MINIMAL SEGMENTATIONS FOR THE MUMFORD-SHAH FUNCTIONAL

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Main theme: Regularity properties of the singular set for minimizers, and a few techniques of analysis or elementary geometric measure theory to get them.

1. The Mumford-Shah functional

We are given a simple domain $\Omega \subset \mathbb{R}^n$, a bounded function $g \in L^\infty(\Omega)$, and we set

$$(1) \quad J_g(u,K) = H^{n-1}(K) + \int_{\Omega\setminus K} |\nabla u|^2 + \int_{\Omega\setminus K} |u-g|^2$$

for $(u,K) \in \mathcal{A}$, the set of acceptable pairs (u,K) such that $K \subset \Omega$ is closed in Ω, and $u \in W^{1,2}(\Omega \setminus K)$ has one derivative in L^2 on $\Omega \setminus K$.

Here $H^{n-1}(K)$, the Hausdorff measure, is the correct analogue of $(n-1)$-dimensional surface measure of K, defined as soon as $K \subset \mathbb{R}^n$ is Borel-measurable.

Introduced by Mumford and Shah ($\leq$ 1989), at least in dimension $n = 2$, for image segmentation. Was also considered as a tool for modelling cracks when $n = 3$.

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For image segmentation, Ω is a screen, g is a given image, and u defines a segmentation for g. If (u,K) minimizes J, u should give a good compromise between three constraints:

- $u - g$ should be small
- u is simple (varies slowly), but may have jumps along a singular set K (which we see as describing edges in the picture), but
- K is not too complicated.

Comments:
- Segmentation $\neq$ compression: it is also fine if u and K only give some simplified idea of g.
- We could give different weights to the three terms, but the difference can be scaled out by multiplying u and g by a constant, and composing with a dilation
- Lots of variants exist, but often with a term like $H^{n-1}(K)$.
- This works fine because, as conjectured by Mumford and Shah, K is automatically regular (instead of just being short) when (u,K) is a minimal pair.
- Good and bad thing: automatic and context free!

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Empirical considerations

Assuming that minimizers exist, what can we expect?

Making K larger allows more jumps for u, hence helps reduce the tension and make $\int_{\Omega\setminus K} |\nabla u|^2$ smaller. But this is only efficient if K has good local separation properties, which:
- forces $H^{n-1}(K \cap B(x,r))$ to be of the order of r^{n-1} (more would be inefficient, less would allow too much passage; see later)
- gives the homogeneity of the problem: when there is a real competition between the first two terms in $B(x,r)$, r small, we expect both terms to give contributions of roughly r^{n-1} in $B(x,r)$
- shows that the third term, which contributes less than Cr^n in $B(x,r)$, plays a minor role locally.

In addition, the expected separation properties of K are a main reason why K is regular.

The Mumford-Shah conjecture

Observe that if g and K are given, it is easy to minimize $\int_{\Omega\setminus K} |\nabla u|^2 + \int_{\Omega\setminus K} |u-g|^2$ with respect to u. This is a convex problem, there is a unique solution u, and u is better than C^1 away from K (elliptic equation $\Delta u = u - g$). So the question is K (and u near K).

First, Mumford and Shah conjectured the existence of minimal pairs. True (see page 6).

But the (main) Mumford-Shah conjecture is the following: if (u, K) is a reduced minimizer for J_g in dimension 2, then K is a finite union of C^1 curves, which can only meet by sets of 3 and with 120° angles*.

- See the next page for "reduced".
- C^1 implies more when g is regular. Up to analytic [Koch-Leoni-Morini].
- If Ω is nice, regularity near $\partial\Omega$ is known [Maddalena-Solimini ; D.-Léger]. Otherwise, make the conclusion local in Ω.
- K is allowed to have tips (but incidentally, we still do not know whether they can really occur).
- Hence (u, K) looks like a decent segmentation, except that T-junctions are destroyed at small scales*.
- In dimension 3, some regularity is known, but we do not know a precise conjectural list of local behaviours.

Definition. The pair (u, K) is called reduced when, if $\widetilde{K} \subset K$ is a proper closed subset of K, the function u has no extension $\widetilde{u} \in W^{1,2}(\Omega \setminus \widetilde{K})$.

