Parameterizations of sets

Palo Alto, January 2012

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Essentially, the two algorithms that I know about:

Reifenberg' topological disk and Carleson's corona construction.

1. REIFENBERG'S THEOREM (SIMPLEST FORM)

We start with notation. We consider sets of dimension d in $\mathbb{R}^n$. $E \subset \mathbb{R}^n$ is a closed set, and for Reifenberg's theorem we assume that E is flat at all scales. Set

$$d_{x,r}(E,F) = \frac{1}{r} \sup_{y \in E \cap B(x,r)} \mathrm{dist}(y,F) + \frac{1}{r} \sup_{y \in F \cap B(x,r)} \mathrm{dist}(y,E)$$

when $F \subset \mathbb{R}^n$ is closed, $x \in \mathbb{R}^n$, and $r > 0$.
[We take $\sup_{y \in E \cap B(x,r)} \mathrm{dist}(y,F) = 0$ if $E \cap B(x,r) = \emptyset$.] Then set

$$\gamma(x,r) = \gamma_E(x,r) = \inf_P d_{x,r}(E,P)$$

where the inf is taken over all d-planes P through x. Measures how flat E is in $B(x,r)$); big holes are not allowed either.

Theorem 1 (Reifenberg). Simple assumptions on E (closed in $\mathbb{R}^n$):
1. Let P_0 be a d-plane, and assume that $\operatorname{dist}(x, P_0) \leq \varepsilon$ for $x \in E$ and $\operatorname{dist}(x, E) \leq \varepsilon$ for $x \in P_0$.
2. Also assume that $\gamma_E(x, r) \leq \varepsilon$ for $x \in E$ and $0 < r \leq 10$.
Conclusion (if ε is small, depending on n and the small τ below):
There is a biHölder bijection $f : \mathbb{R}^n \to \mathbb{R}^n$, such that

$$f(P_0) = E,$$

$$|f(x) - x| \leq \tau \quad \text{for } x \in \mathbb{R}^n,$$

$$(1-\tau)|x-y|^{1+\tau} \leq |f(x) - f(y)| \leq (1+\tau)|x-y|^{1-\tau} \quad \text{for } |x-y| \leq 1.$$

Comments: can be localized; bilipschitz not true for snowflakes; quasisymmetric not true for a line times a snowflake; this gives topological information about E and how it is embedded; the statement is quite flexible, the algorithm is the main thing. Nice like John and Nirenberg.

Preparation for the proof

We just construct the restriction of f to P_0 for the moment.

Fix $k \geq 0$, and choose a maximal collection $\{x_j\}$, $j \in J_k$, of points $x_j \in E$, with $|x_j - x_i| \geq 2^{-k}$.

Set $B_j = B(x_j, 2^{-k})$. Thus $E \subset \bigcup_{j \in J_k} \overline{B}_j$.

Also set $r_j = 2^{-k}$.

Set $P_j = P(x_j, 10 \cdot 2^{-k})$ for $j \in J_k$, so that $d_{x_j,10r_j}(E, P_j) \leq \varepsilon$.

Call π_j the orthogonal projection on P_j, and $D\pi_j$ its differential (the projection on the vector space P'_j).

Construct a partition of unity

$$1 = \theta_0(x) + \sum_{j \in J_k} \theta_j(x)$$

where θ_0 is supported on $\{x\,;\, \mathrm{dist}(x, E) \geq 2^{-k}\}$, θ_j is supported on $3B_j$ for $j \in J_k$, and $|D^l \theta_j| \leq C_l 2^{kl}$ for $l \geq 0$.

We shall take

$$f = \lim_{k \to +\infty} f_k$$

with

$$f_0(x) = x \quad \text{on } P_0,$$

$$f_{k+1} = g_k \circ f_k$$

and

$$\begin{aligned} g_k(x) &= x + \sum_{j \in J_k} \theta_j(x)[\pi_j(x) - x] \\ &= \theta_0(x)x + \sum_{j \in J_k} \theta_j(x)\pi_j(x). \end{aligned}$$

[We just push points in the direction of the P_j (and hence E).]

Some verifications and lemmas.

