SCATTERING FOR THE GEODESIC FLOW ON SURFACES WITH BOUNDARY

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ABSTRACT. These are lecture notes based on a mini-course given in the 2015 summer school *Théorie spectrale géométrique et computationnelle* in CRM, Montréal.

1. Introduction

In these notes, we discuss some geometric inverse problems in 2-dimension that have been studied since the eighties, and we review some results on these questions.

The problem we consider consists in recovering a Riemannian metric on a surface with boundary from measurements at the boundary: the lengths of geodesics relating boundary points and their tangent directions at the boundary. This is a non-linear problem called the *lens rigidity problem*, which has a gauge invariance (pull-backs by diffeomorphisms fixing the boundary). The associated linear problem consists in the analysis of the kernel of the geodesic X-ray transform, a curved version of the Radon transform.

The tools to study the X-ray transform are of analytic nature, more precisely a combination of analysis of transport equations with some energy identity. Microlocal methods have also been very powerful in that study, but we won't review this aspect in these notes. The techniques presented in these notes are quite elementary and give a short introduction to that area of research.

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2. Geometric background

Let (M, g) be a smooth oriented compact Riemannian surface with boundary ∂M and let M° be its interior. In local coordinates $x = (x_1, x_2)$ the metric will be written

$$g = \sum_{i,j=1}^{2} g_{ij}(x) dx_i dx_j,$$

where $(g_{ij}(x))_{ij}$ are symmetric positive definite matrices smoothly depending on x. We will write ∇ the Levi-Civita connection of g on M. The tangent bundle of M is denoted TM and the projection on the base is written

$$\pi_0 : TM \to M.$$

The second fundamental form II is the symmetric tensor

$$\mathrm{II} : T\partial M \times T\partial M \to \mathbb{R}, \quad \mathrm{II}(u,w) := -g(\nabla_u \nu, w),$$

where ν is the interior pointing unit normal vector to ∂M. We say that ∂M is *strictly convex* for g if II is positive definite and we will assume along this course that this property holds.

Exercise: Show that a neighborhood of ∂M in M is isometric to $[0,\epsilon]_r \times \partial M$ with metric $dr^2 + h_r$ where h_r is a smooth 1-parameter family of metrics on ∂M. In these *geodesic normal coordinates*, we have $\nu = \partial_r|_{r=0}$ and $\mathrm{II} = -\frac{1}{2}\partial_r h_r|_{r=0}$.

2.1. **Geodesic flow.** A *geodesic* on M is a C^2 curve on M such that $\nabla_{\dot{x}}\dot{x} = 0$ where $\dot{x}(t) \in T_{x(t)}M$ is the tangent vector to the curve, i.e $\dot{x}(t) := \partial_t x(t)$. In local coordinates $x = (x_1, x_2)$, we can write $\nabla_{\partial_{x_i}}\partial_{x_j} = \sum_{k=1}^2 \Gamma_{ij}^k \partial_{x_k}$ for some smooth functions Γ_{ij}^k called *Christoffel symbols*. The geodesic equation in these coordinates is

$$\ddot{x}_j(t) = -\sum_{k,\ell=1}^{2} \Gamma_{k\ell}^j(x(t))\dot{x}_k(t)\dot{x}_\ell(t), \quad j = 1, 2.$$

By standard arguments of ordinary differential equations – Cauchy-Lipschitz–, this second order equation has a solution $x(t)$ in some interval $t \in [0,\epsilon)$ if we fix an initial condition $(x(0), \dot{x}(0)) = (x_0, v_0) \in TM^\circ$, and the solution can be extended until $x(t)$ reaches ∂M.

The geodesics are minimizers of the *energy* and of the *length functionals*: if p, q are two points in M and if $\gamma : [0,1] \to M$ is a C^2 curve such that $\gamma(0) = p$ and $\gamma(1) = q$, then the energy $E_{p,q}(\gamma)$ and the length $L_{p,q}(\gamma)$ are defined by

$$E_{p,q}(\gamma) = \int_0^1 |\dot{\gamma}(t)|^2_{g(\gamma(t))}dt, \quad L_{p,q}(\gamma) = \int_0^1 |\dot{\gamma}(t)|_{g(\gamma(t))}dt.$$

Then the minimum of $E_{p,q}(\gamma)$ and $L_{p,q}(\gamma)$ among curves as above are obtained by a geodesic. In fact, one can show using variational methods and a compactness argument that in each homotopy class of curves with endpoints p, q, there is a mimimizer for $E_{p,q}$ and $L_{p,q}$ which is a geodesic. The miminizer is in general not unique.

Definition 2.1. *The geodesic flow at time $t \in \mathbb{R}$ is the map φ_t defined by*

$$\varphi_t : U(t) \to TM, \quad \varphi_t(x,v) := (x(t), \dot{x}(t)),$$

if $x(t)$ is the geodesic with initial condition $(x(0), \dot{x}(0)) = (x, v)$, where $U(t) \subset TM$ is the set of points $(x,v) \in TM$ such that the geodesic $x(s)$ with initial condition (x,v) exists in M for all $s \in [0,t]$.

The *exponential map* at a point $x \in M$ is the map

$$\exp_x : U_x \subset T_xM \to M, \quad \exp_x(v) = \pi_0(\varphi_1(x, v)),$$

where U_x is the set of vector $v \in T_xM$ so that $\varphi_t(x, v) \in M$ for all $t \in [0, 1)$. The map $\exp_x$ is a local diffeomorphism near $v = 0$ at each $x \in M^\circ$.

Notice that for $x(t)$ a geodesic on M, $\partial_t(g_{x(t)}(\dot{x}(t), \dot{x}(t))) = 2g_{x(t)}(\nabla_{\dot{x}(t)}\dot{x}(t), \dot{x}(t)) = 0$ and therefore φ_t acts on the unit tangent bundle

$$SM := \{(x, v) \in TM; g_x(v, v) = 1\}.$$

The vector field generating the flow φ_t is a smooth vector field on SM defined by

$$Xf(x, v) = \partial_t f(\varphi_t(x, v))|_{t=0}.$$

The manifold SM is compact and has boundary $\partial SM = \pi_0^{-1}(\partial M)$. This boundary splits into the disjoint parts

$$\partial_- SM := \{(x, v) \in SM; x \in \partial M, g_x(v, \nu) > 0\},$$
$$\partial_+ SM := \{(x, v) \in SM; x \in \partial M, g_x(v, \nu) < 0\},$$
$$\partial_0 SM := \{(x, v) \in SM; x \in \partial M, g_x(v, \nu) = 0\}.$$

We call $\partial_- SM$ the *incoming boundary*, $\partial_+ SM$ the *outgoing boundary* and $\partial_0 SM$ the *glancing boundary*.

2.2. **Hamiltonian approach.** The cotangent bundle T^*M is a symplectic manifold, with symplectic form $\omega = d\alpha$ where $\alpha \in C^\infty(T^*M, T^*(T^*M))$ is the Liouville 1-form defined by $\alpha_{(x,\xi)}(W) = \xi(d\pi_0(x, \xi).W)$ if $\pi_0 : SM \to M$ is the projection on the base. In local coordinates $x = (x_1, x_2)$, $\xi = \xi_1 dx_1 + \xi_2 dx_2$, we have

$$\alpha = \sum_{i=1}^{2} \xi_i dx_i, \quad \omega = \sum_{i=1}^{2} d\xi_i \wedge dx_i.$$

The function $p : T^*M \to \mathbb{R}$ defined by $p(x, \xi) = \frac{1}{2} g_x^{-1}(\xi, \xi)$ where g^{-1} is the metric induced on T^*M by g has a Hamiltonian vector field X_p defined by $\omega(X_p, \cdot) = dp$ and X_p is tangent to $S^*M := p^{-1}(1/2)$.