Given $(u, K) \in \mathcal{A}$, say with $H^{n-1}(K) < +\infty$ (and hence $|K| = 0$), we can always find $K' \subset K$ such that $u \in W^{1,2}(\Omega \setminus K')$ and (u, K') is reduced.

Then (u, K') is equivalent to (or even better than) (u, K) for J_g. So it is enough to consider reduced pairs. We shall do that.

This allows a better description of K: we avoid problems that come from adding to K an ugly set of vanishing H^{n-1}-measure.

Existence of minimal segmentations is a theorem of Ambrosio and De Giorgi-Carriero-Leaci.

The "stupid way" (taking a minimizing sequence (u_j, K_j) and letting K_j tend to a limit K through a subsequence) does not work trivially, because $H^{n-1}(K)$ could be much larger than the limit of the $H^{n-1}(K_j)$. Think about dotted lines*.

The proof uses a weak formulation of J_g in the subclass $SBV \subset BV$ of functions of bounded variation, where $u \in SBV$, and $K = K_u$ is now the singular set of u (not necessarily closed).

It uses two facts: the nice compactness properties of BV extend to SBV; and minimizers for the SBV analogue of J_g are so regular that $H^{n-1}(\overline{K}_u \setminus K_u) = 0$, so they also provide minimal pairs for J_g.

There is also a direct proof, based on the concentration property of Dal Maso, Morel, and Solimini. By DMS when $n = 2$, Maddalna and Solimini for n large. [Maybe two words about it later.]

No uniqueness in general because there may be a brutal change of strategy (a circle vanishes*) or by rupture of symmetry (checkerboard*).

Even less continuous dependence on parameters.

But could it be that uniqueness is generic (in g)?

Local almost-minimizers

Easy to check: if (u, K) is a reduced minimizer for J_h, then it is a (reduced) almost minimizer, with gauge function $h(r) = C||g||_\infty^2 r$.

Definition. A (local) almost minimizer with gauge function h is a pair $(u, K) \in \mathcal{A}$ such that, whenever $(\widetilde{u}, \widetilde{K}) \in \mathcal{A}$ coincides with (u, K) out of some ball $\overline{B} = \overline{B}(x, r)$ (such that $\overline{B} \subset \Omega$),

$$\begin{aligned} &H^{n-1}(K \cap \overline{B}) + \int_{\Omega \cap B \setminus K} |\nabla u|^2 \\ &\quad \leq H^{n-1}(\widetilde{K} \cap \overline{B}) + \int_{\Omega \cap B \setminus \widetilde{K}} |\nabla \widetilde{u}|^2 + h(r) r^{n-1}. \end{aligned}$$

Proof: By a cut-off argument, $||u||_\infty \leq ||g||_\infty$ and we can assume that $||\widetilde{u}||_\infty \leq ||g||_\infty$, so

$$\begin{aligned} LHS &\leq J_g(u, K) \leq J_g(\widetilde{u}, \widetilde{K}) \\ &= H^{n-1}(\widetilde{K} \cap \overline{B}) + \int_{\Omega \cap B \setminus \widetilde{K}} |\nabla \widetilde{u}|^2 + \int_{\Omega \cap B \setminus \widetilde{K}} |\widetilde{u} - g|^2 \\ &\leq RHS. \qquad \square \end{aligned}$$

Comments: other definitions exist; nice way to say that in J_g the third term matters less at small scales.

2. Regularity properties of K

From now on, (u, K) is a reduced (local) almost minimizer, with gauge function h (nondecreasing, and such that $\lim_{r\to 0} h(r) = 0$).

We shall mostly worry about local properties (far from $\partial\Omega$), and often in dimension 2.

We start with the **trivial estimate**:

$$(2)\quad H^{n-1}(K \cap \overline{B}(x,r)) + \int_{\Omega\cap B(x,r)\setminus K} |\nabla u|^2 \leq C r^{n-1}$$

for $r \leq 1$ (and if $\overline{B} = \overline{B}(x,r) \subset \Omega$).

