Regularity of minimal and almost minimal sets and cones: J. Taylor's theorem for beginners

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Abstract. We discuss various settings for the Plateau problem, a proof of J. Taylor's regularity theorem for 2-dimensional almost minimal sets, some applications, and potential extensions of regularity results to the boundary.

1. Introduction

The main purpose of this text is to present some of the techniques used to prove the local regularity of minimal and almost minimal sets in a domain of $\mathbb{R}^n$. The notion of minimality that we shall use is a minor modification of Almgren's notion of "restricted sets" (see [Al3]), which seems to be very good to describe soap films or bubbles away from the boundary sets where the films are attached.

We shall first present a natural context where Almgren minimal sets arise (typically, Plateau problems for soap films). This will provide a motivation for the rest of the paper, but we shall refer to an oncoming paper in a conference proceedings [D8] for more detail about the different settings for Plateau problems.

In the present text, we shall insist more on regularity properties of minimizers away from the boundary, and in particular we shall explain a proof of Jean Taylor's theorem from [T2], which says that if E is a two-dimensional soap film or bubble in $\mathbb{R}^3$ (we shall say, an almost minimal set), then each interior point of E has a neighborhood where E is C^1-equivalent to a minimal cone. In this case, the list of minimal cones is known; there are only three simple types, which can be easily seen in real soap films and bubbles.

Incidentally, we give a slightly different meaning to films and bubbles here. For soap films, we think about sets that locally minimize the area, which in the case of smooth surfaces would imply that the mean curvature vanishes. For soap bubbles, there may be air trapped inside, which leads to minimizing a slightly more complicated functional and would in the smooth case yield surfaces with constant mean curvature (given by the difference between two pressures). In the case of bubbles, and also if we add smaller forces, our sets will only be almost minimal (but most of the known theorems will still be true with similar proofs).

After the initial presentation of Plateau problems (and some easier variants) in Section 2, we shall define minimal and almost minimal sets, and list the known general interior regularity results, which happen to be valid in larger classes of

2010 *Mathematics Subject Classification.* 49Q20.

quasiminimal sets: these sets (of all dimensions d) are locally Ahlfors-regular and uniformly rectifiable, the Hausdorff measure $\mathcal{H}^d$ is lowersemicontinuous along sequences of reduced (uniformly) quasiminimal sets, and finally the various classes are stable under limits (Section 3).

Then we shall state J. Taylor's result and a partial generalization to two-dimensional almost minimal sets in higher ambient dimensions, and describe the main ingredients of a proof (a long Section 4).

In Section 5 we shall address a Bernstein problem (if E is a reduced minimal set of dimension 2 in the whole $\mathbb{R}^3$, is it a cone?), and its variant in the slightly different Mumford-Shah setting.

In Section 6 we shall mention two cases of simpler variants of a Plateau problem (but without boundary conditions) for which the regularity results presented above lead to existence theorems.

We shall also rapidly present a program of extension of some regularity results, all the way up to the boundary, for potential solutions of some type of Plateau problem, with sliding boundary conditions. See Section 7.

This text partially relies on transparencies that the author used for series of lectures in Evian (2008) and Grenoble (2010), and its content is somewhat different from the lectures in Montreal. The author wishes to thank the organizers of the seminar for their kind welcome and perfect organization, Ken Brakke for the authorization to use pictures from his site, T. De Pauw, F. Morgan, and V. Feuvrier for useful discussions and help with the references, and the referee for a careful and indulgent reading.

2. Plateau problems

Joseph Plateau (1801-1883) was really interested in soap films, and in particular described the typical singularities that will be mentioned below. See the book [Pl] from 1873, which is also very interesting for the description of lots of other funny experiments in physics. Some people also mention Lagrange (near 1760) in connection with the Plateau problem; he was clearly interested in variational problems, but I am not sure about soap films in particular.

Plateau's problem usually refers to the question of existence of sets of minimal area that are bounded by a given set (typically, a curve in $\mathbb{R}^3$). Everyone agrees on this, but the precise meaning of the words "area" and "bounded by" may differ considerably. This is why I decided to talk about Plateau problems in the plural.

In this section we shall mention a few ways to state a Plateau problem; this will provide a good motivation for the rest of the paper, but we shall announce no new solution to any Plateau problem. Please forgive the very schematic descriptions below; the author will try to be more precise and give more simple examples in [D8].

2.1. Douglas and the parameterized surfaces. Let $\Gamma \subset \mathbb{R}^n$ be a reasonably smooth simple loop. Denote by D the unit disk in $\mathbb{R}^2$, and parameterize Γ by $g : \partial D \to \mathbb{R}^n$. The simplest way to state a Plateau problem is to look for Lipschitz mappings $f : D \to \mathbb{R}^n$, such that $f_{|\partial D} = g$, and which minimize the area

$$A(f) = \int_D J_f(x)dx, \tag{2.1}$$

where $J_f(x)$ denotes the positive Jacobian of f at x.

This looks nice, but there are two problems with this approach. The first one is that many functions g and even more functions f may describe the same objects Γ and $f(D)$, and it is not always clear which ones to choose. Also, and as a consequence of this too, we get the following obvious complication when we try to prove an existence result. Take a minimizing sequence $\{f_k\}$; this means that

$$\lim_{k\to+\infty} A(f_k) = \inf\big\{A(f)\,;\, f : D \to \mathbb{R}^n \text{ is Lipschitz and } f_{|\partial D} = g\big\}. \tag{2.2}$$

We cannot assume that the Lipschitz constant for f_k stays bounded (we would be solving a quite different problem). So, even if (usually after extracting a subsequence) the sets $f_k(D)$ converge nicely to a set E, it is unlikely a priori that the f_k will converge to an acceptable limit f, instead of having badly degenerating Lipschitz constants, or even that E will remember something useful from (2.2).

In the present case where Γ is a curve and we look for a 2-dimensional surface, we can decide to use conformal parameterizations, which have the good properties to exist in dimension 2, and to have good compactness properties. This approach was taken (among many others) by R. Garnier [Ga], and Tibor Radó [Ra] (1930). In 1931, J. Douglas [Do] obtained an optimal existence theorem, using the following simple but bright idea. We decide that f will be the harmonic extension of g in D (not unrealistic given what conformal parameterizations can be), translate everything in terms of g, and get that we just need to minimize the functional

$$B(g) = \int_0^{2\pi}\int_0^{2\pi} \frac{\sum_{j=1}^n |g_j(\theta) - g_j(\varphi)|^2}{\sin^2\left(\frac{\theta-\varphi}{2}\right)}\, d\theta d\varphi, \tag{2.3}$$

where the g_j are just the coordinates of g. It turns out that this is easy to do.

This is beautiful, and the paper [Do] is very pleasant to go through, but the problem that Douglas solves is not exactly what Plateau intended; the surface $g(D)$ that he gets may cross itself, and near a crossing point everything happens as if $f(D)$ was composed of completely independent pieces that happen to pass through the same point. Soap films do not do that: the different pieces would interact and give different singularities. Typically, they would do one of two things. Either the two pieces would merge along some set (think about pinching two pieces of surfaces together), taking advantage that this makes the surface measure of the set $f(D)$ (which is what the soap film really minimizes) strictly smaller than $A(f)$. Or some hole would open, giving a minimal set that has a different topology and is no longer injectively parameterized by a disk.

2.2. Reifenberg and the homology groups. Even if our boundary data Γ is a nice curve, we do not necessarily want to assume that our solutions are smooth surfaces, so we shall use Hausdorff measures to define their area. Let us recall that for $d \geq 0$ and $E \subset \mathbb{R}^n$, the d-dimensional (exterior) Hausdorff measure of E is

$$\mathcal{H}^d(E) = \lim_{\delta\to 0} \mathcal{H}^d_\delta(E), \tag{2.4}$$

where

$$\mathcal{H}^d_\delta(E) = c_d \inf\Big\{\sum_{j\in\mathbb{N}} \mathrm{diam}(D_j)^d\Big\}, \tag{2.5}$$

c_d is a normalizing constant, and the infimum is taken over all coverings of E by a countable collection $\{D_j\}$ of sets, with $\mathrm{diam}(D_j) \leq \delta$ for all j.

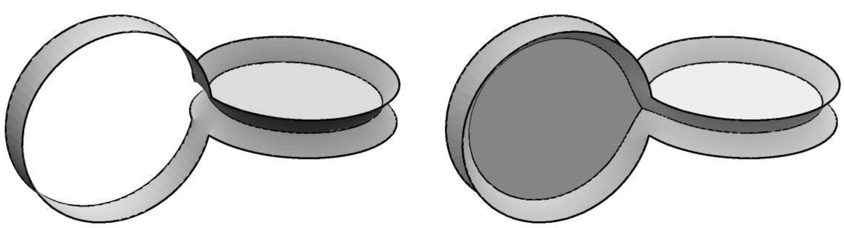

FIGURE 1. Two films that don't fit Reifenberg's setting (K. Brakke)

Let us choose the normalizing constant c_d so that $\mathcal{H}^d$ coincides with the Lebesgue measure on subsets of $\mathbb{R}^d$; then $\mathcal{H}^d(E)$ is the total surface of E when E is a smooth d-dimensional surface. The advantage is that $\mathcal{H}^d(E)$ is defined for all $E \subset \mathbb{R}^n$, and its restriction to Borel sets is a Borel measure (not σ-finite, but this is all right).

Return to the Plateau problem. Let $1 \leq d < n$ and a boundary set Γ be given (for instance, $d = 2$ and Γ is a nice closed curve). We want to find a "surface" E "spanned" by Γ, and for which $\mathcal{H}^2(E)$ is minimal.

For Reifenberg [R1] (1960), this last condition means that E is a compact set that contains Γ, and the boundary condition is stated in terms of Čech homology on some commutative group G. We require the inclusion $i : \Gamma \to E$ to induce a trivial homomorphism from $\check{H}_{d-1}(\Gamma; G)$ to $\check{H}_{d-1}(E; G)$. Or we could also take a subgroup of $\check{H}_{d-1}(\Gamma; G)$, and require that its image vanishes in $\check{H}_{d-1}(E; G)$. In other words, we select a certain number of elements in $\check{H}_{d-1}(\Gamma; G)$, and we want these elements to be boundaries inside E. In the simple case when $d = 1$ and Γ is a curve, we can take the obvious generator γ of $\check{H}_1(\Gamma; G)$, and require that E contains (the support of) a chain that fills γ.

Then we minimize the area $\mathcal{H}^d(E)$ under these constraints. Reifenberg proves the existence of minimizers in all dimensions, but when $G = \mathbb{Z}_2$ or $G = \mathbb{R}/\mathbb{Z}$. This is a Beautiful (although technical) proof by hands, with minimizing sequences and initial haircuts to make the sets look nicer. Recently, De Pauw obtained the existence in the 2-dimensional case when the boundary is a finite union of curves and $G = \mathbb{Z}$ (using currents); he also proved that in that case the infimum for this problem is the same as for the size-minimizing currents of the next subsection. But even then it is not known whether we can define size-minimizing currents supported on the sets that he gets. See [Dp].

Reifenberg's solutions are nice and seem to give a good description of many soap films. Using finite groups G like $\mathbb{Z}_2$, one can even get non-orientable sets E like Möbius strips. But there are some "real-life" soap films spanned by a curve that cannot be obtained as Reifenberg solutions. The two films of Figure 1 are like this (there happens to be a problem with the orientation); see K. Brakke's home page for more examples, and maybe [D8] for a short discussion.

Anyway, a more general existence result (as in [Dp] but in the general case) would be very welcome.

2.3. Plateau problems for currents. The most celebrated and successful model is probably the description of films in terms of currents, initiated by Federer, Fleming, De Giorgi, and others. See for instance [FF], [Fe1].

The most logical way to try to solve a Plateau problem would be to take a minimizing sequence of (smooth) surfaces, take a subsequence that converges in some reasonable topology, and show that the limit is a solution to our problem. The obvious difficulty is that when we choose a topology that is so rough that subsequences converge, the limit is very unlikely to be smooth.

The approach with currents is in the same spirit as for weak solutions for PDE's: first set the initial problem on a much larger class where pleasant compactness theorems exist, prove existence theorems in this class, and then show that the weak solutions that we just found are in fact much more regular than expected, and are acceptable solutions for our initial problem. We need a few definitions.

A d-dimensional current is a continuous linear form on the space of smooth d-forms. This is thus the same as a d-vector valued distribution. In fact, most of the distributions that will be used here are (d-vector valued) measures.

The most basic example is the current S' of integration on any smooth, oriented surface S of dimension d, which is simply defined by $\langle S', \omega \rangle = \int_S \omega$ for every d-form ω. But the point of the approach is that currents provide a much larger class of objects, with good compactness properties.

Another standard example that is relevant here is the rectifiable current T defined on a d-dimensional rectifiable set E such that $\mathcal{H}^d(E) < +\infty$, on which we choose a measurable orientation τ and an integer-valued multiplicity m.

Recall that a rectifiable set of dimension d is a set E such that $E \subset N \cup \bigcup_{j \in \mathbb{N}} G_j$, where $\mathcal{H}^d(N) = 0$ and each G_j is a C^1 embedded submanifold of dimension d. But we could have said that G_j is the Lipschitz image of a subset of $\mathbb{R}^d$, and obtained an equivalent definition. We shall only consider sets E such that $\mathcal{H}^d(E) < +\infty$ here. For such a set E and $\mathcal{H}^d$-almost every $x \in E$, E has what is called an approximate tangent d-plane $x + V(x)$ at x, which of course coincides with the usual tangent plane in the smooth case. A measurable orientation can be defined as the choice of a d-vector $\tau(x)$ that spans $V(x)$, which is defined $\mathcal{H}^d$-almost everywhere on E and measurable. We set

$$\langle T, \omega \rangle = \int_E m(x)\, \omega(x) \cdot \tau(x)\, d\mathcal{H}^d(x), \tag{2.6}$$

where we let the reader guess the precise definition of the number $\omega(x) \cdot \tau(x)$ when ω is a (smooth) d-form.

The boundary of a d-dimensional current T is defined by duality with the exterior derivative d on forms, by

$$\langle \partial T, \omega \rangle = \langle T, d\omega \rangle \quad \text{for every } (d-1)\text{-form } \omega. \tag{2.7}$$

When S is a smooth oriented surface with boundary Γ, Green's theorem says that $\partial S' = \Gamma'$. Notice that $\partial\partial = 0$, just because $dd = 0$.

A normal current is a rectifiable current T such that ∂T is rectifiable too. We will like to work with normal currents here; the additional constraint on ∂T will not disturb us, because we are interested in solutions of the equation $\partial T = S$, where S defines the boundary constraint and is assumed to be nice.

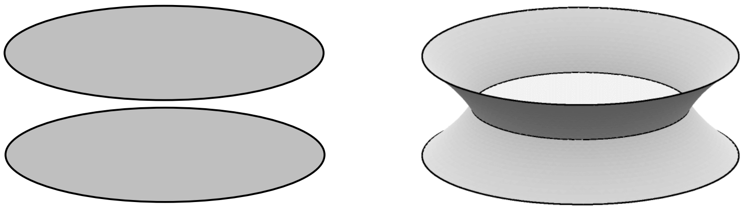

FIGURE 2. A mass minimizer and a size minimizer with the same boundary (composed of 2 circles oriented the same way)

There are two interesting quantities associated to a current. The first one is the mass of T, which is just the operator norm of T, where we put a L^∞-norm on the vector space of d-forms. That is, we see T as a measure, and compute its total mass. When T is the rectifiable current given by (2.6),

$$\text{Mass}(T) = \int_E |m(x)|\, d\mathcal{H}^d(x). \tag{2.8}$$

But we shall also consider the size of T, which when T is given by (2.6) is equal to

$$\text{Size}(T) = \mathcal{H}^d\big(\{x \in E\,;\, m(x) \neq 0\}\big). \tag{2.9}$$

Return to Plateau's problem. The classical way to state it in the setting of currents is to take a $(d-1)$-dimensional integral current S, with $\partial S = 0$, and minimize Mass(T) among all the d-dimensional currents T such that $\partial T = S$. Of course the condition $\partial S = 0$ is needed if we want to solve $\partial T = S$, since $\partial\partial = 0$.

With this setting, the use of currents is a great success. There is a compactness theorem that says the following. Let $\{T_k\}$ be a sequence of normal currents of dimension d, with supports in a fixed compact set in $\mathbb{R}^n$, and assume that Mass(T_k) + Mass$(\partial T_k) \leq M$ for some fixed $M < +\infty$. Then there is a subsequence that converges (in some weak norm) to a normal current T. Moreover,

$$\text{Mass}(T) \leq \liminf_{k\to+\infty} \text{Mass}(T_k) \quad \text{and} \quad \text{Mass}(\partial T) \leq \liminf_{k\to+\infty} \text{Mass}(\partial T_k). \tag{2.10}$$

We apply this to a minimizing sequence for the problem above (assuming that S is an integral current that lives on a compact set), and get a current T that solves $\partial T = S$ and minimizes Mass(T). See [FF] and [Fe1].

Moreover, there are good regularity theorems for mass-minimizing currents, and the solution T that we find is quite nice. For instance, if $d = n-1 \leq 7$, the closed support of T is a smooth manifold away from the support of S [Fe2].

Unfortunately, mass-minimizing is not a great model for soap films. We can guess that, because we know from experience that unlike mass minimizers, 2-dimensional soap films can have singularities. The problem is the same as for the solutions of Douglas: rather than minimizing the mass of T, we should probably minimize its size Size(T). The difference is illustrated in Figure 2, where the current suggested on the right has multiplicity 2 on the central disk, and a smaller size than the sum of the two disks on the left.

But the situation for size minimizers is far from clear. Even when $d = 2$ and S is the current of integration on a smooth curve Γ, there is no general existence result for an integral current T such that $\partial T = S$ and Size(T) is minimal. The main difficulty is that if we want to apply the compactness theorem above to a minimizing sequence $\{T_k\}$, it is not clear that the masses $\mathrm{Mass}(T_k)$ will stay quietly bounded, because we only control the sizes.

There are some encouraging recent partial results by T. De Pauw and R. Hardt [DpH] where the scheme would be to minimize $\mathrm{Size}(T) + \varepsilon \mathrm{Mass}(T)$, get some uniform control on the minimizers, and let ε tend to 0. Also, when $d = 2$ and S comes from a finite collection of smooth closed curves Γ, De Pauw showed that the infimum for the problem is the same as for Reifenberg's problem above, but for the moment failed to show that the solutions for Reifenberg's problem, which exist, give size-minimizing currents.

We should also mention that some real soap films are not orientable (they may be Möbius bands, for instance), and then the problem $\partial T = \Gamma$ does not fit well. Some ad hoc solutions to this exist (see for instance the clever constructions in [Br4], with covering spaces), but apparently no general scheme is available.

One can go further in the direction of weak solutions, where the idea is that maybe currents are not general enough to describe all the soap films. Almgren [Al2] used varifolds to give another proof of existence for the Reifenberg solutions. J. Harrison and H. Pugh state the problem and get existence results in terms of even more general objects called differential k-chains (see [Ha1], [Ha2]); the next question in such a context is probably a description of what the obtained minimizers look like, and hence whether they also minimize in a smaller category.

2.4. Sliding Almgren minimizers. Let us propose yet another Plateau problem, where we return to sets and a possibly more natural existence problem, but also where we have no good existence result so far.

We give ourselves a finite collection of simple compact boundary sets $\Gamma_j \subset \mathbb{R}^n$, $0 \leq j \leq j_{max}$, and an initial candidate E_0 (a compact set such that $\mathcal{H}^d(E_0) < +\infty$).

We want to minimize $\mathcal{H}^d$ in a class $\mathcal{F}(E_0)$ of deformations of E_0 that we define now. We shall say that E is a sliding deformation of E_0, relative to the boundary pieces L_j, when $F = \varphi_1(E_0)$ for a one-parameter family $\{\varphi_t\}$, $0 \leq t \leq 1$, of mappings such that

$$(t,x) \to \varphi_t(x) : [0,1] \times E_0 \to \mathbb{R}^n \text{ is continuous,} \tag{2.11}$$

$$\varphi_0(x) = x \text{ for } x \in E_0, \tag{2.12}$$

$$\varphi_t(x) \in \Gamma_j \text{ when } 0 \leq j \leq j_{max} \text{ and } x \in \Gamma_j, \tag{2.13}$$

and

$$\varphi_1 \text{ is Lipschitz.} \tag{2.14}$$

We decided to require (12.14) mostly by tradition, but this is negotiable and anyway we do not require any Lipschitz bound for φ_1. Let us denote by $\mathcal{F}(E_0)$ the class of sliding deformations of E_0 (relative to the boundary pieces L_j), and set

$$m = \inf \big\{ \mathcal{H}^d(E) \, ; \, E \in \mathcal{F}(E_0) \big\}. \tag{2.15}$$

Obviously $m \leq \mathcal{H}^d(E_0) < +\infty$, but we have to choose E_0 and the L_j so that $m > 0$, because otherwise the problem below is not too interesting. Then we ask for the existence of $E \in \mathcal{F}(E_0)$ such that $\mathcal{H}^d(E) = m$.

A few comments are in order. First, the use of deformations as above is not really new: the definition by Almgren of restricted sets in [Al3] uses competitors for E_0 that are Lipschitz images $\varphi(E_0)$, even though there is no boundary piece L_j that one wants to preserve there. A similar problem is implicitly mentioned in [T2], in the proof of minimality for the cone $\mathbb{T}$ over the edges of a regular tetrahedron. The Surface Evolver software [Br2] also allows this as an option.

We think of our initial set E_0 as an elastic shower curtain, which is attached to the boundary pieces so that it can slide along them, but not be detached.

The most obvious case is when there is a single Γ_j, which is a nice curve, but we can also take 2-dimensional boundaries (think about a soap film leaning on a tube), or mixed problems. Incidentally, 2-dimensional boundaries in $\mathbb{R}^3$ seem interesting too, and could be easier to treat than curves.