Exercise: Show that the duality isomorphism $SM \to S^*M$ given by the metric g conjugates $\varphi_t = e^{tX}$ to e^{tX_p}.

2.3. **Geometry of SM.** There are particular sets of coordinates $x = (x_1, x_2)$ near each points $x_0 \in M$, called *isothermal coordinates*, such that in these coordinates, the metric has the form

$$g = e^{2\rho}(dx_1^2 + dx_2^2)$$

in these coordinates, for some smooth function $\rho(x)$ near x_0. The metric is conformal to the Euclidean metric in these coordinates, which will be a useful fact fow hat follows.

These coordinates can be obtained by solving an elliptic equation, and more precisely a Beltrami equation (see [Ta, Chapter 5.10]). These coordinates induce a diffeomorphism

$$\Omega \times \mathbb{R}/2\pi\mathbb{Z} \to \pi_0^{-1}(\Omega) \subset SM, \quad (x,\theta) \mapsto (x, v = e^{-\rho(x)}(\cos(\theta)\partial_{x_1} + \sin(\theta)\partial_{x_2})),$$

where Ω is the neighborhood of x_0 where the isothermal coordinates are valid. In these coordinates, the vector field X becomes

$$X = e^{-\rho}\Big(\cos(\theta)\partial_{x_1} + \sin(\theta)\partial_{x_2} + (-\sin(\theta)\partial_{x_1}\rho + \cos(\theta)\partial_{x_2}\rho)\partial_\theta\Big). \tag{2.1}$$

We start by analyzing the flow of X near the boundary, under the assumption that ∂M is strictly convex.

Lemma 2.2. *Geodesics in M° intersect ∂M transversally, i.e. a geodesic coming from M° and touching ∂M at a point x can not be tangent to ∂M*

Proof. There are isothermal coordinates $x = (x_1, x_2)$ near each $x_0 \in \partial M$ such that a neighborhood of x_0 in M correspond to a neighborhood of 0 in the half-plane $\{x_2 \geq 0\}$ and the metric is of the form $e^{2\rho}(dx_1^2 + dx_2^2)$, and we can assume that x_0 is mapped to $x = 0$ by this chart. One has $\partial_{x_2}\rho|_{x_2=0} < 0$ if ∂M is strictly convex. If $x(t)$ is a geodesic for $t \leq t_0$ with $x_2(t) > 0$ for $t < t_0$ and $x_2(t_0) = 0$, then if $\dot{x}_2(t_0) = 0$, we get by (2.1) that $\theta(t_0) = 0$ (or π) and $\dot{\theta}(t_0) = e^{-\rho(0)}(\cos(\theta_0)\partial_{x_2}\rho(0))$. Let us consider the case $\theta(t_0) = 0$ (the case $\theta(t_0) = \pi$ is similar): then $\dot{\theta}(t_0) < 0$ and θ decreases as $t \to t_0$, and since $\dot{x}_2(t) = e^{-\rho}\sin\theta(t)$, we get $\ddot{x}_2(t_0) = e^{\rho(0)}\dot{\theta}(t_0) < 0$. A Taylor expansion gives

$$x_2(t) = \tfrac{1}{2}(t - t_0)^2\ddot{x}_2(t_0) + \mathcal{O}((t - t_0)^3),$$

which is negative near t_0, leading to a contradiction. □

Define Θ_t the rotation of angle $+t$ in the fibers of SM; in the coordinates above Θ_t is just $(x,\theta) \mapsto (x, \theta + t)$. This smooth 1-parameter family of diffeomorphisms of SM induces a smooth vector field V defined by

$$Vf(x,v) = \partial_t f(\Theta_t(x,v))|_{t=0}, \quad \forall f \in C^\infty(SM).$$

In the coordinates (x,θ), $V = \partial_\theta$. Next we define another vector field

$$X_\perp := [X, V],$$

which in the coordinates (x,θ), is given by

$$X_\perp = -e^{-\rho}\Big(-\sin(\theta)\partial_{x_1} + \cos(\theta)\partial_{x_2} - (\cos(\theta)\partial_{x_1}\rho + \sin(\theta)\partial_{x_2}\rho)\partial_\theta\Big).$$

It it an elementary computation to check that the three vector fields $(X, X_\perp, V)$ form a global basis of $T(SM)$ (we recover that SM is trivialisable) and satisfy the commutation relations

$$[X_\perp, V] = -X, \quad [X, X_\perp] = -\kappa(x)V, \tag{2.2}$$

where $\kappa(x)$ is the Gaussian curvature of g at x. In isothermal coordinates, a computation yields

$$\kappa(x) = e^{-2\rho(x)}\Delta_x \rho(x)$$

where $\Delta_x = -(\partial_{x_1}^2 + \partial_{x_2}^2)$.

We define the *Sasaki metric* of g as the metric G on SM so that $(X, X_\perp, V)$ is an orthonormal basis, and its volume form dv_G is also equal to the Liouville measure $d\mu_L$ obtained from the symplectic form $\omega = d\alpha$ by setting $d\mu_L = |\alpha \wedge d\alpha|$ when we use the identification $S^*M \to SM$. In isothermal coordinates, one has

$$d\mu_L(x, \theta) = e^{2\rho(x)} dx d\theta.$$

If $W = aX + bX_\perp + cV$, we have

$$G(W, W) = a^2 + b^2 + c^2 = g(d\pi_0(W), d\pi_0(W)) + c^2.$$

The Sasaki metric is usually defined using the splitting of the vertical bundle and horizontal bundle (see [Pa]), but it coincides in our case with the definition above. Here $(X, X_\perp)$ span the horizontal bundle while V span the vertical bundle of the fibration $\pi_0 : SM \to M$.

Exercise: Check, using Cartan formula, that the following Lie derivatives vanish:

$$\forall Z \in \{X, X_\perp, V\},\ \mathcal{L}_Z d\mu_L = 0.$$

As a consequence, we have $X^* = -X$, $V^* = -V$ and $X_\perp^* = -X_\perp$ on $C_0^\infty(SM)$ with respect to the $L^2(SM, d\mu_L)$ product, where $C_0^\infty(SM)$ is the set of smooth functions on SM vanishing at the boundary $\partial(SM)$ of SM.

