem 2 in the more general setting of admissible lifts to vector bundles, as defined in Section 2.2

Here $K(\lambda)$ is a holomorphic family of compact operators on $\mathcal{H}_{NG}\otimes\Lambda\mathfrak{a}_{\mathbb{C}}^*$ for λ in a suitable domain of $\mathfrak{a}_{\mathbb{C}}^*$ that can be made arbitrarily large letting $N\to\infty$. Even after having this parametrix construction, the fact that the joint spectrum is discrete and intrinsic (i.e. independent of the precise construction of the Sobolev spaces) is more difficult than for a single Anosov flow (the rank 1 case): this is because holomorphic functions in $\mathbb{C}^\kappa$ do not have have discrete zeros when $\kappa\geq 2$ and we are lacking a good notion of resolvent, while for one operator the resolvent is an important tool. Due to the link with the theory of the Taylor spectrum, we call $\lambda\in\mathfrak{a}_{\mathbb{C}}^*$ a *Ruelle-Taylor resonance* for the Anosov action if for some $j=0,\dots,\kappa$ the j-th cohomology is non-trivial

$$\ker d_{(X+\lambda)}|_{C^{-\infty}_{E_u^*}(\mathcal{M})\otimes\Lambda^j\mathfrak{a}_{\mathbb{C}}^*}/\operatorname{ran} d_{(X+\lambda)}|_{C^{-\infty}_{E_u^*}\otimes\Lambda^{j-1}\mathfrak{a}_{\mathbb{C}}^*}\neq 0,$$

and we call the non-trivial cohomology classes *Ruelle-Taylor resonant states*. Note that the definition of joint Ruelle resonances precisely means that the 0-th cohomology is non-trivial. Thus, any joint Ruelle resonance is a Ruelle-Taylor resonance. The converse statement is not obvious but turns out to be true, as we will prove in Proposition 4.15.

We continue with the discussion of the *leading resonances*. In view of (1.3) a resonance is called a leading resonance when its real part vanishes. We show that this spectrum carries important information about the dynamics: it is related to a special type of invariant measures as well as to mixing properties of these measures.

First, let v_g be the Riemannian measure of a fixed metric g on $\mathcal{M}$. We call a τ-invariant probability measure μ on $\mathcal{M}$, a *physical measure* if there is $v\in C^\infty(\mathcal{M})$ non-negative such that for any continuous function f and any proper open cone $\mathcal{C}\subset\mathcal{W}$,

$$\mu(f)=\lim_{T\to\infty}\frac{1}{\mathrm{Vol}(\mathcal{C}_T)}\int_{A\in\mathcal{C}_T,}\int_{\mathcal{M}}f(\varphi_1^{-X_A}(x))v(x)dv_g(x)dA \tag{1.6}$$

where $\mathcal{C}_T:=\{A\in\mathcal{C}\,|\,|A|\leq T\}$. In other words, μ is the weak Cesaro limit of a Lebesgue type measure under the dynamics. We prove the following result:

Theorem 3. *Let τ be a smooth abelian Anosov action with generating map X and let $\mathcal{W}$ be a positive Weyl chamber.*

(i) The linear span over $\mathbb{C}$ of the physical measures is isomorphic (as a $\mathbb{C}$ vector space) to $\ker d_X|_{C^{-\infty}_{E_u^*}}$, the space of joint Ruelle resonant states at $\lambda=0\in\mathfrak{a}_{\mathbb{C}}^*$; in particular, it is finite dimensional.

(ii) A probability measure μ is a physical measure if and only if it is τ-invariant and μ has wavefront set $\mathrm{WF}(\mu)\subset E_s^*$, where $E_s^*\subset T^*\mathcal{M}$ is defined by $E_s^*(E_s\oplus E_0)=0$.

(iii) Assume that there is a unique physical measure μ (or by (i) equivalently that the space of joint resonant states at 0 is one dimensional). Then the following are equivalent:

- *The only Ruelle-Taylor resonance on $i\mathfrak{a}^*$ is zero.*
- *There exists $A\in\mathcal{W}$ such that $\varphi_t^{X_A}$ is* weakly *mixing with respect to μ.*
- *For any $A\in\mathfrak{a}$, $\varphi_t^{X_A}$ is* strongly *mixing with respect to μ.*

(iv) $\lambda\in i\mathfrak{a}^*$ is a joint Ruelle resonance if and only if there is a complex measure μ_λ with $\mathrm{WF}(\mu_\lambda)\subset E_s^*$ satisfying for all $A\in\mathcal{W},t\in\mathbb{R}$ the following equivariance under push-forwards of the action: $(\varphi_t^{X_A})_*\mu_\lambda=e^{-i\lambda(A)t}\mu_\lambda$. Moreover, such measures are

absolutely continuous with respect to the physical measure obtained by taking $v = 1$ in (1.6).

(v) If $\mathcal{M}$ is connected and if there exists a smooth invariant measure μ with $\mathrm{supp}(\mu) = \mathcal{M}$*, we have for any $j = 0, \dots, \kappa$*

$$\dim\left(\ker d_X|_{C^{-\infty}_{E_u^*}(\mathcal{M})\otimes\Lambda^j\mathfrak{a}^*_{\mathbb{C}}} / \operatorname{ran} d_X|_{C^{-\infty}_{E_u^*}\otimes\Lambda^{j-1}\mathfrak{a}^*_{\mathbb{C}}}\right) = \binom{\kappa}{j}.$$

We show that the isomorphism stated in (i) and the existence of the complex measures in (iv) can be constructed explicitly in terms of spectral projectors built from the parametrix (1.5). We refer to Propositions 5.4 and 5.10 for these constructions and for slightly more complete statements.

In the case of a single Anosov flow, physical measures are known to coincide with SRB measures (see e.g. [You02] and references therein). The latter are usually defined as invariant measures that can locally be disintegrated along the stable or unstable foliation of the flow with absolutely continuous conditional densities.

We shall prove in a subsequent article that also in the case of Anosov actions the microlocal characterization Theorem 3(ii) of physical measures via their wavefront set implies that the physical measures of an Anosov action are exactly those invariant measures that allow an absolutely continuous disintegration along the stable manifolds. Furthermore, we will show that for each physical/SRB measure, there is a basin $B \subset \mathcal{M}$ of positive Lebesgue measure such that for all $f \in C^0(\mathcal{M})$, all proper open subcones $\mathcal{C} \subset \mathcal{W}$ and all $x \in B$, we have the convergence

$$\mu(f) = \lim_{T\to\infty} \frac{1}{\mathrm{Vol}(\mathcal{C}_T)} \int_{A\in\mathcal{C}_T,} f(e^{-X_A}(x))dA. \tag{1.7}$$

1.2. **Relation to previous results.** The notion of resonances for certain particular Anosov flows appeared in the work of Ruelle [Rue76], and was later extended by Pollicott [Pol85]. The introduction of a spectral approach based on anisotropic Banach and Hilbert spaces came later and allowed to the definition of resonances in the general setting, first for Anosov/Axiom A diffeomorphisms [BKL02, GL06, BT07, FRS08], then for general Anosov/Axiom A flows [Liv04, BL07, FS11, DZ16, DG16]. It was also applied to the case of pseudo Anosov maps [FGL19], Morse-Smale flows [DR18] and geodesic flows for manifolds with cusps [GBW17]. This spectral approach has been used to study SRB measures [BKL02, BL07] but it led also to several important consequences on dynamical zeta function [GLP13, DZ16, FT17a, DG16] of flows, and links with topological invariants [DZ17, DR17, DGRS20].

Higher rank $\mathbb{R}^\kappa$-Anosov actions have in particular been studied mostly for their rigidity: they are conjectured to be always smoothly conjugated to several models, mostly of algebraic nature (see e.g. the introduction of [SV19] for a precise statement and a state of the art on this question). The local rigidity of $\mathbb{R}^\kappa$-Anosov actions near *standard Anosov* *actions*[2] was proved in [KS94], and an important step of the proof relies on showing

$$\ker d_X|_{C^\infty(\mathcal{M})\otimes\Lambda^1\mathfrak{a}^*}/\operatorname{ran} d_X|_{C^\infty(\mathcal{M})} = \mathbb{C}^\kappa.$$

The main tools are based on representation theory to prove fast mixing with respect to the canonical invariant (Haar) measure. It is also conjectured in [KK95] that, more generally, for such standard actions, one has for $j = 1, \dots, \kappa - 1$

$$\ker d_X|_{C^\infty(\mathcal{M})\otimes\Lambda^j\mathfrak{a}^*}/\operatorname{ran} d_X|_{C^\infty(\mathcal{M})\otimes\Lambda^{j-1}\mathfrak{a}^*} = \mathbb{C}^{\binom{\kappa}{j}}.$$

This can be compared to (v) in Theorem 3, except that there the functional space is different. Having a notion of Ruelle-Taylor resonances provides an approach to obtain exponential mixing for more general Anosov actions by generalizing microlocal techniques for spectral gaps [NZ13, Tsu10] to a suitable class of higher rank Anosov action, and by using the functional calculus of Taylor [Tay70a, Vas79]. We believe that such tools might be very useful to obtain new results on the rigidity conjecture.

We would like to conclude by pointing out a different direction: on rank $\kappa > 1$ locally symmetric spaces $\Gamma\backslash G/K$, there is a commuting algebra of invariant differential operators that can be considered as a quantum analog to the Weyl chamber flows. If the locally symmetric space is compact, this algebra always has a discrete joint spectrum of L^2-eigenvalues. Its joint spectrum and relations to trace formulae have been studied in [DKV79]. In the forthcoming work [HWW20], it is shown that a subset of the Ruelle-Taylor resonances for the Weyl chamber flow are in correspondence with the joint discrete spectrum of the invariant differential operators on $\Gamma\backslash G/K$, giving a generalization of the classical/quantum correspondence of [DFG15, GHW18] to higher rank.

1.3. **Outline of the article.** In Section 2 we introduce the geometric setting of Anosov actions and the admissible lifts that we study. In Section 3 we explain how to define the Taylor spectrum for a certain class of unbounded operators and discuss some properties of this Taylor spectrum. In Section 4 we prove Theorem 1 and Theorem 2, using microlocal analysis. A sketch of the central techniques is given at the beginning of Section 4. The last Section 5 is devoted to the proof of Theorem 3. In Appendix A, we recall some classical results of microlocal analysis needed in the paper.

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[2]This class, defined in [KS94], consists ot Weyl chamber flows associated to rank κ locally symmetric spaces and variations of those.

2. Geometric preliminaries

2.1. **Anosov actions.** We first want to explain the geometric setting of Anosov actions and the admissible lifts that we will study.

Let $(\mathcal{M}, g)$ be a closed, smooth Riemannian manifold (normalized with volume 1) equipped with a smooth locally free action $\tau : \mathbb{A} \to \mathrm{Diffeo}(\mathcal{M})$ for an abelian Lie group $\mathbb{A} \cong \mathbb{R}^\kappa$. Let $\mathfrak{a} := \mathrm{Lie}(\mathbb{A}) \cong \mathbb{R}^\kappa$ be the associated commutative Lie algebra and $\exp : \mathfrak{a} \to \mathbb{A}$ the Lie group exponential map. After identifying $\mathbb{A} \cong \mathfrak{a} \cong \mathbb{R}^\kappa$, this exponential map is simply the identity, but it will be quite useful to have a notation that distinguishes between transformations and infinitesimal transformations. Taking the derivative of the $\mathbb{A}$-action one obtains the infinitesimal action which is an injective Lie algebra homomorphism

$$X : \begin{cases} \mathfrak{a} & \to C^\infty(\mathcal{M}; T\mathcal{M}) \\ A & \mapsto X_A := \frac{d}{dt}_{|t=0} \tau(\exp(At)) \end{cases}. \tag{2.1}$$

By commutativity of $\mathfrak{a}$, $\mathrm{ran}(X) \subset C^\infty(\mathcal{M}; T\mathcal{M})$ is a κ-dimensional subspace of commuting vector fields which span a κ-dimensional smooth subbundle which we call the *neutral subbundle* $E_0 \subset T\mathcal{M}$. Note that this subbundle is tangent to the $\mathbb{A}$-orbits on $\mathcal{M}$. It is often useful to study the one-parameter flow generated by a vector field X_A which we denote by $\varphi_t^{X_A}$. One has the obvious identity $\varphi_t^{X_A} = \tau(\exp(At))$ for $t \in \mathbb{R}$. The Riemannian metric on $\mathcal{M}$ induces norms on $T\mathcal{M}$ and $T^*\mathcal{M}$, both denoted by $\|\cdot\|$.

Definition 2.1. An element $A \in \mathfrak{a}$ and its corresponding vector field X_A are called *transversely hyperbolic* if there is a continuous splitting

$$T\mathcal{M} = E_0 \oplus E_u \oplus E_s, \tag{2.2}$$

that is invariant under the flow $\varphi_t^{X_A}$ and such that there are $\nu > 0, C > 0$ with

$$\|d\varphi_t^{X_A} v\| \leq Ce^{-\nu|t|}\|v\|, \quad \forall v \in E_s,\ \forall t \geq 0, \tag{2.3}$$

$$\|d\varphi_t^{X_A} v\| \leq Ce^{-\nu|t|}\|v\|, \quad \forall v \in E_u,\ \forall t \leq 0. \tag{2.4}$$

We say that the $\mathbb{A}$-action is *Anosov* if there exists an $A_0 \in \mathfrak{a}$ such that X_{A_0} is transversely hyperbolic.

Given a transversely hyperbolic element $A_0 \in \mathfrak{a}$ we define the *positive Weyl chamber* $\mathcal{W} \subset \mathfrak{a}$ to be the set of $A \in \mathfrak{a}$ which are transversely hyperbolic with the same stable/unstable bundle as A_0.

Lemma 2.2. *Given an Anosov action and a transversely hyperbolic element $A_0 \in \mathfrak{a}$, the positive Weyl chamber $\mathcal{W} \subset \mathfrak{a}$ is an open convex cone.*

Proof. Let us first take the $\varphi_t^{X_{A_0}}$-invariant splitting $E_0 \oplus E_u \oplus E_s$ and show that it is in fact invariant under the Anosov action τ: let $v \in E_u$ and $A \in \mathfrak{a}$. Using $[X_{A_0}, X_A] = 0$, for each $t_0 \in \mathbb{R}$ fixed and all $t \in \mathbb{R}$ we find

$$d\varphi_{-t}^{X_{A_0}} d\varphi_{t_0}^{X_A} v = d\varphi_{t_0}^{X_A} d\varphi_{-t}^{X_{A_0}} v. \tag{2.5}$$

In particular, $\|d\varphi_{-t}^{X_{A_0}} d\varphi_{t_0}^{X_A} v\|$ decays exponentially fast as $t \to +\infty$. This implies that $d\varphi_{t_0}^{X_A} v \in E_u$ and the same argument works with E_s. Next, we choose an arbitrary norm on $\mathfrak{a}$. There exist $C, C' > 0$ such that for each $v \in E_u$ we have for $t \geq 0$

$$\|d\varphi_{-t}^{X_A} v\| \leq \|d\varphi_{-t}^{X_{A-A_0}} d\varphi_{-t}^{X_{A_0}} v\| \leq C\|v\| e^{-\nu t} \|d\varphi_{-t}^{X_{A-A_0}}\| \leq C\|v\| e^{-\nu t} e^{C' t\|A-A_0\|}.$$

This implies that by choosing $\|A - A_0\|$ small enough, E_u is an unstable bundle for A as well. The same construction works for E_s and we have thus shown that $\mathcal{W}$ is open.

By re-parametrization, it is clear that $\mathcal{W}$ is a cone, so that only the convexity is left to be proved. Now, take $A_1, A_2 \in \mathcal{W}$ and let C_1, ν_1, C_2, ν_2 be the corresponding constants for the transversal hyperbolicity estimates (2.3) and (2.4). Then for $s \in [0,1]$ and $v \in E_u$ we can again use the commutativity and obtain,

$$\|d\varphi_{-t}^{X_{sA_1+(1-s)A_2}} v\| \leq C_1 C_2 e^{-\nu_1 st - \nu_2 (1-s)t} \|v\| \tag{2.6}$$

and this shows that $sA_1 + (1-s)A_2 \in \mathcal{W}$. □

There is an important class of examples given by the Weyl chamber flow on Riemannian locally symmetric spaces.

Example 2.3. Consider a real semi-simple Lie group $\mathbb{G}$, connected and of non-compact type, and let $\mathbb{G} = \mathbb{K}\mathbb{A}\mathbb{N}$ be an Iwasawa decomposition with $\mathbb{A}$ abelian, $\mathbb{K}$ the compact maximal subgroup and $\mathbb{N}$ nilpotent. Then $\mathbb{A} \cong \mathbb{R}^\kappa$ and κ is called the *real rank* of $\mathbb{G}$. Let $\mathfrak{a}$ be the Lie algebra of $\mathbb{A}$ and consider the adjoint action of $\mathfrak{a}$ on $\mathfrak{g}$ which leads to the definition of a finite set of *restricted roots* $\Delta \subset \mathfrak{a}^*$. For $\alpha \in \Delta$ let $\mathfrak{g}_\alpha$ be the associated root space. It is then possible to choose a set of positive roots $\Delta_+ \subset \Delta$ and with respect to this choice there is an algebraic definition of a positive Weyl chamber

$$\mathcal{W} := \{A \in \mathfrak{a} \,|\, \alpha(A) > 0 \text{ for all } \alpha \in \Delta_+\}.$$

If one now considers $\Gamma < \mathbb{G}$ a torsion free, discrete, co-compact subgroup one can define the biquotient $\mathcal{M} := \Gamma\backslash\mathbb{G}/\mathbb{M}$ where $\mathbb{M} \subset \mathbb{K}$ is the centralizer of $\mathbb{A}$ in $\mathbb{K}$. As $\mathbb{A}$ commutes with $\mathbb{M}$, the space $\mathcal{M}$ carries a right $\mathbb{A}$-action. Using the definition of roots, it is direct to see that this is an Anosov action: all elements of the positive Weyl chamber $\mathcal{W}$ are transversely hyperbolic elements sharing the same stable/unstable distributions given by the associated vector bundles:

$$E_0 = \mathbb{G} \times_{\mathbb{M}} \mathfrak{a}, \;\; E_s = \mathbb{G} \times_{\mathbb{M}} \mathfrak{n}, \;\; E_u = \mathbb{G} \times_{\mathbb{M}} \overline{\mathfrak{n}}.$$

Here $\mathfrak{n} := \sum_{\alpha\in\Delta_+} \mathfrak{g}_\alpha$ and $\overline{\mathfrak{n}} := \sum_{-\alpha\in\Delta_+} \mathfrak{g}_\alpha$ are the sums of all positive, respectively negative root spaces, and $\mathfrak{n}$ coincides with the Lie algebra of the nilpotent group $\mathbb{N}$.

Note that there are various other constructions of Anosov actions and we refer to [KS94, Section 2.2] for further examples.

2.2. **Admissible lifts.** We want to establish the spectral theory not only for the commuting vector fields X_A that act as first order differential operators on $C^\infty(\mathcal{M})$ but also for first order differential operators on Riemannian vector bundles $E \to \mathcal{M}$ which lift the Anosov action.

Definition 2.4. Let $\mathcal{M}$ be a closed manifold with an Anosov of $\mathbb{A} \cong \mathbb{R}^\kappa$ and generating map x. Let $E \to \mathcal{M}$ be the complexification of a smooth Riemannian vector bundle over $\mathcal{M}$. Denote by $\mathrm{Diff}^1(\mathcal{M}; E)$ the Lie algebra of first order differential operators with smooth coefficients acting on sections of E. Then we call a Lie algebra homomorphism

$$\mathbf{X} : \mathfrak{a} \to \mathrm{Diff}^1(\mathcal{M}; E),$$

an *admissible lift* of the Anosov action if for any section $s \in C^\infty(\mathcal{M}; E)$ and any function $f \in C^\infty(\mathcal{M})$ one has

$$\mathbf{X}_A(fs) = (X_A f)s + f\mathbf{X}_A s. \tag{2.7}$$

A typical example to have in mind would be when E is a tensor bundle, (e.g. exterior power of the cotangent bundle $E = \Lambda^m T^*\mathcal{M}$ or symmetric tensors $E = \otimes_S^m T^*\mathcal{M}$), and

$$\mathbf{X}_A s := \mathcal{L}_{X_A} s$$

where $\mathcal{L}$ denotes the Lie derivative. This admissible lift can be restricted to any subbundle that is invariant under the differentials $d\varphi_t^{X_A}$ for all $A \in \mathfrak{a}, t > 0$. More generally, the example of a Lie derivative can be seen as a special case where the $\mathbb{A}$-action τ on $\mathcal{M}$ lifts to an action $\widetilde{\tau}$ on E which is fiberwise linear. Then one can define an infinitesimal action

$$\mathbf{X}_A s(x) := \partial_t \tilde{\tau}(\exp(-At)) s(\tau(\exp(At))x)|_{t=0} \tag{2.8}$$

which is an admissible lift.

3. Taylor spectrum and Fredholm complex

The Taylor spectrum was introduced by Taylor in [Tay70b, Tay70a] as a joint spectrum for commuting bounded operators, using the theory of Koszul complexes. While there are different competing notions of joint spectra (see e.g. the lecture notes [Cur88]), the Taylor spectrum is from many perspectives the most natural notion. Its attractive feature is that it is defined in terms of operators acting on Hilbert spaces and does not depend on a choice of an ambient commutative Banach algebra. Furthermore, it comes with a satisfactory analytic functional calculus developed by Taylor and Vasilescu [Tay70a, Vas79].

3.1. **Taylor spectrum for unbounded operators.** Most references introduce the Taylor spectrum for tuples of bounded operators. In our case, we need to deal with unbounded operators. Additionally, working with a tuple implies choosing a basis, which should not be necessary. Let us thus explain how the notion of Taylor spectrum can easily be extended to an important class of abelian actions by unbounded operators.

We start with $E \to \mathcal{M}$ a smooth complex vector bundle over a smooth manifold $\mathcal{M}$, $\mathfrak{a} \cong \mathbb{R}^\kappa$ an abelian Lie algebra and $\mathbf{X} : \mathfrak{a} \to \mathrm{Diff}^1(\mathcal{M}; E)$ a Lie algebra morphism. For the moment we do not have to assume that $\mathcal{M}$ possesses an Anosov action nor that $\mathbf{X}$ is an admissible lift. By linearity $\mathbf{X}$ extends to a Lie algebra morphism $\mathbf{X} : \mathfrak{a}_\mathbb{C} \to \mathrm{Diff}^1(\mathcal{M}; E)$ and for the definition of the spectra we will need to work with this complexified version. Using the Lie algebra morphism $\mathbf{X}$ we define

$$d_\mathbf{X} : \begin{cases} C_c^\infty(\mathcal{M}; E) & \to C_c^\infty(\mathcal{M}; E) \otimes \mathfrak{a}_\mathbb{C}^* \\ u & \mapsto \mathbf{X}u \end{cases}$$

where we have set $(\mathbf{X}u)(A) := \mathbf{X}_A u$ for each $A \in \mathfrak{a}_{\mathbb{C}}$. This will be the central ingredient to define the Koszul complex which will lead to the definition of the Taylor spectrum. In order to do this we need some more notation: we denote by $\Lambda\mathfrak{a}^*_{\mathbb{C}} := \bigoplus_{\ell=0}^{\kappa} \Lambda^\ell \mathfrak{a}^*_{\mathbb{C}}$ the exterior algebra of $\mathfrak{a}^*_{\mathbb{C}}$ — this is just a coordinate-free version of $\Lambda\mathbb{C}^\kappa$. Given a topological vector space V we use the shorthand notation $V\Lambda^\ell := V \otimes \Lambda^\ell \mathfrak{a}^*_{\mathbb{C}}$ and $V\Lambda := V \otimes \Lambda\mathfrak{a}^*_{\mathbb{C}}$. As $\Lambda\mathfrak{a}^*_{\mathbb{C}}$ is finite dimensional $V\Lambda$ is again a topological vector space. We have the *contraction* and *exterior product* maps

$$\iota : \begin{cases} \mathfrak{a}_{\mathbb{C}} \times V\Lambda^\ell & \to V\Lambda^{\ell-1} \\ (A, v \otimes \omega) & \mapsto \iota_A(v \otimes \omega) := v \otimes (\iota_A \omega) \end{cases} \text{ and } \wedge : \begin{cases} V\Lambda^\ell \times \Lambda^r \mathfrak{a}^*_{\mathbb{C}} & \to V\Lambda^{\ell+r} \\ (v \otimes \omega, \eta) & \mapsto v \otimes (\omega \wedge \eta). \end{cases}$$

We can then extend $d_{\mathbf{X}}$ to a continuous map on the spaces $C^\infty_c\Lambda := C^\infty_c(\mathcal{M}; E) \otimes \Lambda\mathfrak{a}^*_{\mathbb{C}}$ (resp. $C^{-\infty}\Lambda := C^{-\infty}(\mathcal{M}; E) \otimes \Lambda\mathfrak{a}^*_{\mathbb{C}}$) by setting for each $u \in C^\infty_c(\mathcal{M}; E)$ (resp. $u \in C^{-\infty}(\mathcal{M}; E)$) and $\omega \in \Lambda^\ell \mathfrak{a}^*_{\mathbb{C}}$

$$d_{\mathbf{X}} : u \otimes \omega \mapsto (d_{\mathbf{X}} u) \wedge \omega.$$

Similarly, for each $A \in \mathfrak{a}$ we will also extend $\mathbf{X}_A$ on these spaces by setting

$$\mathbf{X}_A(u \otimes \omega) := \mathbf{X}_A u \otimes \omega.$$

Remark 3.1. Choosing a basis $A_1, \dots, A_\kappa \in \mathfrak{a}$ provides an isomorphism $\Lambda\mathfrak{a}^* \cong \Lambda\mathbb{R}^\kappa$. One checks that under this isomorphism the coordinate free version $d_{\mathbf{X}} : V \otimes \Lambda^\ell \mathfrak{a}^* \to V \otimes \Lambda^{\ell+1}\mathfrak{a}^*$ of the Taylor differential transforms to the Taylor differential $d_X : V \otimes \Lambda^\ell \mathbb{R}^\kappa \to V \otimes \Lambda^{\ell+1}\mathbb{R}^\kappa$ of the operator tuple $X = (\mathbf{X}_{A_1}, \dots, \mathbf{X}_{A_\kappa})$ defined as

$$d_X(u \otimes e_{i_1} \wedge \dots \wedge e_{i_j}) := \sum_{k=1}^{\kappa} (\mathbf{X}_{A_k} u) \otimes e_k \wedge e_{i_1} \wedge \dots \wedge e_{i_j} \tag{3.1}$$

if the basis $(e_j)_j$ of $\mathbb{R}^\kappa$ is identified to the dual basis of $(A_j)_j$ in $\mathfrak{a}^*$.

Lemma 3.2. *For each $A \in \mathfrak{a}_{\mathbb{C}}$ one has the following identities as continuous operators on $C^\infty_c\Lambda$ and $C^{-\infty}\Lambda$:*

(i) $\iota_A d_{\mathbf{X}} + d_{\mathbf{X}} \iota_A = \mathbf{X}_A$,
(ii) $\mathbf{X}_A d_{\mathbf{X}} = d_{\mathbf{X}} \mathbf{X}_A$,
(iii) $d_{\mathbf{X}} d_{\mathbf{X}} = 0$.