Lemma 1. *For* $k \geq 0$, $i, j \in J_k \cup J_{k+1}$ *such that* $3B_j \cup 3B_i \neq \emptyset$,

$$||D\pi_i - D\pi_j|| \leq C\gamma(x_i, 10r_i) + C\gamma(x_j, 10r_j) \leq C\varepsilon$$

and, for $x \in 10B_i \cup 10B_j$

$$|\pi_i(x) - \pi_j(x)| \leq C(\gamma(x_i, 10r_i) + \gamma(x_j, 10r_j))2^{-k} \leq C\varepsilon 2^{-k}.$$

So P_i is close to P_j in $3B_j \cup 3B_i$.
Proof : use the fact $d_{x_j, 10\cdot 2^{-k}}(E, P_j) \leq \varepsilon$ and $d_{x_i, 10\cdot 2^{-k}}(E, P_i) \leq \varepsilon$.
Easy, but uses the bilateral approximation (P_j is rather well determined by E). □

Lemma 2. *Set* $\Gamma_k = f_k(P_0)$*. Then*

$$\text{(1)} \qquad \operatorname{dist}(x, E) \leq C\varepsilon 2^{-k} \quad \textit{for } x \in \Gamma_k.$$

Fairly easy by induction. Let $k \geq 0$ be given, and assume (1). Then $\theta_0(x) = 0$, and hence $\sum_{j \in J_k} \theta_j(x) = 1$ near Γ_k. So

$$\text{(2)} \qquad g_k(x) = \sum_{j \in J_k} \theta_j(x) \pi_j(x) \quad \text{near } \Gamma_k.$$

Now we can control $\Gamma_{k+1} = g_k(\Gamma_k)$. Let $x \in \Gamma_k$ be given. Pick $j_0 \in$ such that $x \in 2B_j$. Then by Lemma 1

$$g_k(x) - \pi_{j_0}(x) = \sum_{j \in J_k} \theta_j(x)[\pi_j(x) - \pi_{j_0}(x)] = O(\varepsilon 2^{-k})$$

and $\operatorname{dist}(g_k(x), E) \leq C\varepsilon 2^{-k-1}$ since $\pi_{j_0}(x) \in P_{j_0} \cap 3B_{j_0}$ lies close to E.

Easy consequences of the proof of Lemma 2:

$$|g_k(y) - y| \leq C\varepsilon 2^{-k} \quad \text{on } \Gamma_k \tag{3}$$

$$|f_{k+1}(x) - f_k(x)| = |g_k(f_k(x)) - f_k(x)| \leq C\varepsilon 2^{-k} \quad \text{on } P_0 \tag{4}$$

$$f(x) = \lim_{k \to +\infty} f_k(x) \quad \text{exists, and } f(x) \in E \tag{5}$$

$$|f(x) - f_k(x)| \leq 2C\varepsilon 2^{-k} \quad \text{on } P_0. \tag{6}$$

More work will be needed to check that f is injective on P_0, and that $f(P_0) = E$.

Lemma 3. *For each $j \in J_k$, Γ_k coincides in $2B_j$ with the graph of a $C\varepsilon$-Lipschitz function $\varphi_{k,j} : P_j \to P_j^\perp$, that meets $B(x_j, 2^{-k-10})$.*

Proof by induction on k. Recall that

$$g_k(x) = \sum_{j \in J_k} \theta_j(x)\pi_j(x) \text{ near } \Gamma_k. \tag{2}$$

Let $j \in J_{k+1}$ be given, choose $i \in J_k$ such that $x_j \in \overline{B}(x_i, 2^{-k})$, and use the induction assumption to describe Γ_k as a small Lipschitz graph over P_i, in $3B_i \supset 4B_j$.