On $SM \subset TM$, we can consider functions which are restrictions to SM of homogeneous polynomials of order $m \in \mathbb{N}_0$ in the fibers of TM, i.e. symmetric tensors defined as sections of $\otimes_S^m T^*M$. There is a natural map for each $m \in \mathbb{N}_0$

$$\pi_m^* : C^\infty(M, \otimes_S^m T^*M) \to C^\infty(SM), \quad \pi_m^* f(x, v) := f(x)(\otimes^m v). \tag{2.3}$$

2.4. **Conjugate points.** Geodesic flows can have 1-parameter families of geodesics with the same endpoints x_-, x_+: this is related to the existence of *conjugate points*. We say that $x_\pm \in M$ are conjugate points if there exist $v_\pm \in S_{x_\pm}M$ and t_0 so that $\varphi_{t_0}(x_-, v_-) = (x_+, v_+)$, and if

$$d\varphi_{t_0}(x_-, v_-).V \in \mathbb{R}V$$

where V is the vertical vector field. Equivalently, we say that $x_\pm$ are conjugate if there is an *orthogonal Jacobi field* $J = J(t)$ along the geodesic $(x(t))_{t\in[0,t_0]}$ vanishing at x_- and x_+. Recall that an orthogonal Jacobi field is a vector field along $x(t)$, orthogonal to $\dot{x}(t) = v(t)$. If we write $v^\perp(t) = R_{\pi/2}(v(t))$ the unit orthogonal vector obtained by a rotation of angle $\pi/2$ of $v(t)$, such a field would be of the form $J(t) = a(t)v^{\perp}(t)$, and for it to be a Jacobi field the function $a(t)$ has to satisfy:

$$a(0) = a(t_0) = 0, \quad \ddot{a}(t) + \kappa(x(t))a(t) = 0.$$

Exercise: Show that when the Gauss curvature κ is non-positive, there are no conjugate points.

By Gauss-Bonnet theorem, for a geodesic triangle with interior angles $\alpha_1, \alpha_2, \alpha_3$, one has

$$\int_M \kappa \, \mathrm{dvol}_g + \pi = \sum_{i=1}^{3} \alpha_i.$$

If two geodesics have the same endpoints x_-, x_+, we obtain a triangle with angles π, α_2, α_3, and this forces by Gauss-Bonnet to have some positive curvature somewhere. In fact, it can be proved that absence of conjugate points implies that between two points x_-, x_+, there is a unique geodesic – this is done using the *index form*, see [Mi, Sections 14 & 15].

3. Scattering map, length function and X-ray transform

3.1. **Lens rigidity problem.** We start by making the *non-trapping* assumption on the geodesic flow. That is, for each $(x, v) \in SM^\circ$ there is a unique $\ell_+(x, v) \geq 0$ and $\ell_-(x, v) \leq 0$ so that $\varphi_{\ell_\pm(x,v)}(x, v) \in \partial SM$, which means that each geodesic of SM has finite length.

Exercise: Prove, using the strict convexity of ∂M and the implicit function theorem, that $\ell_\pm$ are smooth in SM° and that they extend smoothly to $SM \setminus \partial_0 SM$. Show also that $\ell_\pm$ extend continuously to SM in a way that $\ell_\pm|_{\partial_\pm SM \cup \partial_0 SM} = 0$. We will still call $\ell_\pm$ these continous extensions.

Definition 3.1. *The function ℓ_+ is called the length function and $\ell_g := \ell_+|_{\partial SM}$ is called the* boundary length function. *The map*

$$S_g : \partial_- SM \to \partial_+ SM, \quad S_g(x, v) = \varphi_{\ell_+(x,v)}(x, v)$$

is called the scattering map *of g.*

We introduced these objects to formulate some questions in the realm of inverse problems. They are quantities that can be measured from the boundary. The boundary length function contains the set of Riemannian distances between boundary points, but it does contain a priori more information in the case where there are several geodesics between boundary points. The scattering map tells where geodesics leave SM.

Definition 3.2. *The X-ray transform on functions on SM is defined as the operator*

$$I : C^\infty(SM) \to C^\infty(\partial_- SM), \quad If(x,v) = \int_0^{\ell_+(x,v)} f(\varphi_t(x,v))dt.$$

The X-ray on symmetric tensors of order m is the operator

$$I_m := I\pi_m^* : C^\infty(M, \otimes_S^m T^*M) \to C^\infty(\partial_- SM).$$

The two main inverse problems related to these objects are:

Problem 1: Determine the kernel of I_m and invert I_m on $\mathrm{Ran}(I_m|_{(\ker I_m)^\perp})$.

Problem 2: Do (S_g, ℓ_g) determine the metric g up to Gauge invariance ?

The natural *Gauge invariance* in Problem 2 is the diffeomorphism action: let $\mathrm{Diff}_{\partial M}(M)$ be the group of diffeomorphism of M which are equal to the identity on ∂M, then one has for each $\psi \in \mathrm{Diff}(M)$

$$S_{\psi^* g} = S_g, \quad \ell_{\psi^* g} = \ell_g.$$

Problem 2 is called the *lens rigidity problem* and (S_g, ℓ_g) is the *lens data.*

There is a link between these inverse problems. Indeed, one has

Lemma 3.3. *Let $g_s = e^{2\rho_s} g_0$ be some smooth 1-parameter family of non-trapping metrics with strictly convex boundary and no conjugate points, where ρ_s is a smooth family of smooth functions on M such that $\rho_0 = 0$. If $(\ell_{g_s}, S_{g_s}) = (\ell_{g_0}, S_{g_0})$ for each $s \in (-\epsilon, \epsilon)$, then $I_0(\rho_0') = 0$ if I_0 is the X-ray transform on functions for g_0 and $\rho_0' := \partial_s \rho_s|_{s=0}$.*

Proof. Fix $(x,v) \in \partial_- SM$ and $(x', v') = S_{g_0}(x,v) = S_{g_s}(x,v)$. We differentiate

$$\ell_{g_s}(x,v) = \int_0^{\ell_{g_0}(x,v)} e^{\rho_s(\gamma_s(t,x,v))} |\dot\gamma_s(t,x,v)|_{g_0(\gamma_s(t,x,v))} dt$$

with respect to s, where $\gamma_s(t,x,v)$ is the unique geodesic for g_s relating x and x'. Using the fact that $\gamma_0(t,x,v)$ is the unique geodesic for g_0 relating x and x', it minimizes the length functional among curves with endpoints x, x' and thus it is direct to see that

$$0 = \int_0^{\ell_{g_0}(x,v)} \rho_0'(\gamma_0(t,x,v))dt + \partial_s(L^{g_0}_{x_-,x_+}(\gamma_s))|_{s=0} = I_0(\rho_0')(x,v)$$

where $L^{g_0}_{p,q}(\gamma)$ denotes the length of a curve γ joining p, q for the metric g_0. □

This corresponds to analyzing *deformation lens rigidity* within a conformal class. More generally we have (with essentially the same proof):

Lemma 3.4. *Let g_s be some smooth 1-parameter family of non-trapping metrics with strictly convex boundary and no conjugate points. If $(\ell_{g_s}, S_{g_s}) = (\ell_{g_0}, S_{g_0})$ for each $s \in (-\epsilon, \epsilon)$, then $I_2(g_0') = 0$ if I_2 is the X-ray transform on symmetric 2-tensors for g_0 and $g_0' := \partial_s g_s|_{s=0} \in C^\infty(M; \otimes_S^2 T^*M)$.*

These two lemmas are folklore and the ideas appear already in Guillemin-Kazhdan [GuKa] and the Siberian school (Mukhometov, Anikonov, Sharafutdinov, etc).