Proof. Let $u \otimes \omega \in C^\infty_c\Lambda$ or $u \otimes \omega \in C^{-\infty}\Lambda$. Then by definition

$$\iota_A d_{\mathbf{X}}(u \otimes \omega) = \iota_A((d_{\mathbf{X}} u) \wedge \omega) = (\mathbf{X}_A u) \otimes \omega - d_{\mathbf{X}} u \wedge (\iota_A \omega) = (\mathbf{X}_A - d_{\mathbf{X}} \iota_A)(u \otimes \omega)$$

which yields (i). In order to prove (ii) it suffices, by definition of $d_{\mathbf{X}}$, to prove the identity as a map $C^\infty_c(\mathcal{M}; E) \to C^\infty_c(\mathcal{M}; E) \otimes \mathfrak{a}^*_{\mathbb{C}}$. Take an arbitrary $A' \in \mathfrak{a}_{\mathbb{C}}$ and $u \in C^\infty_c(\mathcal{M}; E)$, then

$$\iota_{A'}(\mathbf{X}_A d_{\mathbf{X}} - d_{\mathbf{X}} \mathbf{X}_A) u = (\mathbf{X}_A \mathbf{X}_{A'} - \mathbf{X}_{A'} \mathbf{X}_A) u = 0$$

which proves the statement. Note that we crucially use the commutativity of the differential operators $\mathbf{X}_A$ in this step.

For (iii) we first conclude from (i) and (ii) that $\iota_A d_{\mathbf{X}} d_{\mathbf{X}} = d_{\mathbf{X}} d_{\mathbf{X}} \iota_A$. Using this identity we deduce that for $u \in C_c^\infty \Lambda^\ell$ and arbitrary $A_1, \dots A_{\ell+1} \in \mathfrak{a}_{\mathbb{C}}$

$$\iota_{A_1} \dots \iota_{A_{\ell+1}} d_{\mathbf{X}} d_{\mathbf{X}} u = 0,$$

which implies $d_{\mathbf{X}} d_{\mathbf{X}} u = 0$. $\square$

As a direct consequence of Lemma 3.2(iii) we conclude that

$$0 \longrightarrow C_c^\infty \Lambda^0 \xrightarrow{d_{\mathbf{X}}} C_c^\infty \Lambda^1 \xrightarrow{d_{\mathbf{X}}} \dots \xrightarrow{d_{\mathbf{X}}} C_c^\infty \Lambda^\kappa \longrightarrow 0 \tag{3.2}$$

and

$$0 \longrightarrow C^{-\infty} \Lambda^0 \xrightarrow{d_{\mathbf{X}}} C^{-\infty} \Lambda^1 \xrightarrow{d_{\mathbf{X}}} \dots \xrightarrow{d_{\mathbf{X}}} C^{-\infty} \Lambda^\kappa \longrightarrow 0 \tag{3.3}$$

are complexes.

We now want to construct a complex of bounded operators on Hilbert spaces which lies between the complexes on $C_c^\infty \Lambda$ and $C^{-\infty}\Lambda$. For this, we consider $\mathcal{H}$ a Hilbert space with continuous embeddings $C_c^\infty(\mathcal{M}; E) \subset \mathcal{H} \subset C^{-\infty}(\mathcal{M}; E)$ such that $C_c^\infty(\mathcal{M}; E)$ is a dense subspace of $\mathcal{H}$. If we fix a non-degenerate Hermitian inner product $\langle \cdot, \cdot \rangle_{\mathfrak{a}_{\mathbb{C}}^*}$, then this induces a scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}\Lambda}$ and gives a Hilbert space structure on $\mathcal{H}\Lambda$. While the precise value of $\langle \cdot, \cdot \rangle_{\mathcal{H}\Lambda}$ obviously depends on the choice of the Hermitian product on $\mathfrak{a}_{\mathbb{C}}^*$, the finite dimensionality of $\mathfrak{a}_{\mathbb{C}}^*$ implies that all Hilbert space structures on $\mathcal{H}\Lambda$ obtained in this way are equivalent. Note that on the Hilbert spaces $\mathcal{H}\Lambda^\ell$ the operators $d_{\mathbf{X}}$ will in general be unbounded operators. However, we have:

Lemma 3.3. *For any choice of a non-degenerate Hermitian product on $\mathfrak{a}_{\mathbb{C}}^*$, the vector space $\mathcal{D}(d_{\mathbf{X}}) := \{u \in \mathcal{H}\Lambda \,|\, d_{\mathbf{X}} u \in \mathcal{H}\Lambda\}$ becomes a Hilbert space when endowed with the scalar product*

$$\langle \cdot, \cdot \rangle_{\mathcal{D}(d_{\mathbf{X}})} := \langle \cdot, \cdot \rangle_{\mathcal{H}\Lambda} + \langle d_{\mathbf{X}} \cdot, d_{\mathbf{X}} \cdot \rangle_{\mathcal{H}\Lambda}. \tag{3.4}$$

Furthermore, all scalar products obtained this way are equivalent and induce the same topology on $\mathcal{D}(d_{\mathbf{X}})$. Finally, $d_{\mathbf{X}}$ is bounded *on $\mathcal{D}(d_{\mathbf{X}})$.*

Proof. We have to check that $\mathcal{D}(d_{\mathbf{X}})$ is complete with respect to the topology of $\langle \cdot, \cdot \rangle_{\mathcal{D}(d_{\mathbf{X}})}$: suppose u_n is a Cauchy sequence in $\mathcal{D}(d_{\mathbf{X}})$, then u_n and $d_{\mathbf{X}} u_n$ are Cauchy sequences in $\mathcal{H}$ and we denote by $v_0, v_1 \in \mathcal{H}\Lambda$ their respective limits. By the continuous embedding $\mathcal{H} \subset C^{-\infty}(\mathcal{M}; E)$ and the continuity of $d_{\mathbf{X}}$ on $C^{-\infty}(\mathcal{M}; E)$ we deduce

$$v_1 = \lim_{n \to \infty} d_{\mathbf{X}} u_n = d_{\mathbf{X}} \lim_{n \to \infty} u_n = d_{\mathbf{X}} v_0,$$

in $C^{-\infty}(\mathcal{M}; E)$ which proves the completeness. For the boundedness, we take $u \in \mathcal{D}(d_{\mathbf{X}})$ and we compute

$$\|d_{\mathbf{X}} u\|^2_{\mathcal{D}(d_{\mathbf{X}})} = \|d_{\mathbf{X}} u\|^2_{\mathcal{H}\Lambda} + \|d_{\mathbf{X}} d_{\mathbf{X}} u\|^2_{\mathcal{H}\Lambda} \leq \|u\|^2_{\mathcal{D}(d_{\mathbf{X}})}.$$ $\square$

To be able to use the usual techniques, it is crucial that $C^\infty(\mathcal{M}; E)$ is not only dense in $\mathcal{H}$ but also in $\mathcal{D}(d_{\mathbf{X}})$ — on this level of generality, this is not a priori guaranteed. For this reason, we say the $\mathfrak{a}$-action $\mathbf{X}$ has a *unique extension* to $\mathcal{H}$ if

$$\overline{C_c^\infty(\mathcal{M}; E \otimes \Lambda)}^{\mathcal{D}(d_{\mathbf{X}})} = \mathcal{D}(d_{\mathbf{X}}). \tag{3.5}$$

We note that by [FS11, Lemma A.1], if $\mathcal{M}$ is a closed manifold and if $\mathcal{H} = \mathcal{A}(L^2(\mathcal{M}, E))$ for some invertible pseudo-differential operator $\mathcal{A}$ on $\mathcal{M}$ so that $\mathcal{A}^{-1} d_{\mathbf{X}} \mathcal{A} \in \Psi^1(\mathcal{M}; E)$ (see Appendix A for the notation), then $C^\infty(\mathcal{M}; E \otimes \Lambda)$ is dense in the domain $\mathcal{D}(d_{\mathbf{X}})$ and there is only one closed extension for $d_{\mathbf{X}}$.

In order to finally define the Taylor spectrum in an invariant way, we consider $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ as a Lie algebra morphism

$$\lambda : \mathfrak{a}_{\mathbb{C}} \to \mathrm{Diff}^0(\mathcal{M}; E) \subset \mathrm{Diff}^1(\mathcal{M}; E), \quad \lambda(A)(u) := \lambda(A)u.$$

In this way we can define $\mathbf{X} - \lambda : \mathfrak{a}_{\mathbb{C}} \to \mathrm{Diff}^1(\mathcal{M}; E)$ and the associated operator $d_{\mathbf{X}-\lambda}$ on $C_c^\infty \Lambda$ and $C^{-\infty}\Lambda$. Since $d_{\mathbf{X}-\lambda} = d_{\mathbf{X}} - d_\lambda$, and d_λ is bounded on $\mathcal{H}\Lambda$, $\mathcal{D}(d_{\mathbf{X}-\lambda})$ does not depend on λ. For $k = 0, \dots, \kappa$, we write $\mathcal{D}^k(d_{\mathbf{X}}) := \mathcal{D}(d_{\mathbf{X}}) \cap \mathcal{H}\Lambda^k$ and we gather the results above in the following

Lemma 3.4. *For an $\mathfrak{a}$-action with unique closed extension to $\mathcal{H}$, for any $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$*

$$0 \longrightarrow \mathcal{D}^0(d_{\mathbf{X}}) \xrightarrow{d_{\mathbf{X}-\lambda}} \mathcal{D}^1(d_{\mathbf{X}}) \xrightarrow{d_{\mathbf{X}-\lambda}} \dots \xrightarrow{d_{\mathbf{X}-\lambda}} \mathcal{D}^\kappa(d_{\mathbf{X}}) \longrightarrow 0 \tag{3.6}$$

defines a complex of bounded operators, and the operators $d_{\mathbf{X}-\lambda}$ depend holomorphically on $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.

We introduce the notation

$$\begin{aligned} \ker_{\mathcal{H}\Lambda} d_{\mathbf{X}-\lambda} &:= \ker_{\mathcal{D}(d_{\mathbf{X}}) \to \mathcal{D}(d_{\mathbf{X}})} d_{\mathbf{X}-\lambda}, \quad \mathrm{ran}_{\mathcal{H}\Lambda} d_{\mathbf{X}-\lambda} := \mathrm{ran}_{\mathcal{D}(d_{\mathbf{X}}) \to \mathcal{D}(d_{\mathbf{X}})} d_{\mathbf{X}-\lambda} \\ \ker_{\mathcal{H}\Lambda^j} d_{\mathbf{X}-\lambda} &:= \ker_{\mathcal{D}^j(d_{\mathbf{X}}) \to \mathcal{D}^{j+1}(d_{\mathbf{X}})} d_{\mathbf{X}-\lambda}, \quad \mathrm{ran}_{\mathcal{H}\Lambda^j} d_{\mathbf{X}-\lambda} := \mathrm{ran}_{\mathcal{D}^{j-1}(d_{\mathbf{X}}) \to \mathcal{D}^j(d_{\mathbf{X}})} d_{\mathbf{X}-\lambda}. \end{aligned} \tag{3.7}$$

Now, following the previous discussion of the Taylor spectrum, we can define

Definition 3.5. Let $\mathbf{X} : \mathfrak{a} \to \mathrm{Diff}^1(\mathcal{M}; E)$ be a Lie algebra morphism. Then we define the *Taylor spectrum* $\sigma_{\mathrm{T},\mathcal{H}}(\mathbf{X}) \subset \mathfrak{a}_{\mathbb{C}}^*$ by

$$\lambda \in \sigma_{\mathrm{T},\mathcal{H}}(\mathbf{X}) \Longleftrightarrow \mathrm{ran}_{\mathcal{H}\Lambda}(d_{\mathbf{X}-\lambda}) \neq \ker_{\mathcal{H}\Lambda}(d_{\mathbf{X}-\lambda}).$$

This is equivalent to saying that the complex (3.6) is not exact. The complex is said to be *Fredholm* if $\mathrm{ran}_{\mathcal{H}\Lambda}(d_{\mathbf{X}-\lambda})$ is closed and the cohomology $\ker_{\mathcal{H}\Lambda}(d_{\mathbf{X}-\lambda}) / \mathrm{ran}_{\mathcal{H}\Lambda}(d_{\mathbf{X}-\lambda})$ has finite dimension. In this case we say that λ is not in the *essential Taylor spectrum* $\sigma_{\mathrm{T},\mathcal{H}}^{\mathrm{ess}}(\mathbf{X})$ of $\mathbf{X}$ and define the *index* by

$$\mathrm{index}(\mathbf{X} - \lambda) := \sum_{\ell=0}^{\kappa} (-1)^\ell \dim(\ker_{\mathcal{H}\Lambda^\ell} d_{\mathbf{X}-\lambda} / \mathrm{ran}_{\mathcal{H}\Lambda^\ell} d_{\mathbf{X}-\lambda}). \tag{3.8}$$

As the usual Fredholm index, it is also a locally constant function of λ (see Theorem 6.6 in [Cur88]).

Note that the non-vanishing of the 0-th cohomology $\ker_{\mathcal{H}\Lambda^0} d_{\mathbf{X}-\lambda}$ of the complex is equivalent to

$$\exists u \in \mathcal{D}^0(d_{\mathbf{X}}) \setminus \{0\}, \quad (\mathbf{X}_{A_j} - \lambda_j)u = 0,$$

which corresponds to $(\lambda_1, \dots, \lambda_\kappa)$ being a joint eigenvalue of $(\mathbf{X}_{A_1}, \dots, \mathbf{X}_{A_\kappa})$. Obviously, on infinite dimensional vector spaces the joint eigenvalues do not provide a satisfactory notion of joint spectrum. Recall that for a single operator, $\lambda \in \mathbb{C}$ is in its spectrum if $\mathbf{X} - \lambda$ is either not injective or not surjective. In terms of the Taylor complex for a single operator ($\kappa = 1$) the non-injectivity corresponds to the vanishing of the zeroth cohomology group whereas the surjectivity corresponds to the vanishing of the first cohomology group. For several commuting operators the vanishing of the higher cohomology groups can thus be interpreted as a replacement of the surjectivity condition for a single operator.

3.2. **Useful observations.** For the reader not familiar with the Taylor spectrum, and for our own use, we have gathered in this section several observations that are helpful when manipulating these objects.

As usual with differential complexes, we have a dual notion of divergence complex. For this, we need a way to identify $\mathfrak{a}$ with $\mathfrak{a}^*$, i.e. a scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{a}$, extended to a $\mathbb{C}$-bilinear two form. If one chooses a basis, the implicit scalar product is given by the standard one in that basis. In any case, $A \mapsto A' := \langle A, \cdot \rangle$ is an isomorphism between $\mathfrak{a}$ and $\mathfrak{a}^*$. If Y is another abelian $\mathfrak{a}$-action and $\mathbf{Y}$ an admissible lift on E, we can define the action $\mathbf{Y}' : \mathfrak{a} \mapsto \mathrm{Diff}^1(\mathcal{M}; E)$ by: for $u \in C_c^\infty(\mathcal{M}; E)$ and $A \in \mathfrak{a}$

$$A'(\mathbf{Y}'u) := \mathbf{Y}u(A).$$

In this fashion, $d_{\mathbf{Y}'}u := \mathbf{Y}'u$ is an element of $C_c^\infty(\mathcal{M}; E) \otimes \mathfrak{a}$ while $d_{\mathbf{Y}}u$ is an element of $C_c^\infty(\mathcal{M}; E) \otimes \mathfrak{a}^*$. We can thus define the operator

$$\delta_{\mathbf{Y}} : \begin{cases} C_c^\infty(\mathcal{M}; E) \otimes \Lambda^j \mathfrak{a}^*_{\mathbb{C}} & \rightarrow & C_c^\infty(\mathcal{M}; E) \otimes \Lambda^{j-1} \mathfrak{a}^*_{\mathbb{C}} \\ u \otimes \omega & \mapsto & \imath_{\mathbf{Y}'u}\omega \end{cases}.$$

In an orthogonal basis $(e_j)_j$ of $\mathfrak{a}$ for $\langle \cdot, \cdot \rangle$ and $(e_j')_j$ the dual basis in $\mathfrak{a}^*$, we get for $u \in C_c^\infty(\mathcal{M}; E)$ and $\omega = e'_{i_1} \wedge \cdots \wedge e'_{i_\ell}$

$$\delta_{\mathbf{Y}}(u \otimes \omega) = \sum_{j=1}^{\ell} (-1)^j (\mathbf{Y}_{e_{i_j}} u) e'_{i_1} \wedge \cdots \wedge \widehat{e'_{i_j}} \wedge \cdots \wedge e'_{i_\ell}.$$

We get directly that for $A' \in \mathfrak{a}^*$

$$A' \wedge \delta_{\mathbf{Y}}(u \otimes \omega) + \delta_{\mathbf{Y}}(A' \wedge (u \otimes \omega)) = A' \wedge \imath_{\mathbf{Y}'u}\omega + \imath_{\mathbf{Y}'u}(A' \wedge \omega) = (A'(\mathbf{Y}'u)) \otimes \omega.$$

It follows from similar arguments as before that

$$\mathbf{Y}_A \delta_{\mathbf{Y}} = \delta_{\mathbf{Y}} \mathbf{Y}_A, \quad \delta_{\mathbf{Y}} \delta_{\mathbf{Y}} = 0.$$

We have the following

Lemma 3.6. *Let $\mathbf{X}$ and $\mathbf{Y}$ be two admissible lifts of abelian actions of $\mathfrak{a}$, with common dense domain $F \subset \mathcal{H}$, such that $\mathbf{XY}, \mathbf{YX} : F \otimes \mathfrak{a}^{\otimes 2} \to \mathcal{H}$ are well-defined and $\mathbf{X}_A \mathbf{Y}_B = \mathbf{Y}_B \mathbf{X}_A$ for $A, B \in \mathfrak{a}$. For $(e_j)_j$ an orthonormal basis of $\mathfrak{a}$ for $\langle \cdot, \cdot \rangle$ and if $\mathbf{X}_i := \mathbf{X}_{e_i}$ and $\mathbf{Y}_j := \mathbf{Y}_{e_j}$, we then have*

$$\delta_{\mathbf{Y}} d_{\mathbf{X}} + d_{\mathbf{X}} \delta_{\mathbf{Y}} = -\Big(\sum_{k=1}^{\kappa} \mathbf{X}_k \mathbf{Y}_k \Big) \otimes \mathrm{Id},$$

taking an orthonormal basis for $\langle \cdot, \cdot \rangle$. The sum does not depend on the choice of basis, because it is the trace of the matrix representing $\mathbf{XY}$ with $\langle \cdot, \cdot \rangle$.

Proof. We compute in an orthogonal basis for $\langle\cdot,\cdot\rangle$, for $u\in F$, $\omega = e'_{i_1}\wedge\cdots\wedge e'_{i_\ell}$

$$d_{\mathbf{X}}\delta_{\mathbf{Y}}(u\otimes\omega) = \\ -\Big(\sum_{k\in I}(\mathbf{X}_k\mathbf{Y}_k u)\otimes e_I + \sum_{k\notin I,j}(-1)^{j-1}(\mathbf{X}_k\mathbf{Y}_{i_j}u)\otimes e'_k\wedge e'_{i_1}\wedge\cdots\wedge\widehat{e'_{i_j}}\wedge\cdots\wedge e'_{i_\ell}\Big),$$

$$\delta_{\mathbf{Y}}d_{\mathbf{X}}(u\otimes\omega) = \\ -\Big(\sum_{k\notin I}(\mathbf{Y}_k\mathbf{X}_k u)\otimes e_I + \sum_{k\notin I,j}(-1)^{j}(\mathbf{Y}_{i_j}\mathbf{X}_k u)\otimes e_k\wedge e_{i_1}\wedge\cdots\wedge\widehat{e}_{i_j}\wedge\cdots\wedge e_{i_\ell}\Big).$$

Using the commutation of $[\mathbf{X}_i,\mathbf{Y}_j]=0$, we obtain the result. □

As an illustration, let us recall the following classical fact:

Lemma 3.7. *Let $X_1,\dots,X_\kappa$ be commuting operators on a finite dimensional vector space V. Then $\sigma_{T,V}(X) = \{\text{joint eigenvalues of } X_1,\dots,X_\kappa\}\subset\mathbb{C}^\kappa$.*

Proof. By the basic theory of weight spaces (see e.g. [Kna02][Proposition 2.4]) V can be decomposed into generalized weight spaces, i.e. there are finitely many $\lambda^{(j)} = (\lambda^{(j)}_1,\dots\lambda^{(j)}_\kappa)\in\mathbb{C}^\kappa$ and a direct sum decomposition $V=\oplus_j V_j$ which is invariant under all $X_1,\dots,X_\kappa$ and there are n_j such that

$$(X_i-\lambda_i^{(j)})^{n_j}_{|V_j} = 0$$

Commutativity and the Jordan normal form then implies that the $\lambda^{(j)}$ are precisely the joint eigenvalues of the tuple X. Now let $\mu\neq\lambda^{(j)}$ for all j. We have to prove that $\mu\notin\sigma_{\mathrm{T},V}(X)$. By $\mu\neq\lambda^{(j)}$ we deduce that for any j there is at least one $1\le k_j\le\kappa$ such that $\mu_{k_j}\neq\lambda^{(j)}_{k_j}$ and again by Jordan normal form $(X_{k_j}-\mu_{k_j}): V_j\to V_j$ is invertible. Setting $\widetilde{V}_k := \oplus_{k_j=k}V_j$, we can thus find an X invariant decomposition $V=\oplus_{k=0}^{\kappa}\widetilde{V}_k$ such that $X_k-\mu_k:\widetilde{V}_k\to\widetilde{V}_k$ is invertible. Let Π_k be the projection onto V_k and set $Y_k := (X_k-\mu_k)^{-1}\Pi_k : V\to V$. Then the Y_k satisfy all the assumptions of Lemma 3.6 and

$$\delta_Y d_{(X-\mu)} + d_{(X-\mu)}\delta_Y = -\operatorname{Id}.$$

Consequently the *Taylor complex* (3.6) is exact. □

In the particular case that $X=(X_1,\dots,X_n)$ are symmetric matrices, using the spectral theorem, we can assume that they are just scalars. From this we deduce that for $\lambda\in\sigma_{\mathrm{T}}(X)$, if m is the dimension of the corresponding common eigenspace,

$$\dim(\ker_{\Lambda^k} d_{X-\lambda}/\operatorname{ran}_{\Lambda^k} d_{X-\lambda}) = \dim(\mathbb{R}^m\otimes\Lambda^k\mathbb{R}^n) = m\binom{n}{k},$$

and we check that

$$\operatorname{index}(X-\lambda) = m\sum_{k=1}^{n}(-1)^k\binom{n}{k} = 0.$$

Our next step is to give a criterion for $d_{\mathbf{X}-\lambda}$ to be Fredholm. We first notice that since $\operatorname{ran} d_{\mathbf{X}-\lambda}\subset\ker d_{\mathbf{X}-\lambda}$, the closedness of $\operatorname{ran} d_{\mathbf{X}-\lambda}$ in $\mathcal{D}(d_{\mathbf{X}})$ or in $\mathcal{H}\Lambda$ is equivalent. We shall use the following criterion for the $d_{\mathbf{X}}$-complex to be Fredholm.

Lemma 3.8. *Let* $\mathbf{X}$ *be an* $\mathfrak{a}$*-action with unique extension to* $\mathcal{H}$*. Assume that there are bounded operators* Q*,* R *and* K *on* $\mathcal{H}\Lambda$*, acting continuously on* $C^{-\infty}(\mathcal{M};E)\Lambda$*, such that* K *is compact,* $\|R\|_{\mathcal{L}(\mathcal{H}\Lambda)} < 1$*, and*

$$Qd_{\mathbf{X}} + d_{\mathbf{X}}Q = \mathrm{Id} + R + K.$$

Then the complex defined by $d_{\mathbf{X}}$ *is Fredholm. Denote by* Π_0 *the projector on the eigenvalue* 0 *of* $\mathrm{Id} + R + K$*, which is bounded on* $\mathcal{D}(d_{\mathbf{X}})$ *and commutes with* $d_{\mathbf{X}}$*. Then the map* $u \mapsto \Pi_0 u$ *from* $\ker d_{\mathbf{X}} \cap \mathcal{D}(d_{\mathbf{X}})$ *to* $\ker d_{\mathbf{X}} \cap \operatorname{ran}\Pi_0$ *factors to an isomorphism*

$$\Pi_0 : \ker_{\mathcal{D}(d_{\mathbf{X}})} d_{\mathbf{X}} / \operatorname{ran}_{\mathcal{D}(d_{\mathbf{X}})} d_{\mathbf{X}} \to \ker_{\operatorname{ran}\Pi_0} d_{\mathbf{X}} / \operatorname{ran}_{\operatorname{ran}\Pi_0} d_{\mathbf{X}}. \tag{3.9}$$

Proof. First, since Q, R and K are continuous on distributions, it makes sense to write $d_{\mathbf{X}}Q + Qd_{\mathbf{X}} = \mathrm{Id} + R + K$ in the distribution sense. Further, from this relation, we deduce that Q is bounded on $\mathcal{D}(d_{\mathbf{X}})$. Additionally, without loss of generality (by modifying R) we can assume that K is a finite rank operator.

Let us prove that the range of $d_{\mathbf{X}}$ is closed. Consider $u \in (\ker d_{\mathbf{X}})^{\perp} \cap \mathcal{D}(d_{\mathbf{X}})$. Since $d_{\mathbf{X}}Qu \in \operatorname{ran}(d_{\mathbf{X}}) \subset \ker d_{\mathbf{X}}$

$$\langle (\mathrm{Id} + R + K)u, u\rangle_{\mathcal{H}\Lambda} = \langle Qd_{\mathbf{X}}u, u\rangle_{\mathcal{H}\Lambda}. \tag{3.10}$$

We get that there is $C > 0$ such that for each $u \in (\ker d_{\mathbf{X}})^{\perp} \cap \mathcal{D}(d_{\mathbf{X}})$ we have

$$(1 - \|R\|)\|u\|_{\mathcal{H}\Lambda} - \|Ku\|_{\mathcal{H}\Lambda} \leq C\|d_{\mathbf{X}}u\|_{\mathcal{H}\Lambda}. \tag{3.11}$$

Since K is of finite rank, we obtain by a standard argument that $d_{\mathbf{X}}$ has closed range (both in $\mathcal{H}\Lambda$ and $\mathcal{D}(d_{\mathbf{X}})$).

The operator $F := \mathrm{Id} + R + K$ is Fredholm of index 0 and, since $Fd_{\mathbf{X}} = d_{\mathbf{X}}Qd_{\mathbf{X}} = d_{\mathbf{X}}F$ *on distributions*, we deduce that F is also bounded on $\mathcal{D}(d_{\mathbf{X}})$. Since F is Fredholm of index 0, we know that for $s \in \mathbb{C}^*$ close to 0, $F - s$ is invertible on $\mathcal{H}\Lambda$. As a consequence, we have $(F-s)^{-1}d_{\mathbf{X}} = d_{\mathbf{X}}(F-s)^{-1}$ on $\mathcal{D}(d_{\mathbf{X}})$ (here we are using the continuity of F on distributions). In particular, this implies that $(F-s)^{-1}$ is itself bounded on $\mathcal{D}(d_{\mathbf{X}})$.