Proof: just try $(\widetilde{u}, \widetilde{K}) = (u, K)$ out of $\overline{B}$, $K \cap \overline{B} = \partial B$, and $\widetilde{u} = 0$ in B. Here and below, C depends on n and h, not on (u, K). □

Next, K is **locally Ahlfors-regular**:

$$(3)\qquad C^{-1} r^{n-1} \leq H^{n-1}(K \cap B(x,r)) \leq C r^{n-1}$$

for $x \in K$ and $r < 1$ such that $\overline{B}(x,r) \subset \Omega$.

Proof by Dal Maso, Morel, Solimini 89 ($n = 2$) and Carriero, Leaci ($n \geq 2$). Idea: if K is too thin, it cannot separate enough to release the tension. Estimate the loss in energy when we remove a piece of K: integrate by parts and estimate the jump of u.

Local Ahlfors regularity is not so easy to get, but very useful. It allows us to use analysis techniques like

Carleson measures.

Often a good idea on spaces of homogeneous type: define functions on the space of balls

$$\Delta = \big\{(x,r) \in K \times (0,1]\,;\, \overline{B}(x,r) \subset \Omega\big\}. \tag{4}$$

Let us measure the normalized local energy with

$$\omega(x,r) = r^{1-n} \int_{B(x,r)\setminus K} |\nabla u|^2 \tag{5}$$

for $(x,r) \in \Delta$, and its L^p generalization for $1 \leq p \leq 2$

$$\omega_p(x,r) = r \left\{ \frac{1}{r^n} \int_{B(x,r)\setminus K} |\nabla u|^p \right\}^{\frac{2}{p}} \tag{6}$$

Note that $\omega(x,r) = \omega_2(x,r) \leq C$ by the trivial estimate, and then $\omega_p(x,r) \leq C$ by Hölder.

But $\omega_p(x,r)$ is often much smaller, to the point of being integrable against the locally infinite invariant measure $dH^{n-1}(x)\frac{dr}{r}$. So we can trade the optimal power aganist better integrability:

[D-Semmes, 96]: for $1 \le p < 2$, there exists $C_p \ge 0$ such that for $(x,r) \in \Delta$,

$$\int_{y\in B(x,r/2)} \int_{0<t<r/2} \omega_p(y,t)\, \frac{dH^{n-1}(y)dt}{t} \le C_p r^{n-1}.$$

Thus $\omega_p(x,r)\, \frac{dH^{n-1}(x)dr}{r}$ is a Carleson measure on Δ. The result is interesting but the proof is not: use the trivial bound, Hölder, Fubini, and the local Ahlfors-regularity to compute interior integrals.

Corollary: for each $1 \le p < 2$ and $\varepsilon > 0$, there exists $C(\varepsilon,p)$ such that, for every $(x,r) \in \Delta$, we can find $(y,t) \in \Delta$, with $y \in K \cap B(x,r/2)$ and $C(\varepsilon,p)^{-1}r \le t \le r/2$, and $\omega_p(y,t) \le \varepsilon$.

Thus each ball contains not-much-smaller good balls.

Proof by Chebyshev: otherwise the integral above is

$$\begin{aligned}
&\ge \int_{y\in B(x,r/2)} \int_{C(\varepsilon,p)^{-1}r<t<r/2} \varepsilon\, \frac{dH^{n-1}(y)dt}{t} \\
&\ge \varepsilon H^{n-1}(K \cap B(x,r/2)) \int_{C(\varepsilon,p)^{-1}r<t<r/2} \frac{dt}{t} \\
&\ge C^{-1}\varepsilon r^{n-1} \log(C(\varepsilon,p)/2) > C_p r^{n-1}
\end{aligned}$$

if $C(\varepsilon,p)$ is large enough [a contradiction]. □

The concentration property: For each small $\tau > 0$, there exists $C(\tau)$ such that, for all $(x,r) \in \Delta$ with $h(r) \le C(\tau)^{-1}$, we can find $(y,t) \in \Delta$, such that $y \in K \cap B(x, r/2)$, $C(\tau)^{-1} r \le t \le r/2$, and

(7) $$H^{n-1}(K \cap B(y,t)) \ge (1-\tau) H^{n-1}(P \cap B(y,t)),$$

where P is any hyperplane through y.

Thus K has almost optimal density in $B(y,t)$.

Theorem of Dal Maso, Morel, Solimini when $n = 2$, Maddalena and Solimini when $n > 2$.