Remark that if $\psi_s : M \to M$ is a smooth family of diffeomorphisms which are equal to Id on ∂M, we have an associated vector field $Z(x) = \partial_s \psi_s(x)|_{s=0}$ with $Z|_{\partial M} = 0$ and, according to Lemma 3.4, $\mathcal{L}_Z g_0 = \partial_s(\psi_s^* g_0)|_{s=0}$ satisfies $I_2(\mathcal{L}_Z g_0) = 0$ if I_2 is the X-ray transform for g_0 on symmetric 2-tensors. The natural question about the kernel of X-ray transform is then: under which conditions on g_0 do we have

$$\ker I_0 = 0 \text{ ? and } \ker I_2 = \{\mathcal{L}_Z g_0; Z \in C^\infty(M, TM), Z|_{\partial M} = 0\} \text{ ?}$$

3.2. **Boundary rigidity problem.** When the metric g has strictly convex boundary, is non-trapping and satisfies that between each pair of points $x, x' \in M$ there is a unique geodesic, we say that g is *simple*. In this case, knowing the lens data is equivalent to knowing the restriction $d_g|_{\partial M \times \partial M}$ of the Riemannian distance $d_g : M \times M \to \mathbb{R}^+$. The lens rigidity problem is called *boundary rigidity problem* in that setting.

4. Resolvents and boundary value problems for transport equations

For a general metric g on SM with strictly convex boundary, we define the incoming tail Γ_- and outgoing tails as the sets

$$\Gamma_- := \{(x, v) \in SM; \ell_+(x, v) = +\infty\}, \quad \Gamma_+ := \{(x, v) \in SM; \ell_-(x, v) = -\infty\}.$$

These sets correspond to the set of points which are on geodesics that are trapped inside SM° in forward (for Γ_-) and backward (for Γ_+) time. They are closed sets in SM°. The trapped set is defined by

$$K := \Gamma_+ \cap \Gamma_-,$$

it corresponds to trajectories contained entirely in the interior SM°. It is a closed set in SM, invariant by the geodesic flow, and by the strict convexity of the boundary ∂M, we actually have $K \subset SM^\circ$, since each point $(x, v) \in \partial_\mp SM$ is not in $\Gamma_\pm$ and each point $(x, v) \in \partial_0 SM$ has $\ell_+(x, v) = \ell_-(x, v) = 0$.

4.1. **Resolvents in physical half-planes.** Assume that $\Gamma_\pm = \emptyset$, i.e the metric is non-trapping. There are two natural boundary value problems for the transport equations associated to X. For $f \in C^\infty(SM)$, find $u_\pm$ in some fixed functional space solving (in the distribution sense)

$$\begin{cases} -X u_\pm = f \\ u_\pm|_{\partial_\pm SM} = 0 \end{cases}$$

One is an incoming Dirichlet type boundary condition and the other one is an ougoing Dirichlet type boundary condition. Moreover, it is easy to check that

$$u_+(x,v) = \int_0^{\ell_+(x,v)} f(\varphi_t(x,v))dt, \quad u_-(x,v) = -\int_{\ell_-(x,v)}^0 f(\varphi_t(x,v))dt$$

are solutions, and in fact they are the only continuous solutions: indeed the difference of two solutions would be constant along flow lines and, under the non-trapping condition each point in SM is on a flow line with two endpoints in ∂_+SM and ∂_-SM. We see in particular that

$$f \in C^\infty(SM) \Longrightarrow u_\pm \in C^\infty(SM \setminus \partial_0 SM).$$

Without assuming the non-trapping condition, we can proceed using the resolvent of X.

Lemma 4.1. *For* $\mathrm{Re}(\lambda) > 0$*, there exist two operators* $R_\pm(\lambda) : C^\infty(SM) \to C^0(SM)$ *satisfying for all* $f \in C^\infty(SM)$ *(in the distribution sense)*

$$\begin{cases} (-X \pm \lambda)R_\pm(\lambda)f = f, \\ R_\pm(\lambda)f|_{\partial_\pm SM} = 0. \end{cases}$$

They are given by the expressions

$$\begin{aligned} R_+(\lambda)f(x,v) &= \int_0^{\ell_+(x,v)} e^{-\lambda t} f(\varphi_t(x,v))dt, \\ R_-(\lambda)f(x,v) &= -\int_{\ell_-(x,v)}^0 e^{\lambda t} f(\varphi_t(x,v))dt. \end{aligned} \tag{4.1}$$

Proof. Lebesgue theorem shows directly that $R_\pm(\lambda)f$ are continuous if f is continuous. The fact that they solve the desired boundary value problem for $\mathrm{Re}(\lambda)$ large enough follows from the fact that $X(f(\varphi_t(x,v))) = \partial_t(f(\varphi_t(x,v))$ and the estimate

$$|d(f \circ \varphi_t)|_G \leq |df|_G |d\varphi_t|_G, \quad |d\varphi_t|_G \leq Ce^{Ct}$$

for some $C > 0$ depending on X. Then, we get that for each $f' \in C_c^\infty(SM^\circ)$ and $\mathrm{Re}(\lambda) > 0$

$$\langle (-X \pm \lambda)R_\pm(\lambda)f, f' \rangle = \langle R_\pm(\lambda)f, (X \pm \lambda)f' \rangle$$

where the pairing uses the measure μ_L, and the left hand side is equal to $\langle f, f' \rangle$ for $\mathrm{Re}(\lambda) > C$, thus by using that the right hand side is analytic in $\mathrm{Re}(\lambda) > 0$, the right hand side is equal to $\langle f, f' \rangle$ in that half-plane. ☐

Exercise: Show that the operators $R_\pm(\lambda)$ extend analytically in $\lambda \in \mathbb{C}$ as operators

$$R_\pm(\lambda) : C_c^\infty(SM^\circ \setminus \Gamma_\pm) \to C^\infty(SM).$$

If we assume that $\mu_L(\Gamma_\pm)=0$, we can then expect to extend $R_\pm(0)$ to some L^p space by some density argument. In fact, observe that

$$|R_+(0)f(x,v)|\leq ||f||_{L^\infty}\ell_+(x,v),$$

thus $\ell_+\in L^p(SM)$ implies that $R_+(0):C^0(SM)\to L^p(SM)$ is bounded.

4.2. **Santalo formula.** The Santalo formula describes the desintegration of the measure μ_L along flow lines.

Lemma 4.2. *Assume that $\mu_L(\Gamma_\pm)=0$ and let $f\in L^1(SM)$, then the following formula holds*

$$\int_{SM} f d\mu_L=\int_{\partial_- SM}\int_0^{\ell_+(x,v)} f(\varphi_t(x,v))dt\, d\mu_\nu(x,v),$$

where $d\mu_\nu(x,v)=\langle v,\nu\rangle_g\iota^*_{\partial SM}\mathrm{dv}_G$ is a measure on ∂SM, dv_G being the Riemannian measure of the Sasaki metric viewed as a 3-form.

Proof. We first take $f\in C_c^\infty(SM^\circ\setminus\Gamma_+)$ and write $f=-XR_+(0)f$ with $R_+(0)f=0$ on ∂_+SM. Thus using Green's formula

$$\int_{SM} fd\mu_L=-\int_{SM} X(R_+(0)f)d\mu_L=\int_{\partial SM}(R_+(0)f)|_{\partial SM}\langle X,N\rangle_G\,\iota^*_{\partial SM}\mathrm{dv}_G$$

if N is the unit inward pointing normal to ∂SM for G. The unit normal N satisfies $d\pi_0(N)=\nu$ and $G(N,V)=0$ if ν is the exterior pointing unit normal to ∂M for g, since the vertical bundle $\mathbb{R}V=\ker d\pi_0$ is tangent to ∂SM. Then we get $\langle N,X\rangle_G=\langle \nu,v\rangle_g$ and we use a density argument for the general case to end the proof. Note that an alternative definition of $d\mu_\nu$ is $d\mu_\nu=\iota^*_{\partial SM}i_Xd\mu_L$. □