The author likes this problem because it has a nice physical flavor, and also because it is simple and seems to give a lot of flexibility. We do not need to think too much about orienting sets, choosing a group, or defining acceptable multiplicities on them, the initial set E_0 will decide about many things for us, and various choices of E_0 will probably lead to most realistic examples.

We could also ask for local minimizers, where we would only allow as competitors the sets $E \in \mathcal{F}(E_0)$ for which $\mathcal{H}^d(\varphi_t(E_0))$ stays below a certain threshold for all t. Of course it is fairly easy to conceive situations where a real soap film E exists and is stable, even though there is a long homotopy that deforms E into a point while preserving the boundary pieces.

But the main problem with this definition of the Plateau problem is that so far there is no good existence result. Again we are messing around with parameterizations (why should there be a nice parameterization φ_1 for the limit of a minimizing sequence?), but we can always hope that we will be able to find minimizing sequences with a good control on parameterizations. See Section 6 for a simpler example where something like this can be done.

In spite of the potential difficulties with this problem, we would like to convince the reader that it is probably interesting to study the regularity of its solutions. Firstly, it is probably a good idea to study the boundary regularity of solutions of Plateau problems in general; to the author's knowledge, very little has been done in this direction, and the sliding setting that we just described seems to be fairly appropriate, both because we can keep some control on the minimizers (because we authorize some sliding of the solutions at the boundary), and because solutions to other Plateau problems (Reifenberg solutions, or the closed supports of size-minimizing currents) may be sliding minimizers as well (when we choose them as the initial E_0). [See [D8] for a rapid verification of this fact.] Also, understanding the boundary behavior is possibly an important step if we want to produce solutions, both because we will know better what to expect and because the proofs of regularity for minimizers often come with interesting ways to produce better competitors (and hence better minimizing sequences). We shall give an example of this, in a much simpler context where there is no boundary condition, in Section 6.

Let us say a little more on this topic. One nice way to try and produce existence results is to use weak solutions and an appropriate compactness result. But then it is important to see how regular these weak solutions are, and it would be nice to do this even near the boundary. It is also possible that the compactness result, or the verification of boundary conditions for the weak limit of a minimizing sequence, will be easier if we know that we can restrict to some subclass where we already have some control near the boundary. At the opposite end, it is possible (and in principle much easier) to produce what the author would like to call lazy solutions to the Plateau problem. In the context of sets (as in this section), a lazy solution would be a set E such that $\mathcal{H}^d(E) \leq \mathcal{H}^d(F)$ for every $F = \varphi_1(E)$, but only for all the deformations $\{\varphi_t\}$ as above that fix every point of the given boundary set Γ, or even possibly of a neighborhood of Γ. It is easier to produce such solutions, because you just need to keep a control of a locally minimizing sequence away from Γ, and you don't even need to look precisely at what happens along Γ. But then we cannot say exactly which Plateau problem is solved by our set E, and at the same time its boundary behavior will probably be much harder to predict. In spite of this, the author does not know of any example of a lazy minimizer that would look suspiciously irregular near the boundary, and after all the minimality away from Γ could imply some regularity at the boundary too.

See Section 7 for some first regularity results for sliding minimizers.

2.5. Other minimization problems. The Plateau problem has the advantage of being simple and celebrated, but there are other problems with the same flavor, and which are possibly easier to solve. Here we partially repeat suggestions made in [D5]. We shall also return to this in Section 6, with just a little more detail.

For instance, fix a simple domain $\Omega \subset \mathbb{R}^n$, preferably closed (so that our sets E can lay along a boundary) and with some holes (so that the problem below is not trivial). Also let $M > 0$ and a continuous bounded function $g : \Omega \to [1, M]$ be given, and set

$$J_g(E) = \int_E g(x)\, d\mathcal{H}^d(x) \text{ for closed sets } E \subset \Omega. \tag{2.16}$$

Then fix a class $\mathcal{F}$ of closed subsets, and look for $E \in \mathcal{F}$ that minimize $J_g(E)$.

It is reasonable to restrict to classes that are stable under small deformations in Ω. That is, if $E \subset \mathcal{F}$ and if $\{\phi_t\}$, $t \in [0, 1]$, is a continuous one-parameter family of mappings $\varphi_t : E \to \Omega$ such that (2.11)-(2.14) hold with the single $\Gamma_j = \Omega$, then $\varphi_1(E) \in \mathcal{F}$.

Maybe also, if we want to restrict to localized deformations, let $r_0 > 0$ be given and only require that $\varphi_1(E) \in \mathcal{F}$ for families $\{\phi_t\}$ such that in addition there are balls B_t of radius r_0 such that for $0 \leq t \leq 1$,

$$\varphi_t(x) = x \text{ for } x \in E \setminus B_t \text{ and } \varphi_t(E \cap B_t) \subset B_t. \tag{2.17}$$

The additional flexibility provided by the weight g is not so much the issue here; we added it to make the existence and computation of minimal sets less trivial in the following examples. Also, we put this example in a different subsection because the only left boundary piece Ω plays a much smaller role than above, and we expect to get existence results more easily than with the standard Plateau problems.

Now there are different types of classes $\mathcal{F}$ that one can use, with possibly different answer. We expect this to depend mostly on how easy it will be to prove that the limit of a minimizing sequence in $\mathcal{F}$ will lie in $\mathcal{F}$ as well.

Example 2.18: separation conditions. When $d = n - 1$, we can give ourselves two or more sets A_j in $\mathbb{R}^n \setminus \Omega$, and take for $\mathcal{F}$ the class of sets that separate them. A simple case is when Ω is an annulus, or the difference between a big ball and a smaller solid torus, and the A_j are the components of the complement. Here and below, we need to select Ω and the A_j so that $\inf_{E \in \mathcal{F}} J_g(E) > 0$, which is why we

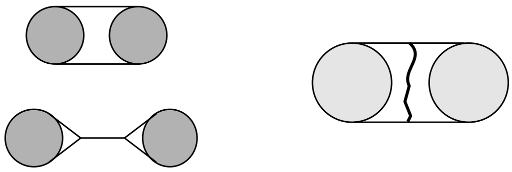

FIGURE 3. Suggested solutions for Example 2.20. On the left: 2 possibilities for E (d=2). On the right: a section of the solution with a wire (d=3)

prefer when Ω is not trivial topologically. For such separation problems, we can state the problem in terms of the separated components, use the compactness properties of BV to find sets of finite perimeter that minimize, and obtain minimizers for our initial problem. We do this in [DS2], for instance, but this is very classical.

Example 2.19: homology conditions. We try to do the same thing as before, but with possibly higher codimensions. We select smooth surfaces ω_j in $\mathbb{R}^n \setminus \Omega$, that represent non trivial elements in the $(n-d-1)$-dimensional homology of $\mathbb{R}^n \setminus \Omega$, and we let $\mathcal{F}$ be the collection of closed sets $E \subset \Omega$ for which ω_j still represents a nonzero element in the homology of $\mathbb{R}^n \setminus E$. Existence results exist, following X. Y. Liang's thesis [Li1]. See Section 6 and [Li3].

Example 2.20: deformations of a given set. We just proceed as in the last section: we give ourselves an initial closed set $E_0 \subset \Omega$, and call $\mathcal{F}(E_0)$ the class of continuous deformations $E = \varphi_1(E_0)$, where the φ_t are as in (2.11)-(2.13) with the single $\Gamma_j = \Omega$. That is, we are allowed to deform E_0, but points are not allowed to leave Ω. We may (or not) require that φ_1 be Lipschitz, as in (2.14), and the author does not know wether this changes the problem.

For instance, we can take $n = 2$, $d = 1$, $\Omega = \mathbb{R}^2 \setminus [B_1 \cup B_2]$ (two disjoint open balls), $g = 1$, and $E_0 = \partial B(0, R)$ (with R large). Here the minimizers are easy to guess; two cases can occur, depending on whether the balls are far from each other; see the left part of Figure 3.

Or we can take $n = 3$, $d = 2$, $\Omega = \mathbb{R}^3 \setminus A$ for some open solid torus A, $g = 1$, and $E_0 = \partial A$. Again two cases occur; when A is quite thin compared to its diameter, the minimizer should be ∂A itself; when A is quite thick, the minimizer should be the boundary of the convex hull of A, plus an additional wire, which is part of the topological problem but plays no role in the minimization. See the right part of Figure 3.

Example 2.20 may look like Examples 2.18 and 2.19, but here we do not say what topological properties of E_0 prevent it from disappearing into a lower-dimensional subset; thus the minimization problem of Example 3 may be more precise than in the previous example. Also, in more complicated situations, it may ve very hard for a nonspecialist to decide whether different topological conditions will lead to really different classes $\mathcal{F}$ or minimizers.

We shall return to this in Section 6 and say why in some cases, existence results can be derived from regularity results for minimizers.

3. Almost minimal sets; general regularity results and limits

We shall now focus on the interior regularity of potential solutions of Plateau problems for sets. We expect the boundary regularity to be somewhat more complicated. We shall only say a few fords about it in Section 7, and in the meantime, we stay inside.

3.1. Almost minimal sets, reduced sets. The best descriptions of soap films or bubbles away from the boundary seem to given by Almgren's notion of restricted sets, or variants. The definition of almost minimal sets that we give now is a minor modification of Almgren's definition, which we just simplify a little.

We work in an open set $U \subset \mathbb{R}^n$, to make the definition local and avoid boundary problems. We also use a small gauge function $h : \mathbb{R}_+ \to \mathbb{R}_+$ to account for perturbations (small additional forces, gently inhomogeneous space, etc.), but please feel free to take $h = 0$ and concentrate on minimal sets. For the moment we just assume that h is continuous, nondecreasing, and such that

$$\lim_{r \to 0} h(r) = 0, \tag{3.1}$$

but a standard choice would be $h(r) = Cr^\alpha$, with $\alpha > 0$ and $C \geq 0$.

We consider closed sets $E \subset U$ with locally finite $\mathcal{H}^d$-measure and (to make things simpler) only define competitors F for E in compact balls $B \subset U$.

Definition 3.2. Let $B = \overline{B}(x, r) \subset U$ be a compact ball in U. A competitor for E in B is a set $F = \varphi(E)$, where $\varphi : U \to U$ is Lipschitz (but no bounds are required), with

$$\varphi(y) = y \text{ for } y \in U \setminus B, \text{ and } \varphi(B) \subset B. \tag{3.3}$$

And we say that the closed set $E \subset U$ is an almost minimal set in U, with gauge function h, when for every compact ball $B = \overline{B}(x, r) \subset U$, $\mathcal{H}^d(E \cap B) < +\infty$ and

$$\mathcal{H}^d(E \cap B) \leq \mathcal{H}^d(F \cap B) + r^d h(r) \tag{3.4}$$

for every competitor F for E in B.

Remarks. Again this is a minor variation of Almgren's definition, which also allowed competitors in other compact subsets of U, and used an accounting that was slightly different from (3.4). A few other variants exist, but they would not be significantly different for what we want to say here.

It is an important feature of the definition that φ is **not** required to be injective. That is, we are allowed to compare E with other sets obtained from E by pinching and merging pieces. Generally speaking, in these problems the definition of the competitors is as important as the accounting.

Note that since B is convex, $\varphi = \varphi_1$ for a one-parameter family of continuous mappings φ_t that satisfy (3.3) (take $\varphi_t(x) = t\varphi(x) + (1-t)x$).

It is then clear that solutions of the sliding Plateau problem mentioned in Section 2.4 are minimal sets in the complement of the Γ_j, or rather of the $\partial\Gamma_j$ (when some of the Γ_j are n-dimensional), but this stays true for other Plateau problems. For instance, the closed support of a size minimizing current T is minimal in the complement of the (closed) support of the current S that defines the boundary condition $\partial T = S$. See for instance [D8].

The fact that we require φ to be Lipschitz is useful to prove this last fact, and does not make the proofs of regularity of E any harder; otherwise the Lipschitzness of φ is not so important. In the opposite direction, J. Harrison observed that it may be useful to show that we get the same class of almost minimizers if we require the functions φ in Definition 3.2 to be smooth. This seems true, but we did not write the proof yet.

A certain number of (unfortunately not so precise) regularity results hold in all dimensions and codimensions, which we shall state soon.

But let us first observe that even minimal sets of dimension 1 may have singularities. Indeed, let $Y \subset \mathbb{R}^2$ be the union of three half lines emanating from the origin, and making 120° angles. It is fairly easy to see that Y is a minimal set in $\mathbb{R}^2$, and hence also $Y \cap U$ is minimal in any open set U.

Smooth minimal surfaces are minimal in small domains, but not necessarily in big ones. For instance, let $E \subset \mathbb{R}^3$ be a catenoid. Then $E \cap B$ is minimal in B when B is a small enough ball, but this is not true when B is a very large ball with the same center as E: in very large balls $B(0, R)$, E looks a lot like the union of two parallel very large disks of radius R, that lie at distance $\log(R)$ from each other, and pinching these disks in the middle gives a significantly better competitor.

There is a minor detail that we need to address. If we add to the almost minimal set E any (closed) set of vanishing $\mathcal{H}^d$-measure, we get another almost minimal set E' because (3.4) does not change. This could make our set E look unnecessarily ugly, so we shall focus our attention to the so-called reduced almost minimal sets.

Definition 3.5. We say that the closed set $E \subset U$ is reduced, or coral, when $E = E^*$, where

$$E^* = \left\{x \in U\,;\, \mathcal{H}^d(E \cap B(x,r)) > 0 \text{ for all } r > 0\right\} \tag{3.6}$$

is the closed support of the restriction $\mathcal{H}^d_{|E}$ of $\mathcal{H}^d$ to E.

It is fairly easy to see that $\mathcal{H}^d(E \setminus E^*) = 0$, and that if E is almost minimal in U, then E^* is a reduced almost minimal set in U, with the same gauge function as E. Because of this, we shall be able to restrict to reduced sets without loss of generality. See for instance Remark 2.14 in [D6].

In other sources such as [D2, 6, 7], we may say coral rather than reduced, because there was an earlier notion of reduction with a different definition. The name "core" for E^* was introduced in [D2], and is not really widespread. But we shall mostly say reduced because there is no danger of confusion here.

Notice also that when E minimizes $\mathcal{H}^d$ or some functional like J_g above in a class of $\mathcal{F}$ of competitors, we do not say that E^* lies in the same class, but only that it is an almost minimal set too. We shall not get any information on $E \setminus E^*$, but such information would be virtually impossible to get anyway.

3.2. Ahlfors regularity and uniform rectifiability. The first important property of our almost minimal sets is their local Ahlfors regularity.

Theorem 3.7 [Al3], [DS3]. There exist constants $\varepsilon > 0$ and $C \geq 1$, that depend only on n and d, such that if E is a reduced almost minimal set of dimension d in $U \subset \mathbb{R}^n$, with the gauge function h, and if $x \in E$ and $r > 0$ are such that

$$B(x, 2r) \subset U \quad \text{and} \quad h(r) \leq \varepsilon, \tag{3.8}$$

then

$$C^{-1}r^d \le \mathcal{H}^d(E \cap B(x,r)) \le Cr^d. \tag{3.9}$$

We just gave two references here because Almgren did not explicitly state (3.9), but his statement was very close and his proof essentially works.

Note that local Ahlfors regularity is not really a regularity property, but just a size condition that says that E is neither too large, nor too small locally. But it turns out to be quite useful. The upper bound in (3.9) is not surprising (otherwise, one imagines that we could deform a part of E into a somewhat smaller d-dimensional skeleton); for the lower bound, one proves that if E gets too thin near a point, we can even deform it locally into a $(d-1)$-dimensional skeleton and make $\mathcal{H}^d(E)$ smaller. For both estimates, deformations known as Federer-Fleming projections are used.

To the author's knowledge, the strongest regularity property that holds for general almost minimal sets of any dimension is the following uniform rectifiability result.

Theorem 3.10 [DS3]. There exist constants $\varepsilon > 0$, $\theta > 0$ and $N \ge 0$, that depend only on n and d, such that if E is a reduced almost minimal set of dimension d in $U \subset \mathbb{R}^n$, with the gauge function h, and if $x \in E$ and $r > 0$ are such that (3.8) holds, then we can find an N-Lipschitz graph G of dimension d such that

$$\mathcal{H}^d(E \cap G \cap B(x,r)) \ge \theta r^d. \tag{3.11}$$

Here "N-Lipschitz graph" means that Γ is the image under an isometry of $\mathbb{R}^n$ of the set $G_A = \big\{(x, A(x))\,;\, x \in \mathbb{R}^d\big\}$, where $A : \mathbb{R}^d \to \mathbb{R}^{n-d}$ is such that $|A(x) - A(y)| \le N|x-y|$ for $x, y \in \mathbb{R}^d$.

In the appropriate technical language, E is locally uniformly rectifiable, with big pieces of Lipschitz graphs (in short, BPLG). We shall not try to discuss here what it means to be uniformly rectifiable; let us just say that Lipschitz graphs and images of $\mathbb{R}^d$ by bilipschitz maps are uniformly rectifiable, and that the general uniformly rectifiable set is not much less regular than that. See [DS1] or [D3] for lots of information on uniform rectifiability. But logically we could expect more regularity from almost minimal sets than BPLG, even if we can't prove it.

Almgren [Al3] already knew the rectifiability, and at the time uniform rectifiability did not exist. But Theorem 3.10 is significantly more complicated to prove than the rectifiability of E.

3.3. Limits of almost minimal sets. The next result helps proving the stability almost minimality under limits. It is proved in [D1], as a consequence of Theorem 3.10. But in fact, to the author's recent surprise, we could also deduce it from the rectifiability of almost minimal sets and a compactness argument.

Theorem 3.12 [D1]. For each $\delta > 0$, there exist constants $\varepsilon > 0$, and $c > 0$, that depend only on n, d, and δ, such that if E is a reduced almost minimal set of dimension d in $U \subset \mathbb{R}^n$, with gauge function h, and if $x \in E$ and $r > 0$ are such that (3.8) holds, then we can find $y \in E$ and $t > 0$ such that

$$t \ge cr\,, \quad B(y,t) \subset B(x,r)\,, \text{ and} \tag{3.13}$$

$$\mathcal{H}^d(E \cap B(y,t)) \ge (1-\delta)\mathcal{H}^d(P \cap B(y,t)) \tag{3.14}$$

for any d-plane P through y.

Of course $\mathcal{H}^d(P\cap B(y,t))$ does not depend on P. In the language of Dal Maso, Morel and Solimini, E satisfies a uniform concentration property ($\mathcal{H}^d$ is almost as concentrated on $E\cap B(y,t)$ as on $P\cap B(y,t)$). This property was introduced in [DMS], to prove the lowersemicontinuity of Hausdorff measure along some minimizing sequences, and then get an existence result for minimizers of the Mumford-Shah functional for image segmentation. And it is tempting to use it here for the same sort of purposes. We first need to define local Hausdorff distances and limits.

When E,F are closed sets in U and for $\overline{B}(x,r)\subset U$, we set

$$\begin{aligned} d_{x,r}(E,F) = r^{-1}\sup\big\{\mathrm{dist}(y,F)\,;\, y\in E\cap B(x,r)\big\} \\ + r^{-1}\sup\big\{\mathrm{dist}(y,E)\,;\, y\in F\cap B(x,r)\big\}, \end{aligned} \tag{3.15}$$

with the convention that $\sup\big\{\mathrm{dist}(y,F)\,;\, y\in E\cap B(x,r)\big\} = 0$ when $E\cap B(x,r)$ is empty, and similarly for $F\cap B(x,r)$.

Next let $\{E_k\}_{k\geq 0}$ be a sequence of closed sets in U, and E be a closed set in U. We say that $\{E_k\}$ converges to E in U when

$$\lim_{k\to+\infty} d_{x,r}(E_k,E) = 0 \quad \text{for every choice of } x,r \text{ such that } \overline{B}(x,r)\subset U. \tag{3.16}$$

The reader may easily check that this is equivalent to various other natural definition of local convergence in E for the Hausdorff distance. For instance, we could have used an exhaustion of U by compact subsets instead of our balls $\overline{B}(x,r)$. The advantage of this notion of convergence is that for each sequence $\{E_k\}_{k\geq 0}$, we can extract a subsequence that converges to a limit E. Maybe see Section 34 in [D2] for additional (easy) detail. But of course we expect more difficulties when we try to prove that E inherits good properties of the E_k. Unless we start with a sequence $\{E_k\}$ with some uniform good properties.

Theorem 3.17 [D1]. *Let $\{E_k\}_{k\geq 0}$ be a sequence of d-dimensional reduced almost minimal sets in U, with the same gauge function h, and suppose that $\{E_k\}$ converges to E in U. Then*

$$\mathcal{H}^d(E\cap V) \leq \liminf_{k\to+\infty} \mathcal{H}^d(E_k\cap V) \quad \text{for every open set } V\subset U \tag{3.18}$$

and E is a reduced almost minimal set in U, with the same gauge function h as the E_k.

The main ingredient in the proof is the lowersemicontinuity estimate (3.18), which is a consequence of Theorem 3.12 and [DMS] (but see [MoS] for a more general statement); but some amount of local cutting and pasting is needed too. Observe that if we did not assume something on the sets E_k, (3.18) would fail miserably. For instance, E_k could be a dotted line segment of total length 1/2, which converges to $E=[0,1]$; the uniform concentration property above precisely prevents this behavior.

In the world of mass minimizing currents, the analogue of Theorem 3.17 is a consequence of lowersemicontinuity of the mass, and is easy to apply because of the compactness theorem for integral currents. Almgren [Al2] proved a similar theorem on limits of varifolds, which may be the reason why he apparently did not try (or care) to prove Theorem 3.17.

Theorem 3.17 will be used a lot in the next section, because it allows easy compactness arguments. It also looks like a good tool to prove existence results, and this was the initial motivation in [D1]. We shall see in Section 6 two examples where it can be used this way, but often the difficulty will be that E does not necessarily lie in the same class of competitors as the E_k.

