# LOCAL REGULARITY PROPERTIES OF ALMOST- AND QUASIMINIMAL SETS WITH A SLIDING BOUNDARY CONDITION

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**Résumé.** On s'intéresse à la régularité jusqu'à la frontière des ensembles presque minimaux et quasiminimaux sous une condition de glissement. Les compétiteurs d'un ensemble $E$ y sont de la forme $F = \varphi_1(E)$, où $\{\varphi_t\}$ est une famille à un paramètre d'applications continues définies sur $E$, et qui préservent des ensembles frontières donnés à l'avance. On généralise des résultats connus à l'intérieur, et on démontre notamment l'Ahlfors régularité, la rectifiabilité et parfois l'uniforme rectifiabilité locales des ensembles quasiminimaux, la stabilité des classes considérées par limites, et la presque monotonie de la densité des ensembles presque minimaux sur des boules centrées à la frontière.

**Abstract.** We study the boundary regularity of almost minimal and quasiminimal sets that satisfy sliding boundary conditions. The competitors of a set $E$ are defined as $F = \varphi_1(E)$, where $\{\varphi_t\}$ is a one parameter family of continuous mappings defined on $E$, and that preserve a given collection of boundary pieces. We generalize known interior regularity results, and in particular we show that the quasiminimal sets are locally Ahlfors-regular, rectifiable, and some times uniformly rectifiable, that our classes are stable under limits, and that for almost minimal sets the density of Hausdorff measure in balls centered on the boundary is almost nondecreasing.

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**Key words.** Minimal sets, Almost minimal sets, Almgren restricted or quasiminimal sets, Sliding boundary condition, Hausdorff measure.

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FREQUENTLY USED NOTATION

$B(x,r) = \{y\,;\,|y-x| < r\}$ is the open ball centered at $x$ and with radius $r > 0$.
$\mathcal{H}^d$ is the $d$-dimensional Hausdorff measure. See [Fe] or [Ma].
$GSAQ = GSAQ(U, M, \delta, h)$ is a class of quasiminimal sets; see Definition 2.3.
$W_t = \{y \in E \cap B\,;\varphi_t(y) \neq y\}$ and $\widehat{W} = \bigcup_{0<t\leq 1} W_t \cup \varphi_t(W_t)$; see (2.1).
$E^* = \{x \in E\,;\,\mathcal{H}^d(E \cap B(x,r)) > 0 \text{ for every } r > 0\}$ is the core of $E$; see (3.2).
$d_{x,r}(E,F)$ is almost a normalized Hausdorff distance in $B(x,r)$; see (10.5).
† † delimits a proof or comment that concerns the Lipschitz assumption only.
$W_f = \{x \in \mathbb{R}^n\,;\,f(x) \neq x\}$; see (11.19).
$\widetilde{f}(x) = \psi(\lambda f(x))$ (used in Part IV, in the Lipschitz case); see (11.50), (12.75).
$B_j = B(x_j, t)$, $j \in J_1$, is our first collection of balls (Part IV); see (12.8)-(12.9).
$B_j = B(x_j, r_j)$, $j \in J_2$, is the second collection of balls; see Lemma 14.6.
$D_j = B(y_j, r_j)$, $j \in J_3$, balls in the image, are used with the $B_{j,x}$; see (15.12)-(15.14).
$B_{j,x}$, $x \in Z(y_j)$, is our third collection of balls; see (15.19) and (15.1).
$h(r)$ is a gauge function that measures almost minimality; see (20.1) and Definition 20.2.
$\mathcal{I}(U,a,b)$, $\mathcal{I}_l(U,a,b)$, and $\mathcal{I}^+(U,a,b)$ are classes of elliptic integrands; see Definition 25.3, Claim 25.89, and (25.94).

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PART I : INTRODUCTION AND DEFINITIONS

## 1. Introduction

The main purpose of this paper is to study the boundary regularity properties of minimal, almost minimal, and quasiminimal sets, subject to sliding boundary conditions that we will explain soon.

A long term motivation is to study various types of Plateau problems, but where the objects under scrutiny are a priori just sets (rather than currents or varifolds), and we want to assume as little structure on them as possible. In this respect, the sliding conditions below seem natural to the author, and should be flexible enough to allow for a variety of applications.

Let us give a very simple example of a Plateau problem that we may want to study, and for which we do not have an existence result yet. Let $\Gamma \subset \mathbb{R}^n$ be a smooth closed curve, and let $E_0 \subset \mathbb{R}^n$ be a compact set that contains $\Gamma$. For instance, parameterize $\Gamma$ by the unit circle, extend the parameterization to the closed unit disk, and let $E_0$ be the image of the disk. Many other examples are possible, but with this one we should not get a trivial problem for which the infimum is zero. Our Plateau problem consists in minimizing $\mathcal{H}^2(E)$ among all sets $E$ that can be written $E = \varphi_1(E_0)$, where $\{\varphi_t\}$, $0 \leq t \leq 1$, is a continuous, one parameter family of continuous mappings from $E_0$ to $\mathbb{R}^n$, with $\varphi_0(x) = x$ for $x \in E_0$ and $\varphi_t(x) \in \Gamma$ for $0 \leq t \leq 1$ when $x \in E_0 \cap \Gamma$. Thus, along our deformation of $E_0$ by the $\varphi_t$, we allow the points of $\Gamma$ to move, but only along $\Gamma$; this is why we shall use the term "sliding boundary condition".

Minimizers of this problem, if they exist, will be among our simplest examples of minimal sets with a sliding boundary condition. But solutions of other types of Plateau problems (Reifenberg minimizers as in [R1,2], [De], or [Fa], or size minimizing currents under the boundary constraint $\partial T = G$, where $G$ denotes the current of integration along $\Gamma$, when they exist, also yield minimal sets with a sliding boundary condition. Thus regularity results for sliding minimal sets may be useful for a variety of problems, and we can also hope that they may help with existence results.

Let us first give some definitions, and then discuss these issues a little more. The sets that we want to study are variants of the Almgren minimal, almost minimal, or quasiminimal sets (he said "restricted sets"), as in [A2], but where we add boundary constraints and are interested in the behavior of these sets near the boundary.

We work in a closed region $\Omega$ of $\mathbb{R}^n$, which may also be $\mathbb{R}^n$ itself, and we give ourselves a finite collections of closed sets $L_j \subset \Omega$, $0 \leq j \leq j_{max}$, that we call boundary pieces. It will make our notation easier to consider $\Omega$ as our first boundary piece, i.e., set

$$L_0 = \Omega. \tag{1.1}$$

For the elementary Plateau problem suggested above, for instance, we would work with $L_0 = \Omega = \mathbb{R}^n$ and $L_1 = \Gamma$.

We are also given an integer dimension $d$, with $0 \leq d \leq n-1$, and we consider closed sets $E \subset \Omega$, whose $d$-dimensional Hausdorff measure is locally finite, i.e., such that

$$\mathcal{H}^d(E \cap B(x,r)) < +\infty \tag{1.2}$$

for $x \in \Omega$ and $r > 0$. The next definition explains what we mean by a deformation of $E$ that preserves the boundary pieces.

**Definition 1.3.** *Let $B = \overline{B}(y,r)$ be a closed ball in $\mathbb{R}^n$. We say that the closed set $F \subset \Omega$ is a* <u>*competitor*</u> *for $E$ in $B$, with sliding conditions given by the closed sets $L_j$, $0 \leq j \leq j_{max}$, when $F = \varphi_1(E)$ for some one-parameter family of functions $\varphi_t$, $0 \leq t \leq 1$, with the following properties:*

$$(t,x) \to \varphi_t(x) \text{ is a continuous mapping from } [0,1] \times E \text{ to } \mathbb{R}^n, \tag{1.4}$$

$$\varphi_t(x) = x \text{ for } t = 0 \text{ and for } x \in E \setminus B, \tag{1.5}$$

$$\varphi_t(x) \in B \text{ for } x \in E \cap B \text{ and } t \in [0,1], \tag{1.6}$$

*and, for $0 \leq j \leq j_{max}$,*

$$\varphi_t(x) \in L_j \text{ when } t \in [0,1] \text{ and } x \in E \cap L_j \cap B. \tag{1.7}$$

*We also require that*

$$\varphi_1 \text{ be Lipschitz,} \tag{1.8}$$

*but with no bounds required.*

We shall sometimes say "sliding competitor in $B$" instead of "competitor for $E$ in $B$, with sliding conditions given by the $L_j$, $0 \leq j \leq j_{max}$", especially when our choice of $\Omega$ and the list of $L_j$ are clear from the context.

We shall soon discuss minimality, almost minimality, and quasiminimality relative to this notion of sliding competitors, but since the class of competitors is often the most important part of the definitions, a number of general comments on Definition 1.3 will be helpful.

It is important here that $\varphi_1$ is allowed not to be injective. So we are allowed to merge different portions of $E$, or contract them to a point, or pinch them in some other way. This, together with the fact that we shall not count measure with multiplicity, is why the union of two parallel disks that lie close to each other will not be minimal.

We added the last requirement (1.8) because Almgren put it in his definitions, and because this will not disturb. If we drop it, we get more competitors for $E$, which means that the almost- and quasiminimality properties are harder to get. Hence the regularity results proved here are also valid in the context where we drop (1.8). On the other hand, (1.8) will often be easy to prove, so it does not bother us much. The author suspects that the reason why Almgren added (1.8) may be the following. Suppose you want to show that the support of a size minimizing current $T$ is a minimal set and, to simplify the discussion, that you are proceeding locally, in the complement of the boundary sets. You are given a deformation $\{\varphi_t\}$ as in Definition 1.3, and of course the simplest way to use it is to show that pushing $T$ by the $\varphi_t$, and in particular $\varphi_1$, defines an acceptable competitor for $T$ (with the same boundary constraints). The constraint (1.8) just makes it possible to define the pushforward of $T$ by $\varphi_1$, so it is convenient. See [D8] for details on this argument and its extension to the boundary.

In the other direction, J. Harrison and H. Pugh once asked wether requiring $\varphi_1$, or even all the $\varphi_t$, to be smooth, would lead to the same classes of almost- and quasiminimal sets. The question was raised in the local context with no boundaries, but it also makes sense in the present context. The answer is yes under suitable conditions on the $L_j$, and if smooth means $C^1$. For higher regularity, a proof seems to be manageable, but quite ugly, and so we only give a very rough sketch of how we would proceed, using the construction of Part IV. This is discussed in Section 27.

We are allowed to take $\Omega = \mathbb{R}^n$, and then (1.7) for $j = 0$ is just empty and if there is no other boundary piece we get a minor variant of Almgren's definition of competitors in $\mathbb{R}^n$. Of course we can still restrict the list of competitors like he did, by requiring that $B$ lies in a fixed open set $U$, or that its diameter be less than some $\delta > 0$; we shall do this when we discuss our classes of almost- and quasiminimal sets, but let us not worry for the moment.

The main difference with Almgren's definition comes from the sliding boundary constraint (1.7), and this is also why we insist on the fact that $\varphi_1$ is the endpoint of a continuous deformation. If we did not require (1.7), and we were given a continuous mapping $\varphi_1$ such that $\varphi_1(x) = x$ for $x \in E \setminus B$ and $\varphi_1(x) \in B$ for $x \in B$, we could define the $\varphi_t$ by $\varphi_t(x) = t\varphi_1(x) + (1-t)x$, and it is easy to check that (1.4)-(1.6) would hold (because $B$ is convex). We could also extend $\varphi_1$ to $\mathbb{R}^n$, which fits with the fact that $\varphi_1$ is traditionally defined on $\mathbb{R}^n$, not just on $E$. But in the present situation we want points of the boundary $L_j$ to stay in $L_j$ (hence, (1.7)), and then it seems natural to say that the deformation condition in (1.7) only concerns points of $E$: we do not want to say that the air besides our soap film $E$ is also concerned by the sliding boundary constraint. Notice that the $\varphi_t$ can be extended to $\mathbb{R}^n$ (but in a way that may not preserve the $L_j$), so we do not have to worry about the case where our deformations would yield a tearing apart (cavitation) of the air besides the soap film.

Notice that with our convention that $L_0 = \Omega$, the set $\varphi_t(E)$ stays in $\Omega$, i.e.,

$$\varphi_t(x) \in \Omega \text{ for } x \in E \text{ and } t \in [0,1], \tag{1.9}$$

either because $x \in E \setminus B$ and $\varphi_t(x) = x \in E \subset \Omega$ by (1.5), or else by (1.1) and (1.7) with $j = 0$.

The author thinks that Definition 1.3 is a nice way to encode boundary constraints, for instance that would be satisfied when $E$ is a soap film in a domain. A Plateau boundary constraint could for instance be associated to one or a few curves $L_j$, but we could also think about $L_1 = \partial\Omega$ (or some other surface) as being a boundary along which the soap film may slide (as if loosely attached to a wall). It is quite probable that such boundary conditions were studied in the past, but the author does not know where.

Once we have a notion of competitors, we can define a corresponding notion of minimal sets. Let us say, for the moment, that the closed set $E \subset \Omega$ is minimal, with the sliding boundary conditions defined by the $L_j$, $0 \leq j \leq j_{max}$, if $H^d(E) < +\infty$ and

$$\mathcal{H}^d(E) \leq \mathcal{H}^d(F) \quad \text{whenever } F \text{ is a sliding competitor for } E \text{ in some ball } B, \tag{1.10}$$

where we allow $B$ to depend on $F$. Many variants of this definition will be proposed, where one may localize the definition to an open set $U$, or add a small error term to the right-hand side in (1.10) (this is how we will define almost minimal sets), or even allow stronger distortions (this will give rise to quasiminimal sets). We shall give the main definitions in Section 2 (for the generalized quasiminimal sets) and later in Section 20 (for almost minimal sets), but for the moment the sliding minimal sets that satisfy (1.10) will give a fair idea of what we want to study.

Of course our notion of competitors can be used to define Plateau problems, as we did earlier with a single curve. Given a collection of boundary pieces $L_j$, and a closed set $E_0$, we can try to minimize $\mathcal{H}^d(E)$ among all the sets $E$ that are sliding competitors of $E_0$ (in some ball $B$ that depends on $E$, or in some fixed huge ball that contains $\Omega$). If $E_0$ is badly chosen (for instance, if some sliding competitors of $E_0$ are reduced to a point), the problem may not be interesting, but it is easy to produce lots of examples where the infimum will be finite and positive. For most of these examples, we do not have an existence result. But it is clear that if minimizers for this Plateau problem exist, they are sliding minimal sets.

The main point of this paper is to study the general (hence often rather weak) regularity properties of the minimal sets, and their almost minimal and quasiminimal variants, in particular when we approach the boundary pieces $L_j$. In practical terms, this means that we will take many interior regularity results for Almgren minimal (or quasiminimal) sets, and try to adapt their proofs so that they work all the way to the boundary. But before we say more about this, let us comment a little more on Definition 1.3 and our motivations.

The word sliding may be misleading in some cases, as some sets $L_j$ may be reduced to points, where in effect no sliding will be allowed. Our assumptions on the $L_j$ will only allow a finite number of points where $E$ is fixed. So, for instance, we do not consider the case where $\Gamma$ is a simple curve and we require that $\varphi_t(x) = x$ for every point $x \in E \cap \Gamma$. This will not bother us, and probably such a condition would make it too hard to produce competitors and get information on $E$ near $\Gamma$ when $E$ is a minimal set with these constraints. Of course we could always say that $E$ is locally minimal (for instance) in the domain $U = \mathbb{R}^n \setminus \Gamma$, and get some information from this, but this is not the point of this paper. On the contrary, the author believes that because we allow our competitors to slide along the $L_j$, we will have an amount of flexibility in the construction of competitors, which we can use to prove some decent regularity results. And at the same time (1.7) looks like a reasonable constraint, for instance, if we want to model the behavior of soap films.

We believe that in addition to being interesting by themselves, regularity results for sliding minimal or almost sets could be useful to prove existence results (in very simple cases) for the Plateau problems discussed above, and also for other similar problems, because some other types of minimizers also yield sliding minimal sets. Let us give two examples.

In [R1], Reifenberg proposed a Plateau problem where we are given a compact boundary set $L \subset \mathbb{R}^n$ of dimension $d-1$, and we minimize $\mathcal{H}^d(E)$ among compact sets $E$ that bound $L$, in the sense that $L \subset E$ and the natural map induced by the inclusion, from the $(d-1)$-dimensional Čech homology group of $L$ to the $(d-1)$-dimensional Čech homology group of $E$, is trivial. He also proves a fairly general existence result, and good interior regularity results for the minimizers (see [R1,2]). These results were generalized by various authors; see for instance [A1], [De], and more recently [Fa] for a quite general existence result. Also see [HP] for a simpler variant of [R1] in codimension 1, where one replaces the computation of Čech homology groups with a simpler linking condition, and which comes with a simpler proof and is related to differential chains.

It is easy to see that if the boundary set $L$ is not too ugly, the minimizing sets that are obtained in these papers are sliding minimal sets associated to $L_0 = \mathbb{R}^n$ and $L_1 = L$. See [D8] for the rather easy verification, whose main point is just that if $E$ bounds $L$ and $F$ is a sliding competitor for $E$, then $F$ bounds $L$ too.

Reifenberg' homological Plateau problem and its minimizers are very nice, and give good descriptions of many soap films, but some people prefer the related problem of size minimizers. That is, we are given a $(d-1)$-dimensional integral current $S$, with $\partial S = 0$, and we look for a $d$-dimensional integral current $T$ such that $\partial T = S$ and whose size (understand, the $H^d$-measure of the set where the multiplicity is nonzero, but we shall be slightly sloppy on the definitions) is minimal. If $d = 2$, $L$ is a nice closed curve in $\mathbb{R}^3$, and $S$ is the current of integration on $L$, T. De Pauw showed in [De] that the infimum for this problem is the same as for Reifenberg's homological problem (where Čech homology is computed over the group $\mathbb{Z}$); but even though De Pauw showed that Reifenberg homological minimizers exist, size minimizers are not known yet to exist. Anyway, size minimizers, if they exist, are also supported (under reasonable conditions) on sliding minimal sets. The point now is that if $T$ is supported by the closed set $E$ and $F$ is a sliding competitor for $E$, then we can use $\varphi_1$ to push $T$ and get another solution of $\partial T = S$, which is supported on $F$. See [D8] for the fairly easy verification of this, and variants where $\partial T$ is only required to be homological to $S$ in a boundary set $L$.

So we have at least two potentially interesting other examples of sliding minimal sets. To the author's knowledge, not much is known on the boundary behavior of these sets, and the results in this paper are probably a good start. A natural question is whether, if we decide to study them by saying they are sliding minimal sets and forgetting about the initial problem they solve, we lose important information that we may have used profitably.

Most of the results of this paper concern the weaker notion of sliding quasiminimal sets, but let us make two short remarks on sliding minimal sets. An important tool that we can still use in some cases is Allard's regularity theorem from [All], which applies to more general stationary varifolds and goes all the way to the boundary. But this result uses some initial flatness assumption that we may not want to assume.

In the special case when $L_0 = \mathbb{R}^3$ and $L_1$ is a nice curve, G Lawler and F. Morgan propose a conjectural list of 10 boundary behaviors for minimal sets bounded by $L_1$; see [LM] and [Mo3], and in particular Figure 13.9.3 (on page 137 of the third edition). The present paper tries to go in such directions, so far in more general contexts but with less precision.

We may now start a description of the results in this paper. Generally speaking, we shall take local regularity results that we like, and try to extend or modify the proofs so that they work also near the boundary pieces.

Many of our results are about what we call generalized quasiminimal sets, which are defined in Section 2 (see Definition 2.3). In the special case without boundary pieces, the notion is just a little bit more general than the quasiminimal sets that Almgren studied in [A2] under the name of "restricted sets". One advantage of the notion is that it is rather weak and quite flexible. For instance, it is stable under bilipschitz mappings (the quasiminimality constant $M$ just gets larger), and contains minimizers of functionals like $\int_E f(x) d\mathcal{H}^d(x)$, where we just need to know that $f$ is bounded and bounded from below, and under the same sort of boundary conditions as above. Thus the graph of any Lipschitz function $F : \mathbb{R}^d \to \mathbb{R}^{n-d}$ is locally quasiminimal (with no boundary condition). Of course this means that we cannot expect better regularity than Lipschitz, but this will already be a good start, and in effect we shall not get so far from that.

We shall work locally, in an open set $U$, and with two set of assumptions on the boundary pieces. In the first one, which we shall call the rigid assumption, $U$ is the unit ball $B_0$, we choose a dyadic grid of $\mathbb{R}^n$, and we require all the sets $L_j$ to coincide in $U$ with a finite union of faces of cubes of our grid. We do not even require all these faces to be of the same dimension.

This already gives some choice, but we do not necessarily want all the faces to be smooth, and we expect some bilipschitz invariance, so we also allow a weaker set of assumptions, which we call the Lipschitz assumption, where $U$ and the $L_j$ are obtained from the previous case by composing with a bilipschitz mapping from $B_0$ to $U$. We even allow an additional dilation that we shall skip here for simplicity. See Definition 2.7. Some times the regularity results in this second case will require more complicated proofs, but we decided to include them anyway.

Even this set of assumption is not entirely satisfactory, because for instance it puts some small bounds on the number of faces that may touch a given point, but the dyadic combinatorics are pleasant to use, and the author was afraid of the complications that may arise in a more general case.

Let us give a rough description of our plan.

Part I deals with the setup and definitions. After the definitions of Section 2, we check that sliding quasiminimal sets in $U$, under the Lipschitz assumption, are just the images by our bilipschitz parameterizaton of sliding quasiminimal sets in $B_0$, with the rigid assumption. See Proposition 2.8.

In Section 3, we introduce the core $E^*$ of a closed set $E$ (our name for the closed support of the restriction of $\mathcal{H}^d$ to $E$, see Definition 3.1), and show that the core of a (generalized sliding) quasiminimal set is quasiminimal with the same constants. See Proposition 3.3 (and before, Proposition 3.27 in the simpler rigid setting). The proof is a little unpleasant (because our boundary constraint (1.7) does not obviously cooperate with removing some parts of $E$), but afterwards we feel better because we can forget about the fuzzy set $E \setminus E^*$, and restrict our attention to coral sets, i.e., sets such that $E^* = E$.

Part II contains our first regularity results for generalized sliding quasiminimal set.

In Section 4, we show that the core $E^*$ of such a set $E$ is locally Ahlfors regular. This means that, if $x \in E^*$, $B(x, 2r) \subset U$ (the open set where we work), $r > 0$ is smaller than the scale constant $\delta$ in the definition 2.3 of quasiminimal sets, and the small parameter $h > 0$ in Definition 2.3 is small enough, then

$$C^{-1} r^d \leq \mathcal{H}^d(E \cap B(x,r)) = \mathcal{H}^d(E^* \cap B(x,r)) \leq C r^d. \tag{1.11}$$

See Proposition 4.1 (under the rigid assumption) and Proposition 4.74 (for the Lipschitz case). The proof relies on comparison arguments based on Federer-Fleming projections. It follows the proof of [DS4] (for the case without boundaries), which itself looks a lot like the proof in [A2] of almost the same result.

Section 5 continues along the same lines. Its main result is Theorem 5.16, which says that quasiminimal sets (with a small enough constant $h$) are rectifiable. We still prove this with a Federer-Fleming projection, and the proof is probably similar to Almgren's original proof (away from the boundaries). The main point is that near a point of density of the unrectifiable part of $E$, we could project $E$ on a small subset of $d$-faces, so small that an additional projection on faces of dimension $d-1$ is possible and allows us to make it essentially disappear. It is interesting that the rectifiability of quasiminimal sets (and their limits, see Part IV) was neglected in [DS4], just because we could prove stronger properties, while here we will have to rely more on it in the cases where we don't get uniform rectifiability.

On a slightly more technical level, Proposition 5.1 says that for $B(x,r)$ as above (i.e., $E$ is quasiminimal, $x \in E^*$, $B(x,2r) \subset U$, $r < \delta$, and $h$ is small enough), there is a Lipschitz mapping $F : E \cap B(x,r) \to \mathbb{R}^d$ such that $\mathcal{H}^d(F(E \cap B(x,r))) \geq C^{-1} r^d$; this is a technical lemma that can be used in later proofs (typically, for uniform rectifiability). Then Proposition 5.7 is a trick from [DS4] that allows us to pretend that $F$ is the orthogonal projection on some $d$-plane.

Part III deals with the local uniform rectifiability of the core $E^*$ when $E$ is quasiminimal (and $h$ is small enough, as always).

The main result of this part says that if $E$ is a quasiminimal set (and $h$ is small enough), and if some technical condition on the dimension of the faces is satisfied, then

$E^*$ is locally uniformly rectifiable, with big pieces of Lipschitz graphs. See Theorem 6.1 under the rigid assumption, and Theorem 9.81 under the Lipschitz assumption.

Uniformly rectifiable with big pieces of Lipschitz graphs means that there are constants $A \geq 0$ and $\theta > 0$ such that, if $B(x,r)$ is as above, then we can find an $A$-Lipschitz graph $\Gamma$ of dimension $d$ such that

$$\mathcal{H}^d(E \cap B(x,r)) \cap \Gamma \geq \theta r^d. \tag{1.12}$$

Thus in $B(x,r)$, a substantial part of $E$ lies in the nice $A$-Lipschitz graph $\Gamma$. Recall that by definition, $\Gamma$ is the graph of some $A$-Lipschitz function that is defined on some $d$-plane $P \subset \mathbb{R}^n$, and with values in the orthogonal $(n-d)$-plane $P^\perp$. In this statement, $A$ and $\theta$ depend only on the dimensions $n$ and $d$, the quasiminimality constant $M$ in Definition 2.3, and the bilipschitz constant $\Lambda$ in Definition 2.7.

Unfortunately, we we only get this under the technical condition (6.2) (or its analogue (9.83) when we use the Lipschitz assumption). It is satisfied if, except for the supporting domain $L_0 = \Omega$, all the boundary pieces $L_j$ are composed of faces of dimensions at most $d$. This takes care of many interesting examples, but it is nonetheless frustrating that we have to assume this. Of course we do not have a counterexample; the main problem could even be that even in the case without boundary, we have only one proof of uniform rectifiability, and this proof is complicated and fails badly when we deal with boundaries.

The positive point of uniform rectifiability is that it has the right invariance under bilipschitz mappings, and that it is, to the author's knowledge, the best very general (weak) regularity result for our quasiminimal sets.

Most of Part III is devoted to a proof of Theorems 6.1 and 9.81 on the local uniform rectifiability of $E^*$. We essentially take the long and complicated proof from [DS4], try to adapt it, and see where it fails.

At the start, Propositions 5.1 and 5.7 allow us to assume that for some orthogonal projection $\pi$ on some $d$-plane, $\mathcal{H}^d(\pi(E \cap B(x,r))) \geq C^{-1} r^d$; the whole proof then consists in showing that we can find a large subset of $E \cap B(x,r)$ where $\pi$ is bilipschitz. Section 6 describes the general scheme of a stopping time argument which is designed to select the large subset, why it fails in general, and why it still works in some limited cases (but really, not so many new things happen, compared to the previous situation with no boundary). We end up, in Proposition 6.41, with a result that says that in some cases, $E \cap B(x,r)$ contains a significant part which is bilipschitz-equivalent to a subset of $\mathbb{R}^d$.

In addition to the stopping time argument described in Section 6, Theorems 6.1 and 9.81 use some amount of general uniform rectifiability theory which is done, when we work under the rigid assumption, in Sections 7 and 8.

The uniform rectifiability of an Ahlfors regular set $E$ can be defined in lots of (eventually equivalent) ways, and in Section 7 we discuss two of them. The first one, called BPBI, asks for the existence, in each ball $B(x,r)$ centered on $E$, of a substantial part of $E \cap B(x,t)$ that can be send to a subset of $\mathbb{R}^d$ by a bilipschitz mapping. In the case of quasiminimal sets, we first restrict to the core $E^*$ and work only locally, i.e., on balls such that $B(x,2r) \subset U$, but let us forget these details. Now Proposition 6.41 gives something like this, but not in enough balls $B(x,r)$, so one has to work more, and in effect go through the BWGL below.

A second definition of uniform rectifiability is by the bilateral weak geometric lemma (BWGL), which asks that for most balls $B(x,r)$ (defined in terms of Carleson measures but please don't mind), there is a $d$-plane $P$ such that $E \cap B(x,r)$ is $\varepsilon r$-close to $P \cap B(x,r)$ (in Hausdorff distance, and where $\varepsilon > 0$ is a fixed small constant). It turns out that this one is easier to get.

The only place in Section 7 where the quasiminimality of $E$ is used directly (as opposed to, via a regularity result of a previous section) is to show that if all the points of $E \cap B(x,r)$ lie within $\varepsilon r$ of some $d$-plane $P$, then the converse is also true: all the points of $P \cap B(x, 3t/2)$ lie within $\varepsilon r$ of $E$, and in addition the orthogonal projection from $E$ to $P$ is locally surjective (see (7.46)). See Lemma 7.38 for a more precise statement that takes into account the position of the boundary pieces $L_j$, and Lemma 9.14 for a generalization of this first statement.

This lemma helps because it is relatively easy to find balls where $E$ stays close to a plane, but the BWGL requires a bilateral approximation that Lemma 7.38 then provides. The rest of Section 7 consists in playing with bad sets of balls and various definitions of uniform rectifiability, to get the BPBI property (for every small ball, not just the good ones in Proposition 6.41). See Proposition 7.85. So $E$ is locally uniformly rectifiable.

In Section 8, we keep the rigid assumption and go from the BPBI to the BPLG, i.e., the existence of big pieces of Lipschitz graphs, as in the statement of Theorem 6.1. For this, the general theory says that we have to find big projections (see Theorem 8.5) and, roughly speaking, this is provided by the BWGL or even its unilateral version the WGL, plus another application of Lemma 7.38 (and (7.46) in particular).

In Section 9 we prove the analogue of Theorem 6.1 under the more general, but some times more painful, Lipschitz assumption. The relevant statements are now Lemma 9.14 (for the generalization of Lemma 7.38) and Theorem 9.81 (for the main uniform rectifiability result).

A consequence of the uniform rectifiability of $E^*$, that has been quite useful for the study of limits far from the boundaries, is the concentration property introduced by Dal Maso, Morel, and Solimini [DMS] in the context of the Mumford-Shah functional. The point is that for any sequence $\{F_k\}$ of sets that satisfies this property (with uniform constants) and converges to $F$ in Hausdorff distance, and any open set $V$, we have the lower semicontinuity property $\mathcal{H}^d(F \cap V) \leq \liminf_{k \to +\infty} \mathcal{H}^d(F_k \cap V)$.

We prove this property in Corollary 8.55 (under the Lipschitz assumption) and Corollary 9.103 (under the Lipschitz assumption), as simple consequences of the local uniform rectifiability, but then with the additional technical assumption (9.2) or (9.105). Fortunately, there is another proof of uniform concentration along sequences that does not use these assumptions; see Proposition 10.82.

Most of this Part III is not needed for the next ones; the failure of Theorems 6.1 and 9.81 in some cases lead the author to finding ways to prove the subsequent theorems (and in particular the results on limits, see Part V) that would not use uniform rectifiability. So the reader will get something positive out of the weakness of this part.

Part IV contains our main results on the limits of quasiminimal sets. The main statement for this part is Theorem 10.8, which says that if $E$ is the Hausdorff limit (locally in the open set $U$) of the sequence $\{E_k\}$ of coral (i.e., $E_k^* = E_k$) quasiminimal sets which all lie in a class $GSAQ(U, M, \delta, h)$, with $h$ small enough, then $E$ lies in the same quasiminimal class $GSAQ(U, M, \delta, h)$ as the $E_k$.

Here again, when we work under the Lipschitz assumption, we only prove this under a minor additional regularity assumption on the faces that compose the $L_j$. Typically, when such a face is more than $d$-dimensional, we require the face to be $C^1$ in a neighborhood of $\mathcal{H}^d$-almost each of its interior points. See (10.7), or Remark 19.52 for a weaker condition.

The main ingredient for Theorem 10.8 is the lower semicontinuity estimate in Theorem 10.97, which says that for $\{E_k\}$ as above,

$$\mathcal{H}^d(E \cap V) \leq \liminf_{k\to+\infty} \mathcal{H}^d(E_k \cap V) \text{ for every open set } V \subset U. \tag{1.13}$$

This is deduced from Dal Maso, Morel, and Solimini's result [DMS] and the fact that the sets $E_k$ are uniformly concentrated, as in Proposition 10.82. In turn Proposition 10.82 is obtained a little bit like Corollaries 8.55 and 9.103, but instead of uniform rectifiability, we use the fact that the limit $E$ is rectifiable (as in Proposition 10.15), and a compactness argument (Proposition 10.21). The surprising part, at least to the author, is the rectifiability of the limit, which is just proved like Theorem 5.16 (the rectifiability of a single $E_k$), with suitable modifications.

Even though Theorem 10.97 is the main ingredient in Theorem 10.8, the full proof takes the rest of Part IV (Sections 11-19). It follows the argument of [D2], but unfortunately with many small modification that force us to give a full proof.

Perhaps we should mention that it is important to prove limiting results like Theorem 10.8 and Theorem 10.97 in the context of sets. In the context of integral currents, for instance, the lower semicontinuity of the mass and strong compactness theorems exist, that have been used very profitably. Here we get an acceptable substitute for some of that. Without this, it would be hard to say much about the blow-up limits of almost minimal sets, for instance.

In Part V we study the stronger notion of almost minimality, and extend the stability results of the previous parts to them. A few different definitions are possible, but let us give a simple one that works when we do not localize. In addition to the list of boundary pieces $L_j$ (which we keep as above), we give ourselves a gauge function $h : (0, +\infty) \to [0, +\infty]$, such that $\lim_{r\to 0} h(r) = 0$. Often we also ask $h$ to be nondecreasing and continuous from the right, and for some results to have a sufficient decay near 0. A typical choice would be to pick $\alpha > 0$ and take $h(r) = r^\alpha$ for $0 \leq r < \delta$ and $h(r) = +\infty$ for $r \geq \delta$. A sliding almost minimal set (of type $A'$) in $\mathbb{R}^n$ is then a closed set $E$ such that (1.2) holds, and for which

$$\mathcal{H}^d(E \cap B) \leq \mathcal{H}^d(F \cap B) + h(r)r^d \tag{1.14}$$

for each closed ball $B = \overline{B}(x, r)$ and each sliding competitor $F$ for $E$ in $B$. When $h(r) \equiv 0$, we recover the definition of sliding minimal sets defined by (1.10). This notion can be localized to an open set $U$, and three slightly different types of sliding almost minimal sets (called $A_+$, $A$, $A'$) are introduced in Definition 20.2. Of course we expect better regularity properties for the sliding almost minimal sets, especially when $h$ is small; here we shall not really look for such properties, but rather prepare the ground with some preliminary results on limits of almost minimal sets and monotonicity properties for their density.

In Section 20 we give the three definitions of sliding almost minimal sets (Definition 20.2), but then prove that the two last ones ($A$ and $A'$) are equivalent. This is Proposition 20.9; the proof follows [D5], were similar notions were defined (to try to unify some definitions with Almgren's initial ones).

In section 21 we use our limiting theorem on quasiminimal sets (Theorem 10.8) to show that limits of coral sliding almost minimal sets (of a given type) with a given gauge function $h$ are also coral sliding almost minimal sets, of the same type and with the same gauge function. This is Theorem 21.3. Also see Remark 21.7 and Corollary 21.15 that say that the limit of a locally minimizing sequence of uniformly quasiminimal sets is locally minimal.

In Section 22 we prove an upper semicontinuity result for $\mathcal{H}^d$: if the sequence $\{E_k\}$ of coral sliding almost minimal sets in $U$ converges to $E$ (as in Theorem 21.3), then for each compact set $H$ in $U$,

$$\mathcal{H}^d(E \cap H) \geq \limsup_{k\to+\infty} \mathcal{H}^d(E_k \cap H); \tag{1.15}$$

see Theorem 22.1, which is specific to the case when $h(r)$ tends 0. For quasiminimal sets, we cannot expect such a neat estimate, but we still have the less precise

$$(1 + Ch)M\mathcal{H}^d(E \cap H) \geq \limsup_{k\to+\infty} \mathcal{H}^d(E_k \cap H) \tag{1.16}$$

which is proved in Lemma 22.3 and is often useful too. Again similar results were proved in [D5] Lemma 13.12, and probably many more places before. Surprisingly, the proof only uses the rectifiability of the limit $E$, some covering lemmas, and an application of the definition of quasiminimality in some flat balls.

Theorems 10.8 and 21.3 have an obvious defect: in many situations, such as for blow-up limits with boundaries $L_j$ that are not cones, we may want to take limits in situations where the domains, and more importantly the boundary sets $L_j$, change mildly. We do this in Theorem 23.8, but rather than redoing the whole proof, we reduce to the previous statements by composing with a variable change of variables that sends us back to a fixed domain (the limit). Our proof forces us to restrict to variable domains that are close to the limiting domain in the bilipschitz category, which is probably not optimal.

We apply this in Section 24 to blow-up limits. Under reasonably mild flatness conditions on the sets $L_j$ at the origin (see Definitions 24.8 and 24.29, and Proposition 24.35 that says that the individual flatness of faces (as in Definition 24.29) is enough), we show that the blow-up limits at the origin of a sliding almost minimal set, are sliding minimal sets in $\mathbb{R}^n$, associated to boundary sets $L_j^0$ obtained from the $L_j$ by the same blow-up. See Theorem 24.13.

Part VI deals with two extensions of our notions of quasiminimality and almost minimality. The main one is related to elliptic integrands. Instead of using the Hausdorff measure $\mathcal{H}^d(E)$ in our various definitions, we may want to use slightly distorted versions like $\int_E f(x)d\mathcal{H}^d(x)$, where $f:\mathbb{R}^n \to [1,M]$ is a continuous function, or even

$$J_f(E) = \int_E f(x, T_xE)d\mathcal{H}^d(x), \tag{1.17}$$

where $f$ is now defined on $\mathbb{R}^n \times G(n,d)$, $G(n,d)$ denotes the Grasssman manifold of vector $d$-planes in $\mathbb{R}^n$, and we restrict to rectifiable sets so that the approximate tangent plane $T_xE$ is defined almost everywhere on $E$ (see (25.2) for a slightly artificial definition, but that would also work on $d$-sets that are not rectifiable). See Definition 25.3 for an acceptable class of elliptic integrands, which is just a little larger than the one introduced by Almgren [A1], [A3].

The main point of Section 25 is that the technique of [DMS] also allows us to prove lower semicontinuity results like (1.13), but for integrals like $J_f(E)$. This was noticed by Yangqin Fang [Fa], who wanted such a result to extend Reifenberg's existence theorem for his homological Plateau problem to the context of elliptic integrands, and Fang's proof is so simple that it would have been stupid not to give it here.

In Theorem 25.7, we prove that if the sequence $\{E_k\}$ of sliding quasiminimal sets in $U$ satisfies the main assumptions of our limiting Theorem 10.8, and if the integrand $f:\mathbb{R}^n \times G(n,d) \to [a,b]$ satisfies the condition of Definition 25.3, then

$$J_f(E\cap V) \leq \liminf_{k\to+\infty} J_f(E_k \cap V) \ \text{ for every open set } V \subset U, \tag{1.18}$$

where as usual $E$ is the limit of the $E_k$.

The proof contains the lower semicontinuity result that we used for Theorems 10.8 and 21.3, so the reader that would not be familiar with [DMS] can read Section 25 instead and get a slightly more direct proof, even for $f \equiv 1$. We still kept the reference to [DMS] for the other readers, and also because this is really where the ideas are coming from.

The notions of quasiminimality and almost minimality can also be defined in terms of an elliptic integrand $f$ as above. Since $a \leq f \leq b$ for some $a, b > 0$, the list of quasiminimal sets is the same, only the constants are different. This is why we do not need to be careful when we state Theorem 25.7. In Section 26 we explain how to extend Theorem 10.8 to limits of $f$-quasiminimal sets; see Claim 26.4. The same thing would happen with other results of Part V, but we omit the details.

We included Section 27 to answer partially a question of J. Harrison (initially raised far from the boundary), and to say that the question is probably not as simple as it seems. Suppose, in the definition of competitors (Definition 1.3), that we only included competitors for which $\varphi_1$ is smooth; would the resulting sets of quasiminimal (or almost minimal) sets be different? We discuss some partial positive results, and a possible strategy for further ones, in Section 27.

Part VII deals with the monotonicity, or near monotonicity, of the density

$$\theta(r) = r^{-d}\mathcal{H}^d(E \cap B(x,r)) \tag{1.19}$$

for sliding minimal or almost minimal sets, but only for balls $B(x,r)$ centered on the boundary pieces.

The simplest result is Theorem 28.4, which says that if $x = 0$, $E$ is coral and locally sliding minimal near 0, and the $L_j$ are cones, $\theta$ is nondecreasing near $r = 0$. When instead $E$ is only almost minimal with a gauge function $h$ that satisfies a Dini condition, and in addition $0 \in E$ (at least, if we deal with $A$-almost minimal sets), Theorem 28.7 says that $\theta$ is nearly monotone, i.e., that we can multiply it by a continuous function with a nonzero limit at the origin and get a nondecreasing function.

The case when the $L_j$ are not exactly cones centered at $x$ is discussed in Remark 28.11 and Theorem 28.15.

The case of equality in Theorem 28.4, i.e., when $E$ is minimal, the $L_j$ are cones, and $\theta$ is constant on some interval, is treated in Section 29. Theorem 29.1 says that in this case $E$ coincides, in the corresponding annulus, with a minimal cone with the same sliding boundary conditions. We use the proof of [D5], by lack of a better idea.

We apply this to blow-up limits of coral sliding almost minimal sets and show in Corollary 29.53 that, under reasonable assumptions, they are sliding minimal cones associated to the blow-up limits of the $L_j$.

We also use the case of equality above, and a compactness argument, to find situations where, if the function $\theta$ is nearly constant on an interval, then $E$ can be well approximated by a minimal cone, both in terms of Hausdorff distance and measure. See Proposition 30.3 for a general statement with annuli, and Proposition 30.19 for a simpler case in a ball.

In a last Section 31, we rapidly discuss a few directions in which this work could be continued or used.

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## 2. Generalized sliding quasiminimal sets

In this section we give the definition of our most general class of quasiminimizers (the sets for which we shall prove most of our regularity results), and also describe the two standard sets of assumptions on the boundary pieces $L_j$ that will be allowed.

The following notion comes from [D5], where it was introduced to generalize both the notion of Almgren quasiminimal set (or "restricted set", see [A2]) and some simpler notions of almost minimal sets.

For the next definition, we shall use a quasiminimality constant $M \geq 1$, a diameter $\delta \in (0, +\infty]$, and a small number $h \in [0, 1)$. We want to be able to localize our definitions, which forces us to work in an open set $U$; but of course we are free to take $U = \mathbb{R}^n$.

Given a closed set $E$, with $\mathcal{H}^d(E \cap H) < +\infty$ for every compact set $H \subset U$, and a one-parameter family $\{\varphi_t\}$, $0 \leq t \leq 1$, such that (1.4)-(1.8) hold, we set

$$W_t = \big\{ y \in E \cap B\, ; \varphi_t(y) \neq y \big\} \tag{2.1}$$

for $0 < t \leq 1$, and then

$$\widehat{W} = \bigcup_{0<t\leq 1} W_t \cup \varphi_t(W_t). \tag{2.2}$$

Note that $W_t \subset \widehat{W} \subset B$, where $B$ is as in (1.4)-(1.8), but they may be smaller and in particular we shall not force $B$ to be contained in $U$.

**Definition 2.3.** *Let $\Omega \subset \mathbb{R}^n$ and the $L_j$, $0 \leq j \leq j_{max}$, be as above. Let $M \geq 1$, $\delta \in (0, +\infty]$, $h \in [0, 1)$, and the open set $U$ be given. Let $E \subset \Omega$ be a closed set in $U$ such $\mathcal{H}^d(E \cap B) < +\infty$ for every closed ball $B \subset U$. We say that $E \in GSAQ(U, M, \delta, h)$ when, for every choice of closed ball $B = \overline{B}(x, r)$ such that $0 < r < \delta$, and every one-parameter family $\{\varphi_t\}$, $0 \leq t \leq 1$, such that (1.4)-(1.8) hold and*

$$\widehat{W} \subset\subset U \tag{2.4}$$

*(i.e., $\widehat{W}$ is contained in a compact subset of $U$), we have*

$$\mathcal{H}^d(W_1) \leq M\mathcal{H}^d(\varphi_1(W_1)) + hr^d, \tag{2.5}$$

*where as before $W_1 = \big\{y \in E \cap B\,; \varphi_1(y) \neq y\big\}$.*

Here $GSAQ$ stands for generalized sliding Almgren quasiminimal set; we should probably have mentioned $\Omega$ and the $L_j$ in the notation, but this could have been too heavy.

Definition 2.3 is the sliding analogue of Definition 2.10 in [D5]. The case when $h = 0$ corresponds to quasiminimal sets, as in [A2] and [DS4], except that here we insist that our final deformation $\varphi_1$ comes as the end of a one-parameter family of continuous maps that satisfy the constraints (1.7). Without these constraints and if $U$ were convex, it would not have been necessary to mention this (because we could take $\varphi_t(x) = (1-t)x + t\varphi_1(x)$), but here we need to be more careful.

Notice that we allow competitors of $E$ in balls $B$ that are not necessarily contained in $U$, but only require (2.4). It would have been essentially as reasonable to restrict to $B \subset U$; this would have given an apparently larger classes $GSAQ$, and probably our main results are still valid in that class. Here we opted for the definition which is closest to Almgren, also because the invariance under changes of variables is a little better (Proposition 2.8 below would not work as nicely). If the reader ever encounters a $GSAQ$ set for the weaker version, but not the one we give, she will probably get the desired results inside $U$ by noticing that it is also a $GSAQ$ set (official definition) in a slightly smaller open set.

Notice also that our Lipschitz mappings $\varphi_t$ are only defined on $E$. If they were allowed to take values in $\mathbb{R}^n$, this would not matter because we could extend them. Here we also require in (1.7) that our set $\varphi_1(E)$ is a deformation of $E$, with the constraints mentioned above, but we see no need to require that $\varphi_t$ extends to a mapping from $\Omega$ to $\Omega$, for instance, and requiring boundary constraints like (1.7) on the $L_j \setminus E$ seems really unnatural. We want to say that the soap is attached in some way to the boundaries, not that every deformation comes from some global deformation in space.

When we take $M = 1$ and $h$ small, we get a notion which is closer to the notions of almost minimality used in [D5]. We are allowed to take $\delta = +\infty$, but often taking $\delta < +\infty$ will help. For instance, sliding almost-minimal sets will be sets $E$ that lie in $GSAQ(1, \delta, h(\delta))$ for $\delta$ small, and with an $h(\delta)$ that tends to 0 with $\delta$.

The difference between (2.5) and its analogue for quasiminimal sets (i.e., when $h = 0$) is not enormous; the only situations where we expect (2.5) to be harder to use are when

$\mathcal{H}^d(W_1)$ and $\mathcal{H}^d(\varphi_1(W_1))$ are very small, i.e., when $\varphi_1$ only moves very few points of $E$. The point of a good part of Section 2 in [D5] was to show that these situations can be avoided when we prove regularity theorems. Here we shall also need to check that we can adapt the proofs to the case of sliding boundary conditions.

We shall work with reasonably strong assumptions on $\Omega$ and the $L_j$, and already this will give us some notational trouble. Let us distinguish between two sets of assumptions.

We introduce first a set of assumptions for $\Omega$ and the $L_j$, which we shall call the "rigid assumption". Its main advantage is its simplicity, and many results will be proved first under the rigid assumption, and generalized (some times painfully) to the Lipschitz assumption below. Again set $\Omega = L_0$, as in (1.1), to simplify the notation.

We shall say that the rigid assumption is satisfied when there is an integer $m \geq 0$ such that, for each $0 \leq j \leq j_{max}$,

$$\text{(2.6)} \qquad \begin{array}{l} L_j \text{ coincides in the unit ball with the union of a finite} \\ \text{number of faces } F_{j,l} \text{ of dyadic cubes of side length } 2^{-m}. \end{array}$$

We shall sometimes refer to the largest $2^{-m}$ such that (2.6) holds as the rigid scale of the $L_j$.

Our cubes and faces will always be closed, by convention. When we say dyadic cube of side length $2^{-m}$, we mean a set $[0, 2^{-m}]^n + 2^{-m}k$, with $k \in \mathbb{Z}^n$. The dimensions of the faces $F_{j,l}$ may be anything from 0 to $n$, and they may be different from each other, even for a fixed $j$. With this definition, it happens that the origin plays a special role (it lies on the boundary of all the faces of dimension $\geq 1$ that touch it), but we shall never need this coincidence (and it will disappear in the next definition).

In terms of combinatorics, this definition still allows a lot of different possibilities. We also authorize sets $L_j$ that are unions of faces of large dimensions, connected to each other by lower dimensional faces, for instance, or that just meet at one point.

For us the rigid assumption is a toy model for more general Plateau problems with boundary conditions of mixed dimensions. We decided to work with faces of dyadic cubes because this will make our life much easier in some case, at least in terms of notation but maybe not only. There are two main objections with this. The first one is the rigidity of the faces, and the next definition will take care of this. The second one is that the dyadic structure puts some constraints on the combinatorics of our boundary sets (for instance, it gives a small bound on the number of 2-dimensional faces that touch a given 1-dimensional face), and this will not be addressed. See Remarks 2.12 and 2.13.

So we want to be able to use less rigid faces, which are fairly smooth but not completely flat. Also, at least as far as quasiminimal sets are concerned, we expect some biLipschitz invariance of our results, so we introduce the following weaker "Lipschitz assumption", where we keep the same structure for the $L_j$, but allow Lipschitz faces.

**Definition 2.7.** *We say that the Lipschitz assumption is satisfied in the open set $U$ when there is a constant $\lambda > 0$ and a bilipschitz mapping $\psi : \lambda U \to B(0,1)$ such that the sets $\psi(\lambda(L_j \cap U))$, $0 \leq j \leq j_{max}$, satisfy the rigid assumption.*

Obviously, in this definition $\lambda U = \psi^{-1}(B(0,1))$ needs to be bilipschitz equivalent to $B(0,1)$, but this will not be a problem. In fact, all our conditions and results will be local, so even if our initial domain $U$ is not a nice open set, we can try to apply our results to a smaller domain $V \subset U$ (such that $\lambda V$ is bilipschitz equivalent to $B(0,1)$), using the fact that $E \in GSAQ(V, M, \delta, h)$ as soon as $E \in GSAQ(U, M, \delta, h)$.

Notice that the Lipschitz assumption comes with two important constants: the bilipschitz constant for $\psi$, and the rigid scale $2^{-m}$ above. The last constant $\lambda > 0$ is just a normalization, and should never play a serious role in the estimates. In fact, we could have decided to take $\lambda = 1$ in the definition, and this would only have forced us to apply our results to dilations of the considered sets and domains.

As far as quasiminimal sets are concerned, there will not be too much difference between our two assumptions; the following proposition will allow us to to go from the rigid assumption to the Lipschitz assumption, at the price of making some constants larger.

**Proposition 2.8.** *Suppose that the Lipschitz assumption is satisfied in the open set $U$, and let $\lambda$ and $\psi$ be as in Definition 2.7. Also denote by $\Lambda \geq 1$ the bilipschitz constant for $\psi$. Then, for each $E \in GSAQ(U, M, \delta, h)$, the set $\psi(\lambda E)$ lies in $GSAQ(B(0,1), \Lambda^{2d}M, \Lambda^{-1}\lambda\delta, \Lambda^{2d}h)$.*

Indeed, let $\{\varphi_t\}$, $0 \leq t \leq 1$ be as in Definition 1.3 (relative to the definition of a competitor for $\psi(\lambda E)$), and also assume that $\widehat{W} \subset\subset B(0,1)$. Set $\widetilde{\psi}(x) = \psi(\lambda x)$; thus $\widetilde{\psi}$ is the natural mapping from $U$ to $B(0,1)$. Then set $\widetilde{\varphi}_t = \widetilde{\psi}^{-1} \circ \varphi_t \circ \widetilde{\psi}$ for $0 \leq t \leq 1$. It is easy to see that the $\{\widetilde{\varphi}_t\}$, $0 \leq t \leq 1$, satisfy the conditions of Definition 1.3, except that $B$ should be replaced with $\widetilde{\psi}^{-1}(B)$, which itself is contained in a ball $B'$ of radius $\widetilde{r} \leq \Lambda\lambda^{-1}r$, where $r$ is the radius of $B$.

In addition, the analogue for the $\{\widetilde{\varphi}_t\}$ of $\widehat{W}$ is $\widetilde{W} = \widetilde{\psi}^{-1}(\widehat{W})$, which is compactly contained in $U$ because $\widehat{W} \subset\subset B(0,1)$.

If $r < \Lambda^{-1}\lambda\delta$, then $\widetilde{r} < \delta$, and the analogue of (2.5) yields

$$\mathcal{H}^d(\widetilde{W}_1) \leq M\mathcal{H}^d(\widetilde{\varphi}_1(\widetilde{W}_1)) + h\widetilde{r}^d, \tag{2.9}$$

with $\widetilde{W}_1 = \big\{y \in \widetilde{\psi}^{-1}(E)\,;\, \widetilde{\varphi}_1(y) \neq y\big\} = \widetilde{\psi}^{-1}(W_1)$. We apply $\widetilde{\psi}$ and get that

$$\begin{aligned}\mathcal{H}^d(W_1) &= \mathcal{H}^d(\widetilde{\psi}(\widetilde{W}_1)) \leq \lambda^d\Lambda^d\mathcal{H}^d(\widetilde{W}_1)\\ &\leq \lambda^d\Lambda^d M\mathcal{H}^d(\widetilde{\varphi}_1(\widetilde{W}_1)) + \lambda^d\Lambda^d h\widetilde{r}^d\\ &= \lambda^d\Lambda^d M\mathcal{H}^d(\widetilde{\varphi}_1(\widetilde{W}_1)) + \Lambda^{2d}hr^d\end{aligned} \tag{2.10}$$

by (2.9). In addition, $\widetilde{\varphi}_1(\widetilde{W}_1) = \widetilde{\psi}^{-1} \circ \varphi_1 \circ \widetilde{\psi}(\widetilde{W}_1) = \widetilde{\psi}^{-1} \circ \varphi_1(W_1)$, so (2.10) says that

$$\mathcal{H}^d(W_1) \leq \lambda^d\Lambda^d M\mathcal{H}^d(\widetilde{\psi}^{-1} \circ \varphi_1(W_1)) + \Lambda^{2d}hr^d \leq \Lambda^{2d}M\mathcal{H}^d(\varphi_1(W_1)) + \Lambda^{2d}hr^d, \tag{2.11}$$

as needed for Proposition 2.8. □

Because of Proposition 2.8, we shall sometimes be able to deduce local regularity properties for the quasiminimal sets under the Lipschitz assumption from their counterparts under the rigid assumption. This will work fine for regularity properties that are invariant under bilipschitz mappings (local Ahlfors regularity, rectifiability, or even uniform rectifiability), but for more sensitive properties, or when we want a precise dependence on the quasiminimality constants, we shall often need to conjugate our rigid proofs and check painfully that they extend to the Lipschitz assumption.

**Remark 2.12.** Our sets of boundaries are not nearly as general as they should be (for the weak regularity properties that we shall prove). There should not be anything so special about dyadic cubes, and we should probably have considered more general nets constructed with convex polyhedra, with a lower bound on the angles in the subfaces. But then the notations would have been somewhat worse, and the author was just afraid. Possibly the difficulty is only a matter of organization, but the reader should be warned that in a few places, we shall use the description of the $L_j$ with standard dyadic cubes to give short proofs, and the author did not even think about how these proofs could be adapted to more general nets. We explain about this a few times, but when other things become more complicated (for instance, in Part IV), we simply forget the issue.

Hopefully, the lack of generality of our rigid and Lipschitz assumptions will be slightly reduced by the fact that we allowed bilipschitz images. But on the other hand, we are missing many simple combinatorial cases. For instance, if we want to allow an $L_j$ where 20 faces of dimension 2 bound a single segment, we will have to adapt the definitions and proofs below, or play a dirty trick such as pretending we live in $\mathbb{R}^{10}$.

When we deal with more precise regularity properties that are not invariant under bilipschitz mappings, we may have to choose new sets of assumptions that are not as restrictive as the rigid assumption (which forces angles between faces to be multiples of $90°$, for instance), and not as lenient as the Lipschitz assumption (which allows ugly Lipschitz faces). Typically, this will happen in Section 24, when we study blow-up limits, and where we will allow $C^1$ faces that make different angles.

**Remark 2.13.** On the other hand, at first sight it looks like we are making our life more complicated than needed, by allowing large integers $m \geq 1$. Let us discuss this in the simple case of the rigid assumption. We are interested in local regularity properties of an almost- or quasi minimal set $E$ near a point $x_0$. If we concentrate on balls of size smaller than $2^{-m-2}$, we are reduced to the situation where each $L_j$ is a cone, centered at the origin (or at the point of the dyadic grid of size $2^{-m}$ that lies closest to $x_0$). This seems simpler than the situation we described, but in fact the difference is not enormous because the combinatorics of the intersections of our cones with a small sphere are not much simpler than the combinatorics of the intersections of small dyadic cubes in one less dimension. So we would essentially win a dimension, but we should not expect drastic simplifications in the combinatorics. Also, and this is the main reason, allowing $m$ to be large will not complicate our proofs.

**Remark 2.14.** The following convention may be useful. We shall say that out list of boundaries $\{L_j\}$ is complete when

(2.15) $\qquad$ for every choice of $0 \leq i, k \leq j_{max}$, $L_i \cap L_k$ is one of the $L_j$

and also

(2.16) $\qquad$ each $L_j$ is connected.

Replacing the initial list of $L_j$ with a complete one costs us nothing. Indeed, adding $L_i \cap L_j$ to our list does not upset (1.7), because (1.7) for $L_i \cap L_k$ is an immediate consequence of (1.7) for $L_i$ and (1.7) for $L_k$. And (1.7) for $L_j$ is equivalent to (1.7) for each of its component, because of (1.4); since we shall only consider sets $L_j$ which have a finite number of connected components, the new collection of sets $L_j$ stays finite. We may also assume that $\Omega$ is connected, because otherwise we could study minimal or almost minimal sets component by component.

## 3. Coral GSAQ and Lipschitz retractions on the $L_j$

In this section we deal with two technical problems. First, we shall later find it more reassuring to restrict our attention to "coral" quasiminimal sets, defined as follows.

**Definition 3.1.** *For $E \subset \mathbb{R}^n$ closed, with locally finite $\mathcal{H}^d$ measure, we denote by $E^*$ the closed support of the restriction of $\mathcal{H}^d$ to $E$; thus*

(3.2)
$$E^* = \big\{x \in E\,;\, \mathcal{H}^d(E \cap B(x,r)) > 0 \text{ for every } r > 0\big\}.$$

*We say that $E$ is coral when $E^* = E$.*

The definition comes from [D4], where $E^*$ was also called the core of $E$, and we wanted to distinguish coral from a slightly different notion of "reduced". The main goal of this section is to check that if $E \in GSAQ(M, \delta, h)$, then automatically $E^* \in GSAQ(M, \delta, h)$, but since this unexpectedly does not seem to follow too obviously from the definitions, we shall restrict to the Lipschitz setting for the sets $L_j$ that was described in Section 2.

**Proposition 3.3.** *Suppose that $E \in GSAQ(U, M, \delta, h)$ and the Lipschitz assumption is satisfied on the open set $U$. Then $E^* \in GSAQ(U, M, \delta, h)$.*

Observe that we do not say that $E^*$ is a competitor for $E$, and indeed it is not always true: it may happen that $E$ is a nice $d$-dimensional surface, plus a $(d-1)$-dimensional handle that cannot be deformed away (or to a subset of $E^*$) inside $\Omega$. The proof of Proposition 3.3 will be slightly complicated because when some part of $E \setminus E^*$ lies on the $L_j$, it adds some constraints on the competitors that we want to use. Put in another way, we have to show that if $E^*$ is not a GSAQ set because of some deformation $\{\varphi_t\}$, we cannot add a set of vanishing measure to $E^*$, in particular on the $L_j$, in such a clever way that we would not be able to extend $\varphi_t$ so that (1.7) holds also on $E \setminus E^*$.

Before we really start the proof, we want to construct Lipschitz retractions from a neighborhood of each $L_j$ onto $L_j$. In fact we shall do this for any finite union of faces of dyadic cubes of the same side length.

**Lemma 3.4.** *Let $L$ be a finite union of faces of dyadic cubes of side length 1, possibly of different dimensions, and set*

(3.5)
$$L^\eta = \big\{y \in \mathbb{R}^n\,;\, \mathrm{dist}(y, L) \le \eta\big\},$$

*where in fact we shall take $\eta = 1/3$. There is a Lipschitz mapping $\pi = \pi_L : L^\eta \to L$ such that*

$$\pi(x) = x \text{ for } x \in L \tag{3.6}$$

*and $\pi(F) \subset F$ for each face $F$ (of any dimension) of each dyadic cube of side length 1. The Lipschitz constant for $\pi$ is less than $C$, where $C$ depends only on $n$.*

We shall construct $\pi$ as a composition of mappings $\rho_m$.

For $m \geq 0$, denote by $A_m$ the set of faces of dimension $n - m$ (of dyadic cubes of side length 1) which touch $L$ but are not contained in $L$.

We may stop at $m = n - 1$, because $A_n = \emptyset$ by definition. Then set

$$T_m = L \cup \Big[ \bigcup_{F \in A_m} (F \cap L^\eta) \Big]. \tag{3.7}$$

We shall define $\rho_m$ on $T_m$, also as a composition of simpler mappings. But let us first check a few facts about distances. We shall often use the fact that

$$\begin{gathered}\text{if } F,\ F' \text{ are faces of unit dyadic cubes and } F \text{ is neither a point}\\ \text{nor contained in } F', \text{ then } \operatorname{dist}(y, F') \geq \operatorname{dist}(y, \partial F) \text{ for } y \in F.\end{gathered} \tag{3.8}$$

Here and below, $\partial F$ is the boundary of the face $F$; it is thus the union of some sub faces of dimension one less (except if $F$ is a point and $\partial F = \emptyset$). Now (3.8) can be deduced from simple considerations of Euclidean geometry; if we were dealing with faces of polyhedra, we would merely get that $\operatorname{dist}(y, F') \geq \eta_0 \operatorname{dist}(y, \partial F)$, where $\eta_0$ depends on the smallest angles that adjacent faces of polyhedra can make, and on the smallest distance between non adjacent faces, and this would only force us to take $\eta$ smaller in Lemma 3.4. But let us just check (3.8) for faces of dyadic cubes.

Let $l$ be the dimension of $F$; thus $l \geq 1$. Without loss of generality, we may assume that $F$ is given by the equations $0 \leq y_j \leq 1$ for $1 \leq j \leq l$, and $y_j = 0$ for $j > l$. Since (3.8) is trivial for points of $\partial F$, we just consider points $y \in F$ such that $0 < y_j < 1$ for $j \leq l$. Notice that $\operatorname{dist}(y, \partial F)$ is the smallest of the $\operatorname{Min}(y_j, 1 - y_j)$, $1 \leq j \leq l$.

Let $z \in F'$ minimize $|z - y|$. If $z_j \neq y_j$ for some $j \leq l$, then $z_j \notin (0, 1)$, because otherwise we could replace $z_j$ with $y_j$, and get a new point $z'$ that still lies in $F'$, but is strictly closer to $y$. In this case, $|z - y| \geq |z_j - y_j| \geq \operatorname{Min}(y_j, 1 - y_j) \geq \operatorname{dist}(y, \partial F)$, as needed. So we may assume that $z_j = y_j$ for $1 \leq j \leq l$. If $|z_j| \geq 1$ for some $j > l$, then $|z - y| \geq |z_j - y_j| = |z_j| \geq 1 \geq \operatorname{dist}(y, \partial F)$, which is fine. Otherwise, we can replace all $z_j$, $j > l$, with 0, and get a new point $z' \in F'$. But $z' = y$, hence $y \in F'$. This is impossible, because $0 < y_j < 1$ for $1 \leq i \leq l$ (recall that $y \in F \setminus \partial F$), and this would force $F \subset F'$ (because $F$ is the smallest face that contains $y$). This proves (3.8).

Let us deduce from (3.8) that when $F \in A_m$,

$$\operatorname{dist}(y, L) \geq \operatorname{dist}(y, \partial F) \quad \text{for } y \in F. \tag{3.9}$$

Indeed, if $z \in L$ and $F'$ is a face of $L$ that contains $z$, we know that $F'$ does not contain $F$ by definition of $A_m$, and also that $F$ is not reduced to one point because $m < n$ (recall that $A_n = \emptyset$), so (3.8) says that $|y - z| \geq \mathrm{dist}(y, F') \geq \mathrm{dist}(y, \partial F)$. Similarly,

$$\mathrm{dist}(y, T_m \setminus F) \geq \mathrm{dist}(y, \partial F) \quad \text{for } F \in A_m \text{ and } y \in F, \tag{3.10}$$

because if $z \in T_m \setminus F$, then either $z \in L$ and we can apply (3.9), or else $z$ lies in some other face $F' \in A_m$, and we can apply (3.8) because $F \neq F'$ and they have the same dimension.

For each face $F \in A_m$, denote by $x_F$ the center of $F$ and by $p_F$ the radial projection from $F \setminus \{x_F\}$ to $\partial F$. That is, $p_F(y)$ is the point $z \in \partial F$ such that $y \in [x_F, z]$. By (3.9), $\mathrm{dist}(x_F, L) \geq 1/2$, hence $p_F$ is defined and Lipschitz on $F \cap L^\eta$.

Extend $p_F$ to $T_m$ by setting $p_F(y) = y$ for $y \notin F$. This is coherent, because if $F'$ is a different face of $A_m$, then $F \cap F' \subset \partial F$ (recall that $F$ and $F'$ have the same dimension), and similarly $L \cap F \subset \partial F$ by (3.9); hence both definitions yield $p_F(y) = y$ on these sets.

Observe that $p_F$ respects the faces, i.e., $p_F(G \cap T_m) \subset G$ for every face $G$ of any dimension of a dyadic cube of side length 1. This is clear when $G$ does not meet the interior of $F$, because then $p_F(y) = y$ on $G$; otherwise, when $G$ meets the interior of $F$, $G$ contains $F$ and we just need to know that $p_F(F) \subset F$. Next let us check that

$$p_F \text{ is } 6\sqrt{n}\text{-Lipschitz on } T_m. \tag{3.11}$$

Recall that $\mathrm{dist}(x_F, L) \geq 1/2$ by (3.9), so $\mathrm{dist}(x_F, L^\eta) \geq 1/2 - \eta = 1/6$, and hence $p_F$ is $6\sqrt{n}$-Lipschitz on $T_m \cap F$. It is trivially 1-Lipschitz on $T_m \setminus F$, and for $y \in T_m \cap F$ and $z \in T_m \setminus F$,

$$\begin{aligned} |p_F(y) - p_F(z)| &= |p_F(y) - z| \leq |p_F(y) - y| + |y - z| \\ &\leq \sqrt{n}\,\mathrm{dist}(y, \partial F) + |y - z| \\ &\leq \sqrt{n}\,\mathrm{dist}(y, T_m \setminus F) + |y - z| \leq (1 + \sqrt{n})|y - z| \end{aligned} \tag{3.12}$$

by (3.10). Thus (3.11) holds.

Now define $\rho_m$ on $T_m$ to be the composition of all the $p_F$, $F \in A_m$. Notice that since $p_F$ only moves the interior points of $F$, which lie out of $L$ by (3.9) and out of the other $F' \in A_m$ because distinct faces of the same dimension have disjoint interiors, we see that the order of composition does not matter (each point is moved at most once), and in fact $\rho_m(y) = p_F(y)$ on $F$ for each $F \in A_m$, and $\rho_m(y) = y$ on $L$. Also, $\rho_m$ is $C$-Lipschitz, with $C \leq 36n$ (refine the proof of (3.11), or brutally observe that on $\{x, y\}$, $\rho_m$ is the composition of two $6\sqrt{n}$-Lipschitz mappings).

Next we want to compose the $\rho_m$. Let us first check that

$$\mathrm{dist}(p_F(y), L) \leq \mathrm{dist}(y, L) \leq \eta \tag{3.13}$$

for $F \in A_m$ and $y \in F \cap L^\eta$. If we were working with polyhedra, we would use (3.9) to show that $\mathrm{dist}(p_F(y), L) \leq C\,\mathrm{dist}(y, L)$, and this would be fine too, except that we would need to choose a smaller $\eta$ at the end.

In order to prove (3.13), we may assume that

$$F = \{y \in \mathbb{R}^n\,;\, 0 \le y_i \le 1 \text{ for } 1 \le i \le m-n \text{ and } y_i = 0 \text{ for } i > m-n\} \tag{3.14}$$

and, by symmetry, that all the coordinates of $y$ lie in $[0, 1/2]$. Set $\widetilde{y} = p_F(y)$; then $0 \le \widetilde{y}_i \le y_i$ for $1 \le i \le m-n$, because $0 \le y_i \le 1/2$, the coordinate of $x_F$.

Let $z \in L$ lie closest to $y$; we just want to find $\widetilde{z} \in L$ such that $|\widetilde{z} - \widetilde{y}| \le |z - y|$. Since $|z - y| \le 1/3$, all the coordinates $z_i$ lie in $[-1/3, 5/6]$. Let $i$ be such that $z_i \le 0$; we keep $\widetilde{z}_i = z_i$, and obviously $|\widetilde{z}_i - \widetilde{y}_i| = |z_i - \widetilde{y}_i| \le |z_i - y_i|$. For the other $i$, we know that $0 < z_i < 5/6$, and we just set $\widetilde{z}_i = \widetilde{y}_i$; notice that the point $\widetilde{z}$ that we get this way lies in the same faces as $z$, because we only replaced some coordinates that lie in $(0, 1)$ with other ones in $[0, 1]$, and this operation preserves any face. Thus $\widetilde{z} \in L$, just like $z$, and since by construction $|\widetilde{z}_i - \widetilde{y}_i| \le |z_i - y_i|$ for all $i$, we completed the proof of (3.13).

Since $p_F$ is the identity out of $F$, (3.13) is also valid for $y \in T_m \subset L^\eta$. We claim that

$$\rho_m(T_m) \subset L \cup \Big[ \bigcup_{F \in A_m} (\partial F \cap L^\eta) \Big] \subset T_{m+1}. \tag{3.15}$$

Let $w \in \rho_m(T_m)$ be given, and let $y \in T_m$ be such that $w = \rho_m(y)$. If $y \in L$, then $p_F(y) = y$ for all $F$ (because $F \cap L \subset \partial F$, by (3.9)), hence $w = y \in L$. If $y \in F$ for some $F \in A_m$, then $p_F(y) \in \partial F$ by construction, and then all the other $p_{F'}$ preserve $\partial F$, because they preserve every face of every cube; thus $w = \rho_m(y)$ lies in $\partial F$ too (recall that we can compose the $p_F$ in any order that we like). Also, $w \in L^\eta$ by successive applications of (3.13).

For the second inclusion, let $F \in A_m$ and $w \in \partial F \cap L^\eta$ be given. Let $H$ be a $(m-n-1)$-dimensional face of $\partial F$ that contains $w$. If $H \subset L$, we are happy because $L \subset T_{m+1}$. Otherwise, as soon as we prove that $H$ meets $L$, we will know that $H \in A_{m+1}$ (by definition of $A_{m+1}$), hence $w \in H \cap L^\eta \subset T_{m+1}$, as needed. Now $\mathrm{dist}(w, L) \le \eta$ because $w \in L^\eta$, so we can find $z \in L$ such that $|z_i - w_i| \le 1/3$ for all $i$. When $i$ is such that $z_i \ne w_i$, we can replace both $z_i$ and $w_i$ with some integer $n_i$ which is close to both of them, without changing the fact that $w \in H$ and $z \in L$; this way we get a point of $H \cap L$. This completes the proof of our claim (3.15).

Now we set $\pi = \rho_{n-1} \circ \ldots \circ \rho_0$. This is a Lipschitz mapping which is defined on $T_0 = L^\eta$ and takes values in $T_n = L$. Thus $\pi(L^\eta) \subset L$. Next, (3.6) holds because $p_F(y) = y$ on $L$ for al $F$; finally, $\pi$ preserves the faces because it is a composition of mappings that preserve the faces of all dimensions. Thus $\pi$ is the desired mapping, and Lemma 3.4 follows. □

**Remark 3.16.** For $0 \le \eta \le 1/3$, we also get a mapping $\pi_L$ as in Lemma 3.4, which is just the restriction to $L^\eta$ of the mapping that we construct with $\eta = 1/3$. That is, we always use the same formulas, only the domains of definition differ.

The retraction from Lemma 3.4 is the endpoint of a deformation; we shall not need this fact before Lemma 8.8, but let us check it now before we forget the notation.

**Lemma 3.17.** *Let $L$, $0 < \eta \le 1/3$, and $L^\eta$ be as in Lemma 3.4. Then there is a Lipschitz mapping $\Pi_L : L^\eta \times [0, 1] \to \mathbb{R}^n$ such that*

$$\Pi_L(x, t) = x \quad \textit{for } x \in L \textit{ and for } t = 0, \tag{3.18}$$

$$\Pi_L(x,1) = \pi_L(x) \quad \textit{for } x \in L^\eta, \tag{3.19}$$

$$|\Pi_L(x,t) - \Pi_L(x,s)| \le C \operatorname{dist}(x,L)|t-s| \quad \textit{for } x \in L^\eta \textit{ and } 0 \le s,t \le 1, \tag{3.20}$$

$$|\Pi_L(x,t) - \Pi_L(y,t)| \le C|x-y| \quad \textit{for } x,y \in L^\eta \textit{ and } 0 \le t \le 1, \tag{3.21}$$

*and $\Pi_L$ also preserves the faces of all dimensions, i.e.,*

$$\begin{aligned} &\Pi_L(x,t) \in F \textit{ whenever } F \textit{ is any face (of any dimension)} \\ &\qquad\qquad \textit{of a dyadic cube of side length } 1,\ x \in F, \textit{ and } 0 \le t \le 1. \end{aligned} \tag{3.22}$$

*The constant $C$ in (3.20) and (3.21) depends only on $n$.*

To see this, observe that $\pi_L$ is obtained by composing a bounded number of Lipschitz mappings $p_F$, where $F \in \cup_m A_m$ is some face of dyadic cube. Recall from the definition below (3.10) that when $F \in A_m$, $F$ is of dimension $n-m$, $p_F$ is defined on the set $T_m$ of (3.7), and is equal to the identity everywhere, except on $F$ itself, where it is a radial projection on $\partial F$. We easily go from the identity to $p_F$ by setting

$$p_F(x,t) = tp_F(x) + (1-t)x \quad \text{for } x \in T_m \text{ and } 0 \le t \le 1; \tag{3.23}$$

then the $p_F(\cdot,t)$ also preserve the faces of all dimensions, are $6\sqrt{n}$-Lipschitz like $p_F$, and

$$|p_F(x,t) - p_F(x,s)| \le C\eta|t-s| \quad \text{for } x \in T_m \text{ and } 0 \le s,t \le 1, \tag{3.24}$$

because $|p_F(x,t) - x| \le C\eta$.

When we used $p_F$, we composed it with a previous mapping $h$, which maps $L^\eta$ to $T_m$ by (3.15) and because the other $p_{F'}$, $F' \subset A_m$, map $T_m$ to $T_m$. Then we can go from $h$ to $p_F \circ h$ by setting $h_F(x,t) = p_F(h(x),t)$ for $x \in L^\eta$ and $0 \le t \le 1$. The mapping $h_F$ is $C$-Lipschitz in $x$ and $C\eta$-Lipschitz in $t$, because $h$ is $C$-Lipschitz.