4.3. **Boundedness in (weighted) L^2 spaces and limiting absorbtion principle.** The operators $R_\pm(\lambda)$ are also bounded on $L^2(SM)$ for $\mathrm{Re}(\lambda)>0$. Indeed, using the Santalo formula and defining $f(\varphi_t(x,v)):=0$ when $t\geq\ell_+(x,v)$, we have

$$\begin{aligned}\int_{SM}|f(\varphi_t(x,v))|^2d\mu_L(x,v)=&\int_{\partial_-SM}\int_0^{\ell_+(x,v)}|f(\varphi_{t+s}(x,v))|^2ds\,d\mu_\nu(x,v)\\ =&\int_{\partial_-SM}\int_t^{\ell_+(x,v)}|f(\varphi_u(x,v))|^2du\,d\mu_\nu(x,v)\\ \leq&||f||_{L^2(SM)}.\end{aligned}$$

We can write using Minkowski inequality

$$\begin{aligned}||R_+(\lambda)f||_{L^2(SM)}\leq&\Big(\int_{SM}\Big(\int_0^{+\infty}e^{-\mathrm{Re}(\lambda)t}|f(\varphi_t(x,v))|dt\Big)^2d\mu_L(x,v)\Big)^{\frac12}\\ \leq&\int_0^{+\infty}e^{-\mathrm{Re}(\lambda)t}||f\circ\varphi_t||_{L^2(SM)}dt\leq\frac{||f||_{L^2(SM)}}{\mathrm{Re}(\lambda)}.\end{aligned}$$

The question of the extension of $R_\pm(\lambda)$ up to the imaginary line in a certain functional space is very similar to the so-called *limiting absorbtion principle* in quantum scattering theory. To make the parallel, recall that the Laplacian on $L^2(\mathbb{R}^3)$ is self-adjoint as an unbounded operator, its spectrum is $[0,\infty)$ and its resolvent is the operator defined for $\mathrm{Im}(\lambda) > 0$ by

$$R_\Delta(\lambda) = (\Delta - \lambda^2)^{-1} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$$

and there is an explicit formula for its integral kernel

$$R_\Delta(\lambda; x, x') = C\frac{e^{i\lambda|x-x'|}}{|x-x'|}$$

for some explicit constant $C \in \mathbb{R}$. Now when $\lambda \in \mathbb{R}$, this is not anymore a bounded operator on L^2 but it makes sense as an operator

$$R_\Delta(\lambda) : \langle x\rangle^{-1/2-\epsilon} L^2(\mathbb{R}^3) \to \langle x\rangle^{1/2+\epsilon} L^2(\mathbb{R}^3) \tag{4.2}$$

and both $R_\Delta(\lambda)$ and $R_\Delta(-\lambda)$ are inverses for $(\Delta - \lambda^2)$ (here $\langle x\rangle := (1+x^2)^{1/2}$). They are called the incoming and outgoing resolvents on the spectrum, and they need to be applied on functions which have some decay near infinity. Such property also holds in higher dimension. A result of Kenig-Ruiz-Sogge [KRS] shows that for $\lambda \in \mathbb{R}^*$

$$R_\Delta(\lambda) : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n), \quad p = \frac{2n}{n+2}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

We will discuss similar properties for the resolvent of the flow vector field $-X$. The first boundedness property we describe is comparable to (4.2).

Lemma 4.3. *For each $\lambda \in i\mathbb{R}$ and $\epsilon > 0$, the resolvent $R_\pm(\lambda)$ is bounded as a map*

$$R_\pm(\lambda) : \langle \ell_+\rangle^{-1/2-\epsilon} L^2(SM) \to \langle \ell_+\rangle^{1/2+\epsilon} L^2(SM).$$

Proof. We do the case $R_+(0)$, the other frequencies λ are similar. First we notice that $\ell_+(\varphi_s(y)) = \ell_+(y) - s$. Then for $f \in C^\infty(SM)$ and $\widetilde{f} := \langle \ell_+\rangle^{-1/2-\epsilon} f$ we have

$$\int_{SM} |R_+(0)\widetilde{f}(y)|^2 \langle \ell_+(y)\rangle^{-1-2\epsilon} d\mu_L(y)$$
$$\leq \int_{\partial_- SM} \int_0^{\ell_+(y)} \langle \ell_+(y) - s\rangle^{-1-2\epsilon} \Big(\int_0^{\ell_+(y)-s} \widetilde{f}(\varphi_{t+s}(y))dt\Big)^2 ds d\mu_\nu(y)$$
$$\leq \int_{\partial_- SM} \int_0^{\ell_+(y)} \langle \ell_+(y) - s\rangle^{-1-2\epsilon} \int_0^{\ell_+(y)-s} \langle \ell_+(y) - s - t\rangle^{-1-2\epsilon} dt \int_0^{\ell_+(y)-s} |f(\varphi_{t+s}(y))|^2 dt d\mu_\nu(y) ds$$
$$\leq C \int_{\partial_- SM} \int_0^{\ell_+(y)} \langle \ell_+(y) - s\rangle^{-1-2\epsilon} \int_0^{\ell_+(y)-s} |f(\varphi_{t+s}(y))|^2 dt d\mu_\nu(y) ds$$
$$\leq C \int_{\partial_- SM} \int_0^{\ell_+(y)} \langle \ell_+(y) - s\rangle^{-1-2\epsilon} \int_0^{\ell_+(y)} |f(\varphi_u(y))|^2 du d\mu_\nu(y) ds$$
$$\leq C^2 \int_{\partial_- SM} \int_0^{\ell_+(y)} |f(\varphi_u(y))|^2 du d\mu_\nu(y) \leq C^2 ||f||^2_{L^2(SM)}$$

where we have used that there is $C > 0$ depending only on ϵ so that

$$\int_0^{a-s} \langle a - s - t\rangle^{-1-2\epsilon} dt \leq C.$$

To complete the proof, we use the density of smooth functions in L^2. □

A priori, it is not clear that the space $\langle \ell_+\rangle^{1/2+\epsilon} L^2(SM)$ can be embedded into the space of distributions on SM, since the function ℓ_+ could explode quite drastically at Γ_-. To quantify this, we define the *non-escaping mass function*:

$$V(T) := \mathrm{Vol}_{\mu_L}(\{y \in SM^\circ; \varphi_t(y) \in SM^\circ, \forall t \in [0, T]\}) = \mathrm{Vol}_{\mu_L}(\ell_+^{-1}([T, +\infty]))$$

which is like the repartition function of ℓ_+. The Cavalieri principle gives

$$||\ell_+||^p_{L^p} \leq C\Big(1 + \int_1^\infty t^{p-1} V(t) dt\Big).$$

Recall that $\ell_+(x, -v) = -\ell_-(x, v)$ thus the L^p norms of ℓ_+ and ℓ_- are the same since the involution $(x, v) \to (x, -v)$ preserves μ_L.