In the context of existence theorems, it may be hard to find an appropriate sequence of almost minimal sets, but fortunately Theorem 3.17 also works with the following less restrictive notion of quasiminimal sets.

3.4. Quasiminimal sets. In some contexts it is useful to consider the larger class of quasiminimal sets, which was introduced by Almgren [Al3], who called them "restricted sets".

This time let us give a definition which is essentially Almgren's. As before, E is a closed set in U, with $\mathcal{H}^d(E \cap B) < +\infty$ for every compact ball $B \subset U$. We still compare E with sets $F = \varphi(E)$, where φ is Lipschitz, but now we only require that if

$$W_\varphi = \big\{x \in \mathbb{R}^n \,;\, \varphi(x) \neq x\big\}, \tag{3.19}$$

then $W_\varphi \cup \varphi(W_\varphi) \subset\subset U$. Also, the accounting will be different.

Definition 3.20. We say that E is a quasiminimal set with constants M and scale δ_0 if

$$\mathcal{H}^d(E \cap W_\varphi) \leq M \mathcal{H}^d(\varphi(E \cap W_\varphi)) \tag{3.21}$$

whenever $\varphi : U \to U$ is as above and $\mathrm{diam}(W_\varphi \cup \varphi(W_\varphi)) \leq \delta_0$.

Before, we only allowed mappings φ such that $W_\varphi \cup \varphi(W_\varphi)$ is contained in a compact ball $\overline{B} \subset U$. We could also do this here, this would give an apparently weaker notion of quasiminimal set, but all the known regularity results would remain true with the same proofs: we only use mappings φ such that $W_\varphi \cup \varphi(W_\varphi) \subset \overline{B} \subset U$ in the proofs.

In addition, we could also add an error term and replace (3.21) with

$$\mathcal{H}^d(E \cap W_\varphi) \leq M \mathcal{H}^d(\varphi(E \cap W_\varphi)) + h(\delta)\delta^d, \tag{3.22}$$

where h is a gauge function as near (3.1), and where $\delta = \mathrm{diam}(W_\varphi \cup \varphi(W_\varphi)$. The corresponding class of generalized quasiminimal sets is introduced in [D6], and the main point is that the usual proofs go through in this case, with only minor modifications.

The accounting in (3.21) may seem a little strange, but at least it fits well with functionals like J_g in (2.16). If E minimizes such a functional in a class of sets that is stable under mappings φ as above, then E is a quasiminimal set with constant M as soon as $1 \leq g(x) \leq M$ everywhere. This is easy to see: if $F = \varphi(E)$ is an acceptable competitor,

$$
\begin{aligned}
\mathcal{H}^d(E \cap W_\varphi) &\leq \int_{E\cap W_\varphi} g(x)d\mathcal{H}^d(x) = J_g(E) - \int_{E\setminus W_\varphi} g(x)d\mathcal{H}^d(x) \\
&\leq J_g(\varphi(E)) - \int_{E\setminus W_\varphi} g(x)d\mathcal{H}^d(x) \\
&= \int_{\varphi(E)} g(x)d\mathcal{H}^d(x) - \int_{E\setminus W_\varphi} g(x)d\mathcal{H}^d(x) \\
&\leq \int_{\varphi(E\cap W_\varphi)} g(x)d\mathcal{H}^d(x) \leq M\mathcal{H}^d(\varphi(E\cap W_\varphi))
\end{aligned}
\tag{3.23}
$$

by (2.16), because E is minimal, and because

$$
\varphi(E) = \varphi(E \cap W_\varphi) \cup \varphi(E \setminus W_\varphi) = \varphi(E \cap W_\varphi) \cup (E \setminus W_\varphi) \tag{3.24}
$$

(recall that $\varphi(x) = x$ on $E \setminus W_\varphi$). The same argument shows that E is almost minimal when g is continuous, with a relation between the gauge function h and the modulus of continuity of g.

Theorems 3.7, 3.10, 3.12, and 3.17 are still true for generalized quasiminimal sets. We just need to let ε, C, θ, N, and c depend on M as well, and in Theorem 3.18 the conclusion is that E is quasiminimal with the same M and h as the elements E_k of the sequence. See [DS3], [D1], and [D6].

The advantage of quasiminimality is a greater flexibility, which will be used in Section 6. Also see [DS2] for a use of this flexibility in codimension 1 and in a slightly different context. Finally observe that Theorem 3.10 (the uniform rectifiability of E) is a little less far from optimality in the context of quasiminimal sets, because Lipschitz graphs and bilipschitz images of d-planes are easily shown to be quasiminimal (the class of quasiminimal sets is closed under bilipschitz mappings).

4. Jean Taylor's regularity theorem

Recall that one of the main goals of this paper is to present a proof of Jean Taylor's celebrated theorem on the local regularity of 2-dimensional almost minimal sets.

4.1. Minimal cones. A minimal cone is just a cone which is also a minimal set. As before, we restrict to reduced sets to avoid ugly additional sets of vanishing measure. But we shall also accept the cones that are not necessarily centered at the origin. As we shall see soon, a good knowledge of the minimal cones is important if we want to understand the local behavior of the minimal sets of the same dimension.

It is easy to check that the (nonempty) minimal cones of dimension 1 in $\mathbb{R}^n$ are the lines, and the sets Y composed of three half lines with the same origin and that make 120° angles at that point. Thus Y is contained in a plane. Also, the union of two perpendicular lines in $\mathbb{R}^2$ is not minimal (pinch near the center to make a better competitor). Similarly, the union in $\mathbb{R}^3$ of four half lines with the same origin is not minimal either (pinch gain). See for instance [M4] or Section 10 of [D6] for the fairly easy description above.

The list of 2-dimensional minimal cones in $\mathbb{R}^3$ is also known, and there are exactly three types of (nonempty) minimal cones. These are the planes (which we shall also call cones of type $\mathbb{P}$), the cones of type $\mathbb{Y}$ obtained as unions of three half planes with a common boundary line L and that make 120° angles along L,

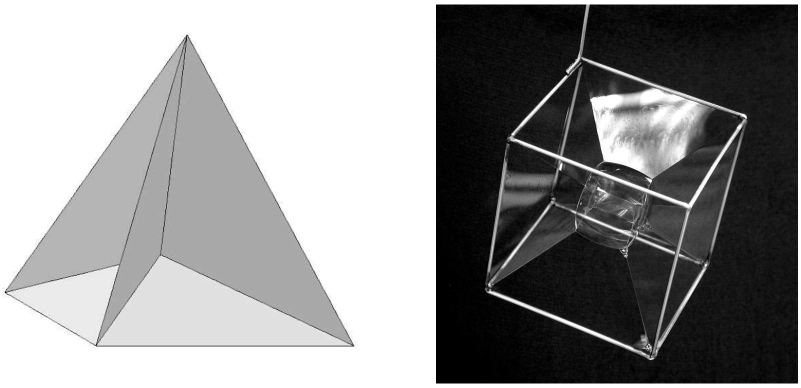

FIGURE 4. Left: A piece of cone of type $\mathbb{T}$. Right: A soap bubble where the two types of singularities are visible

and the cones of type $\mathbb{T}$ obtained as the (positive) cone over the union of the edges of a regular tetrahedron, centered at the center of the tetrahedron (see the left of Figure 4). Thus a cone of type $\mathbb{T}$ has six faces, that meet by sets of three and with $120°$ angles along four edges (half lines emanating from the center).

We can observe these minimal cones as tangent objects in soap films and bubbles; this is not a surprise because we shall see soon that every blow-up limit of an almost minimal set at one of its points is a minimal cone (see the right part of Figure 4).

The fact that cones of types $\mathbb{P}$ and $\mathbb{Y}$ are minimal is easy; for cones of type $\mathbb{T}$, the best is to use a calibration argument [M2] (see also [LM1]) (integrate against a carefully chosen vector field and use Green's theorem to proove that any competitor (that separates) has a large measure).

The fact that no other cone in $\mathbb{R}^3$ is minimal is more complicated. It is not too hard to see that the intersection of such a cone with the unit sphere would be composed of arcs of great circles that make $120°$ angles, but there are quite a few possible configurations, and one needs to describe them and remove the ones for which the corresponding cone is not minimal. For instance, the cone over the union of the edges of a cube is not minimal, but again there are other examples. This discussion was done by Plateau, Lamarle [La], Heppes [He], and Taylor [T2]. Also see Ken Brakke's home page (http://www.susqu.edu/brakke/) for pictures of the non-minimal cones and better competitors.

The list of 2-dimensional minimal cones in $\mathbb{R}^n$ is not known yet. It is fairly easy to see that the cones of type $\mathbb{P}$, $\mathbb{Y}$, and $\mathbb{T}$ are still minimal in $\mathbb{R}^n$, but many more may exist.

The first new candidates are unions $P_1 \cup P_2$ of two planes of dimension 2. If P_1 is orthogonal to P_2, a small projection argument shows that $P_1 \cup P_2$ is minimal. If P_1 and P_2 make a small angle (and in particular if they are not transverse), we can pinch $P_1 \cup P_2$ and get a better competitor, so it is not minimal. A recent theorem of X. Y. Liang [Li1,2] says that if P_1 and P_2 are almost orthogonal (i.e., assuming that $0 \in P_1 \cup P_2$, if for some small constant $\varepsilon > 0$, $|u \cdot v| \leq \varepsilon |u||v|$ for $u \in P_1$ and $v \in P_2$), then $P_1 \cup P_2$ is minimal. There is a conjecture of Morgan [M3] on the precise values of the angles that d-planes P_1 and P_2 need to make when $P_1 \cup P_2$ is minimal, but (when $d = 2$) only the necessary condition was proved [Lw].

The last known new candidate is the product $Y \times Y$ of two one-dimensional sets Y contained in orthogonal 2-planes. This cone is composed of 9 faces (that correspond to 9 arcs of circles in $\partial B(0,1)$), which meet along 6 half lines (that correspond to the vertices where the arcs meet). At the time of the lectures it was not known whether it is minimal, but this was recently proved by Liang [Li4].

Other than that, we only have a general knowledge on what the minimal cones of dimension 2 in $\mathbb{R}^n$, $n \geq 4$, look like. Let E be such a cone (centered at 0), and set $K = E \cap \partial B(0,1)$. Then K is a finite union of great circles (circles centered at 0), and of arcs of great circles. The great circles do not meet each other or the rest of K, and the arcs of great circles meet at their endpoints, by sets of three and with 120° angles. And in addition we have a lower bound on the length of each arc of circle. See [D6].

For instance, $\mathbb{Y}$ corresponds to 3 half circles, $\mathbb{T}$ to 6 shorter arcs, and $P_1 \cup P_2$ to 2 disjoint great circles. But even in $\mathbb{R}^4$ we do not have a full list of candidates for K and E.

Even less is known for higher-dimensional minimal cones. Almgren [Al1] proved that if K is a smooth 2-dimensional surface in the 3-sphere, then the cone over K is not minimal, except if it is a hyperplane. Also, the cone over the union of the $(n-2)$-dimensional faces of the hypercube in $\mathbb{R}^n$ is minimal when $n \geq 4$ [Br1].

4.2. A statement of Jean Taylor's theorem (1976), and an extension.

THEOREM 4.1 [T2]. *Let E be a reduced local almost minimal set of dimension 2 in some open set $U \subset \mathbb{R}^3$, with gauge function $h(r) \leq Cr^\alpha$ (for some choice of $\alpha > 0$ and $C \geq 0$). Then for each $x \in E$, there is a ball $B(x,r)$ where E is the image of a minimal cone centered at x by a C^1-diffeomorphism of $\mathbb{R}^3$ that fixes x.*

See Definitions 3.2 and 3.5 for reduced almost minimal sets. Also recall that there are only three types of minimal cones: the planes and the cones of type $\mathbb{Y}$ or $\mathbb{T}$.

We can make the conclusion a little more precise. There is a minimal cone Z centered at x and a $C^{1+\beta}$-diffeomorphism $\phi : \mathbb{R}^3 \to \mathbb{R}^3$, such that $\phi(x) = x$, $D\phi(x)$ is the identity mapping, and $\phi(Z) \cap B(x,r) = E \cap B(x,r)$. Here $\beta > 0$ is a small constant, that can be computed in terms of α, but, as far as the author knows, the optimal β is not known. We have uniform bounds on $r\phi(r^{-1}x)$, but an unfortunate point is that we do not have a good lower bound for r; in particular, we cannot prove yet that if E is very close to a minimal cone of type $\mathbb{T}$ in $B(x,r) \subset U$, and if $h(r) = 0$, then E is C^1-equivalent to a $\mathbb{T}$ in $B(x, r/100)$.

The reader should probably not pay too much attention to the C^1 diffeomorphism ϕ, the main point of the conclusion is that near x, E is composed of C^1 faces, that meet along C^1 curves with 120° angles and with the same combinatorics as for a minimal cone; the rest would follow from the implicit function theorem anyway.

The two types of singularities ($\mathbb{Y}$ and $\mathbb{T}$) really occur in soap films and bubbles, where they are very easy to produce. See the right side of Figure 4 again.

Our assumption on h is not optimal; if

$$\int_0^1 h(r)\frac{dr}{r} < +\infty \tag{4.2}$$

we get a biHölder equivalence, and if we ask a little more (for instance, that $h(r) \leq C[\log(A/r)]^{-b}$ for some $b > 30$) we obtain the C^1 equivalence, but the conditions that we get are probably not optimal. See [D7], Section 13.

The author does not know what is the optimal regularity near the singularities of E, but away from them (at points where E is locally equivalent to a plane), E is a standard minimal surface, and hence real-analytic!

Let us mention a slightly more recent generalization of Theorem 4.1 to higher ambient dimensions.

Theorem 4.3 [D6]. *Let E be a reduced local almost minimal set of dimension 2 in some open set $U \subset \mathbb{R}^n$, with a gauge function h such that (4.2) holds. Then for each $x \in E$, there is a ball $B(x,r)$ in which E coincides with the image, under a bi-Hölder diffeomorphism of $\mathbb{R}^n$, of a minimal cone.*

This looks like Theorem 4.1, and certainly many of the estimates are similar, but the author suspects that there are also differences in the proof, and does not really know wether J. Taylor's proof can be made to work in $\mathbb{R}^n$.

The statements too are different, and Theorem 4.3 is far from being as perfect as Theorem 4.1. Its main defect is that when $n \geq 4$, we do not know the list of minimal cones. Our statement is less precise because of this: we get that E is close to a minimal cone, and we can use the general description of minimal cones given in the last subsection to say that E decomposes into faces, with some combinatorial and angle constraints, but this is not such a precise information after all.

The bi-Hölder exponent can be taken to be as close to 1 as we want, but even if $h(r) \leq Cr^{\alpha}$ as above we do not get a C^1 equivalence in general. We only prove the C^1 equivalence when some bow-up limit of E at x is a minimal cone with the so-called "full length" property (see [D7]). This property holds for all the minimal cones of dimension 2 that we know so far, but other minimal cones could exist, that maybe don't satisfy it.

Let us give a definition of the full length property, just to give an idea. Write $K = X \cap \partial B(0,1)$ as a union of great circles or arcs of great circles. Cut them in 2 or 3 pieces when their length is larger than $9\pi/10$, so as to get a collection of arcs γ of lengths less than $9\pi/10$; thus $K = \cup_\gamma \gamma$. Call V the set of vertices where the different arcs meet.

Consider any mapping $\varphi : V \to \partial B(0,1)$, with $\sup_{x\in V} |\varphi(x) - x|$ small. When γ is an arc of K with endpoints x and $y \in V$, call $\varphi_*(\gamma)$ the geodesic from $\varphi(x)$ to $\varphi(y)$; it is unique because $\mathcal{H}^1(\gamma) \leq 9\pi/10$. Set $\varphi_*(K) = \cup_\gamma \varphi_*(\gamma)$ and call $\varphi_*(X)$ the cone over $\varphi_*(K)$.

We say that the cone E satisfies the full length property when there exists $c > 0$ such that, whenever φ above is such that $\mathcal{H}^1(\varphi_*(K)) > \mathcal{H}^1(K)$, there is a competitor $\widetilde{X}$ for $\varphi_*(X)$ in $B(0,1)$ (as in Definition 3.2) such that

$$\mathcal{H}^2(\widetilde{X} \cap B(0,1)) \leq \mathcal{H}^2(\varphi_*(X) \cap B(0,1)) - c[\mathcal{H}^1(\varphi_*(K)) - \mathcal{H}^1(K)]. \tag{4.4}$$

There are slightly simpler sufficient conditions, but anyway it seems complicated to check (4.4) without knowing more precisely what K is. The minimal cones in $\mathbb{R}^3$ and the known minimal cones in $\mathbb{R}^n$ have the full length property (and the verification is not hard), but the author did not have the courage to check whether the cone $Y \times Y \subset \mathbb{R}^4$ has the full length property. Recall that we do not know whether it is minimal either.

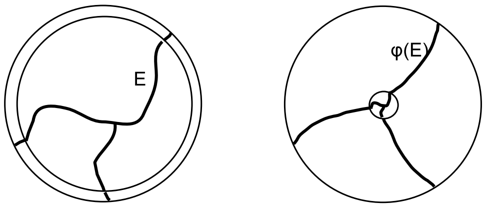

FIGURE 5. How to find a deformation of E (on the left) that looks like the cone over $E \cap \partial B(0,r)$. The thin annulus is mapped to the thick one

4.3. The density is (almost) monotone. From now on we describe the main points of the proof of Theorem 4.1 that was given in [D6], which again is probably not so different from the original proof. As we shall see, the details are sometime a little long, but the scheme is fairly easy to understand.

The first main ingredient is a standard of minimal surfaces: the monotonicity of density.

PROPOSITION 4.5. *Let E be a reduced minimal set of dimension d in $U \subset \mathbb{R}^n$, and set*

$$\theta(x,r) = r^{-d}\mathcal{H}^d(E \cap B(x,r)) \tag{4.6}$$

for $x \in E$ and $0 < r \leq \mathrm{dist}(x, \mathbb{R}^n \setminus U)$. Then

$$\theta(x,\cdot) \ \text{ is nondecreasing.} \tag{4.7}$$

The idea of the proof is simple. Observe that $r \to \mathcal{H}^d(E \cap B(x,r))$ is nondecreasing, so it is the integral of its derivative (seen as a Stiljes measure), which is at least as large as its almost-everywhere derivative. Because of this (and a small computation), it is enough to check that for a.e. $r \leq \mathrm{dist}(x, \mathbb{R}^n \setminus U)$,

$$r^{-d}\,\frac{\partial}{\partial r}\big(\mathcal{H}^d(E \cap B(0,r))\big) \geq d\,r^{-d-1}\mathcal{H}^d(E \cap B(0,r)). \tag{4.8}$$

But for almost every r,

$$\frac{\partial}{\partial r}\big(\mathcal{H}^d(E \cap B(0,r))\big) \geq \mathcal{H}^{d-1}(E \cap \partial B(0,r)) \tag{4.9}$$

[we know that E is rectifiable, so we can reduce to C^1 surfaces, where eventually we just need to compute the contribution of a small element of surface to both sides of (4.9)], so it is enough to show that

$$\mathcal{H}^d(E \cap B(0,r)) \leq \frac{r}{d}\,\mathcal{H}^{d-1}(E \cap \partial B(0,r)). \tag{4.10}$$

Then we observe that

$$\frac{r}{d}\,\mathcal{H}^{d-1}(E \cap \partial B(0,r)) = \mathcal{H}^d(\Gamma \cap B(0,r)), \tag{4.11}$$

where Γ denotes the cone over $E \cap \partial B(0,r)$). Now $[\Gamma \cap B(0,r)] \cup E \setminus B(0,r)]$ is not directly a competitor for E, but we can approximate it by Lipschitz deformations of E in $B(0,r)$. [Expand a lot a small annulus near $\partial B(0,r)$ and contract most of $B(0,r)$ to the origin, as suggested by Figure 5.] The comparison and a limiting argument yield (4.10) and the monotonicity of θ. □

Proposition 4.5 has two useful extensions. If E is merely almost minimal in E, but (4.2) holds, a similar argument yields that $\theta(x,\cdot)$ is almost nondecreasing, in the sense that

(4.12)
$$\theta(x,r)\exp\Big\{C\int_0^r h(2t)\frac{dt}{t}\Big\} \text{ is a nondecreasing function of } r\leq \mathrm{dist}(x,\mathbb{R}^n\setminus U).$$

Notice that by (4.2) the exponential tends to 1 when r tends to 0, so (4.12) really means that $\theta(x,\cdot)$ is almost nondecreasing. Also, the nondecreasing function in (4.12) has a limit when r tends to 0, and so

$$\theta(x)=\lim_{r\to 0}\theta(x,r) \text{ exists for every } x\in E \tag{4.13}$$

when (4.2) holds.

Besides, the case when $\theta(x,\cdot)$ is locally constant in Proposition 4.5 is under control:

$$\begin{aligned}&\text{if } x\in E \text{ and } E \text{ is a (reduced) minimal set in } B(x,r_0)\\ &\text{and } \theta(x,\cdot) \text{ is constant on }]0,r_0[\text{, then } E \text{ coincides}\\ &\text{in } B(x,r_0) \text{ with a (reduced) minimal cone centered at } x.\end{aligned} \tag{4.14}$$

Surprisingly (and maybe out of clumsiness) this requires some additional work in [D6]; it is not hard to see that equality almost-everywhere in (4.9) implies that for almost every $y\in E\cap B(x,r_0)$, the tangent plane to E at y goes through x, but then we still need to show that E contains the line segment $[x,y]$, and this takes some time and the complicated construction of a competitor.

We shall often use the following consequence of (4.14).