In that case, the spectral projector Π_0 of F for the eigenvalue 0 commutes with $d_{\mathbf{X}}$, is bounded on $\mathcal{D}(d_{\mathbf{X}})$, and since $\mathcal{D}(d_{\mathbf{X}})$ is dense in $\mathcal{H}\Lambda$ and Π_0 has finite rank, its image is contained in $\mathcal{D}(d_{\mathbf{X}})$. Further, we can write $F = (F + \Pi_0)(\mathrm{Id} - \Pi_0)$, and $\widetilde{F} := F + \Pi_0$ is invertible on $\mathcal{H}\Lambda$ and $\mathcal{D}(d_{\mathbf{X}})$, and commuting with $d_{\mathbf{X}}$, so that

$$d_{\mathbf{X}}\widetilde{F}^{-1}Q + \widetilde{F}^{-1}Qd_{\mathbf{X}} = \mathrm{Id} - \Pi_0. \tag{3.12}$$

In particular, for $u \in \ker d_{\mathbf{X}} \cap \mathcal{D}(d_{\mathbf{X}})$, we have

$$u = d_{\mathbf{X}}\widetilde{F}^{-1}Qu + \Pi_0 u. \tag{3.13}$$

Since Π_0 and $d_{\mathbf{X}}$ commute, $u \mapsto \Pi_0 u$ factors to a homomorphism between the cohomologies in (3.9). This map in cohomologies is obviously surjective since $\operatorname{ran}\Pi_0 \subset \mathcal{D}(d_{\mathbf{X}})$. To prove that the map is injective, we need to prove that if $\Pi_0 u \in d_{\mathbf{X}} \operatorname{ran}\Pi_0$ for $u \in \ker d_{\mathbf{X}} \cap \mathcal{D}(d_{\mathbf{X}})$, then $u \in d_{\mathbf{X}}\mathcal{D}(d_{\mathbf{X}})$. This fact actually follows directly from (3.13) by using that both $\widetilde{F}^{-1}$ and Q are bounded on $\mathcal{D}(d_{\mathbf{X}})$. $\square$

We can also deduce the following:

Lemma 3.9. *Under the assumptions of Lemma 3.8, if $F = F' \otimes \mathrm{Id}$ where F' is an operator on $\mathcal{H}$ (i.e. F is* scalar*), then $0 \in \sigma_{T,\mathcal{H}}(\mathbf{X})$ if and only if there exists a non-zero $u \in \mathcal{D}(d_{\mathbf{X}})$ such that $\mathbf{X}u = 0$.*

Proof. From Lemma 3.8, we deduce that $0 \in \sigma_{\mathrm{T},\mathcal{H}}(\mathbf{X})$ if and only if the complex given by $d_{\mathbf{X}}$ is not exact on $\operatorname{ran}\Pi_0$. However, if F is scalar, then $\Pi_0 = \Pi_0' \otimes \mathrm{Id}$ with Π_0' the spectral projector at 0 of F' on $\mathcal{H}$. It follows that $d_{\mathbf{X}}$ restricted to $\operatorname{ran}\Pi_0$ is the Taylor complex of $\mathbf{X}_{|\operatorname{ran}\Pi_0'}$. We are thus reduced to finite dimension and we can apply Lemma 3.7. □

The version of the Analytic Fredholm Theorem for the Taylor spectrum is the following statement:

Proposition 3.10. *Let $\mathbf{X}$ be an $\mathfrak{a}$-action with unique extension to $\mathcal{H}$. Then $\sigma_{T,\mathcal{H}}(\mathbf{X}) \setminus \sigma^{\mathrm{ess}}_{\mathrm{T},\mathcal{H}}(\mathbf{X})$ is a complex analytic submanifold of $\mathbb{C}^\kappa \setminus \sigma^{\mathrm{ess}}_{\mathrm{T},\mathcal{H}}(\mathbf{X})$.*

Proof. As the complex (3.6) is an analytic Fredholm complex of bounded operators on $\mathbb{C}^\kappa \setminus \sigma^{\mathrm{ess}}_{\mathrm{T},\mathcal{H}}(\mathbf{X})$ the statement is classical and a proof can be found in [Mül00, Theorem 2.9]. □

In general, the question of whether the spectrum is discrete does not seem to have a very simple answer. For example, a characterization can be found in [AM09, Corollary 2.6 and Lemma 2.7]. Such a criterion is particularly adapted to microlocal methods and it can actually be used in our setting. However, it turns out that an even simpler criterion is sufficient for us:

Lemma 3.11. *Under the assumptions of Lemma 3.8, assume in addition that Q is a divergence associated with an n-tuple of bounded operators $Q_1, \dots, Q_\kappa$, commuting pairwise and with $\mathbf{X}$. Then, Lemma 3.9 applies, and the Taylor spectrum of $\mathbf{X}$ on $\mathcal{H}$ is discrete in a neighbourhood of 0.*

Proof. We observe that

$$d_{\mathbf{X}-\lambda}Q + Qd_{\mathbf{X}-\lambda} = (\underbrace{-\mathbf{X}_{A_1}Q_1 - \cdots - \mathbf{X}_{A_\kappa}Q_\kappa}_{=F'} + \underbrace{\lambda_1 Q_1 + \cdots + \lambda_\kappa Q_\kappa}_{=\lambda\cdot Q}) \otimes \mathrm{Id}\,.$$

Thus, denoting $F'(\lambda) := F' + \lambda \cdot Q$ on $\mathcal{H}$ and $F(\lambda) := F'(\lambda) \otimes \mathrm{Id}$ on $\mathcal{H}\Lambda$, we see that Lemma 3.9 indeed applies.

Next, we observe two things. The first is that F' and $\lambda \cdot Q$ commute. The second is that for λ small enough, $F'(\lambda)$ can still be decomposed in the form $\mathrm{Id} + R(\lambda) + K(\lambda)$ with $\|R(\lambda)\|_{\mathcal{L}(\mathcal{H})} < 1$ and $K(\lambda)$ compact, because Q is bounded. It follows that $d_{\mathbf{X}-\lambda}$ is Fredholm for λ close enough to 0.

From the previous results, we know that the cohomology of $d_{\mathbf{X}-\lambda}$ on $\mathcal{D}(d_{\mathbf{X}})$ is isomorphic to

$$\ker_{\operatorname{ran}\Pi_0(\lambda)} d_{\mathbf{X}-\lambda} / \operatorname{ran}_{\operatorname{ran}\Pi_0(\lambda)} d_{\mathbf{X}-\lambda},$$

and the isomorphism is given by $[u] \mapsto [\Pi_0(\lambda)]$. Let us now describe a sort of *sandwiching procedure*. Assume that we have a projector Π_2 bounded on $\mathcal{D}(d_{\mathbf{X}})$, commuting with $d_{\mathbf{X}}$. Then, the mapping $[u] \mapsto [\Pi_2 u]$ is well-defined and surjective as a map

$$\ker_{\mathcal{D}(d_{\mathbf{X}})} d_{\mathbf{X}-\lambda} / \operatorname{ran}_{\mathcal{D}(d_{\mathbf{X}})} d_{\mathbf{X}-\lambda} \to \ker_{\operatorname{ran}\Pi_2} d_{\mathbf{X}-\lambda} / \operatorname{ran}_{\operatorname{ran}\Pi_2} d_{\mathbf{X}-\lambda}.$$

In general, there is no reason for this map to be *injective*. However, if we further assume that Π_2 and $\Pi_0(\lambda)$ commute, and that $\mathrm{ran}(\Pi_0(\lambda)) \subset \mathrm{ran}(\Pi_2)$, then we can see $\Pi_0(\lambda)$ as a projector on $\mathrm{ran}(\Pi_2)$. Since the mapping $[\Pi_2 u] \mapsto [\Pi_0(\lambda)u]$ has to be surjective, we deduce that it is actually an isomorphism.

Let us write, with $\widetilde{F'} := F' + \Pi_0'$ where Π_0' is the spectral projector of F' at 0,

$$\widetilde{F'}^{-1} F'(\lambda) = \mathrm{Id} - \Pi_0' + \widetilde{F'}^{-1}\lambda \cdot Q.$$

For $u \in \ker F'(\lambda)$, we have $(\mathrm{Id} - \Pi_0')u = -\widetilde{F'}^{-1}\lambda \cdot Qu$. Since $\widetilde{F'}^{-1}\lambda \cdot Q$ commutes with Π_0' (as F' does commute with $\lambda \cdot Q$), we obtain for $u \in \ker F'(\lambda)$

$$(\mathrm{Id} - \Pi_0')u = (\mathrm{Id} - \Pi_0')^2 u = \widetilde{F'}^{-1}\lambda \cdot Q(\mathrm{Id} - \Pi_0')u.$$

For λ small enough $\mathrm{Id} - \widetilde{F'}^{-1}\lambda \cdot Q$ is invertible on $\mathcal{H}$, which implies that $(\mathrm{Id} - \Pi_0')u = 0$. In particular, $u \in \mathrm{ran}\, \Pi_0'$, so that $\ker F'(\lambda) \subset \ker F'$ and $\mathrm{ran}\, \Pi_0'(\lambda) \subset \mathrm{ran}\, \Pi_0'$. But certainly, Π_0' and $\Pi_0'(\lambda)$ commute. So we can apply the argument above, and deduce that for λ sufficiently small,

$$\ker_{\mathcal{D}(d_{\mathbf{X}})} d_{\mathbf{X}-\lambda} / \mathrm{ran}_{\mathcal{D}(d_{\mathbf{X}})} d_{\mathbf{X}-\lambda} \simeq \ker_{\Lambda\, \mathrm{ran}\, \Pi_0'} d_{\mathbf{X}-\lambda} / \mathrm{ran}_{\Lambda\, \mathrm{ran}\, \Pi_0'} d_{\mathbf{X}-\lambda}.$$

Since $\mathrm{ran}\, \Pi_0'$ is a fixed finite dimensional space, the Taylor spectrum of $\mathbf{X}$ is discrete near 0 by Lemma 3.7. □

4. Discrete Ruelle-Taylor resonances via microlocal analysis

Given a vector bundle $E \to \mathcal{M}$ and an admissible lift $\mathbf{X}$ of an Anosov action (see Definition 2.4), we have seen in Section 3.1 how to define the Taylor differential $d_{\mathbf{X}}$ which acts in its coordinate free form on $C^\infty(\mathcal{M}; E) \otimes \Lambda\mathfrak{a}^*$. We have furthermore seen how $d_{\mathbf{X}}$ can be used to define a Taylor spectrum $\sigma_{\mathrm{T},\mathcal{H}}(\mathbf{X}) \subset \mathfrak{a}^*_{\mathbb{C}}$. We take coordinates whenever it is convenient. In that case, we will use the notation d_X to avoid confusions. In the sequel it will be convenient to pass back and forth between these versions and we will mostly use the shorthand notation $C^\infty\Lambda$, leaving open which version we currently consider.

The Ruelle-Taylor resonances that we will introduce will correspond to a discrete spectrum of $-\mathbf{X}$ on some anisotropic Sobolev spaces. From a spectral theoretic point of view this sign convention might seem unnatural. However, from a dynamical point of view this convention is very natural: given the flow φ_t^X of a vector field X, the one-parameter group that propagates probability densities with respect to an invariant measure is given by $(\varphi_{-t}^X)^*$ and thus generated by the differential operator $-X$. We will therefore from now on consider the holomorphic family of complexes generated by $d_{\mathbf{X}+\lambda}$ for $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ (respectively $\lambda \in \mathbb{C}^\kappa$ after a choice of coordinates). The goal of this section is to show the following:

Theorem 4. *Let τ be a smooth abelian Anosov action with generating map X and $\mathbf{X}$ an admissible lift. Let $A_0 \in \mathcal{W}$ be in the positive Weyl chamber. There exists $c > 0$, locally uniformly with respect to A_0, such that for each $N > 0$, there is a Hilbert space $\mathcal{H}_N$ containing $C^\infty(\mathcal{M})$ and contained in $C^{-\infty}(\mathcal{M})$ such that the following holds true:*

1) $-X$ has no essential Taylor spectrum on the Hilbert space $\mathcal{H}_N$ in the region

$$\mathcal{F}_N := \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \,|\, \mathrm{Re}(\lambda(A_0)) > -cN + C_{L^2}(A_0)\}$$

where $C_{L^2}(A) := \inf\{C \geq 0 \,|\, \|e^{-t\mathbf{X}_A}\|_{\mathcal{L}(L^2)} \leq e^{Ct}$ for all $t > 0\}$.

2) For each $\lambda \in \mathcal{F}_N$ one has an isomorphism of finite dimensional spaces

$$\ker d_{\mathbf{X}+\lambda}|_{\mathcal{D}_N^j(d_{\mathbf{X}})}/\operatorname{ran} d_{\mathbf{X}+\lambda}|_{\mathcal{D}_N^{j-1}(d_{\mathbf{X}})} = \ker d_{\mathbf{X}+\lambda}|_{C^{-\infty}_{E_u^*}(\mathcal{M})\otimes\Lambda^j\mathfrak{a}^*_{\mathbb{C}}}/\operatorname{ran} d_{\mathbf{X}+\lambda}|_{C^{-\infty}_{E_u^*}(\mathcal{M})\otimes\Lambda^{j-1}\mathfrak{a}^*_{\mathbb{C}}}$$

with $\mathcal{D}_N^j(d_{\mathbf{X}}) := \{u \in \mathcal{H}_N \otimes \Lambda^j\mathfrak{a}_{\mathbb{C}}^* \,|\, d_{\mathbf{X}}u \in \mathcal{H}_N \otimes \Lambda^{j+1}\mathfrak{a}_{\mathbb{C}}^*\}$, showing that the cohomology dimension is independent of N and A_0.

3) The Taylor spectrum of $-\mathbf{X}$ contained in $\mathcal{F}_N$ is discrete and contained in

$$\bigcap_{A\in\mathcal{W}} \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \,|\, \mathrm{Re}(\lambda(A)) \leq C_{L^2}(A)\}.$$

4) An element $\lambda \in \mathcal{F}_N$ is in the Taylor spectrum of $-\mathbf{X}$ on $\mathcal{H}_N$ if and only if λ is a joint Ruelle resonance of $\mathbf{X}$.

The Hilbert space $\mathcal{H}_N$ will be rather written $\mathcal{H}_{NG}$ below, where G is a certain scaling function on $T^*\mathcal{M}$ giving the rate of Sobolev differentiability in phase space. We use this notation in order to emphasize the dependence of the space on G.

The central point of the proof will be a parametrix construction for the exterior differential $d_{\mathbf{X}+\lambda}$. We will prove in Proposition 4.6 that there are holomorphic families of operators $Q(\lambda), F(\lambda) : C^{-\infty}\Lambda \to C^{-\infty}\Lambda$ such that

$$Q(\lambda)d_{\mathbf{X}+\lambda} + d_{\mathbf{X}+\lambda}Q(\lambda) = F(\lambda).$$

The operators $Q(\lambda)$ and $F(\lambda)$ will be Fourier integral operators and independent of any Hilbert space on which the operators act. However, the crucial fact is that for these operators there exists a scale of Hilbert spaces $C^\infty \subset \mathcal{H}_{NG} \subset C^{-\infty}$ (with $N \geq 0$ and $G \in C^\infty(T^*\mathcal{M})$ a weight function) and domains $\mathcal{F}_{NG} \subset \mathfrak{a}_{\mathbb{C}}^*$ with $\mathfrak{a}_{\mathbb{C}}^* = \cup_{N>0}\mathcal{F}_{NG}$ such that for $\lambda \in \mathcal{F}_{NG}$ the operators $Q(\lambda) : \mathcal{H}_{NG} \to \mathcal{H}_{NG}$ are bounded and the operators $F(\lambda) : \mathcal{H}_{NG} \to \mathcal{H}_{NG}$ are Fredholm and can be decomposed as $F(\lambda) = \mathrm{Id} + R(\lambda) + K(\lambda)$ with $K(\lambda)$ compact and $\|R(\lambda)\|_{\mathcal{L}(\mathcal{H}_{NG}\Lambda)} < 1/2$. Then by Lemma 3.8 we directly conclude that the Taylor complex on $\mathcal{H}_{NG}\Lambda$ is Fredholm on $\lambda \in \mathcal{F}_{NG}$. The fact that the construction of the operator family $F(\lambda) : C^\infty\Lambda \to C^{-\infty}\Lambda$ is independent of the specific Hilbert spaces on which they act will be the key for proving in Section 4.3 that the Taylor spectrum of $d_{\mathbf{X}+\lambda}$ is intrinsic to the Anosov action, i.e. independent of the constructed spaces $\mathcal{H}_{NG}$. The flexibility which we will have in the construction of the escape function G will furthermore allow to identify this intrinsic spectrum with the spectrum of $d_{\mathbf{X}+\lambda}$ on the space $C^{-\infty}_{E_u^*}\Lambda$ of distributions with wavefront set contained in the annihilator $E_u^* \subset T^*\mathcal{M}$ of $E_u \oplus E_0$ (see Proposition 4.9). Finally, we will see that the choice of $Q(\lambda)$ can be made more *geometric*, to enable the use of Lemma 3.11 and prove that this intrinsic spectrum is discrete in $\mathfrak{a}_{\mathbb{C}}^*$.

The construction of the parametrix $Q(\lambda)$ and the Hilbert spaces $\mathcal{H}_{NG}$ will be done using microlocal analysis. Appendix A contains a brief summary of the necessary microlocal tools. Section 4.1 will be devoted to the construction of the anisotropic Sobolev spaces.

With these tools at hand we will construct the parametrix (Section 4.2), and prove that the spectrum is intrinsic (Section 4.3) as well as discrete (Section 4.4).

4.1. **Escape function and anisotropic Sobolev space.** In this section we define the anisotropic Sobolev spaces. Their construction will be based on the choice of an escape function for the given Anosov action. We first give a definition for such an escape function and then prove the existence of escape functions with additional useful properties.

Given any smooth vector field $X \in C^\infty(\mathcal{M}; T\mathcal{M})$ with flow φ_t^X we define the *symplectic lift* of the flow and the corresponding vector field by

$$\Phi_t^X : \begin{cases} T^*\mathcal{M} & \to T^*\mathcal{M} \\ (x,\xi) & \mapsto (\varphi_t^X(x), ((d\varphi_t^X)^{-1})^T\xi) \end{cases} \quad \text{and} \quad X^H := \frac{d}{dt}_{|t=0} \Phi_t^X \in C^\infty(T^*\mathcal{M}; T(T^*\mathcal{M})). \tag{4.1}$$

The notation X^H is chosen because it is the Hamilton vector field of the principal symbol $\sigma_p^1(X)(x,\xi) = i\xi(X(x)) \in C^\infty(T^*\mathcal{M})$ of X (see Example A.2). Recall from Example A.2 that for an admissible lift of an Anosov action, the principal symbols of the lifted differential operator $\mathbf{X}_A$ and that of the vector field X_A tensorized with Id_E coincide. This will turn out to be the reason why we do not have to care about the admissible lifts for the construction of the escape function. We will denote by $\{0\} := \{(x,0) \in T^*\mathcal{M}\}$ the zero section.

Definition 4.1. Let $c_X > 0$, $A \in \mathcal{W}$, $\Gamma_{E_0^*} \subset T^*\mathcal{M}$, an open cone containing E_0^* satisfying $\overline{\Gamma}_{E_0^*} \cap (E_u^* \oplus E_s^*) = \{0\}$. Then a function $G \in C^\infty(T^*\mathcal{M}, \mathbb{R})$ is called an *escape function* for A compatible with c_X, $\Gamma_{E_0^*}$ if there is $R > 0$ so that

(1) for $|\xi| \leq R/2$ one has $G(x,\xi) = 1$ and for $|\xi| > 1$ one can write, $G(x,\xi) = m(x,\xi)\log(1 + f(x,\xi))$. Here $m \in C^\infty(T^*\mathcal{M}; [-1/2, 8])$ and for $|\xi| > R$, m is homogeneous of degree 0, with $m \leq -1/4$ in a conic neighborhood of E_u^* and $m \geq 4$ in a conic neighborhood of E_s^*. Furthermore, $f \in C^\infty(T^*\mathcal{M}, \mathbb{R}^+)$ is positive homogeneous of degree 1 for $|\xi| > R$. We call m the *order function* of G.
(2) $X_A^H m(x,\xi) \leq 0$ for all $|\xi| > R$,
(3) for $\xi \notin \Gamma_{E_0^*}, |\xi| > R$ one has $X_A^H G(x,\xi) \leq -c_X$.

Below (see Proposition 4.3), we will prove the existence of escape functions for Anosov actions. Before coming to this point let us explain how we can build the anisotropic Sobolev spaces based on the escape function: given an escape function G, Property (1) of Definition 4.1 implies that $m \in S_1^0(\mathcal{M})$ and for any $N > 0$, $e^{NG} \in S_{1-}^{Nm}(\mathcal{M})$ is a real elliptic symbol. According to [FRS08, Lemma 12 and Corollary 4] there exists a pseudodifferential operator

$$\hat{\mathcal{A}}_{NG} \in \Psi_{1-}^{Nm}(\mathcal{M}; E) \tag{4.2}$$

such that

(1) $\sigma_p^{Nm}(\hat{\mathcal{A}}_{NG}) = e^{NG}\,\mathrm{Id}_E \mod S_{1-}^{Nm-1+}$,
(2) $\hat{\mathcal{A}}_{NG} : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; E)$ is invertible,
(3) $\hat{\mathcal{A}}_{NG}^{-1} \in \Psi_{1-}^{-Nm}(\mathcal{M}; E)$ and $\sigma_p^{-Nm}(\hat{\mathcal{A}}_{NG}^{-1}) = e^{-NG}\,\mathrm{Id}_E \mod S_{1-}^{-Nm-1+}$.

We can now define the *anisotropic Sobolev spaces*

$$\mathcal{H}_{NG} := \hat{\mathcal{A}}_{NG}^{-1} L^2(\mathcal{M}; E) \text{ with scalar product } \langle u, v\rangle_{\mathcal{H}_{NG}} := \langle \hat{\mathcal{A}}_{NG} u, \hat{\mathcal{A}}_{NG} v\rangle_{L^2}.$$

Note that the scalar product $\langle u, v\rangle_{\mathcal{H}_{NG}}$ depends not only on the choice of the escape function but also on the choice of its quantization $\hat{\mathcal{A}}_{NG}$. However, by L^2-continuity (Proposition A.9), these different choices all yield equivalent scalar products on the given vector space $\mathcal{H}_{NG}$. For that reason we can suppress this dependence in our notation.

We want to study the Taylor spectrum of the admissible lift of the Anosov action on these anisotropic Sobolev spaces. Recall from Section 3.1 that due to the unboundedness of the differential operators we have to verify the unique extension property:

Lemma 4.2. *For any escape function G the $\mathfrak{a}$-action of an admissible lift has a unique extension (in the sense that Equation (3.5) holds) to the anisotropic Hilbert space $\mathcal{H}_{NG}$.*

Proof. Let us consider the Taylor differential $d_{\mathbf{X}}$ as an unbounded operator on $\mathcal{H}_{NG}\Lambda$ with domain $C^\infty(\mathcal{M}; E\otimes\Lambda)$. Then, in the language of closed extensions, the desired equality (3.5) corresponds to the uniqueness of possible closed extensions. By unitary equivalence we can study the conjugate operator $P := \hat{A}_{NG}\, d_{\mathbf{X}} \hat{A}_{NG}^{-1}$ acting as an unbounded operator on $L^2(\mathcal{M}; E\otimes\Lambda)$, instead. We want to apply [FS11, Lemma A.1] which states that any operator in $\Psi^1_1(\mathcal{M}; E\otimes\Lambda)$ has a unique closed extension as an unbounded operator on L^2 with domain C^∞. Since $\hat{A}_{NG}$ has scalar principal symbol we can write $P = d_{\mathbf{X}} + [\mathcal{A}_{NG}, d_{\mathbf{X}}]\mathcal{A}_{NG}^{-1}$, where the first summand is obviously in $\Psi^1_1(\mathcal{M}; E\otimes\Lambda)$ and the second one, by Proposition A.3, in $\Psi^{0+}_{1-}(\mathcal{M}; E\otimes\Lambda)$. Now, by Definition A.1 of symbol spaces, one checks that $S^{0+}_{1-}(\mathcal{M}; E\otimes\Lambda) \subset S^1_1(\mathcal{M}; E\otimes\Lambda)$. We conclude that $P \in \Psi^1_1(\mathcal{M}; E\otimes\Lambda)$ and are able to apply [FS11, Lemma A.1] which completes the proof. ☐

Let us now come to the existence of the escape function:

Proposition 4.3. *Fix an arbitrary $A_0 \in \mathcal{W} \subset \mathfrak{a}$, an open cone $\Gamma_{\mathrm{reg}} \subset T^*\mathcal{M}$ which is disjoint from E_u^*, and Γ_0 a small conic neighborhood of E_0^* so that $\overline{\Gamma}_0 \cap (E_s^* \oplus E_u^*) = \{0\}$. Then there is a $c_X > 0$, an open conic neighborhood $\Gamma_{E_0^*} \subset \Gamma_0$ of E_0^*, and $R > 0$ such that there is an escape function G for A_0 compatible with c_X and $\Gamma_{E_0^*}$ with the additional property that the order function satisfies*

$$m(x,\xi) \geq 1/2 \text{ for } (x,\xi) \in \Gamma_{\mathrm{reg}} \text{ and } |\xi| > R. \tag{4.3}$$

Proof. The proof follows from [DGRS20, Lemma 3.3]: indeed, first we note that the proof there only uses the continuity of the decomposition $T^*\mathcal{M} = E_0^* \oplus E_u^* \oplus E_s^*$ and the contracting/expanding properties of E_s^*, E_u^* but not the fact that E_0^* is one dimensional. It suffices to take, in the notations of [DGRS20], $N_1 = 4$, $N_0 = 1/4$ and $\Gamma_{\mathrm{reg}} = T^*\mathcal{M} \setminus C^{uu}(\alpha_0)$ with $\alpha_0 > 0$ small enough. Although it is not explicitly written in the statement of [DGRS20, Lemma 3.3], the order function m constructed there satisfies $X^H_{A_0} m \leq 0$ and $\Gamma_{E_0^*}$ is arbitrarily small if $\alpha_0 > 0$ is small (see [DGRS20, Section A.2]). ☐

4.2. **Parametrix construction.** The goal of this section is to construct an operator $Q(\lambda)$ as in Lemma 3.8 for the complex $d_{\mathbf{X}+\lambda}$, and so that Q will be bounded on the anisotropic Sobolev spaces $\mathcal{H}_{NG}\Lambda$. The construction will be microlocal in the elliptic region and dynamical near the characteristic set. In Section 4.4 we will provide an alternative construction of a $Q(\lambda)$ which is purely dynamical, i.e. which is a function of the operators $\mathbf{X}_{A_j}$.

Recall the notation $E\otimes\Lambda = E\otimes\Lambda\mathfrak{a}^*$. We will call an operator $A : C^\infty(\mathcal{M};E\otimes\Lambda)\to C^\infty(\mathcal{M};E\otimes\Lambda)$ Λ-*scalar* if there is $A' : C^\infty(\mathcal{M};E)\to C^\infty(\mathcal{M};E)$ such that for any $u\in C^\infty(\mathcal{M};E), \omega\in\Lambda\mathfrak{a}^*$ we have $A(u\otimes\omega) = (A'u)\otimes\omega$. We will also freely identify operators $A : C^\infty(\mathcal{M};E)\to C^{-\infty}(\mathcal{M};E)$ with their Λ-scalar extension on sections of $E\otimes\Lambda$.

Lemma 4.4. *Let $P\in\Psi^0(\mathcal{M};E)$ be such that $\mathrm{WF}(\mathrm{Id}-P)$ does not intersect a conic neighborhood of $E_u^*\oplus E_s^*$, and we make it act as a Λ-scalar operator. There exists a holomorphic family of pseudo-differential operators $Q_{\mathrm{ell}}(\lambda)\in\Psi^{-1}(\mathcal{M};E\otimes\Lambda)$ for $\lambda\in\mathfrak{a}^*_{\mathbb{C}}$ such that*

$$d_{\mathbf{X}+\lambda}Q_{\mathrm{ell}}(\lambda)+Q_{\mathrm{ell}}(\lambda)d_{\mathbf{X}+\lambda} = (\mathrm{Id}-P)+S_1(\lambda)+S_2(\lambda) \tag{4.4}$$

with $S_1(\lambda)\in\Psi^{-1}(\mathcal{M},E\otimes\Lambda)$ holomorphic in λ satisfying $\mathrm{WF}(S_1(\lambda))\subset\mathrm{WF}(P)\cap\mathrm{WF}(\mathrm{Id}-P)$ and $S_2(\lambda)\in\Psi^{-\infty}(\mathcal{M};E\otimes\Lambda)$, also holomorphic in λ. If furthermore $P : C^\infty(\mathcal{M};E\otimes\Lambda^k\mathfrak{a}^*)\to C^\infty(\mathcal{M};E\otimes\Lambda^k\mathfrak{a}^*)$ for all k, then $Q_{\mathrm{ell}}(\lambda) : C^\infty(\mathcal{M};E\otimes\Lambda^k\mathfrak{a}^*)\to C^\infty(\mathcal{M};E\otimes\Lambda^{k-1}\mathfrak{a}^*)$ for all k.