Lemma 4.4. *For $p \in [1, \infty)$ we have the boundedness properties*

$$R_\pm(0) : C^0(SM) \to L^p(SM) \text{ if } \int_1^\infty V(t) t^{p-1} dt < \infty,$$

and for $p > 1$

$$R_\pm(0) : L^p(SM) \to L^1(SM) \text{ if } \int_1^\infty V(t) t^{1/(p-1)} dt < \infty.$$

Proof. Left as an exercise. Use Hölder and Santalo formula. □

4.4. **Return on X-ray transform.** First we remark that the X-ray transform can be defined also in the trapping case as a map

$$I : C_c^\infty(SM \setminus \Gamma_+) \to C^\infty(\partial_- SM);$$

and for a function $f \in C_c^\infty(SM \setminus \Gamma_+)$ this can be written as

$$If(x,v) = (R_+(0)f)(x,v) \quad (x,v) \in \partial_- SM.$$

If $\widetilde{f} = \langle \ell_+ \rangle^{-1/2-\epsilon/2} f$, we have

$$\begin{aligned}\int_{\partial_- SM} |I\widetilde{f}(y)|^2 d\mu_\nu(y) =& \int_{\partial_- SM} \Big| \int_0^{\ell_+(y)} \langle \ell_+(y) - t\rangle^{-1/2-\epsilon/2} f(\varphi_t(y)) dt \Big|^2 d\mu_\nu(y) \\ \leq& C \int_{\partial_- SM} \int_0^{\ell_+(y)} |f(\varphi_t(y))|^2 dt d\mu_\nu(y) = C||f||^2_{L^2(SM)}\end{aligned}$$

and therefore

$$I : \langle \ell_+ \rangle^{-1/2-\epsilon} L^2(SM) \to L^2(\partial_- SM, d\mu_\nu)$$

is bounded for each $\epsilon > 0$. It is a straightforward consequence of Santalo formula that

$$I : L^1(SM) \to L^1(\partial_- SM, d\mu_\nu)$$

is bounded. We get, just as for $R_\pm(0)$, the following boundedness property

Lemma 4.5. *When $p > 2$, the X-ray transform is bounded as a map $L^p(SM) \to L^2(\partial_- SM, d\mu_\nu)$ if $\int_1^\infty t^{p/(p-2)} V(t) dt < \infty$*

Proof. Use the Hölder and Santalo formulas. □

As a consequence, taking the adjoint, we obtain the boundedness

$$I^* : L^2(\partial_- SM, d\mu_\nu) \to L^{p'}(SM)$$

under the assumption $\int_1^\infty t^{p/(p-2)} V(t) dt < \infty$ and in general the boundedness of

$$I^* : L^2(\partial_- SM, d\mu_\nu) \to \langle \ell_+ \rangle^{1/2+\epsilon} L^2(SM)$$

holds for all $\epsilon > 0$. We would like to characterize what operator is I^*. For f, f' smooth supported oustide $\Gamma_- \cup \Gamma_+$, we have

$$\int_{\partial_- SM} If.f' d\mu_\nu = \int_{\partial_- SM} R_+(0)f.f' d\mu_\nu = \int_{SM} -X(R_+(0)f).\mathcal{E}_-(f') d\mu_L = \langle f, \mathcal{E}_-(f')\rangle$$

where $\mathcal{E}_-(f')$ solves

$$\begin{cases} X\mathcal{E}_-(f') = 0, \\ \mathcal{E}_-(f')|_{\partial_- SM} = f' \end{cases}$$

We thus get $I^* = \mathcal{E}_-$. Notice that $\mathcal{E}_-(f')(y) = f'(\varphi_{\ell_-(y)}(y))$ is constant on flow lines.

4.5. **The normal operator.** Assume that $\mu_L(\Gamma_-\cup\Gamma_+)=0$. We define the normal operator on SM as the operator given by the expression

$$\Pi : \langle \ell_+\rangle^{-1/2-\epsilon}L^2(SM)\to \langle \ell_+\rangle^{1/2+\epsilon}L^2(SM), \quad \Pi := I^*I.$$

By Lemma 4.5, we see that if $\int_1^\infty V(t)t^{p/(p-2)}dt<\infty$ for some $p>2$, then Π extends as a bounded operator

$$\Pi : L^p(SM)\to L^{p'}(SM).$$

We can relate Π to the operators $R_\pm(0)$.

Lemma 4.6. *The following identity holds true if $\mu_L(\Gamma_-\cup\Gamma_+)=0$*

$$\Pi = R_+(0)-R_-(0).$$

Proof. Note that as operators acting on $C_c^\infty(SM^\circ)$, we have $R_+(0)^*=-R_-(0)$ thus we need to prove

$$\langle I^*If,f\rangle = 2\langle R_+(0)f,f\rangle.$$

But we have for each $f\in C_c^\infty(SM\setminus\Gamma_+)$

$$\begin{aligned}\int_{SM} R_+(0)f.f d\mu_L &= -\int_{SM} X(R_+(0)f).R_+(0)fd\mu_L = -\frac{1}{2}\int_{SM} X((R_+(0)f)^2)d\mu_L\\ &=\frac{1}{2}\int_{\partial SM}|R_+(0)f|^2d\mu_\nu = \frac{1}{2}\int_{\partial_-SM}|If|^2d\mu_\nu.\end{aligned}$$

and we complete the proof using a density argument. □

Using this, let us characterize the kernel of I:

Lemma 4.7. *Assume $\int_1^\infty V(t)t^{p/(p-2)}dt<\infty$ for some $p>2$, then $f\in\ker I\cap C^0(SM)$ if and only there exists a unique $u\in L^p(SM)\cap C^0(SM\setminus K)$ such that*

$$Xu=f, \quad u|_{\partial SM}=0,$$

K being the trapped set. If $K=\emptyset$ and if $f\in C^\infty(SM)$ vanishes to infinite order at ∂SM, then $u\in C^\infty(SM)$ and u vanishes to infinite order at ∂SM.

Proof. Assume that $If=0$. Set $u=-R_+(0)f$, then $Xu=f$, $u\in C^0(SM\setminus\Gamma_-)$ and satisfies $u|_{\partial_+SM}=0$. We have $\Pi f=0$, thus by Lemma 4.6, $u=-R_+(0)f=-R_-(0)f$. Thus u is actually in $C^0(SM\setminus\Gamma_-)$ and vanishes on ∂_-SM. Since there is no solution of $Xu=0$ with $u|_{\partial SM}=0$ and $u\in L^1(SM)\cap C^0(SM\setminus K)$, we have proved one direction. Conversely, if $u\in L^1(SM)\cap C^0(SM\setminus K)$ satisfies $Xu=f$ in the distribution sense, then $u=-R_+(0)f$ by uniqueness of solutions with $u|_{\partial_+SM}=0$, and similarly $u=-R_-(0)f$. Thus $I^*If=\Pi f=0$ and therefore $If=0$ since $\langle \Pi f,f\rangle=|If|^2_{L^2}$. The fact that u is smooth when g has no trapped set and $f\in C^\infty(SM)$ vanishes to infinite order at ∂SM is direct from the expression (4.1). Notice that if f is smooth but does not vanish to infinite order at ∂SM, then it is not clear that u is smooth at $\partial_0 SM$. □

We have just seen that f being in $\ker I$ can be interpreted in terms of properties of solutions of the transport equation $Xu = f$. In fact, the regularity of the solutions u to $Xu = f$ will be very important for what follows, and this leads us to define the following

Definition 4.8. *We shall say that a metric g with strictly convex boundary has the smooth Livsic property if for each $f \in C^\infty(SM)$ satisfying $If = 0$, there exists a unique $u \in C^\infty(SM)$ such that $Xu = f$ and $u|_{\partial SM} = 0$.*

Pestov-Uhlmann [PeUh] show the following result

Theorem 1. *If g is a non-trapping metric on a surface M with and strictly convex boundary, then it satisfies the smooth Livsic property.*

The main difficulty is the regularity at the glancing region $\partial_0 SM$, which is studied in the work of Pestov-Uhlmann using fold theory - we refer to [PeUh] for the interested reader (where fold theory is recalled). We notice that the presence or absence of conjugate points is not relevant here, since this result is really a property on the flow on SM and has not much to do with the fact that we are working with a geodesic flow.