Lemma 4.15. For each small $\delta>0$, we can find $\varepsilon>0$ (that depends on δ and the dimensions n and d) such that the following holds. Still assume that (4.2) holds, and that E is a reduced almost minimal set in U, with gauge function h. Let $x\in E$ and $B(x,2r)\subset U$ be such that $h(2r)\leq\varepsilon$ and

$$\theta(x)-\varepsilon\leq\theta(x,\rho)\leq\theta(x)+\varepsilon \quad \text{for } 0<\rho\leq 2r, \tag{4.16}$$

where $\theta(x)=\lim_{t\to 0}\theta(x,t)$ as above. Then there is a minimal cone Z centered at x such that

$$d_{x,r}(E,Z)\leq\delta \tag{4.17}$$

and, for $y\in\mathbb{R}^n$ and $t>0$ such that $B(y,t)\subset B(x,r)$,

$$\big|\mathcal{H}^d(E\cap B(y,t))-\mathcal{H}^d(Z\cap B(y,t))\big|\leq\delta r^d. \tag{4.18}$$

See (4.6), (4.13), and (3.15) for the definitions. When E is minimal, instead of (4.16) we just need to say that $\theta(x,2r)\leq\theta(x)+\varepsilon$, because of (4.7), but in the almost minimal case, it is simpler to require (4.16). Also, do not be shocked by (4.18); when t/r is very small, $\mathcal{H}^d(E\cap B(y,t))$ is much smaller than r^d anyway, and (4.18) does not say much.

We shall sometimes refer to Lemma 4.15 as the almost-constant density property. Lemma 4.15 is deduced from (4.14) and Theorem 3.17 (about limits) by a standard compactness argument; let us just sketch the proof, and refer to Proposition 7.24 (which follows from Proposition 7.1) in [D6] for details.

We proceed by contradiction and suppose that for some small $\delta>0$, Lemma 4.15 does not hold with $\varepsilon_k=2^{-k}$. Pick an almost minimal set E_k in a domain $U_k\subset\mathbb{R}^n$, and a ball $B(x_k, 2r_k) \subset U_k$ which satisfy the hypotheses of the lemma, but not the conclusion. By dilation invariance, we can assume that $x_k = 0$, $r_k = 1$, and $U_k \supset B(0,2)$. Then replace $\{E_k\}$ with a subsequence that converges to some limit E in $B(0,2)$. By Theorem 3.17, E is minimal in $B(0,2)$. More precisely, for each k_0 we can apply Theorem 3.17 with the gauge function $\widetilde{h}_{k_0} = \sup_{k \geq k_0} h_k$; we use the fact that $h_k(2) \leq \varepsilon_k = 2^{-k}$ to show that $\widetilde{h}_{k_0}$ is a gauge function, and apply Theorem 3.17 to show that E is almost minimal with the gauge $\widetilde{h}_{k_0}$. Then we observe that $\widetilde{h}_{k_0}(2) \leq 2^{-k_0}$ and get that in fact E is minimal in $B(0,2)$.

By (3.18),

$$\mathcal{H}^d(E \cap B(y,t)) \leq \liminf_{k \to +\infty} \mathcal{H}^d(E_k \cap B(y,t)) \quad \text{when } B(y,t) \subset B(0,2). \tag{4.19}$$

But the proof of Theorem 3.17 also gives the uppersemicontinuity estimate

$$\mathcal{H}^d(E \cap \overline{B}(y,t)) \geq \limsup_{k \to +\infty} \mathcal{H}^d(E_k \cap \overline{B}(y,t)) \quad \text{when } \overline{B}(y,t) \subset B(0,2); \tag{4.20}$$

the idea of the proof is that if (4.20) did not hold, some Hausdorff measure would drop when we pass to the limit, and we could use this to construct a strictly better competitor for E_k. But the proof relies on the construction of Theorem 3.17 and the fact that the E_k are asymptotically minimal.

Let θ denote the density function associated to E (as in (4.6)) and θ_k its analogue for E_k. Notice that for almost every $t \in (0,2)$, $\mathcal{H}^d(E \cap \partial B(0,t)) = 0$ (the set of t where this fails is at most countable, because the sets $E \cap \partial B(0,t)$ are disjoint and $\mathcal{H}^d(E \cap B(0,r)) < +\infty$ for $r < 2$). For such t,

$$\begin{aligned} \limsup_{k \to +\infty} \theta_k(0,t) &= t^{-d} \limsup_{k \to +\infty} \mathcal{H}^d(E_k \cap B(0,t)) \leq t^{-d} \mathcal{H}^d(E \cap \overline{B}(0,t)) \\ &= t^{-d} \mathcal{H}^d(E \cap B(0,t)) = \theta(0,t) \leq \liminf_{k \to +\infty} \theta_k(0,t) \\ &\leq \liminf_{k \to +\infty} \theta_k(0) \end{aligned} \tag{4.21}$$

by (4.20), (4.19), and (4.16). Set $L = \liminf_{k \to +\infty} \theta_k(0)$. Since $L \leq \limsup_{k \to +\infty} \theta_k(0,t)$ by (4.16) again, (4.21) says that $\limsup_{k \to +\infty} \theta_k(0,t) = L$, and then (looking at the intermediate inequalities)

$$\lim_{k \to +\infty} \theta_k(0,t) = L \quad \text{and} \quad \theta(0,t) = L. \tag{4.22}$$

This holds for almost every $t \in (0,2)$, hence $\theta_k(0,t) = L$ for all $t \in (0,2)$, by (4.7) and because E is minimal. Then (4.14) says that E coincides, in $B(0,2)$, with a minimal cone Z centered at 0.

Since (our subsequence of) $\{E_k\}$ converges to E, we get that (4.17) holds for k large. We can also use (4.19) and (4.20) to get that (4.18) holds for some large k; the verification takes a little time, but is not too complicated (see the proof of Proposition 7.1 in [D6] for details); the desired contradiction and Lemma 4.15 then follow. □

4.4. Blow up limits. Let E be a reduced almost minimal set in U, and pick $x \in E$. A blow-up limit of E at x is any set

$$E_\infty = \lim_{k \to +\infty} \frac{1}{r_k} [E - x], \tag{4.23}$$

where $\{r_k\}$ is a sequence in $(0, +\infty)$ that tends to 0. Here the definition of convergence is slightly different, because the domains vary. That is, $U_k = \frac{1}{r_k} [U - x]$

tends to $\mathbb{R}^n$, in the sense that $\mathrm{dist}(0,\mathbb{R}^n \setminus U_k) = \frac{1}{r_k}\,\mathrm{dist}(x,\mathbb{R}^n \setminus U)$ tends to $+\infty$. Then (4.23) is defined by requiring that

$$\lim_{k\to+\infty} d_{0,R}(E_\infty, \frac{1}{r_k}\,[E-x]) = 0 \ \text{ for every } R > 0. \tag{4.24}$$

By (the proof of) the standard fact that the set of compact subsets of any $\overline{B}(0,R)$, with the Hausdorff distance, is itself compact, it is easy to see that from any sequence $\{r_k\}$ that tends to 0, we can extract a subsequence such that the $\frac{1}{r_k}\,[E-x]$ converge. So some blow-up limits of E at a given $x \in E$ exist. But, without additional information, we have to expect that E has more than one blow-up limit at x. We claim that if E is a reduced almost minimal set and E_∞ is a blow-up limit of E at some $x \in E$,

$$\begin{gathered} E_\infty \text{ is a reduced minimal cone in } \mathbb{R}^n, \text{ centered at } 0, \\ \text{and } \mathcal{H}^d(E_\infty \cap B(0,1)) = \theta(x). \end{gathered} \tag{4.25}$$

Indeed, let $\{r_k\}$ be such that $E_k = \frac{1}{r_k}\,[E-x]$ converges to E_∞. Notice that E_k is almost minimal in the domain U_k above, and with the gauge function $h_k(r) = h(rr_k)$ (just apply Definition 2.3 and conjugate with a dilation). Then apply Theorem 3.7 in any fixed ball $B(0,R)$, and observe that the h_k tend to 0 uniformly in $[0,2R]$. We get that E_∞ is minimal in $\mathbb{R}^n$ (because, as above (4.19), it is almost minimal in any $B(0,R)$ and with arbitrarily small gauge functions). By (3.18) and as for (4.19), we get that for $t > 0$,

$$\begin{aligned} \mathcal{H}^d(E_\infty \cap B(0,t)) &\leq \liminf_{k\to+\infty} \mathcal{H}^d(E_k \cap B(0,t)) \\ &= \liminf_{k\to+\infty} r_k^{-d} \mathcal{H}^d(E \cap B(x,tr_k)) \\ &= t^d \lim_{k\to+\infty} \theta(x,tr_k) = t^d\theta(x). \end{aligned} \tag{4.26}$$

Similarly, the same uppersemicontinuity estimate that is hidden in the proof of Theorem 3.7 and that we used for (4.20) shows that

$$\begin{aligned} \mathcal{H}^d(E_\infty \cap \overline{B}(0,t)) &\geq \limsup_{k\to+\infty} \mathcal{H}^d(E_k \cap \overline{B}(y,t)) \\ &= \limsup_{k\to+\infty} r_k^{-d} \mathcal{H}^d(E \cap \overline{B}(x,tr_k)) \\ &\geq t^d \limsup_{k\to+\infty} \theta(x,tr_k) = t^d\theta(x). \end{aligned} \tag{4.27}$$

So $\mathcal{H}^d(E_\infty \cap B(0,t)) = t^d\theta(x)$ for a.e. $t > 0$, hence for all $t > 0$. That is, the density associated to E_∞ at the origin is constant on $(0,+\infty)$, and (4.14) says that E_∞ is a reduced minimal cone. We already checked that $\mathcal{H}^d(E_\infty \cap B(0,t)) = t^d\theta(x)$ for all t, so (4.25) holds.

Of course we expect the minimal cones to be significantly easier to study than the minimal sets of the same dimension. Thus the almost monotonicity of density implies a better control on blow-up limits of minimal sets, that in turn we can try to use to get a good local control of the minimal sets themselves. One could even dream of using this control to get regularity results for minimal cones of one more dimension, and prove regularity results by induction. To some extent, this is what we will do for the 2-dimensional minimal sets, but the program seems hard to continue, because when we try to go from 3-dimensional minimal cones to

3-dimensional minimal sets, we apparently need topological information that we can't get easily. See Remark 4.78.

Let us say two words about blow-in limits. Suppose that E is a reduced minimal set in the whole $\mathbb{R}^n$. A blow-in limit of E is any limit of a sequence $\{\frac{1}{r_k} E\}$, where this time $\lim_{k\to+\infty} r_k = +\infty$. We could have used $\frac{1}{r_k}[E-x]$ for some x, but it is easy to see that we would have obtained the same limit. The same proof as for (4.25) yields that if E_∞ is a blow-in limit of E,

$$\begin{gathered} E_\infty \text{ is a minimal cone in } \mathbb{R}^n, \text{ centered at } 0, \text{ and} \\ \mathcal{H}^d(E_\infty \cap B(0,1)) = \lim_{r\to+\infty} \theta(0,r). \end{gathered} \tag{4.28}$$

4.5. Reifenberg's topological disk theorem. The following extension of Reifenberg's topological disk theorem [R1] will allow us to parameterize our set E, as in the statements of Theorems 4.1 and 4.3, once we have significant information on its geometry. It is nice to know that Reifenberg initially proved this theorem in connection to minimal sets. But, at least in the author's opinion, we should not put too much stress on the use of this result, especially for the C^1 regularity theorem; it is natural that a good control on the distance between E and minimal cones, at all scales and locations, should yield the existence of a nice parameterization. On the other hand, Reifenberg's theorem is quite nice in itself, and will be quite agreeable to use, because it will spare us the tedious construction of a parameterization. See [To] and [DT] for some other applications and generalizations of Reifenberg's argument.

Theorem 4.29 [DDT]. For each $\tau > 0$, we can find $\varepsilon_0 > 0$, that depends only on τ and the dimension n, such that if $E \subset \mathbb{R}^n$ is a closed set, with $0 \in E$ and such that for each $x \in E \cap B(0,2)$ and $r \in (0,2]$, we can find a minimal cone $Z = Z(x,r)$ of dimension 2 such that

$$d_{x,r}(E,Z) \leq \varepsilon_0, \tag{4.30}$$

then there is a minimal cone Z_0 of dimension 2 and a bi-Hölder homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$|f(x) - x| \leq \tau \quad \text{for } x \in B(0,2), \tag{4.31}$$

$$(1-\tau)|x-y|^{1+\tau} \leq |f(x)-f(y)| \leq (1+\tau)|x-y|^{1-\tau} \quad \text{for } x,y \in B(0,2), \tag{4.32}$$

and

$$E \cap B(0,1-\tau) \subset f(Z_0 \cap B(0,1)) \subset E \cap B(0,1+\tau). \tag{4.33}$$

In the sketch of proof below, we shall just use the special case when $n = 3$. Then the minimal cones are known to be of type $\mathbb{P}$, $\mathbb{Y}$, or $\mathbb{T}$. But the general case is used for Theorem 4.3.

In Reifenberg's original theorem, one only uses affine d-planes, but on the other hand all the dimensions and codimensions are allowed.

Notice that $Z(x,r)$ is not necessarily centered at x, (which is better because otherwise the assumptions would not even hold when E is a cone of type $\mathbb{Y}$ centered near the origin).

We would have preferred to have a quasiconformal mapping f, but even in the original theorem of Reifenberg this is not always possible, for instance, when E is the product of a line with a flat snowflake.

Theorem 4.29 has a C^1 variant where one assumes that $d_{x,r}(E,Z) \leq \varepsilon_0 r^\alpha$ for some $\alpha > 0$, and one gets a C^1 parameterization at the end. This is the version that we use in [D7], but although it is obtained in [DDT] as a corollary of the proof of Theorem 4.29, it could probably be obtained directly and with somewhat less work.

Reifenberg's theorem is a nice example where some small local information on E (namely (4.30)), but known at every scale and location, gives a much better global control on the set that one would expect a priori. For instance, (4.30) for the pair (x,r) seems to allow small holes in E, of size $\varepsilon_0 r$, but in fact, due to the fact that potential holes have to be small at every scale, they just don't exist. A similar theorem that the author likes a lot is the John and Nirenberg theorem on the exponential integrability of functions in BMO.

An important information that we use many times in the proof is that (4.30) also forces $Z(x,r)$ to depend fairly nicely on x and r. Because of this, we can rapidly decompose $E \cap B(0,2)$ into the sets of points of type $\mathbb{P}$, $\mathbb{Y}$, and $\mathbb{T}$, depending on the closeness of $Z(x,r)$ to a cone of type $\mathbb{P}$, $\mathbb{Y}$, and $\mathbb{T}$ at small scales r, and one can show, for instance, that $B(0,1)$ contains at most one point of type $\mathbb{T}$.

The proof in [DDT] uses the same ideas as in Reifenberg's original theorem. It is a typical "top-down" algorithm. We start from an initial cone Z_0 (typically, $Z_0 = Z(0,2)$), and construct the final mapping $f : Z_0 \to \mathbb{R}^n$ as an infinite composition of mappings g_k that we construct at geometrically decaying scales r_k. At the scale r_k, E is locally approached by minimal cones, and we construct g_k so that it pushes the points of the image of Z_0 in the direction of E (and of these minimal cones). Technical details (with partitions of unity and distortion estimates) are needed because we need to make sure that our mappings are injective, and make the proof a little long, but the idea is fairly simple.

4.6. J. Taylor's theorem near a point of type $\mathbb{P}$. Let us now try to describe the proof of the biHölder part of J. Taylor's theorem that was given in [D6].

Of course many technical details will be ignored, and we should also say that getting the local C^1 equivalence is much more difficult than the mere biHölder equivalence, but at least the scheme of the proof, and some of the challenges for future generalizations, are reasonably clear in this context. Also, one can argue that the biHölder equivalence should be enough for many purposes.

So let E be a reduced almost minimal set in U, and let $x_0 \in E$ be given. We want to find a small neighborhood of x_0 where Theorem 4.29 can be used, and for this we need to show that E is well approximated by minimal cones near x_0. Let us state this more precisely. For $x \in E$ and $r > 0$, set

$$\beta_c(x,r) = \inf \big\{ d_{x,r}(E,Z)\,;\, Z \text{ is a minimal cone} \big\}; \tag{4.34}$$

we want to find $r_0 > 0$ such that

$$\beta_c(x,r) \leq \varepsilon_0 \ \text{ for } x \in E \cap B(x_0, 2r_0) \text{ and } 0 < r \leq 2r_0, \tag{4.35}$$

with ε_0 as in Theorem 4.29. Once this is done, we just have to apply Theorem 4.29 to $\frac{1}{r_0}[E - x_0]$ and conclude.

Now something needs to be done. It is true that

$$\lim_{r \to 0} \beta_c(x,r) = 0 \ \text{ for each } x \in E. \tag{4.36}$$

Indeed, otherwise we could find $\varepsilon > 0$ and a sequence $\{r_k\}$ that tends to 0, such that $\beta_c(x, r_k) \geq \varepsilon$. Then we could extract a subsequence for which the sets $E_k = \frac{1}{r_k}[E-x]$ converge to a limit E_∞, and by (4.25) E_∞ is a minimal cone. By definition, $d_{0,1}(E_\infty, E_k)$ tends to 0 for this sequence, but since $\beta_c(x, r_k) \leq d_{x,r}(E, x+r_k E_\infty) = d_{0,1}(E_\infty, E_k)$ by definitions, we get a contradiction that proves (4.36).

But (4.36) is not enough for (4.35); we also need some uniform estimates, and this is the whole point of the arguments that we shall describe below.

To make things simpler, we shall assume that E is actually minimal near x; the difference is not enormous, but this way we won't have to deal with error terms. We shall only describe the argument in $\mathbb{R}^3$; the proof extends to $\mathbb{R}^n$, with some modifications in the topological parts, but the general idea is the same. Finally, we may assume that $x_0 = 0$.

Recall from (4.25) that all the blow-up limits of E at 0 are minimal cones, with the same density $\theta(0) = \lim_{r\to 0} r^{-2}\mathcal{H}^2(E \cap B(0,r))$. There are three types of minimal cones ($\mathbb{P}$, $\mathbb{Y}$, or $\mathbb{T}$), with different densities π, $\frac{3\pi}{2}$, and some number $d_T > \frac{3\pi}{2}$, so all the blow-up limits of x_0 are of the same type, determined by the density $\theta(0)$.

In this subsection we just deal with the easiest case when 0 is of type $\mathbb{P}$, which means that $\theta(0) = \pi$ and all the blow-up limits of E at 0 are planes.

Let $\varepsilon_1 > 0$ be small, to be chosen later, and chose a small radius $r_1 > 0$ such that E is minimal in $B(0, 2r_1) \subset U$ and

$$\theta(0, 2r_1) \leq \theta(0) + \varepsilon_1 = \pi + \varepsilon_1. \tag{4.37}$$

Let ε_2 be very small, to be chosen later. If ε_1 is small enough (depending on ε_2), the almost constant density property Lemma 4.15, applied with $\delta = \varepsilon_2$, says that we can find a minimal cone Z centered at 0 such that

$$d_{0,r_1}(E, Z) \leq \varepsilon_2 \tag{4.38}$$

and

$$\left|\mathcal{H}^d(E \cap B(y,t)) - \mathcal{H}^d(Z \cap B(y,t))\right| \leq \varepsilon_2 r_1^d. \tag{4.39}$$

for $y \in \mathbb{R}^n$ and $t > 0$ such that $B(y,t) \subset B(0, r_1)$. We apply this with $B(y,t) = B(0, r_1)$ and get that

$$\begin{aligned} \mathcal{H}^d(Z \cap B(0,r_1)) &\leq \mathcal{H}^d(E \cap B(0,r_1)) + \varepsilon_2 r_1^d = [\theta(0,r_1) + \varepsilon_2]\, r_1^d \\ &\leq [\theta(0,2r_1) + \varepsilon_2]\, r_1^d \leq [\pi + \varepsilon_1 + \varepsilon_2]\, r_1^d \end{aligned} \tag{4.40}$$

because $\theta(0,\cdot)$ is nondecreasing (by (4.7)), and by (4.37). Hence Z is a plane (cones of type $\mathbb{Y}$ and $\mathbb{T}$ have a larger density).

Now let $y \in B(0, r_1/2)$ be given. Apply (4.39) with $t = r_1/2$ to get that

$$\begin{aligned} \theta(y, r_1/2) &= (r_1/2)^{-d}\mathcal{H}^d(E \cap B(y, r_1/2)) \\ &\leq (r_1/2)^{-d}\mathcal{H}^d(Z \cap B(y, r_1/2)) + 2^d\varepsilon_2 \leq \pi + 2^d\varepsilon_2 \end{aligned} \tag{4.41}$$

because $\mathcal{H}^d(Z \cap B(y, r_1/2)) \leq \pi (r_1/2)^d$ (recall that Z is a plane). By monotonicity (i.e., (4.7)),

$$\theta(y, r) \leq \pi + 2^d\varepsilon_2 \quad \text{for } 0 < r \leq r_1/2. \tag{4.42}$$

Then $\theta(y) = \lim_{r\to 0}\theta(y,r) \leq \pi + 2^d\varepsilon_2$ by (4.7). If ε_2 is small enough, $\theta(y) < 3\pi/2$ (the density of cone of type $\mathbb{Y}$), y is a point of type $\mathbb{P}$, and $\theta(y) = \pi$.

Now apply Lemma 4.15 (the almost constant density property) with $\delta = \varepsilon_0$. If ε_2 is small enough, we deduce from that lemma that there exists a minimal cone $Z = Z(y, r/2)$, such that $d_{y,r/2}(E, Z) \leq \varepsilon_0$. Therefore, $\beta_c(y, r/2) \leq \varepsilon_0$ for $0 < r \leq r_1/2$. Since y was any point of $E \cap B(0, r_1/2)$, we just proved (4.35) with $r_0 = r_1/8$.

Notice that we could fairly easily show that $Z(y, r/2)$ is a plane, as we did for Z near (4.40), and so the standard Reifenberg theorem would have been enough in this case.

Anyway, this concludes our description of Theorem 4.1, but with a biHölder equivalence only and when E is locally minimal and x is a point of type $\mathbb{P}$.