We control the next set $\Gamma_{k+1} = g_k(\Gamma_k)$ with the differential

$$Dg_k(x) = \sum_{l \in J_k} \theta_l(x) D\pi_l + \sum_{l \in J_k} \pi_l(x) D\theta_l(x). \tag{7}$$

Recall that (on Γ_n)

$$(7) \qquad Dg_k(x) = \sum_{l \in J_k} \theta_l(x) D\pi_l + \sum_{l \in J_k} \pi_l(x) D\theta_l(x).$$

Recall we have $j \in J_{k+1}$ and we want a description in $2B_j$. So we just need to control Dg_k in $3B_j$. And, since $\sum_{l \in I_k} \theta_l = 1$ near Γ_k,

$$Dg_k(x) = D\pi_j + \sum_{l \in J_k} \theta_l(x)[D\pi_l - D\pi_j] + \sum_{l \in J_k} [\pi_l(x) - \pi_j(x)] D\theta_l(x).$$

By Lemma 1,

$$(8) \qquad |Dg_k(x) - D\pi_j| \leq C\varepsilon.$$

The desired Lipschitz control on Γ_{k+1} follows because g_k can be controlled by integrating Dg_k on Γ_k.

Surjectivity comes from a little bit of degree theory; Γ_{k+1} then meets $B(x_j, 2^{-k-11})$ because it stays close to P_j. Lemma 3 follows. □

Conclusion.

First, f is surjective: fix $z \in E$. For each k, choose $j \in J_k$ such that $z \in \overline{B}_j$. By Lemma 3, Γ_k meets $B(x_j, 2^{-k-10})$. So there exists $w_k \in P_0$ such that $|f_k(w_k) - z| \leq 2^{-k+1}$. Then use compactness.

We are left with the biHölder property. We shall use the following distortion estimate that we deduce from (8):

$$\text{for } x, y \in \Gamma_k, \text{ with } |x-y| \leq 2^{-k-1},$$

$$(1 - C\varepsilon)|x-y| \leq |g_k(x) - g_k(y)| \leq (1 + C\varepsilon)|x-y| \tag{9}$$

(use the fundamental theorem of calculus between x and y along Γ_n).

Now pick $x_0, y_0 \in P_0$. We want to control $|f(x_0) - f(y_0)|$. Set $x_k = f_k(x_0)$ and $y_k = f_k(y_0)$.

We may assume that $|x_0 - y_0| \le 10^{-1}$. As long as $|x_k - y_k| \le 2^{-k-1}$, we use (9) which gives

$$1 - C\varepsilon \le \frac{|x_{k+1} - y_{k+1}|}{|x_k - y_k|} \le 1 + C\varepsilon. \tag{10}$$

The first time $|x_k - y_k| > 2^{-k-1}$ (which occurs because 2^{-k} decreases faster!), just say that $|f(x_0) - x_k| \le C\varepsilon 2^{-k}$ and $|f(y_0) - y_k| \le C\varepsilon 2^{-k}$, so

$$1 - C\varepsilon \le \frac{|f(x_0) - f(y_0)|}{|x_k - y_k|} \le 1 + C\varepsilon. \tag{11}$$

Then compute that we use (10) about $\log_2(|x_0 - y_0|)$ times, and get the Hölder distortion estimates. $\square$

2. REIFENBERG'S THEOREM (VARIANTS)

Recall:

$$d_{x,r}(E,F) = \frac{1}{r}\sup_{y\in E\cap B(x,r)} \operatorname{dist}(y,F) + \frac{1}{r}\sup_{y\in F\cap B(x,r)} \operatorname{dist}(y,E)$$

$$\gamma(x,r) = \gamma_E(x,r) = \inf_P d_{x,r}(E,P)$$

Theorem 1 (Reifenberg). Let E be closed in $\mathbb{R}^n$, and assume that:
1. Let P_0 be a d-plane, and assume that $\operatorname{dist}(x,P_0) \le \varepsilon$ for $x \in E$ and $\operatorname{dist}(x,E) \le \varepsilon$ for $x \in P_0$.
2. Also assume that $\gamma_E(x,r) \le \varepsilon$ for $x \in E$ and $0 < r \le 10$.

Then (if ε is small, depending on n and the small τ below):
There is a biHölder bijection $f : \mathbb{R}^n \to \mathbb{R}^n$, such that $f(P_0) = E$, and

$$|f(x) - x| \le \tau \quad \text{for } x \in \mathbb{R}^n,$$

$$(1-\tau)|x-y|^{1+\tau} \le |f(x) - f(y)| \le (1+\tau)|x-y|^{1-\tau} \quad \text{for } |x-y| \le 1.$$

Soon: Many variants exist, but we often use the same algorithm.