In the recent work [Gu], we show the following result by using techniques of microlocal analysis and anisotropic Sobolev spaces:

Theorem 2. *If the curvature of g near the trapped set K is negative (or more generally if K is a hyperbolic set for the geodesic flow of g), then g has the smooth Livsic property.*

We notice that $V(t) = \mathcal{O}(e^{-Qt})$ for some $Q > 0$ if K is a hyperbolic set by [BoRu] (see [Gu, Proposition 2.4]), and this implies that $\ell_\pm \in L^p$ for all $p < \infty$.

5. Injectivity of X-ray transform for tensors

In this section, we use the Pestov identity (that we will explain below) as in [PSU1] and the results of previous section to prove the injectivity of X-ray transform on functions and divergence-free 1-forms when the metric has no conjugate points. For 2-tensors, we use the method of [GuKa] to determine the kernel of I_2 when the curvature κ is non-positive.

First, we denote by D the symmetrized covariant derivative mapping 1-forms to symmetric 2-tensors by

$$Dw(Y_1, Y_2) := \tfrac{1}{2}((\nabla_{Y_1} w)(Y_2) + (\nabla_{Y_2} w)(Y_1)). \tag{5.1}$$

Its L^2-adjoint is denoted D^* and is called the divergence operator on symmetric 2-tensors. We notice that if $w^\sharp$ is the vector field dual to w, then Dw can be written in terms of Lie derivative of the metric

$$Dw = \tfrac{1}{2}\mathcal{L}_{w^\sharp} g. \tag{5.2}$$

It will be convenient to use the Fourier decomposition in the circle fibers of SM. Using the vertical vector field, we can decompose each function $u\in C^\infty(SM)$ uniquely as a converging sum (in any C^k norms)

$$u=\sum_{k\in\mathbb{Z}} u_k, \quad \text{with } Vu_k=iku_k$$

where $u_k\in C^\infty(SM)$. This gives an orthogonal (with respect to L^2) decomposition

$$C^\infty(SM)=\bigoplus_{k\in\mathbb{Z}}\Omega_k$$

where $\Omega_k=\ker(V-ik)$. In isothermal coordinates $x=(x_1,x_2)$ near a point x_0, one has associated coordinates (x,θ) on SM near $S_{x_0}M$ with $\theta\in\mathbb{R}/2\pi\mathbb{Z}$, see Section 2.3. Then the functions u_k can be written locally as $u_k(x,\theta)=\tilde{u}_k(x)e^{ik\theta}$ for some $\tilde{u}_k$ smooth on M. In fact, when $k\geq 0$, Ω_k can be identified to the space of smooth sections of the k-th tensor power of the complex line bundle $\mathcal{K}:=(T^*M)^{1,0}\subset\mathbb{C}T^*M$ in the sense that $u_k=\pi_k^*s_k$ for some section $s_k\in C^\infty(M;\otimes_S^k\mathcal{K})$ where π_k^* is the map (2.3). Similalry when $k<0$, Ω_k can be identitifed as the space of smooth sections of the k-th tensor power of the bundle $\overline{\mathcal{K}}:=(T^*M)^{0,1}$.

There are two natural operators on $C^\infty(SM)$ called raising and lowering operators, introduced by [GuKa], and defined by

$$\eta_+:=\tfrac{1}{2}(X+iX_\perp), \quad \eta_-:=\tfrac{1}{2}(X-iX_\perp).$$

They satisfy $\eta_\pm:\Omega_k\to\Omega_{k\pm1}$, $X=\eta_++\eta_-$ and $\eta_+^*=-\eta_-$ when acting on smooth functions vanishing at ∂SM.

Theorem 3. *Assume that (M,g) has strictly convex boundary and that g has the smooth Livsic property and no conjugate points. Then we have:*
1) I_0 is injective on $C^\infty(M)$.
2) If $a\in C^\infty(M;T^*M))\cap\ker I_1$, then there exists $f\in C^\infty(M)$ such that $a=df$ and $f|_{\partial M}=0$.
3) Assume that $\kappa\leq 0$. If $h\in C^\infty(M;\otimes_S^2T^*M)\cap\ker I_2$, then there exists a 1-form $w\in C^\infty(M;T^*M)$ such that $h=Dw$ and $w|_{\partial M}=0$, where D is the symmetrized covariant derivative.

Proof. If $f\in\ker I_0$, by Theorem 2 there is $u\in C^\infty(SM)$ such that $u|_{\partial SM}=0$ and $Xu=\pi_0^*f$. Thus $VXu=0$ since $d\pi_0(V)=0$. We have $(VX)^*=XV$ on smooth functions vanishing at ∂SM by using that V,X preserve μ_L and that V is tangent to ∂SM. Then we get

$$||VXu||_{L^2}^2=||XVu||_{L^2}^2+\langle[XV,VX]u,u\rangle$$

and

$$[XV,VX] = XV^2X - VX^2V = VXVX + X_\perp VX - VXVX - VXX_\perp$$
$$= V[X_\perp, X] - X^2 = -X^2 + V\kappa V.$$

This implies the *Pestov identity* for each $u \in C^\infty(SM)$ vanishing at ∂SM

$$||XVu||^2_{L^2} - \langle \kappa Vu, Vu\rangle + ||Xu||^2_{L^2} - ||VXu||^2_{L^2} = 0. \tag{5.3}$$

We conclude that since $VXu = 0$, $\kappa \leq 0$ implies $Xu = \pi_0^* f = 0$. In fact, if there are no conjugate points, we claim that for each smooth function h on SM vanishing on ∂SM

$$||Xh||^2_{L^2} - \langle \kappa h, h\rangle \geq 0$$

and this is equal to 0 only if $h = 0$. This is proved by using Santalo formula to decompose the integral along geodesics with initial points on $\partial_- SM \setminus \Gamma_-$ and then by using that the index form is non-negative for each of these geodesics, when there is no conjugate points along these geodesics (see for example [PSU2, Lemma 11.2]). This completes the proof of the injectivity of I_0 by taking $h = Vu$.

Now if $I_1 a = 0$, we have $u \in C^\infty(SM)$ such that $Xu = \pi_1^* a$ and $u|_{\partial SM} = 0$. We apply the Pestov identity (5.3): since a is a 1-form and V acts as the Hodge operator on pull-backs of 1-forms, we have $||V\pi_1^* a||^2_{L^2(SM)} = ||\pi_1^* a||^2_{L^2(SM)}$ and (5.3) becomes

$$||XVu||^2_{L^2(SM)} - \langle \kappa Vu, Vu\rangle = 0.$$

This implies that $Vu = 0$ and thus $u = \pi_0^* f$ for some $f \in C^\infty(M)$ vanishing on ∂M. Then $Xu = \pi_1^* df$ and this completes the proof since this is equal to $\pi_1^* a$.