We now concatenate all the deformations $h_F$, reparameterize by the unit interval, and get a mapping $\Pi_L$ that satisfies (3.18)-(3.22), except that in (3.20) we only get $\eta$ instead of $\operatorname{dist}(x,L)$. But Remark 3.16 extends to our mapping $p_F(x,t)$ and $\Pi_l :$ the mapping that we would construct on $L^{\eta'}$, with $\eta' = \operatorname{dist}(x,L)$, is just the restriction to $L^{\eta'}$ of the mapping that we constructed here on $L^\eta$. Therefore, (3.20) is just the same thing as (3.24) for $x$, but applied to the mapping $\Pi_L$ associated to $\eta'$. □

**Remark 3.25.** Of course we can also define $\pi_L$ and $\Pi_L$ when $L$ is a finite union of faces of dyadic cubes, not necessarily of size one. That is, if $L$ is a finite union of faces dyadic cubes of size $2^{-m}$ (as in the definitions of our $L_j$), we define $\pi_L$ by

$$\pi_L(z) = 2^{-m}\pi_{2^m L}(2^m z), \tag{3.26}$$

and use a similar definition for $\Pi_L$. When we define the $\pi_{L_j}$ associated to our boundary pieces $L_j$, we shall use this convention; if by luck some $L_j$ are also unions of dyadic faces of larger diameters, we shall ignore that fact and stay with the same $m$.

Next we want to prove Proposition 3.3 in a simpler setting. We shall later see how the proof of Proposition 2.8 allows us to reduce to this case.

**Proposition 3.27.** *Suppose $E \in GSAQ(B_0, M, \delta, h)$, where we set $B_0 = B(0,1)$, and that the rigid assumption is satisfied. Then $E^* \in GSAQ(B_0, M, \delta, h)$.*

So let $E \in GSAQ(B_0, M, \delta, h)$ be given. We shall go from $E$ to $E^*$ in a finite number of steps, where each time we remove a set in some $L_j$. We may assume that the set of $L_j$ is complete, as in Remark 2.14, and that when we enumerate the various $L_j$, we start with the largest ones for the inclusion relation.

We shall first define some intermediate sets $E_j$. Set

$$Z' = E \setminus E^* \text{ and } Z_j = Z' \setminus [\cup_{i \geq j} L_i] \text{ for } 0 \leq j \leq j_{max} + 1. \tag{3.28}$$

This is a nondecreasing sequence of open subsets of $E$. The first one is $Z_0 = \emptyset$, because $L_0 = \Omega$ contains $E$, and the last one is $Z_{j_{max}+1} = Z'$. Note that

$$\mathcal{H}^d(Z') = \mathcal{H}^d(E \setminus E^*) = 0, \tag{3.29}$$

by definition of $E^*$ (see (8.26) on page 58 of [D4], for instance, for the elementary proof). Then set

$$E_j = E \setminus Z_j \text{ for } 0 \leq j \leq j_{max} + 1. \tag{3.30}$$

This is a nonincreasing sequence of closed sets, with $E_0 = E$ and $E_{j_{max}+1} = E^*$. We want to prove by induction that $E_j$ lies in $GSAQ(B_0, M, \delta, h)$, just like $E$. The induction assumption holds for $j = 0$, so let us assume that $0 \leq j \leq j_{max}$ and that

$$E_j \in GSAQ(B_0, M, \delta, h), \tag{3.31}$$

and prove that

$$E_{j+1} \in GSAQ(B_0, M, \delta, h). \tag{3.32}$$

Set

$$\begin{aligned} Z = E_j \setminus E_{j+1} = Z_{j+1} \setminus Z_j &= \{Z' \setminus [\cup_{i \geq j+1} L_i]\} \setminus \{Z' \setminus [\cup_{i \geq j} L_i]\} \\ &= [Z' \cap L_j] \setminus [\cup_{i \geq j+1} L_i] = [E \cap L_j \setminus E^*] \setminus [\cup_{i \geq j+1} L_i]. \end{aligned} \tag{3.33}$$

So we want to prove (3.32). We take a sliding competitor $F = \varphi_1(E_{j+1})$ for $E_{j+1}$ in some closed ball $B$, and we want to prove the analogue of (2.5) for $E_{j+1}$. It is tempting to use the same one-parameter family $\{\varphi_t\}$ and apply (2.5) to it, but since it is only defined for $x \in E_{j+1}$, we have to extend it to $x \in E_j$. The difficult part will be to make sure that we still have (1.7), in particular at points of $Z$. At any rate, we want to define $\widetilde{\varphi}_t(x)$ for $x \in E_j$ and $0 \leq t \leq 1$, and logically we would like to keep

$$\widetilde{\varphi}_t(x) = \varphi_t(x) \text{ for } x \in E_{j+1} \tag{3.34}$$

(so that we would only need to define $\widetilde{\varphi}_t(x)$ when $x \in Z$); we also would like to keep $\widetilde{\varphi}_t(x) = x$ when $t = 0$ and when $x \in E_j \setminus B'$, where $B'$ is a closed ball with the same center as the ball $B$ of Definition 1.3, but just a tiny bit larger, so we should mostly worry about $Z \cap B'$.

We extend $\varphi_1$ first, in a Lipschitz way. That is, we have a Lipschitz mapping $\varphi_1$, defined on $E_{j+1}$, and we first extend it to $E_{j+1} \cup [E_j \setminus B']$ by setting

$$\varphi_1(x) = x \quad \text{for } x \in E_j \setminus B'. \tag{3.35}$$

This map is still Lipschitz, although with a possibly very large constant (but we don't care). Indeed, since $\varphi_1$ is Lipschitz on $E_{j+1}$ and on $E_j \setminus B'$, we only need to estimate $|\varphi_1(x) - \varphi_1(y)|$ when $x \in E_{j+1}$ and $y \in E_j \setminus B'$ say; if $x \in B$ and $y \in 2B'$, we say that $|\varphi_1(x) - \varphi_1(y)| \leq |\varphi_1(x)| + |\varphi_1(y)| \leq 2C \leq 2C(r' - r)^{-1}|x - y|$, where $C$ is a bound for $\varphi$ on $2B'$, and $r, r'$ denote the radii of $B$ and $B'$. If $x \notin B$, $\varphi_1(x) - \varphi_1(y) = x - y$ by definition, and the last case when $x \in B$ and $y \notin 2B'$ is even easier. So $\varphi_1$ is Lipschitz on $E_{j+1} \cup [E_j \setminus B']$. Now we use the Whitney extension theorem to get a Lipschitz extension $\varphi_1 : E_j \to \mathbb{R}^n$.

Next we set $\varphi_0(x) = x$, as we should do, and get a mapping $(x, t) \to \varphi_t(x)$ defined on $[E_j \times \{0, 1\}] \cup [E_{j+1} \times [0, 1]]$. This mapping is continuous; this is clear at points $(x, t)$ where $0 < t < 1$; when $x \in E_{j+1}$ and $t \in \{0, 1\}$, we use the continuity of the previous $\varphi_t$ and the fact that $\varphi_0$ and $\varphi_1$ are Lipschitz; finally when $x \in E_j \setminus E_{j+1}$, we use the fact that $E_{j+1}$ is closed, hence far from $x$.

We can also set $\varphi_t(x) = x$ for $0 \leq t \leq 1$ and $x \in E_j \setminus B'$. The two definitions coincide when $t = 1$, by (3.35), when $t = 0$ by definition of our first extension, and when $x \in E_{j+1}$ by (1.5); we now get a mapping $(x, t) \to \varphi_t(x)$ which is defined on $[E_j \times \{0, 1\}] \cup [E_{j+1} \times [0, 1]] \cup [(E_j \setminus B') \times [0, 1]]$. Let us check that this mapping is continuous. We just need to check this at points $(x, t)$ that lie in the intersection of the closures of our two sets (where we already know that the function is continuous). When $x \in E_{j+1}$, our first extension was already defined at $x$, with $\varphi_t(x) = x$ because $x \in E_{j+1} \cap (E_j \setminus B')^- \subset E_{j+1} \setminus B$, and by (1.5). Since the second definition also yields $\varphi_t(x) = x$, we get the continuity at $(x, t)$. Now suppose $x \in E_j \setminus E_{j+1}$. As before, since $E_{j+1}$ is closed, $x$ is far from $E_{j+1}$ and this forces $t \in \{0, 1\}$. In this case too, $\varphi_t(x)$ was already defined for the first extension, with $\varphi_t(x) = x$, so $\varphi_t$ is continuous at $(x, t)$.

We now use the Titze extension theorem. This gives a continuous mapping $(x, t) \to \varphi_t(x)$, from $E_j \times [0, 1]$ to $\mathbb{R}^n$. This mapping satisfies the continuity condition in (1.4), (1.5) (with $B$ replaced with $B'$), and (1.8). Since we also want (1.6), we compose its values on $B'$ with the 1-Lipschitz radial projection from $\mathbb{R}^n$ to $B'$; this does not change the values on $E_{j+1}$, by (1.6), and does not destroy (1.4), (1.5), or (1.8). Of course our last property (1.7) is not automatically satisfied, so we'll need to modify the $\varphi_t$ again.

Define a slightly better $f_t$ for $0 \leq t \leq 1$ by

$$f_t(x) = \varphi_{t\psi(x)}(x), \tag{3.36}$$

where $\psi(x)$ is a Lipschitz function of $d(x) = \mathrm{dist}(x, E_{j+1})$ such that

$$\begin{gathered} \psi(x) = 1 \text{ when } d(x) \leq \varepsilon, \\ 0 \leq \psi(x) \leq 1 \text{ when } \varepsilon \leq d(x) \leq 2\varepsilon, \\ \psi(x) = 0 \text{ when } d(x) \geq 2\varepsilon, \end{gathered} \tag{3.37}$$

and where $\varepsilon > 0$ is a very small number that will be chosen soon. Near $E_{j+1}$, we do not change anything (because $\psi(x) = 1$). Let $\varepsilon_0 > 0$ be given. Let us check that if $\varepsilon > 0$ is small enough, then

$$\mathrm{dist}(x, E_{j+1} \cap L_j) \leq \varepsilon_0 \text{ for } x \in Z \text{ such that } d(x) \leq 2\varepsilon. \tag{3.38}$$

Otherwise, we could find a sequence $\{x_k\}$ in $Z$, with $d(x_k)$ tending to 0, and that stays $\varepsilon_0$-far from $E_{j+1} \cap L_j$. Replace $\{x_k\}$ with a subsequence with some limit $x$; then $x \in L_j$ because $Z \subset L_j$ by (3.33), and $L_j$ is closed. In addition, $x \in E_{j+1}$ because $d(x) = 0$ and $E_{j+1}$ is closed, a contradiction with the fact that $\mathrm{dist}(x_k, E_{j+1} \cap L_j) \geq \varepsilon_0$; (3.38) follows.

Next observe that

$$\varphi_t(x) \in L_j \quad \text{for } x \in E_{j+1} \cap L_j \text{ and } t \in [0,1], \tag{3.39}$$

by (1.7). Hence, by (3.38) and the uniform continuity of $\varphi_t$ on $[0,1] \times E_j$, we get that for each $\varepsilon_1 > 0$, we can find $\varepsilon > 0$ such that

$$\mathrm{dist}(\varphi_t(x), L_j) \leq \varepsilon_1 \text{ for } t \in [0,1] \text{ and } x \in Z \text{ such that } d(x) \leq 2\varepsilon. \tag{3.40}$$

Then we also get that

$$\mathrm{dist}(f_t(x), L_j) \leq \varepsilon_1 \text{ for } t \in [0,1] \text{ and } x \in Z \text{ such that } d(x) \leq 2\varepsilon \tag{3.41}$$

by (3.36) and because $t\psi(x) \in [0,1]$. But then

$$\mathrm{dist}(f_t(x), L_j) \leq \varepsilon_1 \text{ for } t \in [0,1] \text{ and } x \in Z, \tag{3.42}$$

because when $x \in Z$ and $d(x) > 2\varepsilon$, $\psi(t) = 0$, hence $f_t(x) = \varphi_0(x) = x$, so $f_t(x) \in L_j$ too.

This proximity to $L_j$ is the reason why $f_t$ is better than $\varphi_t$. On the other hand, observe that $f_1(x) = \varphi_{\psi(x)}(x)$ is not necessarily Lipschitz, because we did not require $\varphi_t$ to be Lipschitz in $t$, or even in $x$ when $t < 1$, so we shall fix this now and construct new functions $g_t$. The first step is, given a small $\varepsilon_1 > 0$, to choose a Lipschitz function $\varphi : E_j \times [0,1] \to \mathbb{R}^n$, such that

$$\varphi(x,t) = x \quad \text{for } x \in E_j \setminus B' \text{ and for } t = 0 \tag{3.43}$$

and

$$|\varphi(x,t) - \varphi_t(x)| \leq \varepsilon_1 \quad \text{for } x \in E_j \cap B' \text{ and } 0 \leq t \leq 1. \tag{3.44}$$

This is easy to do, and we just sketch the proof. On the set $A_1 = [E_j \setminus B'] \times [0,1] \bigcup E_j \times \{0\}$, we simply take $\varphi(x,t) = \varphi_t(x) = x$. Then we use the uniform continuity of $(x,t) \to \varphi_t(x)$ on $E_j \times [0,1]$ (which is easy because we just need to consider the compact set $E_j \cap 2B' \times [0,1]$) to get $\tau > 0$ such that

$$|\varphi_t(x) - \varphi_s(y)| < \varepsilon_1 \tag{3.45}$$

for $x, y \in E_j$ and $t, s \in [0,1]$ such that $|x-y| + |t-s| \leq 100\tau$. We select a maximal subset $A_2$ of $\big\{(x,t) \in E_j \times [0,1]\,;\, \mathrm{dist}((x,t), A_1) \geq \tau\big\}$, whose points lie at mutual distances at least $\tau$, and decide that $\varphi(x,t) = \varphi_t(x)$ for $(x,t) \in A_2$. Finally we use a partition of unity to complete the definition of $\varphi$. We get that $\varphi$ is Lipschitz on $E_j \times [0,1]$ by construction, with a very large constant that depends on $\tau$ (but this is all right), and (3.44) holds because for each $(x,t) \in E_j \times [0,1]$, $\varphi(x,t)$ is an average of values of $\varphi(y,s) = \varphi_s(y)$ on $A_1 \cup A_2$ at nearby points $(y,s)$, and by (3.45). If by bad luck $\varphi(x,t)$ falls out of $B'$ for some pairs $(x,t) \in B' \times [0,1]$, compose again with the radial Lipschitz projection onto $B'$, without altering (3.44) or the Lipschitz constants.

Next we choose a Lipschitz function $h(x)$ of $d(x) = \mathrm{dist}(x, E_{j+1})$, such that $0 \leq h \leq 1$ everywhere, $h(x) = 1$ when $d(x) = 0$ (i.e., when $x \in E_{j+1}$), and $h(x) = 0$ when $d(x) \geq \varepsilon/2$. And we set

$$\varphi'(x,t) = h(x)\varphi_t(x) + (1 - h(x))\varphi(x,t) \tag{3.46}$$

on $E_j \times [0,1]$. Notice that $\varphi'(x,t) = \varphi_t(x)$ on $E_{j+1}$ (because $h(x) = 1$ there), and that

$$|\varphi'(x,t) - \varphi_t(x)| \leq \varepsilon_1 \tag{3.47}$$

by (3.44). Set

$$g_t(x) = \varphi'(x, t\psi(x)) \tag{3.48}$$

(compare with (3.36)). Then

$$|g_t(x) - f_t(x)| = |\varphi'(x, t\psi(x)) - \varphi_{t\psi(x)}(x)| \leq \varepsilon_1 \ \text{ for } (x,t) \in E_j \times [0,1], \tag{3.49}$$

by (3.47). We still have that $g_t(x) \in B'$ when $x \in B'$ (because of similar properties for $\varphi_t$ and $\varphi(x,\cdot)$, and since $B'$ is convex), and

$$g_t(x) = x \ \text{ for } x \in E_j \setminus B' \text{ and } t = 0, \tag{3.50}$$

because $\varphi_t(x) = x$ by construction (see above (3.36)), $\varphi(x,t) = x$ by (3.43), $\varphi'(x,t) = x$ by (3.46), and finally $g_t(x) = x$ by (3.48). Next,

$$g_t(x) = \varphi_t(x) \ \text{ on } E_{j+1} \times [0,1], \tag{3.51}$$

because $d(x) = 0$, hence $h(x) = \psi(x) = 1$ and $\varphi'(x,t) = \varphi_t(x)$ by (3.46). Note that $(t,x) \to g_t(x)$ is continuous because all the ingredients in (3.48) and (3.46) are continuous. Let us also check that

$$g_1 \text{ is Lipschitz on } E_j. \tag{3.52}$$

In the region where $d(x) \leq \varepsilon$, (3.37) says that $\psi(x) = 1$, so $g_1(x) = \varphi'(x,1) = h(x)\varphi_1(x) + (1-h(x))\varphi(x,1)$ by (3.48) and (3.46), which is Lipschitz in particular because $\varphi_1$ is Lipschitz. In the region where $d(x) \geq \varepsilon/2$, $h(x) = 0$ so $\varphi'(x,t) = \varphi(x,t)$ and $g_1(x) = \varphi(x, \psi(x))$, which is Lipschitz by definition of $\varphi$. This proves (3.52) because we have more than enough room for the gluing.

We still have

$$g_0(x) = \varphi'(x,0) = h(x)\varphi_0(x) + (1-h(x))\varphi(x,0) = x \quad \text{for } x \in E_j, \tag{3.53}$$

by (3.48), (3.46), (3.43), and the definition of the extension of $\varphi_t$, but we are still missing (1.7). We shall now set

$$\widetilde{\varphi}_t(x) = \pi_j(g_t(x)) \quad \text{for } x \in Z \text{ and } t \in [0,1], \tag{3.54}$$

where we denote by $\pi_j$ the Lipschitz retraction $\pi_{L_j}$ onto $L_j$ that we constructed with the help of Lemma 3.4, after scaling as in (3.26). Recall that $\pi_j$ is defined on a $\eta$-neighborhood of $L_j$, where $\eta = 2^{-m}/3$ is now the third of the side length of the dyadic cubes that compose $L_j$. Note that the definition in (3.54) makes sense because for $x \in Z$ and $0 \leq t \leq 1$,

$$\mathrm{dist}(g_t(x), L_j) \leq \mathrm{dist}(f_t(x), L_j) + \varepsilon_1 \leq 2\varepsilon_1 < \eta \tag{3.55}$$

by (3.49), (3.42), and if we choose $\varepsilon_1 < \eta/2$. Recall from (3.34) that we would like to set $\widetilde{\varphi}_t(x) = \varphi_t(x)$ on $E_j \setminus Z = E_{j+1}$, but the desired Lipschitzness of $\widetilde{\varphi}_1$ at the interface between $Z$ and $E_{j+1}$ will force us to modify this slightly on a small region near $L_j$. Let $\varepsilon_2 > 0$ be very small, to be chosen later, and let $\xi$ be defined by

$$\begin{cases} \xi(y) = 1 \quad \text{for } 0 \leq y \leq \varepsilon_2/2, \\ \xi(y) = 0 \quad \text{for } y \geq \varepsilon_2, \\ \xi \ \text{ is affine on } [\varepsilon_2/2, \varepsilon_2]. \end{cases} \tag{3.56}$$

Also set

$$d_t(x) = \mathrm{dist}(\varphi_t(x), L_j \cap B') \quad \text{for } x \in E_{j+1} \text{ and } t \in [0,1]. \tag{3.57}$$

Notice that if $\xi(d_t(x)) \neq 0$, then $\mathrm{dist}(\varphi_t(x), L_j) \leq d_t(x) \leq \varepsilon_2$ and hence (if $\varepsilon_2 < \eta$) $\pi_j(\varphi_t(x))$ is defined. This allows us to set

$$\widetilde{\varphi}_t(x) = t\,\xi \circ d_t(x)\,\pi_j(\varphi_t(x)) + (1 - t\,\xi \circ d_t(x))\,\varphi_t(x) \quad \text{for } x \in E_{j+1} \text{ and } t \in [0,1]. \tag{3.58}$$

We now start checking that the $\widetilde{\varphi}_t$ satisfy the required properties (1.4)-(1.8) on $E_j$. We first show that

$$(x,t) \to \widetilde{\varphi}_t(x) \text{ is continuous on } E_j \times [0,1]. \tag{3.59}$$

Since it is clearly continuous on $Z \times [0,1]$ and on $E_{j+1} \times [0,1]$, the only potential problem is at a point $(x,t)$ such that $x \in \overline{Z} \cap E_{j+1}$ (recall that $E_{j+1}$ is closed), and it is even enough to show that our two continuous definitions (3.54) and (3.58) give the same result at such a point. But then $x \in L_j$ (by (3.33) and because $L_j$ is closed), so $\varphi_t(x) \in L_j$ (by (1.7) and because $x \in E_{j+1}$), and using (3.54) yields the result

$$\pi_j(g_t(x)) = \pi_j(\varphi_t(x)) = \varphi_t(x) \tag{3.60}$$

by (3.51), (3.26), and (3.6). But (3.58) also yields $\varphi_t(x)$, because $\pi_j(\varphi_t(x)) = \varphi_t(x)$ by (3.60). So (3.59) holds.

Next we check that $\widetilde{\varphi}_1$ is Lipschitz. Again $\widetilde{\varphi}_1$ is Lipschitz on $Z$ and on $E_{j+1}$, but we need to be careful about the interface. That is, we just need to estimate $|\widetilde{\varphi}_1(x) - \widetilde{\varphi}_1(y)|$ when $x \in E_{j+1}$ and $y \in Z$. We'll distinguish between a few cases.

When $d_1(x) \leq \varepsilon_2/2$, $\xi \circ d_1(x) = 1$ and (3.58) says that

$$\widetilde{\varphi}_1(x) = \pi_j(\varphi_1(x)) = \pi_j(g_1(x)) \tag{3.61}$$

by (3.51), so $|\widetilde{\varphi}_1(x) - \widetilde{\varphi}_1(y)| = |\pi_j(g_1(x)) - \pi_j(g_1(y))| \leq C|x-y|$ by (3.54) and the fact that $g_1$ and $\pi_j$ are Lipschitz.

We claim that if $\varepsilon > 0$ is chosen small enough (depending on $\varepsilon_2$), we have that $d_1(x) \leq \varepsilon_2/2$ when $x \in E_{j+1}$ and $y \in Z$ are such that $|x-y| \leq \varepsilon$ and $\operatorname{dist}(x, B') \leq \varepsilon$. Indeed, otherwise we can find sequences $\{x_k\}$ in $E_{j+1}$ and $\{y_k\}$ in $Z$, such that $|x_k - y_k| \leq 2^{-k}$ and $\operatorname{dist}(x_k, B') \leq 2^{-k}$ but $d_1(x_k) \geq \varepsilon_2/2$. We can extract a subsequence so that $\{x_k\}$ converges to a limit $x \in E_{j+1} \cap B'$ (recall that we chose $B'$ closed). Then $x \in L_j$, because all the $y_k$ lie in $Z \subset L_j$, so $\varphi_1(x) \in L_j$ by (1.7). In addition, $\varphi_1(x) \in B'$ by (1.6) (if $x \in B$) or by (1.5) (if $x \in B' \setminus B$), and so $d_1(x) = 0$, which contradicts the fact that $d_1(x_k) \geq \varepsilon_2/2$ because $d_1$ is continuous. This proves our claim.

If $|x-y| \geq \varepsilon$, we simply use the fact that $|\widetilde{\varphi}_1(x) - \widetilde{\varphi}_1(y)| \leq C \leq C\varepsilon^{-1}|x-y|$ (because $\varphi_1$ is Lipschitz on each piece, hence bounded) to get a (very bad) Lipschitz bound. So we may assume that $|x-y| \leq \varepsilon$ and, since we already treated the case when $d_1(x) \leq \varepsilon_2/2$, our claim allows us to suppose that $\operatorname{dist}(x, B') \geq \varepsilon$. Then $\varphi_1(x) = x$ by (1.5), and $y \in Z \setminus B'$ because $|x-y| \leq \varepsilon$.

Let us compute $\widetilde{\varphi}_1(y)$. First observe that $\varphi_t(y) = y$ for $0 \leq t \leq 1$, because $y \in Z \setminus B' \subset E_j \setminus B'$ and by the definition of $\varphi_t$ below (3.35) (just before we use the Titze extension theorem). Also, $\varphi(y,t) = y$ for $0 \leq t \leq 1$, by (3.43), and hence $g_1(y) = \varphi'(y, \psi(y)) = y$, by (3.48) and (3.46). And also $\widetilde{\varphi}_1(y) = \pi_j(g_1(y)) = \pi_j(y)$ by (3.54). In addition, observe that

$$\pi_j(y) = y \quad \text{for } y \in Z \tag{3.62}$$

because $Z \subset L_j$ by (3.33), and then by (3.26) and (3.6). So here $\widetilde{\varphi}_1(y) = \pi_j(y) = y$ and now

$$\begin{aligned} |\widetilde{\varphi}_1(x) - \widetilde{\varphi}_1(y)| &= |\xi(d_1(x))\,\pi_j(\varphi_1(x)) + (1 - \xi(d_1(x)))\,\varphi_1(x) - y| \\ &\leq |\pi_j(\varphi_1(x)) - y| + |\varphi_1(x) - y| = |\pi_j(x) - y| + |x-y| \\ &= |\pi_j(x) - \pi_j(y)| + |x-y| \leq C|x-y| \end{aligned} \tag{3.63}$$

by (3.58), because $\varphi_1(x) = x$ by (1.5), and by (3.62). So $\widetilde{\varphi}_1$ is Lipschitz, which takes care of (1.8). Next we check that

$$\widetilde{\varphi}_0(x) = x \quad \text{for } x \in E_j. \tag{3.64}$$

Notice that $\varphi_0(x) = x$ by the definition below (3.35), $\varphi(x,0) = x$ by (3.43), and $g_0(x) = x$ by (3.46) and (3.48). If $x \in Z$, $\pi_j(x) = x$ by (3.62), and (3.54) yields $\widetilde{\varphi}_0(x) = \pi_j(g_0(x)) = x$. If $x \in E_{j+1}$, $t\xi(d_t(x)) = 0$ because $t = 0$, so (3.58) says that $\widetilde{\varphi}_0(x) = \varphi_0(x) = x$; hence (3.64) holds.

For (1.5), we'll need to know that

$$\widetilde{\varphi}_t(x) = x \quad \text{for } x \in Z \setminus B' \text{ and } t \in [0,1], \tag{3.65}$$

and indeed, $g_t(x) = x$ by (3.50) and $\widetilde{\varphi}_t(x) = \pi_j(g_t(x)) = x$ by (3.54) and (3.62). Similarly, let us check that

$$\widetilde{\varphi}_t(x) = x \quad \text{for } x \in E_{j+1} \setminus B'' \text{ and } t \in [0,1], \tag{3.66}$$

where

$$B'' = \big\{x \in \mathbb{R}^n \,;\, \mathrm{dist}(x, B') \leq \varepsilon_2\big\} \tag{3.67}$$

is just a tiny bit larger than $B'$ and $B$. This time $\varphi_t(x) = x$ by (1.5), hence $d_t(x) \geq \mathrm{dist}(x, B') \geq \varepsilon_2$ by (3.57), $\xi \circ d_t(x) = 0$ by (3.56), and hence $\widetilde{\varphi}_t(x) = \varphi_t(x) = x$ by (3.58). Since $E_j = Z \cup E_{j+1}$, we get that (1.5) holds for $B''$ (or any larger ball).

Next we check (1.6), but with an even larger ball $\widetilde{B}$. Let $w \in \mathbb{R}^n$ denote the center of $B$, and let $r$ and $r'$ denote the respective radii of $B$ and $B'$. Set

$$\widetilde{r} = r' + 4C\varepsilon_1 \quad \text{and } \widetilde{B} = \overline{B}(w, \widetilde{r}) \tag{3.68}$$

where $C$ is a bound for the Lipschitz constant for $\pi_j$ and $\varepsilon_1$ is as in (3.44) and (3.55) (in fact, any small number chosen in advance), and let us check that

$$\widetilde{\varphi}_t(x) \in \widetilde{B} \quad \text{for } x \in E_j \cap \widetilde{B} \text{ and } t \in [0,1]. \tag{3.69}$$

First suppose that $x \in Z$. If $x \in Z \setminus B'$, (3.65) says that $\widetilde{\varphi}_t(x) = x \in B'$, and we are done. Otherwise, $g_t(x) \in B'$ (see below (3.49)), and

$$|\widetilde{\varphi}_t(x) - g_t(x)| = |\pi_j(g_t(x)) - g_t(x)| \leq 2C\,\mathrm{dist}(g_t(x), L_j) \leq 4C\varepsilon_1 \tag{3.70}$$

by (3.54), because $\pi_j(z) = z$ on $L_j$ and $\pi_j$ is $C$-Lipschitz, and by (3.55). Then $\widetilde{\varphi}_t(x) \in \widetilde{B}$, as needed.

So we may assume that $x \in E_{j+1}$. If $x \in E_{j+1} \setminus B''$, (3.66) says that $\widetilde{\varphi}_t(x) = x \in B'' \subset \widetilde{B}$. If instead $x \in E_{j+1} \cap B''$, first notice that $\varphi_t(x) \in B''$ (by (1.6) or (1.5)). If in addition $d_t(x) \geq \varepsilon_2$, then $\xi \circ d_t(x) = 0$ by (3.56) and $\widetilde{\varphi}_t(x) = \varphi_t(x) \in B''$ by (3.58). Otherwise

$$|\widetilde{\varphi}_t(x) - \varphi_t(x)| \leq |\pi_j(\varphi_t(x)) - \varphi_t(x)| \leq 2C \operatorname{dist}(\varphi_t(x), L_j) \leq 2C d_t(x) \leq 2C\varepsilon_2 \tag{3.71}$$

by (3.58), again because $\pi_j(z) = z$ on $L_j$, and by (3.57). If $\varepsilon_2$ is small enough, depending on $\varepsilon_1$, we get that $\widetilde{\varphi}_t(x) \in \widetilde{B}$ because $\varphi_t(x) \in B''$. So (3.69) and (1.6) hold.

We finally check (1.7). So we pick $0 \leq i \leq j_{max}$ and want to show that

$$\widetilde{\varphi}_t(x) \in L_i \quad \text{for } x \in E_j \cap L_i. \tag{3.72}$$

We start with the case when $x \in E_{j+1}$. Notice that $\varphi_t(x) \in L_i$, by (1.7) and because $x \in E_{j+1} \cap L_i$. Let $F$ be a face of $L_i$ that contains $\varphi_t(x)$; then $\pi_j(\varphi_t(x)) \in F$ because Lemma 3.4 says that $\pi_j$ respects the faces of all dimensions. Now (3.58) says that $\widetilde{\varphi}_t(x)$ is a convex combination of $\varphi_t(x)$ and $\pi_j(\varphi_t(x))$, hence $\widetilde{\varphi}_t(x) \in F \subset L_i$, as needed.

Now suppose that $x \in Z \cap L_i$. By (3.33),

$$x \in L_j \setminus \bigcup_{k \geq j+1} L_k \tag{3.73}$$

so we have that $i \leq j$. In addition, if $L_i \cap L_j$ were a proper subset of $L_j$, our completeness assumption (2.15) would say that it is one of the $L_k$, and since we enumerated our boundaries in nonincreasing order (see above (3.28)), we would get that $k \geq j+1$, a contradiction with (3.73) since $z \in L_k$. So $L_i \cap L_j = L_j$, i.e., $L_j \subset L_i$ and it is enough to check that $\widetilde{\varphi}_t(x) \in L_j$. But (3.54) says that $\widetilde{\varphi}_t(x) = \pi_j(g_t(x))$, which lies in $L_j$ by definition of $\pi_j$. This completes our verification of (1.7).

We already checked (1.4), (1.5), (1.6), and (1.8) before, so this completes the verification that $F = \widetilde{\varphi}_1(E_j)$ is a sliding competitor for $E_j$ in $\widetilde{B}$. Recall that we may take $\varepsilon_1$ and $r' - r$, and hence also $\widetilde{r} - r$ in (3.68), as small as we want. Now we apply our induction assumption (3.31) and Definition 2.3 to get that

$$\mathcal{H}^d(\widetilde{W}) \leq M\mathcal{H}^d(\widetilde{\varphi}_1(\widetilde{W})) + \widetilde{r}^d h, \tag{3.74}$$

where

$$\widetilde{W} = \big\{x \in E_j \,;\, \widetilde{\varphi}_1(x) \neq x\big\}. \tag{3.75}$$

We are interested in $\mathcal{H}^d(W)$, where

$$W = \big\{x \in E_{j+1} \,;\, \varphi_1(x) \neq x\big\}, \tag{3.76}$$

because we want an analogue of (3.74) for $\varphi_1$, and we'll cut $W$ into pieces. Write $E_{j+1} = A_0 \cup A$, where

$$A_0 = A_0(\varepsilon_2) = \big\{x \in E_{j+1} \,;\, 0 < d_1(x) < \varepsilon_2\big\} \tag{3.77}$$

and $A = E_{j+1} \setminus A_0$. Observe that

$$\lim_{\varepsilon_2 \to 0} \mathcal{H}^d(A_0(\varepsilon_2)) = 0 \tag{3.78}$$

because $A_0(\varepsilon_2) \subset E_{j+1} \cap \widetilde{B}$ for $\varepsilon_2$ small, $\mathcal{H}^d(E_{j+1} \cap \widetilde{B}) < +\infty$, and because the monotone limit of the sets $A_0(\varepsilon_2)$ (when $\varepsilon_2$ tends to 0) is the empty set. So, given $B'$ and a small $\varepsilon_3 > 0$, we know that

$$\mathcal{H}^d(A_0(\varepsilon_2)) < \varepsilon_3 \tag{3.79}$$

for $\varepsilon_2$ small enough. Next let us check that

$$\widetilde{\varphi}_1(x) = \varphi_1(x) \text{ for } x \in A. \tag{3.80}$$

Write $A = A_1 \cup A_2$, with

$$A_1 = \big\{x \in E_{j+1}\,;\, d_1(x) = 0\big\} \text{ and } A_2 = \big\{x \in E_{j+1}\,;\, d_1(x) \geq \varepsilon_2\big\}. \tag{3.81}$$

When $x \in A_1$, $\xi \circ d_1(x) = 1$ by (3.56), so $\widetilde{\varphi}_1(x) = \pi_j(\varphi_1(x))$ by (3.58). In addition, $\varphi_1(x) \in L_j$, because $d_1(x) = 0$ and by (3.57), and $\pi_j(\varphi_1(x)) = \varphi_1(x)$ by (3.26) and (3.6), as needed for (3.80). When $x \in A_2$, $\xi(x) = 0$ and (3.58) directly yields that $\widetilde{\varphi}_1(x) = \varphi_1(x)$. So (3.80) holds.

If $x \in W \cap A$, then $\widetilde{\varphi}_1(x) = \varphi_1(x) \neq x$ by (3.80) and (3.76); hence $W \cap A \subset \widetilde{W}$ and

$$\begin{aligned} \mathcal{H}^d(W) &\leq \mathcal{H}^d(A_0) + \mathcal{H}^d(W \cap A) \leq \varepsilon_3 + \mathcal{H}^d(W \cap A) \\ &\leq \varepsilon_3 + \mathcal{H}^d(\widetilde{W}) \leq \varepsilon_3 + M\mathcal{H}^d(\widetilde{\varphi}_1(\widetilde{W})) + \widetilde{r}^d h \end{aligned} \tag{3.82}$$

because $W \subset E_{j+1}$ and $E_{j+1} = A_0 \cup A$ by definition, by (3.79), and by (3.74).

Next we estimate $\mathcal{H}^d(\widetilde{\varphi}_1(\widetilde{W}))$. Notice that $E_j = Z \cup E_{j+1} = Z \cup A_0 \cup A$ by (3.33), so

$$\widetilde{W} \subset Z \cup \big(\widetilde{W} \cap [A_0 \cup A]\big) \tag{3.83}$$

since $\widetilde{W} \subset E_j$. First,

$$\mathcal{H}^d(\widetilde{\varphi}_1(Z)) = 0 \tag{3.84}$$

because $\widetilde{\varphi}_1$ is Lipschitz and $Z \subset E \setminus E^*$ is negligible (see (3.33), (3.28), and (3.29)). Next, $\widetilde{\varphi}_1(x) = \varphi_1(x)$ on $A$ (by (3.80)), so $\widetilde{W} \cap A = W \cap A$ and $\widetilde{\varphi}_1(\widetilde{W} \cap A) = \varphi_1(W \cap A)$, hence

$$\begin{aligned} \mathcal{H}^d(\widetilde{\varphi}_1(\widetilde{W})) &= \mathcal{H}^d(\widetilde{\varphi}_1(\widetilde{W} \cap [A_0 \cup A]) \leq \mathcal{H}^d(\widetilde{\varphi}_1(\widetilde{W} \cap A_0)) + \mathcal{H}^d(\varphi_1(W \cap A)) \\ &\leq \mathcal{H}^d(\widetilde{\varphi}_1(A_0)) + \mathcal{H}^d(\varphi_1(W)). \end{aligned} \tag{3.85}$$

by (3.83) and (3.84). We have no nice formula for $\widetilde{\varphi}_1$ on $A_0$, but let us check that

$$\widetilde{\varphi}_1 \text{ is } C\text{-Lipschitz on } A_0, \tag{3.86}$$

with a constant $C$ that may be enormous and depend on various Lipschitz constants (in particular for $\varphi_1$ and $\pi_j$), but does not depend on $\varepsilon_2$. Recall that on $A_0$, $\widetilde{\varphi}_1$ is given by (3.58), i.e.,

$$\begin{aligned}\widetilde{\varphi}_1(x) &= \xi \circ d_1(x)\, \pi_j(\varphi_1(x)) + (1 - \xi \circ d_1(x))\, \varphi_1(x) \\ &= \varphi_1(x) + \xi \circ d_1(x)[\pi_j(\varphi_1(x)) - \varphi_1(x)],\end{aligned} \tag{3.87}$$

where $\xi$ and $d_1$ are still given by (3.56) and (3.57), and only the variations of $\xi \circ d_1(x)$ will be dangerous here (because they could involve some $\varepsilon_2^{-1}$). Write, for $x, y \in A_0$,

$$\begin{aligned}|\widetilde{\varphi}_1(x) - \widetilde{\varphi}_1(y)| &\leq |\varphi_1(x) - \varphi_1(y)| + |\xi \circ d_1(x) - \xi \circ d_1(y)||\pi_j(\varphi_1(x)) - \varphi_1(x)| \\ &\quad + \xi \circ d_1(y)\big|\pi_j(\varphi_1(x)) - \varphi_1(x) - \pi_j(\varphi_1(y)) + \varphi_1(y)\big| \\ &\leq C|x-y| + |\xi \circ d_1(x) - \xi \circ d_1(y)||\pi_j(\varphi_1(x)) - \varphi_1(x)|\end{aligned} \tag{3.88}$$

because all the other functions are Lipschitz with estimates that do not depend on $\varepsilon_2$. By (3.56), (3.57), and because $\varphi_1$ is $C$-Lipschitz,

$$|\xi \circ d_1(x) - \xi \circ d_1(y)| \leq 2\varepsilon_2^{-1}|d_1(x) - d_1(y)| \leq C\varepsilon_2^{-1}|x-y|, \tag{3.89}$$

while

$$|\pi_j(\varphi_1(x)) - \varphi_1(x)| \leq C \operatorname{dist}(\varphi_1(x), L_j) \leq C d_1(x) \leq C\varepsilon_2 \tag{3.90}$$

because $\pi_j(z) = z$ on $L_j$, by (3.57), and because $x \in A_0$ (see the definition (3.77)). Altogether, $|\widetilde{\varphi}_1(x) - \widetilde{\varphi}_1(y)| \leq C|x-y|$ by (3.88), (3.89), and (3.90); this proves (3.86). Now

$$\mathcal{H}^d(\widetilde{\varphi}_1(A_0)) \leq C\mathcal{H}^d(A_0) \leq C\varepsilon_3 \tag{3.91}$$

by (3.79), and

$$\begin{aligned}\mathcal{H}^d(W) &\leq \varepsilon_3 + M\mathcal{H}^d(\widetilde{\varphi}_1(\widetilde{W})) + \widetilde{r}^d h \\ &\leq \varepsilon_3 + M\mathcal{H}^d(\widetilde{\varphi}_1(A_0)) + M\mathcal{H}^d(\varphi_1(W)) + \widetilde{r}^d h \\ &\leq M\mathcal{H}^d(\varphi_1(W)) + \widetilde{r}^d h + (1 + MC)\varepsilon_3\end{aligned} \tag{3.92}$$

by (3.82), (3.85), and (3.91). Recall that $\widetilde{r}$ can be chosen as close to $r$ as we want, and that $\varepsilon_3$ can be chosen arbitrarily small. So we get that $\mathcal{H}^d(W) \leq M\mathcal{H}^d(\varphi_1(W)) + r^d h$. That is, (2.5) holds. This completes our proof of (3.32) given (3.31), and at the same time our proof of Proposition 3.27 by induction (recall that $E_{j_{max}+1} = E^*$, see below (3.30)). □

**Proof of Proposition 3.3.** We now assume that the Lipschitz assumption is satisfied on the open set $U$, and want to check that $E^* \in GSAQ(U, M, \delta, h)$ as soon as $E \in GSAQ(U, M, \delta, h)$. We cannot use Proposition 2.8 directly, because it would give us bad constants, but we can change variables and apply the proof above. That is, let $\lambda > 0$ and

$\psi : \lambda U \to B(0,1)$ be as in Definition 2.7, and then define $\widetilde{\psi}$ by $\widetilde{\psi}(x) = \psi(\lambda x)$ and set $F = \widetilde{\psi}(E)$. This is a quasiminimal set in $B(0,1)$ (by Proposition 2.8), but we don't care so much, and the closed support of the restriction of $\mathcal{H}^d$ to $F$ is $F^* = \widetilde{\psi}(E^*)$. We construct a nonincreasing sequence of sets $F_j$, $0 \leq j \leq j_{max} + 1$, as we did near (3.30), and we set $E_j = \widetilde{\psi}^{-1}(F_j)$. Thus $E_0 = E$ and $E_{j_{max}+1} = E^*$, and we want to show by induction that $E_j \in GSAQ(U, M, \delta, h)$.

This is the case for $j = 0$, and for the induction step, we give ourselves a sliding competitor $\varphi_1(E_{j+1})$ for $E_j$ in some ball $B$. We consider the mappings $f_t = \widetilde{\psi} \circ \varphi_t \circ \widetilde{\psi}^{-1}$ on $F_{j+1}$, which define a sliding competitor for $F_{j+1}$ in $H = \widetilde{\psi}(B)$. Of course $H$ is not a ball, but we don't really care, we still can use the proof of Proposition 3.27 to construct mappings $\widetilde{f}_t$, that define a sliding competitor for $F_j$ in a set $H'$ which is just a tiny bit little larger than $H$. Then the $\widetilde{\varphi}_t = \widetilde{\psi}^{-1} \circ \widetilde{f}_t \circ \widetilde{\psi}$ define a sliding competitor for $E_j$, in the set $\widetilde{\psi}^{-1}(H')$ which is a tiny bit larger than $B$. We apply the definition (2.5) to this competitor and get that

$$\mathcal{H}^d(\widetilde{W}) \leq M\mathcal{H}^d(\widetilde{\varphi}_1(\widetilde{W})) + \widetilde{r}^d h, \tag{3.93}$$

where $\widetilde{r}$ is the radius of a ball that contains $\widetilde{\psi}^{-1}(H')$ (and can be taken as close to $r$ as we want), and

$$\widetilde{W} = \{x \in E_j \,;\, \widetilde{\varphi}_1(x) \neq x\}. \tag{3.94}$$

Then we estimate $\mathcal{H}^d(W)$, where $W = \{x \in E_{j+1} \,;\, \varphi_1(x) \neq x\}$ as we did after (3.76); the error terms, like the ones in (3.79), (3.84), and (3.91) become $C$ times larger because we compose with $\widetilde{\psi}$ and $\widetilde{\psi}^{-1}$, but the argument goes through. □

**Remark 3.95.** Here we defined the rigid assumption, and then the Lipschitz assumption, in terms of dyadic cubes, but we could have obtained similar results if we used a net of convex polyhedra instead, with a rotundity assumption where we ask all the angles in the faces of all dimensions to be bounded from below. The only place where the argument needs to be modified is in the proof of Lemma 3.4. See Remark 2.12 and the comments after (3.8) and (3.13).

**Remark 3.96.** In Propositions 3.3 and 3.27, we can get a slightly stronger conclusion under the same assumption, namely that all the closed sets $F$ such that $E^* \subset F \subset E$ lie in the same $GSAQ(U, M, \delta, h)$ as $E$. That is, we never use the fact that $E^*$ is the closed support of $\mathcal{H}^d_{|E}$, but just the fact that $E^* \subset E$ and $\mathcal{H}^d(E \setminus E^*) = 0$.

This could be useful if we tried to extend Proposition 3.27 to a situation where we only assume that $U$ is covered by a finite collection of domains where we have the Lipschitz assumption, and try to go from $E$ to $E^*$ in a finite number of steps. We would need to check what happens to a sliding competitor in a ball that is not entirely contained in one domain, though. We shall not pursue this here, as Proposition 3.27 shall be enough for our purposes.

The conclusion of this section is that we shall feel free to restrict our attention to coral quasiminimal sets, with no apparent loss of generality. We could probably have managed, in the rest of this paper, not to assume that $E$ is coral, and then prove regularity results on $E^*$, and we may even do this in some cases (so as not to rely on the proof of Propositions 3.3 and 3.27), but we shall find it more comfortable to know that we can work work directly with coral sets. Recall that we do not say that every quasiminimal set is a competitor for its core $E^*$, or the other way around (both things are wrong in general), but just that they have the same minimizing properties.

PART II : AHLFORS REGULARITY AND RECTIFIABILITY

In this part we prove basic regularity properties for the (core of) sliding quasiminimal sets. The main ones are their local Ahlfors regularity (Proposition 4.1), which is of constant use, and rectifiability (Theorem 5.16), which will be important for the theorems of Part IV on limits.

Most of the results of this part and the next one (where we prove the local uniform rectifiability in some cases), and their proofs, are generalizations of results of [DS4], except for rectifiability which was proved in [A2] and ignored in [DS4] (because we thought uniform rectifiability was better), but for which the proof of [DS4] works as well.

We cannot repeat all the arguments from [DS4] (this would be too long), but fortunately many of the intermediate results there can be used essentially without modification here, and there are only a few places where we need to be careful, because a competitor for our quasiminimal set is used. We will try to give an idea, but all the details, of the arguments that work with only minor modifications, and be as precise as possible on the differences, i.e., places where a competitor is defined. Hopefully this will make the reading of this text not too unpleasant, probably at the price of often believing the author when he says that some old estimates estimates still apply.

## 4. Local Ahlfors regularity of quasiminimal sets.

We start now our long study of regularity properties of sliding quasiminimal sets with the very convenient local Ahlfors regularity of the core $E^*$. We start with the rigid case.

**Proposition 4.1.** *For each choice of $M \geq 1$, we can find $h > 0$ and $C_M \geq 1$, depending on $M$ and the dimensions $n$ and $d$, so that the following holds. Suppose that $E \in GSAQ(B_0, M, \delta, h)$, where we set $B_0 = B(0,1)$, and that the rigid assumption is satisfied. Let $r_0 = 2^{-m} \in (0,1]$ denote the side length of the dyadic cubes used to define the rigid assumption. Then if $x \in E^* \cap B_0$ and $0 < r < \mathrm{Min}(r_0, \delta)$ are such that $B(x,2r) \subset B_0$, we have that*

$$C_M^{-1} r^d \leq \mathcal{H}^d(E \cap B(x,r)) \leq C_M r^d. \tag{4.2}$$

Recall that $E^*$, the core of $E$, is as in (3.2). We could also have assumed that $E$ is coral, and then obtained that (4.2) holds for $x \in E \cap B_0$ (instead of $E^* \cap B_0$). Also observe that we can replace $E$ with $E^*$ in (4.2), since $E^* \subset E$ and $\mathcal{H}^d(E \setminus E^*) = 0$.

By Proposition 2.8 and the bilipschitz invariance of local Ahlfors regularity, this result implies the corresponding one when the bilipschitz assumption holds. See Proposition 4.74.

We want to say that the standard proof given in [DS4] goes through in the present setting. We cannot repeat it entirely (this will make this paper too huge and boring), so we shall only recall how the proof goes, and concentrate on the minor modifications that we need to make, in particular in the choices of cubes.

There are two main differences, compared to the initial situation in [DS4]. First, the accounting in the definition of general quasiminimal sets is slightly different (and makes the notion of quasiminimal sets more general) than the one we used in [DS4]. This aspect of things is not really important, and was already discussed in [D5]. The second difference, which really concerns us here, is the additional conditions on the competitors that come from the sliding boundary conditions; we may have to go all the way to the boundary, and make sure that all the competitors that we use in the proof satisfy the sliding condition. This will force us to choose more carefully the cubes where we do Federer-Fleming constructions, and this is why we shall be more prudent in the choice of integers $N_k$ below.

We start our reading of [DS4] with the Federer-Fleming construction of Lipschitz projections that is described in Chapter $*3$ (we shall use the convention that $*$ calls a reference in [DS4]). We need the following slight variant of Proposition $*3.1$. Here and below, cubes are systematically assumed to be closed, and the $k$-dimensional skeleton of a cube $R$ is the union of all the faces of dimension $k$ of $R$. Thus the 1-dimensional skeleton of a cube in $\mathbb{R}^3$ is a union of 12 line segments.

**Lemma 4.3.** *Let $N \geq 1$ be an integer, and let $Q \subset \mathbb{R}^n$ be a cube of side length $N2^k$ (for some $k \in \mathbb{Z}$), which is the almost-disjoint union of $N^n$ dyadic cubes of side length $2^k$. Denote by $\mathcal{R} = \mathcal{R}(Q)$ the set of dyadic cubes of side length $2^k$ that are contained in $Q$, and by $\mathcal{S}_d$ the union of the $d$-dimensional skeletons of the dyadic cubes $R \in \mathcal{R}$. Let $E$ be a compact subset of $Q$ such that $\mathcal{H}^d(E) < +\infty$. Then there is a Lipschitz mapping $\phi : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$\phi(x) = x \text{ for } x \in \mathbb{R}^n \setminus Q \text{ and for } x \in \mathcal{S}_d, \tag{4.4}$$

$$\phi(E) \subset \mathcal{S}_d \cup \partial Q, \tag{4.5}$$

$$\phi(R) \subset R \quad \text{for } R \in \mathcal{R}, \tag{4.6}$$

*and*

$$\mathcal{H}^d(\phi(E \cap R)) \leq C\mathcal{H}^d(E \cap R) \text{ for } R \in \mathcal{R}. \tag{4.7}$$

*Here $C$ depends on $n$ and $d$, but not on $N$ or $E$.*

The only difference with Proposition $*3.1$ in [DS4] is that $Q$ is not required itself to be a dyadic cube (and so $N$ is not required to be a power of 2); however, this fact that $Q$ is dyadic (or rather that $N$ is a power of 2) is never used in [DS4], and the proof can be carried out exactly as before. □

Another consequence of the proof, which works by successive "radial" projections onto faces, is that in addition to (4.6),

$$\phi(F) \subset F \text{ when } F \text{ is a face (of any dimension) of a cube } R \in \mathcal{R}. \tag{4.8}$$

This will be used to prove the stability conditions (1.7).

We now turn to Chapter 4 in [DS4], and prove Proposition 4.1. We start with the easier upper bound. We want to find $C_0 \geq 1$ (depending also on $M$) such that

$$\mathcal{H}^d(E \cap Q_0) \leq C_0 l_0^d \tag{4.9}$$

when $Q_0$ is a cube of side length

$$l_0 \leq \mathrm{Min}(2^{-m}, \frac{\delta}{n}) \tag{4.10}$$

which is dyadic (in the same grid that was used in the description of the $L_j$ for the rigid assumption), and such that $2Q_0 \subset B_0$.

Indeed, if we can prove (4.9) for such cubes, and if $x \in B_0$ (we do not need $x \in E^*$ for the lower bound) and $0 < r < \mathrm{Min}(r_0, \delta)$ are such that $B(x, 2r) \subset B_0$, we can easily cover $B(x, r)$ with less than $C$ cubes $Q_0$ as above, with side lengths less than $r$, and then the upper bound in (4.2) follows from (4.9).

So we give ourselves a cube $Q_0$ as above, assume that (4.9) fails, and we shall derive a contradiction if $C_0$ in (4.9) is large enough (depending on $n$, $d$ and $M$). Here we shall only need to assume that $h \leq 1$. We want to construct by induction an increasing sequence of cubes $Q_k$, $k \geq 1$, with the same center as $Q_0$, and whose side lengths $l_k$ are such that

$$l_0 < l_k < 2l_0 \quad \text{for } k \geq 1. \tag{4.11}$$

At the same time, we shall define large integers $N_k$, $k \geq 1$, and for each $k \geq 1$ cut $Q_k$ into $N_k^d$ cubes of the same side length $N_k^{-1} l_k$; we shall call $\mathcal{R}(Q_k)$ the collection of these smaller cubes, in accordance with the notation of Lemma 4.3. When we do this, we want to make sure that for $k \geq 1$,

(4.12) every cube $R \in \mathcal{R}(Q_k)$ is a dyadic (sub)cube of the grid that was used to define the $L_j$ in the rigid assumption.

The fact that these cubes are of a smaller size than the $L_j$ follows from (4.11) and the fact that $l_0 \leq 2^{-m}$, because $N_k \geq 2$, but we typically want any face of $R$ that intersects the interior of a face of some $L_j$ to be entirely contained in that $L_j$. We require this because we want to apply Lemma 4.3 to $Q_k$ to find a competitor for $E$ in $Q_k$; for similar reasons, we want $Q_{k-1}$ to be obtained from $Q_k$ by removing from $\mathcal{R}(Q_k)$ the two exterior layers of cubes, and then taking the union. In other words, we want to have $Q_{k-1} = \frac{N_k - 4}{N_k} Q_k$, or equivalently (since the cubes have the same center)

$$l_{k-1} = \big(1 - \frac{4}{N_k}\big)\, l_k \tag{4.13}$$

for $k \geq 1$. In fact denote by $A(Q_k)$ the union of the cubes $R \in \mathcal{R}(Q_k)$ that lie on the exterior layer (or equivalently that meet $\partial Q_k$); we wanted to make sure that

$$Q_{k-1} \subset Q_k \setminus A(Q_k), \tag{4.14}$$

and (4.13) ensures this (we need to remove a extra layer because $Q_{k-1}$ and $A(Q_k)$ are both closed).

Next let us assume for the moment that we can choose $N_k$ so that (4.11) and (4.12) hold, and let us use this to control

$$m_k = \mathcal{H}^d(E \cap Q_k) \tag{4.15}$$

in terms of $m_{k-1}$. First apply Lemma 4.3 to $Q_k$, the integer $N_k$, and the decomposition coming from $\mathcal{R}(Q_k)$. This gives a Lipschitz mapping $\phi : \mathbb{R}^n \to \mathbb{R}^n$, which we use to define a family $\{\varphi_t\}$ as in Definition 1.3, simply by setting

$$\varphi_t(x) = t\phi(x) + (1-t)x \quad \text{for } x \in \mathbb{R}^n \text{ and } 0 \leq t \leq 1. \tag{4.16}$$

The properties (1.4), (1.5), (1.6), and (1.8) are easily checked, relative to any closed ball that contains $Q_k$, for instance the ball $B$ with the same center $x_0$ as $Q_k$ and $Q_0$, and with radius

$$r = \frac{\sqrt{n}\, l_k}{2} \leq \sqrt{n}\, l_0 < \delta \tag{4.17}$$

by (4.11) and (4.10). We also need to check (1.7). Let $x \in L_j$ be given, and let $F$ be a face of $L_j$ that contains $x$. Let $F' \subset F$ be a dyadic subface of the same dimension as $F$, but of side length $N_k^{-1} l^k$, that contains $x$. We know that such a face exists, because $N_k^{-1} l^k < 2^{-m}$ (see the line below (4.12)) and the ratio is a power of 2. Since the cubes of $\mathcal{R}(Q_k)$ are dyadic in the same grid as $F$ (see (4.12)), we get that $F'$ is a face of the grid defined by the cubes of $\mathcal{R}(Q_k)$. Now either $x \notin Q_k$, and then $\varphi_t(x) = \phi(x) = x$ by (4.4), or else (4.8) says that $\phi(x)$, and hence also (by (4.16) and the convexity of $F'$) $\varphi_t(x)$, lies on $F' \subset F \subset L_j$. This proves (1.7); hence $\phi(E)$ is a sliding competitor for $E$ in $B = \overline{B}(x_0, r)$, with $r$ as in (4.17).

Let us apply Definition 2.3. We get (2.5), with $\varphi_1 = \phi$. That is,

$$\mathcal{H}^d(W_1) \leq M\mathcal{H}^d(\phi(W_1)) + hr^d \leq M\mathcal{H}^d(\phi(W_1)) + n^{d/2} l_0^d \tag{4.18}$$

where $W_1 = \big\{y \in E\,; \phi(y) \neq y\big\}$, if $h \leq 1$, and by (4.17). Observe that $W_1 \subset Q_k$ because $\phi(x) = x$ out of $Q_k$ by (4.4). In addition, (4.5) and (4.6) imply that

$$\phi(x) \in \mathcal{S}_d \quad \text{for } x \in E \cap Q_k \setminus A(Q_k), \tag{4.19}$$

so

$$\mathcal{H}^d(\phi(W_1 \setminus A(Q_k))) \leq \mathcal{H}^d(\mathcal{S}_d) \leq C N_k^{n-d} l_0^d. \tag{4.20}$$

For $E \cap A(Q_k)$, we decompose $A(Q_k)$ into cubes of $\mathcal{R} = \mathcal{R}(Q_k)$ and use (4.7) to say that

$$\begin{aligned} \mathcal{H}^d(\phi(W_1 \cap A(Q_k))) &\leq \sum_{R \in \mathcal{R}; R \subset A(Q_k)} \mathcal{H}^d(\phi(E \cap R)) \\ &\leq C \sum_{R \in \mathcal{R}; R \subset A(Q_k)} \mathcal{H}^d(E \cap R) \leq 2^n C \mathcal{H}^d(E \cap A(Q_k)) \end{aligned} \tag{4.21}$$

because a given point of $E \cap A(Q_k)$ lies in at most $2^n$ cubes $R$. So

$$\mathcal{H}^d(W_1) \leq M \mathcal{H}^d(\phi(W_1)) + n^{d/2} l_0^d \leq CM \mathcal{H}^d(E \cap A(Q_k)) + C(M+1) N_k^{n-d} l_0^d \tag{4.22}$$

by (4.18), (4.20), and (4.21) (and with a constant $C$ that does not depend on $M$). Finally $E \cap Q_k \setminus A(Q_k) \subset \mathcal{S}_d \cup W_1$ because if $x \in E \cap Q_k \setminus A(Q_k)$ lies out of $\mathcal{S}_d$, then (4.19) says that $x \in W_1$. Then

$$\mathcal{H}^d(E \cap Q_k \setminus A(Q_k)) \leq \mathcal{H}^d(\mathcal{S}_d) + \mathcal{H}^d(W_1) \leq CM \mathcal{H}^d(E \cap A(Q_k)) + C(M+1) N_k^{n-d} l_0^d. \tag{4.23}$$

We add $\mathcal{H}^d(E \cap A(Q_k))$ to both sides, recall that $M \geq 1$, and get that

$$\begin{aligned} m_k = \mathcal{H}^d(E \cap Q_k) &\leq CM \mathcal{H}^d(E \cap A(Q_k)) + C' M N_k^{n-d} l_0^d \\ &\leq CM \mathcal{H}^d(E \cap Q_k \setminus Q_{k-1}) + C' M N_k^{n-d} l_0^d \\ &= CM [m_k - m_{k-1}] + C' M N_k^{n-d} l_0^d \end{aligned} \tag{4.24}$$

by (4.15), (4.14), and (4.15) again. That is,

$$[CM - 1] m_k \geq CM m_{k-1} - C' M N_k^{n-d} l_0^d \tag{4.25}$$

or equivalently (dividing by $CM$)

$$m_k \Big(1 - \frac{1}{CM}\Big) \geq m_{k-1} - \frac{C'}{C} N_k^{n-d} l_0^d. \tag{4.26}$$

We shall choose $N_k$ so that

$$\frac{C'}{C} N_k^{n-d} l_0^d \leq \frac{m_{k-1}}{10CM}, \tag{4.27}$$

so that (4.26) implies that

$$m_k \geq \Big(1 - \frac{1}{CM}\Big)^{-1} \Big(1 - \frac{1}{10CM}\Big) m_{k-1} \geq m_{k-1} \Big(1 + \frac{1}{10CM}\Big) \tag{4.28}$$

and, by induction,

$$m_k \geq \Big(1 + \frac{1}{10CM}\Big)^k m_0 \geq C_0 l_0^d \Big(1 + \frac{1}{10CM}\Big)^k \tag{4.29}$$

by (4.15) and because (4.9) is assumed to fail.

Now we want to check that we can choose $C_0$ (large enough) and the $N_k$ (by induction), so that (4.11), (4.12), and (4.27) hold. The desired contradiction will follow, because the fact that $m_k = \mathcal{H}^d(E \cap Q_k) \leq \mathcal{H}^d(E \cap 2Q_0) < +\infty$ (by (4.15), (4.11), and the finite measure condition in Definition 2.3) contradicts (4.29) for $k$ large.

In fact, we shall choose the $N_k$, $k \geq 1$, so that the following constraints hold. Set

$$\lambda_k = \frac{l_0^{-d} m_k}{10C'M} \tag{4.30}$$

for $k \geq 0$, with $C'$ as in (4.27). Thus (4.27) just demands that $N_{k+1}^{n-d} \leq \lambda_k$ for $k \geq 0$, but we shall pick the $N_k$ so that

$$\frac{\lambda_k}{3^{n-d}} \leq N_{k+1}^{n-d} \leq \lambda_k \quad \text{for } k \geq 0, \tag{4.31}$$

and also such that for $k \geq 0$,

$$N_{k+1} \geq N_k + 4 \quad \text{and} \quad \frac{N_{k+1} - 4}{N_k} \text{ is a power of 2,} \tag{4.32}$$

where we set $N_0 = 1$ for $k = 0$.

First we want to check that (4.12) for $k+1$ follows from this and (if $k \geq 1$) (4.12) for $k$. We do not need to check (4.12) for $k = 0$, but recall that $Q_0$ was assumed to be dyadic in the usual grid (the one that was used to define the $L_j$ in the rigid assumption). When $k \geq 1$, denote by $s_k$ the common side length of the cubes of $\mathcal{R}(Q_k)$; thus $s_k = N_k^{-1} l_k$. Also set $s_0 = l_0$ (the side length of $Q_0$). Then, for $k \geq 0$,

$$s_{k+1} = \frac{l_{k+1}}{N_{k+1}} = \big(1 - \frac{4}{N_{k+1}}\big)^{-1} \frac{l_k}{N_{k+1}} = \frac{l_k}{N_{k+1} - 4} = \frac{N_k}{N_{k+1} - 4}\, s_k \tag{4.33}$$

by (4.13). We know (by definition of $Q_0$ or by induction assumption) that $s_k$ is a dyadic number and $s_k \leq 2^{-m}$ (the size of the dyadic cubes that we used to define the $L_j$), and now (4.32) says that $s_{k+1} \leq s_k$ and is a dyadic number too. So the cubes of $\mathcal{R}(Q_{k+1})$ have the right size; we also need to know that they are dyadic (instead of merely translations of dyadic cubes), and for this we use the fact that, as was observed above (4.13), $Q_k$ is obtained, from the decomposition of $Q_{k+1}$ into cubes of side length $s_{k+1}$, by removing the two exterior layers of cubes. By induction assumption, the cubes of side length $s_k$ that compose $Q_k$ are dyadic, hence this is also true for the cubes of side length $s_{k+1}$ that compose $Q_{k+1}$. This proves (4.12) (if we have (4.32)).

Now let $k \geq 0$ be given, assume that the $N_l$, $1 \leq l \leq k$, were chosen so that (4.11), (4.12), (4.31), and (4.32), hold for $1 \leq l \leq k$ (no condition if $k = 0$), and let us choose $N_{k+1}$. We first check that

$$\lambda_k \geq (N_k + 4)^{n-d}. \tag{4.34}$$

Observe that (4.29) holds (because if $k \geq 1$, (4.27) follows from (4.31) for $k-1$), so

$$\lambda_k = \frac{l_0^{-d} m_k}{10C'M} \geq \frac{C_0}{10C'M} \Big(1 + \frac{1}{10CM}\Big)^k. \tag{4.35}$$

By taking $C_0$ large enough, we can thus make sure that $\lambda_k \geq 300^{n-d}$, for instance, hence (by (4.31) for $k-1$) $N_k \geq 100$ if $k \geq 1$. Note that (4.34) follows trivially from (4.35) (or directly from (4.30) and the failure of (4.9)) when $k = 0$. Otherwise, the definition (4.30) and (4.28) say that

$$\lambda_k = \frac{m_k}{m_{k-1}} \lambda_{k-1} \geq \Big(1 + \frac{1}{10CM}\Big) \lambda_{k-1} \geq \Big(1 + \frac{1}{10CM}\Big) N_k^{n-d} \tag{4.36}$$

by (4.31), and (4.34) will follow as soon as we check that $1 + \frac{1}{10CM} \geq \big(\frac{N_k+4}{N_k}\big)^{n-d} = \big(1 + \frac{4}{N_k}\big)^{n-d}$, or equivalently $\frac{4}{N_k} \leq \big(1 + \frac{1}{10CM}\big)^{1/(n-d)} - 1$. But (4.31) for $k-1$ and (4.35) say that

$$\begin{aligned} 4N_k^{-1} &\leq 12\lambda_{k-1}^{-1/(n-d)} \leq 12\Big(\frac{C_0}{10C'M}\Big)^{-1/(n-d)} \Big(1 + \frac{1}{10CM}\Big)^{-(k-1)/(n-d)} \\ &\leq 12\Big(\frac{C_0}{10C'M}\Big)^{-1/(n-d)} < \big(1 + \frac{1}{10CM}\big)^{1/(n-d)} - 1 \end{aligned} \tag{4.37}$$

if $C_0$ is large enough (depending on $M$, $n$, and $d$), so (4.34) holds.

Because of (4.34), picking $N_{k+1} = N_k + 4$ would already yield the second half of (4.31). We take for $N_{k+1}$ the largest integer $N$ such that $N \geq N_k + 4$, $\frac{N-4}{N_k}$ is a power of 2, and $N^{n-d} \leq \lambda_k$ (the second half of (4.31)). We know from (4.34) that $N_{k+1} \geq N + 4$, and so (4.32) holds. The second half of (4.31) holds by definition. By maximality of $N_{k+1}$, $N = 2N_{k+1} - 4$ does not work. Since $\frac{N-4}{N_k} = 2\frac{N_{k+1}-4}{N_k}$ is also a power of 2, this means that $(2N_{k+1} - 4)^{n-d} > \lambda_k$, which implies that $N_{k+1}^{n-d} > \frac{\lambda_k}{3^{n-d}}$ because $N_{k+1} > N_k$ is large, and as needed for (4.31).

We now check that (4.11) holds for $k+1$ with our choice of $N_{k+1}$. By repeated uses of (4.13),

$$l_{k+1} = l_0 \prod_{1 \leq j \leq k+1} \Big(1 - \frac{4}{N_j}\Big)^{-1} \tag{4.38}$$

so it is enough to show that $\sum_{1 \leq j \leq k+1} \frac{1}{N_j} \leq 10^{-2}$, say, which follows from the first line of (4.37) and its analogue for $j \leq k$, provided that $C_0$ is chosen large enough. This completes the verification and the definition of the $N_k$; the expected contradiction follows, and shows that (4.9) holds. The upper bound in (4.2) follows, as explained below (4.9).

Next we want to establish the lower bound in Proposition 4.1. The main step will be the following.

**Lemma 4.39.** *Let $a < 1$ be given. There are constants $\varepsilon_a$ and $C_a$, that depend on $n$, $a$, and $M$, but not on $h$ or $\delta$, with the following property. Let $E$ be as in Proposition 4.1, and let $Q$ be a cube such that $2Q \subset B_0$, whose side length $l(Q)$ is such that*

$$l(Q) \leq \min(\frac{\delta}{n}, 2^{-m}), \tag{4.40}$$

*and for which*

$$\mathcal{H}^d(E \cap Q) \leq \varepsilon_a l(Q)^d. \tag{4.41}$$

*Then*

$$\mathcal{H}^d(E \cap \frac{1}{100}Q) \leq a\mathcal{H}^d(E \cap Q) + C_a h l(Q)^d. \tag{4.42}$$

Lemma 4.39 will be proved soon, but let us first check that it yields the lower bound in Proposition 4.1. Let $x \in E^* \cap B_0$ and $r$ be as in the proposition. We know from general geometric measure theory that there is a constant $c > 0$ (depending at most on $n$) such that

$$\limsup_{\rho \to 0} \rho^{-d} \mathcal{H}^d(E \cap B(x', \rho)) \geq c \tag{4.43}$$

for $\mathcal{H}^d$-almost every point $x' \in E$. See for instance [Ma], Theorem 6.2 on page 89. Since $x \in E^*$, (3.2) says that $\mathcal{H}^d(E \cap B(x,t)) > 0$ for all $t > 0$, so we can choose $x' \in E$, very close to $x$, such that (4.43) holds.

We choose $a = 200^{-d}$, and Lemma 4.39 yields constants $\varepsilon_a$ and $C_a$. We can safely assume that $\varepsilon_a \leq 400^{-d}c$. Next we choose $h$ so small that $C_a h < 200^{-d}\varepsilon_a$ in (4.42). The point is that if $Q$ satisfies the assumptions of Lemma 4.39, then

$$\mathcal{H}^d(E \cap \frac{1}{100}Q) \leq 200^{-d}\mathcal{H}^d(E \cap Q) + 200^{-d}\varepsilon_a l(Q)^d \leq 100^{-d}\varepsilon_a l(Q)^d \tag{4.44}$$

by (4.42), so $\frac{1}{100}Q$ also satisfies the assumptions of Lemma 4.39, and recursively

$$\mathcal{H}^d(E \cap \frac{1}{100^k}Q) \leq 100^{-kd}\varepsilon_a l(Q)^d \tag{4.45}$$

for $k \geq 0$, by (repeated uses of) (4.44). If $Q$ is centered at $x'$, this implies that for $100^{-k-1}l(Q) \leq 2\rho \leq 100^{-k}l(Q)$,

$$\mathcal{H}^d(E \cap B(x', \rho)) \leq \mathcal{H}^d(E \cap \frac{1}{100^k}Q) \leq 100^{-kd}\varepsilon_a l(Q)^d \leq 200^d \varepsilon_a \rho^d \leq c\rho^d/2, \tag{4.46}$$

which is incompatible with (4.43).

Now we may try this with the largest cube $Q$ centered at $x'$, and such that $2Q \subset B(x,r)$ and (4.40) holds. Notice that $l(Q)$ is then comparable to $r$, (because $r < \mathrm{Min}(r_0, \delta) = \mathrm{Min}(2^{-m}, \delta)$) and since (4.41) fails by the discussion above, we get that

$$\mathcal{H}^d(E \cap B(x,r)) \geq \mathcal{H}^d(E \cap Q) \geq \varepsilon_a l(Q)^d \geq C^{-1} r^d, \tag{4.47}$$

as needed for (the lower bound in) (4.2). Since we already established the upper bound, Proposition 4.1 will follow from Lemma 4.39.

We now prove the lemma. We are given a cube $Q$, and we first reduce to dyadic cubes of the usual grid (the one that was used to define the $L_j$). Let $l_0$ denote the largest dyadic number such that $l_0 \leq l(Q)/2$. Then $l_0 \leq 2^{-m}$ by (4.40). Denote by $Q_0'$ any dyadic cube of side length $l_0$ in the usual grid, and then let $Q_0$ be a translation of $Q_0'$ by an element of $2^{-3} l_0 \mathbb{Z}^n$. We choose $Q_0$ such that, if $x_0$ and $x_Q$ denote the centers of $Q_0$ and $Q$, the size of every coordinate of $x_0 - x_Q$ is at most $2^{-4} l_0$. Then

$$\frac{1}{100} Q \subset \frac{1}{2} Q_0 \subset Q_0 \subset Q. \tag{4.48}$$

Thus it will be enough to show that

$$\mathcal{H}^d(E \cap \frac{1}{2} Q_0) \leq a \mathcal{H}^d(E \cap Q_0) + C_a h l_0^d, \tag{4.49}$$

because (4.42) will follow at once.

We shall now proceed a little bit as for the upper bound, and define by induction a decreasing sequence of concentric cubes $Q_k$, $k \geq 0$, such that

$$m_k = \mathcal{H}^d(E \cap Q_k) \tag{4.50}$$

is rapidly decreasing. That is, up until we stop the process, which will in fact happen after a finite number of steps. We take

$$Q_{k+1} = \big(1 - \frac{6}{N_k}\big) Q_k \tag{4.51}$$

for $k \geq 0$, where $N_k$ is a large number that will be chosen later. The main point is that for $k \geq 0$, $Q_{k+1}$ is obtained from $Q_k$ by the following manipulation. First we cut $Q_k$ into $N_k^n$ equal cubes $R$, $R \in \mathcal{R}(Q_k)$, as we did for Lemma 4.3 (with $N = N_k$). Then we remove the three exterior layers, and $Q_{k+1}$ is the closure of what remains. We want to make sure that (as in (4.12)),

$$\text{every cube } R \in \mathcal{R}(Q_k) \text{ is a dyadic cube of the usual grid,} \tag{4.52}$$

and because of this we require (a little as in (4.32)) that $N_0$ be a (large) power of 2 and, for $k \geq 0$,

$$N_{k+1} \geq N_k - 6 \geq 1 \quad \text{and} \quad \frac{N_{k+1}}{N_k - 6} \text{ is a power of 2.} \tag{4.53}$$

As it turns out, we will only need to define a finite sequence of numbers $N_k$ (after which we shall stop), and we shall even manage to take $N_{k+1} = N_k - 6$ for $k \geq 0$, but let us pretend we need more generality and deal with (4.53) for the moment. Let us first check that (4.52) follows if we apply the rule (4.53). Denote by $s_k$ the common side length of the cubes of $\mathcal{R}(Q_k)$. Thus $s_k = N_k^{-1} l_k$. First, $s_0 = N_0^{-1} l_0$ is a dyadic number, because $N_0$ is large dyadic and $l_0$ is dyadic. For $k \geq 0$,

$$s_{k+1} = N_{k+1}^{-1} l_{k+1} = N_{k+1}^{-1} \big(1 - \frac{6}{N_k}\big) l_k = \frac{N_k - 6}{N_{k+1}}\, s_k \tag{4.54}$$

by (4.51) and because $s_k = N_k^{-1} l_k$. Thus $s_{k+1}$ is dyadic if $s_k$ is dyadic, and $s_{k+1} \leq s_k$ by the first part of (4.53). The verification of (4.52), i.e., the fact that the cubes also match the dyadic grid (instead of just having the right size) is now easy, and goes as for (4.12) near (4.33).