I like it because of the following lowersemicontinuity result from [DMS]:

Let $\{K_j\}$ be a sequence of closed sets that satisfy the concentration property with uniform constants $C(\tau)$. Suppose that $\{K_j\}$ converges to the closed set K, locally in Ω for the Hausdorff distance*. Then

(8) $$H^{n-1}(K \cap U) \le \liminf_{j \to +\infty} H^{n-1}(K_j \cap U)$$

for $U \subset \Omega$ open.

Comments:
- Not true without assumption (dotted lines)
- Useful for producing minimizers (see later twice?)
- Proof by definition of H^{n-1} and coverings!

Proof when $n = 2$

[Advertisement for Carleson measures].

Let τ and $(x, r) \in \Delta$ be given. Pick $p < 2$ close to 2 and $\varepsilon > 0$ very small (chosen later), and let (y, t) be as in the corollary with Chebyshev. Thus

$$\omega_p(y,t) = t \left\{ \frac{1}{t^n} \int_{B(y,t)\setminus K} |\nabla u|^p \right\}^{\frac{2}{p}} \leq \varepsilon \tag{9}$$

(here with $n = 2$, so the power of t is $1 - \frac{1}{p}$).
We want to check that $H^1(K \cap B(y,t)) \geq 2(1-\tau)\, t$.
Enough to check that K meets $\partial B(y, \rho)$ at least twice for most $\rho \in (0, t)$.
For instance for all $\rho > \frac{\tau}{2}\, t$ such that

$$\int_{\partial B(y,\rho)\setminus K} |\nabla u|^p \leq C(\tau) \varepsilon^{\frac{p}{2}} \rho^{1-\frac{p}{2}}. \tag{10}$$

We suppose it does not and construct a better competitor $(\widetilde{u}, \widetilde{K})$.

Cover $K \cap \partial B(y,\rho)$ with an arc Z of $\partial B(y,\rho)$ of length $\frac{\rho}{2C}$, with C as in the Ahlfors-regularity condition (3)).

Set $\widetilde{K} = [K \cup Z] \setminus B(y,\rho)$. We pay $H^1(Z) = \frac{\rho}{2C}$, but we win $H^1(K \cap B(y,\rho)) \geq C^{-1}\rho \geq 2H^1(Z)$ by (3).

The main point is that by (10), we can find an extension $\widetilde{u}$ of $u_{\big|\partial B(y,\rho)\setminus Z}$ to $B(y,\rho)$, with

$$\int_{B(y,\rho)} |\nabla \widetilde{u}|^2 \leq C(\tau)\varepsilon\rho. \tag{11}$$

[First estimate the jump across Z, then extend linearly across Z, then use the Poisson kernel.]

Then $(\widetilde{u},\widetilde{K}) \in \mathcal{A}$, and coincides with (u,K) out of $\overline{B}(y,t)$. The definition of local almost minimizer should yield

$$\begin{aligned} H^1(K \cap B(y,\rho)) &\leq H^1(Z) + \int_{B(y,\rho)} |\nabla \widetilde{u}|^2 + rh(r) \\ &\leq H^1(Z) + C(\tau)\varepsilon\rho + rh(r) \end{aligned}$$

a contradiction if ε and $h(r)$ are small enough. □

Already here, the fact that $K \cap \partial B(y,\rho)$ often has at least 2 points allows K to separate $B(y,t)$ into regions. We'll see this again in the next proof.

Uniform rectifiability when n=2

Theorem [D.-Semmes]. For $(x, r) \in \Delta$, there is a regular curve $\Gamma \subset B(x, r)$, with constant $\leq C$, such that $K \cap B(x, r/2) \subset \Gamma$.

Definition. A regular curve is a (connected) curve Γ such that

$$\text{length}(\Gamma \cap B(x, r)) \leq Cr \quad \text{whenever } 0 < r < \text{diam}(\Gamma).$$

But we could also have taken Ahlfors-regular connected sets. The point is that regular curves are almost as nice as Lipschitz graphs or even C^1 curves.

Another way to say state the theorem: K is locally uniformly rectifiable.