Proof. We will use an arbitrary choice of basis $A_1,\dots,A_\kappa$ in $\mathfrak{a}$ and consider the commuting differential operators $\mathbf{X}_{A_1},\dots,\mathbf{X}_{A_\kappa}$. Recall that the corresponding divergence operator $\delta_{\mathbf{X}+\lambda}$ on $C^\infty(\mathcal{M};E)\otimes\Lambda\mathfrak{a}^*$ is defined by

$$\delta_{\mathbf{X}+\lambda}(u\otimes e_{i_1}\wedge\cdots\wedge e_{i_\ell}) := -\sum_{j=1}^{\ell}(-1)^{j-1}(\mathbf{X}_{A_{i_j}}+\lambda_{i_j})u\otimes e_{i_1}\wedge\cdots\wedge\widehat{e}_{i_j}\wedge\cdots\wedge e_{i_\ell},$$

where $\lambda_j := \lambda(A_j)\in\mathbb{C}$ (here $(e_j)_j$ is a dual basis to A_j in $\mathfrak{a}^*$). Thus, using the commutations $[\mathbf{X}_{A_j}+\lambda_j,\mathbf{X}_{A_k}+\lambda_k]=0$ and Lemma 3.6 with $\mathbf{Y}_j=\mathbf{X}_{A_j}+\lambda_j$, we obtain that the operator $\Delta_{\mathbf{X}+\lambda} := d_{\mathbf{X}+\lambda}\delta_{\mathbf{X}+\lambda}+\delta_{\mathbf{X}+\lambda}d_{\mathbf{X}+\lambda}$ is Λ-scalar and given for each $\omega\in\Lambda\mathfrak{a}^*$ by the expression

$$\Delta_{\mathbf{X}+\lambda}(u\otimes\omega) = -\Big(\sum_{k=1}^{\kappa}(\mathbf{X}_{A_k}+\lambda_k)^2u\Big)\otimes\omega.$$

This shows that $\Delta_{\mathbf{X}+\lambda}\in\Psi^2(\mathcal{M};E\otimes\Lambda)$ with principal symbol given by (see Example A.2)

$$\sigma_p^2(\Delta_{\mathbf{X}+\lambda})(x,\xi) = \|\xi_{E_0}\|^2\,\mathrm{Id}_{E\otimes\Lambda}\ \text{ with } \|\xi_{E_0}\|^2 := \sum_{k=1}^{\kappa}\xi(X_{A_k})^2.$$

It is an operator which is microlocally elliptic outside $E_u^*\oplus E_s^*$ (i.e. $\mathrm{ell}^2(\Delta_{\mathbf{X}+\lambda}) = T^*\mathcal{M}\setminus(E_u^*\oplus E_s^*)$). Thus, by Proposition A.7, if $P'\in\Psi^0(\mathcal{M},E\otimes\Lambda)$ has $\mathrm{WF}(P')$ contained in a conic open set of $T^*\mathcal{M}$ not intersecting $E_u^*\oplus E_s^*$, then there exists a pseudo-differential operator $Q_\Delta(\lambda)\in\Psi^{-2}(\mathcal{M};E\otimes\Lambda)$ holomorphic in λ with $\mathrm{WF}(Q_\Delta(\lambda))\subset\mathrm{WF}(P')$ such that

$$\Delta_{\mathbf{X}+\lambda}Q_\Delta(\lambda)=P'+S'(\lambda)$$

with $S'(\lambda)\in\Psi^{-\infty}(\mathcal{M};E\otimes\Lambda)$ holomorphic in λ. We now choose P' so that $\mathrm{WF}(P')\cap(E_u^*\oplus E_s^*)=\emptyset$ and $\mathrm{WF}(\mathrm{Id}-P')\cap\mathrm{WF}(\mathrm{Id}-P)=\emptyset$; in other words, $P'=1$ microlocally on $\mathrm{WF}(1-P)$. Note that $d_{\mathbf{X}+\lambda}\Delta_{\mathbf{X}+\lambda}=\Delta_{\mathbf{X}+\lambda}d_{\mathbf{X}+\lambda}$ implies

$$\Delta_{\mathbf{X}+\lambda}\left(Q_\Delta(\lambda)d_{\mathbf{X}+\lambda}-d_{\mathbf{X}+\lambda}Q_\Delta(\lambda)\right)=[P',d_{\mathbf{X}+\lambda}]+[S'(\lambda),d_{\mathbf{X}+\lambda}].$$

Using microlocal ellipticity of $\Delta_{\mathbf{X}+\lambda}$ outside $E_u^*\oplus E_s^*$, we deduce from (A.1) that $\mathrm{WF}([Q_\Delta(\lambda),d_{\mathbf{X}+\lambda}])\subset\mathrm{WF}(P')\cap\mathrm{WF}(\mathrm{Id}-P')$. In particular, since $P'=1$ microlocally on $\mathrm{WF}(\mathrm{Id}-P)$, this implies that $[Q_\Delta(\lambda),d_{\mathbf{X}+\lambda}](\mathrm{Id}-P)\in\Psi^{-\infty}(\mathcal{M};E\otimes\Lambda)$. Thus, with $Q_{\mathrm{ell}}(\lambda):=\delta_{\mathbf{X}+\lambda}Q_\Delta(\lambda)(\mathrm{Id}-P)$ we obtain

$$\begin{aligned}d_{\mathbf{X}+\lambda}Q_{\mathrm{ell}}(\lambda)+Q_{\mathrm{ell}}(\lambda)d_{\mathbf{X}+\lambda}=&\Delta_{\mathbf{X}+\lambda}Q_\Delta(\lambda)(\mathrm{Id}-P)+\delta_{\mathbf{X}+\lambda}[Q_\Delta(\lambda),d_{\mathbf{X}+\lambda}](\mathrm{Id}-P)\\&+\delta_{\mathbf{X}+\lambda}Q_\Delta(\lambda)[d_{\mathbf{X}+\lambda},P]\\=&(\mathrm{Id}-P)+S_1(\lambda)+S_2(\lambda)\end{aligned}$$

with $S_2(\lambda)\in\Psi^{-\infty}(\mathcal{M};E\otimes\Lambda)$ and

$$S_1(\lambda):=\delta_{\mathbf{X}+\lambda}Q_\Delta(\lambda)[d_{\mathbf{X}+\lambda},P]=\delta_{\mathbf{X}+\lambda}Q_\Delta(\lambda)[d_{\mathbf{X}+\lambda},\mathrm{Id}-P]\in\Psi^{-1}(\mathcal{M};E\otimes\Lambda)$$

has wavefront set contained in $\mathrm{WF}(P)\cap\mathrm{WF}(\mathrm{Id}-P)$. □

A second ingredient for the construction of the parametrix will be the following estimates of the essential spectral radius of the propagator on the anisotropic Sobolev spaces. We recall that if Y is a bounded operator on a Hilbert space $\mathcal{H}$,

$$r_{\mathrm{ess}}(Y):=\max\{|\lambda|\,|\,\lambda\in\sigma_{\mathrm{ess}}(Y)\}.$$

Lemma 4.5. *Let $P\in\Psi^0(\mathcal{M};E)$ be such that $\mathrm{WF}(P)$ is disjoint from E_0^*, and choose an arbitrary constant $C_P'>C_P:=\limsup_{|\xi|\to\infty}\|\sigma_p^0(P)(x,\xi)\|$ and some $T>0$. Let $A_0\in\mathcal{W}\subset\mathfrak{a}$, Γ_{reg} be an open cone disjoint from $E_u^*\subset T^*\mathcal{M}$, and $\Gamma_0\subset T^*\mathcal{M}$ be a small conic neighborhood of E_0^*. Let G be an escape function for A_0 compatible with c_X, and $\Gamma_{E_0^*}\subset\Gamma_0$ be as constructed in Proposition 4.3. If in addition $\overline{\Gamma}_{E_0^*}\cap\Phi_t^{X_A}(\mathrm{WF}(P))=\emptyset$ for all $0\le t\le T$, the operator*

$$e^{-t\mathbf{X}_A}P:\mathcal{H}_{NG}\to\mathcal{H}_{NG}$$

is bounded and has a decomposition

$$e^{-t\mathbf{X}_A}P=R_{N,G}(t)+K_{N,G}(t)$$

with $\|R_{N,G}(t)\|_{\mathcal{L}(\mathcal{H}_{NG})}\le C_P'e^{-c_XNt}\|e^{-t\mathbf{X}_A}\|_{\mathcal{L}(L^2)}$ and $K_{N,G}(t)$ compact on $\mathcal{H}_{NG}$. Both $R_{N,G}(t),K_{N,G}(t)$ depend on N,G.

Proof. Let $m \in C^\infty(T^*\mathcal{M};\mathbb{R})$ be the order function of the escape function G (see Definition 4.1(1)). Instead of studying $e^{-t\mathbf{X}_A}P$ on $\mathcal{H}_{NG}$ we consider the operator $\hat{\mathcal{A}}_{NG}e^{-t\mathbf{X}_A}P\hat{\mathcal{A}}_{NG}^{-1}$ on $L^2(\mathcal{M};E)$ which is a Fourier integral operator. We write this operator as

$$\hat{\mathcal{A}}_{NG}e^{-t\mathbf{X}_A}P\hat{\mathcal{A}}_{NG}^{-1} = e^{-t\mathbf{X}_A}\underbrace{e^{t\mathbf{X}_A}\hat{\mathcal{A}}_{NG}e^{-t\mathbf{X}_A}}_{=:B_t}P\hat{\mathcal{A}}_{NG}^{-1}. \tag{4.5}$$

For the newly introduced operator B_t we apply Egorov's Lemma (Lemma A.8) and deduce that it is a pseudodifferential operator $B_t \in \Psi_{1-}^{N(m\circ\Phi_t^{X_A})}(\mathcal{M};E)$ with principal symbol

$$\sigma_p^{N(m\circ\Phi_t^{X_A})}(B_t) = e^{N(G\circ\Phi_t^{X_A})} \mod S_{1-}^{N(m\circ\Phi_t^{X_A}))-1+}.$$

Consequently, $B_tP\hat{\mathcal{A}}_{NG}^{-1} \in \Psi_{1-}^{N(m\circ\Phi_t^{X_A}-m)}$ and by Definition 4.1(2) $m\circ\Phi_t^{X_A}(x,\xi)-m(x,\xi) \leq 0$ for $|\xi|$ large enough. Thus $B_tP\hat{\mathcal{A}}_{NG}^{-1} \in \Psi_{1-}^{0}(\mathcal{M};E)$ is bounded on L^2, and we can apply Proposition A.9 to this operator. We calculate its principal symbol

$$\sigma_p^0(B_tP\hat{\mathcal{A}}_{NG}^{-1}) = e^{N(G\circ\Phi_t^{X_A}-G)}\sigma_p^0(P).$$

Now, using Definition 4.1(3), our assumption that $\overline{\Gamma}_{E_0^*}\cap\Phi_t^{X_A}(\mathrm{WF}(P)) = \emptyset$ for $0 \leq t \leq T$ insures that, for any $(x,\xi)\in\mathrm{WF}(P)$ and $|\xi|$ sufficiently large, $\partial_t(G\circ\Phi_t^{X_A}) \leq -c_X$ for all $0\leq t\leq T$. Thus

$$\limsup_{R\to\infty}\sup_{(x,\xi)\in\mathrm{WF}(P),|\xi|>R}\|e^{N(G\circ\Phi_t^{X_A}(x,\xi)-G(x,\xi))}\sigma_p^0(P)(x,\xi)\| \leq C_Pe^{-Nc_Xt}.$$

By closedness of $\overline{\Gamma}_{E_0^*}$ and $\mathrm{WF}(P)$ this estimate can also be extended to a small conical neighborhood of $\mathrm{WF}(P)$. On the complement of this neighborhood, by the definition of the wavefront set, we deduce $\limsup_{|\xi|\to\infty}\|\sigma_p^0(P)(x,\xi)\| = 0$. We have seen above that $e^{N(G\circ\Phi_t^{X_A}-G)} \in S_{1-}^0$. In particular this factor is uniformly bounded. Putting everything together we get

$$\limsup_{|\xi|\to\infty}\|\sigma_p^0(B_tP\hat{\mathcal{A}}_{NG}^{-1})(x,\xi)\| \leq C_Pe^{-Nc_Xt}.$$

Using Proposition A.9 we can write $B_tP\hat{\mathcal{A}}_{NG}^{-1} = \widetilde{R}_N(t)+\widetilde{K}_N(t)$ with $\widetilde{K}_N(t)\in\Psi^{-\infty}(\mathcal{M};E)$ and $\|\widetilde{R}_N(t)\|_{\mathcal{L}(L^2)} \leq C_P'e^{-Nc_Xt}$. Now, by (4.5), our operator of interest can be written as

$$\hat{\mathcal{A}}_{NG}e^{-t\mathbf{X}_A}P\hat{\mathcal{A}}_{NG}^{-1} = e^{-t\mathbf{X}_A}(\widetilde{R}_t+\widetilde{K}_t),$$

and we get the desired property by setting $R_N(t) := e^{-t\mathbf{X}_A}\widetilde{R}_t$ and $K_N(t) := e^{-t\mathbf{X}_A}\widetilde{K}_t$. $\square$

Let us define for $A\in\mathcal{W}$:

$$C_{L^2}(A) := \inf\{C\geq 0\,|\,\|e^{-t\mathbf{X}_A}\|_{\mathcal{L}(L^2)} \leq e^{Ct}\text{ for all } t>0\}.$$

We can now come to the construction of our full parametrix for the Taylor complex:

Proposition 4.6. *For any $A_0 \in \mathcal{W}$, any open cone $\Gamma_0 \subset T^*\mathcal{M}$ containing E_0^* and satisfying $\overline{\Gamma_0} \cap (E_u^* \oplus E_s^*) = \{0\}$, there are families of operators $Q(\lambda), F(\lambda) : C^\infty(\mathcal{M}; E \otimes \Lambda) \to C^{-\infty}(\mathcal{M}; E \otimes \Lambda)$ depending holomorphically on $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ such that*

$$Q(\lambda)d_{\mathbf{X}+\lambda} + d_{\mathbf{X}+\lambda}Q(\lambda) = F(\lambda).$$

Furthermore, for any escape function G for A_0 compatible with $c_X > 0$, and $\Gamma_{E_0^*} \subset \Gamma_0$, one has for any $N > 0$ and $\delta > 0$ that:

1. *$Q(\lambda) : \mathcal{H}_{NG}\Lambda^j \to \mathcal{H}_{NG}\Lambda^{j-1}$ is bounded for any $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $0 \le j \le \kappa$,*
2. *$F(\lambda)$ can be decomposed as $F(\lambda) = \mathrm{Id} + R_{N,G}(\lambda) + K_{N,G}(\lambda)$ where $K_{N,G}(\lambda)$ is a compact operator on $\mathcal{H}_{NG}\Lambda$, and $R_{N,G}(\lambda) : \mathcal{H}_{NG}\Lambda \to \mathcal{H}_{NG}\Lambda$ is bounded with norm $\|R_{N,G}(\lambda)\|_{\mathcal{L}(\mathcal{H}_{NG})} < 1/2$ for*
$$\lambda \in \mathcal{F}_{NG,A_0,\delta} := \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \mid \mathrm{Re}(\lambda(A_0)) > -Nc_X + C_{L^2}(A_0) + \delta\} \subset \mathfrak{a}_{\mathbb{C}}^*.$$
Both operators $R_{N,G}(\lambda), K_{N,G}(\lambda)$ depend on N, G, while $Q(\lambda)$ and $F(\lambda)$ do not.

Remark 4.7.

1. If there is a smooth volume density μ preserved by the Anosov action (e.g. the Haar measure for Weyl chamber flows), and if we consider the scalar case $\mathbf{X}_A = X_A$, then e^{tX_A} is unitary on $L^2(\mathcal{M}, \mu)$ and the constant $C_{L^2}(A)$ vanishes.
2. For proving that the Ruelle-Taylor spectrum is independent of the choice of $\mathcal{H}_{NG}$ it will be essential that the operators $Q(\lambda), F(\lambda)$ will only depend on the choice of A_0 and $\Gamma_{E_0^*}$ but not on the choice of the anisotropic Sobolev space $\mathcal{H}_{NG}$ as long as the escape function G satisfies the required compatibility conditions.

Proof of Proposition 4.6. Let us choose some $T \ge \ln(3)/\delta$. For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ we define the operators $\mathbf{X}_{A_0}(\lambda) := \mathbf{X}_{A_0} + \lambda(A_0)$ and let

$$Q_T'(\lambda) := \int_0^T e^{-t\mathbf{X}_{A_0}(\lambda)}dt : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; E).$$

We have the relations

$$\mathbf{X}_{A_0}(\lambda)Q_T'(\lambda) = Q_T'(\lambda)\mathbf{X}_{A_0}(\lambda) = 1 - e^{-T\mathbf{X}_{A_0}(\lambda)}, \quad [\mathbf{X}_A, Q_T'(\lambda)] = 0 \text{ for all } A \in \mathfrak{a}. \tag{4.6}$$

Now we extend $Q_T'(\lambda)$ to an operator on $C^\infty(\mathcal{M}; E) \otimes \Lambda^\ell\mathfrak{a}^* \to C^\infty(\mathcal{M}; E) \otimes \Lambda^{\ell-1}\mathfrak{a}^*$ for each ℓ by the formula

$$Q_T(\lambda)(u \otimes \omega) := (Q_T'(\lambda)u) \otimes \iota_{A_0}\omega$$

for $u \in C^\infty(\mathcal{M}; E)$ and $\omega \in \Lambda^\ell\mathfrak{a}^*$. Using the relations of (4.6) and Lemma 3.6 we get

$$(Q_T(\lambda)d_{\mathbf{X}+\lambda} + d_{\mathbf{X}+\lambda}Q_T(\lambda))(u \otimes \omega) = \big((1 - e^{-T\mathbf{X}_{A_0}(\lambda)})u\big) \otimes \omega. \tag{4.7}$$

We observe that by the commutativity of the Anosov action $[\mathbf{X}_A, e^{-T\mathbf{X}_{A_0}(\lambda)}] = 0$, and therefore on $C^\infty(\mathcal{M}; E \otimes \Lambda)$ we have

$$[d_{\mathbf{X}+\lambda}, e^{-T\mathbf{X}_{A_0}(\lambda)}] = 0. \tag{4.8}$$

Next, we use the microlocal parametrix in the elliptic region from Lemma 4.4 with a carefully chosen microlocal cutoff function: By our assumption that $\overline{\Gamma}_0 \cap (E_u^* \oplus E_s^*) = \{0\}$

and the fact that $E_u^* \oplus E_s^*$ is a $\Phi_t^{X_{A_0}}$-invariant subset, there exists a conic neighborhood $\Gamma_1 \subset T^*\mathcal{M}$ of $E_u^* \oplus E_s^*$ such that $\Phi_t^{X_{A_0}}(\Gamma_1) \cap \overline{\Gamma_0} = \emptyset$ of $0 \le t \le T$. Let us choose a second, smaller conical neighborhood $E_u^* \oplus E_s^* \subset \Gamma_2 \Subset \Gamma_1$. Now we fix a microlocal cutoff $P = \mathrm{Op}(p) \in \Psi^0(\mathcal{M}, \mathbb{C})$ which is microsupported in Γ_1 (i.e. $\mathrm{WF}(P) \subset \Gamma_1$) and microlocally equal to one on Γ_2 (i.e. $\mathrm{WF}(\mathrm{Id} - P) \cap \Gamma_2 = \emptyset$) and which furthermore has globally bounded symbol $\sup_{(x,\xi)} |p(x,\xi)| \le 1$. We apply Lemma 4.4 with this choice of P and multiply (4.4) from the left with $e^{-T\mathbf{X}_{A_0}(\lambda)}$. Using (4.8), we get

$$d_{\mathbf{X}+\lambda} e^{-T\mathbf{X}_{A_0}(\lambda)} Q_{\mathrm{ell}}(\lambda) + e^{-T\mathbf{X}_{A_0}(\lambda)} Q_{\mathrm{ell}}(\lambda) d_{\mathbf{X}+\lambda} = e^{-T\mathbf{X}_{A_0}(\lambda)} \big(\mathrm{Id} - P + S_1(\lambda) + S_2(\lambda)\big).$$

We define $Q(\lambda) := Q_T(\lambda) + e^{-T\mathbf{X}_{A_0}(\lambda)} Q_{\mathrm{ell}}(\lambda)$ and obtain

$$d_{\mathbf{X}+\lambda} Q(\lambda) + Q(\lambda) d_{\mathbf{X}+\lambda} = F(\lambda) \text{ with } F(\lambda) := \mathrm{Id} - e^{-T\mathbf{X}_{A_0}(\lambda)} \big(P - S_1(\lambda) - S_2(\lambda)\big).$$

Let us now show that $Q(\lambda)$ and $F(\lambda)$ have the required properties. By precisely the same argument as in Lemma 4.5 (using that $X_{A_0}^H m \le 0$) we deduce that for all $t \ge 0$, $e^{-t\mathbf{X}_{A_0}}$ is bounded on $\mathcal{H}_{NG}$ for any escape function G associated to A_0 compatible with $c_X > 0$ and $\Gamma_{E_0^*} \subset \Gamma_0$. Consequently $Q_T(\lambda)$ and $e^{-T\mathbf{X}_{A_0}(\lambda)}$ are bounded operators on $\mathcal{H}_{NG}\Lambda$. As $\hat{\mathcal{A}}_{NG} Q_{\mathrm{ell}}(\lambda) \hat{\mathcal{A}}_{NG}^{-1} \in \Psi^{-2}(\mathcal{M}; E \otimes \Lambda)$, this operator is a bounded operator on L^2, thus $Q_{\mathrm{ell}}(\lambda)$ is bounded on $\mathcal{H}_{NG}\Lambda$ as well. Putting everything together we deduce that $Q(\lambda)$ is bounded on $\mathcal{H}_{NG}\Lambda$ for any $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. As $Q_T(\lambda)$ and $Q_{\mathrm{ell}}(\lambda)$ decrease the order in $\Lambda\mathfrak{a}^*$ by one, $Q(\lambda)$ has this property as well.

Let us deal with $F(\lambda)$: by our choice of Γ_1 we can apply Lemma 4.5 to $e^{-T\mathbf{X}_{A_0}(\lambda)} P = e^{-T\lambda(A_0)} e^{-T\mathbf{X}_{A_0}} P$ and deduce that the $e^{-T\mathbf{X}_{A_0}(\lambda)} P = R_N'(\lambda) + K_N'(\lambda)$ for some $R_N'(\lambda)$ bounded on $\mathcal{H}_{NG}$ and $K_N'(\lambda)$ compact on that space, with bound

$$\|R_N'(\lambda)\|_{\mathcal{L}(\mathcal{H}_{NG})} \le (1+\varepsilon) e^{T\left(-Nc_X - \mathrm{Re}(\lambda(A_0)) + C_{L^2}(A_0)\right)},$$

for some $\varepsilon > 0$. Consequently, by our choice of $T > \ln(3)/\delta$ and for $\lambda \in \mathcal{F}_{NG,A_0,\delta}$ we get $\|R_N'(\lambda)\|_{\mathcal{L}(\mathcal{H}_{NG})} \le (1+\varepsilon)/3$. Note that $S_1(\lambda) + S_2(\lambda) \in \Psi^{-1}(\mathcal{M}; E \otimes \Lambda)$ is compact on $\mathcal{H}_{NG}$ (this can be easily checked by conjugating it with $\hat{\mathcal{A}}_{NG}$ to obtain an operator in $\Psi^{-1}(\mathcal{M}; E \otimes \Lambda)$, thus compact on L^2). This completes the proof of Proposition 4.6 by setting $R_N(\lambda) := -R_N'(\lambda)$ and $K_N(\lambda) := -K_N'(\lambda) - e^{-T\mathbf{X}_{A_0}(\lambda)}(S_1(\lambda) - S_2(\lambda))$. ☐

As a consequence of Lemma 3.8, Proposition 4.3, Proposition 4.6 and Lemma 3.9, we get

Proposition 4.8. *For $A_0 \in \mathcal{W}$ there exists an escape function G such that for any $N > 0$ the operator $d_{\mathbf{X}+\lambda}$ on $\mathcal{H}_{NG}\Lambda$ defines a Fredholm complex for $\lambda \in \mathcal{F}_{NG,A_0,0}$, i.e. we have*

$$\sigma_{T,\mathcal{H}_{NG}}^{\mathrm{ess}}(-\mathbf{X}) \cap \mathcal{F}_{NG,A_0,0} = \emptyset.$$

4.3. **Ruelle-Taylor resonances are intrinsic.** So far we have shown that the admissible lift of an Anosov action X acting as differential operators on $\mathcal{H}_{NG}$ has a Fredholm Taylor spectrum on $\mathcal{F}_{NG,A} := \mathcal{F}_{NG,A,0} \subset \mathfrak{a}_{\mathbb{C}}^*$, where $A \in \mathcal{W}$ and G is an admissible escape function associated to A. Further, we have seen that $\mathcal{F}_{NG,A}$ can be made arbitrarily large by letting $N \to \infty$. However, it is not yet clear if this Fredholm spectrum is intrinsic to $\mathbf{X}$ or if it depends on the choice of the anisotropic Sobolev spaces $\mathcal{H}_{NG}$, i.e. in particular on the choices of N or G.