We apply Lemma 4.3 to $Q_k$ (decomposed as the union of the cubes $R \in \mathcal{R}(Q_k)$), and get a Lipschitz mapping $\phi : \mathbb{R}^n \to \mathbb{R}^n$ which preserves the faces as in (4.8). This time, we do not use the function $\phi$ directly to produce a competitor, but instead try to project once more on $(d-1)$-faces when this is possible. Suppose that

$$m_k \leq c N_k^{-d} l_k^d, \tag{4.55}$$

where $c > 0$ will be chosen soon. Notice that $\mathcal{H}^d(\phi(E \cap R)) \leq C\mathcal{H}^d(E \cap R) \leq C m_k$ for $R \in \mathcal{R}(Q_k)$, by (4.7). Hence

$$\mathcal{H}^d(R \cap \phi(E)) \leq 2^n C m_k \ \text{ for } R \in \mathcal{R}(Q_k) \tag{4.56}$$

because if $y = \phi(x)$ lies on $R \cap \phi(E)$, then (4.4) says that $x \in Q_k$, and then, by (4.6), $x$ lies in $R$ or one of its neighbors of $\mathcal{R}(Q_k)$. We take $c$ smaller than $10^{-n} C^{-1}$, and this way (4.56) says that $\phi(E)$ never gets close to filling the central part of a $d$-dimensional face of a cube $R \in \mathcal{R}(Q_k)$. In this case, the proof of Lemma 4.3 (where we just do an additional Federer-Fleming projection on the interior cubes) says that we can obtain a new mapping $\phi$ such that, in addition to the properties above, $\phi(E) \cap [Q_k \setminus A(Q_k)]$ is contained in a $(d-1)$-skeleton, where $A(Q_k)$ still denotes the union of the cubes $R \in \mathcal{R}(Q_k)$ that lie in the exterior layer. Compare with (4.5), and see the discussion in [DS4], below (∗4.22). Thus

$$\mathcal{H}^d(\phi(E) \cap [Q_k \setminus A(Q_k)]) = 0. \tag{4.57}$$

We may now use this $\phi$ to define a family $\{\varphi_t\}$ of mappings, by the same formula (4.16) as before. The properties (1.4)-(1.8) are verified as before (below (4.16)), using in particular (4.8), (4.40), and the fact that the cubes $R \in \mathcal{R}(Q_k)$ lie in the usual dyadic grid. So $\phi(E)$ is a sliding competitor for $E$ in some ball of radius $r = l_0\sqrt{n}/2$. We can apply (2.5), and we get that

$$\mathcal{H}^d(W_1) \leq M\mathcal{H}^d(\phi(W_1)) + hr^d \tag{4.58}$$

as in the first part of (4.18), and with $W_1 = \{x \in E\,; \phi(x) \neq x\}$. Since $\phi(x) = x$ out of $Q_k$ by (4.4), we get that $W_1 \subset Q_k$, and since $\phi(Q_k) \subset Q_k$ by (4.6), $\phi(W_1) \subset Q_k$ too. Hence

$$\mathcal{H}^d(\phi(W_1)) = \mathcal{H}^d(Q_k \cap \phi(W_1)) = \mathcal{H}^d(A(Q_k) \cap \phi(W_1)) \tag{4.59}$$

by (4.57). Denote by $A_1(Q_k)$ the union of the two exterior layers of cubes $R \in \mathcal{R}(Q_k)$. Then

$$A(Q_k) \cap \phi(W_1) \subset \bigcup_{R \in \mathcal{R}(Q_k)\,;\, R \subset A_1(Q_k)} \phi(R \cap W_1) \tag{4.60}$$

by (4.4) and (4.6), hence

$$\begin{aligned} \mathcal{H}^d(\phi(W_1)) = \mathcal{H}^d(A(Q_k) \cap \phi(W_1)) &\leq \sum_{R \in \mathcal{R}(Q_k)\,;\, R \subset A_1(Q_k)} \mathcal{H}^d(\phi(R \cap W_1)) \\ &\leq C \sum_{R \in \mathcal{R}(Q_k)\,;\, R \subset A_1(Q_k)} \mathcal{H}^d(R \cap E) \leq C\mathcal{H}^d(A_1(Q_k) \cap E) \end{aligned} \tag{4.61}$$

by (4.59) and (4.7). Note that $Q_{k+1} \subset Q_k \setminus A_1(Q_k)$ by construction; also, $\mathcal{H}^d$-almost every point $x \in E \cap Q_k \setminus A_1(Q_k)$ lies in $W_1$, by (4.57) (just notice that if $x \notin W_1$, then $x = \phi(x) \in \phi(E) \cap [Q_k \setminus A(Q_k)]$); hence

$$\mathcal{H}^d(E \cap Q_{k+1}) \leq \mathcal{H}^d(W_1) \leq M\mathcal{H}^d(\phi(W_1)) + hr^d \leq CM\mathcal{H}^d(A_1(Q_k) \cap E) + hr^d \tag{4.62}$$

by (4.58) and (4.61). Next,

$$\mathcal{H}^d(A_1(Q_k) \cap E) \leq \mathcal{H}^d(E \cap Q_k \setminus Q_{k+1}) = m_k - m_{k+1} \tag{4.63}$$

by (4.50), so (4.62) says that

$$m_{k+1} \leq CM(m_k - m_{k+1}) + hr^d, \tag{4.64}$$

hence

$$m_{k+1} \leq \frac{CM}{1+CM}\, m_k + \frac{hr^d}{1+CM}. \tag{4.65}$$

Now we want choose the $N_k$ and check the various constraints. We shall first choose $N_0$ dyadic and very large, depending on $a$ and $M$. We also set $N = N_0/16$ and

$$N_k = N_0 - 6k \ \text{ for } 1 \leq k \leq N. \tag{4.66}$$

This is probably far from optimal, but it will work. Observe that (4.53) is then satisfied, and in fact all the sets $\mathcal{R}(Q_k)$ that we construct will be composed of dyadic cubes of the same side. Our last cube is

$$Q_{N+1} = \Big[\prod_{j=0}^{N} \big(1 - \frac{6}{N_j}\big)\Big] Q_0 \tag{4.67}$$

by (4.51). Since $\frac{6}{N_j} \leq \frac{6}{N_N} \leq \frac{10}{N_0}$ by definition of $N$,

$$\sum_{j=0}^{N} \log\big(1-\frac{6}{N_j}\big) \geq (N+1)\log\big(1-\frac{10}{N_0}\big) \geq -\frac{1}{10} \tag{4.68}$$

if $N_0$ is large enough, and hence (because $e^{-1/10} > 1/2$) $Q_{N+1}$ contains $\frac{1}{2}Q_0$. Thus

$$\mathcal{H}^d(E \cap \frac{1}{2}Q_0) \leq m_{N+1}. \tag{4.69}$$

Now we check (4.55). Observe that for $k \geq 0$,

$$\begin{aligned} N_k^d m_k l_k^{-d} &\leq N_0^d m_0 l_k^{-d} \leq 2^d N_0^d m_0 l_0^{-d} \leq 2^d N_0^d \mathcal{H}^d(E \cap Q) l_0^{-d} \\ &\leq 2^d N_0^d \varepsilon_a l(Q)^d l_0^{-d} \leq 8^d \varepsilon_a N_0^d \end{aligned} \tag{4.70}$$

because $N_k \leq N_0$, $m_k \leq m_0$, $l_k \geq l_0/2$ (since $Q_{N+1}$ contains $\frac{1}{2}Q_0$), by (4.48) and (4.50), by (4.41), and because $l_0 \geq l(Q)/4$ by definition of $l_0$ (below (4.47)). If $\varepsilon_a$ is chosen small enough, depending on $M$ and $a$ through $N_0$ (see near (4.72) below for the choice of $N_0$), (4.70) implies (4.55), we can proceed as above, and (4.65) holds for $0 \leq k \leq N$. That is, if we set $\rho = \frac{CM}{1+CM} < 1$ and $\tau = \frac{hr^d}{1+CM}$, then $m_{k+1} \leq \rho m_k + \tau$ for $k \geq 0$, hence (by induction)

$$m_k \leq \rho^k m_0 + \tau(1+\rho+\rho^2+\ldots) \leq \rho^k m_0 + \frac{\tau}{1-\rho} = \rho^k m_0 + hr^d \tag{4.71}$$

for $0 \leq k \leq N+1$. If $N_0$ (and hence also $N = N_0/16$) is chosen large enough, depending on $a$ and $M$, we get that

$$\begin{aligned} \mathcal{H}^d(E \cap \frac{1}{2}Q_0) &\leq m_{N+1} \leq \rho^{N+1} m_0 + hr^d = \rho^{N+1}\mathcal{H}^d(E \cap Q_0) + hr^d \\ &\leq a\mathcal{H}^d(E \cap Q_0) + Chl_0^d \end{aligned} \tag{4.72}$$

by (4.69) and (4.71), and because $l_0 \geq l(Q)/4 \geq C^{-1}r$. So (4.49) holds and (4.42) follows (see below (4.49)). This completes the proof of Lemma 4.39 and also, as was explained just after the statement of the lemma, of Proposition 4.1. □ □

**Remark 4.73.** The author sees no obvious major obstruction to extending the proof above to the case where the rigid assumption is defined in terms of a net of polyhedra with some uniform size rotundity assumption (instead of dyadic cubes of size $2^{-m}$). Still, one would need to construct appropriate subnets, or at least adapt the construction of Federer-Fleming projections to objects that look like thin neighborhoods of a given polyhedron, but making sure that we preserve the faces of our initial net. The author does not claim that this would be pleasant.

Let us now state a local Ahlfors regularity result under the Lipschitz assumption, which will follow easily from Propositions 4.1 and 2.8.

**Proposition 4.74.** *For each choice of $\Lambda \geq 1$ and $M \geq 1$, we can find $h > 0$ and $C_M \geq 1$, depending on $\Lambda$, $M$, and the dimensions $n$ and $d$, so that the following holds. Suppose that $E \in GSAQ(U, M, \delta, h)$, and that the Lipschitz assumption is satisfied on $U$. Then if $x \in E^* \cap U$ and $0 < r < \mathrm{Min}(\lambda^{-1} r_0, \delta)$ are such that $B(x, 2r) \subset U$, we have that*

$$C_M^{-1} r^d \leq \mathcal{H}^d(E \cap B(x,r)) \leq C_M r^d. \tag{4.75}$$

As before, $r_0 = 2^{-m} \in (0,1]$ denotes the side length of the dyadic cubes used to define the rigid assumption, and $\Lambda$ and $\lambda > 0$ are the constants in the Lipschitz assumption (see Definition 2.7).

It is enough to prove (4.75) for slightly smaller balls, i.e. when

$$0 < r < \Lambda^{-2}\,\mathrm{Min}(\lambda^{-1} r_0, \delta) \ \text{ and } \ B(x, 2\Lambda^2 r) \subset U, \tag{4.76}$$

because if $B(x,r)$ is as in the original statement, a lower bound for $\mathcal{H}^d(E \cap B(x, \Lambda^{-2} r))$ implies a lower bound for $\mathcal{H}^d(E \cap B(x,r))$, and for the upper bound we may cover $B(x,r)$ by less than $C$ balls (centered on $E^*$ if we want) that satisfy the stronger condition (4.76).

So let $B(x,r)$ satisfy the stronger condition. Set $F = \psi(\lambda E)$; by Proposition 2.8, $F \in GSAQ(B(0,1), \Lambda^{2d} M, \Lambda^{-1}\lambda\delta, \Lambda^{2d} h)$, and Proposition 4.1 applies to that set; we shall just get a larger constant $C_{\Lambda^{2d}M}$ in (4.2). Set $y = \psi(\lambda x)$. Of course, $y \in F^*$ because $x \in E^*$, and

$$\mathrm{dist}(y, \partial B(0,1)) \geq \lambda\Lambda^{-1}\,\mathrm{dist}(x, \partial U) \geq 2\lambda\Lambda r, \tag{4.77}$$

by (4.76) and the bilipschitz property of $\psi$. Now $\psi(\lambda B(x,r)) \subset B(y, \lambda\Lambda r)$; let us check that we may apply Proposition 4.1 to $B = B(y, \lambda\Lambda r)$. The fact that $B(y, 2\lambda\Lambda r) \subset B(0,1)$ follows from (4.77), and

$$\lambda\Lambda r \leq \lambda\Lambda\Lambda^{-2}\,\mathrm{Min}(\lambda^{-1} r_0, \delta) \leq \mathrm{Min}(r_0, \Lambda^{-1}\lambda\delta) \tag{4.78}$$

by (4.76), so we may apply Proposition 4.1 to $B$ (or a smaller ball centered at $y \in F^*$). We get that

$$\begin{aligned} \mathcal{H}^d(E \cap B(x,r)) &\leq \lambda^{-d}\Lambda^d \mathcal{H}^d(\psi(\lambda(E \cap B(x,r)))) \leq \lambda^{-d}\Lambda^d \mathcal{H}^d(F \cap B(y, \lambda\Lambda r)) \\ &\leq \lambda^{-d}\Lambda^d C_{\Lambda^{2d}M} (\lambda\Lambda r)^d = \Lambda^{2d} C_{\Lambda^{2d}M}\, r^d \end{aligned} \tag{4.79}$$

by (4.2). This is the desired upper bound. For the lower bound, we observe that $\psi(\lambda B(x,r))$ contains $B(y, \lambda\Lambda^{-1} r)$, so

$$\begin{aligned} \mathcal{H}^d(E \cap B(x,r)) &\geq \lambda^{-d}\Lambda^{-d} \mathcal{H}^d(\psi(\lambda(E \cap B(x,r)))) \\ &\geq \lambda^{-d}\Lambda^{-d} \mathcal{H}^d(F \cap B(y, \lambda\Lambda^{-1} r)) \\ &\leq \lambda^{-d}\Lambda^{-d} C_{\Lambda^{2d}M}^{-1} (\lambda\Lambda^{-1} r)^d = \Lambda^{-2d} C_{\Lambda^{2d}M}^{-1} r^d \end{aligned} \tag{4.80}$$

by the lower bound in (4.2), applied to the smaller ball $B(y, \lambda\Lambda^{-1} r)$. This completes the proof of Proposition 4.74. □

## 5. Lipschitz mappings with big images, projections, and rectifiability.

In this section we extend two propositions from [DS4] and prove that quasiminimal sets are rectifiable.

The first proposition, Proposition 5.1 in [DS4] and here, concerns the existence of Lipschitz functions defined on a quasiminimal set, with values in $\mathbb{R}^d$ and with big images. The second one (Proposition 5.7 below) will concern the quasiminimality of a Lipschitz graph over a quasiminimal set. Both will be used to prove uniform rectifiability estimates in the next section. But we shall only be able to do this last under additional assumptions, so it makes sense to prove the plain rectifiability of quasiminimal sets here, because we can prove it in full generality (and the proof is much easier too). See Theorem 5.16 below. In the standard case without boundaries, the rectifiability was known from Almgren [A2], and the proof below is probably quite similar.

We start the section with the existence of Lipschitz functions with big images.

**Proposition 5.1.** *For each choice of $M \geq 1$, we can find $h > 0$ and $C_M \geq 1$, depending on $M$ and the dimensions $n$ and $d$, so that the following holds. Suppose that $E \in GSAQ(B_0, M, \delta, h)$, where we set $B_0 = B(0,1)$, and that the rigid assumption is satisfied. Let $r_0 = 2^{-m} \in (0,1)$ denote the side length of the dyadic cubes used to define the rigid assumption. Then for $x \in E^* \cap B_0$ and $0 < r < \mathrm{Min}(r_0, \delta)$ such that $B(x, 2r) \subset B_0$, we can find a $C_M$-Lipschitz mapping $F : E \cap B(x,r) \to \mathbb{R}^d$ such that*

$$\mathcal{H}^d(F(E \cap B(x,r))) \geq C_M^{-1} r^d. \tag{5.2}$$

By the bilipschitz invariance provided by Proposition 2.8, this result implies the corresponding one when the Lipschitz assumption holds; the argument, which goes as in Proposition 4.74, is left to the reader. Also, we could immediately reduce to the case when $E$ is coral, by Proposition 3.27 and because any Lipschitz mapping $F : E^* \cap B(x,r) \to \mathbb{R}^d$ with a big image could easily be extended to $E$. But this would not help much anyway.

The proof is a minor variation on what we did for the lower bound in Proposition 4.1. Let $N_0$ be a large power of 2, to be chosen soon, and let $l_0$ denote the largest power of 2 such that $l_0 \leq r/2n$. Let $Q'$ be a dyadic cube of length $l_0$, and choose a translation $Q_0$ of $Q'$ by an integer multiple of $N_0^{-1} l_0$, so that if $x_0$ denotes the center of $Q_0$, then the size of each coordinate of $|x - x_0|$ is at most $N_0^{-1} l_0$. Thus $Q_0 \subset B(x,r)$, and (if $h$ is small enough, depending on $M$ and $n$)

$$\mathcal{H}^d(E \cap \frac{1}{3} Q_0) \geq C^{-1} r^d, \tag{5.3}$$

by Proposition 4.1 and where $C$ depends only on $M$ and $n$. Next choose an integer $N \in [N_0/2, N_0]$ such that

$$\mathcal{H}^d(E \cap \frac{N}{N_0} Q_0 \setminus \frac{N-6}{N_0} Q_0) \leq \frac{12}{N_0} \mathcal{H}^d(E \cap Q_0) \leq \frac{C r^d}{N_0}\,; \tag{5.4}$$

the last inequality follows from (4.2), and such an $N$ exists because we have $N_0/2$ choices of $N$ and no point of $E$ lies in more than six annular regions $\frac{N}{N_0} Q_0 \setminus \frac{N-6}{N_0} Q_0$. We can apply a variant of Lemma 4.3 to the cube $Q = \frac{N}{N_0} Q_0$, with its natural decomposition into $N^n$ subcubes $R \in \mathcal{R}(Q)$, which are dyadic of side length $N_0^{-1} l_0$ and lie in the usual grid (recall that $N_0$ and $l_0$ are powers of 2, and that $l_0 \leq r/2n \leq r_0/2n$). This gives a first mapping $\phi : \mathbb{R}^n \to \mathbb{R}^n$, which preserves the faces of all dimensions as in (4.8).

We need a variant because this time we want to say that, in addition to the properties already mentioned in Lemma 4.3, $\phi$ is $C$-Lipschitz, with a constant $C$ that depends on $M$, $n$, and $d$, but not on $E$ or the $B(x,r)$. This can be arranged, because Proposition 4.1 says that $E \cap Q_0$ is semi-regular, which allows us to apply Lemma 3.31 in [DS4]. The proof is the same as in [DS4], and as before we observe that $N$ does not need to be a power of 2. The point of the argument comes when we need to choose points in various faces of the skeleton to perform Federer-Fleming projections centered at these points. The semi-regularity of $E \cap Q_0$ says that it, and its images by the previous Lipschitz mappings that were already constructed, is sufficiently far from dense in any face of dimension $\geq d+1$, so that we can find a new center that is far from it; then the Federer-Fleming projection is Lipschitz, with good bounds, and we can iterate as long as the faces are at least $(d+1)$-dimensional. Of course our bounds get worse and worse with each iteration (when the codimension is large), but this is all right.

If $\phi(E \cap Q)$ contains a full $d$-dimensional face $T$ of side length $N_0^{-1} l_0$, then we can take for $F$ an extension of $\pi \circ \phi$, where $\pi : \mathbb{R}^n \to \mathbb{R}^d$ is the composition of the orthogonal projection onto the vector space $V$ parallel to $T$, and a linear isometry from $V$ to $\mathbb{R}^d$. In this case, (5.2) holds just because

$$\mathcal{H}^d(F(E \cap B(x,r))) \geq \mathcal{H}^d(\pi \circ \phi(E \cap Q)) \geq \mathcal{H}^d(T) = N_0^{-d} l_0^d \geq C^{-1} r^d. \tag{5.5}$$

So we may assume that $\phi(E \cap Q)$ contains no full $d$-dimensional face of a cube $R \in \mathcal{R}(Q)$, and Proposition 5.1 will follow as soon as we derive a contradiction. We then proceed as we did near (4.56): we compose $\phi$ with an additional Federer-Fleming projection, which is obtained by selecting a center $c_T \in T \setminus \phi(E \cap Q)$ in each $d$-face $T$, and projecting on $\partial T$ from there. This gives a new mapping, which we shall still call $\phi$, and which satisfies the conclusions of Lemma 4.3, plus the fact that $\phi(E) \cap [Q \setminus A(Q)]$ is contained in a $(d-1)$-dimensional skeleton, where $A(Q)$ is again the exterior layer of $Q$. The same computations as in (4.57)-(4.62) yield the analogue of (4.62). Here $\frac{N-6}{N_0} Q_0 = \frac{N-6}{N} Q$ plays the role of $Q_{k+1} = (1 - \frac{6}{N_k}) Q_k$ (see (4.51)). Thus we get that, if $A_1(Q)$ denotes the union of the two exterior layers of $Q$,

$$\begin{aligned} \mathcal{H}^d(E \cap \frac{N-6}{N_0} Q_0) &\leq \mathcal{H}^d(W_1) \leq CM\mathcal{H}^d(E \cap A_1(Q)) + hr^d \\ &\leq CM\mathcal{H}^d(E \cap \frac{N}{N_0} Q_0 \setminus \frac{N-6}{N_0} Q_0) + hr^d \leq \frac{CMr^d}{N_0} + hr^d \end{aligned} \tag{5.6}$$

as before (i.e., because points of $E \cap \frac{N-6}{N_0} Q_0 \setminus W_1$ lie in some $(d-1)$-dimensional skeleton, by quasiminimality (the analogues of (4.58) and (4.61)), and by simple geometry), and then by (5.4).

But if $N_0$ is large enough and $h$ is small enough (both depending on $M$, $n$, and $d$), this contradicts (5.3). Proposition 5.1 follows. □

It would be nicer if the mapping $F$ provided by Proposition 5.1 were the orthonormal projection onto a $d$-space. The second proposition from [DS4] that we generalize here gives a trick that sometimes allows us to pretend that this is the case.

**Proposition 5.7.** *Let $U \subset \mathbb{R}^n$ be an open set, $E \in GSAQ(U, M, \delta, h)$ a quasiminimal set, and $F : U \to \mathbb{R}^m$ a Lipschitz function. Then $\widehat{E} = \big\{(x, F(x))\,;\, x \in E\big\} \in GSAQ(Unb \times \mathbb{R}^m, CM, \delta, Ch)$ for some constant $C$ that depends on $d$ and the Lipschitz constant for $F$, and where on $U \times \mathbb{R}^m$, $GSAQ$ is defined with respect to the boundaries $\widehat{L}_j = L_j \times \mathbb{R}^m$.*

This is a minor generalization of Proposition 6.1 in [DS4]. Explicit values for $C$ could easily be derived from the proof below, but we shall not bother to do so.

The main point of the proof is the following. We are given a competitor for $\widehat{E}$ in a ball $\widehat{B}$ and want to construct a competitor for $E$, to which we apply the definition of $GSAQ(U, M, \delta, h)$ to get estimates. More specifically, the competitor is $\widehat{\varphi}_1(\widehat{E})$ for some one-parameter family of mappings $\widehat{\varphi}_t : \widehat{E} \to \mathbb{R}^{n+m}$, with the properties (1.4)-(1.8). We want to define mappings $\varphi_t$, and we take

$$\varphi_t(x) = \pi \circ \widehat{\varphi}_t(x, F(x)) \ \text{ for } x \in E, \tag{5.8}$$

where $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^n$ denotes the natural projection onto $\mathbb{R}^n$.

Let us check that the $\varphi_t$ satisfy (1.4)-(1.8). First, (1.4) and (1.8) are trivial. For (1.5) and (1.6), we take $B = \pi(\widehat{B})$. If $x \in E \setminus B$, then $(x, F(x)) \in \widehat{E} \setminus \widehat{B}$, so $\widehat{\varphi}_t(x, F(x)) = (x, F(x))$ and $\varphi_t(x) = \pi(x, F(x)) = x$ by (5.8); similarly, when $t = 0$, $\widehat{\varphi}_t(x, F(x)) = (x, F(x))$ and $\varphi_t(x) = \pi(x, F(x)) = x$; thus (1.5) holds.

Next, if $x \in B$, then either $(x, F(x)) \in \widehat{E} \setminus \widehat{B}$, and then $\varphi_t(x) = \pi \circ \widehat{\varphi}_t(x, F(x)) = \pi(x, F(x)) = x \in B$ by (5.8) and (1.5), or else $(x, F(x)) \in \widehat{E} \cap \widehat{B}$ and $\widehat{\varphi}_t(x, F(x)) \in \widehat{B}$ by (1.6), so $\varphi_t(x) \in B$, as needed for (1.6). Finally, if $x \in E \cap L_j \cap B$, then $(x, F(x)) \in \widehat{E} \cap \widehat{L}_j$, so $\widehat{\varphi}_t(x, F(x)) \in \widehat{L}_j$ by (1.7) or (1.5), and $\varphi_t(x) = \pi \circ \widehat{\varphi}_t(x, F(x)) \in L_j$, as needed for (1.7). So $\varphi_t(E)$ is a sliding competitor for $E$ in $B$.

Now we check that

$$W_t \subset \pi(\widehat{W_t}) \ \text{ and } \varphi_t(W_t) \subset \pi(\widehat{\varphi}_t(\widehat{W_t})) \tag{5.9}$$

for $0 \leq t \leq 1$, where $W_t = \big\{y \in E \cap B\,;\, \varphi_t(y) \neq y\big\}$ is as in (2.1), and $\widehat{W_t} = \big\{z \in \widehat{E} \cap \widehat{B}\,;\, \widehat{\varphi}_t(z) \neq z\big\}$ is its analogue for $\widehat{\varphi}_t$. If $x \in W_t$, then $\pi \circ \widehat{\varphi}_t(x, F(x)) \neq x$, so in particular $\widehat{\varphi}_t(x, F(x)) \neq (x, F(x))$; then $(x, F(x)) \in \widehat{W_t}$ and $x \in \pi(\widehat{W_t})$. Moreover, $\varphi_t(x) = \pi \circ \widehat{\varphi}_t(x, F(x)) \in \pi(\widehat{\varphi}_t(\widehat{W_t}))$. So (5.9) holds, and the union of the sets $W_t \cup \varphi_t(W_t)$ is contained in the projection of the union of the $\widehat{W_t} \cup \widehat{\varphi}_t(\widehat{W_t})$. If this last union is relatively compact in $U \times \mathbb{R}^m$ (as in the assumption (2.4)), then the first union is relatively compact in $U$, which allows us to apply Definition 2.3 and get (2.5). That is,

$$\mathcal{H}^d(W_1) \leq M\mathcal{H}^d(\varphi_1(W_1)) + hr^d. \tag{5.10}$$

Set $H_1 = \big\{(x, F(x))\,;\, x \in W_1\big\}$. This is a subset of $\widehat{W_1}$, because $\pi \circ \widehat{\varphi}_1(x, F(x)) = \varphi_1(x) \neq x$ for $x \in W_1$. And

$$\mathcal{H}^d(H_1) \leq C\mathcal{H}^d(W_1) \leq CM\mathcal{H}^d(\varphi_1(W_1)) + Chr^d \leq CM\mathcal{H}^d(\widehat{\varphi}_1(\widehat{W_1})) + Chr^d \tag{5.11}$$

because $F$ is Lipschitz, by (5.9), and because $\pi$ is 1-Lipschitz. Now consider $H_2 = \widehat{W}_1 \setminus H_1$. First observe that

$$\mathcal{H}^d(H_2) \leq C\mathcal{H}^d(\pi(H_2)) \tag{5.12}$$

because $H_2$ is contained in $\widehat{E}$, which lies on the graph of the Lipschitz function $F$. In addition,

$$\pi \circ \widehat{\varphi}_1 = \pi \quad \text{on } H_2 \tag{5.13}$$

because, if $(x, F(x)) \in H_2$, then $x \in E \setminus W_1$, so $x = \varphi_1(x) = \pi \circ \widehat{\varphi}_1(x, F(x))$ by (5.8). Thus

$$\mathcal{H}^d(H_2) \leq C\mathcal{H}^d(\pi(H_2)) = C\mathcal{H}^d(\pi \circ \widehat{\varphi}_1(H_2)) \leq C\mathcal{H}^d(\widehat{\varphi}_1(H_2)) \leq C\mathcal{H}^d(\widehat{\varphi}_1(\widehat{W}_1)) \tag{5.14}$$

by (5.12) and because $H_2 \subset \widehat{W}_1$. Finally,

$$\mathcal{H}^d(\widehat{W}_1) \leq \mathcal{H}^d(H_1) + \mathcal{H}^d(H_2) \leq C(M+1)\mathcal{H}^d(\widehat{\varphi}_1(\widehat{W}_1)) + Chr^d \tag{5.15}$$

by (5.11) and (5.14), which is (2.5) for $\widehat{\varphi}_1(\widehat{E})$. Thus $\widehat{E}$ is a quasiminimal set, and this proves Proposition 5.7. □

We end this section with the fact that quasiminimal sets are rectifiable. Recall that this means that such a set $E$ is contained in a countable union of $d$-dimensional Lipschitz graphs (or $C^1$ surfaces, or Lipschitz images of $\mathbb{R}^d$, if you prefer), plus a set of vanishing $\mathcal{H}^d$-measure. Thus $E$ is rectifiable if and only if its core $E^*$ is rectifiable.

**Theorem 5.16.** *For each choice of $M \geq 1$, we can find $h > 0$, depending on $M$, the dimensions $n$ and $d$, and the bilipschitz constant $\Lambda$ of $\psi$ in the definition 2.7 of the Lipschitz assumption, such that if the Lipschitz assumption is satisfied in $U$ and $E \in GSAQ(U, M, \delta, h)$, then $E$ is rectifiable.*

Let $E \in GSAQ(U, M, \delta, h)$ be as in the statement, and let $\psi$ be the bilipschitz function in Definition 2.7. Since we know that any Lipschitz image of a rectifiable set is rectifiable (see 15.3 and 15.21 in [Ma], to which we shall refer for anything that concerns rectifiability), it will be enough to show that $\psi(\lambda E)$ is rectifiable (where $\lambda > 0$ is also as in Definition 2.7). But by Proposition 2.8, $\psi(\lambda E) \in GSAQ(B(0,1), \Lambda^{2d}M, \Lambda^{-1}\lambda\delta, \Lambda^{2d}h)$, with the rigid assumption. So it is enough to prove Theorem 5.16 when $U = B(0,1)$ and the rigid assumption holds.

Recall that $E$, just like any other set of locally finite $\mathcal{H}^d$-measure, can be written as the disjoint union $E = E_r \cup E_s$ of a rectifiable part $E_r$ and a totally non rectifiable (or singular) part $E_s$. We just need to show that $\mathcal{H}^d(E_s) = 0$, because the union of a rectifiable set and an $\mathcal{H}^d$-null set is rectifiable too. So we shall assume that $\mathcal{H}^d(E_s) > 0$ and derive a contradiction.

Since $\mathcal{H}^d(E)$ is locally finite and $E_r$ does not meet $E_s$, a standard density result (see for instance [Ma], Theorem 6.2 (2) on page 89) says that

$$\lim_{\rho \to 0} \rho^{-d}\mathcal{H}^d(E_r \cap B(x, \rho)) = 0 \tag{5.17}$$

for $\mathcal{H}^d$-almost every $x \in E_s$. In addition, $\mathcal{H}^d(E_s \setminus E^*) \leq \mathcal{H}^d(E \setminus E^*) = 0$, so we can pick $x \in E_s \cap E^*$ such that (5.17) holds.

We shall proceed as in the proof of Proposition 5.1. Let $N_0$ be a large power of 2 (a constant to be chosen soon), and let $l_0$ be a very small power of 2 (it will just need to be small enough, depending on $N_0$ too). Let $Q'$ be a dyadic cube of length $l_0$, and choose a translation $Q_0$ of $Q'$ by an integer multiple of $N_0^{-1} l_0$, so that if $x_0$ denotes the center of $Q_0$, then the size of each coordinate of $|x - x_0|$ is smaller than $N_0^{-1} l_0$ and so $E \cap \frac{1}{3} Q_0$ contains $B(x, \frac{l_0}{10})$. As in (5.3), if $h$ is small enough, depending on $M$ and $n$,

$$\mathcal{H}^d(E \cap \frac{1}{3} Q_0) \geq \mathcal{H}^d(E \cap B(x, \frac{l_0}{10})) \geq C^{-1} l_0^d, \tag{5.18}$$

by Proposition 4.1 and where $C$ depends only on $M$ and $n$. In fact, we do not even need Proposition 4.1 here; we could have chosen a point $x \in E$ where the upper density $\limsup_{r\to 0} r^{-d}\mathcal{H}^d(E \cap B(x, r))$ is larger than a geometric constant, and then taken $l_0$ as small as we want and such that (5.18) holds. We choose the integer $N \in [N_0/2, N_0]$ such that

$$\mathcal{H}^d\big(E \cap \frac{N}{N_0} Q_0 \setminus \frac{N-6}{N_0} Q_0\big) \leq \frac{12}{N_0} \mathcal{H}^d(E \cap Q_0) \leq \frac{C l_0^d}{N_0}, \tag{5.19}$$

as in (5.4) and with the same simple proof by Chebyshev. (And again we could also have obtained the last inequality because $\limsup_{r\to 0} r^{-d}\mathcal{H}^d(E \cap B(x, r)) \leq C$ almost everywhere on $E$.)

Then we want to apply the proof of Lemma 4.3 to the cube $Q = \frac{N}{N_0} Q_0$, with its natural decomposition into $N^n$ subcubes $R \in \mathcal{R}(Q)$, which are all dyadic of side length $N_0^{-1} l_0$ and lie in the usual grid if $l_0$ is small enough. [This time, we shall not need $\phi$ to be $C$-Lipschitz (as for Proposition 5.1), so we do not need the variant that uses the local Ahlfors-regularity of $E^*$.]

So we want to mimic the construction of $\phi$ in Lemma 4.3 (or rather Proposition 3.1 in [DS4]), but with a few changes because we want to project away the unrectifiable part. Our mapping $\phi$ will be obtained as the last element of a sequence $\phi_{n+1}, \phi_n, \phi_{n-1}, \cdots, \phi_d$, obtained recursively by composing with mappings $\psi_l$. That is, we shall start from $\phi_{n+1}(z) = z$ and set

$$\phi_l = \psi_l \circ \phi_{l+1} \quad \text{for } d \leq l \leq n. \tag{5.20}$$

For $R \in \mathcal{R}(Q)$ (the set of dyadic cubes $R \subset Q$ of side length $N_0^{-1} l_0$) and $0 \leq l \leq n$, denote by $\mathcal{S}_l(R)$ the union of all the $l$-dimensional faces of $R$. Also set $\mathcal{S}_l = \bigcup_{R\in\mathcal{R}(Q)} \mathcal{S}_l(R)$. We intend to choose the $\psi_l$ in such a way that

$$\phi_l(E \cap Q) \subset \mathcal{S}_{l-1} \cup \partial Q \tag{5.21}$$

for $d \leq l \leq n$.

Let us say how we do this. We start with $n \geq l > d$; the case of $l = d$ is a little special, and will be treated at the end. Assume that the $\psi_k$, $k > l$, were already defined, and satisfy (5.21). This is of course true when $l = n$. First we decide that

$$\psi_l(z) = z \quad \text{for } z \in \partial Q \cup [\mathbb{R}^n \setminus Q], \tag{5.22}$$

because we want (4.4) to hold. Since $\phi_{l+1}(E \cap Q) \subset \mathcal{S}_l \cup \partial Q$ (by (5.21) for $l+1$), the main thing to do now, if we want a definition of $\phi_l = \psi_l \circ \phi_{l+1}$ on $E$, is to define $\psi_l$ on $\mathcal{S}_l \cap Q$. We shall define $\psi_l$ simultaneously on all the $l$-dimensional faces $F$, in such a way that $\psi_l(z) = z$ on $\partial F$. Then there will be no problem about coherence.

So let $F$ be a $l$-dimensional face of a cube $R \in \mathcal{R}(Q)$. If $F \subset \partial Q$, we keep $\psi_l(z) = z$ on $F$ (because of (5.22)). Otherwise, we select an origin $x_F \in F \setminus \phi_{l+1}(E \cap Q)$, near its center. Other constrains will show up soon, but for the moment let us record that $\mathcal{H}^l$-almost every point $x_F$ is like this, because $\phi_{l+1}(E \cap Q)$ is at most $d$-dimensional (all our mappings are Lipschitz), and $l > d$. Notice that $x_F \in F \setminus \phi_{l+1}(E)$ too, because $\phi_{l+1}(E \setminus Q)$ lies far from the center of $F$ (by iterations of (5.22)).

Pick a small ball $B_F$ centered at $x_F$ and such that

$$\operatorname{dist}(B_F, \phi_{l+1}(E)) > 0. \tag{5.23}$$

The small size of $B_F$ will not matter, it will just make the Lipschitz constant for $\phi$ enormous, but we don't care. We decide that

(5.24) for $z \in F \setminus B_F$, $\psi_l(z)$ is the radial projection of $z$ on $\partial F$, centered at $x_F$.

This last just means that $\psi_l(z) \in \partial F$ and $z$ lies on the segment $[x_F, \psi_l(z)]$. With this choice, observe that if $z \in E \cap Q$, then by (5.21) for $l+1$, $\phi_{l+1}(z)$ either lies on $\partial Q$ (and then $\phi_l(z) = \phi_{l+1}(z) \in \partial Q$ by (5.22) and (5.20)), or else lies on some $F \setminus B_F$ (by (5.23)), so $\phi_l(z) = \psi_l(\phi_{l+1}(z)) \in \partial F$ by (5.24). Thus (5.21) holds for $l$.

Now we extend $\psi_l$ in a Lipschitz way, first to $F$ (so that $\psi_l(F) \subset F$), then (after we are finished with all the faces $F$) to faces $G$ of higher dimensions (so that $\psi_l(G) \subset G$ for every face $G$) and eventually the whole $Q$. Thus $\psi_l(Q) \subset Q$. One checks (see the proof in [DS4]) that these definitions give rise to Lipschitz mappings $\psi_l$, which satisfy (4.4)-(4.6), and also (4.8). For the remaining estimate (4.7) on the $\mathcal{H}^d(\phi(E \cap R))$, we need to be more careful about the choice of centers $x_F$, and this is also where we shall not proceed exactly as in [DS4].

We want to treat the rectifiable and singular parts of $E$ separately. We still intend to use Lemma 3.22 in [DS4], which goes as follows.

**Lemma 5.25.** *Let $F$ is an $l$-dimensional face of cube, with $l > d$, and $A \subset F$ a closed set such that $\mathcal{H}^d(A) < +\infty$. For $\xi \in \frac{1}{2}F$, denote by $\theta_{\xi,F} : F \setminus \{\xi\} \to \partial F$ the radial projection on $\partial F$ centered at $\xi$. Then*

$$\mathcal{H}^l(F)^{-1} \int_{\xi \in \frac{1}{2}F \setminus A} \mathcal{H}^d(\theta_{\xi,F}(A))\, d\mathcal{H}^l(\xi) \leq C\mathcal{H}^d(A). \tag{5.26}$$

As the proof will show, the lemma stays true if $A$ is merely Borel-measurable, but its closure has a finite $\mathcal{H}^d$-measure. The main point of the proof is that for a given $\xi \in \frac{1}{2}F \setminus \overline{A}$,

$$\mathcal{H}^d(\theta_{\xi,F}(A)) \leq C \int_A |x-\xi|^{-d} \operatorname{diam}(F)^d d\mathcal{H}^d(x), \tag{5.27}$$

which follows from computing the local Lipschitz constant of $\theta_{\xi,F}$ near $x$. See (3.24) and (3.20) in [DS4]. We integrate this over $\xi \in \frac{1}{2}F \setminus A$, use Fubini, and get that

$$\begin{aligned}
\int_{\xi\in\frac{1}{2}F\setminus A} \mathcal{H}^d(\theta_{\xi,F}(A)) d\mathcal{H}^l(\xi) &\leq C \operatorname{diam}(F)^d \int_{x\in A} \int_{\xi\in\frac{1}{2}F\setminus\overline{A}} |x-\xi|^{-d} d\mathcal{H}^l(\xi) d\mathcal{H}^d(x) \\
&\leq C \operatorname{diam}(F)^d \int_{x\in A} \Big\{ \int_{\xi\in F\cap B(x,2\operatorname{diam}(F))} |x-\xi|^{-d} d\mathcal{H}^l(\xi) \Big\} d\mathcal{H}^d(x) \\
&\leq C \operatorname{diam}(F)^d \int_{x\in A} \operatorname{diam}(F)^{l-d} d\mathcal{H}^d(x) \leq C \operatorname{diam}(F)^l \mathcal{H}^d(A);
\end{aligned} \tag{5.28}$$

(5.26) and the lemma follow. □

In [DS4] and for Proposition 4.1, Lemma 5.25 is used with $A = F \cap \phi_{l+1}(E)$ to choose $x_F$ so that, with the definitions (5.20) and (5.24),

$$\mathcal{H}^d(\psi_l(F \cap \phi_{l+1}(E))) \leq C\mathcal{H}^d(F \cap \phi_{l+1}(E)). \tag{5.29}$$

Such a choice is possible, by Fubini. Let us record here the fact that, by the proof of Lemma 5.25, we can even get that

$$\int_{F\cap\phi_{l+1}(E)} |x-x_F|^{-d} \operatorname{diam}(F)^d d\mathcal{H}^d(x) \leq C\mathcal{H}^d(F \cap \phi_{l+1}(E)), \tag{5.30}$$

which is stronger than (5.29) because of (5.27).

With this choice of $x_F$ for each $F$, we can sum over $F$, compose our mappings, and get that

$$\mathcal{H}^d(\phi_{d+1}(E \cap R)) \leq C\mathcal{H}^d(E \cap R) \quad \text{for } R \in \mathcal{R}(Q) \tag{5.31}$$

as in (4.7). The proof is the same as in [DS4] and for Proposition 4.1.

As we said earlier, here we want to take advantage of the fact that $x_F$ is chosen by a Fubini argument to apply Lemma 5.25 with $A = F \cap \phi_{l+1}(E_r)$ and get, in addition to (5.29), that

$$\mathcal{H}^d(\psi_l(F \cap \phi_{l+1}(E_r))) \leq C\mathcal{H}^d(F \cap \phi_{l+1}(E_r)), \tag{5.32}$$

and finally obtain, after composing, that

$$\mathcal{H}^d(\phi_{d+1}(E_r \cap R)) \leq C\mathcal{H}^d(E_r \cap R) \quad \text{for } R \in \mathcal{R}(Q). \tag{5.33}$$

We sum this and get that

$$\mathcal{H}^d(\phi_{d+1}(E_r \cap Q)) \leq \sum_{R \in \mathcal{R}(Q)} \mathcal{H}^d(\phi_{d+1}(E_r \cap R)) \leq C\mathcal{H}^d(E_r \cap Q) \tag{5.34}$$

because the cubes $R \in \mathcal{R}(Q)$ have bounded overlap.

Let $\varepsilon > 0$ be very small, to be chosen soon. Because $x$ was chosen so that (5.17) holds, we deduce from (5.34) that

$$\mathcal{H}^d(\phi_{d+1}(E_r \cap Q)) \leq \varepsilon l_0^d \tag{5.35}$$

if $l_0$ was chosen small enough.

For the unrectifiable part $E_s$ of $E$, we use the following fact, which is proved in Lemma 4.3.3 on page 111 of [Fv1] or Lemma 6 on page 26 of [Fv3]. If $F$ is a face of dimension $l > d$, and if $A \subset F$ is such that $\mathcal{H}^d(A) < +\infty$ and $A$ is totally non rectifiable (of dimension $d$), then for almost every choice of $x_F$, $\psi_l(A) \subset \partial F$ is also totally non rectifiable. Of course we choose the various $x_F$ so that this happens (we had some latitude left to do this); then when we compose the $\psi_l$ we get that

$$\phi_{d+1}(E_s \cap Q) \text{ is totally non rectifiable.} \tag{5.36}$$

All this information is valid also on the cubes $R$ that meet $\partial Q$; we concentrated on what happens on faces $F$ that are not contained in $\partial Q$, but on $\partial Q$ we simply need to know that all our mappings are the identity.

Next, if $F$ is any $d$-dimensional face of a cube $R \in \mathcal{R}(Q)$, then

$$\mathcal{H}^d(F \cap \phi_{d+1}(E \cap Q)) = \mathcal{H}^d(F \cap \phi_{d+1}(E_r \cap Q)) \leq \varepsilon l_0^d, \tag{5.37}$$

because the totally non rectifiable set $\phi_{d+1}(E_s \cap Q)$ can only meet the rectifiable set $F$ on a $\mathcal{H}^d$-null set, and by (5.35).

Thus $\phi_{d+1}(E \cap Q)$ never fills a $d$-face $F$ (if $\varepsilon$ is small enough), and this allows us to choose, for each $d$-dimensional face $F$ in $Q$ which is not contained in $\partial Q$, a point $x_F$ near the center of $F$ that does not lie on $\phi_{d+1}(E \cap Q)$. We then choose $B_F$ and define $\psi_d$ as we did above, near (5.23). This gives a last mapping $\phi_d = \psi_d \circ \phi_{d+1}$, which still satisfies (5.21).

We shall need in a later section to know that if $\tau > 0$ is small enough (depending also on our choice of mappings $\phi_l$ and their bad Lipschitz constants), and if $H \subset Q$ is a compact set such that

$$\mathrm{dist}(z, E) \leq \tau \quad \text{for } z \in H, \tag{5.38}$$

then

$$\phi_l(H \cap Q) \subset \mathcal{S}_{l-1} \cup \partial Q \quad \text{for } n+1 \geq l \geq d. \tag{5.39}$$

We naturally prove this by descending induction. Obviously this is true for $l = n+1$, because $\phi_{n+1}(z) = z$ and $\mathcal{S}_n = Q$. Let $l \geq d$ be given, and suppose that (5.39) holds for $l+1$. Let $z \in H \cap Q$ be given; by induction assumption, $\phi_{l+1}(z) \in \mathcal{S}_l \cup \partial Q$. If $\phi_{l+1}(z) \in \partial Q$, (5.22) and (5.20) say that $\phi_l(z) = \phi_{l+1}(z) \in \partial Q$, so we may assume that $\phi_{l+1}(z)$ lies in some $l$-face $F$ that is not contained in $\partial Q$. We know, since $\phi_{l+1}$ is Lipschitz (and $\tau$ is as small as we want) that $\phi_{l+1}(z)$ is arbitrarily close to $\phi_{l+1}(E)$, so (5.23) says that $\phi_{l+1}(z)$ lies out of $B_F$, and hence $\psi_l(z)$ is given by (5.24). Then $\phi_l(z) = \phi_{l+1}(z) \in \partial F$, as before, and (5.39) holds for $l$ too. This proves (5.39).

Return to $E$, and set $\phi^* = \phi_d$ (we write $\phi^*$ instead of $\phi$ to make sure that the $\phi_t^*$ below will not be confused with the $\phi_l$ above). Thus

$$\phi^*(E \cap Q) = \phi_d(E \cap Q) \subset \mathcal{S}_{d-1} \cup \partial Q \tag{5.40}$$

by (5.21) with $l = d$. Now we use the quasiminimality of $E$ to get a contradiction. It is easy to construct a one-parameter family $\{\phi_t^*\}$, that satisfies (1.4)-(1.8), and for which $\phi_1^* = \phi^*$; the verification is the same as for Proposition 4.1, for instance near (4.16). Set

$$W_t = \big\{y \in \mathbb{R}^n\,; \varphi_t^*(y) \neq y\big\} \tag{5.41}$$

for $0 < t \leq 1$ and

$$\widehat{W} = \bigcup_{0<t\leq 1} W_t \cup \varphi_t^*(W_t); \tag{5.42}$$

these are well defined here because the mappings $\varphi_t^*$ are defined everywhere. We can easily arrange the interpolation between the identity and $\phi^*$ so that $\phi_t^*(z) = z$ for $z \in \mathbb{R}^n \setminus Q$ and $\phi_t^*(Q) \subset Q$, and so we get that

$$\widehat{W} \subset Q \subset Q_0 \subset \overline{B}(x, 2\sqrt{n}l_0) \tag{5.43}$$

(see the definition of $Q_0$ and $Q$ near (5.18) and (5.19)). If $l_0$ is chosen small enough, $\overline{B}(x, 2\sqrt{n}l_0)$ is arbitrarily small and contained in $U$, so we can apply Definition 2.3. We get that

$$\mathcal{H}^d(E \cap W_1) \leq M\mathcal{H}^d(\phi^*(E \cap W_1)) + hr^d. \tag{5.44}$$

Denote by $\mathcal{R}_{ext}$ the collection of small cubes $R \in \mathcal{R}(Q)$ that touch $\partial Q$ (that is, the $\mathcal{R}_{ext}$ is the outer layer of cubes in $\mathcal{R}(Q)$). Then set

$$Q' = \bigcup_{R \in \mathcal{R}(Q) \setminus \mathcal{R}_{ext}} R. \tag{5.45}$$

Recall from the definition of $Q$ below (5.19) that

$$Q = \frac{N}{N_0}\, Q_0 \ \text{ for some integer } N \in [N_0/2, N_0]. \tag{5.46}$$

Also, the side length of our cubes $R \in \mathcal{R}(Q)$ is $N_0^{-1} l_0$, so

$$Q' = \frac{N-2}{N}\, Q = \frac{N-2}{N_0}\, Q_0 \supset \frac{1}{3}\, Q_0 \tag{5.47}$$

because $N_0$ is very large. Let us check that

$$E \cap Q' \setminus W_1 \subset \mathcal{S}_{d-1}. \tag{5.48}$$

Let $z \in E \cap Q' \setminus W_1$ be given. Then $z = \phi^*(z) \in \mathcal{S}_{d-1}$ by (5.41), (5.40), and because $z \notin \partial Q$; (5.48) follows. Then

$$\mathcal{H}^d(E \cap W_1) \geq \mathcal{H}^d(E \cap Q' \cap W_1) = \mathcal{H}^d(E \cap Q') \geq H^d(E \cap \frac{1}{3} Q_0) \geq C^{-1} l_0^d \tag{5.49}$$

by (5.48), (5.47), and (5.18). On the other hand, by (5.40) (and the first half of (4.4)),

$$\mathcal{H}^d(\phi^*(E \cap W_1)) = \mathcal{H}^d(\partial Q \cap \phi^*(E \cap W_1)). \tag{5.50}$$

Let us check that

$$\partial Q \cap \phi^*(E \cap W_1) \subset \bigcup_{R \in \mathcal{R}_{ext}} \phi^*(E \cap R), \tag{5.51}$$

where $\mathcal{R}_{ext}$ still denotes the outer rim of small cubes $R \in \mathcal{R}(Q)$ that touch $\partial Q$. Let $w \in \partial Q \cap \phi^*(E \cap W_1)$ be given, and let $z \in E \cap W_1$ be such that $\phi^*(z) = w$. Observe that $z$ lies out of $Q'$, because (4.6) says that $\phi^*(Q') \subset Q'$. So $z \in E \cap R$ for some $R \in \mathcal{R}_{ext}$, and (5.51) follows. Next we verify that for $R \in \mathcal{R}_{ext}$,

$$\mathcal{H}^d(\phi^*(E \cap R) \setminus \phi_{d+1}(E \cap R)) = 0. \tag{5.52}$$

Let $w \in \phi^*(E \cap R)$ be given, and choose $z \in E \cap R$ such that $w = \phi^*(z)$. Recall that $w = \phi^*(z) = \phi_d(z) = \psi_d(\phi_{d+1}(z))$ by definition of $\phi^*$ and (5.20). By (5.21), $\phi_{d+1}(z) \in \mathcal{S}_d \cup \partial Q$. If $\phi_{d+1}(z) \in \partial Q$, then $\psi_d$ does not move it (by (5.22)), and so $w = \psi_d(\phi_{d+1}(z)) = \phi_{d+1}(z)$, which is fine for (5.52). Otherwise, $\phi_{d+1}(z)$ lies on some $d$-dimensional face $F$ that is not contained in $\partial Q$, and by construction its image by $\psi_d$ (that is, $w$) lies on $\partial F$, which is $(d-1)$-dimensional. So (5.52) holds. Altogether,

$$\begin{aligned} \mathcal{H}^d(\phi^*(E \cap W_1)) &\leq \sum_{R \in \mathcal{R}_{ext}} \mathcal{H}^d(\phi^*(E \cap R)) \leq \sum_{R \in \mathcal{R}_{ext}} \mathcal{H}^d(\phi_{d+1}(E \cap R)) \\ &\leq C \sum_{R \in \mathcal{R}_{ext}} \mathcal{H}^d(E \cap R)) \leq C\mathcal{H}^d(E \cap Q \setminus \mathrm{int}(Q')) \end{aligned} \tag{5.53}$$

by (5.50), (5.51), (5.52), (5.31), and the fact that the cubes $R$ have bounded overlap. Since

$$Q \setminus \mathrm{int}(Q') = \frac{N}{N_0}\, Q_0 \setminus \mathrm{int}\Big(\frac{N-2}{N_0}\, Q_0\Big) \tag{5.54}$$

by (5.46) and (5.47), it follows from (5.53), (5.54), and (5.19) that

$$
\begin{aligned}
\mathcal{H}^d(\phi^*(E\cap W_1)) &\le C\mathcal{H}^d(E\cap Q\setminus \mathrm{int}(Q')) \\
&\le C\mathcal{H}^d\Big(E\cap \frac{N}{N_0}\,Q_0\setminus \mathrm{int}\Big(\frac{N-2}{N_0}\,Q_0\Big)\Big) \le C\,\frac{l_0^d}{N_0}.
\end{aligned}
\tag{5.55}
$$

If $N_0$ is large enough and $h$ is small enough (depending on $M$ in particular), we get a contradiction with (5.44) or (5.49); thus we could not find our initial point of density $x\in E_s$, and the rectifiability of $E$ follows. This completes our proof of Theorem 5.16. □

PART III : UNIFORM RECTIFIABILITY OF QUASIMINIMAL SETS

This part is largely independent from the next ones, which is probably a good thing because we shall only be able to complete the desired program in some specific cases, depending on the dimensions of the faces of the $L_j$.

The main goal is to prove that sliding quasiminimal sets are locally uniformly rectifiable, with big pieces of Lipschitz graphs.

When we wrote the long paper [DS4], and even for later results, the author thought that the local uniform rectifiability of $E$ was an unavoidable main step for many things, including the stability of our classes of minimizers under limits (as in Part IV below). As we shall see later, this is not the case, and the proof of rectifiability is enough for many purposes.

This is fortunate, because we shall not be able to prove the local uniform rectifiability of $E$ in all the interesting cases, and also because even when it works, the proof is more difficult than usual.

We nonetheless include a part on uniform rectifiability here because the author cannot deny his past, and it is a nice regularity property. It is probably almost the best general result that we can hope to prove for quasiminimal sets. That is, because quasiminimality is bilipschitz invariant (or directly), Lipschitz graphs are quasiminimal, and uniformly rectifiable sets are not so different (in terms of regularity) from Lipschitz graphs. Even for almost minimal or minimal sets, it is not so clear how to get better general regularity results (i.e., that would hold without assuming some a priori flatness, for instance), even though in this case we expect better regularity.

We continue with the same general writing style as in Part II, i.e., giving a rapid general description of [DS4], except at places where modifications are needed (and then we need to be more precise).

**6. Local uniform rectifiability in some cases.**

So we want to prove that sliding quasiminimal sets are locally uniformly rectifiable, with big pieces of Lipschitz graphs, and we shall only be able to do this under an additional assumptions on the dimensions of the faces of the $L_j$. The main result of this section and the next two is the following theorem, and its generalization (Theorem 9.81) under the Lipschitz assumption.

**Theorem 6.1.** *For each choice of $M\ge 1$, we can find $h>0$, $A\ge 0$, and $\theta>0$, depending on $M$ and the dimensions $n$ and $d$, so that the following holds. Suppose that*

*$E \in GSAQ(B_0, M, \delta, h)$, where we set $B_0 = B(0,1)$, and that the rigid assumption is satisfied. Let $r_0 = 2^{-m} \leq 1$ denote the side length of the dyadic cubes used to define the rigid assumption. Let $x \in E^* \cap B_0$ and $0 < r < \mathrm{Min}(r_0, \delta)$ be such that $B(x, 2r) \subset B_0$. Assume in addition that*

$$\text{if } j \in [0, j_{max}] \text{ is such that some face of dimension (strictly) more than } d \text{ of } L_j \text{ meets } B(x,r), \text{ then } E^* \cap B(x,r) \subset L_j. \tag{6.2}$$

*Then we can find a $d$-dimensional $A$-Lipschitz graph $\Gamma \subset \mathbb{R}^n$ such that*

$$\mathcal{H}^d(E \cap \Gamma \cap B(x,r)) \geq \theta r^d. \tag{6.3}$$

By $d$-dimensional $A$-Lipschitz graph, we mean a set $\Gamma$ which is the image, under an isometry of $\mathbb{R}^n$, of the graph of some Lipschitz function from $\mathbb{R}^d$ to $\mathbb{R}^{n-d}$ whose Lipschitz norm is at most $A$. Notice that we do not have so much of a restriction on dimensions when $d = 2$ and $n = 3$, which will probably be our main interest in the future (but even so we do not allow $L_1$ to be a half space in which $E$ is not contained). Also, Theorem 6.1 does not necessarily apply when $d = 2$, $n = 4$, and some $L_j$ are 3-dimensional.

The author does not know whether this additional restriction on the dimensions is really needed.

The restrictions in Theorem 6.1 do not seem too bad, for instance because they allow boundary constraints given by sets $L_j$ of dimensions at most $d$, and the typical setting for a Plateau problem is like this. But in terms of proof, Theorem 6.1 is rather disappointing because it does not contain much more information than what is readily available from the interior uniform rectifiability (away from the $L_j$). For instance, if all the $L_j$ are at most $(d-1)$-dimensional, the local uniform rectifiability of $E^*$ near the $L_j$ follows from the inside uniform rectifiability and the local Ahlfors-regularity given by Proposition 4.1 (there is just not enough room near the $L_j$ for a bad behavior). We will be able to obtain more cases (for instance, increase the dimension of the $L_j$ by one) by various general tricks, but the center of the proof is still the result from [DS4]. That is, a simpler special case will be obtained in Proposition 6.41, with a minor modification of the argument of [DS4], and then the extension of this result that we do in Sections 7 and 8 will mostly use general manipulations of uniform rectifiability and Carleson measures, and for instance we shall only construct competitors once, in Lemma 7.38 or its generalization Lemma 9.14. It would be nice to have a different, simpler proof of the uniform rectifiability of $E^*$ away from the $L_j$, but for moment we only know one (very complicated) proof.

Here is our plan for the rest of this part. We shall start this section with a rapid description of the proof of local uniform rectifiability given in [DS4]. We shall then say (largely for the record) why it does not seem to go through with our sliding conditions. In the last subsection, we prove a weaker variant of Theorem 6.1, Proposition 6.41, which is what we can almost directly obtain from the proof of [DS4]. In Section 7, we shall prove the conclusion of Proposition 6.41 (the existence of a big piece of bilipschitz image of a subset of $\mathbb{R}^d$) under the weaker assumptions of Theorem 6.1; see Proposition 7.85. Theorem 6.1 itself will only be proved in full in Section 8, with a small additional argument on the existence of big projections. Finally, Theorem 6.1 will be generalized to the case of Lipschitz assumption in Section 9. See Theorem 9.81.

**6.a. How we want to proceed, following [DS4]**

So we are given a quasiminimal set $E$ and a ball $B(x,r)$, as in the statement of Theorem 6.1. Because of Proposition 3.3 (which says that $E^*$ is quasiminimal too), we can assume that $E$ is coral (i.e., $E = E^*$); otherwise just prove and apply the theorem for $E^*$.

We first use Proposition 5.1 to find a $C_M$-Lipschitz mapping $F_r : E \cap B(x,r) \to \mathbb{R}^d$ such that $\mathcal{H}^d(F_r(E \cap B(x,r))) \geq C_M^{-1} r^d$, as in (5.2). By Whitney's extension theorem, we can extend $F_r$ into a $CC_M$-Lipschitz mapping defined on $\mathbb{R}^n$.

Next we apply Proposition 5.7, which says that $\widehat{E}$, the graph of $F_r$ over $E$, is a quasiminimal set in $\mathbb{R}^{n+d}$. We shall denote by $\pi_x : \mathbb{R}^{n+d} \to \mathbb{R}^n$ and $\pi : \mathbb{R}^{n+d} \to \mathbb{R}^d$ the two natural projections, and consider the smaller set $\widehat{E}_0 = \widehat{E} \cap \pi_x^{-1}(B(x,r))$. Then $\pi(\widehat{E}_0) = F_r(E \cap B(x,r))$ and

$$\mathcal{H}^d(\pi(\widehat{E}_0)) = \mathcal{H}^d(F_r(E \cap B(x,r))) \geq C_M^{-1} r^d. \tag{6.4}$$

Next we want to use a stopping time argument from [D1] to find a large piece of $\widehat{E}_0$ where $\pi$ is bilipschitz. More precisely, we want to find a closed set $\widehat{\Gamma}_0 \subset \widehat{E}_0$ such that

$$\mathcal{H}^d(\widehat{\Gamma}_0) \geq \theta' r^d \ \text{ and } \ |y - z| \leq A' |\pi(y) - \pi(z)| \text{ for } y, z \in \widehat{\Gamma}_0, \tag{6.5}$$

where $\theta' > 0$ and $A'$ are constants that depend only on $n$, $d$, and $M$.

If we do so, this will not directly give a big piece of Lipschitz graph in $E \cap B(x,r)$, as required in the statement of Theorem 6.1, but the following weaker conclusion: there is a closed set $G_0 \subset E \cap B(x,r)$ and a mapping $\phi : G_0 \to \mathbb{R}^d$ such that

$$\mathcal{H}^d(G_0) \geq \theta r^d \ \text{ and } \ C_M' |y - z| \leq |\phi(y) - \phi(z)| \leq C_M' |y - z| \text{ for } y, z \in G_0, \tag{6.6}$$

where $\theta$ and $C_M'$ depend only on $n$, $d$, and $M$. In other words, instead of a big piece of Lipschitz graph in $E \cap B(x,r)$, we only find a big piece of bilipschitz image of a subset of $\mathbb{R}^d$.

The verification (from (6.5)) is easy: we just try $G_0 = \pi_x(\widehat{\Gamma}_0)$; then (6.6) follows from (6.5) because $\pi_x : \widehat{E} \to E$ is bilipschitz.

In the terminology of [DS1] or [DS3], (6.6) (for all $x$ and $r$) says that locally, $E$ has big pieces of bilipschitz images of $\mathbb{R}^d$ (BPBI), which amounts to saying that $E$ is locally uniformly rectifiable, while in the statement of Theorem 6.1 we claim that if also contains big pieces of Lipschitz graphs (BPLG) locally.

Now we can go from BPBI to BPLG by a general argument on uniformly rectifiable sets, for which we just need to check that $E$ also has "big projections". This will be discussed soon, but anyway the most important part of Theorem 6.1 is the local uniform rectifiability provided by (6.6).

Return to $\widehat{E}_0 = \widehat{E} \cap \pi_x^{-1}(B(x,r))$, our quest of $\widehat{\Gamma}_0 \subset \widehat{E}_0$ such that (6.5) holds, and the stopping time argument from [D1]. We would like to use the proof described in Sections 8 and 9 of [DS4], which we try to explain now.

A first ingredient of the proof is the construction of what we call cubical patchworks on $\widehat{E}$, which are the analogue on $\widehat{E}$ of the standard dyadic cubes on $\mathbb{R}^n$, and which will be very useful because we want to run stopping time arguments on $\widehat{E}$. This construction is done in Section 7 of [DS4], and goes through in the present setting because it only uses the local Ahlfors-regularity of $E$ near $x$. This last holds as soon as $h$ is small enough (depending on $n$ an $M$), by Proposition 4.1 and because we assumed that $E = E^*$. Naturally, we shall always assume that this ($h$ small enough) is the case. Let us say what the cubical patchwork is in the situation of Theorem 6.1. We get a set $F$ and collections $\Sigma_j$, $j \geq 0$, of so-called dyadic cubes, with the following properties. First,

$$\widehat{E} \cap B(\widehat{x}, r/10) \subset F \subset \widehat{E} \cap B(\widehat{x}, r) \subset \widehat{E}_0, \tag{6.7}$$

where we call $\widehat{x} = (x, F_r(x))$ the natural center for $\widehat{E}_0$, and $F$ also is locally Ahlfors-regular, in the sense that

$$C^{-1}t^d \leq \mathcal{H}^d(F \cap B(y,t)) \leq Ct^d \quad \text{for } y \in F \text{ and } 0 < t < r. \tag{6.8}$$

For each $j \geq 0$, $\Sigma_j$ is a collection of measurable subsets $Q$ of $F$, which we shall abusively call cubes, such that $F$ is the disjoint union of the cubes $Q$, $Q \in \Sigma_j$. The cubes have some low regularity properties, and particular they have a center $c_Q$ such that

$$F \cap B(c_Q, C^{-1}2^{-j}r) \subset Q \subset F \cap B(c_Q, C2^{-j}r) \quad \text{for } Q \in \Sigma_j. \tag{6.9}$$

They also have small boundaries (see (7.4) and (7.10) in [DS4]), but we shall not use this here. Finally, the $\Sigma_j$ have the same structure as for the usual dyadic cubes: if $i \leq j$, $Q \in \Sigma_i$, and $R \in \Sigma_j$, then $R \subset Q$ or else $R \cap Q = \emptyset$.

The main property that we need to prove if we want to get (6.5) is a little complicated, and involves a (given) large constant $C_1$, a (given) small constant $\gamma$, and constants $C_2$ (very large) and $\eta$ (very small), to be chosen (depending on $C_1$, $\gamma$, $M$, and $n$). For $y \in F$ and $j \geq 0$, set

$$T_j(y) = \bigcup_{Q \in \Sigma_j \,;\, Q \cap B(y, C_2 2^{-j}r) \neq \emptyset} Q. \tag{6.10}$$

Thus $T_j(y)$ is a little bit like $F \cap B(y, C_2 2^{-j}r)$, but we prefer to cut neatly along dyadic cubes. The stopping time argument from [D1] that we want to use likes the situations (depending on $y \in F$ and $j \geq 0$) when our projection $\pi : \mathbb{R}^{n+d} \to \mathbb{R}^d$ has a local surjectivity property, namely when

$$\pi(T_j(y)) \supset \mathbb{R}^d \cap B(\pi(y), C_1 2^{-j}r). \tag{6.11}$$

It also likes it when there is a cube $R \subset T_j(y)$ that does significantly better than average in terms of projections, i.e., when

(6.12)

$$\text{there exists } R \in \Sigma_j \text{ such that } R \subset T_j(y) \text{ and } \quad \frac{\mathcal{H}^d(\pi(R))}{\mathcal{H}^d(R)} \geq (1+2\eta)\,\frac{\mathcal{H}^d(\pi(T_j(y)))}{\mathcal{H}^d(T_j(y))}.$$

The property that makes things work in [DS4] is the following.

**Definition 6.13.** *We say that the main lemma holds if for each choice of $C_1$ and $\gamma > 0$, we can find $C_2$ and $\eta > 0$, depending on $C_1$, $\gamma$, $M$, $n$, and $d$ (which includes a dependence on the local Ahlfors-regularity and cubical patchwork constants) such that, whenever $y \in F$ and $j \geq 0$ are such that*

$$B(y, 2C_2 2^{-j} r) \subset B(\widehat{x}, r/10) \tag{6.14}$$

*and*

$$\frac{\mathcal{H}^d(\pi(T_j(y)))}{\mathcal{H}^d(T_j(y))} \geq \gamma, \tag{6.15}$$

*then we have (6.11) or (6.12).*

This property is proved in [DS4], as Main Lemma 8.7. The fact that it allows us to apply a theorem from [D1] and get a graph $\widehat{\Gamma}_0$ as in (6.5) is proved in Section 8 of [DS4], and the proof goes through without major modification in the present context. [Again, it only uses the local Ahlfors regularity properties of $E$, and no construction of competitors.]

Thus we want to know whether the main lemma holds in the context of sliding minimizers, and we study the proof given in Section 9 of [DS4].

We assume that we can find $y \in F$ such that (6.14) and (6.15) hold, but not (6.11) or (6.12), and we want to reach a contradiction (for a correct choice of $C_2$ and $\eta$). That is, we want to construct an appropriate deformation of $\widehat{E}$ (which is a quasiminimal set by Proposition 5.7), for which most of the measure near $x$ disappears. A first step in the verification, which is done in Section 9-2 of [DS4], consists in obtaining the following description of $F$ (or equivalently $\widehat{E}$) near $y$.

As in (9.62) of [DS4], we apply a dilation to all our sets so that

$$2C_1 2^{-j} r = 1; \tag{6.16}$$

this normalization will allow us to work with (standard!) dyadic cubes of unit side length in the $d$-plane $P = \mathbb{R}^d$. We can also assume that $y = 0$. Still denote by $\pi$ the orthogonal projection on $P = \mathbb{R}^d$, and by $\pi_x$ the orthogonal projection on $V = \mathbb{R}^n$ (in [DS4] it is called $h$, but we want to avoid a conflict of notation here).

We shall restrict our attention to the box $P_0 \times V_0$, where $P_0 = [-N, N]^d \subset P$ for some large integer $N$, and $V_0 = V \cap \overline{B}(0, \rho_0)$ for some $\rho_0 \in [N, CN]$. Here $N$ will be chosen very large, depending on $C_1$, $\gamma$, $M$, $n$, and $d$, and $C$ is so large (depending on the same constants), that a Chebyshev argument allows us to choose $\rho_0 \in [N, CN]$ so that

$$\mathrm{dist}(P_0 \times (V \cap \partial B(0, \rho_0)), F) \geq N. \tag{6.17}$$

[See (9.77) in [DS4], and we won't need to modify this part of the argument.] Later on, we shall choose $C_2$ (depending also on $N$), so large that

$$P_0 \times V_0 \subset B(y, C_2 2^{-j-1} r) \subset B(\widehat{x}, r/11); \tag{6.18}$$

the first part is easy to arrange (because $y = 0$ and by the normalization (6.16)), and the second inclusion comes from (6.14). Now set

$$F_0 = F \cap (P_0 \times V_0) = \widehat{E} \cap (P_0 \times V_0) \subset \widehat{E}_0, \tag{6.19}$$

where the last part comes from (6.7) and (6.18). Notice that

$$F_0 \subset T_j(y) \tag{6.20}$$

by (6.18) and (6.10), so it will be easy to use our assumption that (6.11) and (6.12) fail.

Denote by $A_i$, $i \in I_0$, the collection of cubes in $P$, contained in $P_0$, that are obtained from the unit cube $[0,1]^d$ by an integer translation in $\mathbb{Z}^d$. That is, we cut $P_0$ into $(2N)^d$ dyadic unit cubes (and the point of the normalization above is that we can use unit cubes).

We finally come to our description of $F_0$. First set

$$I_1 = \big\{ i \in I_0\,;\ \text{there is an } x_i \in int(A_i) \text{ such that } F_0 \cap \pi^{-1}(x_i) = \emptyset \big\}, \tag{6.21}$$

and let us check that

$$I_1 \text{ is not empty.} \tag{6.22}$$

Recall that (6.11) fails, so we can find $w \in P \cap B(y, C_1 2^{-j} r) \setminus \pi(T_j(y))$. By (6.16) and because $y = 0$, $w \in B(0, 1/2)$; by (6.20), $w \in P \setminus \pi(F_0)$. By (6.19), $F_0$ is compact, so a whole neighborhood of $w$ in $P$ lies in $P \setminus \pi(F_0)$. This neighborhood contains an interior point of some $A_i$, $i \in I_0$, and by definition this $i$ lies in $I_1$. This proves (6.22).

Next, for each $i \in I_0$ there is a finite set $\Xi(i) \in F_0 \cap \pi^{-1}(A_i)$, with at most $C$ elements, and such that

$$\mathrm{dist}(z, \Xi(i)) \leq 1 \ \text{ for every } z \in F_0 \cap \pi^{-1}(A_i). \tag{6.23}$$

This is checked in [DS4], and the same proof applies here; see the verification of (∗9.3) (understand, (9.3) in [DS4]) below (∗9.89), which relies a lot on Lemma ∗9.83. Let us just say here why this is not surprising.

Let $\mathcal{R} = \big\{ R \in \Sigma_j\,;\, R \subset T_j(y) \big\}$ denote the set of cubes that compose $T_j(y)$; these cubes are disjoint by definition of $\Sigma_j$. Set $a_0 = \frac{\mathcal{H}^d(\pi(T_j(y)))}{\mathcal{H}^d(T_j(y))}$; thus $a_0 \geq \gamma$ by (6.15). Also set $a(R) = \frac{\mathcal{H}^d(\pi(R))}{\mathcal{H}^d(R)}$ for $R \in \mathcal{R}$, and

$$a_1 = \mathcal{H}^d(T_j(y))^{-1} \sum_{R \in \mathcal{R}} \mathcal{H}^d(\pi(R)) = \mathcal{H}^d(T_j(y))^{-1} \sum_{R \in \mathcal{R}} a(R) \mathcal{H}^d(R); \tag{6.24}$$

thus $a_1$ is a weighted average of the $a(R)$ (with the weights $\mathcal{H}^d(R)$, $R \in \mathcal{R}$), and at the same time

$$a_1 = \mathcal{H}^d(T_j(y))^{-1} \sum_{R \in \mathcal{R}} \mathcal{H}^d(\pi(R)) \geq \mathcal{H}^d(T_j(y))^{-1} \mathcal{H}^d(\pi(T_j(y))) = a_0 \tag{6.25}$$

because $\pi(T_j(y))$ is the union of the $\pi(R)$. Now (6.12) fails, so $a(R) \leq (1+2\eta)a_0 \leq (1+2\eta)a_1$ for $R \in \mathcal{R}$. If $\eta$ is small, this forces all the $a(R)$ to be very close to their average $a_1$, and also $a_1 - a_0$ to be very small. A first consequence is that $a(R) \geq \gamma/2$, and hence $\mathcal{H}^d(\pi(R)) \geq \gamma \mathcal{H}^d(R)/2 \geq C^{-1}$, for $R \in \mathcal{R}$. But also, the various $\pi(R)$, $R \in \mathcal{R}$, are almost disjoint, because when $R_1$, $R_2 \in \mathcal{R}$ are different,

$$\mathcal{H}^d(T_j(y))(a_1 - a_0) = \mathcal{H}^d(\pi(T_j(y))) - \sum_{R\in\mathcal{R}} H^d(\pi(R)) \geq \mathcal{H}^d(\pi(R_1) \cap \pi(R_2)) \tag{6.26}$$

by the proof of (6.25). Then there are at most $C$ such cubes $R \in \mathcal{R}$ such that $\pi(R)$ falls near a given $A_i$, and the existence of $\Xi(i)$ as in (6.23) follows reasonably easily. The bound $C$ on the cardinal of the $\Xi(i)$ depends on $\gamma$ (and the local Ahlfors-regularity constants, as usual).

Finally set $I_2 = I_0 \setminus I_1$. We also prove that for each $i \in I_2$, there is a point $z_i \in F_0$ such that $\pi(z_i) \in int(A_i)$ and

$$|z - z_i| \leq 1 \text{ for all } z \in F_0 \text{ such that } \pi(z) = \pi(z_i). \tag{6.27}$$

This would be obvious if the $\pi(R)$ were disjoint, because $\mathrm{diam}(R) \leq 2C2^{-j}r < 1$ by (6.9), by the normalization (6.16), and because we can assume that $C_1 > 2C$, where $C$, the constant from (6.9), depends only on the Ahlfors regularity constant. Here the $\pi(R)$ are merely nearly disjoint, so we have to work a little more, i.e., use Chebyshev. The verification is done in [DS4], below (∗9.89), proof of (∗9.5). This completes our description of $F_0$.

The next stage in our proof is a deformation lemma (Proposition 9.6 in [DS4]) that sends most of $F_0$ to a $(d-1)$-dimensional set. The proposition concerns a more arbitrary closed set in $\mathbb{R}^{n+d}$, but we apply it to $F_0$, and the main assumptions are (6.22), (6.23) and (6.27), that we just obtained. It yields the following.

**Lemma 6.28.** *There exists a family $\{\phi_t\}$, $0 \leq t \leq 1$, of Lipschitz mappings of $\mathbb{R}^{n+d}$, with the following properties:*

$$(t, z) \to \phi_t(z) \text{ is Lipschitz (from } [0,1] \times \mathbb{R}^{n+d} \to \mathbb{R}^{n+d}); \tag{6.29}$$

$$\phi_t(z) = z \text{ for } t = 0 \text{ and when } \mathrm{dist}(z, P_0 \times V_0) \geq d + 3; \tag{6.30}$$

$$\begin{gathered} \text{if } \phi_t(z) \neq z \text{ for some } z \in \mathbb{R}^{n+d} \text{ and } t \in [0,1], \text{ then} \\ \mathrm{dist}(z, F_0) \leq C \text{ and } \mathrm{dist}(\phi_s(z), F_0) \leq C \text{ for } 0 \leq s \leq 1 \end{gathered} \tag{6.31}$$

*(so our deformation may move some points a lot, but only close to $F_0$ and somehow along $F_0$);*

$$\phi_t(F_0) \subset P_0 \times V_0 \text{ for all } t \in [0,1]; \tag{6.32}$$

$$(6.33) \qquad H^{d-1}(\phi_1(F_0)) < +\infty$$

*(the main point: we essentially make $F_0$ disappear);*

$$(6.34) \qquad \phi_t \text{ is } C\text{-Lipschitz on } \mathbb{R}^{n+d} \setminus (P_0 \times V_0).$$

As usual, the constant $C$ in this statement depends on $n$, $M$, $C_1$, and $\gamma$ through the constants that arise in the description of $F_0$ above. The last property (6.34) is useful because we do not want to lose what we win by (6.32) by increasing too much the Hausdorff measure of $F = \widehat{E}$ near the boundary of $P_0 \times V_0$.