A slightly stronger property, which also makes sense and holds in every dimension $n \geq 2$, is that K locally contains big pieces of Lipschitz graphs, i.e., that

There exist constants $\tau > 0$ and $C \geq 0$ such that, for all $(x, r) \in \Delta$, there is a C-Lipschitz graph $G \subset \mathbb{R}^n$ such that $H^{n-1}(K \cap G \cap B(x, r)) \geq \tau r^{n-1}$.

No proof for $n > 2$ here, no definition of uniform rectifiability when $n > 2$, or further advertisement for uniform rectifiability. Again, separation plays a big role in the proof.

When $n = 2$, the theorem follows from this

Main Lemma (big pieces of connected sets). There exists $C > 0$ such that for $(x, r) \in \Delta$ such that $h(r) \leq C^{-1}$, there is a connected set $F \subset B(x, r)$ with $H^1(F) \leq Cr$ and $H^1(K \cap F \cap B(x, r)) \geq C^{-1} r$.

Please trust: once we know this, the theorem is a consequence of local Ahlfors regularity, iterations, gluing, and optimizing.

Proof of the main lemma. Pick $p < 2$ close to 2, and $\varepsilon > 0$ small. By the corollary with Carleson meaures, we can find $y \in K \cap B(x, r/2)$ and $t \in [C^{-1}, r/2]$ such that $\omega_p(y, t) \leq \varepsilon$.

By Chebyshev, we can choose $\rho \in (t/2, t)$ such that

$$(12) \qquad \int_{\partial B(y,\rho) \setminus K} |\nabla u|^p \leq C \varepsilon^{\frac{p}{2}} \rho^{1-\frac{p}{2}}$$

as for (10) above. Since $H^1(K \cap B(y, t)) \leq 7t$ by the trivial estimate, we can also arrange that

(13) $\qquad K \cap \partial B(y, \rho)$ has at less than 20 points.

Denote by J_j, $1 \leq j \leq L$, the components of $\partial B(y,\rho) \setminus K$, and by m_k the mean value of u on J_k.

Claim. we can find j and $k \neq j$ such that $|J_j| \geq C_1^{-1}\rho$, $|J_k| \geq C_1^{-1}\rho$, and $|m_j - m_k| \geq C_2^{-1}\rho^{1/2}$.

Proof of claim.* Otherwise, cover $K \cap \partial B(y,\rho)$ and the short arcs J_j by a union Z of arcs of length $C_1^{-1}\rho$, and with $H^1(Z) \leq 100C_1^{-1}\rho < H^1(K \cap B(y,\rho))$ (if C_1 is large enough).
Then use (12) to extend $u_{|\partial B(y,\rho)\setminus Z}$ first to $\partial B(y,\rho)$, and then to $\overline{B}(y,\rho)$ with small energy $\int_{B(y,\rho)} |\nabla \widetilde{u}|^2$ (if C_2 is large enough).
Get a contradiction as for the concentration property.
□

Now let j and k be as in the claim.
By (12), $|u(z) - m_j| \leq (10C_2)^{-1}\rho^{1/2}$ for $z \in J_j$, and similarly $|u(z) - m_k| \leq (10C_2)^{-1}\rho^{1/2}$ for $z \in J_k$.
Suppose $m_j < m_k$. There is an interval $[a,b]$ in the middle of $[m_j, m_k]$, with $b - a \geq (2C_2)^{-1}\rho^{1/2}$ and

(14) $\quad u(z) < a < b < u(w)$ for $z \in J_j$ and $w \in J_k$.

(14) $\quad u(z) < a < b < u(w)$ for $z \in J_j$ and $w \in J_k$.

Let us apply the co-area formula to (a smooth modification of) u in $B(y,t) \setminus K$. For $t \in \mathbb{R}$, denote by $\Gamma_t = \big\{z \in B(y,t)\,;\, u(z) = t\big\}$ the level set. Then

$$\int_t H^1(\Gamma_t)\,dt \leq \int_{B(y,t)} |\nabla u| = t^{3/2}\omega_1(y,t)$$
$$\leq C t^{3/2}\omega_p(y,t) \leq C t^{3/2}\varepsilon$$

by Hölder and our choice of (y,t).
By Chebyshev, we can find $s \in [a,b]$ such that

$$H^1(\Gamma_s) \leq C\varepsilon t. \tag{15}$$

By (14), Γ_s separates J_j from J_k in $B(y,t) \setminus K$. Then $K \cup \Gamma_s$ separates J_j from J_k in $B(y,t)$.