2.a. Extension of f to $\mathbb{R}^n$

How do we extend the mapping above? First, we can build orthonormal bases of the tangent plane $T\Gamma_k$ at $f_k(x)$, with some coherence.

Lemma 4. *We can define linear isometries $R_k(x)$ of $\mathbb{R}^n$, $x \in P_0$, such that $R_0(x) = I$ on P_0,*

$$||R_{k+1} - R_k||_\infty \leq C\varepsilon, \tag{12}$$

appropriate upper bounds on $|DR_k|$ hold, and

$$R_k(x)(P_0') = T\Gamma_k(f_k(x)). \tag{13}$$

Proof by successive small modifications and partitions of unity. Compose with the projection on the tangent plane to modify the image, then retract on the set of isometries.

Now the **formula for the extension** F of f.

Call $y = p(z)$ the projection of $z \in \mathbb{R}^n$ on P_0, and $p^\perp = I - p$, and write

$$z = p(z) + p^\perp(z) = x + y \quad \text{for } z \in \mathbb{R}^n. \tag{13}$$

Write $1 = \sum_k \rho_k(r)$ for $r > 0$ (a reasonable partition of 1 on $]0, +\infty)$), with ρ_k supported on $[2^{-k}, 2^{-k+2}]$ for $k \geq 1$.
Finally set

$$F(z) = \sum_{k \geq 0} \rho_k(|y|) \left[f_k(x) + R_k(x)(y) \right] \tag{14}$$

(with $z = x + y$ as above) and check that this works!

Notice that $F(z) = z$ for z far from P_0 (because only $\rho_0(y) = 1$). Otherwise, only two or three terms, where $|y| \sim 2^{-k}$.

2.b. Holes and $\beta(x,r)$-numbers

What if instead of $\gamma(x,r)$ we only control the P. Jones numbers

$$\beta_E(x,r) = \inf_P \frac{1}{r} \sup_{y \in E \cap B(x,r)} \operatorname{dist}(y,P) \tag{15}$$

where the infimum is taken over the d-planes P through x?

That is, we want to allow flat sets with holes.

New assumptions (for $E \subset \mathbb{R}^n$ closed, nonempty):

$$\text{dist}(x, P_0) \leq \varepsilon \quad \text{for } x \in E \tag{16}$$

and, if we define the J_k as above, then for $k \geq 0$ and $j \in J_k$, there is a d-plane P_j through x_j, such that

$$\text{dist}(y, P_j) \leq \varepsilon \quad \text{for } y \in E \cap 10B_j \tag{17}$$

and

$$d_{x_i, 10r_i}(P_i, P_j) \leq \varepsilon \tag{18}$$

whenever $i, j \in I_k \cup I_{k+1}$ are such that $3B_i \cap 3B_j \neq \emptyset$.

Or equivalently, we require the conclusion of Lemma 1, which we could not get automatically (when E stays close to a $(d-1)$-dimensional plane for a long time).

Theorem 2 [D.-Toro]. *(Memoirs of the AMS 2012). Let E satisfy the assumptions (16)-(18). Then there is a bijective BiHölder mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ (as in Theorem 1), such that $f(P_0)$ is a Reifenberg-flat set (satisfying the assumptions of Reifenberg's Theorem 1) and $E \subset f(P_0)$.*

Proof: check that the proof above goes through.

Comments.
The slow-motion condition (18) is needed: example of a flat set that lies close to a circle.
Because of this, this may be hard to apply.
Again easy to localize.

2.c. Approximation by other sets

Planes in Theorems 1 and 2 may be replaced with other objects. For instance minimal cones of dimension 2. See D.-De Pauw-Toro.

Or by Lipschitz graphs with constant ≤ 1, if the vertical direction varies slowly (to be written with Toro).

But we need analogues of the π_j (to define local retractions), and ways to prove (or assume) that the objects (and the π_j) vary slowly (as in Lemma 1).