4.7. A topological lemma for the existence of $\mathbb{Y}$-points. In the previous argument, we were lucky with the densities, because we found approximating cones Z which had the minimal possible density, and this allowed us to apply the almost constant density property (see below (4.42)). Let us see what happens when we assume that 0 is a point of type $\mathbb{Y}$, i.e., when

$$\theta(0) = \lim_{r \to 0} r^{-d} \mathcal{H}^d(E \cap B(0, r)) = \frac{3\pi}{2}. \tag{4.43}$$

We start as above, choose $r_1 > 0$ such that E is minimal in $B(0, 4r_1) \subset U$ and $\theta(0, 4r_1) \leq \theta(0) + \varepsilon_1 \leq \frac{3\pi}{2} + \varepsilon_1$. Since $\theta(0, \cdot)$ is monotone, we also get that

$$\frac{3\pi}{2} = \theta(0) \leq \theta(0, r) \leq \theta(0, 4r_1) \leq \frac{3\pi}{2} + \varepsilon_1 \quad \text{for } 0 < r \leq 4r_1. \tag{4.44}$$

Let ε_2 be small, as above, and apply the almost constant density property to the pair $(0, 2r_1)$. If ε_1 is small enough (depending on ε_2, not on r_1), we obtain a minimal cone Z centered at 0 such that

$$d_{0,2r_1}(E, Z) \leq \varepsilon_2 \tag{4.45}$$

and

$$\left| \mathcal{H}^d(E \cap B(y, t)) - \mathcal{H}^d(Z \cap B(y, t)) \right| \leq \varepsilon_2 2^d r_1^d \tag{4.46}$$

for $y \in \mathbb{R}^n$ and $t > 0$ such that $B(y, t) \subset B(0, 2r_1)$. Apply (4.46) with $y = 0$ and $t = 2r_1$ to get that

$$\begin{aligned} \mathcal{H}^d(Z \cap B(0, 2r_1)) &\leq \mathcal{H}^d(E \cap B(0, 2r_1)) + \varepsilon_2 2^d r_1^d \\ &= \theta(0, 2r_1)\, r_1^d + \varepsilon_2 2^d r_1^d \leq \Big(\frac{3\pi}{2} + \varepsilon_1 + 2^d \varepsilon_2\Big) r_1^d \end{aligned} \tag{4.47}$$

by (4.44). If ε_1 and ε_2 are small enough, $\frac{3\pi}{2} + \varepsilon_1 + 2^d \varepsilon_2 < d_T$ (the density of a cone of type $\mathbb{T}$), and Z, which is centered at the origin, cannot be of type $\mathbb{T}$. By the same argument,

$$\begin{aligned} \mathcal{H}^d(Z \cap B(0, 2r_1)) &\geq \mathcal{H}^d(E \cap B(0, 2r_1)) - \varepsilon_2 2^d r_1^d \\ &= \theta(0, 2r_1) 2^d r_1^d - \varepsilon_2 2^d r_1^d \\ &\geq \Big(\frac{3\pi}{2} - 2^d \varepsilon_2\Big) 2^d r_1^d > \pi 2^d r_1^d \end{aligned} \tag{4.48}$$

by (4.46) and (4.44), and if ε_2 is small enough, so Z is not a plane either. That is, Z is of type $\mathbb{Y}$. Then

$$\mathcal{H}^d(Z \cap B(y, t)) \leq \frac{3\pi}{2}\, t^d \tag{4.49}$$

for every ball $B(y,t)$; the easiest way to see this is to observe that since Z is a minimal set, $t^{-d}\mathcal{H}^d(Z \cap B(y,t))$ is a nondecreasing function of t (in fact, (4.7) is also true with the same proof when the center x lies out of E), whose limit at $+\infty$ is $\frac{3\pi}{2}$. Return to (4.46), and apply it with $y \in E \cap B(0,r_1)$ and $t = r_1$; this yields

$$\mathcal{H}^d(E \cap B(y,r_1)) \leq \mathcal{H}^d(Z \cap B(y,r_1)) + \varepsilon_2 2^d r_1^d \leq \frac{3\pi}{2} r_1^d + \varepsilon_2 2^d r_1^d, \tag{4.50}$$

by (4.49). Then, by (4.7),

$$\theta(y,\rho) \leq \theta(y,r_1) = r_1^{-d}\mathcal{H}^d(E \cap B(y,r_1)) \leq \frac{3\pi}{2} + 2^d\varepsilon_2 \tag{4.51}$$

for $y \in B(0,r_1)$ and $0 < \rho \leq r_1$. Denote by

$$E_Y = \big\{y \in E\,;\, \theta(y) = \frac{3\pi}{2}\big\} \tag{4.52}$$

the set of points of type $\mathbb{Y}$ in E; (4.51) will be easier to use when $y \in E_Y$, because since $\theta(y) = \frac{3\pi}{2}$, we can apply the almost constant density property again, to the pair $(y, \rho/2)$. We get that for

$$y \in E_Y \cap B(0,r_1) \text{ and } 0 < \rho \leq r_1/2, \tag{4.53}$$

we can find a minimal cone $Z(y,\rho)$, centered at y, such that

$$d_{y,\rho}(E, Z(y,\rho)) \leq \varepsilon_3 \tag{4.54}$$

and, as in (4.18),

$$\big|\mathcal{H}^d(E \cap B(x,r)) - \mathcal{H}^d(Z(y,\rho) \cap B(x,r))\big| \leq \varepsilon_3 \rho^d \tag{4.55}$$

for $x \in \mathbb{R}^n$ and $r > 0$ such that $B(x,r) \subset B(y,\rho)$, with an $\varepsilon_3 > 0$ which is as small as we want (provided that we take ε_2 small enough).

This is essentially as good as before, except that it only applies to $y \in E_Y$. If we want to continue the argument, we need to be able to find sufficiently many points of E_Y. The following existence lemma will help.

LEMMA 4.56. *Let E be a reduced and minimal set of dimension 2 in $B(0,4)$, and assume that there is a cone Y of type $\mathbb{Y}$, centered at 0, such that $d_{0,4}(E,Y) \leq \varepsilon$. Then (if ε is small enough) $E \cap B(0,1)$ contains at least a point of type $\mathbb{Y}$.*

Let us sketch the proof. Incidentally, this is one of the places where it is clear that J. Taylor used similar arguments. We first need to check that

$$\theta(x,1) \leq \frac{3\pi}{2} + \eta \ \text{ for } x \in B(0,2), \tag{4.57}$$

where $\eta > 0$ is as small as we want.

We prove this by contradiction and compactness. If this fails, we can find E_k and Y_k, with $d_{0,4}(E_k,Y_k) \leq \varepsilon_k = 2^{-k}$ but for which (4.57) fails. By rotation invariance, we may assume that $Y_k = Y$ for a fixed Y, and then $\{E_k\}$ converges to Y in $B(0,4)$.

Since (4.57) fails, we can find $x_k \in B(0,2)$ such that $\theta_k(x_k,1) \geq \frac{3\pi}{2} + \eta$, where θ_k is the density function for E_k. We may replace $\{E_k\}$ with a subsequence, so that $\{x_k\}$ converges to some $x \in \overline{B}(0,2)$.

Let $\rho > 1$ be very close to 1, to be chosen soon. By the analogue of (4.49) for Y and the uppersemicontinuity estimate that follows from the proof of Theorem 3.17 (and as for (4.20)),

$$\begin{aligned} \frac{3\pi\rho^d}{2} &\geq \mathcal{H}^d(Y \cap \overline{B}(x,\rho)) \geq \limsup_{k\to+\infty} \mathcal{H}^d(E_k \cap \overline{B}(x,\rho)) \\ &\geq \limsup_{k\to+\infty} \mathcal{H}^d(E_k \cap B(x_k,1)) \geq \frac{3\pi}{2} + \eta \end{aligned} \tag{4.58}$$

because $B(x_k, 1) \subset B(x, \rho)$ for k large, and by definition of x_k. This is impossible if ρ is close enough to 1; (4.57) follows.

Return to the sets E and Y of the lemma. Choose η so small that $\frac{3\pi}{2} + \eta < d_T$, where d_T denotes the density of a cone of type $\mathbb{T}$. Then $\theta(x) \leq \theta(x,1) < d_T$ for $x \in E \cap B(0,2)$, by (4.7) and (4.57), and so $E \cap B(0,2)$ contains no point of type $\mathbb{T}$.

Let us proceed by contradiction, and assume that $E \cap B(0,1)$ contains no point of type $\mathbb{Y}$. Then every point $x \in E \cap B(0,1)$ is of type $\mathbb{P}$, and the part of Theorem 4.1 that we already proved implies that x has a neighborhood where E is biHölder equivalent to a plane.

We want to find a curve in $B(0,1)$ which intersects E three times transversally; then we will deform this curve to a point inside $B(0,1)$, say that the number of intersections stays odd, and get the desired contradiction.

Let us assume for simplicity that Y is composed of three vertical half planes. Denote by P the horizontal plane through 0, set $S = P \cap \partial B(0,1/2)$, and denote by a_1, a_2, a_3 the three points of $S \cap Y$. Since $d_{0,4}(E,Y) \leq \varepsilon$, we can find $x_i \in E$ such that $|x_i - a_i| \leq 4\varepsilon$.

Let P_i denote the (vertical) plane parallel to the face of Y that contains a_i, and which passes through x_i. Notice that

$$d_{x_i,10^{-1}}(E, P_i) \leq 100\varepsilon \tag{4.59}$$

by elementary geometry and because $d_{0,4}(E,Y) \leq \varepsilon_k$. Because of (4.59), and by the same proof by compactness as for (4.57), we get that

$$\theta(x_i, \frac{1}{20}) \leq \pi + \eta. \tag{4.60}$$

Then (if η is chosen small enough) our proof of Theorem 4.1 for points of type $\mathbb{P}$ shows that in $B(x_i, 10^{-4})$, E is equivalent to a plane, through a biHölder mapping f_i such that $|f_i(z) - z| \leq 10^{-5}$ for $z \in \mathbb{R}^n$, as in (4.31).

We then use the three f_i to replace three short arcs of S near the a_i with three short curves, obtained as images of line segments by the f_i, and that cut E transversally (in the local coordinates given by the f_i). The rest of S does not meet E (recall that $d_{0,4}(E,Y) \leq \varepsilon$). So we get a curve γ that meets E exactly three times.

We contract γ inside $B(0,1/2)$, to a point of $B(0,1/2) \setminus E$. With a little bit of topology, we show that along the contraction, the number of intersections with E stays odd, which contradicts the fact that it is null at the end. For the topological verification, which we shall not do here, we manage to reduce to the situation where we just need to compare two curves that coincide out of a small neighborhood of a point $x \in E \cap B(0,1)$, where we have seen that E is equivalent to a plane through a biHölder mapping. In addition, we choose our contraction so that in these local coordinates, both curves are piecewise linear and cross E transversally; the fact that the parity of the number of intersections is preserved is then easy.

This completes our description of the proof of Lemma 4.56. In higher codimensions, the lemma stays true, but we need to replace the final argument with curves by a little bit of degree theory. □

4.8. How to end the proof at a point of type $\mathbb{Y}$ or $\mathbb{T}$. We return to the the proof of the biHölder variant of Theorem 4.1 near a point of type $\mathbb{Y}$. As we shall see, nothing very exciting happens after we use Lemma 4.56 to find points in E_Y, but we describe the argument with some detail to convince the reader of precisely that.

Recall that we chose a small radius r_1 below (4.43); we want to take $r_0 = 10^{-3}r_1$, and prove that

$$\beta_c(x,r) \le \varepsilon_0 \ \text{ for } x \in E \cap B(x_0, 2r_0) \text{ and } 0 < r \le 2r_0, \tag{4.61}$$

which means that we want to find a cone $Z(x,r)$ such that $d_{x,r}(E, Z(x,r)) \le \varepsilon_0$.

For points of E_Y, we already found $Z(x,r)$, by (4.53) and (4.54) (and if we choose $\varepsilon_3 < \varepsilon_0$). So let us consider $x \in E \cap B(x_0, 2r_0) \setminus E_Y$, and set

$$\delta(x) = \operatorname{dist}(x, E_Y) \le |x| \le 2r_0 = 2 \cdot 10^{-3} r_1, \tag{4.62}$$

where the inequalities come from the fact that 0 is of type $\mathbb{Y}$ and $x \in B(0, 2r_0)$. Choose $y \in E_Y$ such that

$$|y - x| \le 2\delta(x) \le 4 \cdot 10^{-3} r_1. \tag{4.63}$$

Let $0 < r \le 2r_0$ be as in (4.61), and set

$$\rho = 10\delta(x) + 10r < 10^{-1} r_1; \tag{4.64}$$

then (4.53) holds, and we can find $Z(y,\rho)$ such that (4.54) and (4.55) holds. In particular, $d_{y,\rho}(E, Z(y,\rho)) \le \varepsilon_3$.

If $r \ge \frac{\delta(x)}{20}$, we can use $Z(x,r) = Z(y,\rho)$, because then

$$\begin{aligned} d_{x,r}(E, Z(x,r)) &= d_{x,r}(E, Z(y,\rho)) \le \frac{\rho}{r}\, d_{y,\rho}(E, Z(y,\rho)) \\ &\le \frac{\rho\varepsilon_3}{r} = \frac{10\delta(x) + 10r}{r}\,\varepsilon_3 \le 300\varepsilon_3 \le \varepsilon_0 \end{aligned} \tag{4.65}$$

since $B(x,r) \subset B(y,\rho)$ and if ε_3 is small enough. So we may assume that $0 < r \le \frac{\delta(x)}{20}$.

Denote by L the singular set of $Z(y,\rho)$. Thus L is the line at the intersection of the faces of $Z(y,\rho)$ if it is of type $\mathbb{Y}$, the union of four half lines emanating from the center of $Z(y,\rho)$ if it is of type $\mathbb{T}$, and the empty set if $Z(y,\rho)$ is a plane (which is unlikely). Let us check that

$$\operatorname{dist}(x, L) \ge \frac{\delta(x)}{4}. \tag{4.66}$$

Otherwise, we can find $\xi \in L$ such that $|\xi - x| \le \frac{\delta(x)}{2}$ and, if $Z(y,\rho)$ is of type $\mathbb{T}$, such that the distance from ξ to the center of $Z(y,\rho)$ is larger than $\frac{\delta(x)}{5}$. Because of this, $Z(y,\rho)$ coincides in $B(\xi, \frac{\delta(x)}{20})$ with a minimal cone Y of type $\mathbb{Y}$, centered at ξ. Notice that

$$|y - \xi| + \frac{\delta(x)}{40} \le |y - x| + |x - \xi| + \frac{\delta(x)}{40} \le 2\delta(x) + \frac{\delta(x)}{2} + \frac{\delta(x)}{40} < \frac{\rho}{3} \tag{4.67}$$

by (4.63) and (4.64), so $B(\xi, \frac{\delta(x)}{40}) \subset B(y,\rho)$ and hence

$$\begin{aligned} d_{\xi,\frac{\delta(x)}{40}}(E,Y) &= d_{\xi,\frac{\delta(x)}{40}}(E,Z(y,\rho)) \leq \frac{40\rho}{\delta(x)}\, d_{y,\rho}(E,Z(y,\rho)) \\ &\leq \frac{400(\delta(x)+r)\,\varepsilon_3}{\delta(x)} \leq 500\,\varepsilon_3 \end{aligned} \tag{4.68}$$

by definition of Y, (4.64), (4.54), and because $r \leq \frac{\delta(x)}{20}$. If ε_3 is small enough, (4.68) will allow us to apply Lemma 4.56 to $E' = \frac{160}{\delta(x)}(E-\xi)$. Indeed E is minimal in $U \supset B(0,4r_1)$, and $B(0,4r_1) \supset B(\xi, \frac{\delta(x)}{40})$ because $|\xi| + \frac{\delta(x)}{40} \leq |x| + |\xi - x| + \frac{\delta(x)}{40} \leq 2r_0 + \frac{\delta(x)}{2} + \frac{\delta(x)}{40} \leq 2r_0 + \delta(x) \leq 4\cdot 10^{-3} r_1$ by (4.62). Then E' is minimal in $B(0,4)$, and $d_{0,4}(E', Y-\xi) = d_{\xi,\frac{\delta(x)}{40}}(E,Y) \leq 500\varepsilon_3$ (recall that Y is centered at ξ, and use (4.68)). So Lemma 4.56 gives a point of type $\mathbb{Y}$ in $E' \cap B(0,1)$, and so $E \cap B(\xi, \frac{\delta(x)}{40})$ contains a point of type $\mathbb{Y}$. But then

$$\delta(x) = \operatorname{dist}(x, E_Y) \leq |x-\xi| + \frac{\delta(x)}{40} \leq \frac{\delta(x)}{2} + \frac{\delta(x)}{40} < \delta(x); \tag{4.69}$$

this contradiction proves (4.66). Then $Z(y,\rho)$ coincides with a plane in $B = B(x, \frac{\delta(x)}{10})$. In addition, $B \subset B(y,\rho)$ by (4.63) and (4.64), so we can apply (4.55) to it. We get that

$$\mathcal{H}^d(E \cap B(x, \frac{\delta(x)}{10})) \leq \mathcal{H}^d(Z(y,\rho) \cap B(x, \frac{\delta(x)}{10})) + \varepsilon_3 \rho^d \leq \big(\frac{\delta(x)}{10}\big)^d \pi + \varepsilon_3 \rho^d. \tag{4.70}$$

By the monotonicity property (4.7), we obtain that

$$\theta(x,t) \leq \theta(x, \frac{\delta(x)}{10}) \leq \pi + \varepsilon_3 \big(\frac{10\rho}{\delta(x)}\big)^d \tag{4.71}$$

for $0 < t \leq \frac{\delta(x)}{10}$. Obviously $\theta(x) \geq \pi$ because $x \in E$, so if ε_3 is small enough, we can apply the almost constant density property (Lemma 4.15) one last time, and get that for $0 < r \leq \frac{\delta(x)}{20}$ we can find a minimal cone $Z(x,r)$ such that $d_{x,r}(E, Z(x,r)) \leq \varepsilon_0$. This is exactly what we needed for (4.61) in our last remaining case, and this completes our description of the proof when the origin is a point of type $\mathbb{Y}$.

Let us finally say a few words about the last case when 0 is a point of type $\mathbb{T}$. We do not need an existence lemma for points of type $\mathbb{T}$, because we already have the origin, and we expect no other point of type $\mathbb{T}$.

Let us only sketch the argument, because it is almost the same as when 0 is a point of type $\mathbb{Y}$, but longer. We start as before: we choose r_1 so small that $\theta(0,4r_1) \leq \theta(0) + \varepsilon_1$, and hence also $\theta(0,r) \leq \theta(0) + \varepsilon_1$ for $0 < r \leq 4r_1$. By the almost constant density property, we can find minimal cones $Z(0,r)$, $0 < r \leq 2r_1$, centered at 0, and such that

$$d_{0,r}(E, Z(0,r)) \leq \varepsilon_2 \tag{4.72}$$

and

$$\big|\mathcal{H}^d(E \cap B(y,t)) - \mathcal{H}^d(Z(0,r) \cap B(y,t))\big| \leq \varepsilon_2 r^d \tag{4.73}$$

for $y \in \mathbb{R}^n$ and $t > 0$ such that $B(y,t) \subset B(0,r)$.

When we apply this with $y = 0$ and $t = r$, we see that $\mathcal{H}^d(Z(0,r) \cap B(y,t)) > \frac{3\pi}{2} t^d$ (because $\theta(0,r) \geq \theta(0) = d_T$), hence $Z(0,r)$ is of type $\mathbb{T}$.

When we apply (4.73) to pairs (y,t) such that $\frac{r}{10} \leq |y| \leq \frac{r}{2}$ and $t = \frac{|y|}{10}$, we get that

$$(4.74)\quad \mathcal{H}^d(E\cap B(y,t)) \leq \mathcal{H}^d(Z(0,r)\cap B(y,t))+\varepsilon_2 r^d \leq \frac{3\pi t^d}{2}+\varepsilon_2 r^d \leq \frac{3\pi t^d}{2}+C\varepsilon_2 t^d$$

because $Z(0,r)$ coincides with a cone of type $\mathbb{P}$ or $\mathbb{Y}$ in $B(y,t)$. Thus $\theta(y, \frac{|y|}{10}) \leq \frac{3\pi}{2} + C\varepsilon_2$.

If in addition $y \in E_Y$, this allows us to apply the almost constant density property and find, for $0 < \rho \leq \frac{|y|}{20}$, a minimal cone $Z(y,t)$ such that

$$(4.75)\qquad d_{y,\rho}(E, Z(y,\rho)) \leq \varepsilon_3$$

and

$$(4.76)\qquad \big|\mathcal{H}^d(E \cap B(x,t)) - \mathcal{H}^d(Z(y,\rho) \cap B(x,t))\big| \leq \varepsilon_3 \rho^d$$

for $x \in \mathbb{R}^n$ and $t > 0$ such that $B(x,t) \subset B(y,\rho)$.

Now we claim that we control the balls centered at the origin (by (4.72)), the balls $B(y,\rho)$ centered on E_Y and for which $\rho \leq \frac{|y|}{20}$ (by (4.75)), but also for which $\rho > \frac{|y|}{20}$, because in this last case we can use $Z(y,\rho) = Z(0, 2(|y|+\rho))$, just because

$$(4.77)\qquad d_{y,\rho}(E, Z(y,\rho)) \leq \frac{2(|y|+\rho)}{\rho}\, d_{0,2(|y|+\rho)}(E, Z(y,\rho)) \leq C\varepsilon_2$$

by (4.72).