Lemma 6.28 still holds in our case, with the same proof, but differences will occur in the way it is applied.

As the reader may have guessed, the mappings $\phi_t$ are used in [DS4] to produce a deformation of $\widehat{E}$ which contradicts its quasiminimality. The reader should not worry about the way the Hausdorff measure estimates go, because it will be the same as in [DS4], but let us just say a few words to explain some of our choices. For instance, (6.34) goes with some control on the size of the set

$$(6.35) \qquad H = \big\{ z \in \widehat{E}\,;\, 0 < \mathrm{dist}(z, P_0 \times V_0) \leq d+3 \big\},$$

where the $\phi_t$ may differ from the identity, but we cannot use (6.33). And we required in (6.17) that $\mathrm{dist}(P_0 \times (V \cap \partial B(0,\rho_0)), F) \geq N$ to get an easier control on $H$. Indeed, let $z \in H$ be given. Then $z \in F$ by (6.7), the first part of (6.18), and (6.14). Write $z = (\pi(z), \pi_x(z)) \in P \times V$, then $\mathrm{dist}(\pi(z), P_0) \leq d+3$ by (6.35), so $\mathrm{dist}(\pi_x(z), V \cap \partial B(0,\rho_0))) \geq N - d - 3 > N/2$ by (6.17), and since $V_0 = V \cap \overline{B}(0,\rho_0)$ and $\mathrm{dist}(\pi_x(z), V_0) \leq d+3$ by (6.35), we get that $\pi_x(z) \in V \cap B(0, \rho_0 - N/2)$. Altogether, $H$ is contained in the simpler set

$$(6.36) \qquad H' = \big\{ z \in \widehat{E}\,;\, \pi_x(z) \in B(0, \rho_0 - N/2) \text{ and } 0 < \mathrm{dist}(\pi(z), P_0) \leq d+3 \big\},$$

which is easier to control because we can use the fact that $P$ is $d$-dimensional to cover $H'$ by a $CN^{d-1}$ balls of radius 1, using something like (6.23). Near the end of the argument, $N$ is chosen so large that the contribution of $H$ to the $\mathcal{H}^d$-measure of the image $\phi_1(\widehat{E})$, which is less than $CN^{d-1}$, is negligible compared to the mass of $\widehat{E}$ that we save by (6.33), which larger than $C^{-1}N^d$. So it is important that $C$, in particular in (6.34), does not depend on $N$. But again the reader should not worry too much, the computations are done in [DS4].

Up to now, we did not need to worry, because all our constructions relied on the general properties of $E$ (local Ahlfors-regularity, existence of Lipschitz mappings with a big image), and not on the definition of a quasiminimal set. This changes now.

In [DS4], the $\phi_t$ define a competitor for $\widehat{E}$, which is significantly better than $\widehat{E}$ (by (6.33) and (6.34), and observations as above) and this leads to the desired contradiction. The computations are done at the end of Section $*9.2$, near ($*9.90$) and below. In the case of generalized quasiminimal sets, we need to add a small term $Chr_0^d$ (coming from the $hr^d$

in (2.5)) to the right-hand side of ($*$9.93). But if $h$ is chosen small enough, this does not upset the end of the proof: the additional term is small compared to $M\mathcal{H}^d(\widehat{E}\cap W)$ in the right-hand side, because this last is larger than $C^{-1}r^d$ by ($*$9.105) and the line before.

In the present situation, the difficulty will come from the fact that the $\phi_t$ may fail to define a competitor for $\widehat{E}$, because we don't know whether they respect the boundaries $L_j$ as in (1.7). Note however that the other constraints (1.4)-(1.6) and (1.8) are satisfied, with the same verification as in [DS4].

**6.b. Some bad news**

Let us try to continue, and see whether the $\phi_t$ defined in [DS4] satisfy the last constraint (1.7), or can be modified so that (1.7) holds. The main point of this short subsection is to explain why the author thinks there is a serious difficulty for the brutal extension of the proof of [DS4].

There is a first obvious reason. Suppose $n = 3$, $\Omega = L_0$ is the half space $\{x_1 \geq 0\}$, $L_1 = \partial\Omega$ is the vertical plane $\{x_1 = 0\}$, and $E = P \cap \Omega$ for some 2-plane $P$ perpendicular to $\{x_1 = 0\}$. For instance, $E = \big\{x_1 \geq 0 \text{ and } x_3 = 0\big\}$. In this case, we don't need the trick of replacing $E$ with $\widehat{E}$, because the projection $\pi$ over $P$ already has a big image, but if we did it, we would just replace $\mathbb{R}^3$ with $\mathbb{R}^5 = \mathbb{R}^3 \times P$, and $E = P \cap \Omega$ with the slightly tilted half plane $\big\{(x_1, x_2, 0, x_1, x_2)\,;\, x_1 \geq 0\big\}$, and the discussion would stay the same as below.

Pick a small ball $B(x, r)$ centered at $x = 0$, and try to apply the proof above. Also pick $y = 0$; we would like to say that the main lemma from Definition 6.13 holds, but we can't. Indeed (6.12) never holds, because $\mathcal{H}^d(\pi(R))$ is just proportional to $\mathcal{H}^d(R)$, (6.15) holds for the same reason (the proportionality constant is not small), and (6.11) fails because $\pi(E) = P \cap \Omega$ only covers half the desired ball.

In [DS4], this would never happen, because we would be allowed to deform points of the boundary $\big\{x_1 = x_3 = 0\big\}$ along $E$ into the domain, thus making a good piece of $E$ disappear and contradicting the quasiminimality of $E$. And indeed the proof of [DS4] does something like that, which is not allowed here because of (1.7) for $L_1$.

Let us say a little more about how these things happen in the proof of [DS4]; the reader may also skip the following discussion and turn to the proof of Theorem 6.1 which starts in in the next subsection.

There are three phases in the construction of the $\phi_t$ in [DS4]. In the first one we we move points horizontally (i.e., with trajectories parallel to $P$), independently in each $\pi^{-1}(A_i)$, so as to project on $\pi^{-1}(\partial A_i)$ whenever this is possible. That is, let $\phi_{1/3}$ denote the endpoint of this first phase, and let $z \in \pi^{-1}(A_i)$ be given. When $i \in I_1$, we manage to obtain that $\pi(\phi_{1/3}(z))$ is the radial projection of $\pi(z)$ on $\partial A_i$, centered at the point $x_i$ of (6.21). When $i \in I_2$, we manage to obtain that $\pi(\phi_{1/3}(z))$ is the the radial projection of $\pi(z)$ on $\partial A_i$, centered at the point $\pi(z_i)$ of (6.27), but only when $z \in \pi^{-1}(\partial A_i)$ and when $z$ lies far from $z_i$ (more precisely, when $|\pi_x(z) - \pi_x(z_i)| \geq 2$).

We can keep the first phase as it is, because since we only move points horizontally and the boundaries for $\widehat{E}$ are the sets $\widehat{L}_j = P \times L_j$, the condition (1.7) is automatically satisfied. At the end of this first stage, $\phi_{1/3}(F_0)$, seen from far, looks a little like a piece of graph (over the union of the $A_i$, $i \in I_2$), plus some uncontrolled junk above the $\partial A_i$.

For the second phase (corresponding to $\phi_t$, $1/3 \leq t \leq 2/3$), we move points vertically, so as to merge the various points of $\pi^{-1}(x) \cap \phi_{1/3}(F_0)$ into a single point when this is possible. The difficulty is to make this in a Lipschitz way with respect to $x$. Between $\phi_{1/3}$ and $\phi_{2/3}$, we move the points linearly, i.e., we take

$$\phi_t(z) = (2-3t)\phi_{1/3}(z) + (3t-1)\phi_{2/3}(z) \quad \text{for } 1/3 \leq t \leq 2/3, \tag{6.37}$$

with, in coordinates,

$$\phi_{2/3}(z) = (\pi(\phi_{1/3}(z)), \varphi(\phi_{1/3}(z)) \in P \times V \tag{6.38}$$

for some $\varphi : \mathbb{R}^{n+d} \to V$ that describes the vertical motion. Observe that our notation here is slightly different; what we denote by $(\pi(z), \varphi(z))$ now was called $\phi_2$ in [DS4]); then (6.38) here corresponds to (∗9.48) there.

It is not so important to describe the precise definition of $\varphi$ and $\phi_{2/3}$ here. Let us just say that this is done with partitions of unity, and that the main point is that the resulting set $F_2 = \phi_{2/3}(F_0)$ has the following nicer property.

Recall that for $i \in I_1$, $\phi_{1/3}(F_0) \cap \pi^{-1}(int(A_i)) = \emptyset$, so

$$F_2 \cap \pi^{-1}(int(A_i)) = \emptyset \text{ for } i \in I_1, \tag{6.39}$$

just because our second phase moves points vertically. [Also see (∗9.43) in [DS4]]. When $i \in I_2$, we only know that $|\pi_x(z) - \pi_x(z_i)| \leq 2$ for all $z \in \pi^{-1}(int(A_i)) \cap \phi_{1/3}(F_0)$. But by our our vertical motion, we make sure that

$$F_2 \cap \pi^{-1}(int(A_i)) \subset \Gamma_i \text{ for } i \in I_2, \tag{6.40}$$

where $\Gamma_i$ is the graph over $int(A_i)$ of some Lipschitz function. See (∗9.44) in [DS4]. So the point of the vertical motion is to merge the various points of $\pi^{-1}(w)$, $w \in int(A_i)$; the partitions of unity help us do this in a nice Lipschitz way.

Our control on the sets $F_2 \cap \pi^{-1}(\partial A_i)$ is a little less precise, but still (∗9.45) in [DS4] says that each of them is contained in a finite union of Lipschitz graphs over $\partial A_i$, so their total $\mathcal{H}^d$-measure is null.

In the present situation, there would be a way to modify the construction of $\phi_{2/3}$ and $F_2$, so that we also have the preservation (1.7) of the boundary pieces $L_j$. In other words, the serious problem is not here yet. The idea is to try to favor choices of points with integer coordinates in $V$ in the description $\Xi_i$, but let us not be more precise, because more serious problems will arise in the third phase.

In the third and last phase of the construction of [DS4], points move a lot more. The mappings $\phi_t$, $2/3 \leq t \leq 1$, are obtained by composing successive deformations, each time occurring on $\pi^{-1}(A_i \cup A_j)$ for some pair of adjacent cubes in $P_0$. That is, we set $t_k = 2/3 + 2^{-k}/6$ and construct recursively $\phi_t$, $t \in I_k = [t_k, t_{k+1}]$. At the start our set $F(k) = \phi_{t_k}(F_0)$ is composed of a certain number of Lipschitz graphs $\Gamma_i$, $i \in I(k) \subset I_2$ over the corresponding open squares $int(A_i)$, plus a set $Z(k)$ of finite $H^{d-1}$-measure. Notice that we have such a description for $F(0) = F_2$, where $I(k) = I_2$ and the small set $Z(0)$ lies above the $\partial A_i$.

If $I(k) = \emptyset$, we stop, and keep $\phi_t = \phi_{t_k}$ for $t_k \leq t \leq 1$. Otherwise, we select an $i \in I(k)$ and a $j \in I_0 \setminus I(k)$ that are contiguous, i.e., shares a face $S$ of dimension $d-1$. Such a pair exists, because $I_1 \subset I_0 \setminus I(k)$ is not empty. We first construct our deformation on $\Gamma_i$, so that it moves points inside $\Gamma_i$ so that the final image lies in $\Gamma_i \cap \pi^{-1}(\partial A_i)$ (and even in $\Gamma_i \cap \pi^{-1}(\partial A_i \setminus int(S)))$ and fixes every point of $\Gamma_i \cap \pi^{-1}(\partial A_i \setminus int(S))$. We just use $\pi^{-1}(int(S)) \cap \Gamma_i$ as a base to push the points in the direction of $\pi^{-1}(\partial A_i)$.

Then we extend our Lipschitz deformation into a Lipschitz deformation of $\mathbb{R}^{n+d}$, which leaves $\pi^{-1}(P \setminus A_i \cup A_j)$ alone, and it is easy to see that $F_{k+1} = \phi_{t_{k+1}}(F_0)$ satisfies the induction assumption. At the end of the construction, $I(k) = \emptyset$, $H^{d-1}(F_k) < +\infty$ as in (6.33), and we are happy.

Unfortunately, we cannot arrange (1.7) for the mappings that we just constructed. The main difficulty is when $\pi^{-1}(A_i)$ contains some points of some $L_j$, say, for the only initial index $i \in I_1$. In our construction, these points get pushed to $\pi^{-1}(\partial A_i)$, and then to other boxes. Along the way, they have to stay close to $\widehat{E}$, and this may well be incompatible with (1.7), for instance if $\widehat{E}$ gets away from $L_j$.

Now we could hope to be lucky, and have a sequence of indices $i \in I_0$, that can be removed in the corresponding order, and such that the list of sets $L_j$ that touch $\pi^{-1}(A_i)$ is a nondecreasing function of time, or some similar condition that seems hard to get in practice. But in view of the counterexample (a half plane) given at the beginning of the subsection, this hope looks very optimistic.

### 6.c. What we can say anyway

There is one special case when the proof of [DS4] can easily be adapted, and which we record now.

Recall that $B_0$ is the unit open ball and that $r_0 = 2^{-m}$ is the scale of the dyadic cubes in the description of the $L_j$.

**Proposition 6.41.** *For each choice of $M \geq 1$, we can find $h > 0$, $\theta > 0$, and $C_M \geq 1$, depending on $M$ and $n$, so that the following holds. Suppose that $E \in GSAQ(B_0, M, \delta, h)$ and that the rigid assumption is satisfied, and let $x \in E^* \cap B_0$ and $0 < r < \mathrm{Min}(r_0, \delta)$ be such that $B(x, 2r) \subset B_0$. Also assume that*

$$E \cap B(x,r) \subset L_j \text{ for every } j \text{ such that } L_j \text{ meets } B(x,r). \tag{6.42}$$

*Then there is a closed set $G_0 \subset E^* \cap B(x,r)$ and a mapping $\phi : G_0 \to \mathbb{R}^d$ such that*

$$\mathcal{H}^d(G_0) \geq \theta r^d \text{ and } C_M^{-1}|y-z| \leq |\phi(y) - \phi(z)| \leq C_M|y-z| \text{ for } y, z \in G_0. \tag{6.43}$$

Thus $E^* \cap B(x, 2r)$ contains a big piece of bilipschitz image of a subset of $\mathbb{R}^d$. The case when no $L_j$ meets $B(x,r)$ corresponds to the result of [DS4].

We shall find it more convenient to assume (6.42) as it is, but it would be enough to assume that $E^* \cap B(x,r) \subset L_j$ when $L_j$ meets $B(x,r)$; the stronger corresponding statement is simply deduced by applying Proposition 6.41 to the set $E^*$, which also lies in $GSAQ(B_0, M, \delta, h)$ by Proposition 3.3.

We shall see later how to deduce Theorem 6.1 from the proposition, but for the moment let us prove it. We follow the scheme of [DS4], as explained above, and in particular get mappings $\phi_t$ as in Lemma 6.28; the only thing that we need to do is modify them so that the satisfy (1.7) in addition.

We are only interested in the $L_j$ that meet $B(x, r/5)$. Indeed, for the other ones, the constraint (1.7) is trivially satisfied because we shall only consider competitors for $\widehat{E}$ in $B(\widehat{x}, r/10)$. With the $\phi_t$ that we have so far, this follows from (6.30) and (6.18), and this will stay true after we modify the $\phi_t$ below.

Let $J_0$ denote the set of indices $j$ such that $L_j$ meets $B(x, r/5)$. We may assume that $J_0$ is not empty, because otherwise there is nothing to check since the $\phi_t$ from Lemma 6.28 do the job as in [DS4]. Set

$$L = \bigcap_{j \in J_0} L_j \,; \tag{6.44}$$

observe that

$$E \cap B(x, r) \subset L, \tag{6.45}$$

by (6.42), so $L$ is not empty. Apply Lemma 3.4 to $L$ (after a dilation of factor $r_0^{-1}$, as in (3.26), used to return to faces of unit size). This gives a Lipschitz retraction

$$\pi_L : L^\eta \to L, \text{ with } L^\eta = \big\{ y \in \mathbb{R}^n \,;\, \mathrm{dist}(y, L) \leq \eta \big\}. \tag{6.46}$$

In Lemma 3.4 we could take $\eta = 1/3$, but here, since $L$ is composed of faces of size $r_0$, we take $\eta = r_0/3$ because we conjugate with a dilation. Retraction means that $\pi_L(y) = y$ on $L$, and Lemma 3.4 also says that $\pi_L$ preserves the faces of size $r_0$ of any dimension. Finally define the analogue of $\pi_L$ on $\mathbb{R}^{n+d}$ by

$$\Pi(z) = (p, \pi_L(v)) = (\pi(z), \pi_L(\pi_x(z))) \text{ for } z = (p, v) \in P \times V. \tag{6.47}$$

We are ready to define the mappings $\phi_t^*$ that will replace the $\phi_t$ from Lemma 6.28. We want to set

$$\phi_t^*(z) = \Pi(\phi_t(z)) \text{ for } z \in F, \tag{6.48}$$

so let us check that this makes sense. If $\phi_t(z) = z$, then its $V$-coordinate $\pi_x(z)$ lies in $E \cap B(x, r)$ (by (6.7)), hence also in $L$ by (6.45), so $\pi_L(\pi_x(z)) = \pi_x(z)$ and $\Pi(\phi_t(z))$ is not only defined, but equal to $z$. For the record,

$$\phi_t^*(z) = z \text{ when } z \in F \text{ and } \phi_t(z) = z. \tag{6.49}$$

If $\phi_t(z) \neq z$, (6.31) says that $\mathrm{dist}(\phi_t(z), F_0) \leq C$. Now $F_0 \subset L$ by (6.7) and (6.45), and we claim that $C$ in (6.31) is much smaller than $\eta = r_0/3$. Indeed, the normalization (6.16) says that $2C_1 2^{-j} r = 1$, and (6.18) implies that $C_2 2^{-j-1} r \leq r/11 \leq r_0/11$ (by an assumption of Proposition 6.41). Now $C_2$ is huge, much larger than $C_1$, which proves our claim. Thus $\phi_t(z) \in L^\eta$, $\Pi(\phi_t(z))$ is defined, and (6.48) makes sense.

We select a very small number $\rho > 0$ (which will even depend on the Lipschitz constant for the $\phi_t$), and keep

$$\phi_t^*(z) = \phi_t(z) \text{ when } \mathrm{dist}(z, F) \geq \rho \tag{6.50}$$

and

$$\phi_t^*(z) = \phi_t(z) = z \text{ when } t = 0, \text{ or } \mathrm{dist}(z, P_0 \times V_0) \geq d+3, \text{ or } \mathrm{dist}(z, F_0) \geq C, \tag{6.51}$$

where $C$ is as in (6.31). The fact that $\phi_t(z) = z$ comes from (6.30) in the first two cases, and (6.31) in the last one. Notice that (6.51) and (6.48) are compatible, by (6.49). In addition, $(t, z) \to \phi_t^*(z)$ is Lipschitz (with a very large constant that depends on $\rho$) on the set where we defined it so far, by (6.29) and because $\Pi$ is Lipschitz. Indeed, we need to estimate $|\phi_t^*(z) - \phi_s^*(z')|$, and the only case where we do not already know the Lipschitz estimate is when we use two different definitions, i.e., when $z \in F$ and $\mathrm{dist}(z', F) \geq \rho$, or the other way around.

Next we extend $\phi_t^*$ to $\mathbb{R}^{n+d}$ in a Lipschitz way, using the standard proof with Whitney cubes (here their size is at most $\rho$ because $\phi_t^*(z)$ was defined when $z \in F$ and when $\mathrm{dist}(z, F) \geq \rho$) and partitions of unity. Our extension $\phi_t^*$ thus satisfies (6.29) (by construction) and (6.30) (by (6.51)).

Let us now check (6.31), and so let $z \in \mathbb{R}^{n+d}$ be such that $\phi_t^*(z) \neq z$ for some $t \in [0, 1]$. Because of (6.51), we know that $\mathrm{dist}(z, F_0) \leq C$. If $\mathrm{dist}(z, F) \geq \rho$, (6.50) says that $\phi_t^*(z) = \phi_t(z)$, and then (6.31) says that $\mathrm{dist}(z, F_0) \leq C$ and $\mathrm{dist}(\phi_s^*(z), F_0) = \mathrm{dist}(\phi_s(z), F_0) \leq C$ for $0 \leq s \leq 1$.

So we may assume that $\mathrm{dist}(z, F) < \rho$, and we let $z_0 \in F$ be such that $|z - z_0| \leq \rho$. By construction, every $\phi_s^*(z)$ is a convex combination of various values of $\phi_s(w)$ or $\Pi(\phi_s(w))$, where $w \in B(z_0, 10\rho)$. Since $\phi_s$ and $\Pi \circ \phi_s$ are Lipschitz, $|\Pi(\phi_s(w)) - \Pi(\phi_s(z_0))| \leq C|\phi_s(w) - \phi_s(z_0)| \leq C\rho$, with a very large constant $C$ but that does not depend on $\rho$.

If in addition $\mathrm{dist}(z_0, F_0) \geq d + 3$, then $\phi_s(z_0) = z_0$ by (6.30), hence $\Pi(\phi_s(z_0)) = \phi_s^*(z_0) = z_0$ by (6.49) and (6.48). Then $|\phi_s^*(z) - z_0| \leq C\rho$, and

$$\mathrm{dist}(\phi_s^*(z), F_0) \leq \mathrm{dist}(z_0, F_0) + C\rho \leq \mathrm{dist}(z, F_0) + C\rho \leq C + 1 \tag{6.52}$$

for $0 \leq s \leq 1$, and if $\rho$ is small enough (recall that $\mathrm{dist}(z, F_0) \leq C$ because $\phi_t^*(z) \neq z$ for some $t$).

Otherwise, if $\mathrm{dist}(z_0, F_0) < d + 3$, then $\mathrm{dist}(\phi_s(z_0), F_0) \leq C$ for $0 \leq s \leq 1$, by (6.31) or (if $\phi_s(z_0) = z_0$) simply because $\mathrm{dist}(z_0, F_0) \leq d+3$. Since $\Pi$ coincides with the identity on $F_0$, this implies that $|\Pi(\phi_s(z_0)) - \phi_s(z_0)| \leq C' \,\mathrm{dist}(\phi_s(z_0), F_0) \leq C''$, where now $C'$ and $C''$ also depend on the Lipschitz constant for $\Pi$, which is all right because this Lipschitz constant depends only on the geometry of $L$. In this case all the $\phi_s(w)$ and $\Pi(\phi_s(w))$ lie within $C'' + C\rho$ of $\phi_s(z_0)$, and

$$\mathrm{dist}(\phi_s^*(z), F_0) \leq \mathrm{dist}(\phi_s(z_0), F_0) + C'' + C\rho \leq C + C'' + 1 \tag{6.53}$$

if $\rho$ is small enough. That is, the $\phi_t^*$ satisfy (6.31), even though with a larger geometric constant.

For the analogue of (6.32) we need to check that $\phi_t^*(z) \in P_0 \times V_0$ when $z \in F_0$. We already know from (6.32) that $\phi_t(z) \in P_0 \times V_0$, and (6.48) says that $\phi_t^*(z) = \Pi(\phi_t(z))$. Write $\phi_t(z) = (p, v)$, with $p = \pi(\phi_t(z))$ and $v = \pi_x(\phi_t(z))$. Then $\phi_t^*(z) = (p, \pi_L(v))$ by (6.47). We know that $p \in P_0$, so we just need to check that $\pi_L(v) \in V_0$.

Recall that $\mathrm{dist}(\phi_t(z), F_0) \leq C$, either by (6.31) or else because $\phi_t(z) = z \in F_0$. Choose $w \in F_0$ such that $|w - \phi_t(z)| \leq C$. Observe that $\pi_x(w) \in E \cap B(x, r) \subset L$, by (6.7) and (6.45), and so $\pi_L(\pi_x(w)) = \pi_x(w)$ by definition of $\pi_L$. Now

$$(6.54) \quad |\pi_L(v) - v| \leq |\pi_L(v) - \pi_L(\pi_x(w))| + |\pi_x(w) - v| \leq C|\pi_x(w) - v| \leq C|w - \phi_t(z)| \leq C$$

because $\pi_L$ is Lipschitz and $v = \pi_x(\phi_t(z))$, and where $C$ is a geometric constant that depends on the constant in (6.31) and the Lipschitz constant for $\pi_L$. So we still can choose $N$ in the definition of $P_0$ and $V_0$ (see the discussion below (6.16)) much larger than this.

Now $w \in F_0 = F \cap (P_0 \times V_0)$ (by (6.19)), and (6.17) says that

$$\mathrm{dist}(w, P_0 \times (V \cap \partial B(0, \rho_0)) \geq N, \tag{6.55}$$

so $\mathrm{dist}(\pi_x(w), V \cap \partial B(0, \rho_0)) \geq N$. Recall that $V_0 = V \cap B(0, \rho_0)$, so $\pi_x(w) \in V \cap B(0, \rho_0)$, hence in fact $\pi_x(w) \in V \cap B(0, \rho_0 - N)$. Since

$$\begin{aligned} |\pi_L(v) - \pi_x(w)| &\leq |\pi_L(v) - v| + |v - \pi_x(w)| \leq C + |v - \pi_x(w)| \\ &= C + |\pi_x(\phi_t(z)) - \pi_x(w)| \leq C + |\phi_t(z) - w| \leq 2C \end{aligned} \tag{6.56}$$

by (6.54), because $v = \pi_x(\phi_t(z))$ and by definition of $w$, we get that $\pi_L(v) \in V \cap B(0, \rho_0) = V_0$, as needed. This proves (6.32) for the $\phi_t^*$.

Observe that (6.33) for $\phi_1^*$ follows from its analogue for $\phi_1$, by (6.48) and because $\Pi$ is Lipschitz.

As for (6.34), we just need to know that the $\phi_t^*$ (and in fact $\phi_1^*$ is enough) are Lipschitz on $F \setminus (P_0 \times V_0)$. Indeed, (6.34) (i.e., ($*$ 9.13) in [DS4]) is only used once, near the end of Section 9.2 of [DS4], to prove ($*$9.98), and for this we only need the restriction to $F$. But then we can use (6.48), and (6.34) for the $\phi_t^*$ follow from (6.34) for the $\phi_t$, because $\Pi$ is $C$-Lipschitz.

This completes the verification of (6.29)-(6.34), but recall that in addition we need to check that (1.7) holds, with respect to the quasiminimal set $\widehat{E}$ and the boundaries $\widehat{L}_j = P \times L_j$. That is, we are given $z \in \widehat{E} \cap \widehat{L}_j$ and $0 \leq t \leq 1$, and we want to check that $\phi_t^*(z) \in \widehat{L}_j$.

The conclusion is trivial if $\phi_t^*(z) = z$, so we may assume that $\phi_t^*(z) \neq z$. By (6.51), $\mathrm{dist}(z, P_0 \times V_0) \leq d + 3$, and hence $z \in B(\widehat{x}, r/10)$, by (6.18). In particular, $\pi_x(z) \in B(x, r/10)$. This excludes the case when $L_j$ does not meet $E \cap B(x, r/10)$, because $\pi_x(z) \in E \cap L_j$. In the remaining case, $j \in J_0$ and $L \subset L_j$ by (6.44).

Return to our $z \in \widehat{E} \cap \widehat{L}_j$. We know that $z \in B(\widehat{x}, r/10)$, so (6.7) says that $z \in F$, and $\phi_t^*(z) = \Pi(\phi_t(z))$ by (6.48). Therefore $\phi_t^*(z) \in P \times L \subset P \times L_j = \widehat{L}_j$ by (6.47) and the fact that $\pi_L$ maps $L^\eta$ to $L$ by definition, and as needed for (1.7).

We finally completed our list of verifications; now we can apply the fact that $\widehat{E}$ is quasiminimal, compute as in [DS4], and get the same conclusion as Proposition ∗8.15 there, which happen to be the same as in Proposition 6.41, which follows. □

## 7. The local uniform rectifiability of $E^*$ and bilateral weak geometric lemmas

Our next goal for this section is to extend Proposition 6.41 to the case when only the assumptions of Theorem 6.1 are satisfied; see Proposition 7.85 below. Then, in Section 8, we shall take care of the difference between big pieces of bilipschitz images and big pieces of Lipschitz graphs, and prove Theorem 6.1.

For our first verification, we shall mostly use Proposition 6.41 itself, the smallness or regularity of the faces that compose the $L_j$, and general knowledge on uniformly rectifiable sets; Lemma 7.38 will be the only place where we use the quasiminimality of $E$ in this argument, to show that a quasiminimal set that stays very close to the interior of a $d$-face does not have big holes there.

We shall use a characterization of uniform rectifiability by the so-called bilateral weak geometric lemma. We are given a locally Ahlfors-regular set $F$ of dimension $d$, which we want to study; our main example will be $E^*$, or a piece of $E^*$. First define the standard P. Jones numbers $\beta(x,r)$ by

$$\beta(x,r) = \beta_F(x,r) = \inf_P \Big\{ \frac{1}{r} \sup_{y\in F\cap B(x,r)} \mathrm{dist}(y,P) \Big\}, \tag{7.1}$$

where $x \in F$, $r > 0$, and the infimum is taken over all the affine $d$-planes $P$ through $x$. It is just as convenient here to restrict to planes that contain $x$, even though the other option could be used too, and would give equivalent results. We shall also use the bilateral variant

$$b\beta(x,r) = b\beta_F(x,r) = \inf_P \Big\{ \frac{1}{r} \sup_{y\in F\cap B(x,r)} \mathrm{dist}(y,P) + \frac{1}{r} \sup_{y\in P\cap B(x,r)} \mathrm{dist}(y,F) \Big\}, \tag{7.2}$$

where we also account for big holes in the middle of $F$. We shall be interested in the size of the bad sets

$$\mathcal{B}(\varepsilon) = \mathcal{B}_F(\varepsilon) = \big\{ (x,r) \in F\times(0,+\infty)\,;\, b\beta(x,r) > \varepsilon \big\} \tag{7.3}$$

when $\varepsilon > 0$ is small enough, because the local uniform rectifiability of (Ahlfors regular) sets turns out to be equivalent to Carleson measure estimates on the $\mathcal{B}(\varepsilon)$.

The following result concerns unbounded Ahlfors-regular sets; it will need to be adapted to the present situation, but gives an idea of what we want to do. We consider a closed (unbounded) Ahlfors-regular set $F$ of dimension $d$. This last means that there is a constant $C_0 \geq 1$ such that

$$C_0^{-1} r^d \leq \mathcal{H}^d(F\cap B(x,r)) \leq C_0 r^d \quad \text{for } x\in F \text{ and } 0<r<+\infty. \tag{7.4}$$

We shall say that $F \in BWGL(\varepsilon, C(\varepsilon))$ (or that $F$ satisfies a bilateral weak geometric lemma, with the constants $\varepsilon$ and $C(\varepsilon)$) when

$$\int_{y\in F\cap B(x,r)} \int_{0<t<r} \mathbf{1}_{\mathcal{B}(\varepsilon)}(y,t)\, \frac{d\mathcal{H}^d(y)dt}{t} \leq C(\varepsilon) r^d \tag{7.5}$$

for $x \in F$ and $0 < r < +\infty$.

We say that $F \in BPBI(\theta, C_1)$ (for big pieces of bilipschitz images) when for all $x \in F$ and $r > 0$, we can find a closed set $G_0 \subset F \cap B(x,r)$ and a Lipschitz mapping $\phi : G_0 \to \mathbb{R}^d$ such that

$$\mathcal{H}^d(G_0) \geq \theta r^d \text{ and } C_1^{-1}|y-z| \leq |\phi(y) - \phi(z)| \leq C_1 |y-z| \text{ for } y, z \in G_0. \tag{7.6}$$

Thus this is the same property as in Proposition 6.41, except that there we restricted to the pairs $(x,r)$ such that $0 < r < \mathrm{Min}(r_0, \delta)$ and $B(x, 2r) \subset B_0$, and there was an extra assumption to get it.

**Theorem 7.7.** *Let $F \subset \mathbb{R}^n$ be a closed Ahlfors-regular set of dimension $d$. If $F \in BPBI(\theta, C_1)$ for some choice of $\theta > 0$ and $C_1 \geq 1$, then for every $\varepsilon > 0$, $F \in BWGL(\varepsilon, C(\varepsilon))$ for some $C(\varepsilon)$ that depends only on $n$, $C_0$ (the regularity constant in (7.4)), $\theta$, $C_1$, and $\varepsilon$. Conversely, there exists $\varepsilon > 0$, that depends only on $n$ and $C_0$, such that if $F \in BWGL(\varepsilon, C(\varepsilon))$ for some $C(\varepsilon) \geq 0$, then there exist $\theta > 0$ and $C_1 \geq 1$, that depend only on $n$, $\varepsilon$, and $C(\varepsilon)$, such that $F \in BPBI(\theta, C_1)$.*

Notice that $BWGL(\varepsilon, C(\varepsilon))$ is smaller when $\varepsilon$ is smaller, so that in the converse statement, assuming that $F \in BWGL(\varepsilon', C(\varepsilon'))$ for some $\varepsilon' \leq \varepsilon$ would be enough too.

Theorem 7.7 follows from Theorems 2.4 and 1.57 in [DS3]; see the remark above Theorem 2.4 (for the fact that only one small $\varepsilon$ is needed), Definitions 2.2 and 1.69 (for the BWGL), and (1.60) and (1.61) (for two of the equivalent conditions in Theorem 1.57, one clearly stronger and one clearly weaker than our BPBI here).

We shall use both parts of the equivalence here. We start with the direct part.

**Lemma 7.8.** *Let $E$, $x \in E^*$, and $r$ be as in Proposition 6.41. In particular, assume that (6.42) holds. Then for each $\varepsilon > 0$*

$$\int_{y \in E^* \cap B(x, r/8)} \int_{0<t<r/8} \mathbf{1}_{\mathcal{B}_{E^*}(\varepsilon)}(y,t) \, \frac{d\mathcal{H}^d(y) dt}{t} \leq C(\varepsilon) r^d, \tag{7.9}$$

*where $C(\varepsilon)$ depends only on $n$, $M$, and $\varepsilon$.*

We want to deduce Lemma 7.8 from Proposition 6.41 and Theorem 7.7, but a small localization argument is needed. To this effect, we apply Proposition 7.6 in [DS4]. This is the same proposition that we used to get the set $F$ in (6.7), but here we shall need to use it in a more precise way. We apply it to the set $E^*$ and the ball $B(x, r/2)$; the assumptions follow from the local Ahlfors-regularity of $E^*$ that we proved in Proposition 4.1.

What we get from Proposition 7.6 in [DS4] is a (bounded) Ahlfors-regular set $F$ such that

$$E^* \cap B(x, r/4) \subset F \subset E^* \cap B(x, r/2), \tag{7.10}$$

and a cubical patchwork for $F$, i.e., collections of decompositions of $F$ into pseudocubes $Q$, $Q \in \Sigma_j$, like the one we used near (6.9). But this time we shall find it convenient to use the fact that our patchwork is adapted to $E$, in the sense that for every cube $Q \in \cup_{j\geq 0}\Sigma_j$ and every (small) $\tau > 0$,

$$\mathcal{H}^d\big(\{w \in Q\,;\, \mathrm{dist}(w, E \setminus Q) \leq \tau\,\mathrm{diam}(Q)\}\big) \leq C\tau^{1/C}\,\mathrm{diam}(Q)^d. \tag{7.11}$$

This is part of ($*$7.10) in [DS4], and this is a little more precise than the usual "small boundary condition" on cubes of $F$, because it also controls the difference between $E$ and $F$. Here $C$ in (7.11) depends on $n$ and $M$ (through the local Ahlfors-regularity constant), but not on $\tau$.

We want to apply Theorem 7.7, and since $F$ is not unbounded, we replace it with $F' = F \cup P$, where $P$ is any affine $d$-plane such that $\mathrm{dist}(x, P) = 2r$. It is easy to see that $F'$ is Ahlfors-regular (as in (7.4)), and now we want to check that $F' \in BPBI(\theta, C_1)$, for some choice of $\theta > 0$ and $C_1 \geq 1$ which will be just a little worse than the constants of Proposition 6.41.

So we pick $y \in F'$ and $t > 0$ and try to find a big bilipschitz piece $G_0$ in $F' \cap B(y,t)$, as in (7.6). If $t > 3r$ or $y \in P$, $F'$ contains a $d$-dimensional disk of radius $t/3$ (contained in $P$), which is a nice choice of $G_0$ for (7.6). So assume that $y \in F$ and $t < 3r$, and try to find $G_0 \subset F$. Let $Q$ be a cube of our patchwork such that

$$y \in Q \subset F \cap B(y, t/10), \tag{7.12}$$

and which is maximal with these properties. Thus ($*$7.2) in [DS4] says that $\mathrm{diam}(Q) \geq t/C$, where $C$ depends on $n$ and $M$ (through the local Ahlfors-regularity constant in Proposition 4.1). Also recall from ($*$7.2) that $\mathcal{H}^d(Q) \geq C^{-1}\,\mathrm{diam}(Q)^d$; then choose the constant $\tau \in (0, 1/10)$ so small (again depending on $n$ and $M$ only) that the right-hand side of (7.11) is smaller than $\mathcal{H}^d(Q)$. This allows us to find $w \in Q$ such that

$$\mathrm{dist}(w, E^* \setminus F) \geq \mathrm{dist}(w, E^* \setminus Q) \geq \tau\,\mathrm{diam}(Q). \tag{7.13}$$

We want to apply Proposition 6.41 to the pair $(w, \tau\,\mathrm{diam}(Q))$, so we check the hypotheses. First, $w \in E^*$ because $F \subset E^*$ (see (7.10)). Next,

$$\tau\,\mathrm{diam}(Q) \leq \tau t/5 \leq 3\tau r/5 \leq 3r/50 \tag{7.14}$$

by (7.12), because $t < 3r$, and because we chose $\tau < 1/10$. In particular $\tau\,\mathrm{diam}(Q) < r < \mathrm{Min}(r_0, \delta)$, and also

$$\begin{aligned} B(w, 2\tau\,\mathrm{diam}(Q)) &\subset B(y, 2\tau\,\mathrm{diam}(Q) + t/10) \subset B(x, 2\tau\,\mathrm{diam}(Q) + t/10 + r/2) \\ &\subset B(x, 6r/50 + 3r/10 + r/2) \subset B(x, r) \subset B_0 \end{aligned} \tag{7.15}$$

by (7.12), because $y \in F$, and by (7.10) and (7.14). Finally, (6.42) for $B(w, \tau\,\mathrm{diam}(Q))$ follows from (6.42) for $B(x,r)$, simply because $B(w, \tau\,\mathrm{diam}(Q)) \subset B(x,r)$ by (7.15). So Proposition 6.41 applies, and gives a set $G_0 \subset E^* \cap B(w, \tau\,\mathrm{diam}(Q))$, such that

$$\mathcal{H}^d(G_0) \geq \theta\tau^d\,\mathrm{diam}(Q)^d \ \text{ and } \ C_M^{-1}|y-z| \leq |\phi(y)-\phi(z)| \leq C_M|y-z| \text{ for } y, z \in G_0. \tag{7.16}$$

This set works in the definition (7.6), because $G_0 \subset F$ by (7.13). Note that since $\mathrm{diam}(Q) \geq C^{-1}t$ (see below (7.12)), $\theta\tau^d \,\mathrm{diam}(Q)^d \geq \theta' t^d$ for some $\theta'$ that depends only on $n$ and $M$. Thus $F' \in BPBI(\theta', C_M)$, as needed, and now Theorem 7.7 says that for every choice of $\varepsilon > 0$, we can find $C(\varepsilon)$ so that (7.5) holds for $F'$ (and any ball centered on $F'$). We just apply this to the ball $B(x, r/8)$ (which is centered on $F'$ by (7.10), and get that

$$\int_{y\in F'\cap B(x,r/8)} \int_{0<t<r/8} \mathbf{1}_{\mathcal{B}_{F'}(\varepsilon)}(y,t)\, \frac{d\mathcal{H}^d(y)dt}{t} \leq 8^d C(\varepsilon) r^d. \tag{7.17}$$

But $F'$ coincides with $E^*$ on $B(x, r/4)$ by (7.10), so (7.17) is just the same as (7.9). Lemma 7.8 follows. □.

We slowly return to the extension of Proposition 6.41 to the situation of Theorem 6.1. We fix a small $\varepsilon > 0$, and want to control the size of $\mathcal{B}_{E^*}(\varepsilon)$, by cutting it into smaller pieces that we can control. Denote by

$$\mathcal{A} = \big\{(y,t) \in E^* \times \mathrm{Min}(r_0, \delta)\,;\, B(y, 2t) \subset B_0\big\} \tag{7.18}$$

the set of balls that we like to consider. We first get rid of the balls that lie close to a face of dimension at most $d-1$ in our initial net.

**Lemma 7.19.** *Denote by $F_1$ the union of all the faces of dimension at most $d-1$ of cubes from the dyadic net that was used to define the $L_j$, and set*

$$\mathcal{B}_1 = \big\{(y,t) \in \mathcal{A}\,;\, B(y, 10t) \text{ meets } F_1\big\}. \tag{7.20}$$

*Then*

$$\int_{y\in E^*\cap B(x,r)} \int_{0<t<r} \mathbf{1}_{\mathcal{B}_1}(y,t)\, \frac{d\mathcal{H}^d(y)dt}{t} \leq C_1 r^d \tag{7.21}$$

*for $(x,r) \in \mathcal{A}$, with a constant $C_1$ that depends only on $n$ and $M$.*

For $0 < t \leq r$, cover $F_1 \cap B(x, 20r)$ with balls $B_i$, $i \in I(t)$, of the same radius $10t$. We can do this with less than $C(r/t)^{d-1}$ balls, i.e., so that $\sharp I(t) \leq C(r/t)^{d-1}$. Then the local Ahlfors-regularity given by Proposition 4.1 yields

$$\begin{aligned}
\int_{y\in E^*\cap B(x,r)} \int_{0<t<r} \mathbf{1}_{\mathcal{B}_1}(y,t)\, \frac{d\mathcal{H}^d(y)dt}{t} &= \int_{0<t<r} \mathcal{H}^d(\{y \in E^*\cap B(x,r); \mathrm{dist}(y, F_1) < 10t\})\, \frac{dt}{t} \\
&\leq \int_{0<t<r} \sum_{i\in I(t)} \mathcal{H}^d(\{E^*\cap B(x,r) \cap 2B_i)) \,\frac{dt}{t} \leq C \int_{0<t<r} \sharp I(t)\, t^d\, \frac{dt}{t} \\
&\leq C r^{d-1} \int_{0<t<r} dt \leq C r^d,
\end{aligned} \tag{7.22}$$

where the fact that $\mathcal{H}^d(\{E^*\cap B(x,r)\cap 2B_i)) \leq Ct^d$ comes from Proposition 4.1 and the fact that $B(x, 2r) \subset B_0$; this proves (7.21). □

Next we consider the pairs $(y,t) \in \mathcal{A} \setminus \mathcal{B}_1$ such that $B(y,2t)$ meets a $d$-dimensional face, without staying too close to it.

**Lemma 7.23.** *Denote by $F_2$ the union of all the $d$-dimensional faces of cubes from the dyadic net that was used to define the $L_j$, and set*

$$\begin{aligned} \mathcal{B}_2 = \big\{ (y,t) \in \mathcal{A} \setminus \mathcal{B}_1 \,;\, & B(y,2t) \text{ meets } F_2 \text{ but there exists} \\ & w \in E^* \cap B(y,2t) \text{ such that } \operatorname{dist}(w,F_2) \geq \varepsilon t \big\}. \end{aligned} \tag{7.24}$$

*Then*

$$\int_{y \in E^* \cap B(x,r)} \int_{0<t<r/10} \mathbf{1}_{\mathcal{B}_2}(y,t)\, \frac{d\mathcal{H}^d(y)dt}{t} \leq C_2(\varepsilon) r^d \tag{7.25}$$

*for $(x,r) \in \mathcal{A}$, with a constant $C_2(\varepsilon)$ that depends only on $n$, $M$, and $\varepsilon$.*

Let $(x,r) \in \mathcal{A}$ be given; we want to estimate the left-hand side of (7.25), which we write as

$$\Lambda = \int\int_{(y,t) \in \mathcal{B}_2(x,r)} \frac{d\mathcal{H}^d(y)dt}{t} \tag{7.26}$$

where $\mathcal{B}_2(x,r) = \big\{ (y,t) \in \mathcal{B}_2 \,;\, y \in B(x,r) \text{ and } 0 < t < r/10 \big\}$. For each $(y,t) \in \mathcal{B}_2(x,r)$ we use the definition (7.24) to pick $w \in E^* \cap B(y,2t)$ such that $\operatorname{dist}(w,F_2) \geq \varepsilon t$, and we set

$$Z(y,t) = E^* \cap B(w, \varepsilon t/2). \tag{7.27}$$

Obviously

$$\operatorname{dist}(z,F_2) \geq \varepsilon t/2 \quad \text{for } z \in Z(y,t). \tag{7.28}$$

Also, $|w-y| \leq 2t$ since $w \in B(y,2t)$, then $|w-x| \leq 2t + r \leq 12r/10$, because $y \in B(x,r)$, so

$$B(w,\varepsilon t) \subset B(x, 13r/10) \subset B_0; \tag{7.29}$$

then we can apply Proposition 4.1 to $B(w,\varepsilon t)$ and get that

$$\mathcal{H}^d(Z(y,t)) \geq C^{-1} \varepsilon^d t^d. \tag{7.30}$$

By choosing $w$ out of a fixed countable dense subset of $E^*$, we can make sure that the relation "$z \in Z(y,t)$" is measurable in all variables. Then

$$\Lambda \leq C\varepsilon^{-d} \int\int_{(y,t) \in \mathcal{B}_2(x,r)} \int_{z \in Z(y,t)} t^{-d}\, \frac{d\mathcal{H}^d(y) d\mathcal{H}^d(z) dt}{t}. \tag{7.31}$$

Let us use Fubini. In the domain of integration, $z$ lies in $E^* \cap B(x, 13r/10)$ by (7.29), and then $|y-z| \leq |y-w| + |w-z| \leq 2t + \varepsilon t$. So $y \in E^* \cap B(z, 3t)$, whose $\mathcal{H}^d$-measure is less than $Ct^d$, by Proposition 4.1 and because $B(z, 6t) \subset B(z, 6r/10) \subset B(x, 2r) \subset B_0$. In addition, $\varepsilon t/2 \leq \mathrm{dist}(z, F_2)$ by (7.28), and $\mathrm{dist}(z, F_2) \leq |z-w| + |w-y| + \mathrm{dist}(y, F_2) \leq \varepsilon t/2 + 2t + 2t \leq 5t$ by (7.24) in particular. Therefore, setting $d(z) = \mathrm{dist}(z, F_2) > 0$ to save space,

$$
\begin{aligned}
\Lambda &\leq C\varepsilon^{-d} \int_{z \in E^* \cap B(x,13r/10)} \int_{5^{-1}d(z) \leq t \leq 2\varepsilon^{-1}d(z)} t^{-d}\mathcal{H}^d(E^* \cap B(z, 3t))\, \frac{d\mathcal{H}^d(z)dt}{t} \\
&\leq C\varepsilon^{-d} \int_{z \in E^* \cap B(x,13r/10)} \int_{5^{-1}d(z) \leq t \leq 2\varepsilon^{-1}d(z)} \frac{d\mathcal{H}^d(z)dt}{t} \\
&\leq C\varepsilon^{-d} \log(10/\varepsilon)\, \mathcal{H}^d(E^* \cap B(x, 13r/10)) \leq C\varepsilon^{-d} \log(10/\varepsilon)\, r^d,
\end{aligned}
\tag{7.32}
$$

by Proposition 4.1, this time applied to a few balls $B_i$ of radius $r/10$ that cover $B(x, 13r/10)$, to make sure that the $2B_i$ are contained in $B_0$. This completes the proof of Lemma 7.23. $\square$

**Lemma 7.33.** *Suppose that $\varepsilon$ is small enough, depending on $n$ and $M$. Let $(y,t) \in \mathcal{A} \setminus (\mathcal{B}_1 \cup \mathcal{B}_2)$ be such that $t < r_0/10$ and $B(y, 2t)$ meets $F_2$. Then $b\beta_{E^*}(y,t) \leq 4\varepsilon$.*

Let $(y,t)$ be as in the statement. Since $(y,t) \notin \mathcal{B}_2$, we know that

$$
\mathrm{dist}(w, F_2) < \varepsilon t \text{ for every } w \in E^* \cap B(y, 2t). \tag{7.34}
$$

In particular, $\mathrm{dist}(y, F_2) < \varepsilon t$ and there is a $d$-dimensional face $F$ from our usual net such that $\mathrm{dist}(y, F) < \varepsilon t$. Let us check that

$$
\mathrm{dist}(y, F') \geq 8t \text{ for every face } F' \text{ of the usual net such that } F \not\subset F'. \tag{7.35}
$$

Let $\partial F$ denote the boundary of $F$; this is a union of $(d-1)$-dimensional faces, and $\mathrm{dist}(y, \partial F) \geq \mathrm{dist}(y, F_1) \geq 10t$ by definition of $F_1$ and because $(y,t) \notin \mathcal{B}_1$. Also let $f \in F$ be such that $|y - f| \leq \varepsilon t$; then (3.8) says that

$$
\mathrm{dist}(f, F') \geq \mathrm{dist}(f, \partial F) \geq \mathrm{dist}(y, \partial F) - \varepsilon t \geq 10t - \varepsilon t \geq 9t \tag{7.36}
$$

(in this lemma, we may assume that $\varepsilon$ is as small as we want, but anyway $\varepsilon > 1$ does not make sense because all the $\beta$-numbers are $\leq 1$); (7.35) follows at once. Of course this would be easy to adapt to polyhedral nets.

Notice that (7.35) applies to every $d$-dimensional face $F' \neq F$ of our net, so $\mathrm{dist}(y, F_2 \setminus F) \geq 8t$ and (7.34) implies that

$$
\mathrm{dist}(w, F) < \varepsilon t \text{ for every } w \in E^* \cap B(y, 2t). \tag{7.37}
$$

Now we we need to use the quasiminimality of $E$ to prove that $E$ has no apparent hole in $B(y,t)$; the next lemma is just a little more general than what we need; also, we shall need to return to its proof and generalize it in Lemma 9.14.

**Lemma 7.38.** *Let $C_0 \geq 1$ be given. Let $E \in GSAQ(B_0, M, \delta, h)$, and suppose that the rigid assumption is satisfied and that $h$ is small enough, depending on $n$, $M$, and $C_0$. Let $y \in E^*$ and $t > 0$ be such that $0 < t < \mathrm{Min}(r_0, \delta)$ and $B(y, 2t) \subset B_0$. Let $P \subset \mathbb{R}^n$ be a $d$-plane, and assume that*

$$\mathrm{dist}(w, P) < \varepsilon t \text{ for } w \in E^* \cap B(y, 2t) \tag{7.39}$$

*for some $\varepsilon > 0$ that we assume to be small enough, depending on $n$, $M$, and $C_0$. Also suppose that*

$$P \cap B(y, 2t) \subset L_j \text{ for every } j \text{ such that } L_j \text{ meets } B(y, 2t) \tag{7.40}$$

*and that we have a Lipschitz function $h : E^* \cap B(y, 2t) \times [0, 1] \to \mathbb{R}^n$ such that*

$$h(w, 0) = w \text{ and } h(w, 1) = \pi(w) \text{ for } w \in E^* \cap B(y, 2t), \tag{7.41}$$

*where $\pi$ denotes the orthogonal projection on $P$,*

$$|h(w, s) - h(w, s')| \leq C_0 \varepsilon t |s - s'| \text{ for } w \in E^* \cap B(y, 2t) \text{ and } 0 \leq s, s' \leq 1, \tag{7.42}$$

$$|h(w, s) - h(w', s)| \leq C_0 |w - w'| \text{ for } w, w' \in E^* \cap B(y, 2t) \text{ and } 0 \leq s \leq 1, \tag{7.43}$$

*and finally*

$$h(w, s) \in L_j \text{ for } 0 \leq s \leq 1 \text{ whenever } w \in E^* \cap L_j \cap B(y, 2t). \tag{7.44}$$

*Then*

$$\mathrm{dist}(p, E^*) \leq \varepsilon t \text{ for } p \in P \cap B(y, 3t/2) \tag{7.45}$$

*and*

$$\pi(E^* \cap B(y, 5t/3)) \text{ contains } P \cap B(\pi(y), 3t/2). \tag{7.46}$$

Of course the simplest choice of path $h$ is to take $h(w, s) = s\pi(w) + (1 - s)w$, but it does not always work, because it may be more efficient to follow the faces of dyadic cubes to stay in the $L_j$ and get (7.44). We keep this type of issues for the next sections.

Let us first check that Lemma 7.38 implies Lemma 7.33. Let $(y, t)$ and $F$ be as in Lemma 7.33, let $P$ be the $d$-plane that contains $F$; then (7.39) follows from (7.37).

Next let $j \leq j_{max}$ be such that $L_j$ meets $B(y, 2t)$. Let $F'$ be a face of $L_j$ that meets $B(y, 2t)$; by (7.35), $F'$ contains $F$. Also, $\mathrm{dist}(y, F) \leq \varepsilon t$ by definition of $F$, and $\mathrm{dist}(y, \partial F) \geq \mathrm{dist}(y, F_1) \geq 10t$ because $(y, t) \notin \mathcal{B}_1$ or by the end of (7.36), so $P \cap B(y, 2t) \subset F \subset F' \subset L_j$, and (7.40) holds.

We take $h(w, s) = s\pi(w) + (1-s)w$, then $h$ is Lipschitz, (7.41) holds trivially, (7.42) and (7.43) are true with $C_0 = 1$ because $|\pi(w) - w| \leq \varepsilon t$ for $w \in E^* \cap B(y, 2t)$, by (7.39), and because $\pi$ is 1-Lipschitz.

We finally check (7.44), i.e., that $[w, \pi(w)] \subset L_j$ when $w \in E^* \cap L_j \cap B(y, 2t)$. Observe that $P$ is defined by some equations $w_i = n_i r_0$, with $n_i \in \mathbb{Z}$, and that $\pi(w)$ is obtained from $w$ by replacing the $w_i$ such that $w_i \neq n_i r_0$ with $n_i r_0$. When this happens, $|w_i - n_i r_0| \leq \varepsilon t$, because $|\pi(w) - w| \leq \varepsilon t$. Then $[w, \pi(w)]$ is contained in any face of any dimension that contains $w$ (we just replace some noninteger coordinates of $r_0^{-1} w$ with other ones that lie in the same dyadic intervals). We apply this to any face of $L_j$ that contains $w$ and get that $[w, \pi(w)] \subset L_j$, as needed for (7.44).

So Lemma 7.38 applies. If we could use $P$ in the definition of $b\beta_{E^*}(y, t)$, (7.39) and (7.45) would imply that $b\beta_{E^*}(y, t) \leq 2\varepsilon$. We cannot exactly, because maybe $P$ does not contain $y$, but $\mathrm{dist}(y, P) \leq \varepsilon t$ by (7.39), so we can use a small translation of $P$ that goes through $y$, and we get that $b\beta_{E^*}(y, t) \leq 4\varepsilon$, as needed. Hence Lemma 7.33 will follow from Lemma 7.38 as soon as we prove it.

Lemma 7.38 is a variant of Lemma 10.10 in [DS4], but we need to modify some things because of the boundary constraints (1.7). We define a first family of deformations $\varphi_s$. First let $\psi : [0, +\infty) \to [0, 1]$ be such that

$$
\begin{aligned}
&\psi(\rho) = 1 \text{ for } 0 \leq \rho \leq \frac{5t}{3} + (C_0 + 1)\varepsilon t, \\
&\psi(\rho) = 0 \text{ for } \rho \geq \frac{5t}{3} + (C_0 + 2)\varepsilon t, \text{ and} \\
&\psi \text{ is affine on } [\frac{5t}{3} + (C_0 + 1)\varepsilon t, \frac{5t}{3} + (C_0 + 2)\varepsilon t].
\end{aligned}
\tag{7.47}
$$

Then set

$$
\varphi_s(w) = h(w, s\psi(|w - y|)) \ \text{ for } w \in E^* \text{ and } 0 \leq s \leq 1; \tag{7.48}
$$

the fact that $h(w, s)$ is only defined for $w \in E^* \cap B(y, 2t)$ is not a problem, because we can set

$$
\varphi_s(w) = w \ \text{ for } w \in E^* \setminus B(y, 11t/6) \text{ and } 0 \leq s \leq 1, \tag{7.49}
$$

where the two definitions make coincide on $B(y, 2t) \setminus B(y, 11t/6)$, if $\varepsilon$ is so small that $\frac{5t}{3} + (C_0 + 2)\varepsilon t < 11t/6$, because $s\psi(|w - y|) = 0$ there. To see that $(s, w) \to \varphi_s(w)$ is Lipschitz, we observe that the two definitions yield Lipschitz functions and coincide in $B(y, 2t) \setminus B(y, 11t/6)$.

We shall not use the $\varphi_t$ as they are, but let us check that they satisfy the properties (1.4)-(1.8), with the closed ball

$$
B = \overline{B}(y, 11t/6) \tag{7.50}
$$

and with respect to the set $E^*$. First observe that we just checked (1.4), and that (1.5) and (1.8) are very easy consequences of the definition.

For (1.6), let $w \in B$ and $0 \leq s \leq 1$ be given; we want to check that $\varphi_s(w) \in B$. This is trivial if $\varphi_s(w) = w$, so we may assume that $w \in B(y, 2t)$ and $\varphi_s(w)$ is given by (7.48). Then $s\psi(|w-y|) \neq 0$, and hence $|w - y| < \frac{5t}{3} + (C_0 + 2)\varepsilon t$. Notice that

$$|\varphi_s(w) - w| = |h(w, s\psi(|w-y|)) - w| \leq C_0 \varepsilon t \ \text{ for } w \in E^* \text{ and } 0 \leq s \leq 1 \tag{7.51}$$

by (7.41) and (7.42) if $w \in B(y, 2t)$, and because $\varphi_s(w) = w$ otherwise. If $\varepsilon$ is small enough, $\varphi_s(w) \in B$ when $|w - y| < \frac{5t}{3} + (C_0 + 2)\varepsilon t$, as needed for (1.6).

Finally (1.7) holds because if $x \in E^* \cap L_j \cap B$ and $s \in [0, 1]$, then $x \in B(y, 2t)$ and $\varphi_s(x) \in L_j$ by (7.48) and (7.44).

Now we shall assume that (7.46) fails, use this to construct a deformation that completes the $\varphi_t$ and makes $E^* \cap B(y, t)$ essentially vanish, and get a contradiction. So let us assume that we can find

$$p \in P \cap B(\pi(y), 3t/2) \setminus \pi(E^* \cap B(y, 5t/3)). \tag{7.52}$$

Observe that

$$\pi(\varphi_1(w)) \text{ lies out of } B(\pi(y), 3t/2) \text{ for } w \in E^* \cap B(y, 2t) \setminus B(y, 5t/3) \tag{7.53}$$

just because $|\pi(\varphi_1(w)) - \pi(y)| \geq |w-y| - |\pi(\varphi_1(w)) - w| - |\pi(y) - y| \geq \frac{5t}{3} - |\pi(\varphi_1(w)) - w| - \varepsilon t$ by (7.39) and $|\pi(\varphi_1(w)) - w| \leq |\pi(\varphi_1(w)) - \pi(w)| + |\pi(w) - w| \leq |\varphi_1(w) - w| + |\pi(w) - w| \leq (C_0 + 1)\varepsilon t$ by (7.51) and (7.39). Thus

$$p \in P \cap B(\pi(y), 3t/2) \setminus \pi(E^* \cap B), \tag{7.54}$$

where $B = \overline{B}(y, 11t/6)$ as before, by (7.52) and (7.53). Since $E^* \cap B$ is compact, we can find $\tau > 0$ (possibly extremely small) such that

$$P \cap B(p, \tau) \text{ does not meet } \pi(E^* \cap B). \tag{7.55}$$

Define $g : P \cap B(\pi(y), \frac{5t}{3}) \setminus B(p, \tau) \to P \cap \partial B(\pi(y), \frac{5t}{3})$ as the radial projection centered at $p$, i.e., by the fact that

$$g(w) \in \partial B(\pi(y), \frac{5t}{3}) \text{ and } w \in [p, g(w)]. \tag{7.56}$$

Also set $g(z) = z$ for $z \in \partial B(\pi(y), \frac{5t}{3})$; this gives a Lipschitz mapping defined on the union, and with values in $\overline{B}(\pi(y), \frac{5t}{3})$. We extend $g$ to $\overline{B}(\pi(y), \frac{5t}{3})$ in a Lipschitz way, with values in $\overline{B}(\pi(y), \frac{5t}{3})$ (use the Whitney extension theorem, and compose with the radial projection on $\overline{B}(\pi(y), \frac{5t}{3})$ if needed). Finally extend $g$ to $\mathbb{R}^n$ by setting

$$g(z) = z \ \text{ for } z \in \mathbb{R}^n \setminus B(\pi(y), \frac{5t}{3}). \tag{7.57}$$

This yields a Lipschitz mapping defined on $\mathbb{R}^n$, which we still call $g$.

Now we define the $\varphi_s$, $1 \leq s \leq 2$, by

$$\varphi_s(w) = (2-s)\varphi_1(w) + (s-1)g(\varphi_1(w)) \ \text{ for } w \in \mathbb{R}^n \text{ and } 1 \leq s \leq 2. \tag{7.58}$$

Let us check the analogue of (1.4)-(1.8) for the $\varphi_s$, $0 \leq s \leq 2$, with the same choice of $B = \overline{B}(y, 11t/6)$ and again with respect to $E^*$.

The mapping $(s,w) \to \varphi_s(w)$ is Lipschitz on $[1,2] \times E^*$, so (1.4) and (1.8) hold. We already know that $\varphi_0(w) = w$ for $w \in \mathbb{R}^n$.

Next, if $w \in E^* \setminus B$, we know from our earlier verification of (1.5) that $\varphi_s(w) = w$ for $0 \leq s \leq 1$, and in particular $\varphi_1(w) = w \notin B$, hence $\varphi_1(w) \notin B(\pi(y), \frac{5t}{3})$ (recall that $|\pi(y) - y| \leq \varepsilon t$), and $g(\varphi_1(w)) = \varphi_1(w)$ by (7.57). Then $\varphi_s(w) = w$ for $1 \leq s \leq 2$ by (7.58), and the analogue of (1.5) holds.

If $w \in B$, we know that $\varphi_s(w) \in B$ for $0 \leq s \leq 1$; then $g(\varphi_1(w)) \in B$ because $g(B(\pi(y), \frac{5t}{3})) \subset B(\pi(y), \frac{5t}{3})$ and $g(z) = z$ out of $B(\pi(y), \frac{5t}{3})$. So (1.6) holds because the $\varphi_s(w)$, $s \geq 1$, lie on the segment $[\varphi_1(w), g(\varphi_1(w))] \subset B$.

We are left with (1.7) to check, and again it is nice to do this relatively to $E^*$ (and not the full $E$). Let $j$ and $w \in E^* \cap L_j \cap B$ be given; we want to show that $\varphi_s(w) \in L_j$ for $1 \leq s \leq 2$ (we already know this for $s \leq 1$). First assume that

$$w \in B(y, \frac{5t}{3} + (C_0+1)\varepsilon t). \tag{7.59}$$

Then $\psi(|w-y|) = 1$ by (7.47) and $\varphi_1(w) = h(w,1) = \pi(w)$ by (7.48) and (7.41). In particular, $\varphi_1(w) \in P \setminus B(p,\tau)$, by (7.55)). If $\varphi_1(w) \in B(\pi(y), \frac{5t}{3})$, then $g(\varphi_1(w))$ is the radial projection of $\varphi_1(w)$ on $\partial B(\pi(y), \frac{5t}{3})$ (as in (7.56)); otherwise, $g(\varphi_1(w)) = \varphi_1(w)$ by (7.57); in both cases, $\varphi_s(y) \in [\varphi_1(w), g(\varphi_1(w))] \subset P \cap B$ (recall that $\varphi_s(y) \in B$ by (1.6)). Now $w \in L_j \cap B \subset B(y,2t)$, so $P \cap B \subset P \cap B(y,2t) \subset L_j$ by (7.40) and $\varphi_s(y) \in L_j$ when (7.59) holds.

If (7.59) fails, $w \in B \setminus B(y, \frac{5t}{3} + (C_0+1)\varepsilon t)$. Then $\varphi_1(w)$ lies out of $B(y, \frac{5t}{3} + \varepsilon t)$ by (7.51), hence also out of $B(\pi(y), \frac{5t}{3})$. In this case, $g(\varphi_1(w)) = \varphi_1(w)$, hence (7.58) and (1.7) for $0 \leq s \leq 1$ yield $\varphi_s(w) = \varphi_1(w) \in L_j$, as needed.

This completes our proof of (1.7) for the $\varphi_s$, $0 \leq s \leq 2$. Note also that (2.4) holds, because $\widehat{W} \subset B \subset\subset B_0$ since we assumed that $B(y,2t) \subset B_0$. We can now apply (2.5), because Proposition 3.3 says that $E^*$ is quasiminimal just like $E$. This is one instance where we use Proposition 3.3 for real; of course we could also have assumed that (7.41)-(7.45) hold with the whole $E_k$, or worked more here to extend our $\varphi_t$ correctly. Anyway, we get that

$$\mathcal{H}^d(W_2) \leq M\mathcal{H}^d(\varphi_2(W_2)) + hr^d, \tag{7.60}$$

where we set $W_2 = \big\{w \in E^* \cap B\,;\, \varphi_2(w) \neq w\big\}$ as in Definition 2.3.

Let us first control $\varphi_2$ on $A_1 = \big\{w \in E^* \cap 2B\,;\, \varphi_1(w) \in B(\pi(y), \frac{5t}{3})\big\}$. We claim that

$$\varphi_2(A_1) \subset P \cap \partial B(\pi(y), 5t/3). \tag{7.61}$$

Indeed, let $w \in A_1$ be given. Recall that $|\pi(y) - y| \leq \varepsilon t$ by (7.39) and $|\varphi_1(w) - w| \leq C_0\varepsilon t$ by (7.51), so $w \subset B(y, \frac{5t}{3} + (C_0+1)\varepsilon t)$ because $\varphi_1(w) \in B(\pi(y), \frac{5t}{3})$. Then $\psi(|w-y|) = 1$

by (7.47), so $\varphi_1(w) = h(w,1) = \pi(w)$ by (7.48) and (7.41). Also, $\pi(w) \in P \setminus B(p,\tau)$ by (7.55), so altogether $\varphi_1(w) = \pi(w) \in P \cap B(\pi(y), \frac{5t}{3}) \setminus B(p,\tau)$. This is the case when $g(\varphi_1(w)) = g(\pi(w))$ is the radial projection of $\pi(w)$ on $\partial B(\pi(y), \frac{5t}{3})$, as in (7.56). But $\varphi_2(w) = g(\varphi_1(w))$ by (7.58), so $\varphi_2(w) \in P \cap \partial B(\pi(y), \frac{5t}{3})$, as needed for (7.61).

We like (7.61) because it immediately implies that

$$\mathcal{H}^d(\varphi_2(A_1)) = 0. \tag{7.62}$$

Also,

$$E^* \cap B(y, \frac{5t}{3} - (C_0+1)\varepsilon t) \subset A_1 \cap W_2 \tag{7.63}$$

because if $w \in E^* \cap B(y, \frac{5t}{3} - (C_0+1)\varepsilon t)$, then $\varphi_1(w) \in B(\pi(y), \frac{5t}{3})$ (again by (7.51) and because $|\pi(y) - y| \leq \varepsilon t$), hence $w \in A_1$; in addition $w \in B(\pi(y), \frac{5t}{3})$ and $\varphi_2(w) \in \partial B(\pi(y), \frac{5t}{3})$ by (7.61), so $\varphi_2(w) \neq w$ and $w \subset W_2$.

Next consider

$$A_2 = E^* \cap B(y, \frac{5t}{3} + (C_0+2)\varepsilon t) \setminus A_1. \tag{7.64}$$

Notice that $A_2$ is fairly small, because it is contained in $E^* \cap B(y, \frac{5t}{3} + (C_0+2)\varepsilon t) \setminus B(y, \frac{5t}{3} - (C_0+1)\varepsilon t)$ (by (7.63)), and in an $\varepsilon t$-neighborhood of $P$ by (7.39). So we can cover $A_2$ by less than $C\varepsilon^{-d+1}$ balls $B_l$ of radius $(C_0+10)\varepsilon t$, centered on the $(d-1)$-dimensional sphere $P \cap \partial B(y, \frac{5t}{3})$. Proposition 4.1 says that $H^d(E \cap B_l) \leq C(C_0 \varepsilon t)^d$ for each $l$ (recall that $C_0 \geq 1$), so

$$\mathcal{H}^d(A_2) \leq C C_0^d \varepsilon t^d. \tag{7.65}$$

But we mostly need to control $\varphi_2(A_2)$, so let us prove that

$$\varphi_2 \text{ is } 2C_0\text{-Lipschitz on } A_2. \tag{7.66}$$

First we check that

$$\varphi_2(w) = \varphi_1(w) = h(w, \psi(|w-y|)) \ \text{ for } w \in A_2. \tag{7.67}$$

Indeed $\varphi_1(w) \notin B(\pi(y), \frac{5t}{3})$ since $w \notin A_1$, then $g(\varphi_1(w)) = \varphi_1(w)$ by (7.57), and so $\varphi_2(w) = g(\varphi_1(w)) = \varphi_1(w)$ by (7.58). The last identity comes from (7.48).

Now let $w, w' \subset A_2$, be given, and set $a = \psi(|w-y|)$ and $a' = \psi(|w'-y|)$, where $\psi$ is still as in (7.47) and (7.48). Thus $|a'-a| \leq (\varepsilon t)^{-1}|w'-w|$. Now

$$\begin{aligned} |\varphi_2(w) - \varphi_2(w')| &= |h(w,a) - h(w',a')| \\ &\leq |h(w,a) - h(w,a')| + |h(w,a') - h(w',a')| \\ &\leq C_0 \varepsilon t |a'-a| + C_0|w'-w| \leq 2C_0|w'-w| \end{aligned} \tag{7.68}$$

by (7.67), (7.42), and (7.43). This proves (7.66).

Next we check that

$$\varphi_2(w) = w \ \text{ for } w = E^* \setminus B(y, \frac{5t}{3} + (C_0+2)\varepsilon t) \tag{7.69}$$

Indeed $\psi(|w-y|) = 0$ by (7.47), hence $\varphi_1(w) = w$ by (7.48) or (7.49), and so $\varphi_2(w) = g(\varphi_1(w)) = g(w)$ by (7.58). But $w \subset \mathbb{R}^n \setminus B(y, \frac{5t}{3})$, so $g(w) = w$ by (7.57) and as needed for (7.69).

By (7.69) and (7.64), $W_2 = \{w \in E^* \cap B\,;\, \varphi_2(w) \neq w\} \subset A_1 \cup A_2$, and

$$\mathcal{H}^d(\varphi_2(W_2)) \leq \mathcal{H}^d(\varphi_2(A_2)) \leq 2^d C_0^d \mathcal{H}^d(A_2)) \leq C C_0^{2d} \varepsilon t^d \tag{7.70}$$

by (7.62), (7.66), and (7.65). On the other hand,

$$\mathcal{H}^d(W_2) \geq \mathcal{H}^d(E^* \cap B(y, \frac{5t}{3} - (C_0+1)\varepsilon t)) \geq C^{-1} t^d \tag{7.71}$$

by (7.63) and Proposition 4.1, and so (7.70) and (7.71) contradict (7.60) if $h$ and $\varepsilon$ are chosen small enough, depending on $M$, $n$, and $C_0$. So we were wrong to assume that there exists $p$ so that (7.52) holds, and this proves (7.46).

Now (7.45) follows from (7.46), because for $p \in P \cap B(y, 3t/2)$, we can find $w \in E^* \cap B(y, 5t/3)$ such that $\pi(w) = p$, and $|p-w| = |\pi(w) - w| \leq \varepsilon t$ because $\mathrm{dist}(w, P) \leq \varepsilon t$ by (7.39). Lemma 7.38 follows, and also Lemma 7.33 (see the comments below the statement of Lemma 7.38). □ □

We are finally in position to gather the estimates on the various bad sets and resume our proof of Theorem 6.1. We start with a control on the bad sets $\mathcal{B}_{E^*}(\varepsilon)$ of (7.3).

**Lemma 7.72.** *Let $E$, $x \in E^*$, and $r$ be as in Theorem 6.1. Then for each $\varepsilon > 0$*

$$\int_{y \in E^* \cap B(x, r/4)} \int_{0<t<r/10} \mathbf{1}_{\mathcal{B}_{E^*}(\varepsilon)}(y,t)\, \frac{d\mathcal{H}^d(y)dt}{t} \leq C(\varepsilon)\, r^d, \tag{7.73}$$

*where $C(\varepsilon)$ depends only on $n$, $M$, and $\varepsilon$.*

Obviously it will be enough to prove (7.73) for $\mathcal{B}_{E^*}(4\varepsilon)$ instead of $\mathcal{B}_{E^*}(\varepsilon)$. Also, we may as well suppose that $\varepsilon$ is small (depending on $n$ and $M$), because $\mathcal{B}_{E^*}(4\varepsilon)$ is larger when $\varepsilon$ is smaller (see the definition (7.3)).

Let $x$, $r$, and $\varepsilon$ be as in the statement, and set

$$\mathcal{B} = \mathcal{B}(4\varepsilon, x, r) = \{(y,t) \in \mathcal{B}_{E^*}(4\varepsilon)\,;\, y \in E^* \cap B(x, r/4) \text{ and } 0 < t < r/10\}; \tag{7.74}$$

then (7.73) is the same as

$$\int\int_{(y,t)\in\mathcal{B}} \frac{d\mathcal{H}^d(y)dt}{t} \leq C(\varepsilon) r^d. \tag{7.75}$$

Clearly $\mathcal{B} \subset \mathcal{A}$, with $\mathcal{A}$ as in (7.18), because $t \leq r < \mathrm{Min}(r_0, \delta)$ and $B(y, 2t) \subset B(x, 2r) \subset B_0$. The set $\mathcal{B} \cap \mathcal{B}_1$ is taken care of by Lemma 7.19, and similarly $\mathcal{B} \cap \mathcal{B}_2$ is controlled by Lemma 7.23. So we just need to show that

$$\int\int_{(y,t) \in \mathcal{B}'} \frac{d\mathcal{H}^d(y)dt}{t} \leq C(\varepsilon) r^d, \tag{7.76}$$

with $\mathcal{B}' = \mathcal{B} \setminus (\mathcal{B}_1 \cup \mathcal{B}_2)$. Notice that if $(y, t) \in \mathcal{B}'$, then $b\beta_{E^*}(y, t) > 4\varepsilon$ because $(y, t) \in \mathcal{B}_{E^*}(4\varepsilon)$ (see the definition (7.3)), and then Lemma 7.33 says that $B(y, 2t)$ does not meet $F_2$. That is,

$$\mathcal{B}' \subset \big\{(y, t) \in \mathcal{A}\,;\, y \in E^* \cap B(x, r/4),\ t < r/10, \text{ and } B(y, 2t) \text{ does not meet } F_2\big\}. \tag{7.77}$$

At this point, we want to cut $\mathcal{B}'$ into smaller sets for which we can apply Lemma 7.8, and for this a covering of $E^* \cap B(x, r/4) \setminus F_2$ will be useful. For $z \in E^* \cap B(x, r/4) \setminus F_2$, set

$$d(z) = \mathrm{Min}(r, \mathrm{dist}(z, F_2)) > 0 \tag{7.78}$$

and $B_z = B(y, \frac{d(z)}{100})$. Then select a maximal set $Z \subset E^* \cap B(x, r/4) \setminus F_2$ such that

$$\text{the } B_z,\ z \in Z, \text{ are disjoint.} \tag{7.79}$$

For each $y \in E^* \cap B(x, r/4) \setminus F_2$, we select $z = z(y) \in Z$ so that $B_z$ meets $B_y$; such a $z$ exists by maximality of $Z$, and it is easy to select $z(y)$ in a measurable way, because $Z$ is at most countable. Then cut $\mathcal{B}'$ as

$$\mathcal{B}' = \bigcup_{z \in Z} \mathcal{B}'(z), \tag{7.80}$$

where

$$\mathcal{B}'(z) = \big\{(y, t) \in \mathcal{B}'\,;\, z(y) = z\big\}. \tag{7.81}$$

Fix $z \in Z$ for the moment. We want to apply Lemma 7.8 to the quasiminimal set $E^*$ and the pair $(z, d(z)/2)$, so let us check the hypotheses. We know from Proposition 3.3 that $E^* \in GSAQ(B_0, M, \delta, h)$, just like $E$, but with $E^*$ (6.42) will be easier to check. First recall that $z \in E^* \cap B(x, r/4)$ and $d(z) \leq r$; hence the first assumptions that $z \in E^* \cap B_0$, $0 < d(z)/2 < \mathrm{Min}(r_0, \delta)$, and $B(z, d(z)) \subset B_0$ follow from the similar assumptions for $(x, r)$. Now we check the main assumption (6.42). Let $j$ be such that $L_j$ meets $B(z, d(z)/2)$; we want to show that $E^* \cap B(z, d(z)/2) \subset L_j$.