2.d. Metric spaces (Cheeger-Colding)

Our only excursion in metric spaces, and even this is slightly exagerated, because or assumption is that E looks Euclidean at all scales.

Here E is (contained in) a metric space, but we have assumptions that say that it is locally close to d-planes in Euclidean space. We measure flatness with $\alpha(x,r)$, the infimum of numbers α such that there is a mapping $\varphi : E \cap B(x,r) \to B_d(0,r) \subset \mathbb{R}^d$, with

$$\big||\varphi(y) - \varphi(z)| - \mathrm{dist}_E(y,z)\big| \leq \alpha r \quad \text{for } y,z \in E \cap B(x,r)$$

and

$$\mathrm{dist}(w, \varphi(E \cap B(x,r))) \leq \alpha r \quad \text{for } w \in B(0,r).$$

[We do not require φ to be continuous.]

Then [Cheeger-Colding 1997] there is a Reifenberg theorem in this context: $\alpha(x,r) \leq \varepsilon$ for $x \in E$ and $r \leq 1$ implies the existence of local biHölder parameterizations.

2.e. Lipschitz parameterizations

The mapping f of Theorem 1 or 2 is Lipschitz, under suitable assumptions. For instance, set

$$J(x) = \sum_{k \geq 0} \beta(x, 2^{-k})^2 \tag{19}$$

for $x \in E$ (a Jones-Bishop function). Then:

Theorem 2 [D.-Toro, Memoirs of the AMS 2012]. *Let E be as in Theorem 1. Assume in addition that $J(x) \leq M$ for $x \in E$. Then f in Theorem 1 is bilipschitz (if $\varepsilon > 0$ is small enough, depending on n, and with bilipschitz bounds that depend only on M and n).*

Comments. Many variants exist.

- Previously by Toro, when $\sum_k \left\{ \sup_x \beta(x, 2^{-k})^2 \right\} < +\infty$;

- Cheeger-Colding (with metric spaces), assuming that $\sum_k \left\{ \sup_x \alpha(x, 2^{-k}) \right\} < +\infty$; this fits well (without the square).

- for Ahlfors-regular sets but with numbers $\beta_q(x, r)^2$; Also, sufficient conditions for big bilipschitz pieces.

- With holes, but with a control on the sum of angles between the P_j.

Proof: "just" pay more attention to the distortion estimates like (9) (or directly on the size of Dg_k on $T\Gamma_k$).

The $\gamma(x, r)$ control the angles between the P_j, which are first order, and the squares control the distortion (by Pythagorus).

This is similar to the travelling salesman results of P. Jones, C. Bishop, G. Lerman, and others. Not a surprise.

3. TRAVELLING SALESMAN THEOREMS

What do we do when $\gamma(x,r)$ (or some angles) are sometimes large?

Main option: work in separate regions of $E \times (0,1]$, and glue partial pieces.

Usually we won't get more than a covering of E by a nicely parameterized (but not injectively) surface.

Reference result for this:

Theorem [P. *Jones, K. Okikiolu]. Let $E \subset \mathbb{R}^n$ be compact. There exists a curve γ of finite length which contains E if and only if*

$$\beta_{tot} = \sum_{k \geq 0} t^{1-n} \int_{\mathbb{R}^n} \beta_E(x,r)^2 dx < +\infty.$$

Comments.

This comes with good estimates : $length(\gamma) \sim \text{diam}(E) + \beta_{tot}$.

Proof (of the sufficient condition): rather cover by a connected set. Proceed scale by scale, cover the x_i, $i \in J_k$, add points at each scale, and compute the costs of the detours.

Sometimes, you need to think a little bit ahead.

In the flat situations like Theorem 1, we just replace segments with thin triangles, and loose something like $2^{-k}\beta(x_j, 2^{-k+1})^2$ (by Pythagorus).

Improvement: set $J_E(x) = \int_0^1 \beta_E(x,r)^2 \frac{dr}{r}$. Then Bishop and Jones say that if $J_E(x) \leq M$ on E, then there is a curve γ such that $E \subset \gamma$ and $length(\gamma) \leq Ce^{CM}$. Here C depends on n only.