We are thus left with the balls $B(x,t)$ centered on $E \setminus [E_Y \cup \{0\}]$. For these we proceed as before. We set $\delta(x) = \min(|x|, \mathrm{dist}(x, E_Y))$, and notice that we have a control on $B(x, 10\delta(x))$ because of one of the balls that we already treated. So we get a nearby cone Z, and we can take $Z(x,t) = Z$ if $t \geq 10^{-5}\delta(x)$, say. For $t < 10^{-5}\delta(x)$, we first show that $B(x, \frac{\delta(x)}{100})$ does not meet the singular set of Z, as we did before (4.70), use this to show that $\mathcal{H}^d(Z \cap B(x, \frac{\delta(x)}{1000})) \leq \pi\big(\frac{\delta(x)}{1000}\big)^d + C\varepsilon_3\delta(x)^d$, deduce from this that $\theta(x,2t) \leq \theta(x, \frac{\delta(x)}{1000}) \leq \pi + C\varepsilon_3$, and use the almost constant density property to find the good cone $Z(x,t)$. The details are a little long and boring, but no difficulty appears.

This completes our description of Theorem 4.1, in the simpler case of minimal sets, and with the easier bi-Hölder conclusion. □

4.9. Last comments, C^1 regularity, and epiperimetry.

Remark 4.78. At this date we do not have an existence lemma for points of type $\mathbb{T}$, that would be similar to Lemma 4.56. Such a lemma would say that if E is a reduced almost minimal set in $B(0,4r)$, with a sufficiently small gauge function h, and if $d_{0,4}(E,T) \leq \varepsilon$ for some minimal cone T of type $\mathbb{T}$ centered at the origin, then there is a point of type $\mathbb{T}$ in $B(0,1)$.

We did not need this for J. Taylor's result, but it would be enough to prove that every reduced minimal set in the whole $\mathbb{R}^3$ is a minimal cone (see Section 5). Also, if we want to establish a J. Taylor theorem for 3-dimensional sets in $\mathbb{R}^4$, we will probably need a lemma that finds points of type $\mathbb{T}\times\mathbb{R}$ when E looks a lot like the minimal cone $\mathbb{T}\times\mathbb{R}$ in the unit ball.

The corresponding extension of Lemma 4.56 for points of type $\mathbb{Y}\times\mathbb{R}$ in $\mathbb{R}^4$ is true, and leads to a Hölder regularity result for 3-dimensional almost minimal sets near points of type $\mathbb{Y}\times\mathbb{R}$; see [Lu].

Remark 4.79 on C^1 regularity. Theorem 4.1 is significantly more delicate than its bi-Hölder variant. Here we just needed to check that the numbers $\beta_c(x,r)$ that control the Hausdorff distances to minimal cones stay small. For the C^1 result, we would need to prove that they have some definite decay.

Almost equivalently, we proved that the density $\theta(x,r)$ is a nondecreasing function of r, and for the C^1 result, it seems that we need to prove that the density excess $f(r) = \theta(x,r) - \theta(x)$ decays when r tends to 0, at some definite speed.

As we have seen, the monotonicity of $\theta(x,\cdot)$ is obtained rather easily, essentially by comparing E with the cone over $E \cap \partial B(x,r)$. The necessary decay for the density excess is harder to get. In [T2] it is obtained as the consequence of some anterior epiperimetry result [T1], which I am not sure I understand.

In [D7], the decay for f is obtained in the following way. Because of Theorem 4.3 (the bi-Hölder estimate), we can reduce to situations where E is quite close to a minimal cone X. We suppose that E does not coincide with X, measure the difference in terms of the density excess $f(r) = \theta(x,r) - \theta(x)$, use this difference to construct a competitor which is even better than the cone over $E \cap \partial B(x,r)$, make sure that the improvement over the cone is large enough compared to the density excess, and use this to get a differential inequality on f that implies the desired decay. Finally we check that f controls other geometric quantities like the $\beta_c(x,r)$.

Somehow, the fact that we can suppose that E is close to a the minimal cone X acts as a linearization of our problem: a situation where E is completely wild would be much harder to control at once.

We use X in the following way. Since X is a minimal cone, $X \cap \partial B(x,r)$ is a collection of geodesics g_i on $\partial B(x,r)$ that may meet by sets of three at vertices (see the description in Subsection 4.1). We use the fact that E is close to X to find a net of nearby curves γ_i, with almost the same vertices, and such that the symmetric difference between $E \cap \partial B(x,r)$ and $\cup_i \gamma_i$ has small $\mathcal{H}^1$ measure. We can even modify the γ_i so that they are Lipschitz graphs with small constants.

Recall that we want to construct good competitors for E, better than the cone over $E \cap \partial B(x,r)$, and after a first deformation that is not too costly, we are reduced to constructing good competitors for the cone over $\cup_i \gamma_i$. This is much easier now. For instance, we can proceed independently for each i, find a competitor for the cone Γ_i over γ_i with the same boundary (i.e., γ_i and the two segments from x to the extremities of γ_i), and then glue.

We then observe that for each i, Γ_i is the graph of a radial extension, and the graph of the corresponding harmonic extension does significantly better, with a good control on the difference in terms of how close γ_i is to a geodesic. This is what we mean by a linearization: we are able to reduce to a situation that is so close to a flat case that we can use asymptotic expansions.

This works well when (some of) the γ_i are not too close to geodesics, but we also need to do something when the γ_i are almost geodesics, but not with the same endpoints as that g_i. This situation should be simpler now because we can compute better with the cone over a collection of geodesics, but this is the place where additional information on the metric properties of our cone X comes into play. We want to know how the sum of the lengths of the g_i changes when we move the vertices, and the full length property discussed near (4.4) is the notion that comes out from the computations.

It is probable that the two different ways to get the desired decay estimates for f are similar; in both cases, some specific knowledge of metric properties of the minimal cone X are needed.

5. Mumford-Shah minimal sets and a Bernstein problem

We start with a variant, for minimal sets, of the classical Bernstein problem where you take a graph over the plane, assume that it is a minimal surface, and ask whether it must be a plane.

(5.1) Is every 2-dimensional reduced minimal set in $U = \mathbb{R}^3$ equal to a cone?

See Definitions 3.2 and 3.5 for reduced minimal sets. We expect this to happen because this is the only way for the set not to get worse at infinity. The question would also make sense in higher dimensions, but it should be harder and we don't even know the answer in the present case.

The author discovered this question (and Almgren minimal sets at the same time) because of the Mumford-Shah functional in image segmentation, where a positive answer helps describe the blow-up limits of minimal segmentations, and then get some regularity properties for the minimal segmentations themselves.

Fortunately the answer to this question is easier in the Mumford-Shah context. Let us define the notion of Mumford-Shah minimal set only in the domain $U = \mathbb{R}^n$ (this is simpler), and refer the reader to [AFP], [D2], [D4], [MoS] for information on the Mumford-Shah functional and image processing.

Definition 5.2. Let E be a closed set in $\mathbb{R}^n$, and let $B \subset \mathbb{R}^n$ be a closed ball. An MS-competitor for E in B is a closed set $F \subset \mathbb{R}^n$ such that

$$F \setminus B = E \setminus B \tag{5.3}$$

and, for $x, y \in \mathbb{R}^n \setminus [B \cup E]$,

$$F \text{ separates } x \text{ from } y \text{ if } E \text{ separates } x \text{ from } y. \tag{5.4}$$

Here we say that E separates x from y when x and y lie in different connected components of $\mathbb{R}^n \setminus E$. Clearly, if F is an MS-competitor for E in B, then it is also an MS-competitor for E in any larger ball B. In $\mathbb{R}^2$, for example, $\mathbb{R} \cup \partial B(0,1)$ is a competitor for $\mathbb{R}$ in (any ball that contains) $\overline{B}(0,1)$, while $\mathbb{R} \setminus (-1,1)$ is not a competitor for $\mathbb{R}$ (in any ball).

Definition 5.5. An MS-minimal set in $\mathbb{R}^n$ is a closed set $E \subset \mathbb{R}^n$ such that $\mathcal{H}^{n-1}(E \cap B) < +\infty$ for every ball B, and

$$\mathcal{H}^{n-1}(E \cap B) \leq \mathcal{H}^{n-1}(F \cap B) \tag{5.6}$$

whenever $B \subset \mathbb{R}^n$ is a closed ball and F is an MS-competitor for E in B.

Thus $d = n - 1$ here. Sets of dimensions smaller than $n - 1$ do not separate points in $\mathbb{R}^n$, so the notion would not be relevant for $d < n - 1$. Notice also that (5.6) stays true if we replace B with a larger ball.

In terms of Alexis Bonnet's definitions relative to the Mumford-Shah functional, $E \subset \mathbb{R}^n$ is an MS-minimal set if and only if the pair (u, E), where u is constant on every component of $\mathbb{R}^n \setminus E$, is a global Mumford-Shah minimizer. Of course we shall not use this here.

It is not so hard to check that if E is a closed subset of $\mathbb{R}^n$ and if F is an Almgren competitor for E (i.e., if $F = \varphi(E)$ for some Lipschitz mapping $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\varphi(x) = x$ for $|x|$ large, as in Definition 3.2), then F is an MS-competitor for E. That is, if x, y lie far from the origin and in different connected components of $\mathbb{R}^n \setminus E$, then $\varphi(E)$ also separates them.

Because of this, every MS-minimal set in $\mathbb{R}^n$ is a minimal set as in Definition 3.2, with $h = 0$ (we shall say Al-minimal to distinguish). And it is true that

(5.7) every 2-dimensional reduced MS-minimal set in $\mathbb{R}^3$ is a cone.

Here reduced is still meant as in Definition 3.5. Of course, our set is then a plane or a cone of type $\mathbb{Y}$ or $\mathbb{T}$, as in Section 4.1, because it is Al-minimal. We don't know whether something like this is true in higher dimensions.

Let us describe the proof of (5.7) given in [D6] because it is amusing. Its idea (the amusing part) was known from F. Morgan; see [M1], even though he was working in a different context. If we had an existence lemma for points of type $\mathbb{T}$ (like Lemma 4.56), we would easily get a positive answer to (5.1) by the same argument.

So let E be a reduced Al-minimal set of dimension 2 in $\mathbb{R}^3$ (we shall mention it explicitly when we use the MS-minimality). Pick any origin $x \in E$, and recall from (4.7) that $\theta(x, r) = r^{-2}\mathcal{H}^2(E \cap B(x, r))$ is nondecreasing. Then it has limits $\theta(x) = \lim_{r\to 0} \theta(x, r)$ and $\theta_\infty = \lim_{r\to +\infty} \theta(x, r)$, and the last one is easily seen not to depend on x.

If for some $x \in E$, $\theta(x) = \theta_\infty$, then $\theta(x, \cdot)$ is constant and (4.14) says that E is a minimal cone. So we may assume that

$$\theta(x) < \theta_\infty \text{ for all } x \in E. \tag{5.8}$$

Recall that we can find blow-in limits of E (as above (4.28)), and by (4.28) they are minimal cones with density θ_∞. Let us distinguish cases, depending on the value of θ_∞.

If $\theta_\infty = \pi$, i.e., if the blow-in limits of E are planes, we pick any $x \in E$, and any blow-up limit of E at x. Such blow-up limits exist, and they are cones of type $\mathbb{P}$, $\mathbb{Y}$, or $\mathbb{T}$. Then $\theta(x)$, which is the density of such a cone, cannot be smaller than $\theta_\infty = \pi$. This contradicts (5.8) and we get the result.

Next suppose that $\theta_\infty = \frac{3\pi}{2}$, i.e., that the blow-in limits of E are cones of type $\mathbb{Y}$. Select a sequence $\{r_k^{-1}E\}$, with $\lim_{k\to+\infty} r_k = +\infty$, that tends to a cone Y of type $\mathbb{Y}$. Recall that this implies that for $R > 0$,

$$\lim_{k\to+\infty} d_{0,R}(r_k^{-1}E, Y) = 0, \tag{5.9}$$

as in (4.24). We apply this with $R = 4$, and choose k so large that $d_{0,4}(r_k^{-1}E, Y) \leq \varepsilon$, where ε is as in Lemma 4.56. Then this lemma says that $r_k^{-1}E$ contains a point y of type $\mathbb{Y}$, and of course $w = x + r_k y$ is a point of type $\mathbb{Y}$ in E. This is impossible again, because (5.8) says that $\theta(w) < \theta_\infty = \frac{3\pi}{2}$.

We are left with the case when $\theta_\infty = d_T$, where d_T is the density at the origin of a cone of type $\mathbb{T}$. If we had an existence lemma for points of type $\mathbb{T}$, we could proceed as above, find a point $w \in E$ such that $\theta(w) = d_T$, and get a contradiction with (5.8). And in fact we will essentially prove an existence lemma in the present situation with an MS-minimal set.

So let us assume that E contains no point of type $\mathbb{T}$ (otherwise, we get a contradiction with (5.8)). Let T be a blow-in limit of E; it is a minimal cone of type $\mathbb{T}$, centered at the origin. As before, we can find a radius r_k such that if we set $F = r_k^{-1}E$,

$$\delta_{0,4}(F,T) \leq \varepsilon, \tag{5.10}$$

where $\varepsilon > 0$ is as small as we want. Of course, F is also a minimal set, and it has no point of type $\mathbb{T}$. Denote by F_Y the set of points of type $\mathbb{Y}$ in F.

Denote by L the singular set of T; thus L is the union of four half lines L_i, $1 \leq i \leq 4$. Denote by a_i, $1 \leq i \leq 4$, the point of $L_i \cap \partial B(0,2)$ (you may start to look at Figure 6). If ε is small enough, we can apply Lemma 4.56 to $B(a_i, 10^{-10})$ to find a point y_i of type $\mathbb{Y}$ in F, with $|y_i - a_i| \leq 10^{-10}$.

We shall use Theorem 4.3 (the biHölder equivalence) rather than the full theorem of J. Taylor (Theorem 4.1), because it is easier to prove. But the reader is free to apply Theorem 4.1 secretly and work with C^1 curves below. Near each y_i, F is biHölder equivalent to a minimal cone of type Y, and F_Y is biHölder equivalent to a line. In fact, the proof of Theorem 4.3 also says that this happens in the ball $B(y_i, 10^{-3})$, say, and that in this ball F_Y coincides with a simple curve that really crosses $\partial B(0,2)$, in the sense that one of its ends lies out of $B(0, 2+10^{-6})$ and the other one lies inside $B(0, 2-10^{-6})$.

We also need to know that outside of the $B(y_i, 10^{-6})$, $\partial B(0,2)$ never gets close to F_Y. This is true because near such points of $\partial B(0,2)$, T and F are very close to a plane, and Theorem 4.3 implies that F is equivalent to a plane and contains no point of type $\mathbb{Y}$.

Return to F_Y. We know that near y_j, F_Y coincides with a simple (biHölder) curve γ_i. We want to follow this curve as it gets inside $B(0,2)$, and follow it as long as we can. That is, we try to continue γ_i into a simple curve contained in F_Y, as long as possible and as suggested by Figure 6.

By Theorem 4.3 again, we know that every point $y \in F_Y$ has a small neighborhood where F_Y is a simple curve. We use this to continue γ_i, little by little. We never hit a point x where we cannot continue γ_i, because $x \in F$ (which is closed), x cannot be of type $\mathbb{T}$ (there is no such point), nor a point of type $\mathbb{P}$ (because such points lie at positive distance from F_Y, by Theorem 4.3), and because at a point of type $\mathbb{Y}$ we can continue the curve. Similarly, γ_i does not return to itself (the first point where this happens would not be of type $\mathbb{Y}$), nor accumulate on a complicated set (any point of this set would be of type $\mathbb{Y}$, a contradiction), so the only thing that it can do is eventually leave $B(0,2)$. In addition, this does not happen near the same a_i (as above, the curve does not return to itself). This means that γ_i goes from y_i to some other y_j.

Let y be any point of F_Y. By Theorem 4.3, there is a small neighborhood of y where $\mathbb{R}^n \setminus E$ is equivalent to the complement of a cone of type $\mathbb{Y}$. This means that there are at most three connected components of $B(0,3) \setminus F$ that touch y. Some of these three components may coincide (because we don't know what happens away from y), but this is all right. Let us denote by $H(y)$ this set of at most three components. By definitions and the description of F near y, $H(y') = H(y)$ for $y \in F_Y$ close to y. That is, H is locally constant along F_Y.

Denote by O_k the connected component of $B(0,3) \setminus F$ that contains $-a_k$. Recall that we have a biHölder equivalence in the large ball $B(y_i, 10^{-3})$, and because of

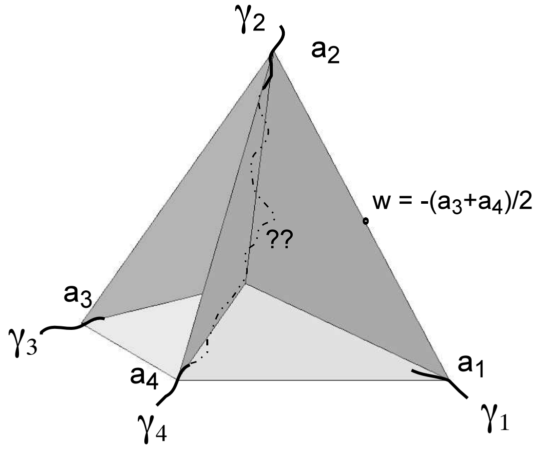

FIGURE 6. The set T and the curves γ_i (imagine that F is very close to T)

this it is fairly easy to check that $H(y_i)$ is composed of the three O_k, $k \neq i$. Again two of these components could be equal.

But when we go from y_i to y_j along γ_i, $H(y)$ stays constant. So $H(y_i) = H(y_j)$. But $H(y_j)$ is composed of the three components O_k, $k \neq j$. Altogether, each of the four components O_k lies in $H(y_i)$, and since $H(y_i)$ has at most three elements, we find that there exist $k \neq l$ such that $O_k = O_l$.

In the context of Al-minimal sets, this would not yet give a topological contradiction: one can produce sets that are locally C^1 equivalent to a cone of type $\mathbb{Y}$ or $\mathbb{P}$ at each point, and that coincide with a cone of type $\mathbb{T}$ on $\mathbb{R}^n \setminus B(0,1)$. We only proved that for such a set F, $\mathbb{R}^n \setminus F$ has at most 3 components (in fact, at most 2 if we use another curve γ_k in F_Y). See [D6] for an example of such a set, which was probably also known before by B. Hardt and F. Morgan. See for instance [M5], page 110. But Liang [Li1] showed that an Al-minimal set which is not a cone cannot be modelled on this example, because the two tubes in $\mathbb{R}^3 \setminus F$ that connect opposite components are not knotted.

In the context of MS-minimal sets, we are happy because we find out that the big wall between $-a_k$ and $-a_l$ is not needed. That is, set $w = -\frac{a_k+a_l}{2}$ (see Figure 6 again). This point lies on the middle of a face of T, the face that separates $-a_k$ from $-a_l$. Theorem 4.3 (applied to a point of F near w) shows that $B(w, 10^{-3})$ does not meet any other component of $B(0,3) \setminus F$ than O_k and O_l. Then set $G = F \setminus B(w, 10^{-3})$. It is not hard to see that G is a competitor for F in $B = \overline{B}(0,4)$ (the only components that could possibly become equal when we replace F with G are O_k and O_l, which happen to be equal before we start). Also, $\mathcal{H}^2(G \cap B) < \mathcal{H}^2(F \cap B)$, so (5.6) fails, F is not minimal, and we get the desired contradiction. This completes our description of the proof of (5.7); see [D6] for details. □

Since the situation is already fairly complicated in $\mathbb{R}^3$, the author does not expect an analogue of (5.7) in higher dimensions to be easy.

Remark 5.11 about topological minimal sets in higher codimensions. The notion of MS-minimal is nice, because of its connections to the Mumford-Shah functional, and also because in this context existence results could be easier. The definition above only makes sense in codimension 1, but a generalization of the notion was recently proposed by Xiangyu Liang [Li1,3]. Let us discuss this rapidly. As above, we only define her topological minimal sets in $\mathbb{R}^n$, because this is simpler.

DEFINITION 5.12. Let E be a closed set in $\mathbb{R}^n$, and let $B \subset \mathbb{R}^n$ be a closed ball. A topological competitor of dimension d for E in B is a closed set $F \subset \mathbb{R}^n$ such that

$$F \setminus B = E \setminus B \tag{5.13}$$

and, for each Euclidean sphere S of dimension $n-d-1$ that is contained in $\mathbb{R}^n \setminus [B \cup E]$, if S represents a nonzero element in the singular homology group $H_{n-d-1}(\mathbb{R}^n \setminus E; \mathbb{Z})$, then S also represents a nonzero element in $H_{n-d-1}(\mathbb{R}^n \setminus F; \mathbb{Z})$.

When $d = n-1$, S is composed of two points, and it represents a nonzero element in $H_0(\mathbb{R}^n \setminus E; \mathbb{Z})$ if and only if these two points lie in different connected components of $\mathbb{R}^n \setminus E$, so Definition 5.12 is really a generalization of Definition 5.2. The corresponding notion of topological minimal set is as follows (we just change the class of competitors).

DEFINITION 5.14. A topological minimal set of dimension d in $\mathbb{R}^n$ is a closed set $E \subset \mathbb{R}^n$ such that $\mathcal{H}^d(E \cap B) < +\infty$ for every ball B, and for which

$$\mathcal{H}^d(E \cap B) \le \mathcal{H}^d(F \cap B) \tag{5.6}$$

whenever $B \subset \mathbb{R}^n$ is a closed ball and F is a topological competitor of dimension d for E in B.

There is an analogue of the Bernstein problems (5.1) and (5.7) for this notion, but even the case of 2-dimensional topological minimal sets in $\mathbb{R}^4$ is unclear.

Topological minimal sets have some nice properties. First, one can show that they are Al-minimal sets. For this, using homology groups was a good idea, because it turns out that the nonvanishing of S in $H_{n-d-1}(\mathbb{R}^n \setminus E; \mathbb{Z})$ is stable under deformations of E in $\mathbb{R}^n$ (away from S); a corresponding attempt with homotopy groups would fail.