Recall from (7.78) that $\mathrm{dist}(z, F_2) \geq d(z)$, where $F_2$ denotes the union of all the $d$-dimensional faces of cubes from our dyadic grid (see Lemma 7.23). This means that the faces of $L_j$ that meet $B(z, d(z)/2)$ are at least $(d+1)$-dimensional. We know that there is at least one face like this, because we assume that $L_j$ meets $B(z, d(z)/2)$. But $B(z, d(z)/2) \subset B(x, r)$ (because $d(z) \leq r$); then the main assumption (6.2) says that

$E^* \cap B(x,r) \subset L_j$, which is enough for (6.42). So we may apply Lemma 7.8, and we get that

$$\int_{y\in E^*\cap B(z,d(z)/16)} \int_{0<t<d(z)/16} \mathbf{1}_{\mathcal{B}_{E^*}(4\varepsilon)}(y,t)\, \frac{d\mathcal{H}^d(y)dt}{t} \leq C(4\varepsilon)d(z)^d. \tag{7.82}$$

Return to $\mathcal{B}'(z)$. If $(y,t) \in \mathcal{B}'(z)$, then $100|z-y| < d(z)+d(y)$ because $B_z$ meets $B_y$ when $z = z(y)$; since $d(y) \leq d(z)+|z-y|$ by (7.78), we get that $100|z-y| < 2d(z)+|z-y|$, and hence $|z-y| < d(z)/49$ and also $d(y) \leq d(z)+|z-y| \leq \frac{50}{49}\,d(z)$.

If in addition $t < d(z)/16$, $(y,t)$ lies in the domain of integration of (7.82) (see (7.74) and the definition of $\mathcal{B}'$ below (7.76)) and it will be taken care of by (7.82). Otherwise, observe that $B(y,2t)$ does not meet $F_2$ (by (7.77)). If $d(y) = \operatorname{dist}(y,F_2)$, this shows that $t \leq d(y)/2 \leq d(z)/3$. Otherwise $d(y) = r$, so $d(z) \geq \frac{49d(y)}{50} \geq \frac{49r}{50}$. In this case too $t \leq d(z)/3$, because $t \leq r/10$ when $(y,t) \in \mathcal{B}'$. Altogether

$$\begin{aligned}\int\!\!\int_{(y,t)\in\mathcal{B}'(z)} \frac{d\mathcal{H}^d(y)dt}{t} &\leq C(4\varepsilon)d(z)^d + \int_{y\in E^*\cap B(z,d(z)/49)} \int_{d(z)/16<t\leq d(z)/3} \frac{d\mathcal{H}^d(y)dt}{t} \\ &\leq C(4\varepsilon)d(z)^d + \ln(16/3)\, \mathcal{H}^d(E^*\cap B(z,d(z)/49)) \leq C'(\varepsilon)d(z)^d\end{aligned} \tag{7.83}$$

by Proposition 4.1 (recall that $B(z,d(z)) \subset B(x,2r)$, so $B(z,d(z)/49)$ is not too large).

We now use (7.80) and sum over $z$:

$$\begin{aligned}\int\!\!\int_{(y,t)\in\mathcal{B}'} \frac{d\mathcal{H}^d(y)dt}{t} &= \sum_{z\in Z} \int\!\!\int_{(y,t)\in\mathcal{B}'(z)} \frac{d\mathcal{H}^d(y)dt}{t} \leq C'(\varepsilon)\sum_{z\in Z} d(z)^d \\ &\leq CC'(\varepsilon)\sum_{z\in Z} \mathcal{H}^d(E^*\cap B_z) \leq CC'(\varepsilon)r^d\end{aligned} \tag{7.84}$$

by Proposition 4.1, and because the $B_z$ are disjoint (by (7.79)) and contained in $B(x,r)$. This is (7.76), and Lemma 7.72 follows. □

Next we use Theorem 7.7 to prove the analogue of Proposition 6.41 under the (weaker) assumptions of Theorem 6.1. A more direct approach to Theorem 6.1 is also possible, using the weak geometric lemma and big projections now, but the next proposition is really a logical consequence of Lemma 7.72.

**Proposition 7.85.** *Let $E$, $x \in E^*$, and $r$ be as in Theorem 6.1 (again with $h$ small enough, depending on $M$ and $n$). Then there is a closed set $G_0 \subset E^* \cap B(x,r)$ and a mapping $\phi : G_0 \to \mathbb{R}^d$ such that*

$$\mathcal{H}^d(G_0) \geq \theta r^d \quad\text{and}\quad C_M^{-1}|y-z| \leq |\phi(y)-\phi(z)| \leq C_M|y-z| \text{ for } y,z \in G_0, \tag{7.86}$$

*where $\theta > 0$ and $C_M$ depend only on $M$ and $n$.*

In other words, we can find, in $E^*\cap B(x,r)$, a big piece of bilipschitz image of a subset of $\mathbb{R}^d$. Proposition 7.85 really follows from Lemma 7.72 and the proof of Theorem 7.7, but we need a localization argument so that we can use the statement of Theorem 7.7 as it is.

Let $E$, $x \in E^*$, and $r$ be as in Proposition 7.85 or Theorem 6.1; let use again Proposition 7.6 in [DS4], as we did for the proof of Lemma 7.8 but applied to a slightly smaller radius, to find a bounded Ahlfors-regular set $F$ such that

$$E^* \cap B(x, r/16) \subset F \subset E^* \cap B(x, r/8). \tag{7.87}$$

Since we want an unbounded Ahlfors-regular set, we consider the set $F' = F \cup H$, where $H$ is a $d$-plane such that $\mathrm{dist}(x, H) = r$. We want to use Theorem 7.7, so let us prove that for every small $\varepsilon > 0$, there is a constant $C(\varepsilon)$, that depends only on $n$, $M$, and $\varepsilon$, such that $F' \subset BWGL(\varepsilon, C(\varepsilon))$.

So we let $(x_1, r_1) \subset F' \times (0, +\infty)$ be given; we want to prove that

$$\int\int_{(y,t)\in\mathcal{B}} \frac{d\mathcal{H}^d(y)dt}{t} \leq C(\varepsilon) r_1^d, \tag{7.88}$$

where $\mathcal{B} = \big\{(y,t) \in (F' \cap B(x_1, r_1)) \times (0, r_1)\,;\, b\beta_{F'}(y,t) > \varepsilon\big\}$. We shall need to cut $\mathcal{B}$ into many pieces, to control various interfaces, but let us start with the most interesting case when $x_1 \in B(x, r/8)$ and $r_1 \leq r/40$. Then the main piece is

$$\mathcal{B}_1 = \Big\{(y,t) \in \mathcal{B}\,;\, \mathrm{dist}(w, F) \leq \frac{\varepsilon t}{2} \text{ for } w \in E^* \cap B(y, 2t)\Big\}. \tag{7.89}$$

We claim that $b\beta_{E^*}(y, 2t) > \varepsilon/4$ for $(y,t) \in \mathcal{B}_1$. Notice that $y \in E^*$ because $F \subset E^*$, so at least $b\beta_{E^*}(y, 2t)$ was officially defined in (7.2). If $b\beta_{E^*}(y, 2t) \leq \varepsilon/4$, there is a $d$-plane $P$ through $y$ such that

$$\sup_{w\in E^*\cap B(y,2t)} \mathrm{dist}(w, P) + \sup_{w\in P\cap B(y,2t)} \mathrm{dist}(y, E^*) \leq \varepsilon t/2; \tag{7.90}$$

but

$$\sup_{w\in F'\cap B(y,t)} \mathrm{dist}(w, P) \leq \sup_{w\in E^*\cap B(y,2t)} \mathrm{dist}(w, P) \tag{7.91}$$

because $F' \cap B(y,t) = F \cap B(y,t) \subset E^* \cap B(y,t)$, and

$$\sup_{w\in P\cap B(y,t)} \mathrm{dist}(y, F') \leq \sup_{w\in P\cap B(y,t)} \mathrm{dist}(y, F) \leq \sup_{w\in P\cap B(y,t)} \mathrm{dist}(y, E^*) + \frac{\varepsilon t}{2} \tag{7.92}$$

because $F \subset F'$ and by definition of $\mathcal{B}_1$, which contradicts the fact that $(y,t) \in \mathcal{B}$. Now

$$\int\int_{(y,t)\in\mathcal{B}_1} \frac{d\mathcal{H}^d(y)dt}{t} \leq \int_{y\in F\cap B(x_1,r_1)} \int_{0<r<r_1} \mathbf{1}_{\mathcal{B}_{E^*}(\varepsilon/4)}(y, 2t)\, \frac{d\mathcal{H}^d(y)dt}{t} \leq C(\varepsilon) r_1^d, \tag{7.93}$$

where the last inequality comes from Lemma 7.72, applied to the pair $(x_1, 20r_1)$.

Then we need to control $\mathcal{B}\backslash\mathcal{B}_1$. To $(y,t) \in \mathcal{B}\backslash\mathcal{B}_1$ we associate $z = z(y,t) \in E^* \cap B(y, 2t)$ such that $\mathrm{dist}(z, F) \geq \frac{\varepsilon t}{3}$, and the set $A(y,t) = E^* \cap B(z, \frac{\varepsilon t}{6})$. We can choose $z(y,t)$ and

$A(y,t)$ in such a way that $(y,t,w) \to \mathbf{1}_{A(y,t)}(w)$ is measurable, for instance by choosing the first available $z$ from a sufficiently dense countable set in $E^*$. Notice that if $w \in A(y,t)$, then $|w-y| \leq 3t$ and $t \leq 6\varepsilon^{-1} \operatorname{dist}(w,F)$. In addition, $t \geq |w-y|/3 \geq \operatorname{dist}(w,F)/3$ because $y \in F$, so that

$$t \in T(w) = (0, r_1] \cap [\operatorname{dist}(w,F)/3, 6\varepsilon^{-1} \operatorname{dist}(w,F)] \tag{7.94}$$

(recall that $0 < t \leq r_1$ for $(y,t) \in \mathcal{B}$). Finally $w \in B(x_1, 4r_1)$ because $y \in B(x_1, r_1)$. Now multiple uses of Proposition 4.1 yield

$$\begin{aligned}
\int\int_{(y,t)\in\mathcal{B}\setminus\mathcal{B}_1} \frac{d\mathcal{H}^d(y)dt}{t} &\leq C \int\int_{(y,t)\in\mathcal{B}\setminus\mathcal{B}_1} \int_{w\in A(y,t)} (\varepsilon t)^{-d} \, \frac{d\mathcal{H}^d(y)d\mathcal{H}^d(w)dt}{t} \\
&\leq C(\varepsilon) \int_{w\in E^*\cap B(x_1,4r_1)} \int_{t\in T(w)} \int_{y\in F\cap B(w,3t)} \frac{\mathcal{H}^d(y)d\mathcal{H}^d(w)dt}{t^{d+1}} \\
&\leq C(\varepsilon) \int_{w\in E^*\cap B(x_1,4r_1)} \int_{t\in T(w)} \frac{d\mathcal{H}^d(w)dt}{t} \\
&\leq C(\varepsilon) \ln(18/\varepsilon) \int_{w\in E^*\cap B(x_1,4r_1)} d\mathcal{H}^d(w) \leq C(\varepsilon) r_1^d,
\end{aligned} \tag{7.95}$$

as needed for (7.88).

This proves (7.88) when $x_1 \in B(x, r/8)$ and $r_1 \leq r/40$. When $x_1 \in H$ and $r_1 \leq r/40$, $F'$ coincides with $H$ on $B(x_1, 2r_1)$, and $\mathcal{B} = \emptyset$. This settles the case when $r_1 \leq r/40$ because $F \subset B(x, r/8)$ by (7.87). So assume now that $r_1 > r/40$.

Note that $b\beta_{F'}(y,t) \leq \varepsilon$ for $y \in F'$ and $t \geq 10\varepsilon^{-1} r$, simply because we can use $H$ in the definition of $b\beta_{F'}(y,t)$, and by (7.87). Thus $\mathcal{B} = \mathcal{B}_2 \cup \mathcal{B}_3$, where $\mathcal{B}_2 = \big\{(y,t) \in \mathcal{B}\,;\, 0 < t < r/40\big\}$ and $\mathcal{B}_3 = \big\{(y,t) \in \mathcal{B}\,;\, r/40 \leq t < 10\varepsilon^{-1} r\big\}$. Since $b\beta_{F'}(y,t) = 0$ when $y \in H$ and $0 < t < r/40$,

$$\begin{aligned}
\int\int_{(y,t)\in\mathcal{B}_2} \frac{d\mathcal{H}^d(y)dt}{t} &\leq \int_{y\in F} \int_{0<t<r/40} \mathbf{1}_{\mathcal{B}_F(\varepsilon)}(y,t) \, \frac{d\mathcal{H}^d(y)dt}{t} \\
&\leq C(\varepsilon) r^d \leq C(\varepsilon) 40^d r_1^d,
\end{aligned} \tag{7.96}$$

where the main inequality comes from the case of $x_1 = x$ and $r_1 = r/40$, which was treated before. And

$$\begin{aligned}
\int\int_{(y,t)\in\mathcal{B}_3} \frac{d\mathcal{H}^d(y)dt}{t} &= \int_{y\in F'\cap B(x_1,r_1)} \int_{r/40\leq t<10\varepsilon^{-1}r} \frac{d\mathcal{H}^d(y)dt}{t} \\
&\leq C(\varepsilon)\mathcal{H}^d(F' \cap B(x_1, r_1)) \leq C(\varepsilon) r_1^d.
\end{aligned} \tag{7.97}$$

This completes our proof of (7.88), from which we deduce that $F' \subset BWGL(\varepsilon, C(\varepsilon))$ and, by Theorem 7.7, that $F' \in BPBI(\theta, C_M)$ for some choice of $\theta > 0$ and $C_M \geq 1$ that depend only on $n$ and $M$. The reader should not be surprised not to see $\varepsilon$ any more; recall that only one value of $\varepsilon$ is needed, that depends on $M$ and $n$ through the Ahlfors-regularity constants for $F'$. We apply the definition (7.6) of $BPBI(\theta, C_M)$ to the pair $(x,r)$ and get a set $G_0 \subset F \cap B(x,r) \subset E^* \cap B(x,r)$ (by (7.87)); then $G_0$ satisfies (7.86), and this completes our proof of Proposition 7.85. $\square$

## 8. Big projections and big pieces of Lipschitz graphs

We shall complete the proof of Theorem 6.1 in this section, with the help of yet another theorem of uniform rectifiability that we state now, in its initial context of unbounded Ahlfors-regular sets. First we need to define the weak geometric lemma, big projections, and big pieces of Lipschitz graphs for such sets.

Let $F \subset \mathbb{R}^n$ be an (unbounded) Ahlfors-regular set of dimension $d$; this means that $F$ is closed and (7.4) holds. Let $\beta_F(x,r)$ be the P. Jones number defined in (7.1), and set

$$\mathcal{M}_F(\varepsilon) = \big\{(x,r) \in F \times (0,+\infty)\,;\, \beta(x,r) > \varepsilon\big\} \tag{8.1}$$

for $0 < \varepsilon < 1$. We say that $F \in WGL(\varepsilon, C(\varepsilon))$ (or that $F$ satisfies a weak geometric lemma, with the constants $\varepsilon$ and $C(\varepsilon)$) when

$$\int_{y\in F\cap B(x,r)} \int_{0<t<r} \mathbf{1}_{\mathcal{M}_F(\varepsilon)}(y,t)\, \frac{d\mathcal{H}^d(y)dt}{t} \leq C(\varepsilon) r^d \tag{8.2}$$

for $x \in F$ and $0 < r < +\infty$. This is the same thing as the bilateral weak geometric lemma, but with the smaller functions $\beta_F(x,r)$ that only check whether every point of $F \cap B(x,r)$ lies near a plane.

We say that $F \in BP(\alpha)$ (for big projections) if for every choice of $x \in F$ and $0 < r < +\infty$, we can find a $d$-plane $P$ such that

$$\mathcal{H}^d(\pi(F \cap B(x,r))) \geq \alpha r^d, \tag{8.3}$$

where $\pi$ denotes the orthogonal projection on $P$.

Finally, we say that $F \in BPLG(\theta, A)$ (for big pieces Lipschitz graphs) when for all $x \in F$ and $r > 0$, we can find a $d$-dimensional $A$-Lipschitz graph $\Gamma \subset \mathbb{R}^n$ (which means, the graph of some Lipschitz function which is defined on a $d$-plane $P$, with values in $P^\perp$, and with a Lipschitz constant at most $A$) such that

$$\mathcal{H}^d(F \cap \Gamma \cap B(x,r))) \geq \theta r^d. \tag{8.4}$$

This property is slightly, but in general strictly stronger than uniform rectifiability (or equivalently $BPBI$). The following is Theorem 1.14 on page 857 of [DS2].

**Theorem 8.5.** *Let $F \subset \mathbb{R}^n$ be a closed unbounded Ahlfors-regular set of dimension $d$. If $F \in BP(\alpha)$ for some $\alpha > 0$ and $F \in WGL(\varepsilon, C(\varepsilon))$ for some small enough $\varepsilon$ (depending only on $n$, $\alpha$, and the constant $C_0$ in (7.4)), then $F \in BPLG(\theta, A)$, where $\theta > 0$ and $A \geq 0$ depend only on $n$, $\alpha$, $\varepsilon$, and $C(\varepsilon)$.*

**Proof of Theorem 6.1.** Let $E$, $x$, and $r$ be as in Theorem 6.1, and let $F$ and $F' = F \cup H$ be the Ahlfors-regular sets that we already used for Proposition 7.85. Thus $E^* \cap B(x, r/16) \subset F \subset E^* \cap B(x, r/8)$ as in (7.87), and $H$ is a $d$-plane such that $\mathrm{dist}(x, H) = r$.

We want to say that $F' \in BPLG(\theta, A)$ for some choice of $\theta$ and $A$, so let us check that $F' \in WGL(\varepsilon, C(\varepsilon)) \cap BP(\alpha)$. In fact, we already know that $F' \in WGL(\varepsilon, C(\varepsilon))$ with $\varepsilon$ as small as we want, because we even checked in (7.88) that $F' \in BWGL(\varepsilon, C(\varepsilon))$, which is obviously stronger. So we just need to check that $F' \in BP(\alpha)$ for some $\alpha > 0$, that depends only on $n$ and $M$.

So let $x_1 \subset F'$ and $r_1 > 0$ be given; we want to find $P = P(x_1, r_1)$ such that, as in (8.3),

$$\mathcal{H}^d(\pi(F' \cap B(x_1, r_1))) \geq \alpha r_1^d, \tag{8.6}$$

where $\pi$ denotes the orthogonal projection on $P$. The case when $x_1 \in H$ or $x_1 \in F$ but $r_1 \geq 4r$ is trivial, because we can take $P = H$, so let us assume that

$$x_1 \in F \text{ and } 0 < r_1 < 4r. \tag{8.7}$$

The idea is to find a pair $(x_2, r_2)$ to which we can apply Lemma 7.38, and deduce (8.6) from (7.46). Let us state what we need.

**Lemma 8.8.** *Theorem 6.1 will follow as soon as we prove the following. For each $\varepsilon > 0$, there is a small constant $c_\varepsilon$, which depend only on $M$, $n$, and $\varepsilon$, such that if $x \in E^*$, $r$, $F$, $x_1 \in F$, and $r_1 < 4r$ are as above, then we can find $x_2 \in F \cap B(x_1, r_1/10)$, $r_2 \in [c_\varepsilon r_1, r_1/10]$, and a $d$-plane $P$ such that the assumptions of Lemma 7.38 (relative to the pair $(x_2, r_2)$) are satisfied, and*

$$H^d(E^* \cap B(x_2, 2r_2) \setminus F) \leq \varepsilon r_2^d. \tag{8.9}$$

To prove the lemma, we assume the existence of $(x_2, r_2)$ and check (8.6). Since Lemma 7.38 applies (if $h$ is small enough, depending on $M$ and $n$), we get a plane $P$ such that (7.46) holds, and then

$$\begin{aligned} \mathcal{H}^d(\pi(F' \cap B(x_1, r_1))) &\geq \mathcal{H}^d(\pi(F \cap B(x_2, 2r_2))) \geq \mathcal{H}^d(\pi(E^* \cap B(x_2, 2r_2))) - \varepsilon r_2^d \\ &\geq \mathcal{H}^d(P \cap B(x_2, 3r_2/2))) - \varepsilon r_2^d \geq C^{-1} r_2^d \geq C(\varepsilon) r_1^d \end{aligned} \tag{8.10}$$

by (8.9), if $\varepsilon$ is small enough, and because $r_2 \geq c_\varepsilon r_1$. Then (8.6) holds, $F'$ has big projections, and Theorem 8.5 says that it contains big pieces of Lipschitz graphs too. We apply the definition (8.4) to the ball $B(x, r/16)$, and the Lipschitz graph $\Gamma$ such that $\mathcal{H}^d(F' \cap \Gamma \cap B(x, r/16))) \geq 16^{-d} \theta r^d$ also works for Theorem 6.1, except that we need to divide $\theta$ by $16^d$. □

So we are left with the assumption of Lemma 8.8 to check. That is, we are given $x_1 \in F$ and $r_1 < 4r$ as above, and we want to find a pair $(x_2, r_2)$. We shall produce first an intermediate pair $(y_0, t_0)$, with better regularity properties than $(x_1, r_1)$. These properties will be stated in terms of a very small $\varepsilon_0 < \varepsilon$, which will be chosen later, depending on $n$, $M$, and $\varepsilon$.

Recall that $F_1$ denotes the union of all the faces of dimension at most $d - 1$ of cubes from our usual dyadic net (see Lemma 7.19), and $F_2$ is the union of all the $d$-dimensional faces of cubes from that net, as in Lemma 7.23.

**Lemma 8.11.** *There exists a constant $c(\varepsilon_0) > 0$, that depends on $n$, $M$, and $\varepsilon_0$, such that for $(x_1, r_1)$ as above, we can find $(y_0, t_0)$ with the following properties:*

$$y_0 \in F \cap B(x_1, r_1/200) \ \text{ and } c(\varepsilon_0) r_1 \leq t_0 \leq r_1/100, \tag{8.12}$$

$$\operatorname{dist}(y_0, F_1) \geq 10 t_0, \tag{8.13}$$

$$\text{if } B(y_0, 2t_0) \text{ meets } F_2, \text{ then } \operatorname{dist}(w, F_2) \leq \varepsilon_0 t_0 \ \text{ for } w \in E^* \cap B(y_0, 2t_0), \tag{8.14}$$

$$b\beta_{E^*}(y_0, t_0) \leq \varepsilon_0, \tag{8.15}$$

*and*

$$\mathcal{H}^d(E^* \cap B(y_0, 2t_0) \setminus F) \leq \varepsilon_0 t_0^d. \tag{8.16}$$

The lemma will follow from more Carleson packing estimates. Let

$$\mathcal{A}_0 = \big\{(y,t)\,;\, y \in F \cap B(x_1, r_1/200) \ \text{ and } 0 < t \leq r_1/200\big\} \tag{8.17}$$

be a little smaller than the set of pairs that we want to pick from; notice that it is contained in the set $\mathcal{A}$ of (7.18). Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be as in (7.20) and (7.24), except that we replace $\varepsilon$ with $\varepsilon_0$ in (7.24); we know from Lemmas 7.19 and 7.23 that

$$\int\int_{(y,t)\in \mathcal{A}_0 \setminus \mathcal{B}_1 \cup \mathcal{B}_2} \frac{d\mathcal{H}^d(y)dt}{t} \leq C(\varepsilon_0) r_1^d. \tag{8.18}$$

Next set

$$\mathcal{B}_3 = \big\{(y,t) \in \mathcal{A}_0\,;\, b\beta_{E^*}(y,t) > \varepsilon_0\big\}. \tag{8.19}$$

Since $x_1 \in F \subset E^* \cap B(x, r/8)$ and $0 < r_1 < 4r$ by (8.7), the pair $(x_1, r_1/8)$ satisfies the assumptions of Lemma 7.72 (in particular because (6.2) is hereditary), so we get from that lemma that

$$\int\int_{(y,t)\in \mathcal{B}_3} \frac{d\mathcal{H}^d(y)dt}{t} \leq C(\varepsilon_0)\, r_1^d. \tag{8.20}$$

Next we want to check that

$$\int\int_{(y,t)\in \mathcal{B}_4} \frac{d\mathcal{H}^d(y)dt}{t} \leq C(\varepsilon_0)\, r_1^d \tag{8.21}$$

for the set

$$\mathcal{B}_4 = \big\{(y,t) \in \mathcal{A}_0\,;\, \mathcal{H}^d(E^* \cap B(y,2t) \setminus F) > \varepsilon_0 t^d\big\}. \tag{8.22}$$

Let us first find, for each pair $(y,t) \in \mathcal{B}_4$, a point $z(y,t)$ such that

$$z(y,t) \in E^* \cap B(y,2t) \ \text{ and } \ \operatorname{dist}(z(y,t),F) \geq 2\gamma t, \tag{8.23}$$

where the very small constant $\gamma > 0$ will chosen soon, depending on $\varepsilon_0$ as well.

For this we need to know that $E^* \setminus F$ is not too dispersed, and we are going to use again the dyadic patchwork that is provided for us by Proposition 7.6 of [DS4], and that we rapidly described near (6.7) and (7.10). This time, we shall need the other part of the small relative boundary condition (*7.10) (that is, (7.10) in [DS4]), namely the fact that for every cube $Q$ of our cubical patchwork for $F$,

$$\mathcal{H}^d\big(\big\{w \in E^* \setminus Q\,;\, \operatorname{dist}(x,Q) \leq \tau \operatorname{diam}(Q)\big\}\big) \leq C\tau^{1/C} \operatorname{diam}(Q)^d \tag{8.24}$$

for every (small) $\tau > 0$, with a constant $C$ that depend only on $M$ and $n$.

Return to the pair $(y,t) \in \mathcal{B}_4$, and denote by $Q_i$, $i \in I$, the collection of maximal cubes of the patchwork such that $Q_i \cap B(y,3t) \neq \emptyset$ and $\operatorname{diam}(Q_i) \leq t$. These sets are disjoint (by (*7.3) and maximality), and since $\operatorname{diam}(Q_i) \geq t/C$ (by maximality and the size property (*7.2)), we get that $\mathcal{H}^d(Q_i) \geq C^{-1}t^d$ (by the second part of (*7.2)) for $i \in I$. But $H^d(\cup_{i\in I} Q_i) \leq \mathcal{H}^d(F \cap B(y,4t)) \leq Ct^d$ by Proposition 4.1, and hence $I$ has at most $C$ elements.

Suppose we cannot find $z(y,t)$ such that (8.23) holds. Then each $z \in E^* \cap B(y,2t) \setminus F$ lies within $2\gamma t$ of $F \cap B(y,3t)$, so $\operatorname{dist}(z,Q_i) \leq 2\gamma t$ for some $i \in I$. Recall that $\operatorname{diam}(Q_i) \geq C^{-1}t$, so $z$ lies in the set of (8.24), with $\tau = 2C\gamma t$. We get that

$$\mathcal{H}^d(E^* \cap B(y,2t) \setminus F) \leq \sum_{i\in I} C\tau^{1/C} \operatorname{diam}(Q)^d \leq C'\tau^{1/C} t^d < \varepsilon_0 t^d \tag{8.25}$$

by (8.24), and if $\tau$ is chosen small enough, depending on $\varepsilon_0$. [And this is how we choose $\gamma$.] This contradiction with the definition (8.22) shows that we can find $z(y,t)$ as in (8.23). We may now prove the Carleson measure estimate (8.21) in the usual way. Denote by $A$ the right-hand side of (8.21); then

$$A \leq C \int\int_{(y,t)\in\mathcal{B}_4} (\gamma t)^{-d} \int_{w \in E^* \cap B(z(y,t),\gamma t)} \frac{d\mathcal{H}^d(y) d\mathcal{H}^d(w) dt}{t} \tag{8.26}$$

because $\mathcal{H}^d(E^* \cap B(z(y,t),\gamma t) \geq C^{-1}(\gamma t)^d$ by Proposition 4.1. Then apply Fubini. Notice that $w \in E^* \cap B(x_1, r_1/10)$ because $z(y,t) \in B(y,2t)$, $y \in F \cap B(x_1, r_1/200)$, and $t \leq r_1/200$ by (8.17). Next $2\gamma t \leq \operatorname{dist}(z(y,t),F) \leq \operatorname{dist}(z(y,t),y) \leq 2t$ by (8.23) and because $y \in F$, so $\gamma t \leq \operatorname{dist}(w,F) \leq 3t$ and $t \in T(w) = (0, r_1/200] \cap [\operatorname{dist}(w,F)/3, \operatorname{dist}(w,F)/\gamma]$. Finally, $|y-w| \leq |y - z(y,t)| + \gamma t \leq 3t$, and (8.26) yields

$$\begin{aligned} A &\leq C\gamma^{-d} \int_{w \in E^* \cap B(x_1,r_1/10)} \int_{t \in T(w)} t^{-d} \int_{y \in F \cap B(w,3t)} \frac{d\mathcal{H}^d(w) dt d\mathcal{H}^d(y)}{t} \\ &\leq C\gamma^{-d} \int_{w \in E^* \cap B(x_1,r_1/10)} \int_{t \in T(w)} \frac{d\mathcal{H}^d(w) dt}{t} \\ &\leq C\gamma^{-d} \ln(3/\gamma)\, \mathcal{H}^d(E^* \cap B(x_1, r_1/10)) \leq C(\varepsilon_0)\, r_1^d \end{aligned} \tag{8.27}$$

by Proposition 4.1 again. This proves (8.21).

Let us finally use our Carleson estimates to find a pair $(y_0, t_0)$ such that

$$(y_0, t_0) \subset \mathcal{A}_0 \setminus \big[\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4\big] \text{ and } t_0 \geq c(\varepsilon_0) r_1 \tag{8.28}$$

where as usual $c(\varepsilon_0) > 0$ depends on $M$, $n$, and $\varepsilon_0$. Observe that

$$\begin{aligned}\int\int_{(y,t)\in\mathcal{A}_0\,;\,t_0\geq c(\varepsilon_0)r_1} \frac{d\mathcal{H}^d(y)dt}{t} &= \mathcal{H}^d(F \cap B(x_1, r_1/200)) \int_{c(\varepsilon_0)r_1\leq t\leq r_1/200} \frac{dt}{t} \\ &\geq C^{-1} r_1^d \ln(200/c(\varepsilon_0))\end{aligned} \tag{8.29}$$

by (8.17) and Proposition 4.1. If we choose $c(\varepsilon)$ small enough, the right-hand side of (8.23) is larger than the sum of the right-hand sides of (8.18), (8.20), and (8.21), and then the domain of integration in (8.29) is not contained in $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4$, which means that we can choose $(y_0, t_0)$ as in (8.28). The pair $(y_0, t_0)$ satisfies (8.12) by (8.17) and (8.28), and (8.13)-(8.16) by definition of the $\mathcal{B}_j$; Lemma 8.11 follows. □

We now use the new pair $(y_0, t_0)$ to find the pair $(x_2, r_2)$ demanded by Lemma 8.8.

A first possibility is that $B(y_0, 2t_0)$ meets $F_2$. Then $\mathrm{dist}(w, F_2) \leq 4\varepsilon_0 t_0$ for $w \in E^* \cap B(y_0, 2t_0)$ (almost as in (7.34)), and the proof of Lemma 7.33 (both before and just after the statement of Lemma 7.38) applies, and says that we can apply Lemma 7.38 to the pair $(y, t) = (y_0, t_0)$ itself, with $C_0 = 1$ and if we took $4\varepsilon_0 \leq \varepsilon$. In this case we may take $(x_2, r_2) = (y_0, t_0)$ in the statement of Lemma 8.8, and (8.9) holds by (8.16).

So we may assume that

$$B(y_0, 2t_0) \cap F_2 = \emptyset. \tag{8.30}$$

Set

$$J = \big\{j \in [0, j_{max}]\,;\, L_j \cap B(y_0, 2t_0) \neq \emptyset\big\} \text{ and } L = \cap_{j\in J} L_j. \tag{8.31}$$

Notice that $J \neq \emptyset$ because $E \subset L_0 = \Omega_0$. Let $j \in J$, and let $A$ be any face of our grid that is contained in $L_j$ and meets $B(y_0, 2t_0)$. Such a face exists by definition of $J$, and by (8.30) $A$ is more than $d$-dimensional. Since $L_j \cap B(x, r) \supset A \cap B(y_0, 2t_0) \neq \emptyset$, (6.2) says that $E^* \cap B(y_0, 2t_0) \subset E^* \cap B(x, r) \subset L_j$. Hence $E^* \cap B(y_0, 2t_0) \subset L_j$ for $j \in J$. That is,

$$E^* \cap B(y_0, 2t_0) \subset L. \tag{8.32}$$

Since there are at most $C$ faces contained in $L$ and that meet $B(y_0, 2t_0)$, we can find such a face $A \subset L$, so that

$$\mathcal{H}^d(E^* \cap B(y_0, t_0/2) \cap A) \geq C^{-1}\mathcal{H}^d(E^* \cap B(y_0, t_0/2)) \geq C^{-1} t_0^d \tag{8.33}$$

by Proposition 4.1. We claim that we can find $d + 1$ points

$$w_0, \ldots, w_d \in E^* \cap B(y_0, t_0/2) \cap A, \tag{8.34}$$

so that for $1 \le l \le d$,

$$\text{(8.35)} \qquad \operatorname{dist}(w_l, P(w_0, \dots, w_{l-1})) \ge ct_0,$$

where $P(w_0, \dots, w_{l-1})$ denotes the affine subspace of dimension $l-1$ spanned by $w_0, \dots, w_{l-1}$ and $c > 0$ is a constant that depends only on $M$ and $n$. Indeed, if we cannot find some $w_l$, the whole $E^* \cap B(y_0, t_0/2) \cap A$ lies within $ct_0$ of $P(w_0, \dots, w_{l-1}) \cap B(y_0, t_0)$ and can be covered by less than $Cc^{-l+1}$ balls of radius $2ct_0$. Then $\mathcal{H}^d(E^* \cap B(y_0, t_0/2) \cap A) \le Cc^{-l+1}(ct_0)^d \le Cct_0^d$ by Proposition 4.1, and this contradicts (8.33) if $c$ is small enough; this proves the claim.

Set $P = P(w_0, \dots, w_d)$, and denote by $W$ the convex hull of the $w_l$; thus

$$\text{(8.36)} \qquad W \subset P \cap A,$$

because $A$ is convex and contains the $w_l$.

Recall that $b\beta_{E^*}(y_0, t_0) \le \varepsilon_0$ by (8.15); so there is a $d$-plane $P'$ through $y_0$ such that

$$\text{(8.37)} \qquad \begin{aligned} &\sup\big\{ \operatorname{dist}(w, P') \,;\, w \in E^* \cap B(y_0, t_0)\big\} \\ &\qquad\qquad + \sup\big\{ \operatorname{dist}(p, E^*) \,;\, w \in P' \cap B(y_0, t_0)\big\} \le \varepsilon_0 t_0. \end{aligned}$$

In particular, we can find $d+1$ points $p_l \in P'$, $0 \le l \le d$, such that $|p_l - w_l| \le \varepsilon_0 t_0$.

By a simple but slightly unpleasant verification using the affine independence estimate (8.35), we get that $P$ is $C\varepsilon_0 t_0$-close to $P'$ in $B(y_0, 10r_0)$, i.e., that

$$\text{(8.38)} \qquad \begin{aligned} &\operatorname{dist}(p, P') \le C\varepsilon_0 t_0 \text{ for } p \in P \cap B(y_0, 10r_0) \\ &\qquad\qquad \text{and } \operatorname{dist}(p', P) \le C\varepsilon_0 t_0 \text{ for } p' \in P' \cap B(y_0, 10r_0). \end{aligned}$$

The idea is simply that we can compute $P$ and $P'$ from the position of the $w_l$ and the $p_l$, in a stable way. If we were to do the computation, we would take coordinates where $P = \big\{x_{d+1} = \dots = x_n = 0\big\}$, observe that the $p_l - p_0$, $1 \le l \le d$, form a basis of $\operatorname{Vect}(P')$, write any unit vector $v \in \operatorname{Vect}(P')$ as $v = \sum_l a_l(p_l - p_0)$, with bounded coefficients $a_l$, get that the coordinates $v_l$, $l \ge d+1$ are all less than $C\varepsilon_0$, and then conclude. From (8.38) and (8.36) we deduce that

$$\text{(8.39)} \qquad \operatorname{dist}(w, P) \le C\varepsilon_0 t_0 \ \text{ for } w \in E^* \cap B(y_0, t_0).$$

Also set $w = \dfrac{1}{d+1} \displaystyle\sum_{l=0}^{d} w_l$ ; obviously

$$\text{(8.40)} \qquad w \in P \cap B(y_0, t_0/2)$$

by (8.34), (8.36), and convexity, but another easy consequence of (8.35) that we leave to the reader is that

$$\text{(8.41)} \qquad B(w, c_0 t_0) \cap P \subset W,$$

where $c_0 \in (0,1)$ is another small constant that depends on $M$ and $n$, but not on $\varepsilon$ or $\varepsilon_0$. We want to choose

$$x_2 \in F \cap B(w, c_0 t_0/100) \tag{8.42}$$

so we pick $p \in P'$ such that $|p - w| \leq C\varepsilon_0 t_0$, (which exists because $w \in P \cap B(y_0, t_0/2)$ and by (8.38)), and then $\xi \in E^*$ such that $|\xi - p| \leq \varepsilon_0 t_0$ (by (8.37)). By Proposition 4.1 and (8.16),

$$\mathcal{H}^d(E^* \cap B(\xi, c_0 t_0/200)) \geq C^{-1} c_0^d t_0^d > \varepsilon_0 t_0^d \geq \mathcal{H}^d(E^* \cap B(y_0, 2t_0) \setminus F) \tag{8.43}$$

if $\varepsilon_0$ is small enough, so we can find $x_2 \subset F \cap B(\xi, c_0 t_0/200)$; then (8.42) holds because $|\xi - w| \leq C\varepsilon_0 t_0 < c_0 t_0/200$.

We choose $x_2$ as in (8.42), and $r_2 = c_0 t_0/10$, and now we need to check that it satisfies the conditions mentioned in Lemma 8.8. First notice that

$$c_0 c(\varepsilon_0) r_1/10 \leq c_0 t_0/10 = r_2 \leq t_0/10 \leq r_1/1000 \tag{8.44}$$

by (8.12), so the size of $r_2$ is correct. Next,

$$|x_2 - y_0| \leq |x_2 - w| + |w - y_0| \leq c_0 t_0/100 + t_0/2 \leq \frac{2t_0}{3} \leq \frac{2r_1}{300} \tag{8.45}$$

by (8.42), (8.40), and (8.12), and since in addition $|y_0 - x_1| \leq r_1/200$ by (8.12), we easily get that $x_2 \subset B(x_1, r_1/10)$ (as needed).

Next we prove (8.9). Notice that

$$B(x_2, 2r_2) \subset B\big(y_0, 2r_2 + \frac{2t_0}{3}\big) \subset B(y_0, \frac{9t_0}{10}) \tag{8.46}$$

by the beginning of (8.45) and because $r_2 = c_0 t_0/10$, so

$$H^d(E^* \cap B(x_2, 2r_2) \setminus F) \leq H^d(E^* \cap B(y_0, t_0) \setminus F) \leq \varepsilon_0 t_0^d = \varepsilon_0 10^d c_0^{-d} r_2^d \leq \varepsilon r_2^d \tag{8.47}$$

by (8.16) and if $\varepsilon_0$ is chosen small enough; so (8.9) holds.

We still need to check that $(x_2, r_2)$ satisfies the hypotheses of Lemma 7.38, with the same $P$ as above. First, (7.39) follows from (8.39) if $\varepsilon_0 \leq c_0\varepsilon/10$, because $r_2 = c_0 t_0/10$ and $B(x_2, 2r_2) \subset B(y_0, t_0)$ by (8.46).

Next, suppose that $L_j$ meets $B(x_2, 2r_2)$. Then $j \in J$ (see (8.46) and (8.31)), and

$$P \cap B(x_2, 3r_2) \subset P \cap B(w, c_0 t_0) \subset W \subset A \subset L \subset L_j \tag{8.48}$$

by (8.42) and because $r_2 = c_0 t_0/10$, then by (8.41), (8.36), the definition of $A$, and (8.31). This proves (7.40).

Now we construct $h : E^* \cap B(x_2, 2r_2) \times [0,1] \to \mathbb{R}^n$, with the properties (7.41)-(7.44).

Let $w \in E^* \cap B(x_2, 2r_2)$ be given. By (8.39) and (8.46), $|\pi(w) - w| \leq C\varepsilon_0 t_0$. Then

$$\pi(w) \in P \cap B(x_2, 2r_2 + C\varepsilon_0 t_0) \subset A \tag{8.49}$$

by (8.48), so $\operatorname{dist}(w, A) \leq |\pi(w) - w| \leq C\varepsilon_0 t_0$. With the notation of (3.5), $w \in A^\eta$ with $\eta = C\varepsilon_0 t_0$.

Let us apply Lemma 3.17 to $L = r_0^{-1} A$ (we need to normalize as in Remark 3.25). We get a mapping $\Pi_A$, obtained from the one we get from Lemma 3.17 by a formula like (3.26). For $w \in E^* \cap B(x_2, 2r_2)$, set

$$h(w, s) = \Pi_A(w, 2s) \quad \text{for } 0 \leq s \leq 1/2 \tag{8.50}$$

and

$$\begin{aligned} h(w, s) &= (2s-1)\pi(w) + (2-2s)h(w, 1/2) \\ &= (2s-1)\pi(w) + (2-2s)\Pi_A(w, 1) \quad \text{for } 1/2 \leq s \leq 1. \end{aligned} \tag{8.51}$$

The fact that $h(w, 0) = w$ follows from (3.18), and $h(w, 1) = \pi(w)$ from (8.51), so (7.41) holds. Next (3.20) says that for $0 \leq s, s' \leq 1/2$,

$$|h(w,s) - h(w,s')| = |\Pi_A(w, 2s) - \Pi_A(w, 2s')| \leq C \operatorname{dist}(w, A)|s - s'| \leq C\varepsilon_0 t_0 |s - s'| \tag{8.52}$$

(the two renormalizations cancel). For $1/2 \leq s, s' \leq 1$,

$$\begin{aligned} |h(w,s) - h(w,s')| &= 2|s-s'||\pi(w) - h(w, 1/2)| \\ &\leq 2|s-s'|\big(|\pi(w) - h(w,0)| + |h(w,0) - h(w,1/2)|\big) \\ &= 2|s-s'|\big(|\pi(w) - w)| + |h(w,0) - h(w,1/2)|\big) \\ &\leq C|s-s'|\varepsilon_0 t_0 \end{aligned} \tag{8.53}$$

because $h(w, 0) = w$ and $|\pi(w) - w| \leq C\varepsilon_0 t_0$, and by (8.52). Altogether

$$|h(w,s) - h(w,s')| \leq C\varepsilon_0 t_0 |s - s'| = 10Cc_0^{-1}\varepsilon_0 r_2 |s - s'| \leq \varepsilon r_2 |s - s'| \tag{8.54}$$

for $0 \leq s, s' \leq 1$, and if $\varepsilon_0$ is small enough. So we get (7.42) with $C_0 = 1$.

Next each $h(\cdot, s)$ is $C$-Lipschitz because the $\Pi_A(\cdot, 2s)$ are $C$-Lipschitz, so (7.43) holds for some $C_0$ that depends only on $n$.

Finally we need to check that $h(w, s) \in L_j$ when $w \in E^* \cap L_j \cap B(x_2, 2r_2)$. For $0 \leq s \leq 1/2$, we use the fact that $\Pi_A$ preserves the faces (as in (3.22)), so that $h(w, s) = \Pi_A(w, 2s)$ lies in any of the faces of $L_j$ that contains $w$. For $s \geq 1/2$, we use the convexity of $A$, the fact that $\Pi_A(w, 1) = \pi_A(w) \in A$ by (3.19) and Lemma 3.4, and the fact that $\pi(w) \in A$ by (8.49), to get that $h(w, s) \in A \subset L \subset L_j$.

Thus (7.44) holds too, the pair $(x_2, r_2)$ satisfies the hypotheses of Lemma 7.38, and this completes the verification of the hypothesis of Lemma 8.8. Finally Theorem 6.1 follows, by Lemma 8.8. □

It is now very easy to prove that, still under the rigid assumption, our quasiminimal sets have the property of concentration, as in the following corollary. Incidentally, this corollary will be improved in Corollary 9.103 and Proposition 10.82. We include it here because it is easy to get now, but the proof of Section 10 is both globally simpler and more general. The property of concentration, introduced in [DMS], is a very nice tool to get the lower semicontinuity of $\mathcal{H}^d$, restricted to sequences of quasiminimal sets; this will be discussed again in Section 10, and even generalized slightly in Section 25.

**Corollary 8.55.** *For each choice of $n$, $M \geq 1$, and $\varepsilon > 0$, we can find $h > 0$ and $c_\varepsilon > 0$ such that the following holds. Suppose that $E \in GSAQ(B_0, M, \delta, h)$, with $B_0 = B(0,1)$, and that the rigid assumption is satisfied. Let $r_0 = 2^{-m} \leq 1$ denote the side length of the dyadic cubes of the usual grid. Let $x \in E^* \cap B_0$ and $0 < r < \mathrm{Min}(r_0, \delta)$ be such that $B(x, 2r) \subset B_0$. Assume in addition that (6.2) holds. Then we can find a pair $(y, t)$, such that $y \subset E^* \cap B(x, r/100)$, $c_\varepsilon r \leq t \leq r/100$, and*

$$\mathcal{H}^d(E^* \cap B(y,t)) \geq (1-\varepsilon)\omega_d t^d, \tag{8.56}$$

*where $\omega_d$ denotes the $d$-dimensional Hausdorff measure of the unit ball in $\mathbb{R}^d$.*

Indeed, let $(x, r)$ be as in the statement; we want to follow the end of the proof of Theorem 6.1. We do not need to define $F$ and $F'$, as we did after the statement of Theorem 8.5; instead, we start with $x_1 = x$ and $r_1 = r$, and check that we can find $(x_2, r_2)$ as in Lemma 8.8. That is, the proof of Lemma 8.11 gives a pair $(y_0, t_0)$, with the properties (8.12)-(8.15), where we do not need (8.16) and we can replace $F$ with $E^*$ in (8.12), and then we use the pair $(y_0, t_0)$, as we did after Lemma 8.11, to find a pair $(x_2, r_2)$ that satisfies the assumptions of Lemma 7.38. We take $y = x_2$ and $t = r_2$. Thus there is a $d$-plane $P$ such that

$$\mathrm{dist}(w, P) < \varepsilon t \text{ for } w \in E^* \cap B(y, 2t) \tag{8.57}$$

because (7.39) holds, and

$$\pi(E^* \cap B(y, 5t/3)) \text{ contains } P \cap B(\pi(y), 3t/2), \tag{8.58}$$

by (7.46) and where $\pi$ still denotes the orthogonal projection on $P$. For each $p \in P \cap B(\pi(y), (1-2\varepsilon)t)$, (8.58) gives a point $w \in E^* \cap B(y, 5t/3)$ such that $\pi(w) = p$. But $|p - w| = |\pi(w) - w| = \mathrm{dist}(w, P) \leq \varepsilon t$ by (8.57) and similarly $|\pi(y) - y| \leq \varepsilon t$, so $|w - y| \leq |p - \pi(y)| + |p - w| + |\pi(y) - y| \leq |p - \pi(y)| + 2\varepsilon t \leq t$ and $w \in B(y, t)$. So $P \cap B(\pi(y), (1-2\varepsilon)t) \subset \pi(E^* \cap B(y, t))$, hence

$$\begin{aligned} \mathcal{H}^d(E^* \cap B(y,t)) &\geq \mathcal{H}^d(\pi(E^* \cap B(y,t))) \\ &\geq \mathcal{H}^d(P \cap B(\pi(y), (1-2\varepsilon)t)) \geq \omega_d (1-2\varepsilon)^d t^d, \end{aligned} \tag{8.59}$$

which implies (8.56), even though only for the slightly larger $\varepsilon'$ such that $1 - \varepsilon' = (1-2\varepsilon)^d$; but $\varepsilon'$ is as small as we want and Corollary 8.55 follows. □

## 9. Extension to the Lipschitz assumption.

In this section we intend to generalize Theorem 6.1 and Corollary 8.55 to the case when we only have the Lipschitz assumption. So we shall assume, throughout this section, that

$$E \in GSAQ(U, M, \delta, h) \tag{9.1}$$

for an open set $U \subset \mathbb{R}^n$, and that

$$\text{the Lipschitz assumption is satisfied in } U. \tag{9.2}$$

Recall from Definition 2.7 that this means that there is a constant $\lambda > 0$ and a bilipschitz mapping $\psi : \lambda U \to B(0,1)$ such that the sets $\psi(\lambda L_j \cap U)$, $0 \le j \le j_{max}$ satisfy the rigid assumption described near (2.6).

Recall also that this last comes with the rigid scale $r_0 = 2^{-m}$, which is the side length of the cubes in the usual dyadic grid.

We shall denote by $\Lambda \ge 1$ the bilipschitz constant for $\psi$; thus

$$\Lambda^{-1}|x-y| \le |\psi(x) - \psi(y)| \le \Lambda|x-y| \ \text{ for } x, y \in U \tag{9.3}$$

and we expect our main constants to depend on $M$, $n$, and now $\Lambda$.

We shall also systematically assume that $h$ in (9.1) is sufficiently small, depending on the various constants at hand (such as $M$, $n$, $\Lambda$) for the proofs to work.

We start with an extension of Proposition 7.85, which will be easy because the conclusion of Proposition 7.85 (in essence, uniform rectifiability) is essentially invariant under bilipschitz mappings.

**Proposition 9.4.** *We can find constants $\theta > 0$ and $C(M, \Lambda) \ge 1$, that depend only on $n$, $M$ and $\Lambda$, such that if $h$ is small enough, depending on $n$, $M$ and $\Lambda$,*

$$x \in E^* \cap U, \ \ 0 < r < \mathrm{Min}(\lambda^{-1} r_0, \delta), \ \ B(x, 2r) \subset U \tag{9.5}$$

*and*

$$\begin{gathered}\text{if } j \in [0, j_{max}] \text{ is such that some face of dimension (strictly) more than } d \\ \text{of } L_j \text{ meets } B(x,r), \text{ then } E^* \cap B(x,r) \subset L_j,\end{gathered} \tag{9.6}$$

*then there is a closed set $G_0 \subset E^* \cap B(x,r)$ and a mapping $\phi : G_0 \to \mathbb{R}^d$ such that*

$$\mathcal{H}^d(G_0) \ge \theta r^d \ \text{ and } \ C(M,\Lambda)^{-1}|y-z| \le |\phi(y) - \phi(z)| \le C(M,\Lambda)|y-z| \text{ for } y, z \in G_0. \tag{9.7}$$

By definition, the faces of the $L_j$ are the images by $\lambda^{-1}\psi^{-1}$ of the faces of the boundaries

$$\widetilde{L}_j = \psi(\lambda L_j). \tag{9.8}$$

Of course we want to use Proposition 2.8, which says that

$$\psi(\lambda E) \in GSAQ(B(0,1), \Lambda^{2d}M, \Lambda^{-1}\lambda\delta, \Lambda^{2d}h). \tag{9.9}$$

Let $(x,r)$ be as in the statement; we want to apply Proposition 7.85 to the set $\widetilde{E} = \psi(\lambda E)$ and the pair $(\widetilde{x}, \widetilde{r})$, where $\widetilde{x} = \psi(\lambda x)$ and $\widetilde{r} = \Lambda^{-1}\lambda r$. First observe that $(\widetilde{E})^* = \psi(\lambda E^*)$ (see the definition (3.2)), so $\widetilde{x} \in (\widetilde{E})^* \cap B(0,1)$; it is clear that $0 < \widetilde{r} < \mathrm{Min}(r_0, \Lambda^{-1}\lambda\delta)$ by (9.5) and also

$$B(\widetilde{x}, 2\widetilde{r}) = B(\psi(\lambda x), 2\Lambda^{-1}\lambda r) \subset \psi(B(\lambda x, 2\lambda r)) \subset \psi(\lambda U) = B(0,1). \tag{9.10}$$

Next we check (6.2). Let $j$ be such that $B(\widetilde{x}, \widetilde{r})$ meets some face of dimension larger than $d$ of $\widetilde{L}_j = \psi(\lambda L_j)$. Since $B(\widetilde{x}, \widetilde{r}) \subset \psi(B(\lambda x, \lambda r))$ by the proof of (9.10), $B(\lambda x, \lambda r)$ meets $\lambda L_j$, so $B(x,r)$ meets $L_j$ and (9.6) says that $E^* \cap B(x,r) \subset L_j$ and then

$$\begin{aligned}(\widetilde{E})^* \cap B(\widetilde{x}, \widetilde{r}) &= \psi(\lambda E^*) \cap B(\widetilde{x}, \widetilde{r}) \subset \psi(\lambda E^*) \cap \psi(B(\lambda x, \lambda r)) \\ &= \psi(\lambda E^* \cap B(\lambda x, \lambda r)) \subset \psi(\lambda L_j) = \widetilde{L}_j,\end{aligned} \tag{9.11}$$

as needed for (6.2).

Now Proposition 7.85 gives a closed set $\widetilde{G}_0 \subset (\widetilde{E})^* \cap B(\widetilde{x}, \widetilde{r})$ and a mapping $\widetilde{\phi} : \widetilde{G}_0 \to \mathbb{R}^d$ such that

$$\mathcal{H}^d(\widetilde{G}_0) \geq \widetilde{\theta}\,\widetilde{r}^d \quad \text{and} \quad \widetilde{C}^{-1}|y-z| \leq |\widetilde{\phi}(y) - \widetilde{\phi}(z)| \leq \widetilde{C}|y-z| \text{ for } y,z \in \widetilde{G}_0, \tag{9.12}$$

where the constants $\widetilde{\theta}$ and $\widetilde{C}$ depend only on $M$ and $n$. We set $G_0 = \lambda^{-1}\psi^{-1}(\widetilde{G}_0)$ and $\phi(y) = \lambda^{-1}\widetilde{\phi}(\psi(\lambda x))$ for $y \in G_0$. Then $G_0 \subset E^* \cap B(x,r)$ (because $\lambda^{-1}\psi^{-1}(B(\widetilde{x}, \widetilde{r})) \subset \lambda^{-1}B(\psi^{-1}(\widetilde{x}), \Lambda\widetilde{r}) \subset B(x,r)$), and $\phi$ is $\Lambda\widetilde{C}$-bilipschitz. Finally,

$$\mathcal{H}^d(G_0) = \lambda^{-d}\mathcal{H}^d(\psi^{-1}(\widetilde{G}_0)) \geq \lambda^{-d}\Lambda^{-d}\mathcal{H}^d(\widetilde{G}_0) \geq \lambda^{-d}\Lambda^{-d}\widetilde{\theta}\,\widetilde{r}^d = \Lambda^{-2d}\widetilde{\theta} r^d. \tag{9.13}$$

So (9.6) holds with $C(M,\Lambda) = \Lambda\widetilde{C}$ and $\theta = \Lambda^{-2d}\widetilde{\theta}$; Proposition 9.4 follows. □

For the extension of Theorem 6.1 itself, we need to work a bit more, because the existence of big projections, or of big pieces of Lipschitz graphs, is not bilipschitz invariant. We need the following extension of Lemma 7.38.

**Lemma 9.14.** *There exist $C_0 \geq 1$, that depends only on $n$ and $\Lambda$, and small constants $\eta \in (0,1)$ and $\overline{\varepsilon} > 0$, that depend only on $n$, $M$, and $\Lambda$, such that the following holds if $h$ is small enough, depending only on $n$, $M$, and $\Lambda$. Let $y \in E^*$ and $t > 0$ be such that*

$$0 < t < C_0^{-1}\,\mathrm{Min}(\lambda^{-1}r_0, \delta) \text{ and } B(y, (C_0+1)t) \subset U. \tag{9.15}$$

*Set*

$$J = \big\{j \in [0, j_{max}]\,;\, L_j \text{ meets } B(y,2t)\big\} \quad \text{and} \quad L = \bigcap_{j \in J} L_j \tag{9.16}$$

*and suppose that*

$$\operatorname{dist}(w, L) \leq \eta t \quad \textit{for } w \in E^* \cap B(y, 2t). \tag{9.17}$$

*Finally let $P$ be a $d$-plane, and suppose that*

$$\operatorname{dist}(w, P) < \varepsilon t \quad \textit{for } w \in E^* \cap B(y, 2t) \tag{9.18}$$

*for some $\varepsilon \leq \overline{\varepsilon}$. Then*

$$\operatorname{dist}(p, E^*) \leq \varepsilon t \textit{ for } p \in P \cap B(y, 3t/2) \tag{9.19}$$

*and, if we denote by $\pi$ the orthogonal projection onto $P$,*

$$\pi(E^* \cap B(y, 5t/3)) \textit{ contains } P \cap B(\pi(y), 3t/2). \tag{9.20}$$

We could have taken $L = \mathbb{R}^n$ when $J = \emptyset$, but the simplest is to observe that this does not happen, as $L_0 = \Omega$ meets $B(y, 2t)$ because it contains $E$. We still want to proceed as in Lemma 7.38, but we shall need to be more careful about the way we move points around. Set

$$\widetilde{L} = \bigcap_{j \in J} \widetilde{L}_j = \psi(\lambda L) \quad \text{and} \quad \widehat{L} = r_0^{-1} \widetilde{L}. \tag{9.21}$$

Notice that $\widehat{L}$ is composed of faces of dyadic cubes of unit size. Denote by $\widehat{\Pi}$ the deformation that we get when we apply Lemma 3.17, with $\eta = 1/3$, to $\widehat{L}$. Thus $\widehat{\Pi}(\widehat{w}, s)$ is defined when $\widehat{w} \in \widehat{L}^{1/3}$ and $0 \leq s \leq 1$. Next use Remark 2.25 to define a similar deformation onto $\widetilde{L}$, by

$$\widetilde{\Pi}(\widetilde{w}, s) = r_0 \widehat{\Pi}(r_0^{-1} \widetilde{w}, s) \quad \text{for } \widetilde{w} \subset \widetilde{L}^{r_0/3} \text{ and } 0 \leq s \leq 1. \tag{9.22}$$

Observe that $\widehat{\pi} = \widehat{\Pi}(\cdot, 1)$ is a Lipschitz retraction from $\widehat{L}^{1/3}$ to $\widehat{L}$, and $\widetilde{\pi} = \widetilde{\Pi}(\cdot, 1)$ is a Lipschitz retraction from $\widetilde{L}^{r_0/3}$ to $\widetilde{L}$.

We conjugate once more to get a deformation onto $L$. Set $\eta_0 = \Lambda^{-1} \lambda^{-1} r_0/3$, to make sure that

$$\psi(\lambda w) \in \widetilde{L}^{r_0/3} \quad \text{when } w \in L^{\eta_0} \cap U, \tag{9.23}$$

and then define $\pi_L$ and $\Pi$ by

$$\pi_L(w) = \lambda^{-1} \psi^{-1}\big(\widetilde{\pi}(\psi(\lambda w))\big) \quad \text{for } w \in L^{\eta_0} \cap U \tag{9.24}$$

and

$$\Pi(w, s) = \lambda^{-1} \psi^{-1}\big(\widetilde{\Pi}(\psi(\lambda w), s)\big) \quad \text{for } w \in L^{\eta_0} \cap U \text{ and } 0 \leq s \leq 1. \tag{9.25}$$

Notice also that

$$B(y,3t) \subset L^{\eta_0} \cap U; \tag{9.26}$$

the first inclusion holds because $L$ meets $B(y,2t)$ (by (9.16)) and $t \leq C_0^{-1}\lambda^{-1}r_0 \leq \eta_0/10$ by (9.15) and if $C_0$ is large enough; the second one holds because $B(y,(C_0+1)t) \subset U$ by (9.15).

We shall now assume that (9.20) fails, combine $\Pi$ and a variant of the deformation used in Lemma 7.38 to build mappings $\varphi_s$, $0 \leq s \leq 3$, apply the definition of a quasiminimizer, and get a contradiction.

We start with a first stage, where we try to go from $w \in E^*$ to $\pi_L(w)$, but a first cut-off function $\psi_1$ will be required. Set

$$a = C_1\Lambda^2(\varepsilon+\eta)t, \tag{9.27}$$

where the geometric constant $C_1 \geq 1$, which depends only on $n$ (through the constants of Lemma 3.17), will be chosen soon. Then define $\xi_1 : [0,+\infty) \to [0,1]$ by

$$\begin{aligned} &\xi_1(\rho) = 1 \text{ for } 0 \leq \rho \leq \frac{5t}{3} + 5a, \\ &\xi_1(\rho) = 0 \text{ for } \rho \geq \frac{5t}{3} + 6a, \text{ and} \\ &\xi_1 \text{ is affine on } [\frac{5t}{3} + 5a, \frac{5t}{3} + 6a], \end{aligned} \tag{9.28}$$

and set

$$\varphi_s(w) = \Pi(w, s\xi_1(|w-y|)) \quad \text{for } w \in E^* \cap B(y,2t) \text{ and } 0 \leq s \leq 1. \tag{9.29}$$

We also set

$$\varphi_s(w) = w \quad \text{for } w \in E^* \setminus B(y, \tfrac{5t}{3} + 6a) \text{ and } 0 \leq s \leq 1. \tag{9.30}$$

We shall take $\overline{\varepsilon}$ and $\eta$ so small that when $\varepsilon \leq \overline{\varepsilon}$, $B(y, \frac{5t}{3} + 10a) \subset B(y,2t)$. Notice then that the two definitions above coincide when $w \in E^* \cap B(y,2t) \setminus B(y, \frac{5t}{3} + 6a)$, and hence $\varphi_s(w)$ is a Lipschitz function of $w$ and $s$. In addition,

$$\varphi_0(w) = w \quad \text{for } w \in E^* \tag{9.31}$$

and

$$\varphi_1(w) = \Pi(w,1) = \pi_L(w) \in L \quad \text{for } w \in E^* \cap B(y, \frac{5t}{3} + 5a), \tag{9.32}$$

by (3.19), Lemma 3.4, and the conjugations.

We want to estimate $|\varphi_s(w)-\varphi_{s'}(w)|$ when $w \in E^*\cap B(y,2t)$, and it will be convenient to set

$$\widetilde{w} = \psi(\lambda w) \text{ and } \widehat{w} = r_0^{-1}\widetilde{w} \ \text{ for } w \in E^* \cap B(y,2t). \tag{9.33}$$

Notice that $w \in L^{\eta_0} \cap U$ by (9.26), so $\widetilde{w} \subset \widetilde{L}^{r_0/3}$ and $\widehat{w} \subset \widehat{L}^{1/3}$ by (9.23) and (9.21). Also set $\alpha = s\xi_1(|w-y|)$ and $\alpha' = s'\xi_1(|w-y|)$. With these notations,

$$\begin{aligned}
|\varphi_s(w)-\varphi_{s'}(w)| &= |\Pi(w,s\xi_1(|w-y|)) - \Pi(w,s'\xi_1(|w-y|))| = |\Pi(w,\alpha)-\Pi(w,\alpha')| \\
&= \lambda^{-1}\big|\psi^{-1}\big(\widetilde{\Pi}(\psi(\lambda w),\alpha)\big) - \psi^{-1}\big(\widetilde{\Pi}(\psi(\lambda w),\alpha')\big)\big| \\
&\le \lambda^{-1}\Lambda|\widetilde{\Pi}(\psi(\lambda w),\alpha) - \widetilde{\Pi}(\psi(\lambda w),\alpha')| \\
&= \lambda^{-1}\Lambda|\widetilde{\Pi}(\widetilde{w},\alpha) - \widetilde{\Pi}(\widetilde{w},\alpha')| = \lambda^{-1}\Lambda r_0|\widehat{\Pi}(\widehat{w},\alpha) - \widehat{\Pi}(\widehat{w},\alpha')|,
\end{aligned} \tag{9.34}$$

by (9.32), (9.25), and (9.22). We now apply (3.20), with

$$\eta' =: \operatorname{dist}(\widehat{w},\widehat{L}) = r_0^{-1}\operatorname{dist}(\widetilde{w},\widetilde{L}) \le \lambda\Lambda r_0^{-1}\operatorname{dist}(w,L) \tag{9.35}$$

and also $\eta' \le 1/3$ because $\widehat{w} \subset \widehat{L}^{1/3}$. We get that for $w \in E^* \cap B(y,2t)$ and $0 \le s \le 1$,

$$\begin{aligned}
|\varphi_s(w)-\varphi_{s'}(w)| &\le C\lambda^{-1}\Lambda r_0\eta'|\alpha-\alpha'| \le C\Lambda^2\operatorname{dist}(w,L)|\alpha-\alpha'| \\
&\le C\Lambda^2\eta t|\alpha-\alpha'| \le C\Lambda^2\eta t|s-s'|
\end{aligned} \tag{9.36}$$

by (9.35) and (9.17).

Next, let $0 \le s \le 1$ and $w, w' \in E^* \cap B(y,2t)$ be given; we want to estimate

$$\begin{aligned}
|\varphi_s(w)-\varphi_s(w')| &= |\Pi(w,s\xi_1(|w-y|)) - \Pi(w',s\xi_1(|w'-y|))| \\
&\le |\Pi(w,s\xi_1(|w-y|)) - \Pi(w,s\xi_1(|w'-y|))| \\
&\qquad + |\Pi(w,s\xi_1(|w'-y|)) - \Pi(w',s\xi_1(|w'-y|))|.
\end{aligned} \tag{9.37}$$

If we set $\alpha = s\xi_1(|w-y|)$ and $\alpha' = s\xi_1(|w'-y|)$, the proof of (9.36) yields

$$\begin{aligned}
|\Pi(w,s\xi_1(|w-y|)) - \Pi(w,s\xi_1(|w'-y|))| &= |\Pi(w,\alpha)-\Pi(w,\alpha')| \\
&\le C\Lambda^2\eta t|\alpha-\alpha'| \le C\Lambda^2\eta t|\xi_1(|w-y|)-\xi_1(|w'-y|)| \\
&\le C\Lambda^2\frac{\eta t}{a}|w-w'| \le C_1^{-1}C|w-w'| \le C|w-w'|
\end{aligned} \tag{9.38}$$

by (9.28) and (9.27). The last term in (9.37) is less than $C\Lambda^2|w-w'|$ because the $\widehat{\Pi}(\cdot,s)$ are $C$-Lipschitz and we conjugate with $\psi$ and two dilations. So

$$|\varphi_s(w)-\varphi_s(w')| \le C\Lambda^2|w-w'| \tag{9.39}$$

for $0 \le s \le 1$ and $w, w' \in E^* \cap B(y,2t)$. Let us finally record that

$$|\varphi_s(w)-w| \le C\Lambda^2\eta t < a \ \text{ for } w \in E^* \text{ and } 0 \le s \le 1, \tag{9.40}$$

either by (9.36) (if $w \in E^* \cap B(y, 2t)$) or trivially by (9.30), and (for the second part) by (9.27) and because we choose $C_1$ is large enough now.

We are ready to start the second stage of the deformation. We do not want to change anything outside of $B(y, \frac{5t}{3} + 4a)$, so let immediately set

$$\varphi_s(w) = \varphi_1(w) \text{ for } w \in E^* \setminus B(y, \tfrac{5t}{3} + 4a) \text{ and } 1 \leq s \leq 2. \tag{9.41}$$

Define a second cut-off function $\xi_2 : [0, +\infty) \to [0, 1]$ by

$$\begin{aligned} &\xi_2(\rho) = 1 \text{ for } 0 \leq \rho \leq \frac{5t}{3} + 3a, \\ &\xi_2(\rho) = 0 \text{ for } \rho \geq \frac{5t}{3} + 4a, \text{ and} \\ &\xi_2 \text{ is affine on } [\frac{5t}{3} + 3a, \frac{5t}{3} + 4a]. \end{aligned} \tag{9.42}$$

This time, we try to go from $\pi_L(w)$ to $\pi_L(\pi(w))$. Set

$$\overline{\varphi}_s(w) = (s-1)\xi_2(|w-y|)\pi(w) + [1 - (s-1)\xi_2(|w-y|)]w, \tag{9.43}$$

and then

$$\varphi_s(w) = \pi_L(\overline{\varphi}_s(w)) \tag{9.44}$$

for $w \in E^* \cap B(y, \frac{5t}{3} + 5a)$ and $1 \leq s \leq 2$.

First observe that when $w \in E^* \cap B(y, \frac{5t}{3} + 5a) \setminus B(y, \frac{5t}{3} + 4a)$, $\xi_2(|w-y|) = 0$ by (9.42), so $\overline{\varphi}_s(w) = w$ and $\varphi_s(w) = \pi_L(w) = \varphi_1(w)$ by (9.32). So the two definitions of $\varphi_s(w)$ coincide on $E^* \cap B(y, \frac{5t}{3} + 5a) \setminus B(y, \frac{5t}{3} + 4a)$.

Similarly, when $w \in E^* \cap B(y, \frac{5t}{3} + 5a)$, (9.32) says that $\varphi_1(w) = \pi_L(w)$, and (9.44) gives the same result because $\overline{\varphi}_1(w) = w$. Altogether $\varphi_s(w)$ is a Lipschitz function of $s$ and $w$. But more precisely, if $1 \leq s \leq 2$ and $w, w' \in E^* \cap B(y, \frac{5t}{3} + 5a)$,

$$|\varphi_s(w) - \varphi_s(w')| = |\pi_L(\overline{\varphi}_s(w)) - \pi_L(\overline{\varphi}_s(w'))| \leq C\Lambda^2 |\overline{\varphi}_s(w) - \overline{\varphi}_s(w')| \tag{9.45}$$

and, if we set $\alpha = (s-1)\xi_2(|w-y|)$ and $\alpha' = (s-1)\xi_2(|w'-y|)$,

$$\begin{aligned} |\overline{\varphi}_s(w) - \overline{\varphi}_s(w')| &= |\alpha\pi(w) + (1-\alpha)w - \alpha'\pi(w') - (1-\alpha')w'| \\ &= \big|(\alpha - \alpha')(\pi(w) - w) + \alpha'(\pi(w) - \pi(w')) + (1-\alpha')(w - w')\big| \\ &\leq |\alpha - \alpha'||\pi(w) - w| + \alpha'|\pi(w) - \pi(w')| + (1-\alpha')|w - w'| \\ &\leq |\alpha - \alpha'|\varepsilon t + |w - w'| \\ &= (s-1)\big|\xi_2(|w-y|) - \xi_2(|w'-y|)\big|\varepsilon t + |w - w'| \\ &\leq \frac{\varepsilon t}{a}|w - w'| + |w - w'| \leq 2|w - w'| \end{aligned} \tag{9.46}$$

by (9.18), (9.42), and (9.27). Therefore

$$(9.47)\quad |\varphi_s(w)-\varphi_s(w')|\le C\Lambda^2|w-w'| \text{ for } w,w'\in E^*\cap B(y,\tfrac{5t}{3}+5a) \text{ and } 1\le s\le 2.$$

The variations in $s$ are easier, since for $1\le s,s'\le 2$ and $w\in E^*\cap B(y,\frac{5t}{3}+5a)$,

$$(9.48)\quad \begin{aligned}|\varphi_s(w)-\varphi_{s'}(w)| &\le C\Lambda^2|\overline{\varphi}_s(w)-\overline{\varphi}_{s'}(w)| = C\Lambda^2\xi_2(|w-y|)|s-s'||\pi(w)-w|\\ &\le C\Lambda^2|s-s'||\pi(w)-w|\le C\Lambda^2|s-s'|\varepsilon t\end{aligned}$$

by (9.44), because $\pi_L$ is $C\Lambda^2$-Lipschitz, and by (9.43) and (9.18). The case when $w\in E^*\setminus B(y,\frac{5t}{3}+5a)$ is even more trivial, because $\varphi_s(w)=\varphi_{s'}(w)=\varphi_1(w)$ by (9.41), so

$$(9.49)\quad |\varphi_s(w)-\varphi_1(w)|\le C\Lambda^2\varepsilon t|s-s'|<a|s-s'| \text{ for } 1\le s,s'\le 2 \text{ and } w\in E^*,$$

where the last inequality comes from (9.27) (if $C_1$ is large enough).

Let us also record the fact that

$$(9.50)\quad \overline{\varphi}_2(w)=\pi(w) \text{ and } \varphi_2(w)=\pi_L(\pi(w)) \text{ for } w\in E^*\cap B(y,\tfrac{5t}{3}+3a),$$

because $\xi_2(|w-y|)=1$ by (9.42), and by the definitions (9.43) and (9.44).

We are now ready for the third stage where we try to move points along $P$ to a lower-dimensional sphere. Let us first decide that

$$(9.51)\quad \varphi_s(w)=\varphi_2(w) \text{ for } w\in E^*\setminus B(y,\tfrac{5t}{3}+3a) \text{ and } 2\le s\le 3.$$

We now assume that (9.20) fails. This means that we can find

$$(9.52)\quad p\in P\cap B(\pi(y),3t/2)\setminus \pi(E^*\cap B(y,5t/3)).$$

Observe that for $w\in E^*\cap B(y,2t)\setminus B(y,5t/3)$, $\pi(w)$ lies outside of $B(\pi(y),3t/2)$ anyway, because $|\pi(w)-w|\le\varepsilon t$ by (9.18). So in fact $p$ lies out of $\pi(E^*\cap B(y,2t))$.

The slightly smaller compact set $\pi(E^*\cap\overline{B}(y,\frac{11t}{6}))$ does not contain $p$ either, so we can find a very small $\tau>0$ such that

$$(9.53)\quad P\cap B(p,\tau) \text{ does not meet } \pi(E^*\cap\overline{B}(y,\tfrac{11t}{6})).$$

We intend to move points inside

$$(9.54)\quad B_1=\overline{B}(\pi(y),5t/3).$$

First define $g:P\cap B_1\setminus B(p,\tau)\to P\cap\partial B_1$, to be the radial projection on $\partial B_1$, centered at $p$. Thus $g(z)$ is characterized by the fact that

$$(9.55)\quad g(z)\in P\cap\partial B_1 \text{ and } z\in[p,g(z)] \text{ for } z\in P\cap B_1\setminus B(p,\tau).$$

We also set

$$g(z) = z \quad \text{for } z \in \mathbb{R}^n \setminus B_1, \tag{9.56}$$

observe that this defines a Lipschitz mapping on $[P \cap B_1 \setminus B(p,\tau)] \cup [\mathbb{R}^n \setminus B_1]$, and extend it to $\mathbb{R}^n$ in a Lipschitz way, so that $g(B_1) \subset B_1$. The Lipschitz constant is very large, because we do not control $\tau$, but we don't care. Now set

$$\overline{\varphi}_s(w) = (s-2)g(\pi(w)) + (3-s)\pi(w) \tag{9.57}$$

and

$$\varphi_s(w) = \pi_L(\overline{\varphi}_s(w)) \tag{9.58}$$

for $w \in E^* \cap B(y, \frac{5t}{3} + 3a)$ and $2 \leq t \leq 3$. Notice that $\pi_L(\overline{\varphi}_s(w))$ is well defined, because $\overline{\varphi}_s(w) \in B(y, 2t) \subset L^{\eta_0} \cap U$ by (9.26) (also see the line below (9.30). For such $w$, (9.50) yields $\overline{\varphi}_2(w) = \pi(w)$ and $\varphi_2(w) = \pi_L(\pi(w))$, so the two definitions of $\varphi_2(w)$ coincide.