Let us denote by $C^{-\infty}_{E_u^*}(\mathcal{M};E)$ the space of distributions in $C^{-\infty}(\mathcal{M};E)$ with wavefront set contained in E_u^*. Equipped with a suitable topology this space becomes a topological vector space [Hör03, Chapter 8], and the lifted Anosov action $\mathbf{X}$ on $C^{-\infty}_{E_u^*}(\mathcal{M};E)$ is by continuous operators. In particular, it makes sense to consider the complex generated by the operator $d_{\mathbf{X}+\lambda}$ on $C^{-\infty}_{E_u^*}(\mathcal{M};E\otimes\Lambda)$. The main result of this section is the following:

Proposition 4.9. *Let $A_0 \in \mathcal{W}$, $N \geq 0$ and G be an escape function for A_0. Then for any $\lambda \in \mathcal{F}_{NG,A_0}$ one has vector space isomorphisms*

$$\ker_{\mathcal{H}_{NG}\Lambda^j} d_{\mathbf{X}+\lambda}/\operatorname{ran}_{\mathcal{H}_{NG}\Lambda^j} d_{\mathbf{X}+\lambda} \cong \ker_{C^{-\infty}_{E_u^*}\Lambda^j} d_{\mathbf{X}+\lambda}/\operatorname{ran}_{C^{-\infty}_{E_u^*}\Lambda^j} d_{\mathbf{X}+\lambda}.$$

Using this result, we see that the Ruelle-Taylor spectrum is independent of A_0 and of the anisotropic space $\mathcal{H}_{NG}\Lambda$ in the region $\mathcal{F}_{NG,A_0}$ of $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ where the Taylor complex $d_{\mathbf{X}+\lambda}$ is Fredholm on $\mathcal{H}_{NG}\Lambda$. We can then define the notion of a Ruelle-Taylor resonance as follows:

Definition 4.10. We define the *Ruelle-Taylor resonances* of $\mathbf{X}$ to be the set

$$\operatorname{Res}_{\mathbf{X}} := \{\lambda \in \mathfrak{a}^*_{\mathbb{C}} \,|\, \exists j, \ker_{C^{-\infty}_{E_u^*}\Lambda^j} d_{\mathbf{X}+\lambda}/\operatorname{ran}_{C^{-\infty}_{E_u^*}\Lambda^j} d_{\mathbf{X}+\lambda} \neq 0\}$$

and the *Ruelle-Taylor resonant cohomology space* of degree j of $\lambda \in \operatorname{Res}_{\mathbf{X}}$ to be

$$\operatorname{Res}_{\mathbf{X},\Lambda^j}(\lambda) := \ker_{C^{-\infty}_{E_u^*}\Lambda^j} d_{\mathbf{X}+\lambda}/\operatorname{ran}_{C^{-\infty}_{E_u^*}\Lambda^j} d_{\mathbf{X}+\lambda}.$$

Another consequence of Proposition 4.9 is:

Corollary 4.11 (Location of Ruelle-Taylor resonances)**.** *One has*

$$\operatorname{Res}_{\mathbf{X}} \subset \bigcap_{A\in\mathcal{W}} \{\lambda \in \mathfrak{a}^*_{\mathbb{C}} \mid \operatorname{Re}(\lambda(A)) \leq C_{L^2}(A)\}$$

Proof. Assume that there exists an $A \in \mathcal{W}$ such that $\operatorname{Re}(\lambda(A)) > C_{L^2}(A)$. Then for some $\delta > 0$, $\lambda \in \mathcal{F}_{0G,A,\delta}$ and consequently $\lambda \in \operatorname{Res}_{\mathbf{X}}$ iff $\ker_{L^2\Lambda} d_{\mathbf{X}+\lambda}/\operatorname{ran}_{L^2\Lambda} d_{\mathbf{X}+\lambda} \neq 0$. However, by (4.7) there is a bounded operator $Q_T(\lambda) : L^2(\mathcal{M};E\otimes\Lambda) \to L^2(\mathcal{M};E\otimes\Lambda)$ such that

$$d_{\mathbf{X}+\lambda}Q_T(\lambda) + Q_T(\lambda)d_{\mathbf{X}+\lambda} = \operatorname{Id} + e^{-T\mathbf{X}_A}e^{-T\lambda(A)}.$$

Since $\operatorname{Re}(\lambda(A)) > C_{L^2}(A)$, the right hand side is invertible on $L^2(\mathcal{M};E\otimes\Lambda)$, so that $\ker_{L^2\Lambda} d_{\mathbf{X}+\lambda}/\operatorname{ran}_{L^2\Lambda} d_{\mathbf{X}+\lambda} = 0$. $\square$

The strategy to prove Proposition 4.9 is to show that in each cohomology class in $\ker_{\mathcal{H}_{NG}\Lambda} d_{\mathbf{X}+\lambda}/\operatorname{ran}_{\mathcal{H}_{NG}\Lambda} d_{\mathbf{X}+\lambda}$ one can find a representative that lies already in $\ker_{C^{-\infty}_{E_u^*}} d_{\mathbf{X}+\lambda}$. To this end we will construct for fixed λ a projector $\Pi_0(\lambda)$ of finite rank such that we can find in each cohomology class a representative in the range of $\Pi_0(\lambda)$. The fact that the range of $\Pi_0(\lambda)$ is independent of the anisotropic Sobolev spaces and contained in $C^{-\infty}_{E_u^*}$ then follows very similarly to the corresponding characterization of Anosov flows [FS11, Theorem 1.7] by the flexibility in the choice of the escape function.

Proof of Proposition 4.9. Given A_0, N, G and $\lambda \in \mathcal{F}_{NG,A_0}$ let us first fix $\delta > 0$ such that $\lambda \in \mathcal{F}_{NG,A_0,\delta}$ and an open cone $\Gamma_0 \subset T^*\mathcal{M}$ containing $\Gamma_{E_0^*}$ and such that $\overline{\Gamma}_0 \cap (E_s^* \oplus E_u^*) = \{0\}$. Then Proposition 4.6 provides operators $Q(\lambda), F(\lambda) : C^{-\infty}(\mathcal{M}; E \otimes \Lambda) \to C^{-\infty}(\mathcal{M}; E \otimes \Lambda)$ which only depend on $\delta, \lambda, A_0, \Gamma_0$ and satisfy

$$d_{\mathbf{X}+\lambda} Q(\lambda) + Q(\lambda) d_{\mathbf{X}+\lambda} = F(\lambda). \tag{4.9}$$

We can thus apply Lemma 3.8, and deduce that if $\Pi_0(\lambda)$ is the spectral projector of $F(\lambda)$ on its kernel,

$$\Pi_0(\lambda) : \ker_{\mathcal{H}_{N'G'}\Lambda} d_{\mathbf{X}+\lambda} / \operatorname{ran}_{\mathcal{H}_{N'G'}\Lambda} d_{\mathbf{X}+\lambda} \to \ker_{\operatorname{ran}\Pi_0(\lambda)} d_{\mathbf{X}+\lambda} / \operatorname{ran}_{\operatorname{ran}\Pi_0(\lambda)} d_{\mathbf{X}+\lambda}$$

is an isomorphism. Here, $\operatorname{ran}(\Pi_0(\lambda)) = \Pi_0(\lambda)\mathcal{H}_{NG}$. But since C^∞ is dense in $\mathcal{H}_{NG}$, and $\Pi_0(\lambda)$ has finite rank, we deduce that it is equal to $\Pi_0(\lambda) C^\infty(\mathcal{M}; E \otimes \Lambda)$. We now need the following lemma:

Lemma 4.12. *The projector $\Pi_0(\lambda)$ satisfies $\Pi_0(\lambda)(C^\infty(\mathcal{M}; E \otimes \Lambda)) \subset C^{-\infty}_{E_u^*}(\mathcal{M}; E \otimes \Lambda)$. Additionally, it has a continuous extension to $C^{-\infty}_{E_u^*}(\mathcal{M}; E \otimes \Lambda)$.*

Proof. Recall that $\Pi_0(\lambda) : \mathcal{H}_{NG}\Lambda \to \mathcal{H}_{NG}\Lambda$ has been defined as the spectral projector at $z = 0$ of $F(\lambda) : \mathcal{H}_{NG}\Lambda \to \mathcal{H}_{NG}\Lambda$, it has finite rank. Since $F(\lambda)$ and its Fredholmness do not depend on the choice of N, G, as long as $\lambda \in \mathcal{F}_{NG,A_0}$, neither does its spectral projector at 0. The image of $\Pi_0(\lambda)$ is thus contained in the intersection of the $\mathcal{H}_{N'G'}\Lambda$ such that $\lambda \in \mathcal{F}_{N'G',A_0}$.

Let us show that this intersection is contained in $C^{-\infty}_{E_u^*}(\mathcal{M}; E \otimes \Lambda)$. We thus take u in all the $\mathcal{H}_{N'G'}$ such that $\lambda \in \mathcal{F}_{N'G',A_0}$. By Proposition 4.3 for an arbitrary cone Γ'_{reg} disjoint from E_u^*, there exists an escape function G' for A_0 compatible with c'_X and $\Gamma'_{E_0^*} \subset \Gamma_0$ such that microlocally on Γ'_{reg}, $\mathcal{H}_{N'G'}$ is contained in the standard Sobolev space $H^{N'/2}(\mathcal{M}; E)$. In particular, taking N' arbitrarily large, $\lambda \in \mathcal{F}_{N'G',A_0}$ and $\mathrm{WF}(u) \cap \Gamma'_{\mathrm{reg}} = \emptyset$. Since Γ'_{reg} was arbitrary, $\mathrm{WF}(u) \subset E_u^*$.

To prove that $\Pi_0(\lambda)$ has a continuous extension to $C^{-\infty}_{E_u^*}(\mathcal{M}; E \otimes \Lambda)$, it suffices to observe that $C^{-\infty}_{E_u^*}(\mathcal{M}; E \otimes \Lambda)$ is also contained in the union of the all the $\mathcal{H}_{N'G'}$ such that $\lambda \in \mathcal{F}_{N'G',A_0}$. This follows from Definition 4.1,(1), since we know that in a conic neighborhood around E_u^* we have $m(x, \xi) \leq -1/4$. As a consequence, $\Pi_0(\lambda)$ is a linear operator from $C^{-\infty}_{E_u^*}(\mathcal{M}; E \otimes \Lambda)$ to $\mathcal{D}'(\mathcal{M}, E \otimes \Lambda)$. It is continuous as it has finite rank. □

To finish the proof of Proposition 4.9, it suffices to apply a variation of the sandwiching trick presented in the proof of Lemma 3.11. Indeed, since $\Pi_0(\lambda)$ is a bounded projector on $C^{-\infty}_{E_u^*}(\mathcal{M}; E \otimes \Lambda)$, commuting with $d_{\mathbf{X}+\lambda}$, $u \mapsto \Pi_0(\lambda)u$ factors to a surjective map

$$\ker_{C^{-\infty}_{E_u^*}\Lambda} d_{\mathbf{X}+\lambda} / \operatorname{ran}_{C^{-\infty}_{E_u^*}\Lambda} d_{\mathbf{X}+\lambda} \to \ker_{\operatorname{ran}\Pi_0(\lambda)} d_{\mathbf{X}+\lambda} / \operatorname{ran}_{\operatorname{ran}\Pi_0(\lambda)} d_{\mathbf{X}+\lambda} \tag{4.10}$$

We need to show the injectivity of this map. This will follow from the fact that $C^{-\infty}_{E_u^*}(\mathcal{M}; E \otimes \Lambda)$ is contained in the union of the $\mathcal{H}_{N'G'}$ such that $\lambda \in \mathcal{F}_{N'G',A_0}$. We consider $u \in C^{-\infty}_{E_u^*}(\mathcal{M}; E \otimes \Lambda)$ such that $d_{\mathbf{X}+\lambda}u = 0$, and $[\Pi_0(\lambda)u] = 0$, i.e, $\Pi_0(\lambda)u =$

$d_{\mathbf{X}+\lambda}\Pi_0(\lambda)v$ for some $v \in C^{-\infty}_{E_u^*}(\mathcal{M}; E \otimes \Lambda)$. Since u belongs to some $\mathcal{H}_{N',G'}$, we then write $\tilde{F}(\lambda) = F(\lambda) + \Pi_0(\lambda)$, and observe, just as in (3.12), that

$$\tilde{F}(\lambda)^{-1}Q(\lambda)d_{\mathbf{X}+\lambda} + d_{\mathbf{X}+\lambda}\tilde{F}(\lambda)^{-1}Q(\lambda) = \mathrm{Id} - \Pi_0(\lambda),$$

so that

$$u = d_{\mathbf{X}+\lambda}\big(\tilde{F}(\lambda)^{-1}Q(\lambda)u + \Pi_0(\lambda)v\big).$$

It remains to check that $\tilde{F}^{-1}(\lambda)Q(\lambda)u \in C^{-\infty}_{E_u^*}(\mathcal{M}; E \otimes \Lambda)$. But this is an element of each $\mathcal{H}_{N'G'}$ such that $\lambda \in \mathcal{F}_{N'G',A_0}$, so it is contained in the intersection thereof. We have seen in the proof of Lemma 4.12 that this intersection is contained in $C^{-\infty}_{E_u^*}(\mathcal{M}; E \otimes \Lambda)$.

Finally, note that the operator $F(\lambda) : \mathcal{H}_{NG}\Lambda \to \mathcal{H}_{NG}\Lambda$ preserves the order in the Koszul complex, i.e. $F(\lambda) : \mathcal{H}_{NG}\Lambda^j \to \mathcal{H}_{NG}\Lambda^j$, and all the subsequent constructions such as $\Pi_0(\lambda)$ do as well. The isomorphism $\Pi_0(\lambda)$ can thus be restricted to the individual cohomology $\ker_{C^{-\infty}_{E_u^*}\Lambda^j} d_{\mathbf{X}+\lambda}/\operatorname{ran}_{C^{-\infty}_{E_u^*}\Lambda^j} d_{\mathbf{X}+\lambda}$ and we have completed the proof of Proposition 4.9. □

4.4. **Discrete Ruelle-Taylor spectrum.** In this section we show that the Ruelle-Taylor resonance spectrum of the admissible lift $\mathbf{X} : \mathfrak{a} \mapsto \mathrm{Diff}^1(\mathcal{M}; E)$ of the Anosov action, for E a Riemannian vector bundle, is discrete in $\mathfrak{a}^*_{\mathbb{C}}$. Our goal is to use Lemma 3.11. In contrast to the Fredholm property, the proof in this section requires a slightly better escape function that provides decay not only in a fixed direction $A_1 \in \mathcal{W}$, but also for all other elements in a small neighbourhood of A_1:

Lemma 4.13. *Let $A_1 \in \mathcal{W}$ be fixed. Then there is an escape function G for A_1, a conic neighborhood $\Gamma^*_{E_0} \subset T^*\mathcal{M}$ of E_0^* such that $\overline{\Gamma}_{E_0^*} \cap (E_u^* \oplus E_s^*) = \{0\}$, a constant $c_X > 0$ and a neighborhood $\mathcal{O} \subset \mathcal{W}$ of A_1 such that G is an escape function for all $A \in \mathcal{O}$ compatible with $c_X > 0$ and $\Gamma_{E_0^*}$. Moreover, G can be chosen to satisfy $X_A^H G \leq 0$ in $\{|\xi| \geq R\}$ for some $R \geq 1$.*

Proof. The construction of the weight function m is written in [GB18, Section 2] in the case of an Anosov flow, but it works mutatis mutandis in our case as the proof simply uses the continuity of the decomposition $T^*\mathcal{M} = E_0^* \oplus E_s^* \oplus E_u^*$ and the expanding/contracting properties of E_s^* and E_u^*, but not the fact that $\dim E_0^* = 1$. The weight function satisfies $X_A^H m \leq 0$ for $|\xi| \geq 1$ for all A close enough to A_1. We can then define the function G as in [FS11, Lemma 1.2] by setting $G(x,\xi) = m(x,\xi)\log(1 + f(x,\xi))$, where $f > 0$ is homogeneous of degree 1 in ξ, satisfies $f(x,\xi) = |\xi(X_{A_1})|$ near $E_0^* \cap \{|\xi| \geq 1\}$, and

$$X_A^H f < -c_1(1+f), \quad (\text{resp. } X_A^H f > c_1/(1+f)) \tag{4.11}$$

in a conic neighborhood of E_s^* (resp. of E_u^*) for some $c_1 > 0$. To construct such f near E_s^*, we can use [DZ16, Lemma C.1]: for (x,ξ) in a conic neighborhood N_s of E_s^*, set

$$f(x,\xi) := \int_0^T |e^{-tX^H_{A_1}}(x,\xi)|dt, \quad T > 0$$

so that, if $A = A_1 + \epsilon A'$ with $|A'|_{\mathfrak{a}} \leq 1$, one has $X_A^H = X_{A_1}^H + \epsilon X_{A'}^H$ and

$$X_A^H f(x,\xi) = |\xi| - |e^{-TX_A^H}(x,\xi)| + \mathcal{O}(\epsilon e^{CT}|\xi|)$$

for some $C>0$ uniform with respect to A' as above. Fix T large enough so that we have $|\xi|-|e^{-TX_A^H}(x,\xi)|\leq -2|\xi|$ for all $|\xi|>1$ in N_s. Once T has been fixed, one can choose $0<\epsilon<e^{-CT}$ so that $X_A^H f(x,\xi)\leq -|\xi|$ in $N_s\cap\{|\xi|>1\}$. Since $(1+f(x,\xi))>c_1^{-1}|\xi|$ in $N_s\cap\{|\xi|>1\}$ for some $c_1>0$, we obtain (4.11). The same construction applies near E_u^*. The proof of [FS11, Lemma 1.2] (using the fact that $X_A^H|\xi(A_1)|=0$) shows that $X_A^H G\leq 0$ for all $|\xi|\geq R$ if R is large enough and that G is an escape function for all $A\in\mathcal{O}:=A_1+\{\epsilon A'\in\mathfrak{a}\,|\,|A'|_{\mathfrak{a}}\leq 1\}$ compatible with some $c_X>0$ and some $\Gamma_{E_0^*}$. ☐

Let us now fix a basis $A_1,\dots,A_\kappa\in\mathcal{O}\subset\mathcal{W}$ of $\mathfrak{a}$ in the positive Weyl chamber. In order to be able to use Lemma 3.11, we will prove the following:

Lemma 4.14. *For each fixed $\lambda\in\mathcal{F}_{NG,A_0,\delta}$ there is a family of commuting operators $Q(\lambda)=(Q_1(\lambda),\dots,Q_\kappa(\lambda))$ bounded on $\mathcal{H}_{NG}$, commuting with $\mathbf{X}+\lambda$, such that if $\delta_{Q(\lambda)}$ is the divergence associated to the family $Q(\lambda)$ we obtain*

$$d_{\mathbf{X}+\lambda}\delta_{Q(\lambda)}+\delta_{Q(\lambda)}d_{\mathbf{X}+\lambda}=\mathrm{Id}+R(\lambda)+K(\lambda)$$

with $R(\lambda),K(\lambda)\in\mathcal{L}(\mathcal{H}_{NG}\Lambda)$, $\|R(\lambda)\|_{\mathcal{L}(\mathcal{H}_{NG})}<1/2$ and $K(\lambda)$ compact on $\mathcal{H}_{NG}\Lambda$.

Proof. recall the basis $A_1,\dots,A_\kappa\in\mathcal{W}$ close to A_1 and write $\lambda_j:=\lambda(A_j)$ as well as $\mathbf{X}_{A_j}(\lambda):=\mathbf{X}_{A_j}+\lambda_j$. Let $T_j>0$ for $j=1,\dots,\kappa$, and consider $\chi_j\in C_c^\infty([0,\infty[;[0,1])$ non-increasing with $\chi_j=1$ in $[0,T_j]$ and $\operatorname{supp}\chi_j\in[0,T_j+1]$. Then we set

$$Q_j'(\lambda):=\int_0^\infty e^{-t_j\mathbf{X}_{A_j}(\lambda)}\chi_j(t_j)dt_j$$

and we make it act on $C^\infty(\mathcal{M})\otimes\Lambda\mathfrak{a}^*$ by $\widetilde{Q}_j(\lambda):u\otimes w\mapsto(Q_j'(\lambda)u)\otimes\iota_{A_j}\omega$. As in Proposition 4.6, we compute

$$d_{\mathbf{X}(\lambda)}\widetilde{Q}_j(\lambda)+\widetilde{Q}_j(\lambda)d_{\mathbf{X}(\lambda)}=\mathrm{Id}+R_j(\lambda),\quad R_j(\lambda)(u\otimes\omega):=\left(\int_0^\infty e^{-t_j\mathbf{X}_{A_j}(\lambda)}u\chi_j'(t_j)dt_j\right)\otimes w.$$

and note that $R_j(\lambda)=R_j'(\lambda)\otimes\mathrm{Id}$ is scalar. We thus have

$$d_{\mathbf{X}(\lambda)}\widetilde{Q}(\lambda)+\widetilde{Q}(\lambda)d_{\mathbf{X}(\lambda)}=F(\lambda),\quad F(\lambda):=\mathrm{Id}-(-1)^\kappa\prod_{\ell=1}^{\kappa}R_k(\lambda). \tag{4.12}$$

with $\widetilde{Q}(\lambda):=\sum_{j=1}^\kappa(-1)^{j-1}\widetilde{Q}_j(\lambda)\prod_{k=1}^{j-1}R_k(\lambda)$. First we note that $\widetilde{Q}(\lambda)=\delta_{Q(\lambda)}$ is the divergence associated to the family of commuting operators $Q(\lambda):=(Q_1(\lambda),\dots,Q_\kappa(\lambda))$ where

$$Q_j(\lambda):=(-1)^jQ_j'(\lambda)\prod_{k=1}^{j-1}R_k'(\lambda).$$

We notice that $Q_j(\lambda)$ commutes with $\mathbf{X}_{A_i}(\lambda)$ for each i,j. As in the proof Proposition 4.6, we have that $Q(\lambda)$ (resp. $\widetilde{Q}(\lambda)$) is bounded on $\mathcal{H}_{NG}$ (resp. on $\mathcal{H}_{NG}\Lambda$): notice that here we use Lemma 4.13 as it is important that the weight function m satifies $X_{A_j}^Hm\leq 0$ for all $j=1,\dots,\kappa$. We take P microsupported in a neighbourhood of $E_u^*\oplus E_s^*$, as in the proof of Proposition 4.6, and follow the arguments given there, which were based on Lemma 4.5: if $T_j := T$ is chosen large enough (as in proof of Proposition 4.6)

$$\begin{aligned}\prod_{k=1}^{\kappa} R_k'(\lambda)P &= \int_{[T,T+1]^\kappa} e^{-\sum_{j=1}^{\kappa} t_j X_{A_j}(\lambda)} P \prod_{j=1}^{\kappa} \chi_j'(t_j)dt \\ &= \int_{[T,T+1]^\kappa} (R(t,\lambda)+K(t,\lambda)) \prod_{j=1}^{\kappa} \chi_j'(t_j)dt\end{aligned},$$

where $\|R(t,\lambda)\|_{\mathcal{L}(\mathcal{H}_{NG})} \prod_j \|\chi_j'\|_{L^\infty} \leq 1/2$ and $K(t,\lambda)$ is compact on $\mathcal{H}_{NG}$ for all $t \in [T, T+1]$ (both depend on N, G). This shows that the operator $(F(\lambda) - \mathrm{Id})P$ decomposes as $(F(\lambda) - \mathrm{Id})P = R(\lambda) + K_1(\lambda)$ with $\|R(\lambda)\|_{\mathcal{L}(\mathcal{H}_{NG}\Lambda)} < 1/2$ and $K_1(\lambda)$ compact on $\mathcal{H}_{NG}\Lambda$. Next, we claim that using that $P \in \Psi^0(\mathcal{M})$ is scalar with $\mathrm{WF}(1-P)$ not intersecting a conic neighborhood of $E_u^* \oplus E_s^*$, we see that $K_2(\lambda) := (F(\lambda) - \mathrm{Id})(1-P)$ is a compact operator on $\mathcal{H}_{NG}\Lambda$. Indeed, let us first take a microlocal partition of $(1-P)$ so that $(1-P) - \sum_{k=1}^{\kappa} P_k \in \Psi^{-\infty}(\mathcal{M})$ with $P_k \in \Psi^0(\mathcal{M})$ and $\mathrm{WF}(P_k)$ not intersecting a conic neighborhood of the characteristic set $\{(x,\xi) \in T^*\mathcal{M} \,|\, \xi(X_{A_k}) = 0\}$. Let us show that $R_k'(\lambda)P_k$ is compact on $\mathcal{H}_{NG}$: first,

$$R_k'(\lambda)P_k\mathbf{X}_{A_k}(\lambda) = \int_T^{T+1} e^{-t_k\mathbf{X}_{A_k}(\lambda)} P_k \chi_k''(t_k)dt_k + R_k'(\lambda)[P_k, \mathbf{X}_{A_k}] \in \mathcal{L}(\mathcal{H}_{NG}), \tag{4.13}$$

where we used that $[P_k, \mathbf{X}_{A_k}] \in \Psi^0(\mathcal{M})$ and that $e^{-t_k\mathbf{X}_{A_k}(\lambda)}$ is bounded on $\mathcal{H}_{NG}$. Since $\mathbf{X}_k(\lambda)$ is elliptic near $\mathrm{WF}(P_k)$, we can construct a parametrix $Z_k(\lambda) \in \Psi^{-1}(\mathcal{M})$ so that $\mathbf{X}_{A_k}(\lambda)Z_k(\lambda) - P_k' \in \Psi^{-\infty}(\mathcal{M})$ for some $P_k' \in \Psi^0(\mathcal{M})$ with $P_k'P_k - P_k \in \Psi^{-\infty}(\mathcal{M})$. We thus obtain that

$$R_k'(\lambda)P_k\mathbf{X}_{A_k}(\lambda)Z_k(\lambda) - R_k'(\lambda)P_k \in \Psi^{-\infty}(\mathcal{M}),$$

but $Z_k(\lambda)$ being compact on $\mathcal{H}_{NG}$, we get that $R_k'(\lambda)P_k$ is compact on $\mathcal{H}_{NG}$ using (4.13). Next, we write

$$\Big(\prod_{k=1}^{\kappa} R_k'(\lambda)\Big)(\mathrm{Id} - P) - \sum_{j=1}^{\kappa}\Big(\prod_{k=1}^{\kappa} R_k'(\lambda)\Big)P_j \in \Psi^{-\infty}(\mathcal{M}).$$

This operator is compact since all the $R_j'(\lambda)$ are bounded on $\mathcal{H}_{NG}$ and commute with each other and $R_k'(\lambda)P_k$ is compact. Putting everything together we deduce that $F(\lambda)$ has the desired properties by setting $K(\lambda) := K_1(\lambda) + K_2(\lambda)$. □

As a corollary, using Lemma 3.11 and Lemma 3.9, we deduce the following:

Proposition 4.15. *For an admissible lift of an Anosov action $\mathbf{X}$, the Ruelle-Taylor resonance spectrum is a discrete subset of $\mathfrak{a}_{\mathbb{C}}^*$. Moreover, $\lambda \in \mathcal{F}_{NG,A_0} \cap \sigma_{T,\mathcal{H}_{NG}}(-\mathbf{X})$ if and only if there is $u \in \mathcal{H}_{NG}$ such that*

$$(\mathbf{X} + \lambda)u = 0.$$

This completes the proof of Theorem 4. In the scalar we will show in Corollary 4.16 below that part 3) of Theorem 4 can be sharpened using the dynamical parametrix $Q(\lambda)$ in Lemma 4.14 (the same argument also works for admissible lifts under the condition $\|e^{-t\mathbf{X}_A} f\|_{\mathcal{L}(L^\infty)} \leq C$ for all $t \in \mathbb{R}$):

Corollary 4.16. *Let X be an Anosov action. Then one has*

$$\mathrm{Res}_X \subset \bigcap_{A\in\mathcal{W}} \{\lambda \in \mathfrak{a}^*_{\mathbb{C}} \mid \mathrm{Re}(\lambda(A)) \leq 0\}.$$

Proof. Let $A \in \mathcal{W}$ and assume that $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ satisfies $\mathrm{Re}(\lambda(A)) > 0$, then we will show that λ can not be a Ruelle-Taylor resonance. We use the parametrix $\widetilde{Q}(\lambda)$ of Lemma 4.14 with $A_1 := A$ and $(A_j)_j \in \mathcal{W}^\kappa$ forming a basis of $\mathfrak{a}$ with A_j in an arbitrarily small neighborhood of A_1 so that $\mathrm{Re}(\lambda(A_j)) > 0$ for all j. We get that (4.12) holds with $F(\lambda)$ having discrete spectrum near $z = 0$. Let $\Pi_0(\lambda)$ be the spectral projector of the Fredholm operator $F(\lambda)$ at $z = 0$, which can be written

$$\Pi_0(\lambda) = \frac{1}{2\pi i}\int_{|z|=\epsilon} (z - F(\lambda))^{-1} dz \tag{4.14}$$

for some small enough $\epsilon > 0$. We notice that for $f \in L^\infty(\mathcal{M})$, we have

$$\begin{aligned}\|(\mathrm{Id} - F(\lambda))f\|_{L^\infty} \leq & \int_{(\mathbb{R}^+)^\kappa} \|e^{-\sum_{j=1}^\kappa t_j \mathbf{X}_{A_j}(\lambda)} f\|_{L^\infty} \prod_{j=1}^{\kappa}(-\chi_j'(t_j)) dt_1 \dots dt_\kappa \\ \leq & \|f\|_{L^\infty} e^{-\sum_{j=1}^\kappa T_j \lambda(A_j))} \int_{(\mathbb{R}^+)^\kappa} \prod_{j=1}^{\kappa}(-\chi_j'(t_j)) dt_1 \dots dt_\kappa \\ \leq & \|f\|_{L^\infty} e^{-\sum_{j=1}^\kappa T_j \lambda(A_j))}.\end{aligned}$$

This shows that by choosing $T_j > 0$ large enough, $\|(\mathrm{Id} - F(\lambda))\|_{\mathcal{L}(L^\infty)} < 1/2$. In particular $F(\lambda)$ is invertible on L^∞ and therefore $\Pi_0(\lambda) = 0$ since the expression (4.14) holds also as a map $C^\infty(\mathcal{M}) \to C^{-\infty}(\mathcal{M})$. This ends the proof. ◻

Let us end the section with a statement about joint Jordan blocks for an admissible lifts $\mathbf{X}$: Therefore given $\alpha \in \mathbb{N}^\kappa$ we define $\mathbf{X}^\alpha(\lambda) := \prod_{j=1}^\kappa (\mathbf{X}_{A_j} + \lambda_j)^{\alpha_j}$.