Liang shows that the almost orthogonal union of two planes of dimension d in $\mathbb{R}^n$ is topologically minimal, and that the product by $\mathbb{R}$ of a topologically minimal set is topologically minimal; note that the corresponding property for Al-minimal sets is not known. A consequence of this is that some non transversal unions of 3-planes in $\mathbb{R}^5$, for instance, are topological, and hence also Al-minimal sets. See [Li1,3] for details.

Another nice feature of the notion is that, as for the MS-minimality, related existence theorem can be proved. See the next section.

6. Existence results for simpler problems

Simpler means, compared to the Plateau problems above. In this section we describe how a construction of V. Feuvrier [Fv1,2] can be used to prove some existence results for minimizers.

We start with notation for a general situation. Let Ω be a simple "domain", which we often want to be closed, so that limits of sets in Ω stay in Ω. Or Ω can be a manifold. Let $\mathcal{F}$ be a class of closed subsets of Ω. Also let $g : \Omega \to [0, +\infty)$ be a given continuous function, and assume that $1 \leq g(x) \leq N$ for some $N \geq 1$. We want to minimize the functional

$$J_g(E) = \int_E g(x)\, d\mathcal{H}^d(x) \tag{6.1}$$

over the class $\mathcal{F}$. Of course we expect many things to depend on the specific choice of $\mathcal{F}$, and for our construction to have a chance to work we will need $\mathcal{F}$ to be closed under some class of deformations.

It is better to choose Ω and $\mathcal{F}$ so that

$$0 < m_g(\mathcal{F}) := \inf \big\{ J_g(E)\,;\, E \in \mathcal{F} \big\} < +\infty, \tag{6.2}$$

because otherwise nothing interesting happens. This is why we like to take domains Ω with some topology, to make sure that sets $E \in \mathcal{F}$ cannot be deformed to one point, for instance.

Here is the most obvious way to try to get a minimizer. Take a minimizing sequence, i.e., a sequence $\{E_k\}$ in $\mathcal{F}$ such that

$$\lim_{k \to +\infty} J_g(E_k) = m_g(\mathcal{F}), \tag{6.3}$$

and extract a subsequence (which will also be denoted by $\{E_k\}$) which converges to a limit E_∞. If we want to be sure that this is possible, we usually have no choice on the notion of convergence: we take the convergence for the Hausdorff distance (locally in Ω if Ω is open). That is, we could also hope to work in a subclass of $\mathcal{F}$ for which we have better compactness properties, but this will probably happen because in that subclass some apparently stronger notion of convergence is implied by the Hausdorff convergence, so let us not bother yet.

There are two main problems now. The first one is that it is not always true that

$$J_g(E_\infty) \leq \lim_{k \to +\infty} J_g(E_k), \tag{6.4}$$

i.e., J_g is not lowersemicontinuous. For instance, dotted lines segments $E_k \subset [0,1]$ may converge to $E_\infty = [0,1]$, while $\mathcal{H}^1(E_k) = 1/2$ for all k. This is the main issue that the Feuvrier construction addresses. The other problem is that we cannot be sure that

$$E_\infty \in \mathcal{F}. \tag{6.5}$$

This part will depend on $\mathcal{F}$; some classes will be easier to deal with than others, and we shall just give two examples where we can get something like (6.5). But in general, we can expect serious difficulties. For instance, if $\mathcal{F}$ is the class of all the sliding deformations of an initial set E_0, as in Example 3 in Section 2.4, we have the usual problem relative to parameterizations: each E''_k is a obtained through a deformation by mappings $\varphi_{t,k}$, but we have no reason to believe that the $\varphi_{t,k}$ will converge to nice mappings $\varphi_{t,\infty}$.

Example 6.6. This example is related to Example 2.20, but we will need extra assumptions. We try to follow [Fv1,3], but where unfortunately the details of this precise example are not carried out; the reader will thus have to trust the sketchy proof below. Here Ω will be a simple manifold without boundary obtained by pasting faces of dyadic cubes in $\mathbb{R}^n$. That is, we start from a finite collection of dyadic cubes Q_j of unit length, and glue some faces together so as to get a manifold without boundary $\mathbf{M}$; it does not matter whether the manifold is orientable or not.

For instance, we could get a torus $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \simeq \mathbb{S}^1 \times \mathbb{S}^1$ from a single square, by gluing the opposite faces correctly. The fact that we have no boundary left is convenient because they would restrict our definition of deformations. The precise form of rigid manifold is also convenient, because polyhedral grids are much easier to construct there.

We also have to assume that $d = 2$, and that the function $g : \mathbf{M} \to [1, N]$ is Hölder-continuous, say. This is because we intend to use Theorem 4.3.

Let us proceed as in Example 2.20. Select a closed set $E_0 \in \Omega$, and denote by $\mathcal{F}(E_0)$ the class of continuous deformations of E_0 inside Ω. That is, $F \in \mathcal{F}(E_0)$ if $F = \varphi_1(E_0)$, for some mappings $\varphi_t : E_0 \to \mathbf{M}$, $0 \leq t \leq 1$, such that $(x,t) \to \varphi_t(x) : E_0 \times [0,1] \to \mathbf{M}$ is continuous, and $\varphi_0(x) = x$ on E_0.

Claim 6.7. Under all these assumptions, we can find $F \in \mathcal{F}(E_0)$ such that

$$J_g(F) = \inf\big\{J_g(E)\,;\, E \in \mathcal{F}(E_0)\big\}. \tag{6.8}$$

Let us explain how we can prove this. The construction may be more general, but it is simpler under our assumptions. Its main point is that we want to have (6.4), for a minimizing sequence that we shall need to select carefully. The idea is not new, this is what Reifenberg did in [R1], already in the context of the Plateau problem (with homology).

We start from any minimizing sequence $\{E_k\}$ as above, and our first idea (that will not work) is to replace each E_k with its Federer-Fleming projection on a dyadic net. Recall that in codimension 1, this would be done in one step, as follows. We cut $\mathbf{M}$ into dyadic cubes Q of sidelength 2^{-m}, for some large integer $m = m(k)$. For each such Q, we select a center $\xi_Q \in Q \setminus E_k$, and use the radial projection centered at ξ_Q to project $E \cap Q$ on ∂Q. (Notice that we do not change E on the boundaries ∂Q.) When $d < n-1$, we have to iterate this procedure, project again on faces of dimension $n-2$, and so on. After a few steps, we get a new candidate which is contained in the d-dimensional skeleton of our dyadic net. Usually, when we get to faces of dimension d, we cannot continue because our set covers the full face. But if by chance it meets a face without containing it, we can iterate once more. Eventually we get a new competitor E'_k, which is a finite union of faces of dimensions at most d.

Unfortunately, $J_g(E'_k)$ may be significantly larger than $J_g(E_k)$ and $m_g(\mathcal{F}(E_0))$; even in the 2-plane, we may have projected the diagonal of a square onto a $\sqrt{2}$ times longer pair of sides. But given E_k, V. Feuvrier constructs a polyhedral net, similar to the usual dyadic net, but which near the flat rectifiable parts of E_k has faces that are almost parallel to E_k, so that after the Federer-Fleming projection on the net, $J_g(E'_k) - J_g(E_k)$ is as small as we want. This also uses the fact that g is continuous (and then we work at scales that are so small that g acts like a constant). The construction of the polyhedral nets is the main part of [Fv1] and [Fv2], and the main difficulty is to be able to complete a collection of non parallel dyadic nets living in disjoint regions, in such a way as to keep a lower bound on all the angles in the various faces of all the dimensions. The reason why we decided to stick to our rigid manifold $\mathbf{M}$ is because Feuvrier's construction of polyhedral nets has not yet been adapted to manifolds. [There are difficulties even with the definition of polyhedral nets, because of the non-affine changes of charts.]

The fact that this lower bound on angles is independent of E_k and the small scale 2^{-m} at which we decide to work is important for the next part of the argument.

We replace E'_k (the Federer-Fleming projection of E_k on our net) with another finite union E''_k of faces of the same net, obtained as a deformation of E'_k in $\mathbf{M}$ (and hence also of E_k, since Federer-Fleming projections are deformations), and for which $J_g(E''_k)$ is minimal. The existence is easy, because there are only finitely many combinations of faces.

When we do this, we do not increase $J_g(E'_k)$, and we gain something because can use the minimality property of E''_k (among unions of faces) to prove that it is quasiminimal (as in Section 3.4.) in our manifold $\mathbf{M}$. The quasiminimality constant M that we get depends only on N and the lower bound for the angles in the grid, and not on k or $m = m(k)$. The proof of quasiminimality uses more Federer-Fleming projections, and is not really difficult (see below (6.39) for some hints); again the main difficulty was the construction of the grid.

There is another issue that needs to be addressed: E''_k is a union of d-dimensional faces, and also of lower-dimensional faces where we were able to contract more, and we can only control the d-dimensional part correctly. That is, set

$$F_k = (E''_k)^* = \big\{x \in E''_k\,;\, \mathcal{H}^d(E''_k \cap B(x,r)) > 0 \text{ for all } r > 0\big\} \tag{6.9}$$

as in (3.6). This is the d-dimensional part that we control, and for the rest we only know that

$$\mathcal{H}^{d-1}(E''_k \setminus F_k) < +\infty, \tag{6.10}$$

but with no control on precisely where $E''_k \setminus F_k$ may lie. So we only want to take a limit of the sets F_k.

Notice that F_k is a reduced quasiminimal set (it is easy to see that E^* is quasiminimal when E is quasiminimal). We replace $\{F_k\}$ with a subsequence for which $\{F_k\}$ converges to a limit F, and use the fact that Theorem 3.17, and in particular the lowersemicontinuity estimate (3.18), stays true for sequences of reduced quasiminimal sets (with uniform estimates, as here). We get that

$$J_g(F) \leq \liminf_{k\to+\infty} J_g(F_k) = \liminf_{k\to+\infty} J_g(E''_k) = \inf\big\{J_g(E)\,;\, E \in \mathcal{F}(E_0)\big\}. \tag{6.11}$$

[It is not hard to see that the continuous function g in (6.1) does not disturb the lowersemicontinuity estimate.]

So this takes care of (6.4) and (6.8), but we still need to show that some minor modification of F lies $\mathcal{F}(E_0)$, and this is where we shall use Theorem 4.3 and need more assumptions.

By the proof of Theorem 3.17, and if g was identically equal to 1, we would get that F is locally minimal in $\mathbf{M}$. Here g may vary, and we only get, by an argument similar to what we did near (3.23), that F is almost minimal, with a gauge function h that depends on the modulus of continuity of g. If g is Hölder-continuous, $h(r) \leq Cr^\alpha$ for some $\alpha > 0$, which is more than enough to apply Theorem 4.3.

We get that near each point $x \in F$, F is biHölder equivalent to a minimal cone. Then, near x, there is a Hölder retraction onto F, defined on a small neighborhood of F. We claim that we can make this global, i.e., that there is an ε-neighborhood V_ε of F, and a Hölder retraction $R : V_\varepsilon \to F$, so that $R(x) = x$ on F.

Let us check this in a probably too brutal way. For each $x \in F$, there is a radius $r(x) > 0$ such that the conclusion of Theorem 4.3 holds in $B(x, 10r(x))$, with uniform estimates. We use the biHölder equivalence provided by Theorem 4.3 to get a Hölder retraction $R_x : B(x, 5r(x)) \to F$, which we obtain by conjugating a

Lipschitz retraction on the minimal cone provided by Theorem 4.3 with the biHölder mapping also provided by that theorem. Thus

$$R_x(y) = y \quad \text{for } y \in F \cap B(x, 5r(x)), \tag{6.12}$$

and

$$|R_x(y) - R_x(z)| \leq C|y-z|^\alpha \quad \text{for } y, z \in B(x, 5r(x)), \tag{6.13}$$

where the exponent $\alpha < 1$ can be chosen as close to 1 as we want, because we can take the exponent in Theorem 4.3 arbitrarily close to 1.

We don't like the fact that the radius $r(x)$ depends on x, so we cover the compact set F by a finite number of balls $B(x_i, r(x_i))$, and then set $r = \inf_i r(x_i)$. For each $x \in F$, we now pick i such that $x \in B(x_i, r(x_i))$, and denote by S_x the restriction of R_{x_i} to $B(x, 4r) \subset B(x_i, 5r(x_i))$. Obviously

$$S_x(y) \in F \quad \text{for } y \in B(x, 4r), \tag{6.14}$$

$$S_x(y) = y \quad \text{for } y \in F \cap B(x, 4r), \tag{6.15}$$

and

$$|S_x(y) - S_x(z)| \leq C|y-z|^\alpha \quad \text{for } y, z \in B(x, 4r). \tag{6.16}$$

We need mappings defined on the whole $\mathbf{M}$, so we define $T_x : \mathbf{M} \to \mathbf{M}$ by

$$T_x(y) = y \quad \text{for } y \in \mathbf{M} \setminus B(x, 3r), \tag{6.17}$$

$$T_x(y) = S_x(y) \quad \text{for } y \in B(x, 2r), \tag{6.18}$$

and, for $y \in B(x, 3r) \setminus B(x, 2r)$,

$$T_x(y) = \frac{3r - |y-x|}{r}\, S_x(y) + \frac{|y-x| - 2r}{r}\, y. \tag{6.19}$$

There is a small abuse of notation here, because in principle $\mathbf{M}$ is a manifold, but we can choose r so small that the convex combination in (6.19) makes sense.

Pick a (new) maximal collection of points $x_i \in F$, $i \in I$, so that $|x_i - x_j| \geq r/2$ when $i \neq j$. Then the $B(x_i, r)$ cover F and the $B(x_i, 5r)$ have bounded overlap. Set $T_i = T_{x_i}$ for $i \in I$. We take for R the composition of all the T_i (taken in any order that we like), restricted to $V_\varepsilon = \big\{y \in \mathbf{M}\,;\, \mathrm{dist}(y, F) \leq \varepsilon\big\}$, and where ε is very small, to be chosen soon.

Let us verify that R is the desired retraction. Let $y \in V_\varepsilon$ be given, and let us see what happens when we apply the successive T_i to it. To simplify the notation, assume that $I = \{1, \cdots, m\}$, and that we apply the T_i in the natural order. Set $y_0 = y$ and, by induction, $y_i = T_i(y_{i-1})$ for $i \geq 1$. We want to check that

$$\mathrm{dist}(y_i, F) \leq \varepsilon_i, \tag{6.20}$$

for some small numbers ε_i that we shall compute along the way. This is the case when $i = 0$, with $\varepsilon_0 = \varepsilon$. Suppose $i \geq 1$ and this is true for $i-1$. If $y_{i-1} \in \mathbf{M} \setminus B(x_i, 3r)$, then $y_i = T_i(y_{i-1}) = y_{i-1}$ by (6.17), and (6.20) holds (if $\varepsilon_i \geq \varepsilon_{i-1}$). So we may assume that $y_{i-1} \in B(x_i, 3r)$. Pick $z_{i-1} \in F$ such that $|z_{i-1} - y_{i-1}| \leq \varepsilon_{i-1}$.

Then $z_{i-1} \in B(x_i, 4r)$ (if $\varepsilon_{i-1} \leq r$; we shall take care of this later). Notice that $T_i(y_{i-1}) \in [S_{x_i}(y_{i-1}), T_i(y_{i-1})]$ by (6.17)-(6.19); then

$$\begin{aligned} |y_i - y_{i-1}| &= |T_i(y_{i-1}) - y_{i-1}| \leq |S_{x_i}(y_{i-1}) - y_{i-1}| \\ &\leq |S_{x_i}(y_{i-1}) - S_{x_i}(z_{i-1})| + |z_{i-1} - y_{i-1}| \\ &\leq C|z_{i-1} - y_{i-1}|^\alpha + |z_{i-1} - y_{i-1}| \leq C\varepsilon_{i-1}^\alpha + \varepsilon_{i-1} \end{aligned} \tag{6.21}$$

by (6.20), because $S_{x_i}(z_{i-1}) = z_{i-1}$ by (6.15), and by (6.16). Then

$$\mathrm{dist}(y_i, F) \leq \mathrm{dist}(y_{i-1}, F) + C\varepsilon_{i-1}^\alpha + \varepsilon_{i-1} \leq C\varepsilon_{i-1}^\alpha + 2\varepsilon_{i-1} \tag{6.22}$$

by (6.20) for $i-1$. We choose

$$\varepsilon_i = C\varepsilon_{i-1}^\alpha + 2\varepsilon_{i-1} \tag{6.23}$$

and get (6.20). This gives a definition of all ε_i in terms of $\varepsilon_0 = \varepsilon$, and now we choose ε so small that $\sum_{i=0}^m \varepsilon_i \leq r/2$, say. This way we can be sure that

$$|y_i - y| \leq \sum_{j \leq i} |y_j - y_{j-1}| \leq r/2 \ \text{ for } 1 \leq i \leq m \tag{6.24}$$

and (as was used before) $\varepsilon_i < r$ for $0 \leq i \leq m$.

A consequence of (6.24) is that $y_i = y_{i-1}$ most of the time. Even more, let $w \in V_\varepsilon \cap B(y, r/2)$ and define $w_i = T_i \circ \cdots \circ T_1(w)$ as we did for y_i. By (6.17), $w_i = w_{i-1}$ as soon as w_{i-1} lies out of $B(x_i, 3r)$; by (6.24) (applied to w) this happens as soon as $|w - x_i| > \frac{7r}{2}$, hence also as soon as $|y - x_i| > 4r$. This means that in $V_\varepsilon \cap B(y, r/2)$, R is also the composition of the mappings T_i, $i \in I(y)$, where $I(y) = \big\{i\,;\, |x_i - y| \leq 4r\big\}$. Because of this (and the fact that the balls $B(x_i, 5r)$ have bounded overlap), R is still Hölder-continuous, with a new exponent that is as close to 1 as we want. We don't care about the Hölder constant for that exponent, just the exponent. Also, it appears that we have been too conservative in our choice of ε (because we could have counted how many times y_i changes), but this is all right.

The fact that $R(y) \in F$ for $y \in V_\varepsilon$ is easy. Since the $B(x_i, r)$ cover F, we can find $i \geq 1$ such that $y \in B(x_i, r + \varepsilon)$, and then $y_{i-1} \in B(x_i, 2r)$ by (6.24). Then $y_i = T_i(y_{i-1}) = S_{x_i}(y_{i-1}) \in F$ by (6.18) and (6.14). The ulterior mappings T_j leave y_j alone, by (6.15) and (6.17)-(6.19), so $R(y) = y_m \in F$. Finally, $R(y) = y$ for $y \in F$, for the same reason using (6.15). This completes our definition of the Hölder retraction R.

We want to use R to define a one parameter family of deformations g_t. Set $d_F(x) = \mathrm{dist}(x, F)$, and let $\tau > 0$ be much smaller than ε. We want to keep

$$g_t(x) = x \text{ when } d(x) \geq 2\tau, \tag{6.25}$$

and otherwise we set

$$g_t(x) = \psi_t(x)R(x) + (1 - \psi_t(x))x \text{ when } d(x) \leq 2\tau, \tag{6.26}$$

where

$$\psi_t(x) = t\,\mathrm{Min}\big(1, \frac{2\tau - d(x)}{\tau}\big) \text{ when } d_F(x) \leq 2\tau. \tag{6.27}$$

We abuse notation slightly again, because $\mathbf{M}$ is a manifold. But it is locally the same as $\mathbb{R}^n$, which is enough to define the intermediate points $\psi_t(x)R(x) + (1 - \psi_t(x))x$ and compute with them locally. Notice that $g_0(x) = x$ for $x \in \Omega_\varepsilon$, and $g_t(x)$ is a continuous function of x and t.

Return to the sets E''_k. By construction, E_k, E'_k, and finally E''_k are deformations of E_0, which means that $E''_k = \varphi_1(E_0)$, for some acceptable family of continuous mappings φ_t, $0 \leq t \leq 1$, as above. Now we define a new family $\{f_t\}$, $0 \leq t \leq 1$, by

$$f_t(x) = \varphi_{2t}(x) \text{ for } x \in E_0 \text{ and } 0 \leq t \leq 1/2 \tag{6.28}$$

and

$$f_t(x) = g_{2t-1}(\varphi_1(x)) \text{ for } x \in E_0 \text{ and } 0 \leq t \leq 1/2. \tag{6.29}$$

Again f_t is a continuous function of $x \in E_0$ and $t \in [0,1]$, and $f_0(x) = x$ for $x \in E_0$. Set

$$G_k = f_1(E_0) = g_1 \circ \varphi_1(E_0) = g_1(E''_k) \tag{6.30}$$

by definitions; then $G_k \in \mathcal{F}(E_0)$. We want to check that for k large,

$$J_g(G_k) \leq \inf\big\{J_g(E)\,;\, E \in \mathcal{F}(E_0)\big\}, \tag{6.31}$$

because as soon as we do this, we will know that G_k is a minimizer, and Claim 6.7 will follow.

The main piece of E''_k is $F_k = (E''_k)^*$; since F is the limit of the F_k, we get that for k large, $d_F(y) \leq \tau$ for $y \in F_k$. Then (6.27) and (6.26) yield $\psi_1(y) = 1$ and $g_1(y) = R(x) \in F$. Thus

$$g_1(F_k) \subset F. \tag{6.32}$$

The remaining part is $Z_k = E''_k \setminus F_k$. By (6.10), $\mathcal{H}^{d-1}(Z_k) < +\infty$. But g_1 is Hölder-continuous with the same exponent as R, which can be taken as close to 1 as we want. Then $\mathcal{H}^d(g_1(Z_k)) = 0$. Altogether,

$$\begin{aligned} J_g(G_k) &= \int_{G_k} g(x)d\mathcal{H}^d(x) = \int_{g_1(F_k\cup Z_k)} g(x)d\mathcal{H}^d(x) \\ &= \int_{g_1(F_k)} g(x)d\mathcal{H}^d(x) \leq \int_F g(x)d\mathcal{H}^d(x) \\ &= J_g(F) \leq \inf\big\{J_g(E)\,;\, E \in \mathcal{F}(E_0)\big\} \end{aligned} \tag{6.33}$$

because $G_k = g_1(E''_k) = g_1(F_k \cup Z_k)$ by (6.30), and by (6.32) and (6.11). Thus (6.31) holds, G_k is a minimizer, and our proof of existence is complete. □

Remarks 6.34. The reader noticed that our assumptions were probably not optimal. Perhaps we can allow simple domains Ω (rather than our flat manifolds without boundary), but at least the argument used for the next example would have to be modified.