If $w \in E^* \cap B(y, \frac{5t}{3} + 3a) \setminus B(y, \frac{5t}{3} + a)$, then in addition

$$|\pi(w) - \pi(y)| \geq |w-y| - 2\varepsilon t \geq \frac{5t}{3} + a - 2\varepsilon t \geq \frac{5t}{3} + a/2, \tag{9.59}$$

by (9.18), (9.27), and if $C_1 \geq 4$; then $\pi(w) \in P \setminus B_1$ and (9.56) says that $g(\pi(w)) = \pi(w)$. In this case $\overline{\varphi}_s(w) = \pi(w)$ and $\varphi_s(w) = \pi_L(\pi(w)) = \varphi_2(w)$. We claim that because of this,

$$\varphi_s(w) = \varphi_2(w) \quad \text{for } w \in E^* \setminus B(y, \tfrac{5t}{3} + a) \text{ and } 2 \leq s \leq 3. \tag{9.60}$$

We just checked this when $w \in E^* \cap B(y, \frac{5t}{3} + 3a)$, but otherwise this is just (9.51).

We have two Lipschitz definitions of the $\varphi_s$ (by the formulas (9.51) and (9.57)) that overlap on the annulus $B(y, \frac{5t}{3} + 3a) \setminus B(y, \frac{5t}{3} + a)$, hence $(w,s) \to \varphi_s(w)$ is Lipschitz on $E^* \times [2,3]$.

We just constructed Lipschitz mappings $\varphi_s : E^* \times [0,3] \to \mathbb{R}^n$, and we want to check the properties (1.4)-(1.8), for the longer interval $[0,3]$, and with respect to the ball

$$B = \overline{B}(y, C_0 t), \tag{9.61}$$

where the value of $C_0 \geq 2$ will be decided soon. We already know that $(w,s) \to \varphi_s(w)$ is Lipschitz, so (1.8) holds. Also, $\varphi_0(w) = w$ by (9.31). Next,

$$\varphi_s(w) = w \quad \text{for } w \in E^* \setminus B(y, \tfrac{5t}{3} + 6a) \text{ and } 0 \leq s \leq 3; \tag{9.62}$$

by (9.30), (9.41), and (9.51). This takes care of (1.5) because $\overline{B}(y, \frac{5t}{3} + 6a) \subset B(y, 2t) \subset B$ if $\varepsilon$ and $\eta$ are small enough (see (9.27)).

For (1.6), we just need to check that

$$\varphi_s(w) \in B \quad \text{for } w \in E^* \cap B(y, \tfrac{5t}{3} + 6a) \text{ and } 0 \leq s \leq 3 \tag{9.63}$$

because $\varphi_s(w) = w \in B$ when $w \in B \setminus B(y, \frac{5t}{3} + 6a)$. For $0 \leq s \leq 2$, (9.40) and (9.49) say that $|\varphi_s(w) - w| \leq 2a$, and then $\varphi_s(w) \in B(y, \frac{5t}{3} + 8a) \subset B$.

So we may assume that $s \geq 2$, and that $\varphi_s(w) \neq \varphi_2(w)$. This implies that $w \in B(y, \frac{5t}{3} + a)$, by (9.60), and that $g(\pi(w)) \neq \pi(w)$ (because (9.57) and (9.58) apply). Then $\pi(w) \in B_1$ by (9.56), and also $\pi(w) \in P \setminus B(p, \tau)$ by (9.53). Hence (9.55) applies and $g(\pi(w)) \in P \cap \partial B_1$. By (9.57), $\overline{\varphi}_s(w) \in [\pi(w), g(\pi(w))] \subset P \cap B_1$ and

$$(9.64) \quad |\varphi_s(w) - \varphi_2(w)| = |\pi_L(\overline{\varphi}_s(w)) - \pi_L(\overline{\varphi}_2(w))| \leq C\Lambda^2 |\overline{\varphi}_s(w)) - \overline{\varphi}_2(w)| \leq 4C\Lambda^2 t$$

because $\mathrm{diam}(B_1) \leq 4t$. Since $\varphi_2(w) \in B(y, \frac{5t}{3} + 8a) \subset B(y, 2t)$, we simply choose $C_0 \geq 2 + 4C\Lambda^2$, and (9.63) follows from (9.64).

Finally we check (1.7). Let $j \leq j_{max}$ and $w \in E^* \cap L_j$ be given; we want to check that $\varphi_s(w) \in L_j$ for $0 \leq s \leq 3$. We may assume that $w \in E^* \setminus B(y, \frac{5t}{3} + 6a)$, because otherwise (9.62) says that $\varphi_s(w) = w$.

We first consider $s \leq 1$. Then $\varphi_s(w) = \Pi(w, s\xi_1(|w - y|))$ by (9.29). Set $s' = s\xi_1(|w - y|)$, and recall that the mapping $\widehat{\Pi}(\cdot, s')$ of Lemma 3.17 preserve all the faces of unit dyadic cubes. This is also true for $\Pi(\cdot, s')$ and the faces of the $L_j$. Then $\varphi_s(w)$ lies is in any of the faces of $L_j$ that contains $x$.

Next consider $s \in (1, 2]$. If $w \in E^* \setminus B(y, \frac{5t}{3} + 4a)$, then $\varphi_s(w) = \varphi_1(w) \in L_j$ by (9.41) and the previous case. Otherwise, $\varphi_s(w) = \pi_L(\overline{\varphi}_s(w)) \in L$ by (9.44) and the definition of $\pi_L$. Since $w \in E^* \cap L_j \cap B(y, \frac{5t}{3} + 4a)$, (9.16) says that $j \in J$ and $L \subset L_j$, so $\varphi_s(w) \in L_j$.

We are left with the case when $s > 2$. If $w \in E^* \setminus B(y, \frac{5t}{3} + 3a)$, (9.51) says that $\varphi_s(w) = \varphi_2(w) \in L_j$. Otherwise, $\varphi_s(w) = \pi_L(\overline{\varphi}_s(w)) \in L \subset L_j$ by (9.58) and the same argument as above.

This completes the verification of (1.4)-(1.8), relative to the set $E^*$ and the ball $B = \overline{B}(y, C_0 t)$. The condition (2.4) is also satisfied, since the set $\widehat{W}$ of (2.2) is contained in $B \subset\subset U$ by (9.61) and (9.15). In addition $C_0 t < \delta$ by (9.15). Finally recall that $E^* \in GSAQ(U, M, \delta, h)$, by (9.1) and Proposition 3.3. By Definition 2.3, (2.5) holds, i.e.,

$$(9.65) \qquad \mathcal{H}^d(W) \leq M\mathcal{H}^d(\varphi_3(W)) + hr^d,$$

where

$$(9.66) \qquad W = \big\{w \in E^* \cap B\,;\, \varphi_3(y) \neq y\big\}.$$

First we consider

$$(9.67) \qquad A_1 = E^* \cap B(y, \frac{5t}{3} - a)$$

and

$$(9.68) \qquad A_2 = \big\{w \in E^* \cap B(y, \frac{5t}{3} + 3a)\,;\, \varphi_3(w) \neq \varphi_2(w)\big\}.$$

Let us check that

$$(9.69) \qquad \varphi_3(w) \in \pi_L(P \cap \partial B_1) \ \text{ for } w \in A_1 \cup A_2.$$

First let $w \in A_1$ be given. Notice that $|\pi(y) - \pi(w)| \leq |y-w| + |\pi(y) - y| + |\pi(w) - w| \leq |y-w| + 2\varepsilon t < \frac{5t}{3}$ by (9.18), so $\pi(w) \in B(\pi(y), \frac{5t}{3}) \subset B_1$. But $\pi(w) \in P$ by definition of $\pi$, and $\pi(w) \notin P \cap B(p,\tau) \subset B_1$ by (9.53), so (9.55) says that $g(\pi(w)) \in P \cap \partial B_1$. Then

$$\varphi_3(w) = \pi_L(\overline{\varphi}_3(w)) = \pi_L(g(\pi(w))) \in \pi_L(P \cap \partial B_1) \tag{9.70}$$

by (9.58) and (9.57). Similarly, let $w \in A_2$ be given. Since $\varphi_2(w)$ and $\varphi_3(w)$ are still given by (9.58) and (9.57) in this case, the fact that they are different implies that $g(\pi(w)) \neq \pi(w)$. Then $\pi(w) \in B_1$ by (9.56). Again, $\pi(w) \notin P \cap B(p,\tau)$ by (9.53), so $g(\pi(w)) \in P \cap \partial B_1$ by (9.55), and (9.70) holds as above. This proves (9.69).

Recall that $\mathcal{H}^d(P \cap \partial B_1) = 0$ (this is a $(d-1)$-dimensional set), so

$$\mathcal{H}^d(\varphi_3(A_1 \cup A_2)) \leq \mathcal{H}^d(\pi_L(P \cap \partial B_1)) = 0, \tag{9.71}$$

by (9.69) and because $\pi_L$ is Lipschitz. Since by (9.66) $A_1 \setminus W \subset \varphi_3(A_1)$, we get that

$$\mathcal{H}^d(A_1 \setminus W) = 0 \tag{9.72}$$

by (9.71). Next set

$$A_3 = E^* \cap B(y, \frac{5t}{3} + 6a) \setminus [A_1 \cup A_2], \tag{9.73}$$

and let $w \in A_3$ be given. If $w \in E^* \cap B(y, \frac{5t}{3} + 3a)$, then $\varphi_3(w) = \varphi_2(w)$ because $w \notin A_2$. Otherwise, $\varphi_3(w) = \varphi_2(w)$ by (9.60). Thus

$$\varphi_3(w) = \varphi_2(w) \quad \text{for } w \in A_3. \tag{9.74}$$

We cut $A_3$ into two pieces. On $A_{31} = A_3 \setminus B(y, \frac{5t}{3} + 4a)$, (9.41) says that $\varphi_2(w) = \varphi_1(w)$, and (9.39) says that $\varphi_1$ is $C\Lambda^2$-Lipschitz. On $A_{32} = A_3 \cap B(y, \frac{5t}{3} + 4a)$, (9.47) says that $\varphi_2$ is $C\Lambda^2$-Lipschitz. Altogether,

$$\mathcal{H}^d(\varphi_3(A_3)) = \mathcal{H}^d(\varphi_2(A_3)) = \mathcal{H}^d(\varphi_1(A_{31})) + \mathcal{H}^d(\varphi_2(A_{32})) \leq C\Lambda^{2d}\mathcal{H}^d(A_3). \tag{9.75}$$

Notice that $A_3 \subset E^* \cap B(y, \frac{5t}{3} + 6a) \setminus B(y, \frac{5t}{3} - a)$ (by (9.67)), which by (9.18) is contained in a $(6a + \varepsilon t)$-neighborhood of $P \cap \partial B(y, \frac{5t}{3})$. Also recall that $\varepsilon t < a$, by (9.27). Thus we can cover $A_3$ by less than $C(\frac{t}{a})^{d-1}$ balls $B_i$ of radius $10a$. By Proposition 4.1, $\mathcal{H}^d(A_3 \cap B_i) \leq \mathcal{H}^d(E^* \cap B_i) \leq Ca^d$ because $A_3 \subset E^*$. Altogether,

$$\mathcal{H}^d(A_3) \leq \sum_i \mathcal{H}^d(A_3 \cap B_i) \leq C\Big(\frac{t}{a}\Big)^{d-1} a^d = C\,\frac{a}{t}\,t^d = CC_1\Lambda^2(\varepsilon + \eta)\,t^d \tag{9.76}$$

by (9.27), and (9.75) yields

$$\mathcal{H}^d(\varphi_3(A_3)) \leq C\Lambda^{2d}\mathcal{H}^d(A_3) \leq C\Lambda^{2d}C_1\Lambda^2(\varepsilon + \eta)\,t^d. \tag{9.77}$$

Notice that if $w \in W$, (9.62) says that $w \in E^* \cap B(y, \frac{5t}{3} + 6a)$. Thus $W \subset A_1 \cup A_2 \cup A_3$, by (9.73).

We may now return to (9.65). First observe that

$$\begin{aligned} \mathcal{H}^d(\varphi_3(W)) &\leq \mathcal{H}^d(\varphi_3(A_1 \cup A_2)) + \mathcal{H}^d(\varphi_3(W \cap A_3)) \\ &= \mathcal{H}^d(\varphi_3(W \cap A_3)) \leq CC_1 \Lambda^{2d+2} (\varepsilon + \eta)\, t^d \end{aligned} \tag{9.78}$$

by (9.71) and (9.77). On the other hand, Proposition 4.1 yields

$$C^{-1} t^d \leq \mathcal{H}^d(A_1) = \mathcal{H}^d(A_1 \cap W) \leq \mathcal{H}^d(W) \tag{9.79}$$

by (9.67) and (9.72). We now apply (9.65) and get that

$$C^{-1} t^d \leq \mathcal{H}^d(W) \leq M \mathcal{H}^d(\varphi_3(W)) + hr^d \leq CC_1 M \Lambda^{2d+2} (\varepsilon + \eta)\, t^d + hr^d. \tag{9.80}$$

If $\eta$, $\varepsilon$, and $h$ are small enough, depending on $M$, $n$, and $\Lambda$ (recall that $C_1$ depends only on $n$), we get the desired contradiction that proves (9.20).

We still need to prove (9.19), but it follows from (9.20) and (9.18), with the same short proof as in Lemma 7.38, a little below (7.71).

This completes our proof of Lemma 9.14. □

Let us now state and prove the generalization of Theorem 6.1.

**Theorem 9.81.** *For each choice of constants $n$, $M \geq 1$ and $\Lambda \geq 1$, we can find $h > 0$, $A \geq 0$, and $\theta > 0$, depending only on $n$, $M$, and $\Lambda$, such that if $E \in GSAQ(U, M, \delta, h)$ is a quasiminimal set in $U \subset \mathbb{R}^n$ (as in (9.1)), and if the pair $(x, r)$ is such that*

$$x \in E^* \cap U, \;\; 0 < r < \mathrm{Min}(\lambda^{-1} r_0, \delta), \;\; B(x, 2r) \subset U, \tag{9.82}$$

*and*

$$\begin{gathered} E^* \cap B(x,r) \subset L_j \text{ for every } j \in [0, j_{max}] \text{ such that some face} \\ \text{of } L_j, \text{ of dimension (strictly) more than } d, \text{ meets } B(x,r), \end{gathered} \tag{9.83}$$

*then we can find a $d$-dimensional $A$-Lipschitz graph $\Gamma \subset \mathbb{R}^n$ such that*

$$\mathcal{H}^d(E \cap \Gamma \cap B(x,r)) \geq \theta r^d. \tag{9.84}$$

By $d$-dimensional $A$-Lipschitz graph, we still mean the image, under an isometry of $\mathbb{R}^n$, of the graph of some $A$-Lipschitz function from $\mathbb{R}^d$ to $\mathbb{R}^{n-d}$.

We want to copy the proof that we did for Theorem 6.1 (see below Theorem 8.5). We localize as usual: we define an unbounded Ahlfors-regular set $F' = F \cup H$ such that $E^* \cap B(x, r/16) \subset F \subset E^* \cap B(x, r/8)$, as we did near (7.87). This, and most of our Carleson estimates, depend only on the local Ahlfors-regularity of $E^*$ (Proposition 4.1 in the rigid case, Proposition 4.74 in the Lipschitz case).

Next we check that $F'$ is uniformly rectifiable, and more precisely that $F' \in BPBI(\theta, C)$ (see (7.6) for the definition), for some constants $\theta$ and $C$ that are a little worse than those of Proposition 9.4, but still depend only on $n$, $M$, and $\Lambda$. For this we shall reduce to Proposition 9.4, as in the proof of Lemma 7.8; we sketch the argument, but the reader may return to that proof for additional detail.

We are given a ball $B(y,t)$ centered on $F'$, and we look for a bilipschitz image of a piece of $\mathbb{R}^d$ inside $F \cap B(y,t)$. When $y \in H$ or $y \in F$ and $t \geq 3r$, we can easily find this bilipschitz image inside $H$, so we may assume that $y \in F$ and $t \leq 3r$. Then we find a reasonably large cube $Q$, of the cubical patchwork for $F$, such that $y \in Q \subset F \cap B(y, t/10)$ (as in (7.12)). And inside $Q$, we find a point $w \in F$ such that $\mathrm{dist}(w, E^* \setminus F) \geq \tau \,\mathrm{diam}(Q)$ (for some constant $\tau > 0$ that depends on the local Ahlfors regularity constant), as in (7.13). This part uses the fact that our cubical patchwork for $F$ is adapted to the set $E^*$, as in (7.11). Then we apply Proposition 9.4 to the ball $B(w, \tau \,\mathrm{diam}(Q))$, and find a set $G \subset E^* \cap B(w, \tau \,\mathrm{diam}(Q))$ which is the bilipschitz image of a piece of $\mathbb{R}^d$; this set is contained in $F \cap B(y,t)$, in particular because $\mathrm{dist}(w, E^* \setminus F) \geq \tau \,\mathrm{diam}(Q)$, and it is large enough because $\mathrm{diam}(Q) \geq C^{-1} t$; see (7.14)-(7.16) for the verification. Thus $F' \in BPBI(\theta, C)$, as announced.

By Theorem 7.7, $F' \in BWGL(\varepsilon, C(\varepsilon))$ for every $\varepsilon$, where as usual $C(\varepsilon)$ depends also on $n$, $M$, and $\Lambda$. As in the rigid case, it is enough to show that $F'$ has big projections, because then Theorem 8.5 will say that $F'$ contains big pieces of Lipschitz graphs, and we can use one of these pieces (contained in $F \subset E^* \cap B(x, r/8)$) in the statement of Theorem 9.81.

Again we are given a ball $B(x_1, r_1)$ centered on $F'$, and now we want to find a $d$-plane $P$ such that

$$\mathcal{H}^d(\pi(F' \cap B(x_1, r_1))) \geq \alpha r_1^d, \tag{9.85}$$

where $\pi$ denotes the orthogonal projection on $P$. When $x_1 \in H$ or $x_1 \in F$ and $r_1 \geq 3r$, we easily get this with $P = H$, because $\mathcal{H}^d(\pi(F' \cap B(x_1, r_1))) \geq \mathcal{H}^d(H \cap B(x_1, r_1)) \geq C^{-1} r_1^d$. So we may assume that $x_1 \in F$ and $r_1 \leq 3r$.

Let $\varepsilon_0 > 0$ be very small, to be chosen later. We proceed as in Lemma 8.11 to find a pair $(y_0, t_0)$, with the properties (8.12)-(8.16), except that in (8.12) we replace $r_1/100$ with the smaller $(100C_0)^{-1} r_1$, with $C_0$ as in (9.15), that in (8.13) we require that $\mathrm{dist}(y_0, F_1) \geq 10\Lambda^2 t_0$, and that in (8.15) and (8.19) we replace $b\beta_{E^*}$ with $b\beta_F$. The reader recalls that in Lemma 8.11, the pair $(x_1, r_1)$ was also such that $x_1 \in F$ and $r_1 \leq 3r$. Most of the proof of Lemma 8.11 can be repeated here, because it relies on Carleson measure computations based on the local Ahlfors regularity of $E^*$ and $F$, and simple distance estimates with faces of our dyadic grid on $U$. There is just one exception, which is the Carleson measure estimate (8.20) on the bad set $\mathcal{B}_3$ where $E^*$ is not flat. Here we replace $b\beta_{E^*}$ with $b\beta_F$ in (8.15) and (8.19), and we get the analogue of (8.20) because $F$ is uniformly rectifiable, hence satisfies a bilateral weak geometric lemma (i.e, $F' \in BWGL(\varepsilon, C(\varepsilon))$ as above).

So we get the pair $(y_0, t_0)$, and we now check that Lemma 9.14 applies to the pair $(y_0, t_0/3)$. The first part of (9.15) holds because

$$\frac{t_0}{3} \leq \frac{r_1}{100C_0} \leq \frac{4r}{100C_0} < \frac{4}{100C_0} \,\mathrm{Min}(\lambda^{-1} r_0, \delta) \tag{9.86}$$

by the modified (8.12), (8.7), and (9.82). For the second part, observe that

$$|y_0 - x| \le |y_0 - x_1| + |x_1 - x| \le \frac{r_1}{200} + \frac{r}{8} \le \frac{r}{4} \tag{9.87}$$

by (8.12) and because $x_1 \in F \subset B(x, r/8)$ and $r_1 \le 3r$; then (9.86) and (9.82) yield

$$B(y_0, \frac{(C_0+1)t_0}{3}) \subset B(x, \frac{r}{4} + \frac{(C_0+1)t_0}{3}) \subset B(x, r/2) \subset U. \tag{9.88}$$

Recall that $J$ in (9.16) is not empty (because $y \in \Omega = L_0$), and let us check (9.17). A first possibility is that for each $j \in J$, $L_j$ has a face of dimension larger than $d$ that meets $B(y_0, 2t_0/3)$. Since $B(y_0, 2t_0/3) \subset B(x, r)$, (9.83) says that $E^* \cap B(y_0, 2t_0/3) \subset E^* \cap B(x, r) \subset L_j$ for each $j$, hence $E^* \cap B(y_0, 2t_0/3) \subset L$ and (9.17) holds with $\eta = 0$.

So let us assume that for some $j \in J$, no face of $L_j$ of dimension larger than $d$ meets $B(y_0, 2t_0/3)$. Since $j \in J$, $L_j$ meets $B(y_0, 2t_0/3)$, which means that some face $H$ of $L_j$ meets $B(y_0, 2t_0/3)$. Then $H$ is of dimension at most $d$.

It will be easier to make our metric computations with standard dyadic cubes, so we set $h(y) = r_0^{-1}\psi(\lambda y)$ for $y \in \mathbb{R}^n$, and observe that $h(H)$ is a standard unit dyadic face by construction. Notice that

$$\mathrm{dist}(h(y_0), h(H)) \le r_0^{-1}\lambda\Lambda\,\mathrm{dist}(y_0, H) \le r_0^{-1}\lambda\Lambda t_0 \le \frac{1}{10}, \tag{9.89}$$

where the last inequality comes from (9.86) if we took $C_0 \ge \Lambda$. But by (8.13) with the new constant $10\Lambda^2$,

$$\mathrm{dist}(h(y_0), h(F_1)) \ge r_0^{-1}\lambda\Lambda^{-1}\,\mathrm{dist}(y_0, F_1) \ge 10 r_0^{-1}\lambda\Lambda t_0, \tag{9.90}$$

so (by (9.89)) $H$ is not contained in $F_1$, and it is $d$-dimensional. By (8.14),

$$\mathrm{dist}(h(w), h(F_2)) \le r_0^{-1}\lambda\Lambda\,\mathrm{dist}(w, F_2) \le r_0^{-1}\lambda\Lambda\varepsilon_0 t_0 \tag{9.91}$$

for $w \in E^* \cap B(y_0, 2t_0)$. We use (9.89) to find $\xi \in h(H)$ such that $|h(y_0) - \xi| \le r_0^{-1}\lambda\Lambda t_0$; then if $\widehat{H}$ is any other $d$-dimensional face of the dyadic net (i.e., $\widehat{H} \ne h(H)$),

$$\begin{aligned} \mathrm{dist}(h(y_0), \widehat{H}) &\ge \mathrm{dist}(\xi, \widehat{H}) - r_0^{-1}\lambda\Lambda t_0 \ge \mathrm{dist}(\xi, \partial(h(H)) - r_0^{-1}\lambda\Lambda t_0 \\ &\ge \mathrm{dist}(h(y_0), \partial(h(H)) - 2r_0^{-1}\lambda\Lambda t_0 \ge 8r_0^{-1}\lambda\Lambda t_0, \end{aligned} \tag{9.92}$$

where the main inequalities come from (3.8) and (9.90). If $w \in E^* \cap B(y_0, 2t_0)$, this yields

$$\mathrm{dist}(h(w), \widehat{H}) \ge \mathrm{dist}(h(y_0), \widehat{H}) - 2r_0^{-1}\lambda\Lambda t_0 \ge 6r_0^{-1}\lambda\Lambda t_0. \tag{9.93}$$

In other words, all the other faces that compose $h(F_2)$ are too far, and (9.91) implies that

$$\mathrm{dist}(h(w), h(H)) \le r_0^{-1}\lambda\Lambda\,\mathrm{dist}(w, F_2) \le r_0^{-1}\lambda\Lambda\varepsilon_0 t_0 \ \text{ for } w \in E^* \cap B(y_0, 2t_0). \tag{9.94}$$

In fact, (9.93) also implies that $\mathrm{dist}(w, H') \geq 6t_0$ for every $d$-dimensional face $H' = \lambda^{-1}\psi^{-1}(r_0\widehat{H})$ of dimension $d$ of our (distorted) dyadic grid on $U$, other than $H$, and (8.14) (or the second half of (9.91)) now says that

$$\mathrm{dist}(w, H) = \mathrm{dist}(w, F_2) \leq \varepsilon_0 t_0 \ \text{ for } w \in E^* \cap B(y_0, 2t_0). \tag{9.95}$$

Let us now check that

$$L_i \text{ contains } H \ \text{ for } i \in J. \tag{9.96}$$

Let $G$ be a be a face of $L_i$ that meets $B(y_0, 2t_0/3)$; such a face exists by definition of $J$. Also let $\xi \in h(H)$ be, as above, such that $|h(y_0) - \xi| \leq r_0^{-1}\lambda\Lambda t_0$. If (9.96) fails, $G$ does not contain $H$, $h(G)$ does not contain $h(H)$, $h(H)$ is not reduced to one point because it is $d$-dimensional, and so (3.8) says that

$$\begin{aligned} \mathrm{dist}(h(y_0), h(G)) &\geq \mathrm{dist}(\xi, h(G)) - r_0^{-1}\lambda\Lambda t_0 \geq \mathrm{dist}(\xi, \partial(h(H)) - r_0^{-1}\lambda\Lambda t_0 \\ &\geq \mathrm{dist}(h(y_0), \partial(h(H)) - 2r_0^{-1}\lambda\Lambda t_0 \geq 8r_0^{-1}\lambda\Lambda t_0, \end{aligned} \tag{9.97}$$

by (9.92) or directly (9.90). This is impossible, because

$$\mathrm{dist}(h(y_0), h(G)) \leq r_0^{-1}\lambda\Lambda \,\mathrm{dist}(y_0, G) \leq r_0^{-1}\lambda\Lambda t_0 \tag{9.98}$$

since $G$ meets $B(y_0, t_0)$. So (9.95) holds.

By (9.95) and the definition (9.16), $L$ contains $H$. Then $\mathrm{dist}(w, L) \leq \mathrm{dist}(w, H) \leq \varepsilon_0 t_0$ for $w \in E^* \cap B(y_0, 2t_0)$, by (9.95). This proves (9.17), with $\eta = 3\varepsilon_0$.

Finally we need to check (9.18) for some $P$; we know from the modified (8.15) that $b\beta_F(y_0, t_0) \leq \varepsilon_0$, so there is a plane $P$ through $y_0$ such that in particular

$$\mathrm{dist}(w, P) \leq \varepsilon_0 t_0 \text{ for } w \in F \cap B(y_0, t_0). \tag{9.99}$$

If (9.18) fails for this $P$ (and the pair $(y_0, t_0/3)$), we can find $w \in E^* \cap B(y_0, 2t_0/3)$ such that $\mathrm{dist}(w, P) \geq \varepsilon t_0/3$. If $\varepsilon_0 \leq \varepsilon/6$, (9.99) implies that $\mathrm{dist}(w, F) \geq \varepsilon t_0/3$. But then

$$\mathcal{H}^d(E^* \cap B(y_0, 2t_0) \setminus F) \geq \mathcal{H}^d(E^* \cap B(w, \varepsilon t_0/3)) \geq C^{-1}\varepsilon^d t_0^d \tag{9.100}$$

by Proposition 4.74. This contradicts (8.16) if $\varepsilon_0$ is small enough (depending on $\varepsilon$, $M$, and $\Lambda$); thus (9.18) holds.

This completes the verification of the hypotheses of Lemma 9.14 for the pair $(y_0, t_0/3)$. We apply Lemma 9.14 and get that

$$\pi(E^* \cap B(y_0, 5t_0/9) \text{ contains } P \cap B(\pi(y), t_0/2), \tag{9.101}$$

as in (9.20). Then we use the modified (8.16) to get that

$$\begin{aligned} \mathcal{H}^d(\pi(F \cap B(y_0, 5t_0/9)) &\geq \mathcal{H}^d(\pi(E^* \cap B(y_0, 5t_0/9))) - \mathcal{H}^d(E^* \cap B(y_0, 5t_0/9) \setminus F) \\ &\geq \mathcal{H}^d(P \cap B(\pi(y), t_0/2)) - \varepsilon_0 t_0^d \geq C^{-1}t_0^d \end{aligned} \tag{9.102}$$

if $\varepsilon_0$ is small enough. But $B(y_0, 5t_0/9) \subset B(x, r_1)$ by (8.12), so (9.102) implies that $\mathcal{H}^d(\pi(F \cap B(x_1, r_1)) \geq C^{-1}t_0^d$, and (9.85) follows because $t_0 \geq c(\varepsilon_0)r_1$ by (8.12).

Thus $F'$ has big projections, hence contains big pieces of Lipschitz graphs, and we know that Theorem 9.81 follows. □

Let us also state the property of concentration under the Lipschitz assumption. The following is a generalization of Corollary 8.55; it will be further generalized in Section 10, where we shall (simplify the proof and) remove the unnatural assumption (9.105). See Proposition 10.82.

**Corollary 9.103.** *For each choice of $n$, $M \geq 1$, $\Lambda \geq 1$ and $\varepsilon > 0$, we can find $h > 0$ and $c_\varepsilon > 0$ such that the following holds. Suppose that $E \in GSAQ(U, M, \delta, h)$ for some open set $U \subset \mathbb{R}^n$, and that the Lipschitz assumption is satisfied, with the constants $\lambda$ and $\Lambda$ (as in (9.3)). Also denote by $r_0 = 2^{-m} \leq 1$ the side length of the dyadic cubes of the usual grid. Then let $(x, r)$ be such that*

$$x \in E^* \cap U, \;\; 0 < r < \mathrm{Min}(\lambda^{-1}r_0, \delta), \;\; B(x, 2r) \subset U, \tag{9.104}$$

*and*

$$\begin{gathered} E^* \cap B(x,r) \subset L_j \text{ for every } j \in [0, j_{max}] \text{ such that some face} \\ \text{of } L_j, \text{ of dimension (strictly) more than } d, \text{ meets } B(x,r). \end{gathered} \tag{9.105}$$

*Then we can find a pair $(y, t)$, such that $y \subset E^* \cap B(x, r/100)$, $c_\varepsilon r \leq t \leq r/100$, and*

$$\mathcal{H}^d(E^* \cap B(y, t)) \geq (1 - \varepsilon)\omega_d t^d, \tag{9.106}$$

*where $\omega_d$ denotes the $d$-dimensional Hausdorff measure of the unit ball in $\mathbb{R}^d$.*

The proof is the same as for Corollary 8.55, except that we use Lemma 9.14 instead of Lemma 7.38; the point is that given a small constant $\varepsilon > 0$ and a pair $(x, r)$ as in the statement, we can find a new pair $(y, t)$, with $y \in E^* \cap B(x, r/2)$ and $c(\varepsilon) \leq t \leq r/10$, which satisfies the assumptions of Proposition 9.14. For instance, we can proceed as in the final part of the proof of Theorem 9.81 above and pick the pair $(y_0, t_0/3)$ associated to $x_1 = x$ and $r_1 = r$.

Then we just need to use the properties (9.18)-(9.20) to prove (9.106) for this pair (unfortunately with a slightly larger constant $\varepsilon'$, but this does not matter), exactly as we did in (8.57)-(8.59). □

## PART IV : LIMITS OF QUASIMINIMAL SETS

In this part, we want to generalize results that come mainly from [D2], that concern limiting properties of quasiminimal (or almost minimal, or minimal) sets. The main result of this part (Theorem 10.8) is that our various classes of quasiminimal sets are stable under limits, and the main reason why it holds is the lower semicontinuity of $\mathcal{H}^d$, restricted to a sequence of quasiminimal sets with uniform quasiminimality constants; cf. Theorem 10.97. See the general introduction for a discuss of the interest of these results.

In turn the main ingredient in the proof of Theorems 10.97 and 10.8 is the fact that our quasiminimal sets satisfy a concentration property that was introduced by Dal Maso, Morel, and Solimini [DMS] in the different, but related context of minimizers for the Mumford-Shah functional in image processing (see Proposition 10.82 below). They used this property to prove lower semicontinuity results for $\mathcal{H}^d$ (on some minimizing sequences) and get an existence result for minimal segmentations of the functional. At the same time, E. De Giorgi, M. Carriero, and A. Leaci [DeCL] obtained the same existence result, using the weak form of the functional and a compactness result of Ambrosio in the class $SBV$ of special bounded variation functions.

For minimal sets and surfaces, it seems that the idea of using the concentration property to obtain existence results was not considered before [D2], probably because people were very happy with the quite strong compactness properties of integral currents and varifolds. Most often, when a limiting property for minimal sets was needed, people would revert to currents or varifolds, take a limit there, and return to sets. Nonetheless, it is good to have limiting theorems like Theorems 10.97 and 10.8, both because this looks more direct and, for instance, because some minimal sets are hard to describe as supports of currents, typically for orientation reasons or because multiplicities could become too large.

Recall that we already proved the concentration property in some cases, as a consequence of the uniform rectifiability of the quasiminimal sets; see Corollaries 8.55 and 9.103. But the very surprising thing, at least for the author, is that there is a more direct route to this, through the fact that limits of quasiminimal sets (with uniform quasiminimality constants) are rectifiable (Proposition 10.15 below), which gives a simpler and general proof. Also, we shall give a slightly more direct proof of Theorem 10.97 (still based on the same ideas but improved by Y. Fang) in Section 25, that also works when $\mathcal{H}^d$ is multiplied by some elliptic integrands.

We prove Theorem 10.97 in the next section, but the proof of Theorem 10.8 (the quasiminimality of limits) will be quite long, and we split in into smaller pieces (Sections 11-19). The difficulty is that given a sequence $\{E_k\}$ of quasiminimal sets that converge to the set $E$, and a competitor $F = \varphi_1(E)$ for $E$, the obvious competitor $\varphi_1(E_k)$ may be very bad, and we have to spend some energy to make it better, typically by pinching parallel leaves of $\varphi_1(E_k)$ to diminish their total mass. Unfortunately, unlike what happens with uniform rectifiability, we essentially have to redo most of the proof of [D2]. Note that it is far from impossible (since the author worked by himself for all of this) that a better proof exists. But in the mean time we seem to be stuck with a long, technical, but not so inventive proof.

## 10. Limits of quasiminimal sets: the main statement, rectifiability, and l.s.c.

In this section and the next ones, we take a sequence of sets $E_k$, which are quasiminimal in a same domain, with sliding conditions with respect to the same boundary pieces $L_j$, and with uniform quasiminimality constants, and we try to prove that if the cores $E_k^*$ converge (in local Hausdorff distance) to $E$, then $E$ is quasiminimal with the same constants. It is also natural to make the $L_j$ vary as well, but it will be simpler for us not to do this until Section 23.

In this section we shall take care of the (plain) rectifiability of $E$, the uniform concentration property for the $E_k$, and the lower semicontinuity of $\mathcal{H}^d$ along our sequence.

Let us describe our assumptions for the next few sections. We fix an open set $U \subset \mathbb{R}^n$, and boundary pieces $L_j$, $0 \leq j \leq j_{max}$, and we assume that

$$\text{the Lipschitz assumption is satisfied in } U. \tag{10.1}$$

Recall from Definition 2.7 that this assumption comes with a positive dilation constant $\lambda > 0$, and a bilipschitz mapping $\psi : \lambda U \to B(0,1)$; we shall denote by $\Lambda$ (a bound for) the bilipschitz constant of $\psi$, as in (9.3).

Next we are given a sequence $\{E_k\}$ of closed sets in $U$. By this we mean that $E_k$ is contained in $U$ and relatively closed in $U$, but it would make no difference if we just assumed that $E_k \subset \mathbb{R}^n$ and that its intersection with $U$ is closed in $U$, because anyway we shall never look at points that lie outside of $U$. We assume that there are constants $M \geq 1$, $\delta \in (0, +\infty]$, and $h > 0$ (systematically assumed to be small enough, depending on $n$, $M$, and $\Lambda$) such that for all $k$,

$$E_k \in GSAQ(U, M, \delta, h), \tag{10.2}$$

and

$$E_k \text{ is coral, i.e., } E_k^* = E_k \tag{10.3}$$

(see Definitions 2.3 and 3.1). This time there is no point in trying to avoid the assumption (10.3): if the $E_k$ are not coral, the sets $E_k^* \setminus E_k$ may converge to anything, even though they have a vanishing measure. This means that in concrete problems where the sets $E_k^* \setminus E_k$ have some meaning, one may need to do something special about them, probably after taking care of the $E_k^*$.

We also assume that there is a closed set $E$ in $U$, such that

$$\lim_{k \to +\infty} E_k = E \ \text{ locally in } U, \tag{10.4}$$

where the limit is defined as follows. For each choice of $x \in U$ and $r > 0$ such that $\overline{B}(x,r) \subset U$, and of two sets $E$, $F$, which we can assume to be closed in $U$, we set

$$\begin{aligned} d_{x,r}(E,F) = \frac{1}{r} \sup\big\{ \operatorname{dist}(y,F)\,;\, y \in E \cap B(x,r) \big\} \\ + \frac{1}{r} \sup\big\{ \operatorname{dist}(y,E)\,;\, y \in F \cap B(x,r) \big\}, \end{aligned} \tag{10.5}$$

where by convention $\sup\big\{ \operatorname{dist}(y,F)\,;\, y \in E \cap B(x,r) \big\} = 0$ when $E \cap B(x,r)$ is empty, and similarly $\sup\big\{ \operatorname{dist}(y,E)\,;\, y \in F \cap B(x,r) \big\} = 0$ when $F \cap B(x,r)$ is empty. But when $E \cap B(x,r)$ is nonempty but $F$ is empty, for instance, we set $d_{x,r}(E,F) = +\infty$; we include these cases to be able to say that sets that go away to the boundary tend to the empty set, but in fact this situation does not interest us (because we know that $\emptyset$ is minimal, for instance). Now (10.4) means that

$$\lim_{k \to +\infty} d_{x,r}(E_k, E) = 0 \tag{10.6}$$

for all choices of $x \in U$ and $r > 0$ such that $\overline{B}(x,r) \subset U$. This is easily seen to be equivalent to other ways of defining (10.4), for instance where we would replace our family of balls $B(x,r)$ with an exhaustion of $U$ by compact subsets. The main point of using this definition is that given any sequence $\{E_k\}$ of closed sets in $U$, we can always find a subsequence that converges to some closed set.

There is a small technical assumption that we want to make when the Lipschitz assumption holds:

(10.7) for each $0 \leq j \leq j_{max}$ and each face $F \subset U$ of our net such that dimension$(F) > d$ and $F \subset L_j$, but the interior of $F$ (as a face) is not contained in the interior of $L_j$ (as a subset of $\mathbb{R}^n$), we have that for $\mathcal{H}^d$-almost every interior point $y$ of $F$, we can find $t > 0$ such that the restriction of $\psi$ to $\lambda F \cap B(\lambda y, t)$ is of class $C^1$.

This condition is a little strange so let us explain a little. Notice that we require the exceptional set to be small for $\mathcal{H}^d$, regardless of the dimension of the face $F$. The interior of $L_j$ is really taken with the topology of $\mathbb{R}^n$, not with respect to the dimension of some faces: we add this constraint on $F$ because we don't want to put regularity conditions on the faces $F$ that lie in the middle of out initial domain $\Omega = L_0$, for instance. But we require some control on the boundary of $\Omega$.

Let us state (10.7) in a slightly different way. For each $y \in U$, denote by $F(y)$ the smallest face of our grid that contains $y$; thus $y$ is an interior point of $F(y)$ (when $F(y) = \{y\}$, we may say that $\mathrm{int}(F(y)) = \{y\}$, but this case will be rapidly dismissed anyway). We require that for $0 \leq j \leq j_{max}$ and for $\mathcal{H}^d$-almost every point of $L_j$, if dimension$(F(y)) > d$ and $y$ does not lie in the $n$-dimensional interior of $L_j$, we can find $t > 0$ such that the restriction of $\psi$ to $\lambda F(y) \cap B(\lambda y, t)$ is of class $C^1$.

In this second condition also, we only really exclude the case when dimension$(F(y)) = d$, because the lower dimensional skeletons have vanishing $\mathcal{H}^d$-measure anyway.

Let us check that the two conditions are equivalent. If (10.7) holds and $y$ lies in none of the exceptional sets associated to faces $F$, and if $y \in U$ does not lie in the $n$-dimensional interior of $L_j$ and is such that dimension$(F(y)) > d$, we can apply (10.7) to $F = F(y)$ (which is contained in $L_j$ because $y \in L_j$ and because $L_j$ is composed of faces), and we get the desired $t > 0$ because $y$ does not lie in the exceptional set of $F$.

If our second condition holds and the face $F$ is such that $F \subset L_j$, dimension$(F) > d$, and $F$ is not contained in the $n$-dimensional interior of $L_j$, observe that for every interior point $y$ of $F$, we have that $F(y) = F$. Thus, for all the points $y \in \mathrm{int}(F)$ that do not lie in the exceptional set of the second condition, we can find $t > 0$ as needed.

We shall state later a weaker (but a little more complicated to state) condition that works as well (see Remark 19.52), but we did not find any obvious way to get rid of it entirely. Notice that (10.7) does not require anything in faces $F$ of dimension $d$, and that is trivially satisfied under the rigid assumption, or if the bilipschitz mapping $\psi : \lambda U \to B(0,1)$ is of class $C^1$.

Here is our main result about limits.

**Theorem 10.8.** *Let $U$, $\{E_k\}$, and $E$ satisfy the hypotheses above (including (10.7) if the Lipschitz assumption holds). Also suppose that $h$ is small enough, depending only on $n$, $M$, and $\Lambda$. Then $E$ is coral, and*

$$E \in GSAQ(U, M, \delta, h), \tag{10.9}$$

*with the same constants $M$, $\delta$, and $h$.*

See Remark 19.52 for a statement where (10.7) is slightly weakened.

The smallness of $h$ and the additional assumption (10.7) will only be used in the various limiting arguments, but will have no impact on the constants in the conclusion.

In addition, the smallness of $h$ is only used to show that the $E_k$ have, uniformly in $k$, some good regularity properties (that imply, in particular, the lower semicontinuity estimate (10.98)), and then we don't need it any more. So, if we have (10.2) for some combination of $M$, $\delta$, and $h$ for which $h$ is small enough (as required), and (10.2) also holds for some other combination of $M$, $\delta$, and $h$ (this time, with no constraint), then our conclusion (10.9) holds for both combinations.

Theorem 10.8 generalizes Theorem 4.1 on page 126 of [D2]; we shall try to follow the proof, but since many modifications will be needed in the middle of the construction, we shall need to explain things with more detail than in the previous part.

The main goal of the rest of this section is to prove the lower semicontinuity of $\mathcal{H}^d$, when we restrict our attention to sequences of quasiminimal sets that satisfy the assumptions (10.1)-(10.4) of Theorem 10.8 (we shall not need (10.7) for quite some time). This is Theorem 10.97 below, which in a way is the main tool for our proof of Theorem 10.8.

We intend to deduce Theorem 10.97 from a result of Dal Maso, Morel, and Solimini [DMS] which says that $\mathcal{H}^d$ is lower semicontinuous along uniformly concentrated sequences, but we shall follow a different route to the concentration property.

In [D2] we deduced it from the local uniform rectifiability of the $E_k$, but here we were only able to get this under additional (and not too natural) assumptions on the dimensions of the faces of the $L_j$. That is, Corollaries 8.55 and 9.103 have some unnatural assumptions that we want to avoid.

So we shall prove a weaker regularity condition, the existence of reasonably large balls where a given quasiminimal set is approximated by a $d$-plane, and then show that we can use it to prove the desired concentration property. See Lemma 10.21 for the approximation property, and Proposition 10.82 for the concentration property. Also see the later Section 25 for a more direct proof of Theorem 10.97 that works with some elliptic integrands.

Before we get to Lemma 10.21, we shall consider a sequence of quasiminimal sets that satisfy the assumptions (10.1)-(10.4) (again, we do not need (10.7) in this section), and prove various simple properties.

The first ones are the local Ahlfors regularity of the limit (see (10.11)), very rough lower and upper semicontinuity properties of $\mathcal{H}^d$ along the sequence ((10.12) and (10.14)), and the rectifiability of the limit (Proposition 10.15).

This last, which is not a direct consequence of the rectifiability of the $E_k$, but essentially follows from its proof, is useful because we use it to prove the approximation Lemma 10.21 (through a compactness argument), and because we shall use the rectifiability of $E$ to construct the competitors in the next sections.

The construction of a competitor for the main part of the argument will only start in the next section, and estimates will continue up until Section 19.

So let $\{E_k\}$ and its limit $E$ satisfy our assumptions (10.1)-(10.4); we want to derive a few simple properties.

Let us first observe that the $E_k$ are locally Ahlfors-regular, with uniform estimates. This means that there exists a constant $C_M$, that depends only on $n$, $M$ and $\Lambda$, such that

$$C_M^{-1} r^d \leq \mathcal{H}^d(E_k \cap B(x,r)) \leq C_M r^d \tag{10.10}$$

for every pair $(x,r)$ such that $x \in E_k$, $0 < r < \mathrm{Min}(\lambda^{-1} r_0, \delta)$, and $B(x,2r) \subset U$. This is an easy consequence of Propositions 4.1 and 4.74, that we already used a lot in the last sections. We deduce from this that

$$C_M^{-1} r^d \leq \mathcal{H}^d(E \cap B(x,r)) \leq C_M r^d \tag{10.11}$$

when $x \in E$ and $0 < r < \mathrm{Min}(\lambda^{-1} r_0, \delta)$ are such that $B(x,2r) \subset U$, with a possibly larger constant $C_M$, but that still depends only on $n$, $M$ and $\Lambda$. This is easy to check, because the (local) Ahlfors regularity of $E$ follows from the existence of a locally finite Borel measure $\mu$ on $E$ that satisfies (10.11) (where we would replace $\mathcal{H}^d(E \cap B(x,r))$ with $\mu(E \cap B(x,r))$), and such a measure is easy to obtain as a weak limit of the restriction of $\mathcal{H}^d$ to $E_k$, or for a subsequence of $\{E_k\}$. More detail can be found in the proof of Lemma 4.2 in [D2]. Also notice that $E$ is clearly coral because of (10.11).

Let us deduce from (10.10) and (10.11) that

$$\mathcal{H}^d(E \cap V) \leq C_M \liminf_{k \to +\infty} \mathcal{H}^d(E_k \cap V) \ \text{ for every open set } V \subset U, \tag{10.12}$$

where again $C_M$ depends only on $n$, $M$ and $\Lambda$. To do this, cover $E \cap V$ by balls $B(x_j, r_j)$ such that $x_j \in E$, $0 < r_j < 10^{-1} \mathrm{Min}(\lambda^{-1} r_0, \delta)$, and $B(x_j, 10 r_j) \subset V \subset U$. Then use the usual 5-covering lemma to cover $\mathcal{H}^d(E \cap V)$ with a family $B(x_j, 5r_j)$, $j \in J$, such that the $B(x_j, r_j)$ are disjoint.

For each finite subset $J_0$ of $J$, we deduce from (10.4) that for $k$ large enough, every $B(x_j, r_j/2)$ contains a point $y_j \in E_k$. Then

$$\begin{aligned}\sum_{j \in J_0} \mathcal{H}^d(E \cap B(x_j, 5r_j)) &\leq C \sum_{j \in J_0} r_j^d \leq C \sum_{j \in J_0} \mathcal{H}^d(E_k \cap B(y_j, r_j/2)) \\ &\leq C \sum_{j \in J_0} \mathcal{H}^d(E_k \cap B(x_j, r_j)) \leq C \mathcal{H}^d(E_k \cap V)\end{aligned} \tag{10.13}$$

by (10.11) and (10.10). Thus $\sum_{j \in J_0} \mathcal{H}^d(E \cap B(x_j, 5r_j)) \leq C \liminf_{k \to +\infty} \mathcal{H}^d(E_k \cap V)$ for each finite set $J_0 \subset J$. We take the supremum, observe that $\mathcal{H}^d(E \cap V) \leq \sum_{j \in J} \mathcal{H}^d(E \cap$

$B(x_j, 5r_j))$ because the $B(x_j, 5r_j)$ cover $E \cap V$, and get (10.12). A similar argument shows that

$$\limsup_{k \to +\infty} \mathcal{H}^d(E_k \cap H) \leq C_M \mathcal{H}^d(E \cap H) \tag{10.14}$$

whenever $H$ is a compact subset of $U$, and where $C_M$ depends only on $n$, $M$, and $\Lambda$. We skip the details, because this is the same as (3.11) in [D2], and the proof applies here.

In the present context, we cannot really hope for (10.14) to hold with $C_M = 1$: $\mathcal{H}^d(E)$ could be smaller than the limit of the $\mathcal{H}^d(E_k)$ (even if it exists). For instance, with our assumptions, $E_k$ could be the graph of $f_k(x) = 2^{-k}\cos(2^k x)$, which is somewhat longer than its limit (a straight line). Nonetheless, we shall see in Lemma 22.3 that (10.14) holds with the more precise constant $C_M = (1 + Ch)M$.

And in the more restricted context of almost-minimal sets, we will have a much better control on the upper semicontinuity defect, and show that (10.14) holds holds with the optimal constant $C_M = 1$. See Theorem 22.1.

Surprisingly, both result only use very little information: the rectifiability of $E^*$, a covering argument, and an application of the definition of quasiminimality.

Fortunately, the situation is different for (10.12), for which $C_M$ can be removed, even for quasiminimal sets. See Theorem 10.97 below, which is the main goal of this section.

We shall use the fact that $E$ is rectifiable. Notice that in general, limits of rectifiable sets are not always rectifiable, but in the present situation the proof of rectifiability given in Section 5 will kindly pass to the limit.

**Proposition 10.15.** *Let $U$, $\{E_k\}$, and $E$ satisfy the hypotheses (10.1), (10.2), (10.3), and (10.4). Also suppose that $h$ is small enough, depending only on $n$, $M$, and $\Lambda$. Then $E$ is rectifiable.*

Of course the rectifiability of $E$ is compatible with Theorem 10.8 and the fact that quasiminimal sets are rectifiable, but we need to prove it first. If we are in a position to apply Theorem 6.1 or Theorem 9.81 to the $E_k$, we can deduce the rectifiability of $E$ from its uniform rectifiability (because unlike plain rectifiability, uniform rectifiability (with uniform estimates) goes to the limit). But this gives a much longer proof, and that does not even always works.

We start the proof like we did for Theorem 5.16. Let $\lambda > 0$ and $\psi : \lambda U \to B(0,1)$ be as in Definition 2.7. Since the bilipschitz mapping $\psi$ preserves rectifiable sets, it is enough to show that $E' = \psi(\lambda E)$ is rectifiable. But $E'$ is the limit (locally in $B(0,1) = \psi(\lambda U))$ of the sets $E'_k = \psi(\lambda E_k)$, and by Proposition 2.8 each $E'_k$ lies in $GSAQ(B(0,1), \Lambda^{2d}M, \Lambda^{-1}\lambda\delta, \Lambda^{2d}h)$. So it is enough to prove the proposition under the rigid assumption.

As before, we assume that $E$ is not rectifiable to get a contradiction. Let $N_0 \geq 0$ be a large integer. We first find an origin $x \in E_s$ (the singular part of $E$) such that (5.17) holds, pick a cube $Q_0$ and an integer $N \in [N_0/2, N_0]$ such that (5.18) and (5.19) hold, and construct a mapping $\phi^* = \phi_d$ as in (5.20)-(5.40).

Now we cannot use the fact that $E$ is quasiminimal, but instead we shall apply the definition of quasiminimality to $E_k$ for $k$ large, with the same family $\{\phi^*_t\}$, $0 \leq t \leq 1$, as before. The $\{\phi_t^*\}$ satisfy (1.4)-(1.8) with respect to $E_k$ as well as $E$; the main point is that (1.7) only uses the fact that the various mappings $\psi_l$ that we composed to get $\phi^*$ all preserve every face of every cube $R \in \mathcal{R}(Q)$, as in (4.8), and the verification is still the same as below (4.16).

We still can apply Definition 2.3, because of (5.43) (which is the same for $E_k$ as for $E$), and we get that

$$\mathcal{H}^d(E_k \cap W_1) \leq M\mathcal{H}^d(\phi^*(E_k \cap W_1)) + hr^d \tag{10.16}$$

(compare with (5.44)). Next observe that for $k$ large enough, (5.38) holds for $H = E_k \cap Q$, so we get (5.39). That is,

$$\phi_l(E_k \cap Q) \subset \mathcal{S}_{l-1} \cup \partial Q \tag{10.17}$$

for $n+1 \geq l \geq d$, which will be a good replacement for (5.21) and (5.40).

We may now continue as in Section 5, and get that

$$\mathcal{H}^d(E_k \cap W_1) \geq H^d(E_k \cap \frac{1}{3}Q_0) \geq C^{-1} l_0^d \tag{10.18}$$

as in (5.49), and where the last inequality now follows from the first half of (5.18), and the facts that $E_k$ is locally Ahlfors-regular with uniform bounds and $\lim_{k\to+\infty} \operatorname{dist}(x, E_k) = 0$.

The proof of (5.53) also goes through (it just uses (5.21) and (5.40)), and yields

$$\mathcal{H}^d(\phi^*(E_k \cap W_1)) \leq C\mathcal{H}^d(E_k \cap Q \setminus \operatorname{int}(Q')), \tag{10.19}$$

where the slightly smaller $Q'$ is defined in (5.47) (we removed from $Q$ the exterior layer of small cubes). Let us now apply (10.14) to $H = Q \setminus \operatorname{int}(Q')$; we get that for each $\varepsilon > 0$,

$$\mathcal{H}^d(E_k \cap Q \setminus \operatorname{int}(Q')) \leq C\mathcal{H}^d(E \cap Q \setminus \operatorname{int}(Q')) + \varepsilon \leq C\frac{l_0^d}{N_0} + \varepsilon \tag{10.20}$$

for $k$ large, and by the last part of (5.55). We choose $\varepsilon$ small and $N_0$ large, as we did in Section 5, and get a contradiction with (10.16) or (10.18). Proposition 10.15 follows. $\square$

The following lemma is a slightly more uniform version of our rectifiability result for quasiminimal sets; we shall deduce it from Proposition 10.15.

**Lemma 10.21.** *For each choice of $n$, $M \geq 1$, $\Lambda$, and $\varepsilon > 0$, we can find $h > 0$ and $c_\varepsilon > 0$ such that if $U \subset \mathbb{R}^n$ is open, $E \in GSAQ(U, M, \delta, h)$ is a sliding quasiminimal set, and if the pair $(x, r)$ is such that*

$$x \in E^*,\ \ 0 < r < \operatorname{Min}(\lambda^{-1} r_0, \delta),\ \text{and } B(x, 2r) \subset U, \tag{10.22}$$

*then we can find $y \in E^* \cap B(x, r/2)$, $t \in [c_\varepsilon r, r/2]$, and a $d$-plane $P$ through $y$, such that*

$$\operatorname{dist}(z, P) \leq \varepsilon t\ \ \text{for } z \in E^* \cap B(y, t). \tag{10.23}$$

Here $\Lambda$, $\lambda$, and $r_0$ are as in the definition of the rigid and Lipschitz assumptions. Observe that we say that the constant $c_\varepsilon$ does not depend on $r_0$, $\delta$, $\lambda$, or the precise list of boundary pieces $L_j$, which is natural but will cost us a slightly more complicated compactness argument.

The conclusion of Lemma 10.21 is a little more quantitative than rectifiability, and could easily be deduced from the local uniform rectifiability of $E^*$ if we could prove it (we would use Lemma 7.8 and a small Chebyshev argument to find the pair $(y,t)$). But it is not as strong as local uniform rectifiability, for which we would need to know that for most pairs $(y,t) \in E^* \cap B(x,r/2) \times (0,r/2]$ (in the sense that the complement satisfies a Carleson packing condition), we can find $P$ such that (10.23) holds.

The gap could possibly be filled, for instance if we could prove a regularity result that says that if $\varepsilon$ is small enough and (10.23) holds for the pair $(y,t)$, then it also holds for all pairs $(w,s) \in E^* \cap B(y,t/2) \times (0,t/2]$ (with a possibly different, but arbitrarily small $\varepsilon'$, and where the plane $P$ may depend on $(w,s)$). Such a regularity result exists in the standard case without boundaries, and could for instance be deduced from Allard's theorem [All], but the proof is not easy.

Note also that because our proof will use a compactness argument, we shall not get any computable lower bound for $c_\varepsilon$.

Since Proposition 3.3 says that $E^* \in GSAQ(U,M,\delta,h)$, it will be enough to prove Lemma 10.21 when $E$ is coral, i.e., when $E^* = E$.

We shall assume that Lemma 10.21 is false and derive a contradiction. Let $n$, $d$, $M$, $\Lambda$, and $\varepsilon > 0$ be such that the statement fails, and for each $k \geq 0$, choose a domain $U_k$, a coral set $E_k \in GSAQ(U_k, M, \delta_k, h_k)$, and a pair $(x_k, r_k)$ that satisfy the hypotheses of the lemma, with $h_k = 2^{-k}$, but for which but we cannot find $y_k$, $t_k$, and $P_k$ such that

$$y_k \in E_k \cap B(x_k, r_k/2),\ t_k \in [2^{-k} r_k, r_k/2], \tag{10.24}$$

and

$$\operatorname{dist}(z, P_k) \leq \varepsilon t_k \text{ for } z \in E_k \cap B(y_k, t_k). \tag{10.25}$$

The general scheme is quite simple: we want to take a limit, obtain a limiting set which is rectifiable (by Proposition 10.15), find a tangent plane to the limit at some point, and use it to get a contradiction. But our quasiminimality assumption involves a $\Lambda$-bilipschitz mapping $\psi_k : \lambda_k U_k \to B(0,1)$, a scale $\delta_k$, a choice of basic size $r_{0,k}$ for our dyadic grid, and a collection of boundary pieces $L_{j,k}$, that may all depend on $k$; our argument will be a little complicated by the fact that we shall need to make two or three changes of variables so as to be able to apply Proposition 10.15 to a fixed ball and with fixed boundaries.

Here is the first change of variables. Set

$$\widetilde{E}_k = \psi_k(\lambda_k E_k) \ \text{ and } \ \widetilde{x}_k = \psi_k(\lambda_k x_k) \in \widetilde{E}_k \tag{10.26}$$

(because $x_k \in E_k$), and observe that

$$\operatorname{dist}(\widetilde{x}_k, \partial B(0,1)) \geq \Lambda^{-1} \operatorname{dist}(\lambda_k x_k, \partial(\lambda_k U)) \geq 2\Lambda^{-1} \lambda_k r_k \tag{10.27}$$

because $\psi_k$ is $\Lambda$-bilipschitz and by (10.22). Let $a_n$ denote the integer power of 2 such that

$$\sqrt{n} \leq a_n < 2\sqrt{n} \tag{10.28}$$

(we prefer to use dyadic numbers here), and let $m_k$ be the integer such that

$$2^{-m_k} \leq \frac{\lambda_k r_k}{8 a_n \Lambda} < 2^{-m_k+1}. \tag{10.29}$$

Then choose a new origin $o_k \in (2^{-m_k}\mathbb{Z})^n$ such that

$$|o_k - \widetilde{x}_k| \leq 2^{-m_k}\sqrt{n} \leq 2^{-m_k} a_n \leq \frac{\lambda_k r_k}{8\Lambda}. \tag{10.30}$$

Also set

$$\gamma_k = 2^{-m_k+2} a_n \ \text{ and } \widetilde{B} = B(o_k, \gamma_k). \tag{10.31}$$

Observe that

$$B(\widetilde{x}_k, 2^{-m_k} a_n) \subset B(o_k, 2^{-m_k+1} a_n) = B(o_k, \gamma_k/2) \subset \widetilde{B} \tag{10.32}$$

and

$$\widetilde{B} \subset B(\widetilde{x}_k, 2^{-m_k+3} a_n) \subset B(\widetilde{x}_k, \Lambda^{-1}\lambda_k r_k) \subset B(0,1) \tag{10.33}$$

(by (10.29) and (10.27)). Our second change of variable is the dilation $A_k$ given by

$$A_k(z) = \gamma_k^{-1}(z - o_k) = (2^{-m_k+2} a_n)^{-1}(z - o_k) \ \text{ for } z \in \mathbb{R}^n \tag{10.34}$$

which is of course chosen so that

$$A_k(\widetilde{B}) = B(0,1). \tag{10.35}$$

We set

$$\widetilde{E}_k^\sharp = A_k(\widetilde{E}_k) \cap B(0,1) = A_k(\widetilde{E}_k \cap \widetilde{B}) = A_k(\psi_k(\lambda_k E_k) \cap \widetilde{B}) \tag{10.36}$$

and

$$\widetilde{x}_k^\sharp = A_k(\widetilde{x}_k) = A_k(\psi_k(\lambda_k x_k)) \in \widetilde{E}_k^\sharp \tag{10.37}$$

(because $\widetilde{x}_k \in \widetilde{E}_k \cap \widetilde{B}$, and by (10.26)).

Let us check that $\widetilde{E}^\sharp$ is quasiminimal. Recall that $E_k \in GSAQ(U_k, M, \delta_k, h_k)$; then Proposition 2.8 says that

$$\widetilde{E}_k = \psi_k(\lambda_k E_k) \in GSAQ(B(0,1), \widetilde{M}, \widetilde{\delta}_k, \widetilde{h}_k), \tag{10.38}$$

with $\widetilde{M} = \Lambda^{2d} M$, $\widetilde{\delta}_k = \Lambda^{-1}\lambda_k \delta_k$, and $\widetilde{h}_k = \Lambda^{2d} h_k$. For $\widetilde{E}_k$, we have the rigid assumption, and the boundary sets are the $\widetilde{L}_{j,k} = \psi_k(\lambda_k L_{j,k})$, which by assumption are composed of faces of dyadic cubes of side length $r_{0,k}$.

We claim that $\widetilde{E}_k^\sharp$ is also quasiminimal, and more precisely that

$$\widetilde{E}_k^\sharp \in GSAQ(B(0,1), \widetilde{M}, \widetilde{\delta}_k^\sharp, \widetilde{h}_k), \quad \text{with } \widetilde{\delta}_k^\sharp = \gamma_k^{-1}\widetilde{\delta}_k. \tag{10.39}$$

This comes directly from the definitions, using the fact that by (10.38) $\widetilde{E}_k \cap \widetilde{B}$ is quasiminimal in $\widetilde{B} \subset B(0,1)$. We just need to multiply $\widetilde{\delta}_k$ with the dilation factor $\gamma_k^{-1}$; the other constants stay the same when we dilate everything. Also, the boundary constraints are now given by the sets

$$\widetilde{L}_{j,k}^\sharp = A_k(\widetilde{L}_{j,k}) \cap B(0,1) = A_k(\psi_k(\lambda_k L_{j,k})) \cap B(0,1). \tag{10.40}$$

Let us verify that the $A_k(\widetilde{L}_{j,k})$ lie in an acceptable grid. As was said above, the $\widetilde{L}_{j,k}$ are composed of faces of dyadic cubes of side length $r_{0,k}$. Our center $o_k$ lies in $(2^{-m_k}\mathbb{Z})^n$, and

$$2^{-m_k} \leq \frac{\lambda_k r_k}{8 a_n \Lambda} \leq \frac{r_{0,k}}{8 a_n \Lambda} \leq r_{0,k} \tag{10.41}$$

by (10.29) and (10.22), so the sets $\widetilde{L}_{j,k} - o_k$ are composed of faces of dyadic cubes of size $2^{-m_k}$. By (10.34), $A_k$ maps our cubes into dyadic cubes of side length $(2^{-m_k+2}a_n)^{-1}2^{-m_k} = (4a_n)^{-1}$ That is, for $\widetilde{E}_k^\sharp$ we have the rigid assumption, with a scale $\widetilde{r}_0^\sharp = (4a_n)^{-1}$ that does not depend on $k$.

This is good, because there is only a finite number of possibilities for the $\widetilde{L}_{j,k}^\sharp$, and modulo replacing $\{E_k\}$ with a subsequence we may assume that the $\widetilde{L}_{j,k}^\sharp$ are always the same. The scale constant $\widetilde{\delta}_k^\sharp$ will not create trouble either, because

$$\begin{aligned} \widetilde{\delta}_k^\sharp &= \gamma_k^{-1}\widetilde{\delta}_k = \gamma_k^{-1}\Lambda^{-1}\lambda_k\delta_k = (2^{-m_k+2}a_n)^{-1}\Lambda^{-1}\lambda_k\delta_k \\ &\geq (4a_n)^{-1}\,\frac{8a_n\Lambda}{\lambda_k r_k}\,\Lambda^{-1}\lambda_k\delta_k = \frac{2\delta_k}{r_k} \geq 2 \end{aligned} \tag{10.42}$$

by (10.31), (10.29), and our assumption (10.22). Altogether (10.39) simplifies into

$$\widetilde{E}_k^\sharp \in GSAQ(B(0,1), \widetilde{M}, 2, \widetilde{h}_k), \text{ with } \widetilde{M} = \Lambda^{2d}M,\ \widetilde{h}_k = \Lambda^{2d}h_k, \tag{10.43}$$

with the rigid assumption at the fixed scale $(4a_n)^{-1}$ and with a fixed set of boundaries.

Let us replace $\{E_k\}$ with a new subsequence, so that

$$\widetilde{E}_k^\sharp \text{ converges to a limit } \widetilde{E}_\infty^\sharp \tag{10.44}$$

(locally in $B(0,1)$, as in (10.4)-(10.6)). Recall that in (10.43), $\widetilde{h}_k = \Lambda^{2d}h_k$ tends to 0, because we assumed that $h_k = 2^{-k}$. Thus we can apply Proposition 10.15 and get that

$$\widetilde{E}_\infty^\sharp \text{ is rectifiable.} \tag{10.45}$$

Then extract again a subsequence, so that $\widetilde{x}_k^\sharp$ tends to some limit $\widetilde{x}_\infty^\sharp$. Recall that $\widetilde{x}_k^\sharp = A_k(\widetilde{x}_k) \in \widetilde{E}_k^\sharp$ is defined in (10.37), and let us check that

$$\widetilde{x}_\infty^\sharp = \lim_{k\to+\infty} \widetilde{x}_k^\sharp \in \widetilde{E}_\infty^\sharp \cap \overline{B}(0,1/4). \tag{10.46}$$

First of all,

$$\begin{aligned} |\widetilde{x}_k^\sharp| &= |A_k(\widetilde{x}_k)| = |A_k(\widetilde{x}_k) - A_k(o_k)| = (2^{-m_k+2}a_n)^{-1}|\widetilde{x}_k - o_k| \\ &\leq (2^{-m_k+2}a_n)^{-1}\, 2^{-m_k} a_n = \frac{1}{4} \end{aligned} \tag{10.47}$$

by (10.34) and (10.30). So $\widetilde{x}_k^\sharp \in \overline{B}(0,1/4)$, hence $\widetilde{x}_\infty^\sharp = \lim_{k\to+\infty} \widetilde{x}_k^\sharp \in \overline{B}(0,1/4)$ too. Finally, $\widetilde{x}_\infty^\sharp \in \widetilde{E}_\infty^\sharp$ because $\widetilde{x}_k^\sharp \in \widetilde{E}_k^\sharp$, $\widetilde{E}_k^\sharp$ converges to $\widetilde{E}_\infty^\sharp$ locally in $B(0,1)$, and our points stay in $\overline{B}(0,1/4)$. This proves (10.46).

We need a last change of coordinates. Recall that $A_k^{-1}(B(0,1)) = \widetilde{B} \subset B(0,1)$ by (10.35) and (10.33), so we can define $\theta_k$ on $B(0,1)$ by

$$\theta_k(z) = \gamma_k^{-1}\psi_k^{-1}(A_k^{-1}(z)). \tag{10.48}$$

Observe that $\theta_k$ is a $\Lambda$-bilipschitz mapping (we conjugate $\psi_k^{-1}$ by a dilation, and translate). We can extend $\theta_k$ to the closed ball $\overline{B}(0,1)$, either because it is Lipschitz on $B(0,1)$, or simply using (10.48) and the fact that $A_k^{-1}(\overline{B}(0,1)) \subset B(0,1)$. By Arzelà-Ascoli, the collection of mappings $\theta_k(\cdot) - \theta_k(0) : \overline{B}(0,1) \to \mathbb{R}^n$ is totally bounded, so we can extract another subsequence so that

$$\text{the functions } \theta_k(\cdot) - \theta_k(0) \text{ converge, uniformly on } \overline{B}(0,1), \text{ to a limit } \theta_\infty. \tag{10.49}$$

We need to remove the constant $\theta_k(0)$ because the domains $U_k$ could go away to infinity very fast; we could also avoid this minor problem by translating $U_k$, $E_k$, and $x_k$ by a same vector $v_k$, with the effect of precomposing $\psi_k$ with a translation, adding $\lambda_k v_k$ to $\psi_k^{-1}$, keeping $\widetilde{E}_k$ and $A_k$ as they are, adding $\gamma_k^{-1}\lambda_k v_k$ to $\theta_k$, and making $\theta_k(0) = 0$ (if $v_k$ is chosen correctly).

Note that $\theta_\infty$ is $\Lambda$-bilipschitz on $B(0,1)$, because the $\theta_k$ are. Next set

$$E_k^\sharp = \theta_k(\widetilde{E}_k^\sharp),\ E_\infty^\sharp = \theta_\infty(\widetilde{E}_\infty^\sharp), \text{ and } x_\infty^\sharp = \theta_\infty(\widetilde{x}_\infty^\sharp) \in E_\infty^\sharp \tag{10.50}$$

(because $\widetilde{x}_\infty^\sharp \in \widetilde{E}_\infty^\sharp$ by (10.46)). We expect correct translations of $E_k^\sharp$ to converge to $E_\infty^\sharp$, but in fact we shall find it more convenient to work directly on $\widetilde{E}_k^\sharp$ and $\widetilde{E}_\infty^\sharp$. Obviously,

$$E_\infty^\sharp \text{ is rectifiable,} \tag{10.51}$$

by (10.45) and because $\theta_\infty$ is bilipschitz. We need the last change of variable because we want $E_k$ to be close to a plane, not $\widetilde{E}_k$, and for this (10.51) will be more useful.

Let $\varepsilon^\sharp$ be very small (to be chosen near the end); we shall use $\varepsilon^\sharp$ to measure various small quantities that are not necessarily connected to each other. We claim that we can choose

$$y^\sharp_\infty \in E^\sharp_\infty \cap B(x^\sharp_\infty, \varepsilon^\sharp) \tag{10.52}$$

so that

$$E^\sharp_\infty \text{ has a tangent plane } P^\sharp \text{ at } y^\sharp_\infty. \tag{10.53}$$

Indeed, $E^\sharp_\infty$ is rectifiable by (10.51) so it has an approximate tangent plane at $\mathcal{H}^d$-almost every point. In addition, recall from (10.44) that $\widetilde{E}^\sharp_\infty$ is the local limit in $B(0,1)$ of a sequence of reduced quasiminimal sets $\widetilde{E}^\sharp_k$, with uniform constants (see (10.43)). When $k$ is large, $\widetilde{h}_k = \Lambda^{2d} h_k = \Lambda^{2d} 2^{-k}$ is as small as we want; then the assumptions (10.1)-(10.4) are satisfied, and so $\widetilde{E}^\sharp_\infty$ is locally Ahlfors-regular in $B(0,1)$, as in (10.11). Its bilipschitz image $E^\sharp_\infty = \theta_\infty(\widetilde{E}^\sharp_\infty)$ is also locally Ahlfors-regular. Then the approximate tangent planes are automatically tangent planes (see for instance Exercise 41.21 on page 277 of [D4]), and so $E^\sharp_\infty$ admits a (true) tangent plane at $\mathcal{H}^d$-almost every point $y^\sharp_\infty$. Finally, $\mathcal{H}^d(E^\sharp_\infty \cap B(x^\sharp_\infty, \varepsilon^\sharp)) > 0$ (again by local Ahlfors-regularity, and because $x^\sharp_\infty \in E^\sharp_\infty$), so we can choose $y^\sharp_\infty$ in $B(x^\sharp_\infty, \varepsilon^\sharp)$, as needed.

By (10.53), we can find $\rho \in (0,1)$ such that

$$\operatorname{dist}(z, P^\sharp) \le \varepsilon^\sharp \rho \quad \text{for } z \in E^\sharp_\infty \cap B(y^\sharp_\infty, \rho). \tag{10.54}$$

We want to use $y^\sharp_\infty$ and $P^\sharp$ to find, for $k$ large enough, a pair $(y_k, t_k)$ that satisfies (10.24) and (10.25); the desired contradiction will ensue.