Proposition 4.17. *For any Ruelle-Taylor resonance $\lambda \in \mathrm{Res}_{\mathbf{X}}$ there is $J \in \mathbb{N}$ which is the minimal integer such that, whenever for some $u \in C^{-\infty}_{E_u^*}(\mathcal{M})$ and $k \in \mathbb{N}$ one has $\mathbf{X}^\beta(\lambda)u = 0$ for all $|\beta| = k$ then $\mathbf{X}^\alpha(\lambda)u = 0$ for all $|\alpha| = J$. Moreover the space of* generalized joint resonant states *is the finite dimensional space given by*

$$\{u \in C^{-\infty}_{E_u^*}(\mathcal{M}) \,|\, \mathbf{X}^\alpha(\lambda)u = 0 \text{ for all } |\alpha| = J\} \subset \mathrm{ran}\,\Pi_0(\lambda).$$

where $\Pi_0(\lambda)$ is the spectral projector of $F(\lambda)$ at $z = 0$, defined in (4.14).

Proof. Let $\mathcal{H}_{NG}$ be an anisotropic Sobolev space such that $\lambda \in \mathcal{F}_{NG}$. We construct the parametrix from (4.12) with $\chi_j = \chi_1$ for all $j = 1, \dots \kappa$ and $\psi := -\chi_1'$, and we denote by $\Pi_0(\lambda) : \mathcal{H}_{NG} \to \mathcal{H}_{NG}$ the spectral projector on the generalized eigenspace of $F(\lambda)$ for the eigenvalue zero. Note that it commutes with $\mathbf{X}_{A_j}$ for all j, since $F(\lambda)$ does. We now show by induction that for any $u \in C^{-\infty}_{E_u^*}(\mathcal{M})$ with $\mathbf{X}^\alpha(\lambda)u = 0$ for all $|\alpha| = k$ we have $u \in \mathrm{ran}(\Pi_0(\lambda)) \subset \mathcal{H}_{NG}$. The start of the induction is simply deduced from the proof of Proposition 4.9. Next we show that if the property is satisfied at step k then it is at step $k+1$: if for $u \in C^{-\infty}_{E_u^*}(\mathcal{M})$, $\mathbf{X}^\alpha(\lambda)u = 0$ for all $|\alpha| = k+1$ then for any $|\beta| = k$ we have $\mathbf{X}_{A_j}(\lambda)(\mathbf{X}^\beta(\lambda)u) = 0$ for $j = 1, \dots, \kappa$. Thus, again by the proof of Proposition 4.9 we know $\mathbf{X}^\beta(\lambda) \in \mathrm{ran}(\Pi_0(\lambda))$. As $[\Pi_0(\lambda), \mathbf{X}^\beta(\lambda)] = 0$ we conclude $\mathbf{X}^\beta(\lambda)(\Pi_0(\lambda)u - u) = 0$. Consequently, by induction hypothesis, $\Pi_0(\lambda)u - u \in \mathrm{ran}(\Pi_0(\lambda))$ and the claim follows. The statement of the proposition follows because $\mathrm{ran}(\Pi_0(\lambda))$ is a finite dimensional $\mathbf{X}_{A_i}(\lambda)$ invariant subspace. □

5. The leading resonance spectrum

In this section we study the leading resonance spectrum, i.e. those resonances with vanishing real part and show that they give rise to particular measures and are related to mixing properties of the Anosov action.

5.1. **Imaginary Ruelle-Taylor resonances in the non-volume preserving case.** In this section, we investigate the purely imaginary Ruelle-Taylor resonances and in particular the resonance at 0 for the action on functions. We assume that the Anosov action X does not necessarily preserve a smooth invariant measure. We choose a basis $A_1, \dots, A_\kappa$ of $\mathfrak{a}$, with dual basis $(e_j)_j$ in $\mathfrak{a}^*$, and set $X_j := X_{A_j}$, and we use dv_g the smooth Riemannian probability measure on $\mathcal{M}$. Let us choose $i\lambda \in i\mathfrak{a}^*$ and fix non-negative functions $\chi_j \in C_c^\infty(\mathbb{R}^+)$, equal to 1 near 0, with $\chi_j' \leq 0$ and build a parametrix $Q(i\lambda)$ as in Lemma 4.14 so that

$$Q(i\lambda)d_{X+i\lambda} + d_{X+i\lambda}Q(i\lambda) = \mathrm{Id} - R(i\lambda) \otimes \mathrm{Id},$$

and writing $\psi_j := -\chi_j' \in C_c^\infty((0,\infty))$ and $\lambda_j := \lambda(A_j)$

$$R(i\lambda) = \prod_{j=1}^{\kappa} \int e^{-t_j(X_j + i\lambda_j)} \psi_j(t_j) dt_j. \tag{5.1}$$

We proved that $R(i\lambda)$ has essential spectral radius < 1 in the anisotropic space $\mathcal{H}_{NG}$, and the resolvent $(R(i\lambda) - z)^{-1}$ is meromorphic outside $|z| < 1 - \epsilon$ for some ϵ, and the poles in $|z| > 1 - \epsilon$ are the eigenvalues of $R(i\lambda)$. Moreover, for $f \in L^\infty$, one has

$$\|R(i\lambda)f\|_{L^\infty} \leq \|f\|_{L^\infty} \prod_{j=1}^{\kappa} \int_{\mathbb{R}} \psi_j(t_j) dt_j \leq \|f\|_{L^\infty}. \tag{5.2}$$

Since $R(i\lambda)$ is bounded, for $|z|$ large enough one has on $\mathcal{H}_{NG}$

$$(z - R(i\lambda))^{-1} = z^{-1} \sum_{k \geq 0} z^{-k} R(i\lambda)^k \tag{5.3}$$

but the L^∞ estimate (5.2) shows that this series converges in $\mathcal{L}(L^\infty)$ and is analytic for $|z| > 1$. Therefore, using the density of $C^\infty(\mathcal{M})$ in $\mathcal{H}_{NG}$, we deduce that $R(i\lambda)$ has no eigenvalues in $|z| > 1$. We will use the notation $\langle u, v \rangle$ for the distributional pairing associated to the Riemannian measure dv_g fixed on $\mathcal{M}$ that also extends to a complex bilinear pairing $\mathcal{H}_{NG} \times \mathcal{H}_{-NG} \to \mathbb{C}$; in particular if $u, v \in L^2(\mathcal{M})$, this is simply $\int_{\mathcal{M}} uv \, dv_g$. Accordingly, we also write $\langle u, v\rangle_{L^2}$ for the pairing $\int_{\mathcal{M}} u\bar{v} \, dv_g$ and its sesquilinear extension to the pairing $\mathcal{H}_{NG} \times \mathcal{H}_{-NG} \to \mathbb{C}$.

The next three lemmas (Lemma 5.1) characterize the spectral projector of $R(i\lambda)$ onto the possible eigenvalue 1. Keep in mind that by Lemma 3.8 this spectral projector is closely related to the Ruelle Taylor resonant states. Finally in Proposition 5.4 we will use the knowledge about this spectral projector to characterize the leading resonance spectrum and to define SRB measures.

Lemma 5.1. *Let $\lambda \in i\mathfrak{a}^*$. If τ is an eigenvalue of $R(i\lambda)$ with modulus 1, it has no associated Jordan block, i.e. $(z - R(i\lambda))^{-1}$ has at most a pole of order 1 at $z = \tau$*

Proof. We take $u \in \mathcal{H}_{NG}$ such that $(R(i\lambda) - z)^{-1}u$ has a pole of order > 1 at $z = \tau$. By density of C^∞ in $\mathcal{H}_{NG}$, we can always assume that u is smooth. Denoting by $\psi^{(k)} = \psi * \cdots * \psi$ (k-th convolution power), we can write

$$R(i\lambda)^k = \prod_{j=1}^{\kappa} \int_{\mathbb{R}} e^{-t_j X_j(i\lambda)} \psi_j^{(k)}(t_j) dt_j.$$

Note that $R(0)1 = 1$. We take v another smooth function, then

$$\begin{aligned}|\langle R(i\lambda)^k u, v\rangle| =& \Big| \int_{\mathbb{R}^\kappa} \prod_{j=1}^{\kappa} \psi_j^{(k)}(t_j) e^{-i\sum t_j\lambda_j} \Big(\int_{\mathcal{M}} \bar{v} e^{-\sum t_j X_j} u \, dv_g \Big) dt_1 \dots dt_\kappa \Big| \\ \leq& |v|_{L^\infty} |u|_{L^\infty} R(0)^k 1 = |v|_{L^\infty} |u|_{L^\infty}\end{aligned}$$

We deduce that for $|z| > 1$

$$|\langle z(z - R(i\lambda))^{-1}u, v\rangle| \leq \sum |z|^{-k} |v|_{L^\infty} |u|_{L^\infty} \leq C(1 - |z|)^{-1}.$$

This is in contradiction with the assumption that z is a pole is of order > 1. $\square$

Then we can prove the following

Lemma 5.2. *For $\lambda = \sum_{j=1}^{\kappa} \lambda_j e_j \in \mathfrak{a}^*$, $R(i\lambda)$ has an eigenvalue of modulus 1 on $\mathcal{H}_{NG}$ if and only if λ is a Ruelle-Taylor resonance. In that case, the only eigenvalue of modulus 1 of $R(i\lambda)$ in $\mathcal{H}_{NG}$ is $\tau = 1$ and the eigenfunctions of $R(i\lambda)$ at $\tau = 1$ are the joint Ruelle resonant states of X at λ. Moreover, if $\Pi(i\lambda)$ is the spectral projector of $R(i\lambda)$ at $\tau = 1$, one has, as bounded operators in $\mathcal{H}_{NG}$,*

$$\lim_{k\to\infty} R(i\lambda)^k = \Pi(i\lambda). \tag{5.4}$$

Proof. First, let $\Pi(i\lambda)$ be the spectral projector of $R(i\lambda)$ at $\tau \in \mathbb{S}^1$: it commutes with the X_j, so we can use Lemma 3.7 to decompose $\operatorname{ran}\Pi(i\lambda)$ in terms of joint eigenspaces for X_j. By Proposition 4.17, the space of generalized Ruelle resonant states at $i\lambda$ is contained in $\operatorname{ran}\Pi(i\lambda)$, and Lemma 3.8 says that the space of joint Ruelle resonant states is $\ker(X + i\lambda)|_{\operatorname{ran}\Pi(i\lambda)}$. Let u be a joint-eigenfunctions of X_j in $\operatorname{ran}\Pi(i\lambda)$, with $X_j u = \zeta_j u$.

By Lemma 5.1, $R(i\lambda)$ has no Jordan block at τ, thus $u \in \mathcal{H}_{NG}$ is a non-zero eigenfunction of $R(i\lambda)$ with eigenvalue $\tau \in \mathbb{S}^1$. Then

$$R(i\lambda)u = \tau u = u \int_{\mathbb{R}^\kappa} \prod_{j=1}^{\kappa} e^{-t_j(\zeta_j + i\lambda_j)} \psi_j(t_j) dt_j = u \prod_{j=1}^{\kappa} \hat{\psi}_j(\lambda_j - i\zeta_j)$$

For τ to have modulus 1, we need $\prod_{j=1}^{\kappa} |\hat{\psi}_j(\lambda_j - i\zeta_j)| = 1$. But since $\int_{\mathbb{R}} \psi_j = 1$ and the ζ_j's have non-negative real part,

$$|\hat{\psi}_j(\lambda_j - i\zeta_j)| \leq \int_{\mathbb{R}} e^{-t\mathrm{Re}(\zeta_j)} \psi_j(t) dt \leq 1$$

so $\mathrm{Re}(\zeta_j) = 0$ and $|\hat{\psi}_j(\lambda_j - i\zeta_j)| = 1$ for all j. But then there is $\alpha \in \mathbb{R}$ so that $1 = \int_{\mathbb{R}} e^{-it(\lambda_j + \mathrm{Im}(\zeta_j)) - i\alpha} \psi_j(t) dt$ and thus $\zeta_j = -i\lambda_j$, and $\tau = 1$. In particular,

$$R(i\lambda) = \Pi(i\lambda) + K(i\lambda)$$

with $K_i(\lambda)\Pi(i\lambda) = \Pi(i\lambda)K_i(\lambda) = 0$, and $K(i\lambda)$ having spectral radius $r < 1$ on $\mathcal{H}_{NG}$, thus satisfying that for all $\epsilon > 0$, there is n_0 large so that for all $n \geq n_0$

$$\|K(i\lambda)^n\|_{\mathcal{L}(\mathcal{H}_{NG})} \leq (r + \epsilon)^n.$$

We can chose $r + \epsilon < 1$, which implies that

$$\forall n \geq n_0, \ R(i\lambda)^n = \Pi(i\lambda) + K(i\lambda)^n \to \Pi(i\lambda) \text{ in } \mathcal{L}(\mathcal{H}_{NG}) \tag{5.5}$$

proving (5.4).

To conclude the proof, we want to prove that $(X_j + i\lambda_j)\Pi(i\lambda) = 0$ for all $j = 1, \dots, \kappa$. By the discussion above, 0 is the only joint eigenvalue of $(X_1 + i\lambda_1, \dots, X_\kappa + i\lambda_\kappa)$ on $\operatorname{ran}\Pi(i\lambda)$, i.e. there is $J > 0$ such that $\prod_{j=1}^{\kappa} (X_j + i\lambda_j)^{\alpha_j} \Pi(i\lambda) = 0$ for all multi-index $\alpha \in \mathbb{N}^\kappa$ with length $|\alpha| = J$. We already know that R has no Jordan block, and we want to deduce that this is also true for the X_j's. By Proposition 4.17, we get

$$\operatorname{ran}\Pi(i\lambda) = \{u \in C^{-\infty}_{E_u^*}(\mathcal{M}) \mid \prod_{j=1}^{\kappa} (X_j + i\lambda_j)^{\alpha_j} u = 0, \forall \alpha \in \mathbb{N}^\kappa, |\alpha| = J\}.$$

In particular this space does not depend on the choice of χ_j (and thus ψ_j). The operator $e^{-\sum_j t_j(X_j + i\lambda_j)} : \operatorname{ran}\Pi(i\lambda) \to \operatorname{ran}\Pi(i\lambda)$ is represented by a finite dimension matrix $M(t)$ with $t = (t_1, \dots, t_\kappa)$, and $R(i\lambda)|_{\operatorname{ran}\Pi(i\lambda)} = \mathrm{Id}$ (since $R(i\lambda)$ has no Jordan blocks), thus

$$\mathrm{Id} = \int_{\mathbb{R}^\kappa} M(t)\psi_j(t_j) dt_1 \dots dt_\kappa$$

for all choices of χ_j (and $\psi_j = -\chi_j'$). We can thus take, for $T = (T_1, \dots, T_\kappa)$, the family ψ_j converging to the Dirac mass δ_{T_j} and we obtain $M(T) = \mathrm{Id}$. This shows that $M(t) = \mathrm{Id}$ for all $t \in \mathbb{R}^\kappa_+$ and therefore $(X_j + i\lambda_j)\Pi(i\lambda) = 0$ for all j. This implies that $\operatorname{ran}\Pi(i\lambda)$ is exactly the space of Ruelle resonant states for X at $i\lambda$. $\square$

From what we have shown in Lemma 5.2, we deduce that we can write the spectral projector as $\Pi(i\lambda)f = \sum_{k=1}^{J} v_k\langle f, w_k\rangle_{L^2}$ with $v_k \in \mathcal{H}_{NG}$ spanning the space of joint Ruelle resonant states of the resonance $i\lambda$ and $w_k \in \mathcal{H}_{NG}^* \simeq \mathcal{H}_{-NG}$. Recall that we have shown that the space of joint Ruelle resonant states (i.e. the range of $\Pi(i\lambda)$) is intrinsic, i.e. does not depend on the precise form of the parametrix. But surely the operator $R(i\lambda)$ depends on the choice of the cutoff functions ψ_j (see (5.1)) and thus also $\Pi(i\lambda)$ might depend on that choice. In order to see that this is not the case, let us consider $X_j^* = -X_j + \mathrm{div}_{v_g}(X_j)$ which are the adjoints with respect to the fixed measures v_g. Note that by the commutativity of the X_j, the operators X_j^* also commute and are admissible operators (in the sense of Definition 2.4) for the inverted Anosov action $\tau^-(a) := \tau(a^{-1})$ which is obviousely again an Anosov action (with the same positive Weyl chamber after swaping the stable and unstable bundles). Therefore we can apply the results of Section 4 to the admissible operators X_j^*, in particular they a discrete joint spectrum on the spaces $\mathcal{H}_{-NG}$. Using $(X_j + i\lambda_j)\Pi(i\lambda) = 0$ and the fact that $[X_j, \Pi(i\lambda)] = 0$ we deduce that $(X_j^* - i\lambda_j)w_k = 0$ and thus all w_k, $k = 1, \dots, J$ are joint resonant states of the X_j^*. We can even see that they span the space of joint resonant states: one can perform the same parametrix construction Lemma 4.14 to X_j^*

$$Q_{X^*}(i\lambda)d_{X^*+i\lambda} + d_{X^*+i\lambda}Q_{X^*}(i\lambda) = \mathrm{Id} - R_{X^*}(i\lambda) \otimes \mathrm{Id},$$

and if we choose the same cutoff functions as in the parametrix for X_j at the beginning of this section we find

$$R_{X^*}(i\lambda) = \prod_{j=1}^{\kappa} \int e^{-t_j(X_j^*+i\lambda_j)}\psi_j(t_j)dt_j. \tag{5.6}$$

In particular $R_{X^*}(-i\lambda) = (R_X(i\lambda))^*$ and if $\Pi_{X^*}(i\lambda)$ is the spectral projector of $R_{X^*}(i\lambda)$ onto the eigenvalue 1 then we obtain $\Pi_{X^*}(-i\lambda)f = \Pi_X(i\lambda)^*f = \sum_{k=1}^{J} w_k\langle f, v_k\rangle_{L^2}$. By Lemma 3.8 the space of joint resonant states $(X_j^* - i\lambda)w = 0$, $w \in \mathcal{H}_{-NG}$ is in the range of $\Pi_{X^*}(-i\lambda)$, consequently the w_j span the space of joint resonant states of X^* with joint resonance $-i\lambda$ and the w_j span the space of joint resonant states of the resonance $i\lambda$. Putting everything together, we have:

Lemma 5.3. *Let $\lambda \in \mathfrak{a}$ such that $i\lambda$ is a Ruelle-Taylor resonance of X. Then $-i\lambda$ is also a Ruelle-Taylor resonance of X^* and spaces of joint resonant states have the same dimension. If $v_1, \dots, v_J \in C^{-\infty}_{E_u^*}(\mathcal{M})$ and $w_1, \dots, w_J \in C^{-\infty}_{E_s^*}(\mathcal{M})$ are such that they span the space of joint resonant states of X at $i\lambda$ and X^* at $-i\lambda$ respectively and fullfill $\langle v_j, w_k\rangle_{L^2} = \delta_{jk}$, then we can write $\Pi(i\lambda) = \sum_{k=0}^{J} v_k\langle\cdot, w_k\rangle_{L^2}$. In particular $\Pi(i\lambda)$ depends only on the X_j but not on the choice of $R(i\lambda)$.*

We can now identify resonant states on the imaginary axis with some particular invariant measures.

Proposition 5.4.

(1) For each $v \in C^\infty(\mathcal{M}, \mathbb{R}^+)$, the map

$$\mu_v : u \in C^\infty(\mathcal{M}) \mapsto \langle\Pi(0)u, v\rangle$$

is a Radon measure with mass $\mu_v(\mathcal{M}) = \int_{\mathcal{M}} v\, dv_g$*, invariant by* X_j *for all* $j = 1, \ldots, \kappa$ *in the sense* $\mu_v(X_j u) = 0$ *for all* $u \in C^\infty$*.*

(2) *The space*
$$\mathrm{span}\{\mu_v \,|\, v \in C^\infty(\mathcal{M})\} = \Pi(0)^*(C^\infty(\mathcal{M}))$$
is a finite dimensional subspace of $C^{-\infty}_{E_s^*}(\mathcal{M})$ *and it is precisely the space spanned by all finite measures* μ *with* $\mathrm{WF}(\mu) \subset E_s^*$ *that are invariant under the Anosov action.*

(3) *For each open cone* $\mathcal{C} \subset \mathcal{W}$ *in the positive Weyl chamber, and* $u, v \in C^\infty(\mathcal{M})$ *we can express* μ_v *as the Birkhoff sum*
$$\mu_v(u) = \lim_{T\to\infty} \frac{1}{\mathrm{Vol}(\mathcal{C}_T)} \int_{A\in\mathcal{C}_T,} \langle e^{-X_A}u, v\rangle dA \tag{5.7}$$
where dA *is the Lebesgue-Haar measure on* $\mathfrak{a}$ *and* $\mathcal{C}_T := \{A \in \mathcal{C} \,|\, |A| \le T\}$*.*

(4) *Similarly, for* $\lambda \in \mathfrak{a}^*$*,* $v \in C^\infty(\mathcal{M})$ *the map*
$$\mu_v^\lambda : u \in C^\infty(\mathcal{M}) \mapsto \langle \Pi(i\lambda)u, v\rangle$$
is a complex valued measure. The measures are flow equivariant in the sense that $\mu_v^\lambda(X_j u) = -i\lambda_j \mu_v^\lambda(u)$ *and the set* $\{\mu_v^\lambda \,|\, v \in C^\infty(\mathcal{M})\}$ *is finite dimensional and coincides with the space of finite complex measures* μ *with* $\mathrm{WF}(\mu) \subset E_s^*$ *which are equivariant in the above sense.*

(5) *Let* $v_1, v_2 \in C^\infty(\mathcal{M}, \mathbb{R}^+)$ *with* $v_1 \le C v_2$ *for some* $C > 0$ *and* $i\lambda \in i\mathfrak{a}^*$ *a Ruelle Taylor resonance. Then* $\mu_{v_1}^\lambda$ *is absolutely continuous with bounded density with respect to* μ_{v_2}*. In particular any* μ_v^λ *is absolutely continuous with respect to* μ_1*.*

Proof. First $R(0)1 = 1$ is clear and X has a Taylor-Ruelle resonance at $\lambda = 0$ by Lemma 3.9. If $u, v \in C^\infty(\mathcal{M})$ are non-negative, we have $a_k := \langle R(0)^k u, v\rangle \ge 0$ and
$$\lim_{k\to\infty} \langle R(0)^k u, v\rangle = \langle \Pi(0)u, v\rangle \ge 0.$$
Note also that for each k, and each $u \in C^\infty(\mathcal{M})$ non-negative
$$\forall x \in \mathcal{M}, \quad 0 \le (R(0)^k u)(x) \le (R(0)^k 1)\|u\|_{C^0} \le \|u\|_{C^0}.$$
This implies that for each $v \in C^\infty$ with $v \ge 0$, $\mu_v^k : u \mapsto \langle R(0)^k u, v\rangle$ is a Radon measure with finite mass $\mu_v^k(\mathcal{M}) = \int_{\mathcal{M}} v\, dv_g$ and thus μ_v is as well. The invariance of μ_v is a direct consequence of Lemma 5.3. The same holds for property (2). The invariance of the space spanned by these measures with respect to X_j follows from $\Pi(0)X_j = X_j\Pi(0) = 0$, obtained by Lemma 5.2.

Let us next show that for an arbitrary Ruelle-Taylor resonance $i\lambda \in i\mathfrak{a}^*$ we get complex measures μ_v^λ and in the same turn prove the absolute continuity statement (5). We consider $u \in C^\infty(\mathcal{M})$, $v_1, v_2 \in C^\infty(\mathcal{M}, \mathbb{R}^+)$ with $v_1 \le v_2$ and get
$$|\langle R(i\lambda)^k u, v_1\rangle| \le \langle R(0)^k |u|, v_1\rangle \le \langle R(0)^k |u|, v_2\rangle$$
thus $|\mu_{v_1}^\lambda(u)| \le \mu_{v_2}(|u|)$.