By "elementary $2d$-topology", $[K \cup \Gamma_s] \cap B(y,t)$ contains a connected piece F that separates J_j from J_k in $B(y,t)$.

First $H^1(F) \leq H^1(\Gamma_s) + H^1(K \cap B(y,t)) \leq 8t$. Also $H^1(F) \geq \frac{1}{2}\mathrm{Min}\big\{|J_j|, |J_k|\big\} \geq \rho/(2C_1)$, and then $H^1(F \cap K) \geq H^1(G) - H^1(\Gamma_s) \geq \rho/(3C_1) \geq r/C$ (by (15) and if ε is small). $\square$

Lots of C^1 pieces in K

Here we assume that $h(r) \le Cr^\alpha$ for some $\alpha > 0$.

Theorem [Ambrosio, Fusco, Pallara]. For almost every $x \in K$, we can find $r > 0$ such that K coincides in $B(x,r)$ with a C^1 and 10^{-2}-Lipschitz graph.

Also see D. and Bonnet when $n = 2$, and the following improvement* (when $n > 2$) from Rigot:

There exists $C \ge 1$ such that whenever $(x,r) \in \Delta$ and $r \le C^{-1}$, we can find $y \in K \cap B(x,r/2)$ and $t \in [r/C, r/2]$, such that K coincides in $B(y,t)$ with a C^1 and 10^{-2}-Lipschitz graph.

Main point: if $K \cap B(x,r)$ is flat enough, and $\int_{B(x,r)} |\nabla u|^2$ is small enough*, then K coincides with a C^1 and 10^{-2}-Lipschitz graph in $B(x,r/2)$.

Recent improvement by A. Lemenant for $n = 3$: if $K \cap B(x,r)$ is close enough enough to a minimal cone, and $\int_{B(x,r)} |\nabla u|^2$ is small enough, then K is C^1-equivalent to a minimal cone in $B(x,r/2)$.

Comments:
- List of minimal cones below
- Connection with minimal sets and the Jean Taylor theorem
- Proof: control of many constants, improvement from $B(x,r)$ to $B(x,r/2)$, and iteration. Not today.

3. Blow up limits and global minimizers

Important developments, following A. Bonnet.

Again let (u, K) be a reduced local minimizer in $\Omega \subset \mathbb{R}^n$, with gauge function h.

Let $\{x_k\}$ be a sequence in K and $\{r_k\}$ a sequence in $(0, +\infty)$, with $\lim_{k \to +\infty} r_k = 0$. Also assume that $\lim_{k \to +\infty} r_k^{-1} \operatorname{dist}(x_k, \mathbb{R}^n \setminus \Omega) = +\infty$. Then $\Omega_k = r_k^{-1}(\Omega - x_k)$ tends to $\mathbb{R}^n$.
Often we just take $x_k = x$ (blow-up at a given point).

Set $K_k = r_k^{-1}(K - x_k)$ and $u_k(y) = r_k^{-1/2} u(r_k y + x_k)$. Then (u_k, r_k) is a local minimizer in Ω_k, with gauge function $h(r/r_k)$.

We can easly* extract subsequences so that K_k tends to a closed set K and u_k tends to some $u \in W^{1,2}_{loc}(\mathbb{R}^n \setminus K)$, in the sense that for every $\rho > 0$,

$$D_\rho(K, K_k) = \sup_{y \in K \cap B(0,R)} \operatorname{dist}(y, K_k) + \sup_{y \in K_k \cap B(0,R)} \operatorname{dist}(y, K) \tag{16}$$

tends to 0, and, for every connected component W of $\mathbb{R}^n \setminus K$, we can find constants $c_k = c_k(W)$ such that

$$\{u_k - c_k\} \text{ converges to } u \text{ uniformly on compact subsets of } W. \tag{17}$$

[In effect, ∇u_k converges to ∇u and we integrate.]

Theorem [Bonnet + …]: If (u, K) is a limit of the (u_k, K_k) as above, then (u, K) is a global minimizer in $\mathbb{R}^n$.