Since $y^\sharp_\infty \in E^\sharp_\infty = \theta_\infty(\widetilde{E}^\sharp_\infty)$ (by (10.52) and (10.50)), there exists $\widetilde{y}^\sharp_\infty \in \widetilde{E}^\sharp_\infty$ such that $\theta_\infty(\widetilde{y}^\sharp_\infty) = y^\sharp_\infty$. Notice that

$$|\widetilde{y}^\sharp_\infty - \widetilde{x}^\sharp_\infty| \le \Lambda |y^\sharp_\infty - x^\sharp_\infty| \le \Lambda \varepsilon^\sharp \tag{10.55}$$

because $\theta_\infty$ is $\Lambda$-bilipschitz, $x^\sharp_\infty = \theta_\infty(\widetilde{x}^\sharp_\infty)$ by (10.50), and by (10.52). Then $\widetilde{y}^\sharp_\infty \in \widetilde{E}^\sharp_\infty \cap B(0,1/2)$ by (10.46), and by (10.44) we can find points $\widetilde{y}^\sharp_k \in \widetilde{E}^\sharp_k$ so that

$$\widetilde{y}^\sharp_\infty = \lim_{k\to+\infty} \widetilde{y}^\sharp_k. \tag{10.56}$$

Set $y^\sharp_k = \theta_k(\widetilde{y}^\sharp_k) \in E^\sharp_k$ (by (10.50)), $\widetilde{y}_k = A_k^{-1}(\widetilde{y}^\sharp_k) \in \widetilde{E}_k \cap \widetilde{B}$ (by (10.36) and (10.35)), and $y_k = \lambda_k^{-1}\psi_k^{-1}(\widetilde{y}_k) \in E_k$ (by (10.33) and (10.26)). By (10.48),

$$y^\sharp_k = \theta_k(\widetilde{y}^\sharp_k) = \gamma_k^{-1}\psi_k^{-1}(A_k^{-1}(\widetilde{y}^\sharp_k)) = \gamma_k^{-1}\lambda_k y_k. \tag{10.57}$$

For (10.24) we first need to check that for $k$ large,

$$y_k \in E_k \cap B(x_k, r_k/2). \tag{10.58}$$

But $y_k = \lambda_k^{-1}\psi_k^{-1}(A_k^{-1}(\widetilde{y}_k^\sharp))$, and similarly $x_k = \lambda_k^{-1}\psi_k^{-1}(A_k^{-1}(\widetilde{x}_k^\sharp))$ by (10.37), so

$$\begin{aligned}|y_k - x_k| &= |\lambda_k^{-1}\psi_k^{-1}(A_k^{-1}(\widetilde{y}_k^\sharp)) - \lambda_k^{-1}\psi_k^{-1}(A_k^{-1}(\widetilde{x}_k^\sharp))| \\ &\leq \lambda_k^{-1}\Lambda|A_k^{-1}(\widetilde{y}_k^\sharp) - A_k^{-1}(\widetilde{x}_k^\sharp)| = \lambda_k^{-1}\Lambda 2^{-m_k+2}a_n|\widetilde{y}_k^\sharp - \widetilde{x}_k^\sharp| \\ &\leq 4\lambda_k^{-1}\Lambda\,\frac{\lambda_k r_k}{8a_n\Lambda}\,a_n|\widetilde{y}_k^\sharp - \widetilde{x}_k^\sharp| = \frac{r_k}{2}\,|\widetilde{y}_k^\sharp - \widetilde{x}_k^\sharp|\end{aligned} \tag{10.59}$$

because $\psi_k$ is $\Lambda$-bilipschitz, and by (10.34) and (10.29). Now $\widetilde{y}_k^\sharp$ tends to $\widetilde{y}_\infty^\sharp$ by (10.56), $\widetilde{x}_k^\sharp$ tends to $\widetilde{x}_\infty^\sharp$ by (10.46), and so

$$|\widetilde{y}_k^\sharp - \widetilde{x}_k^\sharp| \leq |\widetilde{y}_\infty^\sharp - \widetilde{x}_\infty^\sharp| + |\widetilde{y}_k^\sharp - \widetilde{y}_\infty^\sharp| + |\widetilde{x}_k^\sharp - \widetilde{x}_\infty^\sharp| \leq |\widetilde{y}_\infty^\sharp - \widetilde{x}_\infty^\sharp| + \varepsilon^\sharp \leq (\Lambda+1)\varepsilon^\sharp \tag{10.60}$$

for $k$ large, and by (10.55). By (10.59),

$$|y_k - x_k| \leq \frac{(\Lambda+1)\varepsilon^\sharp\, r_k}{2} \tag{10.61}$$

for $k$ large, and of course (10.58) follows. We choose

$$t_k = \frac{\rho r_k}{20\Lambda^2} \leq \frac{r_k}{20\Lambda^2}, \tag{10.62}$$

with $\rho \in (0,1)$ as in (10.54). Obviously the pair $(y_k, t_k)$ satisfies (10.24) for $k$ large, so we just need to find a $d$-plane $P_k$ through $y_k$ such that (10.25) holds.

So let $z \in E_k \cap B(y_k, t_k)$ be given. We first want to define

$$\widetilde{z} = \psi_k(\lambda_k z) \in \widetilde{E}_k\,,\ \ \widetilde{z}^\sharp = A_k(\widetilde{z}) \in \widetilde{E}_k^\sharp \ \text{ and } \ z^\sharp = \theta_k(\widetilde{z}^\sharp) \in E_k^\sharp\,, \tag{10.63}$$

because (10.54) gives us some control on $E_\infty^\sharp$ and hence probably on $E_k^\sharp$.

We start with $\widetilde{z} = \psi_k(\lambda_k z)$, which is defined because $z \in B(y_k, t_k) \subset U_k$ (recall that $B(y_k, t_k) \subset B(x_k, r_k)$ by (10.61) and (10.62), and use (10.22)). Next, $\widetilde{z} \in \widetilde{E}_k$ by (10.26). Before we switch to $\widetilde{z}^\sharp$, observe that

$$\begin{aligned}|\widetilde{z} - \widetilde{x}_k| &= |\psi_k(\lambda_k z) - \psi_k(\lambda_k x_k)| \leq \Lambda|\lambda_k z - \lambda_k x_k| \leq \Lambda\lambda_k(|z - y_k| + |y_k - x_k|) \\ &\leq \Lambda\lambda_k\Big(t_k + \frac{(\Lambda+1)\varepsilon^\sharp\, r_k}{2}\Big) \leq \frac{\lambda_k r_k}{19\Lambda} < 2^{-m_k}a_n\end{aligned} \tag{10.64}$$

by (10.26), (10.61), if $\varepsilon^\sharp$ is small enough, and by (10.62) and (10.29). Thus

$$\widetilde{z} \in B(\widetilde{x}_k, 2^{-m_k}a_n) \subset B(o_k, \gamma_k/2) = \frac{1}{2}\widetilde{B} \tag{10.65}$$

by (10.32) and (10.31). By (10.34) or (10.35), $A_k(B(o_k, \gamma_k/2)) = B(0, 1/2)$, so

$$\widetilde{z}^\sharp = A_k(\widetilde{z}) \in \widetilde{E}_k^\sharp \cap B(0, 1/2), \tag{10.66}$$

because (10.36) says that $\widetilde{E}_k^\sharp = A_k(\widetilde{E}_k \cap \widetilde{B})$. Now $z^\sharp = \theta_k(\widetilde{z}^\sharp)$ is defined and lies in $E_k^\sharp$ by the definition (10.50) and because $\widetilde{z}^\sharp \in \widetilde{E}_k^\sharp$. This completes our verification of (10.63).

Notice that

$$z^\sharp = \theta_k(\widetilde{z}^\sharp) = \gamma_k^{-1}\psi_k^{-1}(A_k^{-1}(\widetilde{z}^\sharp)) = \gamma_k^{-1}\psi_k^{-1}(\widetilde{z}) = \gamma_k^{-1}\lambda_k z \tag{10.67}$$

by (10.63) and (10.48). Set

$$\varepsilon_k = \sup_{w \in \widetilde{E}_k^\sharp \cap \overline{B}(0,1/2)} \mathrm{dist}(w, \widetilde{E}_\infty^\sharp); \tag{10.68}$$

then $\varepsilon_k$ tends to 0 because $\overline{B}(0,1/2)$ is a compact subset of $B(0,1)$, and by the local convergence of $\widetilde{E}_k^\sharp$ to $\widetilde{E}_\infty^\sharp$ (see (10.44)). Similarly,

$$\varepsilon_k' = \sup_{w \in B(0,1)} |\theta_k(w) - \theta_k(0) - \theta_\infty(w)| \tag{10.69}$$

tends to 0, by the uniform convergence in (10.49).

Return to $\widetilde{z}^\sharp$, and choose $\widetilde{\xi}^\sharp \in \widetilde{E}_\infty^\sharp$ such that

$$|\widetilde{\xi}^\sharp - \widetilde{z}^\sharp| \leq \mathrm{dist}(\widetilde{z}^\sharp, \widetilde{E}_\infty^\sharp) \leq \varepsilon_k \tag{10.70}$$

(compare (10.68) with (10.66)). Set $\xi^\sharp = \theta_\infty(\widetilde{\xi}^\sharp)$; then

$$\xi^\sharp \in E_\infty^\sharp = \theta_\infty(\widetilde{E}_\infty^\sharp) \tag{10.71}$$

(see (10.50)). Also,

$$\begin{aligned} |\xi^\sharp - y_\infty^\sharp| &= |\theta_\infty(\widetilde{\xi}^\sharp) - \theta_\infty(\widetilde{y}_\infty^\sharp)| \leq |\theta_k(\widetilde{\xi}^\sharp) - \theta_k(\widetilde{y}_\infty^\sharp)| + 2\varepsilon_k' \\ &\leq |\theta_k(\widetilde{z}^\sharp) - \theta_k(\widetilde{y}_k^\sharp)| + \Lambda\big(|\widetilde{\xi}^\sharp - \widetilde{z}^\sharp| + |\widetilde{y}_\infty^\sharp - \widetilde{y}_k^\sharp|\big) + 2\varepsilon_k' \\ &\leq |\theta_k(\widetilde{z}^\sharp) - \theta_k(\widetilde{y}_k^\sharp)| + \Lambda\varepsilon_k + \Lambda|\widetilde{y}_\infty^\sharp - \widetilde{y}_k^\sharp| + 2\varepsilon_k' \\ &\leq |\theta_k(\widetilde{z}^\sharp) - \theta_k(\widetilde{y}_k^\sharp)| + \varepsilon^\sharp\rho \end{aligned} \tag{10.72}$$

because $y_\infty^\sharp = \theta_\infty(\widetilde{y}_\infty^\sharp)$ (see above (10.55)) and $\theta_k$ is $\Lambda$-Lipschitz, by (10.70), then by (10.56), because $\varepsilon_k$ and $\varepsilon_k'$ tend to 0, and if $k$ is large enough. Next

$$\theta_k(\widetilde{z}^\sharp) - \theta_k(\widetilde{y}_k^\sharp) = \gamma_k^{-1}\lambda_k(z - y_k) \tag{10.73}$$

by (10.67) and (10.57); since

$$|z - y_k| \leq t_k = \frac{\rho r_k}{20\Lambda^2} \tag{10.74}$$

because $z \in E_k \cap B(y_k, t_k)$ and by (10.62), we deduce from (10.73) that

$$\begin{aligned} |\theta_k(\widetilde{z}^\sharp) - \theta_k(\widetilde{y}_k^\sharp)| &\le \gamma_k^{-1}\lambda_k \,\frac{\rho r_k}{20\Lambda^2} = (2^{-m_k+2}a_n)^{-1}\lambda_k \,\frac{\rho r_k}{20\Lambda^2} \\ &\le \frac{1}{2}\,\frac{8a_n\Lambda}{\lambda_k r_k}\,a_n^{-1}\lambda_k\,\frac{\rho r_k}{20\Lambda^2} = \frac{\rho}{5\Lambda} \end{aligned} \tag{10.75}$$

by (10.31) and (10.29). Therefore

$$|\xi^\sharp - y_\infty^\sharp| \le |\theta_k(\widetilde{z}^\sharp) - \theta_k(\widetilde{y}_k^\sharp)| + \varepsilon^\sharp\rho \le \frac{\rho}{5\Lambda} + \varepsilon^\sharp\rho < \frac{\rho}{4\Lambda} \tag{10.76}$$

by (10.72) and (10.75), and if $\varepsilon^\sharp$ is small enough. Since $\xi^\sharp \in E_\infty^\sharp$ by (10.71), we get that $\xi^\sharp \in E_\infty^\sharp \cap B(y_\infty^\sharp, \rho)$, and (10.54) says that $\mathrm{dist}(\xi^\sharp, P^\sharp) \le \varepsilon^\sharp\rho$. Set

$$P_k' = \gamma_k\lambda_k^{-1}[P^\sharp + \theta_k(0)]; \tag{10.77}$$

then

$$\begin{aligned} \mathrm{dist}(z, P_k') &= \gamma_k\lambda_k^{-1}\,\mathrm{dist}(\gamma_k^{-1}\lambda_k z, P^\sharp + \theta_k(0)) = \gamma_k\lambda_k^{-1}\,\mathrm{dist}(z^\sharp, P^\sharp + \theta_k(0)) \\ &\le \gamma_k\lambda_k^{-1}\big[\,\mathrm{dist}(\xi^\sharp + \theta_k(0), P^\sharp + \theta_k(0)) + |\xi^\sharp + \theta_k(0) - z^\sharp|\big] \\ &= \gamma_k\lambda_k^{-1}\big[\,\mathrm{dist}(\xi^\sharp, P^\sharp) + |\xi^\sharp + \theta_k(0) - z^\sharp|\big] \\ &\le \gamma_k\lambda_k^{-1}\big[\varepsilon^\sharp\rho + |\xi^\sharp + \theta_k(0) - z^\sharp|\big] \end{aligned} \tag{10.78}$$

by (10.67). Recall that $\xi^\sharp = \theta_\infty(\widetilde{\xi}^\sharp)$ (see above (10.71)) and $z^\sharp = \theta_k(\widetilde{z}^\sharp)$ (see (10.63)), so

$$\begin{aligned} |\xi^\sharp + \theta_k(0) - z^\sharp| &= |\theta_\infty(\widetilde{\xi}^\sharp) + \theta_k(0) - \theta_k(\widetilde{z}^\sharp)| \\ &\le |\theta_\infty(\widetilde{\xi}^\sharp) + \theta_k(0) - \theta_k(\widetilde{\xi}^\sharp)| + |\theta_k(\widetilde{\xi}^\sharp) - \theta_k(\widetilde{z}^\sharp)| \\ &\le \varepsilon_k' + \Lambda|\widetilde{\xi}^\sharp - \widetilde{z}^\sharp| \le \varepsilon_k' + \Lambda\varepsilon_k \end{aligned} \tag{10.79}$$

by (10.69) and (10.70). We combine this with (10.78) and get that

$$\begin{aligned} \mathrm{dist}(z, P_k') &\le \gamma_k\lambda_k^{-1}\big[\varepsilon^\sharp\rho + \varepsilon_k' + \Lambda\varepsilon_k\big] = 2^{-m_k+2}a_n\lambda_k^{-1}\big[\varepsilon^\sharp\rho + \varepsilon_k' + \Lambda\varepsilon_k\big] \\ &\le 4\,\frac{\lambda_k r_k}{8a_n\Lambda}a_n\lambda_k^{-1}\big[\varepsilon^\sharp\rho + \varepsilon_k' + \Lambda\varepsilon_k\big] = \frac{r_k}{2\Lambda}\,\big[\varepsilon^\sharp\rho + \varepsilon_k' + \Lambda\varepsilon_k\big] \\ &= \frac{10\Lambda t_k}{\rho}\big[\varepsilon^\sharp\rho + \varepsilon_k' + \Lambda\varepsilon_k\big] \end{aligned} \tag{10.80}$$

by (10.31), (10.29), and (10.62). Recall that $\varepsilon_k$ and $\varepsilon_k'$ tend to 0; then, for $k$ large enough (depending also on $\rho$, but this is all right), we deduce from (10.80) that

$$\mathrm{dist}(z, P_k') \le 20\Lambda\varepsilon^\sharp t_k \le \frac{\varepsilon t_k}{2} \tag{10.81}$$

if $\varepsilon^\sharp$ is chosen small enough.

Now all this is true for all $k$ large (not depending on $z$), and all $z \in E_k \cap B(y_k, t_k)$. In particular, $z = y_k$ yields $\mathrm{dist}(y_k, P'_k) \leq \frac{\varepsilon t_k}{2}$. We choose for $P_k$ a translation of $P'_k$ that contains $y_k$; this is required for (10.23) and (10.25), but fortunately we just need to translate by at most $\frac{\varepsilon t_k}{2}$. Then $\mathrm{dist}(y_k, P_k) \leq \varepsilon t_k$ for $z \in E_k \cap B(y_k, t_k)$, by (10.81) and as needed for (10.25).

We finally found a plane $P_k$ through $y_k$ that satisfies (10.25); as announced earlier, its existence contradicts the definition of our sequence $\{E_k\}$; this completes our proof of Lemma 10.21 by contradiction. $\square$

Our next preparatory result is a (simplified) generalization of Corollaries 9.103 and 8.55; it says that if $E$ is a quasiminimal set, its core $E^*$ is concentrated, with uniform bounds. The terminology comes from [DMS] (and is justified by that fact that (10.84) below says that $E^*$ is almost as concentrated in $B(y, t)$ as a $d$-plane through $y$), and the result is interesting because it will soon allow us to prove the lower semicontinuity of $\mathcal{H}^d$ along convergent sequences of uniformly quasiminimal sets.

**Proposition 10.82.** *For each choice of constants $n$, $M \geq 1$, $\Lambda \geq 1$ and $\varepsilon > 0$, we can find $h > 0$ and $d_\varepsilon > 0$ such that the following holds. Suppose that $E \in GSAQ(U, M, \delta, h)$ for some open set $U \subset \mathbb{R}^n$, and that the Lipschitz assumption are satisfied, with the constants $\lambda$ and $\Lambda$ (as in (9.3)). Also denote by $r_0 = 2^{-m} \leq 1$ the side length of the dyadic cubes of the usual grid. Then let $(x, r)$ be such that*

$$x \in E^*, \quad 0 < r < \mathrm{Min}(\lambda^{-1} r_0, \delta), \quad B(x, 2r) \subset U. \tag{10.83}$$

*Then we can find a pair $(y, t)$, such that $y \subset E^* \cap B(x, r/2)$, $d_\varepsilon r \leq t \leq r/4$, and*

$$\mathcal{H}^d(E^* \cap B(y, t)) \geq (1 - \varepsilon)\omega_d t^d, \tag{10.84}$$

*where $\omega_d$ denotes the $d$-dimensional Hausdorff measure of the unit ball in $\mathbb{R}^d$.*

The proof will be similar to the proof of Corollaries 9.103 and 8.55, but we shall rely on Lemma 10.21 rather than the uniform rectifiability of our quasiminimal sets, which we do not know how to prove with enough generality. This slightly different approach is new, and even in the case of standard quasiminimal sets without boundaries, it has the advantage of not using our complicated proof of uniform rectifiability (with the tough stopping time argument on the projections). But one more compactness argument is used, and we loose an "explicit" control on $d_\varepsilon$.

Compared with Corollary 9.103, we just get rid of the unpleasant additional assumption (9.105) on the dimensions of some faces. Recall that Corollary 8.55 works under the rigid assumption, and also has the unpleasant dimensionality assumption (6.2).

For the proof, first notice that by Proposition 3.3, $E^* \in GSAQ(U, M, \delta, h)$, so it is enough to prove Proposition 10.82 when $E$ is coral, i.e., when $E^* = E$.

Let $E$, $x$, and $r$ be as in the statement. Our goal is to apply Lemma 9.14 to some pair $(y, t)$, because surjective projections will help us find lower bounds on $\mathcal{H}^d(E \cap B(y, t))$. Let $C_0 \geq 1$, $\eta > 0$, and $\overline{\varepsilon} > 0$ be as in the statement of that lemma, and recall that they depend only on $n$, $M$, and $\Lambda$.

Let $\varepsilon_0$ be very small, to be chosen near the end, and apply Lemma 10.21, with the constant $\varepsilon_0$, to the pair $(x, (2C_0)^{-1}r)$; the hypotheses for Lemma 10.21 are the same as for the present proposition, so the pair $(x, (2C_0)^{-1}r)$ satisfies them. We get $(y_0, t_0)$ such that

$$y_0 \in E \cap B(x, \frac{r}{4C_0}) \text{ and } \frac{c_{\varepsilon_0} r}{2C_0} \leq t_0 \leq \frac{r}{4C_0}, \tag{10.85}$$

and a plane $P$ through $y_0$ such that

$$\mathrm{dist}(z, P) \leq \varepsilon_0 t_0 \text{ for } z \in E \cap B(y_0, t_0). \tag{10.86}$$

We would be happy to apply Lemma 9.14 directly to $(y_0, t_0)$, but the unpleasant assumption (9.17) on the proximity to some boundaries $L_j$ may not be satisfied. As in (9.16), set

$$J(y,t) = \big\{ j \in [0, j_{max}] \, ; \, L_j \text{ meets } B(y, 2t) \big\} \quad \text{and} \quad L(y,t) = \bigcap_{j \in J(y,t)} L_j \tag{10.87}$$

for $y \in E$ and $t > 0$. Recall that $J(y,t) \neq \emptyset$ because $y \in E \subset L_0 = \Omega$. We want to find pairs $(y, t)$ such that

$$\mathrm{dist}(w, L(y,t)) \leq \eta t \text{ for } w \in E \cap B(y, 2t), \tag{10.88}$$

as in (9.17). We shall restrict to pairs $(y, t)$ such that $B(y, 2t) \subset B(y_0, t_0)$, near which (10.86) says that $E$ stays very close to $P$.

We shall define a (finite) sequence of pairs $(y_k, t_k)$. Naturally, we start with $(y_0, t_0)$.

Suppose we already defined $(y_k, t_k)$. If the pair $(y_k, t_k/2)$ satisfies (10.88), we stop the construction. Otherwise, we define $(y_{k+1}, t_{k+1})$ as follows.

If $J(y_k, t_k/4) \neq J(y_k, t_k)$ (which by (10.87) means that it is strictly smaller), set $(y_{k+1}, t_{k+1}) = (y_k, t_k/4)$. We are happy, because $J(y_{k+1}, t_{k+1})$ is strictly contained in $J(y_k, t_k)$ and this cannot happen too often.

If $J(y_k, t_k/4) = J(y_k, t_k)$ and the pair $(y_k, t_k/4)$ satisfies (10.88), set $(y_{k+1}, t_{k+1}) = (y_k, t_k/2)$. This time we are happy too because we know that we will stop next time.

In the remaining case, the failure of (10.88) for $(y_k, t_k/4)$ gives a point $w \in E \cap B(y_k, t_k/2)$ such that

$$\mathrm{dist}(w, L(y_k, t_k)) = \mathrm{dist}(w, L(y_k, t_k/4)) \geq \eta t_k/4, \tag{10.89}$$

where the first identity comes from the fact that $J(y_k, t_k/4) = J(y_k, t_k)$. In this last case, we set $(y_{k+1}, t_{k+1}) = (w, a t_k)$, for some small constant $a \in (0, 1/2)$ that will be chosen soon. Notice that $B(y_{k+1}, t_{k+1}) \subset B(y_k, t_k)$ in all cases, so that we know that

$$B(y_k, t_k) \subset B(y_0, t_0) \text{ for all } k \geq 1. \tag{10.90}$$

This completes our definition of the pairs $(y_k, t_k)$. Now we want to show that the construction stops after at most $j_{max} + 2$ steps (where $j_{max} + 1$ still denotes the number of boundary sets $L_j$), and for this it will be enough to show that in our last case,

$$J(w, a t_k) \text{ is strictly contained in } J(y_k, t_k). \tag{10.91}$$

Notice that $J(w, at_k) \subset J(y_k, t_k)$ by (10.87) and because $B(w, 2at_k) \subset B(w, t_k) \subset B(y_k, 2t_k)$, so (10.91) just means that $J(w, at_k) \neq J(y_k, t_k)$. Let us suppose that (10.91) fails, and derive a contradiction.

Then $J(w, at_k) = J(y_k, t_k)$, and for each $j \in J(y_k, t_k)$, the definition (10.87) says that $L_j$ meets $B = B(w, 2at_k)$. Choose a face $F_j \subset L_j$ that meets $B$, and set

$$F = \bigcap_{j \in J(y_k, t_k)} F_j \subset L(y_k, t_k), \tag{10.92}$$

where the inclusion comes from (10.87). We want to show that $F$ is nonempty and meets a larger ball. We return to the standard grid because this will make the computations easier.

Set $w' = \psi(\lambda w)$, $r' = 2\lambda\Lambda a t_k$, and $B' = B(w', r')$. Then $\psi(\lambda B) = \psi(B(\lambda w, 2\lambda a t_k)) \subset B(w', 2\lambda\Lambda a t_k) = B'$, just because $\psi$ is $\Lambda$-lipschitz. Now $B'$ is not very large, because $r' = 2\lambda\Lambda a t_k \leq 2\lambda\Lambda a t_0 \leq \lambda\Lambda a r \leq \Lambda a r_0$ by construction, (10.85), and (10.83).

Also set $F'_j = \psi(\lambda F_j)$ for $j \in J(y_k, t_k)$. The $F'_j$ are now real dyadic faces of side length $r_0$, and they all meet $B'$ because the $F_j$ meet $B$.

We need to know the following geometrical fact about our net. We have a collection of faces, that all meet a small ball $B'$, and we want to know that their intersection meets $CB'$. This is probably true with general polyhedral networks, but here again let us cheat and use the fact that we have a cubical network.

Write things in coordinates. Each $F'_j$ is given by the equations $z_i \in I_{i,j}$, $1 \leq i \leq n$, where each $I_{i,j}$ is either a point or a dyadic interval of size $r_0$. Let $w'_i$ denote the $i$-th coordinate of $w'$. Since $B'$ meets $F'_j$, we get that $\mathrm{dist}(w'_i, I_{i,j}) < r'$ for all $i$. If $a < (3\Lambda)^{-1}$, then $r' < r_0/3$; then for each $i$, either all the $I_{i,j}$ are equal, or else they all have a common endpoint $\xi_i$ which in addition is such that $|w'_i - \xi_i| \leq r'$ (easy proof by induction on $j$).

In all cases, we get $\xi_i \in \bigcap_{j \in J(y_k, t_k)} I_{i,j}$ such that $|w'_i - \xi_i| \leq r'$. Now the point $\xi$ with coordinates $\xi_i$ lies in $\bigcap_{j \in J(y_k, t_k)} F'_j$, and $|\xi - w'| \leq \sqrt{n} r'$. Set $\zeta = \lambda^{-1}\psi^{-1}(\xi)$; then $\zeta \in F$, and

$$|\zeta - w| = |\lambda^{-1}\psi^{-1}(\xi) - \lambda^{-1}\psi^{-1}(w')| \leq \lambda^{-1}\Lambda|\xi - w'| \leq \lambda^{-1}\Lambda\sqrt{n} r' = 2\Lambda^2\sqrt{n} a t_k \tag{10.93}$$

Choose $a = \frac{\eta}{10\Lambda^2\sqrt{n}}$; then $|\zeta - w| < \eta t_k/4$. We also know that $\zeta \in F$, so $\zeta \in L(y_k, t_k)$, by (10.92). This contradicts (10.89).

So (10.91) holds, and our construction stops after at most $j_{max} + 2$ steps. Let $(y_k, t_k)$ be the last pair, where we stop. Set $y = y_k$ and $t = t_k/2$. By definition of stopping, $(y, t)$ satisfies (10.88).

Let us try to apply Lemma 9.14 to the pair $(y, t)$. First we need to check (for (9.15)) that $0 < t < C_0^{-1}\,\mathrm{Min}(\lambda^{-1} r_0, \delta)$, but this is true because $t \leq t_k \leq t_0 \leq \frac{r}{4C_0}$ (by (10.85)), and by (10.83). Also, $B(y, (C_0 + 1)t) \subset 2C_0 B(y, t) \subset 2C_0 B(y_0, t_0) = B(y_0, 2C_0 t_0) \subset B(y_0, r/2) \subset B(x, r) \subset U$ because $B(y_k, t_k) \subset B(y_0, t_0)$ by (10.90), and by (10.85) and (10.83); this proves (9.15).

The ugly condition (9.17) is now satisfied, precisely because $(y, t)$ satisfies (10.88). For the last condition (9.18), let us take the same plane $P$ as in (10.86), and show that

$$\mathrm{dist}(z, P) \leq A\varepsilon_0 t \quad \text{for } z \in E \cap B(y, 2t), \tag{10.94}$$

where $A = 2a^{-j_{max}-2}$ is just another geometric constant, that only depends on $n$ and $\Lambda$.

Let $z \in E \cap B(y, 2t) = E \cap B(y_k, t_k)$ be given. Then $z \in E \cap B(y_0, t_0)$ (again by (10.90)), and $\mathrm{dist}(z, P) \leq \varepsilon_0 t_0$ by (10.86). So we just need to check that $t_0 \leq At$.

But during our construction, we always took $t_{l+1} \geq at_l$. Therefore, $t = \frac{1}{2} t_k \geq \frac{1}{2} a^{j_{max}+2} t_0 = A^{-1} t_0$, and (10.94) follows.

We just proved that (9.17) is satisfied, with the constant $A\varepsilon_0$. We shall of course choose $\varepsilon_0$ so small that $A\varepsilon_0 \leq \overline{\varepsilon}$, where $\overline{\varepsilon}$ is the threshold in Lemma 9.14; then the lemma applies, and we get that (9.20) holds, i.e., that

$$\pi(E \cap B(y, 5t/3)) \text{ contains } P \cap B(\pi(y), 3t/2), \tag{10.95}$$

where we denote by $\pi$ the orthogonal projection onto $P$.

We shall now conclude as in the other corollaries. We want to check that our pair $(y, t)$ satisfies the conclusions of Proposition 10.82. We know that $y \subset E^* \cap B(x, r/2)$ because $y = y_k \in B(y_0, t_0)$ (by (10.90)) and by (10.85). Similarly, $t \leq t_0 \leq r/4$ by (10.85), and $t \geq A^{-1} t_0 \geq \frac{c_{\varepsilon_0} r}{2AC_0}$ (by (10.85) again). So we will be able to take $d_\varepsilon = \frac{c_{\varepsilon_0}}{2AC_0}$, and we just need to check (10.84).

For each $p \in P \cap B(y, (1 - A\varepsilon_0)t)$, (10.95) gives a point $z \in E \cap B(y, 3t/2)$ such that $\pi(w) = p$. Since $|p - w| = |\pi(w) - w| = \mathrm{dist}(w, P) \leq A\varepsilon_0 t$ by (10.94), $w \in B(y, t)$. So $P \cap B(y, (1 - A\varepsilon_0)t) \subset \pi(E \cap B(y, t))$, and

$$\begin{aligned} \mathcal{H}^d(E \cap B(y, t)) &\geq \mathcal{H}^d(\pi(E \cap B(y, t))) \\ &\geq \mathcal{H}^d(P \cap B(y, (1 - A\varepsilon_0)t)) \geq \omega_d (1 - A\varepsilon_0)^d t^d, \end{aligned} \tag{10.96}$$

where $\omega_d$ is the same as in (10.84). We choose $\varepsilon_0$ so small, depending on $\varepsilon$, that $(1 - A\varepsilon_0)^d \leq 1 - \varepsilon$, and then deduce (10.84) from (10.96). This completes our proof of Proposition 10.82. □

We finally come to the lower semicontinuity of $\mathcal{H}^d$ along convergent sequences of uniformly quasiminimal sets, which we will deduce from Proposition 10.82 and the lower semicontinuity result of Dal Maso, Morel, and Solimini [DMS] for the uniformly concentrated set.

**Theorem 10.97.** *Let $U$, $\{E_k\}$, and $E$ satisfy the hypotheses (10.1), (10.2), (10.3), and (10.4). Also suppose that $h$ is small enough, depending only on $n$, $M$, and $\Lambda$. Then*

$$\mathcal{H}^d(E \cap V) \leq \liminf_{k \to +\infty} \mathcal{H}^d(E_k \cap V) \quad \textit{for every open set } V \subset U. \tag{10.98}$$

See Theorem 25.7 for an extension of Theorem 10.97 where we also prove the lower semicontinuity of $\int_{E_k \cap V} f(x) d\mathcal{H}^d(x)$ for some continuous functions $f$ or even elliptic integrands (where $f$ may also depend on the tangent plane to $E_k$ at $x$). The proof of Theorem 25.7, which is based on a recent result of Y. Fang [Fa], can thus be used as an alternative to [DMS] by readers that would not already be familiar with it.

Here again, we do not need (10.7). Of course the major difference with (10.12) is that we removed the ugly constant $C_M$.

The (fairly short) proof of Theorem 10.97 is the same as for Theorem 3.4 in [D2]: the conclusion of Proposition 10.82 is stronger than what we prove in Lemma 3.6 of [D2], which was already more than enough to apply the results of [DMS]. □

Theorem 10.97 will lie at the center of our proof of Theorem 10.8, even though many complications will occur, both in the definition of the competitors (in particular because we have to follow the sliding boundary rules) and in the accounting (because Almgren's definition of quasiminimal sets does not cooperate too well with deformation mappings that are not injective).

## 11. Construction of a stabler deformation: the initial preparation

In this section and the next ones, we continue with the notation and assumptions of Theorem 10.8, except that we don't yet need to assume (10.7), which will only be needed for the final Hausdorff measure computations.

Also, the construction of our main deformation will be a little more unpleasant when we work under the Lipschitz assumption, so in most section we shall first describe the construction under the simpler rigid assumption, and explain the necessary modifications for the general case (in the best cases, this is just a conjugation of some mappings with our bilipschitz mapping $\psi$, but in some cases more work is needed) to the end of sections or subsections, so that the reader may easily skip them. We even put the corresponding text between daggers (†) to make the skipping easier (but it would be a shame).

Most of the next sections consists in describing the construction of a deformation that was done in [D2], and adapting it to the sliding boundary conditions. After the construction itself, we shall complete the argument with some Hausdorff measure estimates. The proof will finally be competed in Sections 18 (under the rigid assumption) and 19 (in the Lipschitz case). We return to an almost self-contained mode, because so many modifications are needed from the original proof in [D2] (after all, most of that paper is the construction of a competitor).

So let $\{E_k\}$ be a sequence of quasiminimal sets, such that (10.1)-10.4) hold for some relatively closed set $E \subset U$. Since we want to show that $E$ is quasiminimal, we give ourselves a one-parameter family of functions $\varphi_t$, $0 \leq t \leq 1$, such that (1.4)-(1.8) hold for some closed ball $B$ and relative to $E$; we assume that

$$B = \overline{B}(X_0, R_0),\ 0 < R_0 < \delta, \quad \text{and} \quad \widehat{W} \subset\subset U \tag{11.1}$$

(as in (2.4), and where $\widehat{W}$ is as in (2.2)). Very often, we shall replace our Lipschitz assumption (10.1) with the rigid assumption, so $U$ will be the unit ball, but this does not matter yet.

We want to prove (2.5), and naturally we would like to use the $\varphi_t$ to construct a competitor for $E_k$ for $k$ large, apply (2.5) to $E_k$, get some information, and take a limit. Our first task will be to extend the $\varphi_t$ to $\mathbb{R}^n$, because they are not defined on $E_k$ yet. We know that we shall probably need to modify the extension slightly, because we want to have (1.7) for $E_k$ and not just $E$. But even in the standard case when we have no $L_j$, we cannot use the $\varphi_t$, or their extension, directly, because of complications with the multiplicities that will be explained soon and will be our main source of trouble.

Anyway, let us first define extensions, that we shall also call $\varphi_t$. We shall find it better to use a specific extension algorithm, because this way it will be easier to derive estimates.

We first set $\varphi_t(x) = x$ near $\partial U$. That is, set

$$\delta_0 = \mathrm{dist}(\widehat{W}, \mathbb{R}^n \setminus U) > 0 \ \text{ and } \ U_{ext} = \big\{x \in \mathbb{R}^n\,;\, \mathrm{dist}(x, \mathbb{R}^n \setminus U) \leq \delta_0/2\big\} \tag{11.2}$$

(we know that $\delta_0 > 0$ because of (11.1)). We set

$$\varphi_t(x) = x \ \text{ for } x \in U_{ext} \text{ and } 0 \leq t \leq 1. \tag{11.3}$$

At this point, we have a definition of $\varphi_t$ on $E \cup U_{ext}$, and it satisfies the properties (1.4), (1.5), (1.6) and (1.8), with $E$ replaced by $E \cup U_{ext}$. For instance, $\varphi_1$ is Lipschitz on $E \cup U_{ext}$ because $\varphi_1(x) = x$ on $E \setminus \widehat{W}$, $\mathrm{dist}(U_{ext}, \widehat{W}) \geq \delta_0 > 0$, and $\varphi_1(x) - x$ is bounded on $E \cap \widehat{W}$.

Now we extend all these mappings to $\mathbb{R}^n$. We shall use the Whitney algorithm, as it is described in Chapter IV.2 of [St], for instance, and we refer to this book for details on the construction that follows. We cover $U \setminus (E \cup U_{ext})$ by Whitney cubes $Q_j \subset \mathbb{R}^n \setminus (E \cup U_{ext})$, $j \in J$, with disjoint interiors, and such that

$$10\,\mathrm{diam}(Q_j) \leq \mathrm{dist}(Q_j, E \cup U_{ext}) \leq 21\,\mathrm{diam}(Q_j). \tag{11.4}$$

This easy to do (use maximal dyadic cubes with the first inequality), and the point is that then the cubes $3Q_j$ have bounded overlap.

Also choose, for each $j$, a point $\xi_j \in E \cup U_{ext}$ such that $\mathrm{dist}(\xi, Q_j) = \mathrm{dist}(Q_j, E \cup U_{ext})$, and construct a partition of unity subordinate to the $Q_j$, which means a collection of smooth functions $\chi_j \geq 0$ such that

$$\sum_j \chi_j = \mathbf{1}_{\mathbb{R}^n \setminus (E \cup U_{ext})}, \tag{11.5}$$

$$0 \leq \chi_j \leq \mathbf{1}_{2Q_j} \ \text{ for each } j, \tag{11.6}$$

and

$$||\nabla \chi_j||_\infty \leq C\,\mathrm{diam}(Q_j)^{-1}. \tag{11.7}$$

We use the Whitney extension formula to extend the function $\varphi_t(x) - x$ from $E \cup U_{ext}$ to $\mathbb{R}^n$. That is, we set

$$\varphi_t(x) = x + \sum_j \chi_j(x)[\varphi_t(\xi_j) - \xi_j] \ \text{ for } x \in \mathbb{R}^n \setminus (E \cup U_{ext}). \tag{11.8}$$

Naturally we keep $\varphi_t$ as it was on the set $E \cup U_{ext}$.

It will be useful to know that if $\chi_j(x) \neq 0$, for some $j$ such that $\xi_j \in E$, then $x \in 2Q_j$ and hence

$$\mathrm{dist}(x, E) \leq |\xi_j - x| = \mathrm{dist}(Q_j, E) \leq \mathrm{dist}(x, E) + \mathrm{diam}(Q_j) \tag{11.9}$$

and hence, since (11.4) says that $10\,\mathrm{diam}(Q_j) \leq \mathrm{dist}(Q_j, E)$,

$$\mathrm{dist}(x, E) \geq \mathrm{dist}(Q_j, E) - \mathrm{diam}(Q_j) \geq 9\,\mathrm{diam}(Q_j), \tag{11.10}$$

which we can plug back in (11.9) to get that

$$|\xi_j - x| \leq \mathrm{dist}(x, E) + \mathrm{diam}(Q_j) \leq \frac{10}{9}\,\mathrm{dist}(x, E). \tag{11.11}$$

The advantage of extending $\varphi_t(x) - x$ is that we more easily spot places where it vanishes. Let us check that

$$\varphi_t(x) = x \ \text{ when } \ \mathrm{dist}(x, E) < \frac{9}{10}\,\mathrm{dist}(x, W_t), \tag{11.12}$$

where as before

$$W_t = \big\{y \in E\,;\, \varphi_t(y) \neq y\big\}. \tag{11.13}$$

This is clear when $x \in E$ (because then $\mathrm{dist}(x, W_t) > 0$), and when $x \in U_{ext}$ (by (11.3)), so let us consider $x \in \mathbb{R}^n \setminus (E \cup U_{ext})$. If $\varphi_t(x) \neq x$, (11.8) says that we can find $j$ such that $\chi_j(x) \neq 0$ and $\varphi_t(\xi_j) \neq \xi_j$. Then $\xi_j \in E$ by (11.3), and $|\xi_j - x| \leq \frac{10}{9}\,\mathrm{dist}(x, E)$ by (11.11); but we assumed that $\mathrm{dist}(x, W_t) > \frac{10}{9}\,\mathrm{dist}(x, E)$, so $\xi_j \notin W_t$, hence $\varphi_t(\xi_j) = \xi_j$, a contradiction which proves (11.12).

We also have that

$$\varphi_0(x) = x \ \text{ for } x \in \mathbb{R}^n, \tag{11.14}$$

just because all the $\varphi_0(\xi_j) - \xi_j$ vanish, that

$$(t, x) \to \varphi_t(x) \ \text{ is continuous on } [0,1] \times \mathbb{R}^n \tag{11.15}$$

(in particular because each fixed $\varphi_t(\xi_j)$ is a continuous function of $t$), and similarly

$$\varphi_1 : \mathbb{R}^n \to \mathbb{R}^n \ \text{ is Lipschitz,} \tag{11.16}$$

because we used the standard formula for the Whitney extension theorem.

Our extensions $\varphi_t$ will be better when we stay close to $E$, for instance because otherwise (11.12) does not give much of a control, but this is all right because we only need them on sets $E_k$ that tend to $E$. But also, the $\varphi_t$ have some defects that we'll need to fix. The main one, which will be explained soon, is that $\varphi_1$ may be one-to-one on the $E_k$, while it is many-to-one on $E$, which possibly makes the $\varphi_1(E_k)$ much worse competitors than $\varphi_1(E)$.

The truth is that we mostly care about $\varphi_1$. We need to keep the $\varphi_t$, $0 \leq t < 1$, because eventually we shall need to check (1.7) on the $E_k$, but the main point of the argument concerns estimates like (2.5), where only $\varphi_1$ counts.

Except for the fact that we have to worry about (1.7) and have a slightly more complicated way to define competitors (we now have a ball $B$ and the open set $U$), we will mostly follow the construction of [D2]. Even though we will change many little things, it will some times be convenient to refer to [D2] for small independent things.

So we are interested in the values of $f = \varphi_1$ near $E$. Even in the standard case with no $L_j$, if by bad luck there is a region where $E$ and $E_k$ are composed of many pieces, $f$ maps all the pieces of $E$ to a same small disk, say, but maps all the $E_k$ to parallel, but disjoint little disks, $f(E)$ may be a quite reasonable competitor (because the measure of the single disk is small), but not $f(E_k)$. Then, when we apply (2.5) to $E_k$ and $f(E_k)$, we won't get much information, not enough to control $f(E)$. What we intend to do in this case, when $f$ sends many pieces to parallel and nearby disks, is to modify $f$ (typically, by composing it with a projection) so that it sends all these pieces to a single disk. This will make $f(E_k)$ a much better competitor, and then we have a good chance to run the usual lower semicontinuity arguments and get the desired inequality (2.5).

A good part of the construction below consists in doing such grouping, but obviously this will require some nontrivial amount of cutting and pasting. As the reader may have guessed, we shall use the fact that $E$ is rectifiable (to show that it has a tangent at most points), the fact that $\varphi_1$ is Lipschitz on $E$ (to show that it is often close to its differential), and lots of covering arguments (to reduce to situations where $\varphi_1$ is almost affine and $E$ is almost flat). The word stability in the title of the section refers to the fact that after this grouping, the total measure of $f(E_k)$ will be much less dependent on $k$.

**11.17. Remark about the many constants.** Since there will be lots of constants in this argument, let us announce here in which order we intend to choose them, so that the reader may more easily check that we do not cheat. We shall systematically denote by $C$ constants that depend only on $n$ $M$, and $\Lambda$ (when we work under the Lipschitz assumption). This includes the local Ahlfors-regularity constants for $E$.

Next observe that from now on, $U$, $B$, the $\varphi_t$, and in particular $f = \varphi_1$ are fixed, so we shall not mind if our constants depend on $r_0$, $\lambda$, of $\psi$ in the Lipschitz assumption, or on $f$, typically through its Lipschitz constant $|f|_{lip}$. Similarly, we can let our constants depend on the number $\mathcal{H}^d(\{x \in E\,;\, f(x) \neq x\})$. In both cases, we shall often indicate this dependence, mostly to comfort the reader, but will not be a real issue.

A first string of constants is

$$\begin{gathered} \gamma > 0 \text{ (small)}, \ a < 1 \text{ (close to 1)}, \ \alpha > 0 \text{ (small)}, \ N \text{ (large)}, \\ \eta > 0 \text{ (small), and } \varepsilon > 0 \text{ (small)} \end{gathered} \tag{11.18}$$

to be chosen in this order. Our small constants $\delta_i$, $1 \leq \delta_i \leq 9$, will be allowed to depend on these constants (even though the first ones don't), and are thus chosen after $\eta$ and $\varepsilon$. Typically, they are chosen smaller and smaller. They act as scale parameters, and force us to work with balls that are small enough for some properties to hold, but they should not have an incidence on the estimates.

Finally, we will choose two last small constant $\varepsilon_0$ and $\varepsilon_*$. They also determine the distance that we allow between $E$ and the $E_k$, and so they could also have been called $\delta_{10}$ and $\delta_{11}$, but we decided to revert to the name $\varepsilon$ to show that they are even smaller.

Let us also mention that our estimates will only be valid for $k$ large, depending on the various constants, and in particular the $\delta_i$ and $\varepsilon_0$.

We cut the construction into a few smaller steps; only the first one will be completed in this section. As explained above, we shall first carry the construction under the rigid assumption, and then we shall explain how to modify things under the Lipschitz assumption.

**Step 1. We remove a few small bad sets.** Before we define balls $B_j$ and modify $f$ on them, we remove some small bad sets from $E$, where $E$ or $f$ is not regular enough. Set

$$W_f = \big\{x \in \mathbb{R}^n\,;\, f(x) \neq x\big\} = \big\{x \in \mathbb{R}^n\,;\, \varphi_1(x) \neq x\big\}; \tag{11.19}$$

our star starting set is

$$X_0 = E \cap W_f = \big\{x \in E\,;\, f(x) \neq x\big\} \subset \widehat{W} \subset\subset U, \tag{11.20}$$

(by (2.2) and (11.1)) and we immediately replace it with a compact subset $X_1 \subset X_0$, such that

$$\mathcal{H}^d(X_0 \setminus X_1) \leq \eta, \tag{11.21}$$

where $\eta$ is some very small positive number, that will tend to 0 at the end of the argument. Set

$$\delta_1 = \mathrm{dist}(X_1, \mathbb{R}^n \setminus W_f) > 0; \tag{11.22}$$

the fact that $\delta_1 > 0$ (because $X_1$ is compact and $W_f$ is open) will make it easier to stay inside $W_f$ when we cover $X_1$ by small balls.

Some manipulations will be easier if we force $f(x)$ to stay far from the boundaries of faces, because it will make the smallest face that contains $f(x)$ locally constant. For $0 \leq l \leq n$, and under the rigid assumption, denote by $\mathcal{S}_l$ the union of all the faces of dimension $l$ of the dyadic cubes of side length $r_0$ of our usual grid. When we work under the Lipschitz assumption, we shall just call "cubes" the images of standard dyadic cubes by $\lambda^{-1}\psi^{-1}$, and similarly for faces, and we shall define the $\mathcal{S}_l$ in terms of these distorted faces. Set

$$X_{1,\delta}(l) = \big\{x \in X_1\,;\, f(x) \in \mathcal{S}_l \setminus \mathcal{S}_{l-1} \;\text{ and }\; \mathrm{dist}(f(x), \mathcal{S}_{l-1}) \geq \delta\big\} \tag{11.23}$$

for $\delta > 0$ and $1 \leq l \leq n$; for $l = 0$, simply put

$$X_{1,\delta}(0) = \big\{x \in X_1\,;\, f(x) \in \mathcal{S}_0\big\}. \tag{11.24}$$

Then set

$$X_{1,\delta} = \bigcup_{0 \leq l \leq n} X_{1,\delta}(l). \tag{11.25}$$

Since $X_1$ is the monotone union of the $X_{1,\delta}$ (when $\delta$ decreases to 0), we can choose $\delta_2 > 0$ so small that $\mathcal{H}^d(X_1 \setminus X_{1,\delta_2}) \leq \eta$. Then we set

$$X_2 = X_{1,\delta_2} = \bigcup_{0\leq l\leq n} X_{1,\delta_2}(l) \text{ and so } \mathcal{H}^d(X_1 \setminus X_2) \leq \eta. \tag{11.26}$$

For each $x \in X_2$, denote by $F(f(x))$ the smallest face of our dyadic grid that contains $f(x)$. We claim that (under the rigid assumption)

$$F(f(x)) = F(f(y)) \text{ for } x, y \in X_2 \text{ such that } |f(x) - f(y)| < \delta_2. \tag{11.27}$$

Indeed, let $l = l(x)$ be such that $x \in X_{1,\delta_2}(l)$, and define $l(y)$ similarly. By symmetry, we may assume that $l \geq l(y)$. By definition, $f(x) \in \mathcal{S}_l$ and (if $l \geq 0$) $\mathrm{dist}(f(x), \mathcal{S}_{l-1}) \geq \delta_2$. In particular, $f(y)$ cannot lie on $\mathcal{S}_{l-1}$, so $l(y) \geq l$ and, by our symmetry assumption, $l(y) = l$. Now $F(f(x))$ and $F(f(y))$ are two faces of the same dimension $l$. Suppose for a moment that they are different. If $l = 0$, this is impossible because $|f(x) - f(y)| < \delta_2$ and so (if we chose $\delta_2 \leq r_0$) $f(x)$ and $f(y)$ cannot both lie on $\mathcal{S}_0$. Otherwise, (3.8) says that

$$\mathrm{dist}(f(x), \partial F(f(x))) \leq \mathrm{dist}(f(x), F(f(y))) \leq |f(x) - f(y)| < \delta_2, \tag{11.28}$$

which contradicts the fact that $\mathrm{dist}(f(x), \mathcal{S}_{l-1}) \geq \delta_2$. This proves (11.27) in the rigid case.

† We can do the same argument under the Lipschitz assumption, where faces are only bilipschitz images of faces of a true rigid grid; we just need to require that $|f(x) - f(y)| < \Lambda^{-2}\delta_2$ in (11.27), because we may lose a constant $\Lambda^2$ in the first inequality of (11.28). †

Next we want to use the rectifiability of $E$, which we deduce from Proposition 10.15. By Theorem 15.21 on page 214 of [Ma], we can find a countable collection of $C^1$ embedded submanifolds $\Gamma_s$ (or, if you prefer, images by rotations of $C^1$ graphs) of dimension $d$, such that

$$\mathcal{H}^d(E \setminus \bigcup_s \Gamma_s) = 0. \tag{11.29}$$

To be fair, we don't really need $C^1$ submanifolds, and Lipschitz graphs would not have required much more work, but on the other hand, we should not pretend that we do not use strong results, when in the proof of Proposition 10.15 we used the much stronger fact that unrectifiable $d$-sets have negligible projections in almost all directions.

Select a finite set $S$ of indices, so that if we set $X_3 = X_2 \cap \left[\bigcup_{s\in S} \Gamma_s\right]$, then

$$\mathcal{H}^d(X_2 \setminus X_3) \leq \eta. \tag{11.30}$$

Put a total order on $S$, and set

$$X_3(s) = X_3 \cap \Gamma_s \setminus \bigcup_{s'<s} \Gamma_{s'} \tag{11.31}$$

for $s \in S$. Thus the $X_3(s)$ are disjoint and cover $X_3$. We know that

$$\lim_{r\to 0} r^{-d} H^d(\Gamma_s \cap B(x,r) \setminus X_3(s)) = 0 \tag{11.32}$$

for $\mathcal{H}^d$-almost every $x \in X_3(s)$, just because $\mathcal{H}^d(\Gamma_s)$ is locally finite and $\Gamma_s \setminus X_3(s)$ does not meet $X_3(s)$; see for instance Theorem 6.2 on page 89 of [Ma]. Similarly,

$$\lim_{r\to 0} r^{-d} H^d(E \cap B(x,r) \setminus X_3(s)) = 0 \tag{11.33}$$

for $\mathcal{H}^d$-almost every $x \in X_3(s)$, this time because $X_3(s)$ has a neighborhood where $\mathcal{H}^d(E)$ is finite (recall that $X_3(s) \subset X_1$ and $X_1$ is a compact subset of $\widehat{W} \subset\subset U$, by (11.20)).

Let us check that if $x \in X_3(s)$ is such that (11.33) holds, and if $P_x$ denotes the tangent plane to $\Gamma_s$ at $x$, then $P_x$ is also a tangent plane to $E$ at $x$, which means that

$$\lim_{y\to x\,;\,y\in E} |y-x|^{-1} \operatorname{dist}(y, P_x) = 0. \tag{11.34}$$

Indeed, if $d(y) = \operatorname{dist}(y, X_3(s))$, then $d(y) \leq |y-x|$ trivially, and

$$\begin{aligned} C^{-1} d(y)^d &\leq \mathcal{H}^d(E \cap B(y, d(y))) = \mathcal{H}^d(E \cap B(y, d(y)) \setminus X_3(s)) \\ &\leq H^d(E \cap B(x, 2|y-x|) \setminus X_3(s)) = o(|y-x|^d) \end{aligned} \tag{11.35}$$

because $E$ is locally Ahlfors-regular and $B(y,d(y)) \subset B(x, 2|y-x|)$, and by (11.33). But then $\operatorname{dist}(y, \Gamma_s) \leq d(y) = o(|x-y|)$ because $X_3(s) \subset \Gamma_s$, and now (11.34) holds because $\Gamma_s$ is tangent to $P_x$ at $x$. Notice that without surprise, $E$ and $\Gamma_s$ share the same tangent plane at $x$ (regardless of the rest of $E$, which anyway does not matter because of (11.33)); the uniqueness of the tangent plane to $E$ follows from the local Ahlfors-regularity of $E$, but we could also deduce it from the fact that by (11.32), most points of $\Gamma_s$ lie in $X_3(s)$, and hence in $E$.

We also want to throw out the points of $X_3(s)$ where $f$ is not differentiable in the direction of $P_x$. That is, denote by $X_4(s)$ the set of points $x \in X_3(s)$ such that (11.32) and (11.33) hold, and there exists an affine function $A_x : \mathbb{R}^n \to \mathbb{R}^n$, of rank at most $d$ and with Lipschitz norm

$$|DA_x| \leq |f|_{lip}, \tag{11.36}$$

and such that

$$\lim_{y\to x\,;\,y\in\Gamma_s} \frac{|f(y) - A_x(y)|}{|y-x|} = 0. \tag{11.37}$$

Let us check that if $x \in X_3(s)$ satisfies (11.32) and (11.33), and if the restriction of $f$ to $\Gamma_s$ is differentiable at $x$, then $x \in X_4(s)$; clearly this will imply that

$$\mathcal{H}^d(X_3(s) \setminus X_4(s)) = 0. \tag{11.38}$$

Let $x \in X_3(s)$ be like this. There is a small neighborhood of $x$ where $\Gamma_s$ is the graph of some $C^1$ function $F$, defined on $P_x$ and with values in $P_x^\perp$, the $(n-d)$-vector space perpendicular to $P_x$. That is, $\Gamma_s$ is locally equal to the set of points $z + F(z)$, $z \in P_x$, and with these conventions $F(x) = 0$ and $Df(x) = 0$.

Set $g(z) = f(z + F(z))$ for $z \in P_x$ near $x$. By assumption, $g$ is differentiable at $z = x$. Denote by $P'_x$ the vector space parallel to $P_x$ and by $A : P'_x \to \mathbb{R}^n$ the differential in question. Obviously, the rank of $A$ is at most $d$, and the norm of $A$ is at most $|f|_{lip}$, because for each $\lambda > |f|_{lip}$, there is a neighborhood of $x$ in $P_x$ where $g$ is $\lambda$-Lipschitz.

Set $A_x(y) = f(x) + A(\pi(y-x))$ for $y \in \mathbb{R}^n$, and where $\pi$ denotes the orthogonal projection on $P'_x$. Then $A_x$ is affine, with rank at most $d$, and (11.36) holds. For $y \in \Gamma_s$ close enough to $x$, write $y = z + F(z)$, where $z = x + \pi(y-x)$ is the projection of $y$ on $P_x$; then

$$\begin{aligned} |f(y) - A_x(y)| &= |g(z) - A_x(y)| = |g(z) - f(x) - A(\pi(y-x))| \\ &= |g(z) - f(x) - A(z-x)| = o(|z-x|) = o(|y-x|) \end{aligned} \tag{11.39}$$

by definition of $A_x$ and $A$ (and because $f(x) = g(x)$ since $F(x) = 0$). So (11.37) holds and $x \in X_4(s)$, as needed for (11.38).

Let us also check that

$$\lim_{y \to x\,;\, y \in E \cup P_x} \frac{|f(y) - A_x(y)|}{|y-x|} = 0 \quad \text{when } x \in X_4(s). \tag{11.40}$$

For $y \in E \cup P_x$ near $x$, choose $w \in \Gamma_s$ such that $|w-y| \leq 2\,\mathrm{dist}(y, \Gamma_s)$; we know from (11.34) or the fact that $P_x$ is tangent to $\Gamma_s$ that $|w-y| = o(|y-x|)$, and then

$$\begin{aligned} |f(y) - A_x(y)| &\leq |f(w) - A_x(w)| + |f(w) - f(y)| + |A_x(w) - A_x(y)| \\ &\leq |f(w) - A_x(w)| + 2|w-y||f|_{lip} \\ &= o(|w-x|) + o(|y-x|) = o(|y-x|), \end{aligned} \tag{11.41}$$

by (11.37); (11.40) follows.

We want to have all the properties above with some uniformity. So we let $\varepsilon > 0$ be small, and denote by $X_5(s)$ the set of points $x \in X_4(s)$ such that the following properties hold. First, there is a $C^1$ mapping $F_x : P_x \to P_x^\perp$, such that

$$F_x(x) = 0\,,\ DF_x(x) = 0\,,\ \text{and } ||DF_x||_\infty \leq \varepsilon \tag{11.42}$$

and

$$z + F_x(z) \in \Gamma_s \quad \text{for } z \in P_x \cap B(x, \delta_3), \tag{11.43}$$

where $\delta_3$ is another small constant, that will be chosen soon, depending on $\varepsilon$. This is just a quantified version of the description of $\Gamma_s$ near $x$ that we used below (11.38). Next,

$$\mathcal{H}^d(B(x,r) \cap [\Gamma_s \cup E] \setminus X_3(s)) \leq \varepsilon r^d \quad \text{for } 0 < r \leq \delta_3 \tag{11.44}$$

(compare with (11.32) and (11.33)), and also

$$\text{dist}(y, P_x) \leq \varepsilon |y - x| \quad \text{for } y \in E \cap B(x, \delta_3) \tag{11.45}$$

as in (11.34). Finally,

$$|f(y) - A_x(y)| \leq \varepsilon |y - x| \quad \text{for } y \in [E \cup P_x \cup \Gamma_s] \cap B(x, \delta_3), \tag{11.46}$$

as in (11.37) and (11.40).

Each set $X_4(s)$ is the monotone union, when $\delta_3$ goes to 0, of the corresponding sets $X_5(s)$ that we just defined. So we can choose $\delta_3 > 0$ so small that if we set

$$X_4 = \bigcup_{s \in S} X_4(s) \quad \text{and } X_5 = \bigcup_{s \in S} X_5(s) \subset X_4, \tag{11.47}$$

then $\mathcal{H}^d(X_4 \setminus X_5) \leq \eta$, and hence

$$\mathcal{H}^d(X_0 \setminus X_5) \leq 4\eta \tag{11.48}$$

by (11.21), (11.26), (11.30), (11.31), and (11.38).

† **Remark 11.49.** In this first step that we just finished, the flatness of the faces does not show up. Under the Lipschitz assumption, we can proceed exactly as we did, except that cubes and faces are more complicated. Things will become more unpleasant when we try to use the existence of the tangent planes $P_x$ to derive information on the closeness of $E$ to the (no longer flat) faces of cubes, and in fact we shall need to require the equivalent of (11.36), (11.37), and (11.40) also for the mapping $\widetilde{f}$ defined by

$$\widetilde{f}(x) = \psi(\lambda f(x)) \in B(0,1) \quad \text{for } x \in E \tag{11.50}$$

(which makes sense because we know that $f = \varphi_1 : E \to U$ and $\psi : \lambda U \to B(0,1)$); see near (12.36). †

## 12. Step 2 of the construction: the places where $f$ is many-to-one

In this section we continue the construction of Section 11, and modify the function $f$ in some balls $B_j$, $j \in J_1$, where we can make $f$ highly non injective. Let $N$ be a large number, and set

$$Y_N = \big\{ y \in f(X_5)\,;\, X_5 \cap f^{-1}(y) \text{ contains at least } N \text{ distinct points} \big\}. \tag{12.1}$$

As before, it will be easier to (demand some uniformity and) control the set

$$\begin{aligned} Y_N(\delta_4) = \big\{ & y \in f(X_5)\,;\, X_5 \cap f^{-1}(y) \text{ contains at least} \\ & N \text{ distinct points at mutual distances} \geq \delta_4 \big\} \end{aligned} \tag{12.2}$$

for some small $\delta_4 > 0$. Since $f^{-1}(Y_N)$ is the monotone union of the $f^{-1}(Y_N(\delta_4))$, we can choose $\delta_4 > 0$ so small, depending on $\eta$ and other constants, that if we set

$$X_N(\delta_4) = X_5 \cap f^{-1}(Y_N(\delta_4)), \tag{12.3}$$

then

$$\mathcal{H}^d\big([X_5 \cap f^{-1}(Y_N)] \setminus X_N(\delta_4)\big) \leq \eta. \tag{12.4}$$

We shall need a covering of $X_N(\delta_4)$.

**Step 2.a. We cover $X_N(\delta_4)$ by balls $B_j = B(x_j, t)$, $j \in J_1$.**

We want to cover $X_N(\delta_4)$ with small balls of the same very small radius $t/2$, but let us first say how small we want $t$ to be. Set

$$\delta_5 = \inf\big\{|f(x) - x|\,;\, x \in X_1\big\}; \tag{12.5}$$

notice that $\delta_5 > 0$ because $X_1$ is compact and $f(x) \neq x$ for $x \in X_1 \subset X_0$ (by (11.20)). Recall that we set

$$\delta_0 = \mathrm{dist}(\widehat{W}, \mathbb{R}^n \setminus U) > 0 \tag{12.6}$$

in (11.2). Pick

$$\delta_6 < \frac{1}{10\Lambda^2(1 + |f|_{lip})}\ \mathrm{Min}\,\big(\lambda^{-1} r_0, \delta, \delta_0, \delta_1, \delta_2, \delta_3, \delta_4, \delta_5\big) \tag{12.7}$$

(where $\lambda = \Lambda = 1$ in the rigid case) and any $t$ such that

$$0 < t < \delta_6. \tag{12.8}$$

Then pick a maximal collection $\{x_j\}$, $j \in J_1$, of points in $X_N(\delta_4)$, that lie at mutual distances at least $t/3$. Thus

$$X_N(\delta_4) \subset \bigcup_{j \in J_1} B(x_j, t/2) \tag{12.9}$$

by maximality, and we claim that

$$J_1 \text{ has at most } Ct^{-d}\mathcal{H}^d(X_0) \text{ elements.} \tag{12.10}$$

Indeed, $x_j \in X_5 \subset X_1$ for $j \in J_1$, so

$$t < \delta_6 < \frac{1}{10}\,\delta_1 = \frac{1}{10}\,\mathrm{dist}(X_1, \mathbb{R}^n \setminus W_f) \leq \frac{1}{10}\,\mathrm{dist}(x_j, \mathbb{R}^n \setminus W_f), \tag{12.11}$$

which implies that $E \cap B(x_j, t/6) \subset E \cap W_f = X_0$ (see the definition (11.20)). Let us also show that

$$\mathcal{H}^d(E \cap B(x_j, t/6)) \geq C^{-1} t^d. \tag{12.12}$$

We want to apply Proposition 4.1 or Proposition 4.74, so we just need to check that $t/6 < \mathrm{Min}(\lambda^{-1} r_0, \delta)$ and $B(x_j, t/3) \subset U$. The first one follows from (12.7) and (12.8), and the second one holds because $x \in X_0 \subset \widehat{W}$ and $t < \delta_6 \leq \delta_0$. So Proposition 4.1 or 4.74 applies and gives (12.12).

The $E \cap B(x_j, t/6)$, $j \in J_1$ are disjoint (by definition of $J_1$), and contained in $X_0$, so (12.10) follows from (12.11).

We agree that (12.10) is not a very good bound, but a large choice of $N$, depending on $\mathcal{H}^d(X_0)$ and $\eta$, will compensate. At least, notice that $\mathcal{H}^d(X_0) < +\infty$ (because $X_0 \subset \widehat{W} \subset\subset U$), and will be taken as a constant (it only depends on $E$ and $f$).

**Step 2.b. We cover $Y_N(\delta_4)$ by balls $D_l$, $l \in \mathcal{L}$.**

We also want a covering of $Y_N(\delta_4)$. So we take a maximal set of points $y_l$, $l \in \mathcal{L}$, in $Y_N(\delta_4)$, at mutual distances at least $t/2$. Then

$$\text{the balls } D_l = B(y_l, t),\ l \in \mathcal{L}, \text{ cover } Y_N(\delta_4). \tag{12.13}$$

Let us prove that the cardinality of $\mathcal{L}$ is

$$|\mathcal{L}| \leq C N^{-1} (1 + |f|_{lip})^d t^{-d} \mathcal{H}^d(X_0). \tag{12.14}$$

For each $l \in \mathcal{L}$, select $N$ points $x_{l,j} \in X_5$, $1 \leq j \leq N$, at mutual distances at least $\delta_4$, such that $f(x_{l,j}) = y_l$. Such points exist, by the definition (12.2), and they lie in $X_N(\delta_4)$, by (12.3).

Set $s = \frac{t}{4(1+|f|_{lip})}$ and $B_{l,j} = B(x_{l,j}, s)$; we claim that

$$B_{l,j} \text{ is disjoint from } B_{l',j'} \text{ when } (l,j) \neq (l',j'). \tag{12.15}$$

When $l = l'$ and $j \neq j'$, this is because $s \leq t/4 \leq \delta_4/4 \leq |x_{l,j} - x_{l,j'}|/4$ (by (12.8), (12.7), and our choice of points $x_{l,j}$); when $l \neq l'$, this is because if $x \in B_{l,j}$ and $x' \in B_{l',j'}$, then

$$|f(x) - f(x')| \geq |f(x_{l,j}) - f(x_{l',j'})| - 2s|f|_{lip} = |y_l - y_{l'}| - 2s|f|_{lip} \geq t/2 - t/4 > 0. \tag{12.16}$$

With the same verification as for (12.12), $\mathcal{H}^d(E \cap B_{l,j}) \geq C^{-1} s^d \geq C^{-1}(1 + |f|_{lip})^{-d} t^d$. Also, $E \cap B_{l,j} \subset X_0$ because $s \leq t/4 \leq \mathrm{dist}(x_{l,j}, \mathbb{R}^n \setminus W_f)$ (because $x_{l,j} \in X_5$ and by (11.20) and the proof of (12.11)). Since there are $N|\mathcal{L}|$ balls $B_{l,j}$, which are disjoint by (12.15) and have a total mass at most $\mathcal{H}^d(X_0)$, we get (12.14).

**Step 2.c. The collection of disks $Q$, $Q \in \mathcal{F}_l$.**

For each $l \in \mathcal{L}$, we want to select a reasonably large collection of affine subspaces $P$, and then disks $Q \subset P$. The general idea is that when we modify $f$ on the $B_j$, $j \in J_1$, we want to send points preferably to these disks $Q$, which will make $f$ more stably many-to-one.

This is a stage of the construction where we need to be careful about the boundary pieces $L_j$, so we shall diverge slightly from [D2]) and also we shall need to modify some things when we work under the Lipschitz assumption, but let us first describe what we do when the rigid assumption hold.

Let us fix $l \in \mathcal{L}$. Denote by $F_l$ the smallest face of our usual grid that contains $y_l$; this makes sense, because if $y_l$ lies on two faces $F$ and $F'$, then $F \cap F'$ is also a face that contains $y_l$. Denote by $d(l)$ the dimension of $F_l$, and by $W(y_l)$ the $d(l)$-dimensional affine subspace that contains $F_l$.

If $d(l) \leq d$, we just choose one affine subspace, namely $P = W(y_l)$, and one disk $Q$, namely

$$Q = P \cap B(y_l, 3(1 + |f|_{lip})t). \tag{12.17}$$

If $d(l) > d$, we choose a whole collection of affine $d$-planes $P$ through $D_l$, all of them contained in $W(y_l)$, with a density property that we shall explain soon. For each $P$ that we choose, we still define $Q$ by (12.17); this gives a collection $\mathcal{F}_l$ of disks $Q$, $Q \in \mathcal{F}_l$, (which is just composed of one disk when $d(l) \leq d$).

The density property is the following. Let $\alpha > 0$ be a very small constant (to be chosen later); we demand that if $d(l) > d$, then for each affine $d$-plane $P'$ through $D_l$ which is contained in $W(y_l)$, we can find $Q \in \mathcal{F}_l$ such that

$$\mathrm{dist}(z, Q) \leq \alpha t \;\text{ for } z \in P' \cap B(y_l, 3(1 + |f|_{lip})t). \tag{12.18}$$

So we choose the set $\mathcal{F}_l$ like this, but with not too many elements, so that

$$|\mathcal{F}_l| \leq C(\alpha, f) \tag{12.19}$$

for some constant $C(\alpha, f)$ that depends on $\alpha$ and $|f|_{lip}$. Now we set $\mathcal{F} = \cup_{l \in \mathcal{L}} \mathcal{F}_l$ and observe that

$$|\mathcal{F}| \leq C(\alpha, f)|\mathcal{L}| \leq C(\alpha, f)N^{-1}t^{-d}\mathcal{H}^d(X_0) \tag{12.20}$$

by (12.14), and with new constants $C(\alpha, f)$ that depend on $\alpha$ and $|f|_{lip}$. Let us record the fact that by construction,

$$Q \subset W(y_l) \cap B(y_l, 3(1 + |f|_{lip})t) \;\text{ for } l \in \mathcal{L} \text{ and } Q \in \mathcal{F}_l. \tag{12.21}$$

We shall see why this is important when we check the boundary condition (1.7).

† When we work under the Lipschitz assumption, we proceed almost the same way. Recall that the dyadic cubes and faces are now obtained from the standard ones in the unit ball, using (the inverse of) the bilipschitz mapping $\psi : \lambda U \to B(0,1)$; thus $F_l$, the smallest face that contains $y_l$, is the image $F_l = \lambda^{-1}\psi^{-1}(\widetilde{F}_l)$ of some true dyadic face $\widetilde{F}_l$

(the smallest one that contains $\widetilde{y}_l = \psi(\lambda y_l)$). When $d(l) \leq d$, we just choose one affine space $\widetilde{P}$, namely the $d(l)$-dimensional affine subspace $W(\widetilde{y}_l)$ that contains $\widetilde{F}_l$, and set

$$\widetilde{Q} = \widetilde{P} \cap B(\widetilde{y}_l, 20\lambda\Lambda^2(1+|f|_{lip})t) \text{ and } Q = \lambda^{-1}\psi^{-1}(\widetilde{Q}). \tag{12.22}$$

When $d(l) > d$, we still define $\widetilde{Q}$ and $Q$ by (12.22), but we let $\widetilde{P}$ run through a fairly large collection of $d$-planes $\widetilde{P}$, such that

$$\widetilde{P} \subset W(\widetilde{y}_l) \text{ and } \widetilde{P} \text{ meets } B(\widetilde{y}_l, 2\lambda\Lambda(1+|f|_{lip})t). \tag{12.23}$$

We choose this collection so dense that, if $\widetilde{P}'$ is any other affine $d$-plane such that

$$\widetilde{P}' \subset W(\widetilde{y}_l) \text{ and } \widetilde{P}' \text{ meets } B(\widetilde{y}_l, 2\lambda\Lambda(1+|f|_{lip})t)), \tag{12.24}$$

then there is a $d$-plane $\widetilde{P}$ in the collection such that

$$\mathrm{dist}(z, \widetilde{P}) \leq \lambda\Lambda^{-1}\alpha t \text{ for } z \in \widetilde{P}' \cap B(\widetilde{y}_l, 10\lambda\Lambda^2(1+|f|_{lip})t). \tag{12.25}$$

We get a collection of plates $Q$, $Q \in \mathcal{F}_l$, and again we can manage so that each plane $\widetilde{P}$ we choose satisfies (12.23), and $|\mathcal{F}_l| \leq C(\alpha, f)$, where here $C(\alpha, f)$ also depends on $\Lambda$. This way we still get (12.19) and (12.20), and the analogue of (12.21), namely the fact that

$$\widetilde{Q} = \psi(\lambda Q) \subset W(\widetilde{y}_l) \cap B(\widetilde{y}_l, 20\lambda\Lambda^2(1+|f|_{lip})t) \text{ for } l \in \mathcal{L} \text{ and } Q \in \mathcal{F}_l. \tag{12.26}$$

Incidentally, notice that none of our main constants will depend on $\lambda$, and in fact we could easily get rid of $\lambda$ with a simple dilation of $E$ and the $E_k$; we shall not do this and keep mentioning $\lambda$ in our estimates, but the reader could also decide not to bother and make $\lambda = 1$ everywhere. †

**Step 2.d. Where do the tangent planes go?**

The following lemma will help us soon in our choice of approximate tangent planes in the image, but we state it independently. As usual, we start under the rigid assumption.

**Lemma 12.27.** *For $x \in X_5$, let $P_x$ and $A_x$ be as in (11.34) and (11.40), denote by $F(f(x))$ the smallest face of our usual grid that contains $f(x)$ and by $W(f(x))$ the smallest affine subspace that contains $F(f(x))$. Then*

$$A_x(P_x) \subset W(f(x)). \tag{12.28}$$

Fix $x$ as in the statement, and denote by $P$ and $W$ the vector spaces parallel to $P_x$ and $W(f(x))$ respectively. Since we already know that $A_x(x) = f(x) \in W(f(x))$, we just need to check that $A(P) \subset W$, where $A$ denotes the linear part of $A_x$.

First we want to find, for each small $\rho > 0$, a collection of affinely independent points $w_k^\rho \in X_3 \cap B(x, \rho)$, $1 \leq k \leq d$, and more precisely such that if we set $w_0 = x$, then for $1 \leq k \leq d$,

$$\mathrm{dist}(w_k^\rho, P(w_0^\rho, \dots, w_{k-1}^\rho)) \geq c\rho, \tag{12.29}$$

where $P(w_0^\rho, \dots, w_{k-1}^\rho)$ denotes the affine subspace of dimension $k-1$ spanned by $w_0^\rho, \dots, w_{k-1}^\rho$ and $c > 0$ is a constant that depends only on $M$ and $n$.

The proof of existence is the same as for (8.35); we just need to know that $E$ is locally Ahlfors-regular and that for $\rho > 0$ small enough,

$$\mathcal{H}^d(X_3 \cap B(x, \rho)) \geq C^{-1} \rho^d. \tag{12.30}$$

This last follows from Proposition 4.1 (the local Ahlfors-regularity of $E$), (11.33), and the fact that $X_3(s) \subset X_3$ by (11.31). Alternatively (if you don't like (8.35)), we could take $d$ affinely independent points in $P_x$ (with a property like (12.29), even with $c = 1/2$), recall that $P_x$ was initially defined as the tangent plane to the $\Gamma_s$ that contains $x$, and use (11.32) to find points of $X_3$ that lie close enough to these points. Either way we can find the $w_k^\rho$.

Take a sequence of radii $\rho$ that tends to 0, and for which

$$\text{each } \rho^{-1}(w_k^\rho - x),\ 0 \leq k \leq d, \text{ has a limit } w_k. \tag{12.31}$$

Then $w_0 = 0$ and (12.29) yields

$$\operatorname{dist}(w_k, P(0, \dots, w_{k-1})) \geq c \tag{12.32}$$

for $1 \leq k \leq d$, and so the $w_k$, $1 \leq k \leq d$, are linearly independent. In addition, $w_k \in P$, because

$$\begin{aligned} \operatorname{dist}(\rho^{-1}(w_k^\rho - x), P) &= \rho^{-1} \operatorname{dist}(w_k^\rho - x, P) = \rho^{-1} \operatorname{dist}(w_k^\rho, P_x) \\ &= \rho^{-1} o(|w_k^\rho - x|) = o(1) \end{aligned} \tag{12.33}$$

by (11.34) (which holds because $x \in X_3$) and because $w_k^\rho \in X_3 \subset E$. So the $w_k$, $1 \leq k \leq d$, are a basis of $P$, and it is enough to check that $A(w_k) \in W$ for $1 \leq k \leq d$. But

$$\begin{aligned} \operatorname{dist}(A(w_k^\rho - x), W) &= \operatorname{dist}(A_x(w_k^\rho) - A_x(x), W) \\ &= \operatorname{dist}(A_x(w_k^\rho), W + A_x(x)) = \operatorname{dist}(A_x(w_k^\rho), W(f(x))) \\ &\leq \operatorname{dist}(f(w_k^\rho), W(f(x))) + |f(w_k^\rho) - A_x(w_k^\rho)| \end{aligned} \tag{12.34}$$

because $A_x$ is affine and $A_x(x) = f(x) \in W(f(x))$, and by definition of $W$ (the vector space parallel to $W(f(x))$). But $|f(w_k^\rho) - f(x)| \leq \rho |f|_{lip} < \delta_2$ if $\rho$ is small enough, so $F(f(w_k^\rho)) = F(f(x))$ by (11.27) (recall that $x \in X_5$ and $w_k^\rho \in X_3$, so they both lie in $X_2$), and then $f(w_k^\rho) \in F(f(w_k^\rho)) = F(f(x)) \subset W(f(x))$, by definitions. Hence (12.34) yields

$$\operatorname{dist}(A(w_k^\rho - x), W) \leq |f(w_k^\rho) - A_x(w_k^\rho)| = o(|w_k^\rho - x|) = o(\rho) \tag{12.35}$$

by (11.40). We divide by $\rho$, take the limit, use (12.31), and get that $A(w_k) \in W$, as needed; Lemma 12.27 follows. □

† Under the Lipschitz assumption, we shall use an analogue of Lemma 12.27 where $f$ is replaced with the mapping $\widetilde{f}$ defined by

$$\widetilde{f}(x) = \psi(\lambda f(x)) \in B(0,1) \quad \text{for } x \in E \tag{12.36}$$

(which is the same as (11.50)). Recall that $f : E \to U$ (because $f = \varphi_1$ on $E$, and because $\widehat{W} \subset\subset U$), and then $\psi : \lambda U \to B(0,1)$ by definition of the Lipschitz assumption. In fact, $\widetilde{f}$ is even defined in a neighborhood of $E$ in $U$.