Even more is true, by P. Jones and G. Lerman:

Let μ be a locally finite Borel measure on $\mathbb{R}^n$. For each cube Q (with faces parallel to the axes), set

$$\beta_\mu(Q) = \frac{1}{\operatorname{diam}(Q)} \inf_P \left\{ \int_Q \operatorname{dist}(y, P)^2 \, \frac{d\mu(y)}{\mu(Q)} \right\}^{1/2}$$

(where the infimum is over all d-planed P), and then

$$J(x) = \sum_{k \in \mathbb{Z}} \sup \left\{ \beta(Q)^2 \, ; \, Q \text{ is a cube that contains } x \right\}$$

for $x \in E$, the closed support of E.

Theorem [Jones-Lerman]. *There exist constants C_1 and C_2 (that depend only on n), so that if Q_0 is the unit cube and*

$$\int_{C_1 Q_0} e^{C_2 J_Q(x)} d\mu(x) \leq A\mu(Q_0)$$

then there is an ω-regular surface Γ, with constant at most $C(A, n, d)$, such that

$$\mu(\Gamma) \geq C_2^{-1} A^{-1} \mu(Q_0).$$

[Sorry: no definition of the (uniformly rectifiable) ω-regular surfaces.]

Long and complicated proof, but not unlike the above; with some amount of gluing too.

4. CORONA DECOMPOSITIONS

Take the set E. Usually, Ahlfors-regular of dimension d, or at least on which a d-dimensional measure μ is given.

Go from large scales to small ones.

Define the stopping time regions $\mathcal{R}$ in $E \times (0,1]$, under a given ball $B_0 = B(x_0, r_0)$, by stopping at the largest balls $B(x,r) \subset B(x_0, r_0)$ such that one of the following bad things happen:

- $r^{-d}\mu(B(x,r)$ is too large, or too small;
- $\beta(x, 10r)$ is too large;
- the good plane $P(x,r)$ makes a big angle with $P(x_0, r_0)$;
- Maybe some other conditions (in addition or instead).

Then for each $\mathcal{R}$ there is a Lipschitz graph $\Gamma_{\mathcal{R}}$, (or a nice set) that approximates E well in $\mathcal{R}$.

Cover $E \times (0,1]$ by regions $\mathcal{R}$. Parameterize each $\Gamma_{\mathcal{R}}$, glue, and this gives a parameterization of a set that contains E.

Main problem: give conditions on E, like uniform rectifiability, that ensure that there are not regions $\mathcal{R}$.

Main advantage: we can use $\Gamma_{\mathcal{R}}$ itself to prove such an estimate! Works like a machine.

An extension theorem for bilipschitz mappings

In fact, a technical lemma in a paper of J. Azzam and R. Schul on big pieces of bilipschitz mappings.

Theorem [Azzam-Schul]. *For all small $\kappa > 0$ we can find $\varepsilon > 0$ such that, if $E \subset \mathbb{R}^n$ us closed and $f : E \to \mathbb{R}^n$ is (κ, ε)-Reifenberg and L-bilipschitz, then it has an extension which is a (bijective) L'-bilipschitz mapping: $\mathbb{R}^n \to \mathbb{R}^n$.*

Comments and definitions.

Here $L' \leq C(n)\kappa^{-1}L$, and $f : E\mathbb{R}^D$, with $D > n$ is possible.

Reifenberg-flat is hard to prove (so Azzam-Schul use this in connection with stopping-time constructions). It is defined as for sets :

$f : E \to \mathbb{R}^n$ is Reifenberg-flat means that:

For every dyadic cube Q such that Q meets E, there is an approximating affine mapping $A_Q : \mathbb{R}^n \to \mathbb{R}^n$ such that:

$$\text{the } n \text{ singular values of } A_Q \text{ are } \geq \kappa$$

$$|f(x) - A_Q(x)| \leq \varepsilon \operatorname{diam}(Q) \text{ for } x \in E \cap 3Q$$

$$||DA_Q - DA_R|| \leq \varepsilon \qquad \text{when } R \text{ is a child of } Q$$

and when they have the same size and touch.

Similar in spirit to Theorem 2 above!
Previously, an extension theorem of Tukia and Vaisälä.

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