Let us finally show (5.7). We fix a small open cone $\mathcal{C} \subset \mathcal{W}$ and choose a basis $(A_j)_{j=1}^\kappa$ of $\mathfrak{a}$ so that $A_1 \in \mathcal{C}$, we then identify $\mathfrak{a} \simeq \mathbb{R}^\kappa$ by identifying the canonical basis $(e_j)_j$ of $\mathbb{R}^\kappa$ with $(A_j)_j$. We let $\Sigma = \mathcal{C} \cap \{A_1 + \sum_{j=1}^\kappa t_j A_j, \,|\, t_j \in \mathbb{R}\}$ be a hyperplane section of the cone

$\mathcal{C}$. Choose $\psi \in C_c^\infty((-1/2,1/2))$ non-negative even with $\int_{\mathbb{R}} \psi = 1$, and for each $\sigma \in \mathbb{R}^\kappa$, define $\psi_\sigma(t) := \prod_{j=1}^\kappa \psi(t_j - \sigma_j)$. The operator $Q(0), R(0)$ constructed in Lemma 4.14 can be defined, for σ close to e_1, with the cutoff function χ_j so that $-\chi_j'(t_j) = \psi(t_j - \sigma_j)$, we then denote Q_σ, R_σ the corresponding operators, which in turn are locally uniform in σ. Then μ_v is given by $\mu_v(u) = \lim_{k\to\infty}\langle R_\sigma(0)^k u, v\rangle$ locally uniformly in σ. This means that viewing Σ as an open subset of $e_1 + \mathbb{R}^{\kappa-1}$ containing e_1, taking any $q \in C_c^\infty(\Sigma)$ with $\int_{\mathbb{R}^{\kappa-1}} q(\bar t)d\bar t = 1$ and any $\omega \in C_c^\infty((0,1))$ with $\int_0^1 \omega = 1$, we have for $\sigma(\bar t) := (1, \bar t) \in \Sigma$

$$\mu_v(u) = \lim_{N\to\infty} \int_\Sigma \frac{1}{N}\sum_{k=1}^N \omega\big(\frac{k}{N}\big)\langle R^k_{\sigma(\bar t)}u, v\rangle q(\bar t)d\bar t.$$

Recall that R_σ^k is given by the expression

$$R^k_\sigma u = \int_{\mathbb{R}^\kappa} (e^{-\sum_{j=1}^\kappa t_j X_j}u)\psi_\sigma^{(k)}(t)dt.$$

where $\psi_\sigma^{(k)}$ is the k-th convolution of ψ_σ. We claim the

Lemma 5.5. *With ω, q as above, and let $\widetilde\omega(r) := r^{1-\kappa}\omega(r)$ the following limit holds as $N\to\infty$*

$$\int_{\mathbb{R}^{\kappa-1}} \frac{1}{N}\sum_{k=1}^N \omega\big(\frac{k}{N}\big)\langle R^k_{\sigma(\bar t)}u, v\rangle q(\bar t)d\bar t - \frac{1}{N^\kappa}\int_0^N \int_{\mathbb{R}^{\kappa-1}} \langle e^{-\sum_j t_j X_j}u, v\rangle)\widetilde\omega\big(\frac{t_1}{N}\big)q\Big(\frac{\bar t}{t_1}\Big)dt_1 d\bar t \to 0$$

This lemma implies (5.7) by approximating the function $t_1^{\kappa-1}\mathbf{1}_{[0,1]}(t_1)\mathbf{1}_\Sigma(\bar t/t_1)$ by some smooth cutoffs $\omega(t_1)q(\bar t/t_1)$. □

Proof of Lemma 5.5. Since for $u \in C^\infty(\mathcal{M})$, $\|e^{-\sum_{j=1}^\kappa t_j X_j}u\|_{L^\infty} \le \|u\|_{L^\infty}$, it suffices to show that

$$t = (t_1, \bar t) \in \mathbb{R}_+ \times \mathbb{R}^{\kappa-1} \mapsto \int_{\mathbb{R}^{\kappa-1}} \frac{1}{N}\sum_{k=1}^N \omega(\tfrac{k}{N})\psi^{(k)}_{\sigma(\theta)}(t)q(\theta)d\theta - \frac{1}{N^\kappa}\widetilde\omega(\tfrac{t_1}{N})q(\tfrac{\bar t}{t_1})$$

converges to 0 in $L^1(\mathbb{R}^\kappa)$. Let $\epsilon > 0$ small so that $\mathrm{supp}(\omega) \subset (\epsilon, 1-\epsilon))$. Scaling $t \to tN$, the above convergence statement is equivalent to show that

$$f_N(t) := N^{\kappa-1}\sum_{k=\epsilon N}^{N(1-\epsilon)} \omega\big(\frac{k}{N}\big)\int_{\mathbb{R}^{\kappa-1}} \psi^{(k)}_{\sigma(\theta)}(tN)q(\theta)d\theta$$

is such that $\lim_{N\to\infty}\|f_N - h\|_{L^1(\mathbb{R}^\kappa)} = 0$ if $h(t) := \widetilde\omega(t_1)q\big(\frac{\bar t}{t_1}\big)$. First, notice that $\mathrm{supp}(\psi^{(k)}_{\sigma(\theta)}(N\cdot)) \subset B(0,2)$ for each $k \le N$, and $\int \psi_\theta^{(k)}(t)dt = 1$. It then suffices to prove that f_N converges in $L^2(\mathbb{R}^\kappa)$ to h. We proceed using the Fourier transform, writing $\xi = (\xi_1, \bar\xi)$

$$\hat f_N(\xi) = \frac{1}{N}\sum_{k=\epsilon N}^{(1-\epsilon)N} \omega\big(\frac{k}{N}\big)\Big(\hat\psi_0\big(\frac{\xi}{N}\big)\Big)^k e^{-i\frac{k}{N}\xi_1}\hat q\big(\frac{k}{N}\bar\xi\big).$$

First, for ξ fixed, since $\hat{\psi}_0(\xi) = 1 + \mathcal{O}(|\xi|^2)$ for small ξ, one has (using Riemann sums) the pointwise convergence

$$\lim_{N\to\infty} \hat{f}_N(\xi) = \int_{\mathbb{R}} \omega(t_1)\hat{q}(t_1\bar{\xi})e^{-it_1\xi_1}dt_1 = \hat{h}(\xi).$$

To prove L^2 convergence, we use that there is $c_0 > 0$ such that

$$|\hat{\psi}_0(\xi)| \leq (1 + c_0|\xi|^2)^{-1}, \tag{5.8}$$

thus for $\delta > 0$ arbitrarily small, there is $C > 0$ depending only on $\|\hat{q}\|_{L^\infty}, \|\hat{\omega}\|_{L^\infty}$ such that

$$\int_{|\xi|\geq N^{1/2+\delta}} |\hat{f}_N(\xi)|^2 d\xi \leq CN^\kappa \int_{|\xi|\geq N^{-1/2+\delta}} |\hat{\psi}_0(\xi)|^{2\epsilon N} d\xi \leq CN^\kappa e^{-c_0\epsilon N^{2\delta}} \int |\hat{\psi}_0(\xi)| d\xi \to 0.$$

Next, we will show that for $\delta > 0$ small and $\ell \in \mathbb{N}$, there is N_ℓ, C_ℓ such that for all $N \geq N_\ell$

$$\forall \xi, |\xi| \in [1, N^{1/2+\delta}], \quad |\hat{f}_N(\xi)| \leq C_\ell(|\xi|^{-\ell} + N^{-\ell(\frac{1}{2}-\delta)}). \tag{5.9}$$

This will prove the convergence of $\hat{f}_N$ to $\hat{h}$ in L^2, since for all $n > 0$, there is T_n and $N_n > 0$ such that for $N \geq N_n$

$$\int_{|\xi|\geq T_n} |\hat{f}_N(\xi) - \hat{h}(\xi)|^2\, d\xi \leq 1/n$$

and, using dominated convergence,

$$\lim_{N\to\infty} \int_{\mathbb{R}^\kappa} |\hat{f}_N(\xi) - \hat{h}(\xi)|^2 \mathbf{1}_{[0,T_n]}(|\xi|) d\xi = 0.$$

We next show (5.9). We will use a discrete integration by parts to gain decay in $\hat{f}_N(\xi)$ in the ξ_1 variable. For $\rho \in C_0^\infty((0,1))$, we can define some sequences a_k^m, b_k^m for $k \in \mathbb{Z}$ and $m \in \mathbb{N}$ by induction. First, $b_k^0 := e^{-i\frac{k}{N}\xi_1}$, $a_k^0 = \rho(k/N)$ for $k \in \mathbb{Z}$. Next, for $m \geq 1$, $k \in \mathbb{Z}$,

$$b_k^m := b_k^{m-1}\frac{e^{-i\frac{\xi_1}{N}}}{1 - e^{-i\frac{\xi_1}{N}}} = e^{-i\frac{k}{N}\xi_1}\Big(\frac{e^{-i\frac{\xi_1}{N}}}{1 - e^{-i\frac{\xi_1}{N}}}\Big)^m, \quad a_k^m := a_k^{m-1} - a_{k+1}^{m-1}.$$

Note also that $a_k^m = 0$ for $k < \epsilon N - m$ and $k > (1-\epsilon)N$ and remark that $b_k^m = b_{k-1}^{m+1} - b_k^{m+1}$. Thus,

$$\sum_{k\in\mathbb{Z}} a_k^m b_k^m = \sum_{k\in\mathbb{Z}} a_k^m(b_{k-1}^{m+1} - b_k^{m+1}) = -\sum_{k\in\mathbb{Z}} b_k^{m+1}(a_k^m - a_{k+1}^m) = -\sum_{k\in\mathbb{Z}} b_k^{m+1}a_k^{m+1}.$$

Since ρ is smooth, a Taylor expansion gives for each m a constant $C_m > 0$ such that for $N \geq 1$, $|a_k^m| \leq C_m\|\rho\|_{C^m}N^{-m}$. Up to increasing the value of C_m, we also have that $|b_k^m| \leq C_m N^m/|\xi_1|^m$. We deduce that for each m, there exists $C_m > 0$ such that for all N large enough,

$$\Big|\sum_{k=1}^N \rho(k/N)e^{-i\xi_1\frac{k}{N}}\Big| \leq \min(C_m\|\rho\|_{C^m}|\xi_1|^{-m}, N\|\rho\|_{L^\infty}).$$

Now, take $\rho(x) := \hat{q}(x\bar{\xi})\omega(x)e^{xN\log(\hat{\psi}_0(\xi/N))}$, which exists since $|\hat{\psi}_0(\xi/N)-1| \leq 1/2$ for N large enough (since $|\xi|/N$ is assumed small). For all $\ell \in \mathbb{N}$, there is $C_{\ell,m} > 0$ such that for all $x \in \operatorname{supp}(\omega)$

$$|\partial_x^m(\hat{q}(x\bar{\xi})\omega(x))| \leq C_{\ell,m}(1+|\bar{\xi}|^2)^{-\ell+m}.$$

Since $|N\log(\hat{\psi}_0(\xi/N))| \leq C|\xi|^2/N$ for some unifom $C>0$ by using (5.8) if $|\xi|/N$ is small, we finally obtain the bound (choosing $\ell = 2m$ above): for all m there is $C_m > 0$ so that for N large enough

$$\|\rho\|_{C^m} \leq C_m(1+|\bar{\xi}|^2)^{-m}(1+|\xi|^{2m}/N^m).$$

This implies that for all m, there is $C_m, C'_m > 0$ so that for all N large enough and $|\xi| \in [1, N^{1/2+\delta}]$

$$|\hat{f}_N(\xi)| \leq C_m \frac{1+|\xi|^{2m}N^{-m}}{(1+|\bar{\xi}|^2)^m(1+|\xi_1|)^m} \leq C'_m\Big(\frac{1}{|\xi|^m} + \frac{|\xi|^m}{N^m}\Big) \leq C'_m\Big(\frac{1}{|\xi|^m} + \frac{1}{N^{m(\frac{1}{2}-\delta)}}\Big)$$

which shows (5.9). □

As noted in the introduction, we will call *physical measures* the measures μ_v, and μ_1 will be called the *full physical measure*.

5.2. **Imaginary Ruelle-Taylor resonances for volume preserving actions.** In this section, we are going to study the dimensions of the Ruelle-Taylor resonance at $\lambda = 0$ in the case where there is a smooth measure preserved by the action. First, we want to prove

Proposition 5.6. *Assume that there is a smooth invariant measure μ for the action, i.e. $\mathcal{L}_{X_A}\mu = 0$ for each $A \in \mathcal{W}$. Then, for each $\lambda \in i\mathfrak{a}^*$ imaginary, there is an injective map*

$$\ker_{C^{-\infty}_{E_u^*}\Lambda^j} d_{X+\lambda}/\operatorname{ran}_{C^{-\infty}_{E_u^*}\Lambda^j} d_{X+\lambda} \to \ker_{C^\infty\Lambda^j} d_{X+\lambda}/\operatorname{ran}_{C^\infty\Lambda^j} d_{X+\lambda}. \tag{5.10}$$

Proof. Fix a basis $A_1,\dots,A_\kappa \in \mathcal{W}$ close to A_1 and write $\lambda_j := \lambda(A_j)$ and $X_{A_j}(\lambda) := X_{A_j}+\lambda_j$. Let $T_j > 0$ for $j = 1,\dots,\kappa$, let $\epsilon > 0$ be small and consider $\chi_j \in C_c^\infty([0,\infty[;[0,1])$ non-increasing with $\chi_j = 1$ in $[0,T_j]$ and $\operatorname{supp}\chi_j \in [0, T_j+\varepsilon]$. We shall use the same parametrix $Q(\lambda)$ as in Lemma 4.14. We set

$$\widetilde{Q}_j(\lambda) := \int_0^\infty e^{-t_jX_{A_j}(\lambda)}\chi_j(t_j)dt_j$$

and we make it act on $C^\infty(\mathcal{M})\otimes\Lambda\mathfrak{a}^*$ by $Q_j(\lambda) : u\otimes w \mapsto (\widetilde{Q}_j(\lambda)u)\otimes\iota_{A_j}\omega$. As in Proposition 4.6, we compute

$$d_{X(\lambda)}Q_j(\lambda) + Q_j(\lambda)d_{X(\lambda)} = 1 + R_j(\lambda), \quad R_j(\lambda)(u\otimes\omega) := \Big(\int_0^\infty e^{-t_jX_{A_j}(\lambda)}u\chi'_j(t_j)dt_j\Big)\otimes w.$$

We thus have

$$d_{X(\lambda)}Q(\lambda) + Q(\lambda)d_{X(\lambda)} = F(\lambda), \quad F(\lambda) = \mathrm{Id} - (-1)^\kappa\prod_{k=1}^\kappa R_k(\lambda). \tag{5.11}$$

with $Q(\lambda) := \sum_{m=1}^\kappa(-1)^{m-1}Q_m(\lambda)\prod_{k=1}^{m-1}R_k(\lambda)$. As in the proof of Lemma 4.14, $F(\lambda) - \mathrm{Id} = R(\lambda) + K(\lambda)$ with $K(\lambda)$ compact on $\mathcal{H}_{NG}$ and $\|R(\lambda)\| < 1/2$. In precisely the same

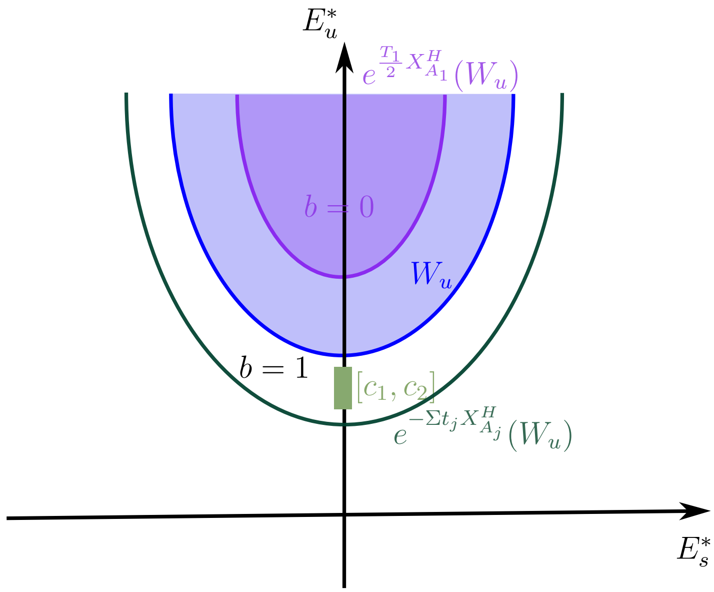

FIGURE 1. Schematic sketch of the phase space regions appearing in the proof of Proposition 5.6.

way as in Lemma 4.12, we deduce $\mathrm{ran}(\Pi_0(\lambda)) \subset C^{-\infty}_{E_u^*}(\mathcal{M};\Lambda\mathfrak{a}^*)$ if $\Pi_0(\lambda)$ is the spectral projector of $F(\lambda)$ at $z=0$. We will show that the range of the spectral projector $\Pi_0(\lambda)$ at $z=0$ of $F(\lambda)$ actually satisfies

$$\mathrm{ran}\,\Pi_0(\lambda) \subset C^\infty(\mathcal{M};\Lambda\mathfrak{a}^*). \tag{5.12}$$

Since $F(\lambda)$ is a scalar operator, we can work on scalar valued distributions, and we shall then identify F with an operator $\mathcal{H}_{NG} \to \mathcal{H}_{NG}$ for some $N>0$ large enough, and fixed. First, we notice that $\|\mathrm{Id} - F(\lambda)\|_{\mathcal{L}(L^2)} \leq 1$: indeed, using $\|e^{-tX_{A_j}(\lambda)}\|_{\mathcal{L}(L^2)} = 1$, one has

$$\|R_j'(\lambda)\|_{\mathcal{L}(L^2)} \leq \int_0^{T_j+1} -\chi_j'(t)dt = \chi_j(0) = 1.$$

Therefore $z=1$ is at most a pole of order 1 of $(\mathrm{Id}-F(\lambda)-z)^{-1}$, so that each $u \in \mathrm{ran}(\Pi_0(\lambda))$ satisfies $F(\lambda)u=0$. Then let $u \in \mathcal{H}_{NG}$ such that $F(\lambda)u=0$. We now consider a semiclassical parameter $h>0$. Recall from [DZ16] that $\mathrm{WF}(u) = \mathrm{WF}_h(u) \cap T^*\mathcal{M} \setminus \{0\}$. We are now going to show that $\mathrm{WF}_h(u) \cap \{(x,\xi) \in E_u^* \,|\, |\xi| \in [c_1,c_2]\} = \emptyset$ for some $0 < c_1 < c_2$ by using the equation $F(\lambda)u=0$, the propagation of semiclassical wavefront sets (Egorov theorem) and the explicit expression of $F(\lambda)$ in terms of the propagators $e^{-tX_{A_j}(\lambda)}$.

For $T_1 > 0$ large enough but fixed and $T_2,\dots,T_\kappa \in [0,\epsilon]$ small enough, one can find a closed neighborhood W_u of $E_u^* \cap \partial\overline{T}^*\mathcal{M}$ in the fiber radial compactification of $T^*\mathcal{M}$, which is conic for $|\xi|$ large, $0<c_1<c_2$ such that for all $t_1 \in [T_1/2, T_1+\epsilon]$ and all $t_j \in [0,2\epsilon]$

when $j \geq 2$ we have

$$W_u \subset e^{-\sum_{j=1}^{\kappa} t_j X^H_{A_j}}(W_u) \text{ and } \{(x,\xi) \in E_u^* \,|\, |\xi| \in [c_1, c_2]\} \subset e^{-\sum_j t_j X^H_{A_j}}(W_u) \setminus W_u.$$

We choose $B = \mathrm{Op}_h(b) \in \Psi_h^0(\mathcal{M})$ where $b \in S^0(\mathcal{M})$ (depending on h) satisfies the following properties. First,

$$b \geq 0, \quad b(x,\xi) = 1 \text{ in } T^*\mathcal{M} \setminus W_u, \quad b(x,\xi) = 0 \text{ in } e^{\frac{T_1}{2} X^H_{A_1}}(W_u).$$

Second, for each $t = (t_1, \dots, t_\kappa)$ with $t_j \in [T_j, T_j + \epsilon]$ the symbol

$$0 \leq b(x,\xi) - b(e^{\sum_{j=1}^{\kappa} t_j X^H_{A_j}}(x,\xi))$$

is equal to 1 on $\{(x,\xi) \in E_u^* \,|\, |\xi| \in [c_1, c_2]\}$. Finally, B is chosen so that $1 - B^*B - C^*C \in h^\infty \Psi_h^0(\mathcal{M})$ for some $C = \mathrm{Op}_h(c) \in \Psi_h^0(\mathcal{M})$. Note that the escape function G can be chosen so that the weight function $m \geq 0$ in the region $T^*\mathcal{M} \setminus W_u$ for $|\xi|$ large enough. Since $u \in \mathcal{H}_{NG}$, we thus have $Bu \in L^2$. Let $\hat{\chi} \in C^\infty(\mathbb{R}^\kappa)$ be given by $\hat{\chi}(t) = (-1)^\kappa \prod_{j=1}^{\kappa} \chi_j'(t_j) \geq 0$ for $t \in \mathbb{R}^\kappa$. We can write, using the semiclassical Egorov Lemma,

$$\begin{aligned} Bu = B(\mathrm{Id} - F(\lambda))u &= \int_{(\mathbb{R}^+)^\kappa} B u e^{-\sum_{j=1}^{\kappa} t_j X_{A_j}(\lambda)} \hat{\chi}(t) dt_1 \dots dt_\kappa \\ &= \int_{(\mathbb{R}^+)^\kappa} e^{-\sum_{j=1}^{\kappa} t_j X_{A_j}(\lambda)} B_t u \hat{\chi}(t) dt_1 \dots dt_\kappa \end{aligned}$$

with $B_t - \mathrm{Op}_h(b \circ e^{\sum_j t_j X^H_{A_j}}) \in h\Psi_h^{-1}(\mathcal{M})$ and $\mathrm{WF}_h(B_t) \subset e^{-\sum_j t_j X^H_{A_j}}(\mathrm{WF}_h(B))$. This gives

$$\|Bu\|_{L^2} = \|B(\mathrm{Id} - F(\lambda))u\|_{L^2} \leq \int_{(\mathbb{R}^+)^\kappa} \|B_t u\|_{L^2} \hat{\chi}(t) dt$$

with $\hat{\chi}(t) > 1$ on a ball of radius $\delta > 0$ centered at $t_0 \in \prod_{j \geq 1} [T_j, T_j + \epsilon]$. We can then write

$$\int_{(\mathbb{R}^+)^\kappa} (\|Bu\|_{L^2} - \|B_t u\|_{L^2}) \hat{\chi}(t) dt \leq 0. \tag{5.13}$$

Next, we claim that there is $e_t \in S^0(\mathcal{M}; [0,1])$ such that we have $B^*B - (B_t^*B_t + E_t^*E_t) \in h^\infty \Psi_h^0(\mathcal{M})$ for $E_t := \mathrm{Op}_h(e_t)$ and such that $e_t(x,\xi) = 1 + \mathcal{O}(h)$ in the region $\{(x,\xi) \in E_u^* \,|\, |\xi| \in [c_1, c_2]\}$. Indeed, E_t is microlocally equal to $C_t := e^{\sum_{j=1}^{\kappa} t_j X_{A_j}(\lambda)} C e^{-\sum_{j=1}^{\kappa} t_j X_{A_j}(\lambda)}$ on $\mathrm{WF}_h(B_t)$ and to B on the complement of $\mathrm{WF}_h(B_t)$. This implies, thanks to (5.13),

$$\int_{(\mathbb{R}^+)^\kappa} \|E_t u\|^2_{L^2} \hat{\chi}(t) dt = \mathcal{O}(h^\infty).$$

There is $f, g_t \in S^0(\mathcal{M}; [0,1])$ with $f = 1 + \mathcal{O}(h)$ on $\{(x,\xi) \in E_u^* \,|\, |\xi| \in [c_1, c_2]\}$ with f independent of t, such that for t near t_0 $E_t^*E_t - (F^*F + G_t^*G_t) \in h^\infty \Psi_h^0(\mathcal{M})$, where $F = \mathrm{Op}_h(f)$ and $G_t = \mathrm{Op}_h(g_t)$. We thus obtain

$$\|Fu\|^2_{L^2} \leq \int_{(\mathbb{R}^+)^\kappa} \|E_t u\|^2_{L^2} \hat{\chi}(t) dt + \mathcal{O}(h^\infty) = \mathcal{O}(h^\infty),$$

which implies that $\mathrm{WF}_h(u) \cap \{(x,\xi) \in E_u^* \,|\, |\xi| \in [c_1, c_2]\} = \emptyset$. We then conclude that $\mathrm{WF}(u) \cap E_u^* = \emptyset$, which also shows that $u \in C^\infty$ and (5.12).

Then we define the following map

$$\mathcal{I}:\begin{cases}\ker_{\operatorname{ran}\Pi_0(\lambda)} d_{X(\lambda)}/\operatorname{ran}_{\operatorname{ran}\Pi_0(\lambda)} d_{X(\lambda)} & \to \ker_{C^\infty\Lambda} d_{X(\lambda)}/\operatorname{ran}_{C^\infty\Lambda} d_{X(\lambda)}\\ u+\operatorname{ran}_{\operatorname{ran}\Pi_0(\lambda)} d_{X(\lambda)} & \mapsto u+\operatorname{ran}_{C^\infty\Lambda} d_{X(\lambda)}\end{cases} \tag{5.14}$$

which is well defined since $\operatorname{ran}\Pi_0(\lambda)\subset C^\infty\Lambda$. We claim that this map is injective: let $u=d_{X(\lambda)}v\in\operatorname{ran}\Pi_0(\lambda)$ with $v\in C^\infty\Lambda^j$, then we need to show that $u=d_{X(\lambda)}w$ for some $w\in\operatorname{ran}\Pi_0(\lambda)$. But is suffices to use $[d_{X(\lambda)},\Pi_0(\lambda)]=0$ to see that $u=\Pi_0(\lambda)u=d_{X(\lambda)}\Pi_0(\lambda)v$. This proves the claim and concludes the proof of the lemma by using also the isomorphism (4.10). □

Lemma 5.7. *Assume that there is a smooth invariant measure μ for the action, i.e. $\mathcal{L}_{X_A}\mu=0$ for each $A\in\mathfrak{a}$, and that $\operatorname{supp}(\mu)=\mathcal{M}$. Then the periodic orbits are dense in $\mathcal{M}$.*

Proof. Since $\mathcal{M}$ is compact, the measure is finite, so we can apply Poincaré's recurrence theorem: almost every point x of $\mathcal{M}$ is recurrent, i.e. its orbit comes back infinitely close to x infinitely many times (and this for each direction of the action). Katok-Spatzier [KS94, Theorem 2.4] proved a closing lemma for Anosov actions: there is $\delta>0$ such that whenever there is $x\in\mathcal{M}$ and $t\in\mathcal{W}$ with $d(\tau(t)x,x)<\delta$, then there is a periodic torus for the action at distance at most $\frac{1}{\delta}d(\tau(t)x,x)$ from x. □

Proposition 5.8. *Assume that there is a smooth invariant measure μ for the action, with* $\operatorname{supp}(\mu)=\mathcal{M}$. *Then*

$$\dim\Big(\ker_{C^{-\infty}_{E_u^*}\Lambda^j} d_X/\operatorname{ran}_{C^{-\infty}_{E_u^*}\Lambda^j} d_X\Big)=\dim\Lambda^j\mathfrak{a}^*=\binom{\kappa}{j}$$

and the cohomology space is generated by the constant forms $e'_{i_1}\wedge\cdots\wedge e'_{i_\kappa}$ if $(e'_j)_j$ is a basis of $\mathfrak{a}^*$.

Proof. In the proof of Proposition 5.6 now with $\lambda=0$, we have defined an operator $F(0)$ that is Fredholm on $\mathcal{H}_{NG}$ and $\Pi_0(0)$ is its spectral projector at $z=0$, with $\operatorname{Im}(\Pi_0(0))\subset C^\infty(\mathcal{M})$. Recall also that $F(0)$ is scalar and can thus be considered as an operator on functions. Let us show that $\operatorname{Im}(\Pi_0(0))=\mathbb{R}$ consists only of constants under our assumptions. Pick $u\in C^\infty(\mathcal{M})$ such that $F(0)u=0$. Let $x\in\mathcal{M}$ belong to a closed orbit, i.e. $\varphi^{X_A}_{t_0}(x)=x$ for some $A\in\mathfrak{a}$ and $t_0>0$. Then the orbit $T_x:=\{\varphi^{X_A}_s(x)\,|\,s\in\mathbb{R},A\in\mathfrak{a}\}$ is a closed κ-dimensional torus isomorphic to $\mathbb{R}^\kappa/\mathbb{Z}^\kappa$ by the map

$$\psi_x: t\in\mathbb{R}^\kappa\mapsto\tau\Big(\sum_{j=1}^{\kappa}t_jX_{A'_j}\Big)(x)$$

for some basis $A'_i\in\mathfrak{a}$. Let us restrict the identity $Fu=0$ on T_x. We can decompose $v:=\psi_x^*u$ into Fourier series

$$t\in\mathbb{R}^\kappa,\quad v(t)=\sum_{k\in\mathbb{Z}^\kappa}e^{2i\pi k.t}v_k.$$

Then for $A_j = \sum_{j,i} M_{ij} A'_i$ (using $\sum_{\ell=1}^{\kappa} s_\ell A_\ell = \sum_{\ell=1}^{\kappa} s_\ell M_{i\ell} A'_i$) the identity $Ru(x) = u(x)$ can be rewritten

$$\sum_{k\in\mathbb{Z}^\kappa} e^{2i\pi k.t} v_k = \sum_{k\in\mathbb{Z}^\kappa} v_k e^{2i\pi k.t} \int_{(\mathbb{R}^+)^\kappa} e^{-2i\pi k.Ms} \hat{\chi}(s) ds$$

with $M = (M_{ij})_{ij}$ real valued. This shows that for each $k \in \mathbb{Z}^\kappa$,

$$v_k = 0 \text{ or } \int_{(\mathbb{R}^+)^\kappa} (e^{-2i\pi k.Ms} - 1)\hat{\chi}(s) ds = 0.$$

Using that $\hat{\chi} \geq 0$ and $\hat{\chi}(s) > 0$ in some open set, we see that either $v_k = 0$ or $k = 0$, i.e. $v(t) = v(0)$ is constant. Therefore u is constant on each periodic torus. Since u is smooth and the periodic tori are dense, this implies that $d_X u = 0$ and $u(\varphi_t^{X_A}(x)) = u(x)$ for each $x \in \mathcal{M}$, $t \in \mathbb{R}$ and $A \in \mathfrak{a}$. Taking $A \in \mathcal{W}$, there is $\nu > 0$ such that for each $t > 0$ large enough so that $|d\varphi_t^{X_A} v| \leq e^{-\nu t}|v|$ for each $v \in E_s$. Thus

$$|du_x(v)| = |du_{\varphi_t^{X_A}(x)} d\varphi_t^{X_A}(x) v| \leq \|du\| e^{-\nu t}|v|.$$

Letting $t \to \infty$, we conclude that $du|_{E_s} = 0$. The same argument with $t < 0$ shows that $du|_{E_u} = 0$ and therefore $du = 0$. Since $F(0)1 = 0$, this shows that, when viewed as an operator on $\Lambda\mathfrak{a}^*$, $\operatorname{ran}\Pi_0(0)$ is exactly the space of constant forms. We can then use the isomorphism (4.10) to conclude the proof since it is direct to see that constant forms $e'_{i_1} \wedge \cdots \wedge e'_{i_j}$ form a basis of $\ker d_X / \operatorname{Im} d_X$ on $\operatorname{Im}(\Pi_0(0))$ (as $d_X|_{\operatorname{ran}\Pi_0(0)} = 0$). ☐

As a corollary of and Proposition 5.8 and the result of Katok-Spatzier [KS94] on $\ker_{C^\infty\Lambda^1} d_X / \operatorname{ran}_{C^\infty\Lambda^1} d_X$ for standard Anosov actions, we obtain

Corollary 5.9. *If the Anosov $\mathbb{R}^\kappa$-action is standard in the sense of* [KS94]*, then the map* (5.10) *is an isomorphism for $j = 1$.*

5.3. **Ruelle-Taylor resonances and mixing properties.** In this section we want to establish the following relation of Ruelle-Taylor resonances and mixing properties

Proposition 5.10. *Let X be an Anosov action on $\mathcal{M}$ then the following are equivalent:*

(1) There is a direction $A_0 \in \mathfrak{a}$ such that $\varphi_t^{X_{A_0}}$ is weakly mixing with respect to the full physical measure μ_1.
(2) 0 is the only Ruelle-Taylor resonance on $i\mathfrak{a}^*$ and there is a unique normalized physical measure μ_1.
(3) For each $A \in \mathcal{W}$, $\varphi_t^{X_A}$ is strongly mixing with respect to the full physical measure μ_1.