Definition of global minimizers soon. The main point of the proof is that K_k is uniformly concentrated, with uniform bounds, so

$$(18) \qquad H^{n-1}(K \cap U) \leq \liminf_{k \to +\infty} H^{n-1}(K_k \cap U)$$

for every open set $U \subset \mathbb{R}^n$. Then we consider a competitor $(\widetilde{u}, \widetilde{K})$ for (u, K), use it to construct a competitor $(\widetilde{u}_k, \widetilde{K}_k)$ for (u_k, K_k), use the almost minimality of (u_k, K_k), and use (18) to get a useful comparizon with (u, K).

Comments: with this sort of argument, a limit of reduced almost minimizers in Ω is a "topological almost minimizer" with the same gauge function h. This allows many compactness arguments.

Also, there is a proof by Dal Maso-Morel-Solimini $(n = 2)$ and Maddalena-Solimini $(n > 2)$ that Mumford-Shah minimizers exist. For instance, when $n = 2$, first minimize under the constraint that K has at most N components, and then take a limit. Not so simple, but it works.

Global minimizers

Denote by $\mathcal{A}$ the set of pairs (u, K) such that $K \in \mathbb{R}^n$ is closed, and $u \in W^{1,2}_{loc}(\mathbb{R}^n \setminus K)$.

A competitor for $(u, K) \in \mathcal{A}$ is a pair $(\widetilde{u}, \widetilde{K}) \in \mathcal{A}$ such that for R large,

$$\widetilde{u} = u \text{ and } \widetilde{K} = K \text{ out of } \overline{B}(0, R) \tag{19}$$

and

$$\begin{aligned} &\text{if } x,\, y \in \mathbb{R}^n \setminus [\overline{B}(0, R) \cup K] \text{ and } K \text{ separates} \\ &\qquad x \text{ from } y, \text{ then } \widetilde{K} \text{ separates } x \text{ from } y. \end{aligned} \tag{20}$$

By "K separates x from y", we mean that x and y lie in different connected components of $\mathbb{R}^n \setminus K$.*

Definition A global minimizer is a reduced pair $(u, K) \in \mathcal{A}$ such that

$$\begin{aligned} &H^{n-1}(K \cap \overline{B}(0, R)) + \int_{B(0,R)\setminus K} |\nabla u|^2 \\ &\qquad \leq H^{n-1}(\widetilde{K} \cap \overline{B}(0, R)) + \int_{B(0,R)\setminus \widetilde{K}} |\nabla \widetilde{u}|^2 \end{aligned}$$

whenever $(\widetilde{u}, \widetilde{K})$ is a competitor for (u, K) and R is so large that (19) and (20) hold. [A Dirichlet condition at infinity]

Expected: the study of global minimizers should be simple (no domain Ω, no image g or gauge function h), to the point that we could even give the full list.

And once we have information on the global minimizers, we shall return and get information on the local minimizers.

Examples of global minimizers when $n = 2$:
- $K = \emptyset$, u is constant;
- K is a line, u is constant on each component of $\mathbb{R}^2 \setminus K$;
- K is a Y, u is constant on each component of $\mathbb{R}^2 \setminus K$;
- The cracktip: $K = (-\infty, 0] \subset \mathbb{R}$ and

$$u(r\cos\theta, r\sin\theta) = C \pm \sqrt{\frac{2}{\pi}}\, r^{1/2}\, \sin\frac{\theta}{2}$$

for $r \geq 0$ and $|\theta| < \pi$.

The $120°$ angle in the Mumford-Shah conjecture comes from the Y (the only global minimizers for which u is locally constant are as above).

The fact that Cracktip is a global minimizer is true, but non trivial [D., Bonnet]. But is it a blow-up limit?

The constant $\sqrt{\frac{2r}{r}}$ is forced by balance betwen length and energy (otherwise, make the crack longer or shorter).

Strong Mumford-Shah conjecture: modulo rotations (for the cracktip), there is no other global minimizer.

What is known in $\mathbb{R}^2$?

Theorem [Bonnet]. If (u, K) is a global minimizer in $\mathbb{R}^2$ and K is connected, then (u, K) is in the list above.

Main ingredient: prove that $r \to \frac{1}{r} \int_{B(x,r) \setminus K} |\nabla u|^2$ is nondecreasing, and use limits.