When $n > 2$, we could be in trouble if our definition of deformations demanded that φ_1 be Lipschitz, because we only know that R is Hölder. When $n = 2$, we can apply Theorem 4.1 to get a Lipschitz retraction, and conclude as above. Otherwise we would need to prove a better regularity or a better retraction theorem.

When $d > 2$, we do not know the existence of neighborhood retractions near almost minimal sets, even though this would look like a weak regularity property to prove, so we don't have an existence theorem.

Example 6.35. We return to Example 2.19 and its homology conditions (which also includes simpler separation conditions, as in Example 2.18). Let Ω be our closed domain. We shall assume that for some $\varepsilon > 0$, there is a Lipschitz retraction

$R : \Omega_\varepsilon \to \Omega$. Here $\Omega_\varepsilon = \big\{x \in \mathbb{R}^n\,;\, \mathrm{dist}(x,\Omega) \leq \varepsilon\big\}$, and retraction means that $R(x) = x$ for $x \in \Omega$ and $R(x) \in \Omega$ for $x \in \Omega_\varepsilon \setminus \Omega$.

Theorem 6.36 [Li3]. Let $\Omega \subset \mathbb{R}^n$ be compact, and suppose that for some $\varepsilon > 0$ there is a Lipschitz retraction $R : \Omega_\varepsilon \to \Omega$. Let $g : \Omega \to [1, N]$ be continuous, and let ω_j, $j \in J$, be a collection of smooth $(n-d-1)$-dimensional surfaces in $\mathbb{R}^n \setminus \Omega_\varepsilon$, such that ω_j represents a nonzero element in the singular homology group $H_{n-d-1}(\mathbb{R}^n \setminus \Omega; \mathbb{Z})$. Set

$$\mathcal{F} = \big\{E \subset \Omega\,;\, E \text{ is closed and each } \omega_j \text{ represents a nonzero element in } H_{n-d-1}(\mathbb{R}^n \setminus E; \mathbb{Z})\big\}. \tag{6.37}$$

Then there is a set $E \in \mathcal{F}$ such that $J_g(E) = \inf\big\{J_g(F)\,;\, F \in \mathcal{F}\big\}$.

The proof is based on the same program as for Example 6.6, but there are minor complications because Ω has a boundary. We explain one way to fix this; it could be that the organization in [Li3] is a little more efficient, but probably not too much.

First extend g to Ω_ε in a continuous way, so that we can talk about $J_g(E)$ when $E \subset \Omega_\varepsilon$. The simplest is to take $g \circ R$ on Ω_ε, and observe that this does not change g on Ω.

Now let $\{E_k\}$ be a minimizing sequence as above, and for each k choose a Feuvrier polyhedral net $\mathcal{R}_k$ with two constraints. The first one is that $\mathrm{diam}(Q) \leq \alpha_k$ for every polyhedron Q of the net, where $\alpha_k << \varepsilon$ will tend to 0. The main one is that $\mathcal{R}_k$ is adapted to E_k, in the sense that if E'_k denotes the Federer-Fleming projection of E_k on this net, then $J_g(E'_k) \leq J_g(E_k) + 2^{-k}$. The fact that g is continuous is used here, because somewhere in the argument we compare $\mathcal{H}^d(E'_k)$ and $J_g(E'_k)$ in small balls. The details are as in Example 6.6.

Notice that $E'_k \subset \Omega_{n\alpha_k} \subset \Omega_\varepsilon$ by construction (we never move a point by more than α_k when we do a step of the Federer-Fleming projection). Thus we shall not change $J_g(E'_k)$ when we replace g with the larger g_k defined by

$$g_k(x) = g(x) \text{ for } x \in \Omega_{n\alpha_k} \quad \text{and} \quad g_k(x) = N_1 \text{ for } x \in \Omega_\varepsilon \setminus \Omega_{n\alpha_k}, \tag{6.38}$$

where N_1 is a very large constant, that depends on N and the Lipschitz constant for the retraction R.

Choose a (fixed) union Ω_1 of (standard) dyadic cubes, that contains a neighborhood of Ω and is contained in Ω_ε. It will be more convenient to work in Ω_1. For k large, $\Omega_{2n\alpha_k} \subset \Omega_1$; we use this and Feuvrier's result to modify our network $\mathcal{R}_k$ so that it stays the same in $\Omega_{n\alpha_k}$, and coincides with a dyadic refinement of the usual dyadic net in $\Omega_1 \setminus \Omega_{2n\alpha_k}$.

Now replace E'_k with E''_k, which is a finite union of faces of our new network $\mathcal{R}_k$ and a deformation of E'_k in Ω_1, and minimizes $J_k(E''_k) = \int_{E''_k} g_k(x) d\mathcal{H}^d(x)$ under these constraints. The existence is easy because the total number of competitors is finite. Then

$$J_k(E''_k) \leq J_k(E'_k) = J_g(E'_k) \leq J_g(E_k) + 2^{-k} \tag{6.39}$$

by definitions.

Now E''_k is a quasiminimal set in Ω_1, with (bad) constants that depend on N_1 and the angles of our networks, but do not depend on k. The verification requires some computations, but the idea is not too complicated. If $F = \varphi_1(E''_k)$ is a deformation of E''_k in Ω_1, we cannot compare F directly to E''_k, because it is not a union of faces in $\mathcal{R}_k$, but we can replace F with its Federer-Fleming projection G on $\mathcal{R}_k$. Then $J_k(E''_k) \leq J_k(G)$ by definition of E''_k. When we replace F with G, we only move the points of $Z = F \setminus \mathcal{S}_d$, where $\mathcal{S}_d$ denotes the d-dimensional skeleton of $\mathcal{R}_k$. Because of the angle conditions on the faces, when we apply the projection we can only multiply the contribution of Z to $J_k(F)$ by CN_1. At the same time $Z \subset \varphi_1(W)$, where $W = \{x \in E''_k\,;\, \varphi_1(x) \neq x\}$, because $E''_k \subset \mathcal{S}_d$ by definition. Now if (3.21) fails, i.e., if $\mathcal{H}^d(\varphi_1(W))$ is much smaller than $\mathcal{H}^d(W)$, then $\mathcal{H}^d(\varphi_1(Z)) \leq \mathcal{H}^d(\varphi_1(W)) << \mathcal{H}^d(W)$, and the contribution of its Federer-Fleming projection to $J_g(G)$ is still small compared to $\mathcal{H}^d(W)$. The rest of $\varphi_1(W)$ stays the same, so its contribution stays very small too. When we sum up, we find that G is strictly better than E''_k, a contradiction.

So the E''_k are quasiminimal with bad, but uniform constants. Set $F_k = (E''_k)^*$; the F_k are also quasiminimal with uniform constants. Replace $\{F_k\}$ with a subsequence for which F_k tends to a limit F. We can apply (3.18), which yields that

$$\begin{aligned} J_g(F) &\leq \liminf_{k\to+\infty} J_g(F_k) \leq \liminf_{k\to+\infty} J_g(E''_k) \\ &\leq \liminf_{k\to+\infty} [J_g(E_k) + 2^{-k}] = \inf\{J_g(F)\,;\, F \in \mathcal{F}\} \end{aligned} \tag{6.40}$$

because the fact that we multiplied $\mathcal{H}^d$ with the continuous function g does not matter for the first inequality, then because $F_k \subset E''_k$, by (6.39), and because $\{E_k\}$ is a minimizing sequence.

We now want to check that $F \in \mathcal{F}$. We first need to see that F_k stays close to Ω. We claim that

$$F_k \subset \Omega_{\beta_k}, \tag{6.41}$$

where β_k can be computed from α_k, and tends to 0 when k tends to $+\infty$. Again the computation is a little long, but the idea is simple: since g_k is so much larger on $\Omega \setminus \Omega_{n\alpha_k}$ (see (6.38)), it is not interesting for F_k to get far from Ω. The existence of the Lipschitz retraction R will be used here, to provide competitors that have a substantial piece in Ω where g_k is so much smaller. But we'll have to try different competitors before we find one for which the thin annulus where we distort things without winning is small.

Set $a_l = 2^l n\alpha_k$ for $l \geq 0$, and

$$\begin{aligned} V_l^- &= \{x \in \Omega_1\,;\, \mathrm{dist}(x,\Omega) \leq a_l\}, \\ V_l &= \{x \in \Omega_1\,;\, a_l < \mathrm{dist}(x,\Omega) \leq 2a_l\}, \text{ and} \\ V_l^+ &= \{x \in \Omega_1\,;\, \mathrm{dist}(x,\Omega) > 2a_l\}. \end{aligned} \tag{6.42}$$

We define a competitor $\varphi(E''_k)$ for E''_k by

$$\begin{aligned} &\varphi(x) = x && \text{for } x \in V_l^-, \\ &\varphi(x) = \frac{2a_l - \mathrm{dist}(x,\Omega)}{a_l}\, x + \frac{\mathrm{dist}(x,\Omega) - a_l}{a_l}\, R(x) && \text{for } x \in V_l, \text{ and} \\ &\varphi(x) = R(x) && \text{for } x \in V_l^+. \end{aligned} \tag{6.43}$$

It would be easy to write down a one-parameter family $\{\varphi_t\}$ of mappings that ends with φ, but let us not bother the reader with this. Notice that we can expect to win a lot when we apply φ in the last region, because $g_k(x) = N_1$ in the last two regions (by (6.38)), while $g_k(R(x)) = g(R(x)) \leq N$ (which is much smaller) because $R(x) \in \Omega$. That is,

$$\begin{aligned} J_k(\varphi(E_k'')) &\leq \int_{\varphi(E_k''\cap V_l^-)} g_k + \int_{\varphi(E_k''\cap V_l)} g_k + \int_{\varphi(E_k''\cap V_l^+)} g_k \\ &\leq \int_{E_k''\cap V_l^-} g_k + N_1\mathcal{H}^d(\varphi(E_k''\cap V_l)) + N\mathcal{H}^d(\varphi(E_k''\cap V_l^+)) \\ &\leq \int_{E_k''\cap V_l^-} g_k + CN_1\mathcal{H}^d(E_k''\cap V_l) + CN\mathcal{H}^d(E_k''\cap V_l^+) \end{aligned} \tag{6.44}$$

by (6.43) and because φ is C-Lipschitz (including on V_l where we can use the fact that $|R(x) - x| \leq C\,\mathrm{dist}(x, \Omega) \leq Ca_l$). At the same time,

$$\begin{aligned} J_k(E_k'') &= \int_{E_k''\cap V_l^-} g_k + \int_{E_k''\cap V_l} g_k + \int_{E_k''\cap V_l^+} g_k \\ &= \int_{E_k''\cap V_l^-} g_k + N_1\mathcal{H}^d(E_k''\cap V_l) + N_1\mathcal{H}^d(E_k''\cap V_l^+) \end{aligned} \tag{6.45}$$

by (6.38), so

$$J_k(\varphi(E_k'')) - J_k(E_k'') \leq CN_1\mathcal{H}^d(E_k''\cap V_l) - \frac{N_1}{2}\mathcal{H}^d(E_k''\cap V_l^+) \tag{6.46}$$

if $\frac{N_1}{N}$ is large enough. If we were allowed to compare $\varphi(E_k'')$ directly to E_k'', we would deduce from (6.46) that

$$\mathcal{H}^d(E_k''\cap V_l^+) \leq 2C\mathcal{H}^d(E_k''\cap V_l). \tag{6.47}$$

Now we cannot do that, but we can compose φ with a Federer-Fleming projection that sends $\varphi(E_k'')$ back to our net, and by the same computations we eventually obtain (6.47) anyway (but with a larger constant).

Observe that (6.47) shows that $\mathcal{H}^d(E_k''\cap V_l^+)$ decays exponentially fast. Indeed, V_l^+ is the disjoint union of V_{l+1} and V_{l+1}^+ by (6.42) and because $a_{k+1} = 2a_k$, so

$$\mathcal{H}^d(E_k''\cap V_l^+) = \mathcal{H}^d(E_k''\cap V_{l+1}) + \mathcal{H}^d(E_k''\cap V_{l+1}^+) \geq (1 + \frac{1}{2C})\,\mathcal{H}^d(E_k''\cap V_{l+1}^+) \tag{6.48}$$

by (6.47) and, after some iterations,

$$\mathcal{H}^d(E_k''\cap V_l^+) \leq \Big(\frac{2C}{2C+1}\Big)^k \mathcal{H}^d(E_k''). \tag{6.49}$$

Now we could modify the definition of the a_l to make the sequence $\{\frac{a_l}{\alpha_k}\}$ bounded (but with very large values), and still get, with the same computations as above, that $\lim_{l\to+\infty} \mathcal{H}^d(E_k''\cap V_l^+) = 0$. Let us cheat slightly instead, and use the local Ahlfors-regularity of $F_k = (E_k'')^*$ to get a less precise estimate which is enough. If F_k meets V_l at some point x, then

$$\mathcal{H}^d(F_k \cap (V_{l-1}\cup V_l \cup V_{l+1})) \geq \mathcal{H}^d(F_k \cap B(x, a_l/2)) \geq C^{-1}a_l^d. \tag{6.50}$$

We are now ready to prove (6.41). Let $\beta > 0$ be small, and denote by l_k the largest integer such that $a_l = 2^l n\alpha_k < \beta$. Obviously l_k is well defined for k large, and tends to $+\infty$ when k tends to $+\infty$. Then (6.49) says that $\mathcal{H}^d(F_k \cap (V_{l_k-1}\cup V_{l_k}\cup V_{l_k+1}))$ tends to 0 (recall that $\mathcal{H}^d(E_k'')$ stays bounded). But the right-hand side of (6.50) is comparable to β^d, so it does not tend to 0. This contradiction proves that for k large, $F_k \cap V_{l_k} = \emptyset$. Then $\mathcal{H}^d(E_k''\cap V_{l_k}) = 0$ by definition of $F_k = (E_k'')^*$, (6.47) implies that $\mathcal{H}^d(E_k'' \cap V_{l_k}^+) = 0$, and (by local Ahlfors-regularity of F_k again) F_k does not meet $V_{l_k}^+$ either. In other words, $F_k \subset V_{l_k}^- \subset \Omega_\beta$ for k large; (6.41) follows.

Recall that we want to show that $F \in \mathcal{F}$. By (6.41), we know that $F \subset \Omega$; we still need to check that for $j \in J$, ω_j represents a nonzero element in $H_{n-d-1}(\mathbb{R}^n \setminus F; \mathbb{Z})$. We know this with E_k (instead of F), just because $E_k \in \mathcal{F}$. This is then true for E_k' and E_k'', because they are deformation of E_k in Ω_1, and it is known that such deformations away from ω_j preserve the fact that $\omega_j \neq 0$ in homology. This is also true for F_k, because removing pieces of $(d-1)$-dimensional faces from E_k'' does not change the fact that ω_j represents a nonzero element in $H_{n-d-1}(\mathbb{R}^n \setminus E_k''; \mathbb{Z})$, this time by a general position argument (we could move the support of a $n-d$ chain that closes ω_j so that it does not meet the faces). Finally, ω_j represents a nonzero element in $H_{n-d-1}(\mathbb{R}^n \setminus F; \mathbb{Z})$, because if the support of a chain that closes ω_j does not meet F, it does not meet F_k for k large. This completes the verification of the fact that $F \in \mathcal{F}$, and the description of the proof of Theorem 6.36. We refer to [Li3] for details. □

7. Boundary regularity

The author believes that the logical way to try to prove existence results under sliding boundary conditions (as in Section 2.4) is to first prove regularity properties for potential solutions. Apparently there has not been so much work on the subject of boundary regularity, but one can hope that the sliding context of Section 2.4 and Definition 7.6 below will provide sufficient flexibility to allow some positive results.

Here we just give definitions, announce some partial results (that still need to be written down or proofread) and mention plans for the future.

We give ourselves a finite collection of simple compact boundary sets $\Gamma_j \subset \mathbb{R}^n$, $0 \leq j \leq j_{max}$; for instance, each Γ_j can be a finite union of (closed) faces (of various dimensions) of dyadic cubes. Or we can choose the Γ_j like this, and consider their images under a (same) biLipschitz mapping. Other options are possible, but the author decided to restrict to these for the moment.

We shall also proceed locally, in an open ball B. Let $E \subset B$ be closed in B, and assume that $\mathcal{H}^d(E \cap K) < +\infty$ for every compact subset K of B. A sliding competitor for E in B is a set $F = \varphi_1(E)$, where the one-parameter family $\{\varphi_t\}$, $0 \leq t \leq 1$, has the following properties (similar to what we asked in Section 2.4):

$$(t,x) \to \varphi_t(x) : [0,1] \times E \to \mathbb{R}^n \text{ is continuous,} \tag{7.1}$$

$$\varphi_0(x) = x \text{ for } x \in E_0, \tag{7.2}$$

$$\varphi_t(x) \in \Gamma_j \text{ when } 0 \leq j \leq j_{max} \text{ and } x \in \Gamma_j, \tag{7.3}$$

$$\varphi_1 \text{ is Lipschitz} \tag{7.4}$$

(which we again keep out of respect for traditions); for the locality, we set

$$W_t = \{x \in E\,;\, \varphi_t(x) \neq x\}, \quad W = \bigcup_{t\in[0,1]} (W_t \cup \varphi_t(W_t)), \tag{7.5}$$

and demand that W be contained in a compact subset of B (in short, $W \subset\subset B$).

Notice that if the first set Γ_0 contains E, it may play the role of a closed domain where everything happens (the set Ω that we used in examples).

Definition 7.6. We say that E is a sliding almost minimal set with gauge function h (and relative to the choice of boundaries Γ_j) when

$$\mathcal{H}^d(E \cap W) \leq \mathcal{H}^d(\varphi(E) \cap W) + h(\mathrm{diam}(W)) \,\mathrm{diam}(W)^d \tag{7.7}$$

when the φ_t and W are as above.

As usual, we would like to write $\mathcal{H}^d(E) \leq \mathcal{H}^d(\varphi_1(E)) + h(\mathrm{diam}(W)) \,\mathrm{diam}(W)^d$, but $\mathcal{H}^d(E)$ may be infinite, and anyway E and $\varphi_1(E)$ coincide out of W. There is also a notion of sliding quasiminimal set, as in Definition 3.20 but with the sliding competitors, but let us not write the definition here.

Of course we set things so that the minimizers of the sliding Plateau problem of Section 2.4, if they exist, are sliding almost minimizers with this definitions. This may also be the case for the solutions of some other problems (for instance, minimize the size of a current under a boundary constraint), but the author did not take the time to check yet.

So we want to see to which extent the interior regularity theory described above extends to the sliding context. Here are a few things that seem to work.

The local Ahlfors-regularity of E for almost minimal and quasiminimal sets (as in Theorem 3.7) seems to extend without real trouble.

The situation for the uniform rectifiability of E (Theorem 3.10) is a little more complicated; the main stopping time argument in the proof does not go through, and we only seem to be only able to prove that E is uniformly rectifiable under additional dimension conditions. The case when $d = 2$ and the boundaries Γ_j are at most 2-dimensional is still all right, but in fact the uniform rectifiability up to the boundary, when we can prove it, does not mean much more than what we knew anyway from the interior regularity.

Fortunately, sliding quasiminimal sets are still rectifiable, and even every Hausdorff limit of reduced quasiminimal sets (with uniform estimates) is rectifiable. This is somewhat easier, and it is still enough to prove the concentration property (Theorem 3.12), and then (with some work) an analogue of Theorem 3.17 (the lowersemicontinuity of $\mathcal{H}^d$ and the stability under limits), which therefore still hold in the sliding context.

The monotonicity of the density $r^{-d}\mathcal{H}^d(E \cap B(0,r))$, when all the Γ_j are cones centered at the origin and E is minimal, seems to be true, and maybe also the fact that E coincides with a cone when the density is constant, yielding the fact that blow-up limits at the boundary are sliding minimal cones.

What should be done next is still a long program. First, we should try to get a list of the sliding minimal cones at a boundary point, depending on the list of boundary cones Γ_j. But also, some more precise understanding of how the set converges to a blow-up limit will be welcome, starting with the simple blow-up limits that we already know.

We should mention that we cannot expect the monotonicity of $r^{-d}\mathcal{H}^d(E \cap B(x,r))$ at some point x near the boundaries to be true as before, so we should then estimate and quantify more precisely the lack of monotonicity (or equivalently define something like a density profile).

Figure 7 (probably the best time of the author's Montreal lectures), which comes from the link on Soap films on the Borromean Rings in K. Brakke's homepage, is supposed to illustrate the fact that it may be hard to study precisely the boundary behaviour of soap films. See also [Br3], [LM2], and Figure 13.9.3 on page 137 in

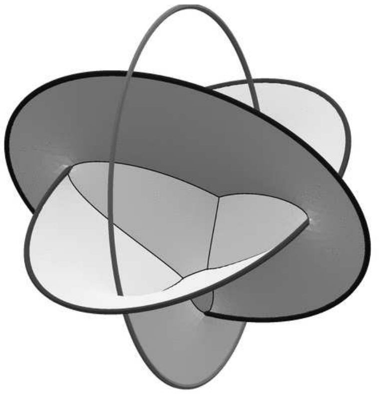

FIGURE 7. A soap film which leaves one of the boundary curves (K. Brakke). Guess how it looks like at small scales; is its behaviour stable when the width of the wires tend to 0?

[M5] for information and conjectures about boundary regularity (maybe of slightly different objects).

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