Let $h \in C^\infty(M; \otimes^2_S T^*M)$, then $\pi_2^* h = h_0 + h_2 + h_{-2}$ with $h_k \in \Omega_k$. If $I_2 h = 0$, there is $u \in C^\infty(SM)$ such that $Xu = \pi_2^* h$ and, after possibly replacing u by $\frac{1}{2}(u(x,v) - u(x,-v))$, we can always assume that u is odd with respect to the involution $A : (x,v) \mapsto (x,-v)$. Indeed, X maps even functions with respect to A to odd functions, and conversely. We will write $u = \sum_k u_k$ with $u_k \in \Omega_k$ and $u_{2k} = 0$ for all $k \in \mathbb{Z}$. Since $X = \eta_+ + \eta_-$, we have $\eta_+ u_{k-1} + \eta_- u_{k+1} = 0$ if $k \notin \{-2, 0, 2\}$. Next, using the commutation relation

$$\eta_-\eta_+ = \eta_+\eta_- - \tfrac{1}{2}\kappa i V$$

which follows from (2.2), the fact that $u_k|_{\partial SM} = 0$ for all k and $\eta_+^* = -\eta_-$, we obtain

$$||\eta_+ u_{k+1}||^2_{L^2} = -\langle \eta_-\eta_+ u_{k+1}, u_{k+1}\rangle = -\langle \eta_+\eta_- u_{k+1}, u_{k+1}\rangle - \tfrac{1}{2}(k+1)\langle \kappa u_{k+1}, u_{k+1}\rangle$$
$$\geq ||\eta_- u_{k+1}||^2$$

for $k+1 \geq 0$, since $\kappa \leq 0$. For $k \geq 3$, this implies that $||\eta_+ u_{k+1}||_{L^2} \geq ||\eta_+ u_{k-1}||_{L^2}$ and therefore $c_k := ||\eta_+ u_k||_{L^2}$ is a non-decreasing series which converges to 0, that is $c_k = 0$ for all $k \geq 2$. A similar argument shows that $\eta_- u_k = 0$ for all $k \leq -2$. This shows that $u = u_1 + u_{-1}$ and $u = \pi_1^* w$ for some smooth 1-form w vanishing at ∂M. Now it is easy to check that $Xu = \pi_2^*(Dw)$. □

Remark that the proof given in 3) actually works as well for $\ker I_m$ with $m \geq 3$, and we obtain that $h \in \ker I_m$ if and only if $h = Dw$ for some $w \in C^\infty(M, \otimes_S^{m-1} T^*M)$ and D is the symmetrized covariant derivative, defined similarly to (5.1). Acting by X on a pull-back (by π_m^*) of an m-symmetric tensor w is equivalent to pull-back Dw on SM by π_{m+1}^*. A proof of the injectivity of I_m on divergence-free tensors for simple metrics was provided recently in [PSU1], without assuming $\kappa \leq 0$.

Combining Theorem 3 with Theorem 2 and Lemma 3.4, we deduce the

Corollary 5.1. *Let g_s be a smooth family of metrics on a surface M with strictly convex boundary, non-positive curvature and with either no trapped set or hyperbolic set. If the lens data (ℓ_{g_s}, S_{g_s}) is constant in s, then one has $g_s = \phi_s^* g_0$ where ϕ_s is a smooth family of diffeomorphism equal to Identity on ∂M.*

Proof. Let $q_s := \partial_s g_s$. By Lemma 3.4, Theorem 3 and Theorem 2, we know that there is w_s so that $q_s = \mathcal{L}_{w_s^\sharp} g_s$ (with $w_s^\sharp$ the dual vector field to w_s though g_s). We claim that $w_s = (\Delta_{D_s})^{-1} D_s^* q_s$ if D_s is the operator D on 1-forms for g_s, D_s^* its adjoint with respect to the L^2-product of g_s and $\Delta_{D_s} := D_s^* D_s$ with Dirichlet condition. Indeed $\Delta_{D_s} w_s = D_s^* q_s = \Delta_{D_s}(\Delta_{D_s})^{-1} D_s^* q_s$ and the Laplacian Δ_{D_s} with Dirichlet condition has no kernel since $\langle \Delta_{D_s} u, u\rangle = ||D_s u||_{L^2}^2$ if $u|_{\partial M} = 0$, and $D_s u = 0$ with $u|_{\partial M} = 0$ implies $u = 0$ (check this as an exercise). Then, since g_s is smooth in s, it can be shown by elliptic theory that the inverse $\Delta_{D_s}^{-1}$ maps smooth functions of (s, x) to smooth functions of (s, x), if x is the variable on M. This implies that w_s is a smooth family in s of smooth 1-forms on M. Integrating the dual vector field $w_s^\sharp$, we can construct a smooth family of diffeomorphism which are the Identity on ∂M by $\partial_s \phi_s(x) = w_s^\sharp(\phi_s(x))$ and $\phi_0(x) = x$. Then ϕ_s satisfies $g_s = \phi_s^* g_0$. □

6. Some references

We haved worked in dimension 2 for simplicity but many results described here are also valid in higher dimension.

We provide a few references on the subject, this is not a comprehensive list.

We first recommend the lecture notes of Merry-Paternain [Pa] and the lecture notes of Sharafutdinov [Sh], which contain a lot of material on the subject. The survey of Croke [Cr3] also contains a nice overview of the subject (up to 2004).

The following articles deal with the boundary rigidity problem or the analysis of X-ray transform.

- For simple metrics in a fixed conformal class, Mukhometov [Mu2] proved that the boundary distance function determines the metric, with a stability estimate (see also the previous works [Mu1, MuRo]). This result was proved later with a simpler method by Croke [Cr2]. These works show the injectivity of the X-ray transform on functions for simple metrics (any dimension).
- The paper of Michel [Mi] established that simple metrics with constant curvature are boundary rigid in dimension 2. Gromov [Gr] proved the same result in higher dimension for flat metrics.
- Croke [Cr1] and Otal [Ot] proved boundary ridAfter idity for simple negatively curved surfaces (dimension 2).
- Pestov-Uhlmann [PeUh] proved boundary rigidity for all simple surfaces. More particularly, they proved that the scattering data determines the conformal class of the surface by relating the scattering map of the geodesic flow to the Dirichlet-to-Neumann map for the Laplacian. Using Mukhometov result, this shows the full boundary rigidity.
- Burago-Ivanov [BuIv] proved that metrics close to flat ones are boundary rigid (any dimension).
- Anikonov-Romanov [AnRo] proved injectivity of the X-ray transform on the space of divergence-free 1-forms for simple metrics (any dimension).
- Pestov-Sharafutdinov [PeSh] proved the injectivity of the X-ray transform on the space of divergence-free symmetric tensors for simple non-positively curved metrics (any dimension). This uses the so-called Pestov identity.
- Stefanov-Uhlmann [StUh] proved injectivity of the X-ray transform on tensors for analytic simple metrics and deduce a local boundary rigidity result for generic metrics.
- Paternain-Salo-Uhlmann [PSU1] proved injectivity of the X-ray transform on all divergence-free symmetric tensors for simple surfaces (dimension 2).
- Guillarmou [Gu] proved injectivity of the X-ray transform on functions and on the space of divergence-free 1-forms for metrics with strictly convex boundary, hyperbolic trapped set and no conjugate points; this setting contains all negatively curved metrics with strictly convex boundary. When the curvature is in addition non-positive, the injectivity on tensors is also proved (any dimension). The same result as Pestov-Uhlmann is shown also in that class of metrics.
- Uhlmann-Vasy [UhVa] proved injectivity of the X-ray transform on functions for manifolds admitting a foliation by convex hypersurfaces (any dimension ≥ 3), and injectivity for the local X-ray transform (i.e. integrals along geodesics almost tangent to boundary)

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