Consequence: If (u, K) is a minimizer of the Mumford-Shah functional in $\Omega \subset \mathbb{R}^2$ and K_0 is an isolated component of K, then K_0 is a finite union of C^1 curves. Use blow-up limits and perturbation results near lines and sets Y.

Similarly, the strong Mumford-Shah conjecture would imply the standard one.

Léger's formula: if (u, K) is a global minimizer in $\mathbb{R}^2$,

$$2 \frac{\partial u}{\partial z}(z) = -\frac{1}{2\pi} \int_K \frac{dH^1(w)}{(z-w)^2}$$

for $z \in \mathbb{R}^2 \setminus K \approx \mathbb{C} \setminus K$ (Beurling transform of $H^1_{|K}$).

In particular, u is essentially unique given K.

Theorem [D., Léger]. If (u, K) is a global minimizer in $\mathbb{R}^2$ and $\mathbb{R}^2 \setminus K$ is not connected, then (u, K) is in the list above.

Etc. . .

What is known in $\mathbb{R}^3$?

Less, but this makes more interesting questions.

Examples of global minimizers in $\mathbb{R}^3$ for which u is locally constant. That is, minimal sets K in $\mathbb{R}^3$, with the topological constraint (20) for competitors:
- $\emptyset$;
- planes;
- products $\mathbb{Y}$ of a Y with an orthogonal line: $\mathbb{Y}$ is the union of three half planes bounded by a common line L and making $120°$ angles along L;
- cones $\mathbb{T}$ over the union of the edges of a regular tetrahedron centered at the origin (six infinite triangular faces bounded by four half lines).

Theorem [J. Taylor+D.] Every Mumford-Shah minimal set K in $\mathbb{R}^3$ is one of these cones.

Curiously recent, and no answer yet for Almgren minimal sets, where we still minimize $H^2(K)$ locally, but competitors for K are sets $\widetilde{K} = \varphi(K)$, where $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ is Lipschitz, with $\varphi(x) = x$ out of some ball.

Recall that Mumford-Shah competitors for K are sets $\widetilde{K}$ such that $\widetilde{K} = K$ out of some big ball $\overline{B}$, and $\widetilde{K}$ separates x and $y \in \mathbb{R}^3 \setminus [K \cup \overline{B}]$ whenever K separates them. Almgren competitors are Mumford-Shah competitors. so there may be more Almgren minimal sets.

So we control the global minimizers for which u is locally constant.

Even locally: A. Lemenant's result says that if (u, K) is a local minimizer in $\Omega \subset \mathbb{R}^3$, with $h(r) \leq Cr$, and if one of the blow-up limits of K at x is one of the cones above, then x has a neighborhood B where K is C^1-equivalent to this cone and u is smooth in each component of $B \setminus K$.

Example where u is not constant: cracktip times a line, so $K = (-\infty, 0] \times \{0\} \times \mathbb{R}$ (a vertical half plane) and

$$u(r\cos\theta, r\sin\theta, z) = C \pm \sqrt{\frac{2}{\pi}}\, r^{1/2} \sin\frac{\theta}{2}.$$

Comments:
- Not too hard to check the minimality, by slicing;
- Here u is essentially unique given K [Lemenant];
- This is the only known (or suspected) global minimizer in $\mathbb{R}^3$ where u is not locally constant.

Questions

- Are there other global minimizers? What happens when you cut cut a $\mathbb{Y}$ locally*? I had suggestions, but B. Bourdin and B. Merlet don't seem to like them.

- Can we first describe (u, K) when K is a cone? [Lemenant: u is homogeneous of degree $1/2$; hence connections with the spectrum of Δ on $\partial B(0,1) \setminus K$.]

- Is u essentially unique given K? [True in the examples above.]

- Suppose K contains a small (flat) disk; is it one of the examples above?

- Suppose u is constant somewhere?

- Is every connected component of $\mathbb{R}^3 \setminus K$ a John domain (true when $n = 3$); how many components?

And, even in dimension 2,

- prove the strong Mumford-Shah conjecture

- Does Cracktip really show up as the blow-up limit at x of (u, K) for some Mumford-Shah minimizer in a domain?

- Suppose it does, can K spiral at x? [I think not, proof by Bonnet.]

- Would the list of global minimizers change if we allowed u to be valued in $\mathbb{R}^k$?

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