Proof. Obviously (3) $\Rightarrow$ (1). So let us prove (1) $\Rightarrow$ (2): Assume that there is either a non-zero Ruelle-Taylor resonance $i\lambda \in i\mathfrak{a}^*$ or a non-unique normalized SRB measure then by Proposition 5.4(5) there is a non-constant bounded density $f \in L^2(\mathcal{M}, \mu_1)$ with $X_A f = i\lambda(A) f$ for all $A \in \mathfrak{a}$ (setting $\lambda = 0$ if the density comes from the non-uniqueness of the SRB measure). As f is non-constant there exists $g \in L^2(M,\mu_1)$ $\int g\, d\mu_1 = 0$ but $\int gf\, d\mu_1 \neq 0$. With these two functions the correlation function

$$C_{f,g}(t;A_0) := \int_{\mathcal{M}} g(\varphi_{-t}^{X_{A_0}})^* f\, d\mu_1 - \int_{\mathcal{M}} g\, d\mu_1 \int_{\mathcal{M}} f\, d\mu_1 = e^{-i\lambda(A_0)t} \int_{\mathcal{M}} gf\, d\mu_1$$

so $\varphi_t^{A_0}$ is obviously not weakly mixing.

We will now prove (2)⇒(3) using the regularity of a joint spectral measure: Let us first introduce these measures: We consider the space $L^2(M,\mu_1)$. Since the measure μ_1 is flow-invariant, the flow acts as unitary operators on $L^2(M,\mu_1)$. In particular, for each $A \in \mathfrak{a}$, X_A is anti self-adjoint when acting on $L^2(M,\mu_1)$ with domain

$$\mathcal{D}(X_A) = \left\{ u \in L^2(M,\mu_1) \;\middle|\; \lim_t \frac{1}{t}(e^{tX_A}u - u) \text{ exists} \right\} = \{u \in L^2(M,\mu_1) \mid X_A u \in L^2(M,\mu_1)\}.$$

Additionally, since the flow commute, the X_A are *strongly commuting*, so that we can apply the joint spectral theorem – see Theorem 5.21 in [Sch12]. There exists a Borel, $L^2(M,\mu_1)$-projector valued, measure ν on $\mathfrak{a}^*$ such that for $u \in L^2(M,\mu_1)$,

$$u = \int_{\mathfrak{a}^*} d\nu(\vartheta)u, \qquad X_A u = \int_{\mathfrak{a}^*} i\vartheta(A) d\nu(\vartheta)u$$

We will prove the following regularity result of these measures below:

Lemma 5.11. *Let X be an Anosov action. Assume that there is no non-zero purely imaginary Ruelle-Taylor resonance and a unique normalized SRB measure. Then for any $f,g \in C^\infty(\mathcal{M})$ with $\int_{\mathcal{M}} f d\mu_1 = \int_{\mathcal{M}} g d\mu_1 = 0$ we consider $\nu_{f,g}(\theta) := \langle \nu(\theta)f, g\rangle_{L^2(\mathcal{M},\mu_1)}$ which are finite complex valued measures on $\mathfrak{a}^*$. Then the wavefront set $\mathrm{WF}(\nu_{f,g}) \subset \mathfrak{a}^* \times \mathfrak{a}$ fulfills*

$$\mathrm{WF}(\nu_{f,g}) \cap (\mathfrak{a}^* \times \mathcal{W}) = \emptyset$$

Before proving this Lemma let us show that it implies (3). Take $A_0 \in W$, f, g as in the above Lemma, then the spectral theorem yields

$$C_{f,g}(t;A_0) = \int_{\mathcal{M}} g(\varphi_{-t}^{X_A})^* f d\mu_1 = \int_{\mathfrak{a}^*} e^{-i\vartheta(A)t} d\nu_{f,\overline{g}}(\vartheta)$$

Given any $\varepsilon > 0$, using the fact that $\nu_{f,\overline{g}}$ is finite, there is a cutoff function $\chi_K \in C_c^\infty(\mathfrak{a}^*, [0,1])$ equal to 1 on a sufficiently large compact set $K \subset \mathfrak{a}^*$ such that $|\int_{\mathfrak{a}^*} e^{-i\vartheta(A)t}(1-\chi_K) d\nu_{f,\overline{g}}(\vartheta)| \leq \varepsilon/2$ uniformly in t. Furthermore by the fact that the wavefront set is empty in the direction of the Weyl chamber $\mathcal{W}$ we deduce that there is T such that $|\int_{\mathfrak{a}^*} e^{-i\vartheta(A)t} \chi_K d\nu_{f,\overline{g}}(\vartheta)| \leq \varepsilon/2$ for any $t > T$ thus $\lim_{t\to\infty} C_{f,g}(t, A_0) = 0$. The passage to arbitrary $L^2(\mathcal{M},\mu_1)$ functions follows by the density of the smooth functions. □

Proof of Lemma 5.11. Let us pick any $A_0 \in \mathcal{W}$ and a basis $A_1, \dots, A_\kappa \in \mathcal{W}$ such that these elements span an open cone around A_0. With this basis we identify the joint spectral measure to a measure on $\mathbb{R}^\kappa$. Recall the definition of $R_\sigma(i\lambda)$ from the proof of Proposition 5.4

which was based on the choice of an even, positive $\psi \in C^\infty((-1/2,1/2))$ with $\int \psi = 1$ and some $\sigma \in \mathbb{R}_+^\kappa$. Using the spectral theorem we calculate for any $f, g \in L^2(\mathcal{M}, \mu_1)$

$$\langle R_\sigma(i\lambda)^k f, g\rangle_{L^2(\mathcal{M},\mu_1)} = \int_{\mathbb{R}^\kappa} \hat{\Psi}(\vartheta+\lambda)^k e^{ik\sigma(\vartheta+\lambda)} d\nu_{f,g}(\vartheta) \tag{5.15}$$

where $\Psi(t) := \prod_{j=1}^\kappa \psi(t_j)$. Now let us define the following closed subspaces $\mathcal{H}_{NG,0} := \{u \in \mathcal{H}_{NG}, \int u d\mu_1 = 0\} \subset \mathcal{H}_{NG}$. Note that these are well defined for sufficiently large N because $\mu_1 \in \mathcal{H}_{-NG}$. Furthermore from the invariance of μ_1 under the Anosov actions the spaces $\mathcal{H}_{NG,0}$ are preserved under $R_\sigma(i\lambda)$. Now the assumption that there is no imaginary Ruelle-Taylor resonance except zero and that there is a unique normalized SRB measure imply (combining the findings of Section 5.1) that $R_\sigma(i\lambda)$ has a spectral radius < 1 on $H_{NG,0}$ for any $\lambda \in \mathbb{R}^\kappa$, and $\sigma \in \mathbb{R}_+^\kappa$ sufficiently large. Thus there are $C_{\sigma,\lambda}, \varepsilon_{\sigma,\lambda} > 0$, locally uniformly in σ, λ such that $\|R_\sigma(i\lambda)^k\|_{\mathcal{H}_{NG,0}} \le C_{\sigma,\lambda} e^{-\varepsilon_{\sigma,\lambda} k}$. Now let f, g be as in the assumption of our Lemma, then we can estimate

$$\langle R_\sigma(i\lambda)^k f, g\rangle_{L^2(\mathcal{M},\mu_1)} \le \|R_\sigma(i\lambda)^k f\|_{\mathcal{H}_{NG,0}} \|g\mu_1\|_{\mathcal{H}_{-NG}} \le C_{f,g,\sigma,\lambda} e^{-\varepsilon_{\sigma,\lambda} k}.$$

Let us come back to the expression (5.15) involving the spectral measures. By the properties of ψ we deduce that near zero $\hat{\Psi}(\xi) = \exp(-S(\xi))$ with some analytic function $S(\xi) = a|\xi|^2 + \mathcal{O}(|\xi|^4)$. Furthermore for any $\delta > 0$, there is $\varepsilon_2 > 0$ such that $\hat{\Psi}(\xi) < e^{-\varepsilon_2}$ for $|\xi| > \delta$. Choosing a cutoff function $\chi \in C_c^\infty((-3\delta, 3\delta)^\kappa)$ with $\chi(\xi) = 1$ for $|\chi| < 2\delta$ we get by the boundedness of $\nu_{f,g}$ for an arbitrary fixed $\lambda_0 \in \mathbb{R}^\kappa$

$$\left| \langle R_\sigma(i\lambda)^k f, g\rangle_{L^2(\mathcal{M},\mu_1)} - \int_{\mathbb{R}^\kappa} \hat{\Psi}(\vartheta+\lambda)^k e^{ik\sigma(\vartheta+\lambda)} \chi(\vartheta+\lambda_0) d\nu_{f,g}(\vartheta) \right| \le C e^{-\varepsilon_2 k}$$

uniformly for $\sigma \in \mathbb{R}_+^\kappa$ $|\lambda - \lambda_0| < \delta$. Putting everything together we get

$$\left| \int_{\mathbb{R}^\kappa} e^{-kS(\vartheta+\lambda) - ik\sigma(\vartheta+\lambda)} \chi(\vartheta+\lambda_0) d\nu_{f,g}(\vartheta) \right| \le \tilde{C} e^{-\tilde{\varepsilon} k}$$

with $\tilde{C}, \tilde{\varepsilon} > 0$ locally uniform in $|\lambda - \lambda_0| < \delta$ and $\sigma \in \mathbb{R}_+^\kappa$. In the expression on the left hand we recognize the Fourier-Bros-Iagolnitzer (FBI) transform (see e.g. [Sjo82])of the distribution $\chi(\vartheta+\lambda_0) d\nu_{f,g}(\vartheta)$ and the exponential decay implies that (λ_0, σ) is not in the wavefront set of $\chi(\vartheta+\lambda_0) d\nu_{f,g}(\vartheta)$. As $\chi = 1$ near zero the statement about the wavefront set transfers to the spectral measure $\nu_{f,g}$ and we have completed the proof of Lemma 5.11. □

Appendix A. Tools from microlocal analysis

We recall here some essentials of microlocal analysis. In the paper, we are working with pseudodifferential operators acting on $C^\infty(\mathcal{M}; E) \otimes \Lambda\mathfrak{a}_{\mathbb{C}}^* \cong C^\infty(\mathcal{M}; E \otimes \Lambda\mathfrak{a}_{\mathbb{C}}^*)$. Note that by fixing an arbitrary scalar product on $\mathfrak{a}^*$ the bundle $E \otimes \Lambda := E \otimes \Lambda\mathfrak{a}_{\mathbb{C}}^* \to \mathcal{M}$ is again a Riemannian bundle. We will therefore introduce notations for pseudodifferential operators on general Riemannian bundles $E \to \mathcal{M}$ over a compact Riemannian manifold $\mathcal{M}$. Only when we want to exploit some specific structures of $E \otimes \Lambda$, will we refer to this particular bundle.

For more details we refer to standard references such as [GS94]. For the details concerning the anisotropic calculus we refer to [FRS08].

Definition A.1. Let $k \in \mathbb{R}$, $1/2 < \rho \leq 1$. Then the *standard symbol space* $S^k_\rho(\mathcal{M}; E)$ is the space of functions $a \in C^\infty(T^*\mathcal{M}; \mathrm{End}(E))$, for which in any local chart $U \subset \mathbb{R}^n$ of $\mathcal{M}$ and any local trivialization of the bundle, for any compact set $K \subset U$ and any two multindices $\alpha, \beta \in \mathbb{N}^n$

$$\sup_{(x,\xi)\in T^*U, x\in K} \|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\| \langle\xi\rangle^{-(k-\rho|\beta|+(1-\rho)|\alpha|)} < \infty.$$

Given a zeroth order symbol $m(x,\xi) \in S^0_1(\mathcal{M})$ then the *anisotropic symbol space* $S^{m(x,\xi)}_\rho(\mathcal{M}; E)$ is the space of functions $a \in C^\infty(T^*\mathcal{M}; \mathrm{End}(E))$ for which in any local chart $U \subset \mathbb{R}^n$ for any compact set $K \subset U$ and any two multiindices $\alpha, \beta \in \mathbb{N}^n$

$$\sup_{(x,\xi)\in T^*U, x\in K} \|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\| \langle\xi\rangle^{-(m(x,\xi)-\rho|\beta|+(1-\rho)|\alpha|)} < \infty.$$

We furthermore set[3]

$$S^{-\infty}(\mathcal{M}; E) := \cap_{k>0} S^{-k}_\rho(\mathcal{M}; E), \quad S^{\infty}(\mathcal{M}; E) := \cup_{k>0} S^{k}_\rho(\mathcal{M}; E),$$
$$S^{m+}_\rho(\mathcal{M}; E) := \cap_{\varepsilon>0} S^{m+\varepsilon}_\rho(\mathcal{M}; E), \quad S^{m}_{\rho-}(\mathcal{M}; E) := \cup_{\varepsilon>0} S^{m}_{\rho-\varepsilon}(\mathcal{M}; E).$$

Note that by setting $m(x,\xi) = k \in \mathbb{R}$ the standard symbols are a special case of anisotropic symbols. We will therefore mostly introduce the notation in the anisotropic setting as it comprises the standard symbols as a special case. Furthermore, note that $x \mapsto \mathrm{Id}_{E_x}$ is a global smooth section of $\mathrm{End}(E) \to \mathcal{M}$ and multiplication with this section yields a canonical embedding $S^\infty_\rho(\mathcal{M}) \hookrightarrow S^\infty_\rho(\mathcal{M}; E)$. We will denote symbols in the image of this embedding as the space of *scalar symbols*.

After fixing a finite atlas and a suitable partition of unity of $\mathcal{M}$ one can define a quantization (see e.g. [DZ19, E.1.7]) that associates to any $a \in S^\infty_\rho(\mathcal{M}; E)$ a continuous operator $\mathrm{Op}(a) : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; E)$ which extends to a continuous operator $\mathrm{Op}(a) : C^{-\infty}(\mathcal{M}; E) \to C^{-\infty}(\mathcal{M}; E)$. We denote by $\Psi^{-\infty}(\mathcal{M}; E)$ the space of smoothing operators $A : C^{-\infty}(\mathcal{M}; E) \to C^\infty(\mathcal{M}; E)$. The quantization has the property that $\mathrm{Op}(S^{-\infty}(\mathcal{M}; E)) \subset \Psi^{-\infty}(\mathcal{M}; E)$. We say that $A \in \Psi^m_\rho(\mathcal{M}; E)$ iff there is $a \in S^m_\rho(\mathcal{M}; E)$ such that $A - \mathrm{Op}(a) \in \Psi^{-\infty}(\mathcal{M}; E)$. When $\rho = 1$, we will drop the ρ index and write $S^m(\mathcal{M}; E)$ and $\Psi^m(\mathcal{M}; E)$ instead of $S^m_1(\mathcal{M}; E)$ and $\Psi^m_1(\mathcal{M}; E)$.

With any $A \in \Psi^m_\rho(\mathcal{M}; E)$ one can associate its principal symbol

$$\sigma^m_p(A) \in S^m_\rho(\mathcal{M}; E)/S^{m-2\rho+1}_\rho(\mathcal{M}; E).$$

The principal symbol is an inverse to Op in the sense that

$$\sigma^m_p \circ \mathrm{Op} : S^m_\rho \to S^m_\rho / S^{m-2\rho+1}_\rho \text{ and } \mathrm{Op} \circ \sigma^m_p : \Psi^m_\rho \to \Psi^m_\rho / \Psi^{m-2\rho+1}_\rho$$

are simply the projections on the respective quotients.

[3]Note that $\cap_{k>0} S^{-k}_\rho(\mathcal{M}; E)$ is independent of $1/2 < \rho \leq 1$ and we therefore drop the index in the notation of $S^{-\infty}(\mathcal{M}; E)$.

Example A.2. Any k-th order differential operator P with smooth coefficients on the bundle $E \to \mathcal{M}$ is in $\Psi_1^k(\mathcal{M}; E)$ and a representative of its principal symbol $\sigma_p^k(P)$ can be calculated by

$$\left[\sigma_p^k(P)(x,\xi)\right] u(x) = \lim_{t\to\infty} t^{-k}\left[e^{-it\phi}P(e^{it\phi}u)\right](x),$$

where $u \in C^\infty(\mathcal{M}; E)$ and $\phi \in C^\infty(\mathcal{M})$ is a phase function with $d\phi(x) = \xi$ (see e.g. [Hör03, (6.4.6')]). As a direct consequence we get:

(1) For any vector field $X \in C^\infty(\mathcal{M}; T^*\mathcal{M}) \subset \Psi_1^1(\mathcal{M})$ we have $\sigma_p^1(X)(x,\xi) = i\xi(X(x))$.
(2) If $\mathbf{X} : \mathfrak{a} \to \mathrm{Diff}^1(\mathcal{M}; E) \subset \Psi_1^1(\mathcal{M}, E)$ is an admissible lift of an Anosov action, then for all $A \in \mathfrak{a}$ one finds that the principal symbol $\sigma_p^1(\mathbf{X}_A)(x,\xi) = i\xi(X_A(x))\,\mathrm{Id}_{E_x}$ is scalar.
(3) In order to express the principal symbol of the exterior derivative $d_{\mathbf{X}} \in \Psi_1^1(\mathcal{M}, E \otimes \Lambda\mathfrak{a}_{\mathbb{C}}^*)$ of X, let us consider the smooth map $T^*\mathcal{M} \ni (x,\xi) \mapsto \xi(X_\bullet(x)) \in \Lambda^1\mathfrak{a}^*$. With its help we calculate for $v \in E_x, \omega \in \Lambda\mathfrak{a}^*$

$$\sigma_p^1(d_{\mathbf{X}})(x,\xi)(v\otimes\omega) = iv \otimes (\xi(X_\bullet(x))\wedge\omega)\,.$$

(Thus $\sigma_p^1(d_{\mathbf{X}})$ is scalar on the E-component but not on the $\Lambda\mathfrak{a}^*$-component as it increases the order of differential forms.)

Proposition A.3. *Let $A \in \Psi_\rho^{m_1(x,\xi)}(\mathcal{M}; E)$ and $B \in \Psi_\rho^{m_2(x,\xi)}(\mathcal{M}; E)$, then $AB \in \Psi_\rho^{m_1+m_2}(\mathcal{M}; E)$ and $\sigma_p^{m_1+m_2}(AB) = \sigma_p^{m_1}(A)\sigma_p^{m_2}(B) \mod S_\rho^{m_1+m_2-2\rho+1}(\mathcal{M}; E)$.*

Definition A.4. Given $a \in S_\rho^{m(x,\xi)}(\mathcal{M}; E)$, we define its *elliptic set* to be the open cone $\mathrm{ell}^{m(x,\xi)}(a) \subset T^*\mathcal{M} \setminus \{0\}$ which consists of all $(x_0,\xi_0) \in T^*\mathcal{M} \setminus \{0\}$ for which there is a $C > 0$ and a function $\chi \in C^\infty(T^*\mathcal{M})$, positively homogeneous of degree zero for $|\xi| \geq C$, and $\chi(x_0, C\xi_0/|\xi_0|) > 0$, such that $a(x,\xi) \in \mathrm{End}(E_x)$ is invertible for all $(x,\xi) \in \mathrm{supp}(\chi)$ and $\chi a^{-1} \in S_\rho^{-m(x,\xi)}(\mathcal{M}; E)$. We call $a \in S_\rho^{m(x,\xi)}(\mathcal{M}, E)$ *elliptic* if $\mathrm{ell}^{m(x,\xi)}(a) = T^*\mathcal{M}\setminus\{0\}$.

As a direct consequence of the chain rule for derivatives and the symbol estimates we get

Lemma A.5. *If $a \in S_\rho^{m(x,\xi)}(\mathcal{M}; E)$ is a scalar symbol, then $(x_0,\xi_0) \in \mathrm{ell}^{m(x,\xi)}(a)$ if there exists an open cone $\Gamma \subset T^*\mathcal{M}$ containing (x_0,ξ_0) and $C > 0$ such that*

$$|a(x,\xi)| \geq \frac{1}{C}\langle\xi\rangle^{m(x,\xi)} \text{ for all } (x,\xi) \in \Gamma \cap \{|\xi| > C\}.$$

One checks that for $a \in S_\rho^{m(x,\xi)}(\mathcal{M}; E)$ and $r \in S_\rho^{m(x,\xi)-\varepsilon}(\mathcal{M}; E)$ one has $\mathrm{ell}^{m(x,\xi)}(a) = \mathrm{ell}^{m(x,\xi)}(a+r)$ which allows to define the elliptic set of an operator $A \in \Psi^{m(x,\xi)}(\mathcal{M}; E)$ via its principal symbol $\mathrm{ell}^{m(x,\xi)}(A) := \mathrm{ell}^{m(x,\xi)}(\sigma_p^{m(x,\xi)}(A))$.

Definition A.6. Given $A = \mathrm{Op}(a) \mod \Psi^{-\infty}(\mathcal{M}; E)$, we define its *wavefront set* to be the closed cone $\mathrm{WF}(A) \subset T^*\mathcal{M} \setminus \{0\}$ which is the complement of all $(x_0,\xi_0) \in T^*\mathcal{M} \setminus \{0\}$ for which there is an open cone $\Gamma \subset T^*\mathcal{M}$ around (x_0,ξ_0) such that for all $N > 0, \alpha, \beta \in \mathbb{N}^n$ there is $C_{N,\alpha,\beta}$ such that

$$\|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)\| \leq C_{N,\alpha,\beta}\langle\xi\rangle^{-N} \text{ for all } (x,\xi) \in \Gamma.$$

The wavefront set has the following property for the product of two pseudodifferential operators $A, B \in \Psi_\rho^\infty(\mathcal{M}; E)$:

$$\mathrm{WF}(AB) \subset \mathrm{WF}(A) \cap \mathrm{WF}(B).$$

We will crucially use the following constructions of microlocal parametrices.

Lemma A.7. *If $A \in \Psi_\rho^{m_1(x,\xi)}(\mathcal{M}; E), B \in \Psi_\rho^{m_2(x,\xi)}(\mathcal{M}; E)$ and $\mathrm{WF}(B) \subset \mathrm{ell}^{m_1(x,\xi)}(A)$, then there is $Q \in \Psi_\rho^{m_2(x,\xi)-m_1(x,\xi)}$ with $\mathrm{WF}(Q) \subset \mathrm{WF}(B)$ such that*

$$AQ - B \in \Psi^{-\infty}(\mathcal{M}; E).$$

If furthermore A and B are holomorphic families of operators, then Q can be chosen to be holomorphic as well.

As a consequence of Lemma A.7, if $A \in \Psi_\rho^{m_1}(\mathcal{M}; E)$ and $B \in \Psi_\rho^{m_2}(\mathcal{M}; E)$, then

$$\mathrm{ell}^{m_1}(A) \cap \mathrm{WF}(B) \subset \mathrm{WF}(AB). \tag{A.1}$$

We also have the following particular case of Egorov's lemma.

Lemma A.8. *Let $F \in \mathrm{Diffeo}(\mathcal{M})$ be a smooth diffeomorphism and let $\widetilde{F} \in \mathrm{Diffeo}(E)$ be a lift of F, i.e. $\widetilde{F}$ acts linearly in the fibers and $\pi \circ \widetilde{F} = F \circ \pi$ for $\pi : E \to \mathcal{M}$ the fiber projection. Define the* transfer operator

$$L_F : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; E), \quad (L_F u)(x) := \widetilde{F}^{-1}(F(x), u(F(x))).$$

Then for each $A \in \Psi_\rho^m(\mathcal{M}; E)$, we have $L_F A L_F^{-1} \in \Psi_\rho^{m\circ\Phi}(\mathcal{M}; E)$ with $\Phi(x,\xi) := (F(x), (dF^{-1})^T \xi)$ and

$$\sigma_p^{m\circ\Phi}(L_F A L_F^{-1})(x,\xi) = \widetilde{F}^{-1}(F(x), \cdot) \circ \sigma_p^m(A)(\Phi(x,\xi)) \circ \widetilde{F}(x, \cdot).$$

Proposition A.9 (L^2-boundedness)**.** *Let $A \in \Psi_\rho^0(\mathcal{M}; E)$, then A can be extended from an operator on $C^\infty(\mathcal{M}; E)$ to a bounded operator on $L^2(\mathcal{M}; E)$. Furthermore, for any*

$$C > \limsup_{|\xi|\to\infty} \|\sigma_p^0(A)(x,\xi)\|,$$

there exists a decomposition $A = K + R$, where $K \in \Psi^{-\infty}(\mathcal{M}; E)$ is a smoothing and hence L^2-compact operator and $\|R\|_{L^2\to L^2} \le C$. If A_t is a smooth family in $\Psi_\rho^0(\mathcal{M}; E)$ for $t \in [t_1, t_2]$, the decomposition $A_t = R_t + K_t$ can be chosen so that $t \mapsto R_t$ and $t \mapsto K_t$ are continuous in t.

Proof. See [FRS08, Lemma 14] for the proof. The dependence in t is straightforward from the proof. □

We conclude this appendix by mentioning that one can use a small semiclassical parameter $h > 0$ in the quantization, in which case we shall write Op_h, by using the expression in a local chart

$$\mathrm{Op}_h(a) f(x) = \frac{1}{(2\pi h)^n} \int e^{\frac{i(x-x')\xi}{h}} a(x,\xi) f(x') d\xi dx'$$

if a is supported in a chart. We do not use this semiclassical quantization except in the two subsections 4.4 and 5.2 and we refer to [DZ19, Appendix E] for the results on semiclassical pseudodifferential operators that we will use. One of the advantages is that one can get the estimate $\|\operatorname{Op}_h(a)\|_{L^2\to L^2} \le \sup_{x,\xi}|a(x,\xi)| + \mathcal{O}(h)$ for small $h>0$ and if $a\in S^0(\mathcal{M};E)$.

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