In the definition of $X_4(s)$, and in addition to the differentiability at $x$ of the restriction of $f$ to $\Gamma_s$ (see below (11.37)), let us also require that the restriction of $\widetilde{f}$ to $\Gamma_s$ be differentiable at $x$. This is also true for $\mathcal{H}^d$ almost every $x \in X_3(s)$, and the proof of (11.36) and (11.37) shows that there is an affine function $\widetilde{A}_x : \mathbb{R}^n \to \mathbb{R}^n$, of rank at most $d$, such that

$$|D\widetilde{A}_x| \leq \lambda\Lambda |f|_{lip}, \tag{12.37}$$

and

$$\lim_{y \to x\,;\, y \in \Gamma_s} \frac{|\widetilde{f}(y) - \widetilde{A}_x(y)|}{|y-x|} = 0, \tag{12.38}$$

where $\widetilde{f}$ is defined, near $x$, by the same formula (12.36) as above. Then the proof of (11.40) yields

$$\lim_{y \to x\,;\, y \in E \cup P_x} \frac{|\widetilde{f}(y) - \widetilde{A}_x(y)|}{|y-x|} = 0 \tag{12.39}$$

when $x \in X_4(s)$, if we add this requirement to the definition of $X_4(s)$. Then we get the following variant of Lemma 12.27.

**Lemma 12.40.** *For $x \in X_5$, let $P_x$ and $\widetilde{A}_x$ be as in (11.34) and (12.39), denote by $F(\widetilde{f}(x))$ the smallest face of our usual rigid dyadic grid that contains $\widetilde{f}(x) = \psi(\lambda f(x))$ and by $W(\widetilde{f}(x))$ the smallest affine subspace that contains $F(\widetilde{f}(x))$. Then*

$$\widetilde{A}_x(P_x) \subset W(\widetilde{f}(x)). \tag{12.41}$$

The proof is the same as for Lemma 12.27; we just replace $f$ with $\widetilde{f}$ and (11.40) with (12.39) in (12.35). □†

**Step 2.e. We choose disks $Q_j$, $j \in J_1$.**

For each $j \in J_1$, we set $B_j = B(x_j, t)$ (see Step 2.a). We want to choose a $Q_j$ in our large collection $\mathcal{F}$, and not so far from $f(E \cap B_j)$ so that composing $f$ with a projection on $Q_j$ will not move points too much. As before, we start with the easier rigid assumption.

So fix $j \in J_1$. Recall that $x_j \in X_N(\delta_4) = X_5 \cap f^{-1}(Y_N(\delta_4))$ (see the line below (12.8), and (12.3)). Then $f(x_j) \in Y_N(\delta_4)$ and by (12.13) we can find $l = l(j) \in \mathcal{L}$ such that $f(x_j) \in D_l = B(y_l, t)$. Since $y_l \in Y_N(\delta_4) \subset f(X_5)$ by (12.1), we can find $x(l) \in X_5$ such that $y_l = f(x(l))$.

Still denote by $F_l$ the smallest face that contains $y_l$ and by $W(y_l)$ the affine space spanned by $F_l$. With the notation of (11.27), $F_l = F(y_l) = F(f(x(l)))$, and (11.27) says that $F_l = F(f(x_j))$ as well, since $x_j$ and $x(l)$ both lie in $X_5 \subset X_2$ and $|f(x_j) - f(x(l))| = |f(x_j) - y_l| \leq t < \delta_6 < \delta_2$ (see (12.8) and (12.7)). Then $W(y_l) = W(f(x_j))$ (the affine space spanned by $F(f(x_j))$ as well.

Set $P_j = P_{x_j}$; then Lemma 12.27 says that

$$A_{x_j}(P_j) \subset W(f(x_j)) = W(y_l). \tag{12.42}$$

We claim that that we can find a disk $Q_j \in \mathcal{F}_l$ such that

$$\mathrm{dist}(z, Q_j) \leq \alpha t \text{ for } z \in A_{x_j}(P_j) \cap B(y_l, 3(1 + |f|_{lip})t). \tag{12.43}$$

Indeed, if $W(y_l)$ is of dimension at most $d$, $\mathcal{F}_l$ is composed of a single element $Q = W(y_l) \cap B(3(1+|f|_{lip})t)$, which satisfies (12.43) by (12.42) and (12.17). Otherwise, (12.42) (and the fact that $A_{x_j}(P_j)$ is at most $d$-dimensional) allows us to pick an affine $d$-plane $P'$ such that $A_{x_j}(P_j) \subset P' \subset W(y_l)$. Observe that $P'$ goes through $D_l$ because $A_{x_j}(x_j) = f(x_j) \in D_l$, so we can choose $Q_j \in \mathcal{F}_l$ so that (12.18) holds; (12.43) follows because $A_{x_j}(P_j) \subset P'$.

Let us check that

$$\mathrm{dist}(f(z), Q_j) \leq \big(2\varepsilon + 2\varepsilon|f|_{lip} + \alpha\big)\, t \text{ for } z \in E \cap 2B_j = E \cap B(x_j, 2t). \tag{12.44}$$

Indeed, $x_j \in X_5$, so (11.46) holds for $x = x_j$. We get that

$$|f(z) - A_{x_j}(z)| \leq \varepsilon|z - x_j| \leq 2\varepsilon t \text{ for } z \in E \cap 2B_j, \tag{12.45}$$

because $|z - x_j| \leq 2t < 2\delta_6 < \delta_3$ by (12.8) and (12.7). Denote by $\pi(z)$ the projection of $z$ on $P_j$; then $|\pi(z) - z| \leq 2\varepsilon t$ by (11.45) (again applied with $x = x_j$ and valid because $|z - x_j| < \delta_3$). By (11.36),

$$|A_{x_j}(z) - A_{x_j}(\pi(z))| \leq |\pi(z) - z|\, |f|_{lip} \leq 2\varepsilon t\, |f|_{lip}. \tag{12.46}$$

In addition,

$$\begin{aligned} |A_{x_j}(\pi(z)) - y_l| &\leq |A_{x_j}(\pi(z)) - f(x_j)| + |f(x_j) - y_l| \\ &= |A_{x_j}(\pi(z)) - A_{x_j}(x_j)| + |f(x_j) - y_l| \\ &\leq |f|_{lip}|\pi(z) - x_j| + t \leq (2|f|_{lip} + 1)t \end{aligned} \tag{12.47}$$

because $A_{x_j}(x_j) = f(x_j)$, $f(x_j) \in D_l = B(y_l, t)$, and $|\pi(z) - x_j| \leq |z - x_j| \leq 2t$. Since $A_{x_j}(\pi(z)) \in A_{x_j}(P_j)$ trivially, (12.47) allows us to apply (12.43) to it; this yields $\mathrm{dist}(A_{x_j}(\pi(z)), Q_j) \leq \alpha t$, and now (12.44) follows from (12.45) and (12.46).

† Under the Lipschitz assumption, we still can define $l = l(j) \in \mathcal{L}$ such that $f(x_j) \in D_l$, and $x(l) \in X_5$ such that $y_l = f(x(l))$. The smallest face $F_l$ that contains $y_l$ is still the same as for $f(x_j)$; equivalently, the smallest face $\widetilde{F}_l$ of the rigid grid that contains $\widetilde{y}_l = \psi(\lambda y_l)$ is the same as for $\widetilde{f}(x_j) = \psi(\lambda f(x_j))$. Then the smallest affine space $W(\widetilde{y}_l)$ that contains $\widetilde{F}_l$ is the same as for $\widetilde{f}(x_j)$. This time, we use Lemma 12.40, which says that

$$\widetilde{A}_{x_j}(P_{x_j}) \subset W(\widetilde{f}(x_j)) = W(\widetilde{y}_l), \tag{12.48}$$

and we want to use this to find a close enough $Q_j \in \mathcal{F}_l$. If $F(y_l)$ is of dimension at most $d$, we pick the only element $Q$ defined by (12.22) with $\widetilde{P} = W(\widetilde{y}_l)$. Otherwise, we want to use the rule (12.25) to select a $\widetilde{P}$. First use (12.48) to choose $\widetilde{P}'$, of dimension $d$, such that $\widetilde{A}_{x_j}(P_{x_j}) \subset \widetilde{P}' \subset W(\widetilde{y}_l)$, and observe that $\widetilde{P}'$ meets $B(\widetilde{y}_l, \lambda\Lambda(1+|f|_{lip})t))$ because $\widetilde{f}(x_j) = \widetilde{A}_{x_j}(x_j) \in \widetilde{A}_{x_j}(P_{x_j})$ and

$$|\widetilde{f}(x_j) - \widetilde{y}_l| = |\psi(\lambda f(x_j)) - \psi(\lambda y_l)| \leq \lambda\Lambda |f(x_j) - y_l| \leq \lambda\Lambda t \tag{12.49}$$

because $f(x_j) \in D_l$. Thus (12.24) holds, and we can find a $d$-plane $\widetilde{P}$ in our collection, so that (12.25) holds. We call that plane $\widetilde{P}_j$. Hence

$$\mathrm{dist}(z, \widetilde{P}_j) \leq \lambda\Lambda^{-1}\alpha t \ \text{ for } z \in \widetilde{A}_{x_j}(P_{x_j}) \cap B(\widetilde{y}_l, 10\lambda\Lambda^2(1+|f|_{lip})t), \tag{12.50}$$

since $\widetilde{A}_{x_j}(P_{x_j}) \subset \widetilde{P}'$. Then let $Q_j \in \mathcal{F}_l$ be the set $Q$ defined by (12.22) with this choice of $\widetilde{P} = \widetilde{P}_j$; we want to check that

$$\mathrm{dist}(f(z), Q_j) \leq 10\Lambda^3(1+|f|_{lip})\varepsilon t + \alpha\, t \ \text{ for } z \in E \cap B(x_j, 10\Lambda t), \tag{12.51}$$

as in (12.44). For this, we assume that in the definition of $X_5(s)$ (near (11.42)), we added to the requirement (11.46) its analogue for $\widetilde{f}$. That is, we first take $\delta_3$ so small that $f(y) \in U$ for $x \in X_1$ and $y \in B(x, \delta_3)$ (this is easy, because $f$ is Lipschitz, $X_1$ is compact, and $f(X_1) \subset U$); this way, we can define $\widetilde{f}(y) = \psi(\lambda f(y))$ for $x \in X_1$ and $y \in B(x, \delta_3)$, as in (12.36). But more importantly, we take $\delta_3$ so small that with this definition,

$$|\widetilde{f}(y) - \widetilde{A}_x(y)| \leq \lambda\varepsilon|y - x| \ \text{ for } y \in [E \cup P_x \cup \Gamma_s] \cap B(x, \delta_3). \tag{12.52}$$

This is possible, for the same reason as for (11.46).

Now let $z \in E \cap B(x_j, 10\Lambda t)$ be given; then

$$|\widetilde{f}(z) - \widetilde{A}_{x_j}(z)| \leq \lambda\varepsilon|z - x_j| \leq 10\lambda\Lambda\varepsilon t \tag{12.53}$$

by (12.52) and because $|z - x_j| \leq 10\Lambda t < 10\Lambda\delta_6 < \delta_3$ by (12.8) and (12.7). Again the projection $\pi(z)$ of $z$ on $P_j$ is such that $|\pi(z) - z| \leq 10\Lambda\varepsilon t$, by (11.45), so

$$|\widetilde{A}_{x_j}(z) - \widetilde{A}_{x_j}(\pi(z))| \leq \lambda\Lambda|f|_{lip}|\pi(z) - z| \leq 10\lambda\Lambda^2|f|_{lip}\varepsilon t \tag{12.54}$$

by (12.37), and also

$$\begin{aligned} |\widetilde{A}_{x_j}(\pi(z)) - \widetilde{y}_l| &\leq |\widetilde{A}_{x_j}(\pi(z)) - \widetilde{f}(x_j)| + |\widetilde{f}(x_j) - \widetilde{y}_l| \\ &= |\widetilde{A}_{x_j}(\pi(z)) - \widetilde{A}_{x_j}(x_j)| + |\psi(\lambda f(x_j)) - \psi(\lambda y_l)| \\ &\leq \lambda\Lambda|f|_{lip}|\pi(z) - x_j| + \lambda\Lambda|f(x_j) - y_l| \\ &\leq \lambda\Lambda|f|_{lip}|z - x_j| + \lambda\Lambda t \leq 10\lambda\Lambda^2(1+|f|_{lip})t \end{aligned} \tag{12.55}$$

because $\widetilde{A}_{x_j}(x_j) = \widetilde{f}(x_j)$, by (12.37) again, because $x_j = \pi(x_j)$ since $P_j$ goes through $x_j$, and because $f(x_j) \in D_l = B(y_l, t)$ and $z \in B(x_j, 10\Lambda t)$.

Now $\widetilde{A}_{x_j}(\pi(z)) \in \widetilde{A}_{x_j}(P_j)$, so by (12.55) we may apply (12.50) to it; we get that $\mathrm{dist}(\widetilde{A}_{x_j}(\pi(z)), \widetilde{P}_j) \leq \lambda\Lambda^{-1}\alpha t$. Since

$$|\widetilde{f}(z) - \widetilde{A}_{x_j}(\pi(z))| \leq 10\lambda\Lambda^2(1 + |f|_{lip})\varepsilon t \tag{12.56}$$

by (12.53) and (12.54), we get that

$$\mathrm{dist}(\widetilde{f}(z), \widetilde{P}_j) \leq 10\lambda\Lambda^2(1 + |f|_{lip})\varepsilon t + \lambda\Lambda^{-1}\alpha t \ \text{ for } z \in E \cap B(x_j, 10\Lambda t). \tag{12.57}$$

In addition, $|\widetilde{f}(z) - \widetilde{y}_l| \leq 11\lambda\Lambda^2(1 + |f|_{lip})t$ by (12.55) and (12.56), so the point of $\widetilde{P}_j$ that minimizes the distance to $\widetilde{f}(z)$ automatically lies in $\widetilde{Q} = \psi(\lambda Q_j)$ by (12.22). Finally,

$$\begin{aligned} \mathrm{dist}(f(z), Q_j) &= \mathrm{dist}(f(z), \lambda^{-1}\psi^{-1}(\widetilde{Q})) \leq \lambda^{-1}\Lambda\, \mathrm{dist}(\widetilde{f}(z), \widetilde{Q}) \\ &= \lambda^{-1}\Lambda\, \mathrm{dist}(\widetilde{f}(z), \widetilde{P}_j) \leq 10\Lambda^3(1 + |f|_{lip})\varepsilon t + \alpha t \end{aligned} \tag{12.58}$$

by (12.22) again and (12.57); (12.51) follows. †

**Step 2.f. We construct mappings $g_j$, $j \in J_1$.**

Return to the rigid assumption. For each $j \in J_1$, we now define a Lipschitz mapping $g_j : U \to \mathbb{R}^n$. We use a new constant $a \in (0, 1)$ quite close to 1. Set

$$g_j(x) = f(x) \ \text{ for } x \in U \setminus B(x_j, \frac{1+a}{2}\, t) \tag{12.59}$$

and

$$g_j(x) = \pi_j(f(x)) \ \text{ for } x \in aB_j = B(x_j, at), \tag{12.60}$$

where $\pi_j$ denotes the orthogonal projection onto the affine plane spanned by $Q_j$. In the middle, interpolate linearly as usual, by setting

$$g_j(x) = \frac{2|x - x_j| - 2at}{(1-a)t}\, f(x) + \frac{(1+a)t - 2|x - x_j|}{(1-a)t}\, \pi_j(f(x)) \tag{12.61}$$

for $x \in B(x_j, \frac{1+a}{2}\, t) \setminus B(x_j, at)$. This gives a Lipschitz function $g_j$, with a quite large Lipschitz norm that we don't want to compute, and such that

$$\begin{aligned} ||g_j - f||_\infty &\leq \sup_{x \in B_j} |\pi_j(f(x)) - f(x)| \leq \sup_{x \in B_j} \mathrm{dist}(f(x), Q_j) \\ &\leq \mathrm{dist}(f(x_j), Q_j) + \sup_{x \in B_j} |f(x) - f(x_j)| \leq \mathrm{dist}(f(x_j), Q_j) + |f|_{lip}\, t \\ &\leq \big(2\varepsilon + 2\varepsilon|f|_{lip} + \alpha\big)\, t + |f|_{lip} t \leq (1 + |f|_{lip})t \end{aligned} \tag{12.62}$$

by (12.44) and if $\varepsilon$ and $\alpha$ are small enough. Fortunately, the estimates get better near $E$. Set

$$E^{\varepsilon t} = \big\{x \in U\,;\, \mathrm{dist}(x,E) \leq \varepsilon t\big\}; \tag{12.63}$$

we claim that

$$|g_j(x) - f(x)| \leq (2\varepsilon + 3|f|_{lip}\,\varepsilon + \alpha)\,t \quad \text{for } x \in E^{\varepsilon t}. \tag{12.64}$$

Indeed, by (12.59) we can assume that $x \in B(x_j, \frac{1+a}{2}\,t)$; choose $z \in E$ such that $|z-x| \leq \varepsilon t$; then $z \in 2B_j$, and

$$\begin{aligned} |g_j(x) - f(x)| &\leq |\pi_j(f(x)) - f(x)| \leq \mathrm{dist}(f(x), Q_j) \\ &\leq \mathrm{dist}(f(z), Q_j) + |f(x) - f(z)| \\ &\leq \big(2\varepsilon + 2\varepsilon|f|_{lip} + \alpha\big)\,t + |f|_{lip}\varepsilon t = (2\varepsilon + 3\varepsilon|f|_{lip} + \alpha)\,t \end{aligned} \tag{12.65}$$

as above, and by (12.44); the claim follows. Similarly, let us check that

$$g_j \ \text{ is } (1 + |f|_{lip})\text{-Lipschitz on } E^{\varepsilon t}. \tag{12.66}$$

Let $x, y \in E^{\varepsilon t}$, and use (12.59)-(12.61) to write

$$g_j(x) = \beta(x)f(x) + (1 - \beta(x))\pi_j(f(x)) \tag{12.67}$$

for some $\beta(x) \in [0,1]$, and similarly $g_j(y) = \beta(y)f(y) + (1-\beta(y))\pi_j(f(y))$. Write

$$g_j(y) = \beta(x)f(y) + (1-\beta(x))\pi_j(f(y)) - [\beta(x) - \beta(y)][f(y) - \pi_j(f(y))], \tag{12.68}$$

and then subtract (12.68) from (12.67) to get that

$$\begin{aligned} g_j(x) - g_j(y) = \beta(x)[f(x) - f(y)] + (1-\beta(x))[\pi_j(f(x)) - \pi_j(f(y))] \\ + [\beta(x) - \beta(y)][f(y) - \pi_j(f(y))]. \end{aligned} \tag{12.69}$$

The first part is at most

$$\beta(x)|f(x) - f(y)| + (1-\beta(x))|\pi_j(f(x)) - \pi_j(f(y))| \leq |f(x) - f(y)| \leq |f|_{lip}|x-y|$$

because $\pi_j$ is 1-Lipschitz. If $y \in B(x_j, \frac{1+a}{2}\,t)$, the proof of (12.65) shows that

$$|\pi_j(f(y)) - f(y)| \leq (2\varepsilon + 3\varepsilon|f|_{lip} + \alpha)\,t \tag{12.70}$$

and since $|\beta(x) - \beta(y)| \leq \dfrac{2|x-y|}{(1-a)t}$ by (12.59)-(12.61), we get that

$$|\beta(x) - \beta(y)||f(y) - \pi_j(f(y))| \leq (2\varepsilon + 3\varepsilon|f|_{lip} + \alpha)\,\frac{2|x-y|}{(1-a)} \leq |x-y| \tag{12.71}$$

if $\varepsilon$ and $\alpha$ are small enough, depending on $|f|_{lip}$, but also on $a$. (This is all right, see Remark 11.17.) Altogether $|g_j(x) - g_j(y)| \leq (1 + |f|_{lip})|x - y|$, as needed for (12.66) in this first case.

If $x \in B(x_j, \frac{1+a}{2}\, t)$, the same argument, with $x$ and $y$ exchanged from the start (i.e., also in (12.68) and (12.69)) gives the desired result. Finally, if both $x$ and $y$ lie out of $B(x_j, \frac{1+a}{2}\, t)$, then $\beta(x) = \beta(y) = 1$ by (12.59), and $|g_j(x) - g_j(y)| \leq |f|_{lip}|x - y|$ directly by (12.70). This completes our proof of (12.66).

**†Step 2.g. The mappings $g_j$, under the Lipschitz assumption.†**

† Now we do the same thing under the Lipschitz assumption. We shall try to do the linear algebra and convex combinations on $B(0,1)$, because we want to preserve the faces when we can, but (later on, when we interpolate between the $g_j$) we shall still use partitions of unity defined on $U$.

Before we define mappings $\widetilde{g}_j$ we need to extend our definition of the $\widetilde{f}$ of (12.36). We are particularly interested in the set

$$U_{int} = \big\{x \in U\,;\, \mathrm{dist}(x, X_0) \leq \frac{\delta_0}{4(1 + |f|_{lip})}\big\}, \tag{12.72}$$

because we shall see that it is so small that (12.36) makes sense on it, and sufficiently large to contain the $2B_j$, $j \in J_1$. Let us first check that

$$\mathrm{dist}(f(x), \widehat{W}) \leq \frac{\delta_0}{3} \quad \text{for } x \in U_{int}. \tag{12.73}$$

For $x \in U_{int}$, pick $y \in X_0$ such that $|y - x| \leq \frac{\delta_0}{3(1+|f|_{lip})}$; then $f(y) \in \widehat{W}$ by (11.20), (2.1), and (2.2), so

$$\mathrm{dist}(f(x), \widehat{W}) \leq |f(x) - f(y)| \leq |f|_{lip}|y - x| \leq \frac{\delta_0}{3} \tag{12.74}$$

as needed. For such $x$, $f(x) \in U$ because $\delta_0 = \mathrm{dist}(\widehat{W}, \mathbb{R}^n \setminus U)$ by (12.6). Hence we can define $\psi(\lambda f(x))$. So we can extend the definition (12.36), and set

$$\widetilde{f}(x) = \psi(\lambda f(x)) \quad \text{for } x \in U_{int} \cup E. \tag{12.75}$$

Note that $\widetilde{f}(x) \in B(0,1)$ automatically, because $\psi(\lambda U) = B(0,1)$. It will also be good to know that

$$2B_j = B(x_j, 2t) \subset U_{int} \quad \text{for } j \in J_1, \tag{12.76}$$

which is true because $x_j \in X_5 \subset X_0$ and $t < \delta_6 \leq \frac{\delta_0}{10(1+|f|_{lip})}$ by (12.8) and (12.7).

Next we define intermediate mappings $\widetilde{g}_j$. Recall that $Q_j$ is defined by (12.22) for some affine plane $\widetilde{P} = \widetilde{P}_j$; we denote by $\widetilde{\pi}_j$ the orthogonal projection onto $\widetilde{P}_j$, and then set

$$\widetilde{g}_j(x) = \widetilde{f}(x) \quad \text{for } x \in U_{int} \setminus B(x_j, \frac{1+a}{2}\, t) \tag{12.77}$$

(a little as in (12.59)),

$$\widetilde{g}_j(x) = \widetilde{\pi}_j(\widetilde{f}(x)) \ \text{ for } x \in aB_j = B(x_j, at), \tag{12.78}$$

(as in (12.60)), and

$$\widetilde{g}_j(x) = \frac{2|x-x_j| - 2at}{(1-a)t}\,\widetilde{f}(x) + \frac{(1+a)t - 2|x-x_j|}{(1-a)t}\,\widetilde{\pi}_j(\widetilde{f}(x)) \tag{12.79}$$

for $x \in B(x_j, \frac{1+a}{2}\,t) \setminus B(x_j, at)$ (as in (12.61)). The simplest for us will be not to define $\widetilde{g}_j$ in $U \setminus U_{int}$.

Let us concentrate on what happens in $2B_j = B(x_j, 2t)$. Recall that $2B_j \subset U_{int}$ by (12.76), so $\widetilde{g}_j$ is defined on $2B_j$. Then, for $x \in 2B_j$,

$$\begin{aligned} |\widetilde{f}(x) - \widetilde{y}_l| &= |\psi(\lambda f(x)) - \psi(\lambda y_l)| \leq \lambda\Lambda |f(x) - y_l| \\ &\leq \lambda\Lambda\big(|f(x) - f(x_j)| + |f(x_j) - y_l|\big) \\ &\leq \lambda\Lambda\big(|f|_{lip}|x - x_j| + t\big) \leq \lambda\Lambda\big(2|f|_{lip} + 1\big)\, t \end{aligned} \tag{12.80}$$

by (12.75), the definition of $\widetilde{y}_l$ above (12.48), and the fact that $f(x_j) \in D_l$. Hence

$$\begin{aligned} |\widetilde{g}_j(x) - \widetilde{f}(x)| &\leq |\widetilde{\pi}_j(\widetilde{f}(x)) - \widetilde{f}(x)| = \mathrm{dist}(\widetilde{f}(x), \widetilde{P}_j) \leq \mathrm{dist}(\widetilde{y}_l, \widetilde{P}_j) + |\widetilde{f}(x) - \widetilde{y}_l| \\ &\leq 2\lambda\Lambda(1 + |f|_{lip})\, t + |\widetilde{f}(x) - \widetilde{y}_l| \leq 4\lambda\Lambda(1 + |f|_{lip})\, t \end{aligned} \tag{12.81}$$

by (12.77)-(12.79), because $\widetilde{P}_j$ was chosen (near (12.49)) so that (12.23) holds, and by (12.80). Let us also check that

$$\widetilde{g}_j(x) \in B(0,1) \ \text{ for } x \in 2B_j. \tag{12.82}$$

By (12.80) and (12.81), $|\widetilde{g}_j(x) - \widetilde{y}_l| \leq 6\lambda\Lambda(1 + |f|_{lip})\, t$. But $y_l \in X_0 \subset W_1$, so

$$\begin{aligned} \mathrm{dist}(\widetilde{y}_l, \partial B(0,1)) &= \mathrm{dist}(\psi(\lambda y_l), \psi(\lambda \partial U)) \geq \lambda\Lambda^{-1}\, \mathrm{dist}(y_l, \partial U) \\ &\geq \lambda\Lambda^{-1}\, \mathrm{dist}(\widehat{W}, \mathbb{R}^n \setminus U) = \lambda\Lambda^{-1}\delta_0 \\ &\geq 10(1 + |f|_{lip})\, \lambda\Lambda\delta_6 \geq 10(1 + |f|_{lip})\, \lambda\Lambda t \end{aligned} \tag{12.83}$$

because $\widetilde{y}_l = \psi(\lambda y_l)$ (see above (12.22)), because $\psi$ has a bilipschitz extension from the closure of $\lambda U$ to $\overline{B}(0,1)$, and by (12.6)-(12.8). Then

$$\mathrm{dist}(\widetilde{g}_j(x), \mathbb{R}^n \setminus B(0,1)) \geq \mathrm{dist}(\widetilde{y}_l, \partial B(0,1)) - |\widetilde{g}_j(x) - \widetilde{f}(x)| \geq 4(1 + |f|_{lip})\, \lambda\Lambda t > 0, \tag{12.84}$$

and (12.82) holds.

We may now define $g_j$ on $2B_j$ by

$$g_j(x) = \lambda^{-1}\psi^{-1}(\widetilde{g}_j(x)) \ \text{ for } x \in 2B_j, \tag{12.85}$$

because $\widetilde{g}_j(x) \in B(0,1)$ and hence $\psi^{-1}(\widetilde{g}_j(x))$ is defined. We also get that $g_j(x) \in U$, because $\psi : \lambda U \to B(0,1)$.

Observe that when $x \in 2B_j \setminus B(x_j, \frac{1+a}{2}\, t)$, (12.85), (12.77) and (12.75) yield $g_j(x) = \lambda^{-1}\psi^{-1}(\widetilde{g}_j(x)) = \lambda^{-1}\psi^{-1}(\widetilde{f}(x)) = f(x)$. So we can safely set

$$g_j(x) = f(x) \ \text{ for } x \in U \setminus B(x_j, \frac{1+a}{2}\, t), \tag{12.86}$$

the two definitions coincide on $2B_j \setminus B(x_j, \frac{1+a}{2}\, t)$, and $g_j$ is Lipschitz on $U$.

Return to $x \in 2B_j$. Since $f(x) = \lambda^{-1}\psi^{-1}(\widetilde{f}(x))$ by (12.75), we see that

$$||g_j - f||_{L^\infty(2B_j)} \leq \lambda^{-1}\Lambda||\widetilde{g}_j - \widetilde{f}||_{L^\infty(2B_j)} \leq 4\Lambda^2(1 + |f|_{lip})\, t, \tag{12.87}$$

by (12.85) and (12.81). This will be a good enough analogue for (12.62).

We also need better estimates when $x \in E^{\varepsilon t}$ (the small neighborhood of $E$ defined in (12.63)). Let $x \in E^{\varepsilon t} \cap B(x_j, \frac{1+a}{2} t)$ be given, and pick $z \in E$ such that $|z - x| \leq \varepsilon t$; then

$$\begin{aligned}
|\widetilde{g}_j(x) - \widetilde{f}(x)| &\leq |\widetilde{\pi}_j(\widetilde{f}(x)) - \widetilde{f}(x)| \leq \operatorname{dist}(\widetilde{f}(x), \widetilde{P}_j) \\
&\leq \operatorname{dist}(\widetilde{f}(z), \widetilde{P}_j) + |\widetilde{f}(x) - \widetilde{f}(z)| \leq \operatorname{dist}(\widetilde{f}(z), \widetilde{P}_j) + |x - z|\, |\widetilde{f}|_{lip} \\
&\leq 10\lambda\Lambda^2(1 + |f|_{lip})\varepsilon t + \lambda\Lambda^{-1}\alpha t + \varepsilon t\lambda\Lambda|f|_{lip} \\
&\leq 11\lambda\Lambda^2(1 + |f|_{lip})\varepsilon t + \lambda\Lambda^{-1}\alpha t
\end{aligned} \tag{12.88}$$

by (12.77)-(12.79), because $\widetilde{\pi}_j$ denotes the projection on the plane $\widetilde{P}_j$ that was used to construct $Q_j$ (see near (12.22) and (12.77)), by (12.57), and because $|\widetilde{f}|_{lip} \leq \lambda\Lambda|f|_{lip}$ by (12.75). Then (12.85) yields

$$|g_j(x) - f(x)| \leq \lambda^{-1}\Lambda|\widetilde{g}_j(x) - \widetilde{f}(x)| \leq 11\Lambda^3(1 + |f|_{lip})\varepsilon t + \alpha t, \tag{12.89}$$

which is a good replacement for (12.65). But $g_j(x) = f(x)$ for $x \in E^{\varepsilon t} \setminus \cap B(x_j, \frac{1+a}{2}\, t)$ (by (12.86), so

$$|g_j(x) - f(x)| \leq 11\Lambda^3(1 + |f|_{lip})\varepsilon t + \alpha t \ \text{ for } x \in E^{\varepsilon t}, \tag{12.90}$$

which is an acceptable analogue of (12.64).

Next we copy the proof of (12.66). Let us estimate $|\widetilde{g}_j(x) - \widetilde{g}_j(y)|$ when $x, y \in E^{\varepsilon t} \cap 2B_j$. Then we can use (12.77)-(12.79); as before, we write $\widetilde{g}_j(x)$ as a linear combination of $\widetilde{f}(x)$ and $\widetilde{\pi}_j(\widetilde{f}(x))$, and similarly for $\widetilde{g}_j(y)$, and then we compute as in (12.67)-(12.71).

As in (12.69), we get that $\widetilde{g}_j(x) - \widetilde{g}_j(y) = A + B$, with

$$A = \beta(x)[\widetilde{f}(x) - \widetilde{f}(y)] + (1 - \beta(x))[\widetilde{\pi}_j(\widetilde{f}(x)) - \widetilde{\pi}_j(\widetilde{f}(y))] \tag{12.91}$$

and

$$B = [\beta(x) - \beta(y)][\widetilde{f}(y) - \widetilde{\pi}_j(\widetilde{f}(y))]. \tag{12.92}$$

As before, $|A| \leq |\widetilde{f}(x) - \widetilde{f}(y)| \leq \lambda\Lambda|f|_{lip}|x-y|$ because $\widetilde{\pi}_j$ is 1-Lipschitz and $\widetilde{f}$ is $\lambda\Lambda|f|_{lip}$ -Lipschitz.

In the first case when $y \in E^{\varepsilon t} \cap B(x_j, \frac{1+a}{2}\, t)$, the proof of the second part of (12.88) yields

$$|\widetilde{f}(y) - \widetilde{\pi}_j(\widetilde{f}(y))| = \mathrm{dist}(\widetilde{f}(y), \widetilde{P}_j) \leq 11\lambda\Lambda^2(1+|f|_{lip})\varepsilon t + \lambda\Lambda^{-1}\alpha t; \tag{12.93}$$

since $|\beta(x) - \beta(y)| \leq \dfrac{2|x-y|}{(1-a)t}$ as before, we get that

$$|B| \leq \big\{11\lambda\Lambda^2(1+|f|_{lip})\varepsilon t + \lambda\Lambda^{-1}\alpha t\big\}\, \frac{2|x-y|}{(1-a)t} \leq \lambda\Lambda^{-1}|x-y| \tag{12.94}$$

if $\varepsilon$ and $\alpha$ are small enough. In this case

$$|\widetilde{g}_j(x) - \widetilde{g}_j(y)| \leq |A| + |B| \leq \lambda\Lambda|f|_{lip}|x-y| + \lambda\Lambda^{-1}|x-y|. \tag{12.95}$$

The other two cases are treated as before, and we get that

$$|\widetilde{g}_j(x) - \widetilde{g}_j(y)| \leq \lambda\, \frac{1+\Lambda^2|f|_{lip}}{\Lambda}|x-y| \quad \text{for } x, y \in E^{\varepsilon t} \cap 2B_j. \tag{12.96}$$

By this and (12.85), we get that

$$g_j \text{ is } (1+\Lambda^2|f|_{lip})\text{-Lipschitz on } E^{\varepsilon t} \cap 2B_j. \tag{12.97}$$

This is a good enough analogue of (12.66), which was the last estimate of Step 2.f. †

## 13. Step 2.h. We glue the mappings $g_j$, $j \in J_1$, and get a first mapping $g$.

We start a new section, but continue to take care of the places where $f$ is very many-to-one. Now we want to use the functions $g_j$ that we just built to modify $f$ on a subset of

$$V = \bigcup_{j \in J_1} B(x_j, t) = \bigcup_{j \in J_1} B_j. \tag{13.1}$$

Part of the difficulty will be that the balls $B(x_j, t)$ are not disjoint (but fortunately they have the same radius). We shall apply the same trick as in [D2], based on adapted partitions of unity. Set

$$R_j = B_j \setminus aB_j = \big\{z \in \mathbb{R}^n \,;\, at \leq |z - x_j| < t\big\} \tag{13.2}$$

for $j \in J_1$, and

$$R = \bigcup_{j \in J_1} R_j\,. \tag{13.3}$$

Then let $\varphi_j$ be a smooth function such that

$$0 \le \varphi_j(x) \le 1 \ \text{ and } |\nabla\varphi_j(x)| \le \frac{2}{(1-a)t} \ \text{ for } x \in \mathbb{R}^n, \tag{13.4}$$

$$\varphi_j(x) = 1 \text{ for } x \in aB_j, \text{ and } \ \varphi_j(x) = 0 \text{ for } x \in \mathbb{R}^n \setminus B_j. \tag{13.5}$$

Next put an arbitrary order on the set $J_1$, and set

$$\psi_j(x) = \sup_{i\in J_1;\, i\le j} \varphi_i(x) - \sup_{i\in J_1;\, i<j} \varphi_i(x) \tag{13.6}$$

for $j \in J_1$ and $x \in \mathbb{R}^n$ (and where the empty sup is zero). Clearly

$$\sum_{i\in J_1;\, i\le j} \psi_i(x) = \sup_{i\in J_1;\, i\le j} \varphi_i(x). \tag{13.7}$$

Finally set

$$\psi(x) = \sum_{i\in J_1} \psi_i(x) = \sup_{i\in J_1} \varphi_i(x); \tag{13.8}$$

then

$$0 \le \psi(x) \le 1 \ \text{ for } x \in \mathbb{R}^n, \tag{13.9}$$

$\psi$ is $C[(1-a)t]^{-1}$-Lipschitz (because the $B_j$ have bounded overlap), and $\psi = 1$ on $\bigcup_{j\in J_1} aB_j$. Because of our particular choice of functions, we get that

$$\psi(x) = 0 \text{ for } x \in \mathbb{R}^n \setminus \big[\bigcup_{j\in J_1} B_j\big] = \mathbb{R}^n \setminus V \tag{13.10}$$

and

$$\begin{gathered}\text{if } x \in aB_j \setminus R \text{ for some } j \in J_1, \text{ then one of the } \psi_i(x) \text{ is equal to } 1\\ \text{and the other ones are equal to } 0\end{gathered} \tag{13.11}$$

(where in fact $i$ is the first index in $J_1$ such that $x \in aB_i$, or equivalently $x \in B_i$).

We now use the $\psi_j$ to construct a mapping $g : U \to \mathbb{R}^n$. As usual, we first do the description under the rigid assumption. We set

$$g(x) = f(x) + \sum_{j\in J_1} \psi_j(x)[g_j(x) - f(x)], \tag{13.12}$$

which we write like this because $\sum_{j\in J_1}\psi_j(x)$ may be smaller than 1 at some points. Notice that

$$||g-f||_\infty \le (1+|f|_{lip})\,t \tag{13.13}$$

by (12.62) and (13.9), and

$$|g(x)-f(x)| \le \sum_{j\in J_1}\psi_j(x)|g_j(x)-f(x)| \le (2\varepsilon+3\varepsilon|f|_{lip}+\alpha)\,t \;\text{ for } x\in E^{\varepsilon t}, \tag{13.14}$$

by (12.64). We also claim that because of (12.66),

$$g \;\text{ is } (2+3|f|_{lip})\text{-Lipschitz on } E^{\varepsilon t}. \tag{13.15}$$

Indeed, since $f$ is Lipschitz, it is enough to show that $g-f$ is $(2+2|f|_{lip})$-Lipschitz on $E^{\varepsilon t}$ and estimate

$$\Delta(x,y)=(g-f)(x)-(g-f)(y)=\sum_{j\in J_1}\Big\{\psi_j(x)[g_j(x)-f(x)]-\psi_j(y)[g_j(y)-f(y)]\Big\} \tag{13.16}$$

for $x,y\in E^{\varepsilon t}$. Since $\Delta(x,y)\le 2(2\varepsilon+3\varepsilon|f|_{lip}+\alpha)\,t$ by the $L^\infty$ bound in (13.14), we may assume that $|x-y|\le t/10$.

Let $j\in J_1$ be such that $g_j(x)-f(x)\neq 0$; then $x\in B(x_j,\frac{1+a}{2}\,t)$ by (12.59), and $y\in\frac{3}{2}B_j$ because $|x-y|\le t/10$. Similarly, if $g_j(y)-f(y)\neq 0$, then $y\in B(x_j,\frac{1+a}{2}\,t)$ and $x\in\frac{3}{2}B_j$. So $x,y\in\frac{3}{2}B_j$ for every $j\in J_1$ that has a contribution to the right-hand of (13.16). There are at most $C$ indices $j$ like this (recall that $B_j=B(x_j,t)$ and that $|x_i-x_j|\ge t/3$ when $i\neq j$; see above (12.9)), and each contribution is estimated as follows. We write

$$\begin{aligned}
&\big|\psi_j(x)[g_j(x)-f(x)]-\psi_j(y)[g_j(y)-f(y)]\big|\\
&\qquad\le \psi_j(x)|g_j(x)-f(x)-g_j(y)+f(y)|+|\psi_j(x)-\psi_j(y)|\,|g_j(y)-f(y)|\\
&\qquad\le \psi_j(x)|g_j(x)-f(x)-g_j(y)+f(y)|+4(1-a)^{-1}t^{-1}|x-y|\,|g_j(y)-f(y)|\\
&\qquad\le \psi_j(x)(1+2|f|_{lip})|x-y|+4(1-a)^{-1}|x-y|(2\varepsilon+3\varepsilon|f|_{lip}+\alpha)
\end{aligned} \tag{13.17}$$

because $\psi_j$ is $4[(1-a)t]^{-1}$-Lipschitz (by (13.6) and (13.4)), and by (12.66), (12.64), and (13.14).

When we sum (13.17) over $j$, the first term gives a total contribution of at most $(1+2|f|_{lip})|x-y|$, by (13.8) and (13.9), and the second one of at most $|x-y|$, if $\varepsilon$ and $\alpha$ are chosen small enough, depending on $n$, $|f|_{lip}$ and $a$. So $|\Delta(x,y)|\le(2+2|f|_{lip})|x-y|$ by (13.16), and our claim (13.15) follows.

Let us record the fact that, by (13.12) and (12.59),

$$g(x)=f(x)\;\text{ on } \mathbb{R}^n\setminus\bigcup_{j\in J_1}B(x_j,\frac{1+a}{2}\,t). \tag{13.18}$$

Also, we claim that

$$g\Big(\bigcup_{j\in J_1} B_j \setminus R\Big) \subset \bigcup_{l\in\mathcal{L}} \bigcup_{Q\in\mathcal{F}_l} Q = \bigcup_{Q\in\mathcal{F}} Q. \tag{13.19}$$

Let $j \in J_1$ and $x \in B_j \setminus R$ be given. Then $x \in aB_j$ (see (13.2) and (13.3)), and by (13.11) exactly one $\psi_i(x)$ is equal to 1, and the other ones are equal to 0. For this $i$, $x \in aB_i$ (see below (13.11)) and $g_i(x) = \pi_i(f(x))$ by (12.60), so $g(x) = g_i(x) = \pi_i(f(x))$ by (13.12), and hence (13.19) will follow as soon as we prove that $\pi_i(f(x)) \in Q_i$.

Obviously $\pi_i(f(x))$ lies on the affine subspace $P$ spanned by $Q_i$ (by definition of $\pi_i$ below (12.60)), so by (12.17) it is enough to show that

$$\pi_i(f(x)) \in B(y_l, 3(1 + |f|_{lip})t), \tag{13.20}$$

where $l = l(i)$ is the index that we used in the definition of $Q_i$, above (12.42). But

$$|f(x) - y_l| \leq |f(x) - f(x_i)| + |f(x_i) - y_l| \leq |f|_{lip} t + t \tag{13.21}$$

because $x \in B_i$ and $f(x_i) \in D_l$ . In addition, if $\pi_i'$ denotes the orthogonal projection onto the affine plane through $y_l$ parallel to $P$, then $||\pi_i' - \pi_i||_\infty \leq t$ because $P$ goes through $D_l = B(y_l, t)$ (by definition of $\mathcal{F}_l$; see below (12.17)). Then

$$|\pi_i(f(x)) - y_l| \leq |\pi_i'(f(x)) - y_l| + t \leq |f(x) - y_l| + t \leq (2 + |f|_{lip})t \tag{13.22}$$

by (13.21), and now (13.20) and (13.19) follow.

Since by definition (12.17), $\mathcal{H}^d(Q) \leq C(1+|f|_{lip})^d t^d$ for all $Q \in \mathcal{F}$, (13.19) and (12.20) imply that

$$\mathcal{H}^d\Big(g\Big(\bigcup_{j\in J_1} B_j \setminus R\Big)\Big) \leq \sum_{Q\in\mathcal{F}} \mathcal{H}^d(Q) \leq C(\alpha, f) N^{-1} \mathcal{H}^d(X_0), \tag{13.23}$$

where $C(\alpha, f)$ depends on $\alpha$ and $|f|_{lip}$. This is still good, because $N$ will be chosen very large, depending on $f$, $\mathcal{H}^d(X_0)$, $\alpha$, and $\eta$ in particular.

Because of (13.23), we shall not need to worry too much about what happens in $\bigcup_{j\in J_1} B_j \setminus R$. The set $R$ will not disturb much either, because $E \cap R$ is small. Indeed, we claim that

$$\mathcal{H}^d\big(E \cap \overline{R}\big) = \mathcal{H}^d\big(E \cap \bigcup_{j\in J_1} [\overline{B}_j \setminus aB_j]\big) \leq C(1 - a)\mathcal{H}^d(X_0), \tag{13.24}$$

where $C$ depends only on $M$ and $n$. First fix $j \in J_1$, and observe that

$$\mathrm{dist}(x, P_{x_j}) \leq \varepsilon |x - x_j| \leq \varepsilon t \ \text{ for } x \in E \cap \overline{B}_j, \tag{13.25}$$

because $|x - x_j| \leq t < \delta_3$ by (12.7) and (12.8), because $x_j \in X_5$, and by (11.45). By elementary geometry, we can cover $P_{x_j} \cap [\overline{B}_j \setminus aB_j]$ by less than $C(1 - a)^{-d+1}$ balls of radius $(1-a)t$. Then the double balls cover $E \cap \overline{B}_j \setminus aB_j$ (if $\varepsilon$ is small enough compared to $1-a$), and the local Ahlfors-regularity of $E$ (with the same justification as for (12.12)) yields

$$\mathcal{H}^d(E \cap \overline{B}_j \setminus aB_j) \leq C(1-a)^{-d+1}[(1-a)t]^d = C(1-a)t^d. \tag{13.26}$$

Next $\overline{R} \subset \bigcup_{j \in J_1} \overline{R}_j = \bigcup_{j \in J_1} [\overline{B}_j \setminus aB_j]$ by (13.2) and (13.3). Also recall from (12.10) that $J_1$ has at most $Ct^{-d}\mathcal{H}^d(X_0)$ elements; then (13.24) follows from (13.26).

† We now switch to the Lipschitz assumption. Set

$$V' = \bigcup_{j \in J_1} 2B_j \subset U_{int}, \tag{13.27}$$

where $U_{int}$ is defined in (12.72) and the inclusion follows from (12.76). We keep the same functions $\psi_j$ as above (not to be confused with our bilipschitz mapping $\psi$), and use the definition of $\widetilde{f}$ in (12.75) to set

$$\widetilde{g}(x) = \widetilde{f}(x) + \sum_{j \in J_1} \psi_j(x)[\widetilde{g}_j(x) - \widetilde{f}(x)] \quad \text{for } x \in V' \tag{13.28}$$

(compare with (13.12); we still want to do the linear algebra on $B(0,1)$ and the partitions of unity on $U$). We intend to set

$$g(x) = \lambda^{-1}\psi^{-1}(\widetilde{g}(x)) \quad \text{for } x \in V', \tag{13.29}$$

so we need to check that $\widetilde{g}(x) \in B(0,1)$. Notice that

$$\widetilde{g}(x) = \widetilde{f}(x) \quad \text{when } x \in V' \setminus \bigcup_{j \in J_1} B(x_j, \frac{1+a}{2}\, t), \tag{13.30}$$

because (12.77) says that $\widetilde{g}_j(x) = \widetilde{f}(x)$ for all $j$, and by (13.28). For such an $x$, $\widetilde{g}(x) = \widetilde{f}(x) = \psi(\lambda f(x))$ by (12.75), $\widetilde{g}(x) \in B(0,1)$ because $\psi$ maps to $B(0,1)$, and so (13.29) makes sense and we get that

$$g(x) = f(x) \quad \text{when } x \in V' \setminus \bigcup_{j \in J_1} B(x_j, \frac{1+a}{2}\, t). \tag{13.31}$$

Next suppose that $x$ lies in some $B(x_j, \frac{1+a}{2}\, t)$. Obviously $\widetilde{f}(x)$ and $\widetilde{g}(x)$ are defined because $x \in V'$. Also observe that in fact, (12.77) says that $x \in B(x_j, \frac{1+a}{2}\, t)$ for all the indices $j$ such that $\widetilde{g}_j(x) - \widetilde{f}(x) \neq 0$, so

$$\begin{aligned} |\widetilde{g}(x) - \widetilde{f}(x)| &\leq \sum_{j \in J_1} \psi_j(x)|\widetilde{g}_j(x) - \widetilde{f}(x)| \\ &\leq 4\lambda\Lambda(1 + |f|_{lip})\, t \sum_{j \in J_1} \psi_j(x) \leq 4\lambda\Lambda(1 + |f|_{lip})\, t \end{aligned} \tag{13.32}$$

by (13.28), (12.81), (13.8) and (13.9). In addition,

$$\begin{aligned}\operatorname{dist}(\widetilde{g}(x), \mathbb{R}^n \setminus B(0,1)) &\geq \operatorname{dist}(\widetilde{y}_l, \partial B(0,1)) - |\widetilde{g}(x) - \widetilde{y}_l| \\ &\geq 10(1+|f|_{lip})\,\lambda\Lambda t - |\widetilde{g}(x) - \widetilde{f}(x)| - |\widetilde{f}(x) - \widetilde{y}_l| \\ &\geq 10(1+|f|_{lip})\,\lambda\Lambda t - 4(1+|f|_{lip})\,\lambda\Lambda t - 2(1+|f|_{lip})\,\lambda\Lambda t \\ &\geq 4(1+|f|_{lip})\,\lambda\Lambda t\end{aligned} \tag{13.33}$$

because $\widetilde{y}_l = \psi(\lambda y_l) \in B(0,1)$, by (12.83), (13.32), and (12.80). In this case too, $\widetilde{g}(x) \in B(0,1)$ and we can define $g(x)$ as in (13.29). This completes the legitimation of (13.29).

We decide to set directly

$$g(x) = f(x) \quad \text{when } x \in U \setminus V'; \tag{13.34}$$

since $2B_j \subset V'$ for all $j$, we see that $\operatorname{dist}(U \setminus V', B(x_j, \frac{1+a}{2}\,t)) > t$ and (13.31) gives us a large enough transition region where the two definitions of $g$ give the same result. But in fact we shall never use that definition outside of $V'$.

By (13.31) and (13.32), $||\widetilde{g} - \widetilde{f}||_{L^\infty(V')} \leq 4\lambda\Lambda(1+|f|_{lip})\,t$, and then (by (13.29) and (13.34))

$$||g - f||_\infty \leq 4\Lambda^2(1+|f|_{lip})\,t. \tag{13.35}$$

Next we restrict to $E^{\varepsilon t}$ and check that

$$|g(x) - f(x)| \leq 11\Lambda^3(1+|f|_{lip})\varepsilon t + \alpha t \quad \text{for } x \in E^{\varepsilon t}. \tag{13.36}$$

By (13.31) and (13.34), we can assume that $x \in \bigcup_{j \in J_1} B(x_j, \frac{1+a}{2}\,t)) \subset V'$. As for (13.32),

$$|\widetilde{g}(x) - \widetilde{f}(x)| \leq \sum_{j \in J_1} \psi_j(x) |\widetilde{g}_j(x) - \widetilde{f}(x)|, \tag{13.37}$$

and the only indices $j \in J_1$ that contribute are such that $x \in E^{\varepsilon t} \cap B(x_j, \frac{1+a}{2}\,t)$ (use (13.28) and (12.77)). For these $j$, (12.88) applies and says that $|\widetilde{g}_j(x) - \widetilde{f}(x)| \leq 11\lambda\Lambda^2(1+|f|_{lip})\varepsilon t + \lambda\Lambda^{-1}\alpha t$. We sum in $j$, use the fact that $\sum_{j \in J_1} \psi_j(x) \leq 1$ by (13.8) and (13.9), and get that $|\widetilde{g}(x) - \widetilde{f}(x)| \leq 11\lambda\Lambda^2(1+|f|_{lip})\varepsilon t + \lambda\Lambda^{-1}\alpha t$. Now (13.36) follows from (13.29).

We also want to check that

$$\widetilde{g} \ \text{ is } \lambda\,\frac{2 + 3\Lambda^2|f|_{lip}}{\Lambda}\text{-Lipschitz on } E^{\varepsilon t} \cap V'. \tag{13.38}$$

We follow the proof of (13.15); given $x, y \in E^{\varepsilon t} \cap V'$, we set

$$\begin{aligned}\widetilde{\Delta}(x,y) &= (\widetilde{g} - \widetilde{f})(x) - (\widetilde{g} - \widetilde{f})(y) \\ &= \sum_{j \in J_1} \Big\{\psi_j(x)[\widetilde{g}_j(x) - \widetilde{f}(x)] - \psi_j(y)[\widetilde{g}_j(y) - \widetilde{f}(y)]\Big\} =: \sum_{j \in J_1} \Delta_j(x,y)\end{aligned} \tag{13.39}$$

as in (13.16); since $\widetilde{f}$ is $\lambda\Lambda$-Lipschitz, we just need to show that

$$|\widetilde{\Delta}(x,y)| \le \lambda \, \frac{2+2\Lambda^2|f|_{lip}}{\Lambda} \, |x-y|. \tag{13.40}$$

When $|x-y| \ge t/10$, (13.40) holds because $|\widetilde{\Delta}(x,y)| \le 22\lambda\Lambda^4(1+|f|_{lip})\varepsilon t + 2\lambda\Lambda\alpha t$ by (13.36) and (13.9) (and because we can choose $\varepsilon$ and $\alpha$ very small), so we may assume that $|x-y| \le t/10$. By the same argument as above, $x, y \in \frac{3}{2}B_j$ for every $j \in J_1$ such that $\Delta_j(x,y) \ne 0$ in (13.39), and there are at most $C$ indices $j$ for which this happens. For such $j$, we proceed as in (13.17) and get that

$$\begin{aligned}
|\Delta_j(x,y)| &\le \psi_j(x)|\widetilde{g}_j(x) - \widetilde{f}(x) - \widetilde{g}_j(y) + \widetilde{f}(y)| + |\psi_j(x) - \psi_j(y)|\, |\widetilde{g}_j(y) - \widetilde{f}(y)| \\
&\le \psi_j(x)|\widetilde{g}_j(x) - \widetilde{f}(x) - \widetilde{g}_j(y) + \widetilde{f}(y)| + 4(1-a)^{-1}t^{-1}|x-y|\, |\widetilde{g}_j(y) - \widetilde{f}(y)| \\
&\le \lambda \, \frac{1+2\Lambda^2|f|_{lip}}{\Lambda} \, \psi_j(x)|x-y| + 4(1-a)^{-1}|x-y|(11\lambda\Lambda^2(1+|f|_{lip})\varepsilon + \lambda\Lambda^{-1}\alpha) \qquad (13.41)
\end{aligned}$$

because $\psi_j$ is still $4[(1-a)t]^{-1}$-Lipschitz, $\widetilde{f}$ is $\lambda\Lambda$-Lipschitz, $\widetilde{g}_j$ is $\lambda\,\frac{1+\Lambda^2|f|_{lip}}{\Lambda}$-Lipschitz on $E^{\varepsilon t} \cap 2B_j$ (by (12.96)), and by (12.88) (if $y \in B(x_j, \frac{1+a}{2}t)$; otherwise $\widetilde{g}_j(y) = \widetilde{f}(y)$ directly by (12.77)).

When we sum this over $j$, the first term gives a total contribution which is bounded by $\lambda\,\frac{1+2\Lambda^2|f|_{lip}}{\Lambda}\,|x-y|$, and the second one contributes at most $\lambda\Lambda^{-1}|x-y|$, if $\varepsilon$ and $\alpha$ are small enough (depending on $a$); (13.40) and (13.38) follow.

We deduce from (13.38) and (13.29) that

$$g \ \text{ is } (2+3\Lambda^2|f|_{lip})\text{-Lipschitz on } E^{\varepsilon t} \cap V'. \tag{13.42}$$

Return to what we did in the rigid case. We still have (13.18) in the present Lipschitz case (see (13.31) and (13.34)). Let us check now that (13.19) also holds now, i.e., that

$$g\Big(\bigcup_{j\in J_1} B_j \setminus R\Big) \subset \bigcup_{l\in\mathcal{L}} \bigcup_{Q\in\mathcal{F}_l} Q = \bigcup_{Q\in\mathcal{F}} Q. \tag{13.43}$$

As before, any $x \in \bigcup_{j\in J_1} B_j \setminus R$ lies in some $aB_j$ (the first one), for which $\widetilde{g}(x) = \widetilde{g}_j(x) = \widetilde{\pi}_j(\widetilde{f}(x))$ (by (13.11) and (13.28)), and it is enough to check that $g(x) \in Q_j$, or equivalently that $\widetilde{\pi}_j(\widetilde{f}(x)) \in \widetilde{Q}_j$, since $Q_j = \lambda^{-1}\psi^{-1}(\widetilde{Q}_j)$ by (12.22) and $g(x) = \lambda^{-1}\psi^{-1}(\widetilde{g}(x))$ by (13.29).

Recall from the definition above (12.77) that $\widetilde{\pi}_j$ is the orthogonal projection onto the plane $\widetilde{P}_j$ that was defined below (12.49), subject to the constraint (12.23) for some $y_l$ such that $f(x_j) \in D_l$ (see above (12.48)). That is,

$$\widetilde{P}_j \text{ meets } B(\widetilde{y}_l, 2\lambda\Lambda(1+|f|_{lip})t)). \tag{13.44}$$

Since $\widetilde{Q}_j = \widetilde{P}_j \cap B(\widetilde{y}_l, 20\lambda\Lambda^2(1+|f|_{lip})t)$ by (12.22), it is enough to check that

$$|\widetilde{\pi}_j(\widetilde{f}(x)) - \widetilde{y}_l| < 20\lambda\Lambda^2(1+|f|_{lip})t. \tag{13.45}$$

But

$$
\begin{aligned}
|\widetilde{f}(x)-\widetilde{y}_l| &= |\psi(\lambda f(x))-\psi(\lambda y_l)| \leq \lambda\Lambda|f(x)-y_l| \\
&\leq \lambda\Lambda(|f(x)-f(x_j)|+|f(x_i)-y_l|) \\
&\leq \lambda\Lambda(a|f|_{lip}t+t) \leq \lambda\Lambda(1+|f|_{lip})\,t
\end{aligned} \tag{13.46}
$$

by (12.75) and the definition of $\widetilde{y}_l=\psi(\lambda y_l)$ above (12.22), and because $x\in aB_i$.

Let $\pi$ denote the orthogonal projection onto the plane parallel to $\widetilde{P}_j$, but through $\widetilde{y}_l$; then $||\pi-\widetilde{\pi}_j||_\infty \leq 2\lambda\Lambda(1+|f|_{lip})t)$ by (13.44), and

$$
\begin{aligned}
|\widetilde{\pi}_j(\widetilde{f}(x))-\widetilde{y}_l| &\leq ||\pi-\widetilde{\pi}_j||_\infty+|\pi(\widetilde{f}(x))-\widetilde{y}_l| \leq ||\pi-\widetilde{\pi}_j||_\infty+|\widetilde{f}(x)-\widetilde{y}_l| \\
&\leq 2\lambda\Lambda(1+|f|_{lip})t)+\lambda\Lambda(1+|f|_{lip})\,t
\end{aligned} \tag{13.47}
$$

by (13.46). This is better than (13.45), and (13.43) follows.

Clearly (13.23) still holds, even though with a larger constant $C(\alpha,f)$, now by (13.43), (12.22), and (as before) (12.20).

The last estimates (13.24)-(13.26) stay the same; they do not even involve $\psi$. †

## 14. Step 3. Places where $f$ has a very contracting direction, and the $B_j$, $j\in J_2$.

At the beginning of Section 12, we were left with a set $X_5\subset X_0$, such that $\mathcal{H}^d(X_0\setminus X_5)\leq 4\eta$ by (11.48). Set (as in (13.1))

$$
V=\bigcup_{j\in J_1} B_j=\bigcup_{j\in J_1} B(x_j,t), \tag{14.1}
$$

which contains $X_5\cap X_N(\delta_4)$ by (12.9). In principle, we already took good care of $V$ in Sections 12 and 13, by (13.23) and (13.26). We also know from (12.3) and (12.4) that

$$
\mathcal{H}^d\big([X_5\cap f^{-1}(Y_N)]\setminus X_N(\delta_4))\big)=\mathcal{H}^d\big([X_5\cap f^{-1}(Y_N)]\setminus f^{-1}(Y_N(\delta_4))\big)\leq\eta, \tag{14.2}
$$

where $Y_N$ is as in (12.1). Next consider

$$
X_6=X_5\setminus\Big[f^{-1}(Y_N)\cup V\Big]. \tag{14.3}
$$

If $x\in X_5\setminus[V\cup X_6]$, then it lies in $f^{-1}(Y_N)$ ($V$ is not allowed) and, since it does not lie in $X_5\cap X_N(\delta_4)$ (which is contained in $V$ too), it lies in the set of (14.2). So

$$
\mathcal{H}^d(X_0\setminus[V\cup X_6])\leq\mathcal{H}^d(X_0\setminus X_5)+\mathcal{H}^d(X_5\setminus[V\cup X_6])\leq 5\eta, \tag{14.4}
$$

and we may now turn to $X_6$.

Our next target is the set of points $x\in X_6$ where $A_x$ has a very contracting direction along $P_x$. That is, we want to control the set

$$
X_7=\big\{x\in X_6\,;\ \text{there is a unit vector } \nu\in P'_x \text{ such that } |DA_x(\nu)|\leq\gamma\big\}, \tag{14.5}
$$

where $P'_x$ denotes the vector space parallel to $P_x$, $DA_x$ is the differential of $A_x$, and $\gamma < 1$ is another very small positive constant, to be chosen later.

The following is very similar to Lemma 4.60 in [D2], whose fairly standard proof applies here too (so we skip it).

**Lemma 14.6.** *We can find a finite collection of balls $B_j = B(x_j, r_j)$, $j \in J_2$, with the following properties:*

$$x_j \in X_7 \text{ and } 0 < r_j \leq \delta_6 \text{ for } j \in J_2, \tag{14.7}$$

*where $\delta_6$ is as in (12.7),*

$$\text{the } \overline{B}_j,\ j \in J_2 \text{ are disjoint, and do not meet } \bigcup_{j \in J_1} B\big(x_j, \frac{(1+a)t}{2}\big), \tag{14.8}$$

*and*

$$\mathcal{H}^d\big(X_7 \setminus \bigcup_{j \in J_2} \overline{B}_j\big) \leq \eta. \tag{14.9}$$

This time, since the $\overline{B}_j$ are disjoint, we shall not need a subtle partition of unity as before, and we can define functions $g_j$ independently. Also, what we intend to do here in the $B_j$, $j \in J_2$, will be independent of what we did in the $B\big(x_j, \frac{(1+a)t}{2}\big)$. Again we start with the rigid assumption.

We set $P_j = P_{x_j}$, $Q_j = A_{x_j}(P_j)$, denote by $\pi_j$ the orthogonal projection on $Q_j$, and define $g_j$ by the same formulae (12.59)-(12.61) as before (with $t$ replaced by $r_j$).

Notice that $g_j(x) \in [f(x), \pi_j(f(x))]$; then

$$|g_j(x) - f(x)| \leq |\pi_j(f(x)) - f(x)| \leq |f|_{lip}|x - x_j| \tag{14.10}$$

because $\pi \circ f - f = (\pi - I) \circ f$ is $|f|_{lip}$-Lipschitz and vanishes at $x_j$ (recall that $Q_j$ goes through $f(x_j)$ because the definition (11.37) says that $A_{x_j}(x_j) = f(x_j)$). When $x \in B_j$, we get that $|g_j(x) - f(x)| \leq |f|_{lip}\, r_j$. When $x \in U \setminus B_j$, the analogue of (12.59) says that $g_j(x) = f(x)$. Altogether,

$$||g_j - f||_\infty \leq |f|_{lip}\, r_j \leq |f|_{lip}\, \delta_6 \tag{14.11}$$

by (14.7) and as in (12.62). Also, the the proof of (12.66) (which could also be simplified here) says that

$$g_j \text{ is } (1 + |f|_{lip})\text{-Lipschitz on } E^{\varepsilon r_j}. \tag{14.12}$$

For $k$ large enough ($k$ is the index in our initial sequence of quasiminimal sets $E_k$, which converges to $E$),

$$\mathrm{dist}(z, P_j \cap aB_j) \leq 2\varepsilon r_j \text{ for } z \in E_k \cap aB_j, \tag{14.13}$$

by (11.45). Set $\mathcal{E} = A_{x_j}(P_j \cap aB_j) \subset Q_j$, let $z \in E_k \cap aB_j$ be given, and let $w \in P_j \cap aB_j$ be such that $|z - w| \leq 3\varepsilon r_j$; then

$$\begin{aligned} \operatorname{dist}(f(z), \mathcal{E}) &\leq |f(z) - f(w)| + |f(w) - A_{x_j}(w)| + \operatorname{dist}(A_{x_j}(w), \mathcal{E}) \\ &= |f(z) - f(w)| + |f(w) - A_{x_j}(w)| \\ &\leq 3\varepsilon r_j |f|_{lip} + \varepsilon |w - x_j| \leq (1 + 3|f|_{lip})\varepsilon r_j \end{aligned} \tag{14.14}$$

because $A_{x_j}(w) \in A_{x_j}(P_j \cap aB_j) = \mathcal{E}$, by (11.46), and again because $w \in P_j \cap aB_j$. By (12.60), $g_j(z) = \pi_j(f(z))$, so it lies in $Q_j$, and (by (14.14)) in a $(1+3|f|_{lip})\varepsilon r_j$-neighborhood $\mathcal{E}'$ of $\mathcal{E}$ in $Q_j$. Now $x_j \in X_7$, so $\mathcal{E}$ is an ellipsoid, with a shortest axis of length at most $2\gamma a r_j$ (by (14.5)), and other axes of length at most $2|f|_{lip} a r_j$. Then

$$\begin{aligned} \mathcal{H}^d(g_j(E_k \cap aB_j)) \leq \mathcal{H}^d(\mathcal{E}') &\leq C(1 + |f|_{lip})^{d-1}(\gamma + (1 + 3|f|_{lip})\varepsilon) r_j^d \\ &\leq C(1 + |f|_{lip})^{d-1} \gamma r_j^d \end{aligned} \tag{14.15}$$

again for $k$ large enough and if $\varepsilon$ is small enough, depending on $\gamma$.

Set $R_j = B_j \setminus aB_j$, as before. Then

$$\mathcal{H}^d(E \cap \overline{R}_j) = \mathcal{H}^d(E \cap \overline{B}_j \setminus aB_j) \leq C(1 - a) r_j^d \tag{14.16}$$

by the same proof as for (13.26). Since

$$r_j < \delta_6 < \frac{1}{10}\delta_1 = \frac{1}{10} \operatorname{dist}(X_1, \mathbb{R}^n \setminus W_f)) \leq \operatorname{dist}(x_j, \mathbb{R}^n \setminus W_f)) \tag{14.17}$$

by (14.7), (12.7), and (11.22) (and as in (12.11)), we also get that

$$r_j^d \leq C\mathcal{H}^d(E \cap B_j) \ \text{ for } j \in J_2, \tag{14.18}$$

by the local Ahlfors-regularity of $E$, and where the use of Proposition 4.1 is justified as for (12.12). Then

$$\sum_{j \in J_2} r_j^d \leq C \sum_{j \in J_2} \mathcal{H}^d(E \cap B_j) \leq C\mathcal{H}^d\big(E \cap \bigcup_{j \in J_2} B_j\big) \leq C\mathcal{H}^d(E \cap W_f) \tag{14.19}$$

by (14.19), (14.8), and (14.17), and now (14.16) implies that

$$\begin{aligned} \mathcal{H}^d(E \cap \bigcup_{j \in J_2} \overline{R}_j) \leq \sum_{j \in J_2} \mathcal{H}^d(E \cap \overline{B}_j \setminus aB_j) &\leq C(1 - a) \sum_{j \in J_2} r_j^d \\ &\leq C(1 - a)\mathcal{H}^d(E \cap W_f) = C(1 - a)\mathcal{H}^d(X_0) \end{aligned} \tag{14.20}$$

(because $E \cap W_f = X_0$ by (11.20)).

So we should not worry too much about the $R_j$, and since we have some control on the $aB_j$ by (14.15) and (14.19), we shall now concentrate on

$$X_8 = X_6 \setminus \big[X_7 \cup \bigcup_{j \in J_2} B_j\big]. \tag{14.21}$$

Let $X_9$ be a compact subset of $X_8$ such that $\mathcal{H}^d(X_8 \setminus X_9) \leq \eta$. Since

$$X_6 \subset X_8 \cup \Big( \bigcup_{j\in J_2} B_j \Big) \cup X_7 \subset X_8 \cup \Big( \bigcup_{j\in J_2} \overline{B}_j \Big) \cup \Big( X_7 \setminus \bigcup_{j\in J_2} \overline{B}_j \Big) \tag{14.22}$$

we get that

$$\mathcal{H}^d\big(X_6 \setminus \big[X_9 \cup \big( \bigcup_{j\in J_2} \overline{B}_j \big)\big]\big) \leq \mathcal{H}^d(X_8 \setminus X_9) + \mathcal{H}^d\big(X_7 \setminus \bigcup_{j\in J_2} \overline{B}_j\big) \leq 2\eta \tag{14.23}$$

by (14.9). Let us deduce from this and (14.4) that

$$\mathcal{H}^d\big(X_0 \setminus \big[X_9 \cup \big( \bigcup_{j\in J_1\cup J_2} \overline{B}_j \big)\big]\big) \leq 7\eta. \tag{14.24}$$

Let $Z, Z', Z''$ the sets in the left-hand sides of (14.24), (14.23) and (14.4) respectively; we want to check that $Z \subset Z' \cup Z''$. Let $x \in Z \setminus Z''$ be given. Then $x \in X_0$, and so $x \in V \cup X_6$. But $x \in V = \bigcup_{j\in J_1} B_j$ is impossible because $x \in Z$ (also see the definition (14.1)), hence $x \in X_6$. Then $x \in Z'$, as needed. So (14.24) holds.

† Under the Lipschitz assumption, we need to modify the definition of $g_j$. We still set $P_j = P_{x_j}$, but we consider $\widetilde{Q}_j = \widetilde{A}_{x_j}(P_j)$ (where $\widetilde{A}_{x_j}$ is the affine approximation of $\widetilde{f}$, as in (12.37)-(12.39)). We denote by $\widetilde{\pi}_j$ the orthogonal projection onto $\widetilde{Q}_j$, and will define $\widetilde{g}_j$ as we did near (12.77). Again we first work in the set $U_{int}$ defined by (12.72), because this is where we extended $\widetilde{f}$ (see (12.75)). Notice that $U_{int}$ contains all the $2B_j$, $j \in J_3$, by proof of (12.76) (just use (14.7) instead of (12.7)).

We define $\widetilde{g}_j$ on $U_{int}$ with the same formulas (12.77)-(12.79) as before, with the choice of $\widetilde{\pi}_j$ that we just made, and $t$ replaced with $r_j$.

We continue, as in Section 12, with estimates for $x \in 2B_j \subset U_{int}$. First observe that

$$|\widetilde{g}_j(x) - \widetilde{f}(x)| \leq |\widetilde{\pi}_j(\widetilde{f}(x)) - \widetilde{f}(x)| \leq \lambda\Lambda |f|_{lip} |x - x_j| \tag{14.25}$$

because $\widetilde{g}_j(x) \in [\widetilde{f}(x), \widetilde{\pi}_j(\widetilde{f}(x))]$ by (12.77)-(12.79), and because $(\widetilde{\pi}_j - I) \circ \widetilde{f}$ is $\lambda\Lambda|f|_{lip}$-Lipschitz and vanishes at $x_j$ by definition of $\widetilde{A}_{x_j}$ and $\widetilde{\pi}_j$. Next

$$\begin{aligned} |\widetilde{g}_j(x) - \widetilde{f}(x_j)| &\leq |\widetilde{g}_j(x) - \widetilde{f}(x)| + |\widetilde{f}(x) - \widetilde{f}(x_j)| \\ &\leq 2\lambda\Lambda|f|_{lip}|x - x_j| \leq 2\lambda\Lambda|f|_{lip}\, r_j \leq 2\lambda\Lambda|f|_{lip}\, \delta_6 \end{aligned} \tag{14.26}$$

by (14.25), because $\widetilde{f}$ is $\lambda\Lambda|f|_{lip}$-Lipschitz, and by (14.7). But

$$\mathrm{dist}(f(x_j), \mathbb{R}^n \setminus U) \geq \mathrm{dist}(\widehat{W}, \mathbb{R}^n \setminus U) = \delta_0 \geq 10\Lambda^2(1 + |f|_{lip})\delta_6 \tag{14.27}$$

because $x_j \in E_0 = E \cap W_f$ (see (11.20) and (11.19)), so $f(x_j) \in \widehat{W}$ (see (2.1) and (2.2)), and by (12.6) and (12.7). Hence

$$\begin{aligned} \mathrm{dist}(\widetilde{g}_j(x), \mathbb{R}^n \setminus B(0,1)) &\geq \mathrm{dist}(\widetilde{f}(x_j), \mathbb{R}^n \setminus B(0,1)) - 2\lambda\Lambda|f|_{lip}\, \delta_6 \\ &\geq \lambda\Lambda^{-1}\, \mathrm{dist}(f(x_j), \mathbb{R}^n \setminus U) - 2\lambda\Lambda|f|_{lip}\, \delta_6 \\ &\geq 10\lambda\Lambda(1 + |f|_{lip})\delta_6 - 2\lambda\Lambda|f|_{lip}\, \delta_6 \geq 8\lambda\Lambda(1 + |f|_{lip})\delta_6 \end{aligned} \tag{14.28}$$

by (14.26), (12.75), and (14.27). Thus $\widetilde{g}_j(x) \in B(0,1)$ when $x \in 2B_j$.

When $x \in U_{int} \setminus 2B_j$, and even when $x \in U_{int} \setminus B(x_j, \frac{1+a}{2} r_j)$, (12.77) and (12.75) say that $\widetilde{g}_j(x) = \widetilde{f}(x) = \psi(\lambda f(x))$; then of course $\widetilde{g}_j(x) \in \psi(\lambda U) = B(0,1)$. Thus in both cases $\widetilde{g}_j(x) \in B(0,1)$, and we can define $g_j$ on $U_{int}$ by

$$g_j(x) = \lambda^{-1} \psi^{-1}(\widetilde{g}_j(x)) \quad \text{for } x \in U_{int}, \tag{14.29}$$

(compare with (12.85)). This formula yields $g_j(x) = f(x)$ for $x \in U_{int} \setminus B(x_j, \frac{1+a}{2} r_j)$, and we may even extend it by deciding that

$$g_j(x) = f(x) \quad \text{for } x \in U \setminus B(x_j, \frac{1+a}{2} r_j) \tag{14.30}$$

(now compare with (12.86)). But in fact the values of $g_j$ on $U \setminus U_{int}$, or even $U \setminus 2B_j$, will never matter.

Return to the modifications concerning this section. The analogue of (14.11) is now

$$||g_j - f||_\infty \leq \Lambda^2 |f|_{lip} r_i \leq \Lambda^2 |f|_{lip} \delta_6, \tag{14.31}$$

which follows from (14.30), (14.25), and (14.29). Then we worry about the Lipschitz estimate (14.12). The fact that

$$\widetilde{g}_j \text{ is } \lambda \, \frac{1 + \Lambda^2 |f|_{lip}}{\Lambda}\text{-Lipschitz on } E^{\varepsilon t} \cap 2B_j \tag{14.32}$$

is proved as (12.96) or (12.66) (with some simplifications), and implies that

$$g_j \text{ is } (1 + \Lambda^2 |f|_{lip})\text{-Lipschitz on } E^{\varepsilon t} \cap 2B_j; \tag{14.33}$$

will be good enough to take replace (14.12).

Observe that (14.13) still holds with the same proof. Next we generalize (14.14) and (14.15). Set $\widetilde{\mathcal{E}} = \widetilde{A}_{x_j}(P_j \cap aB_j) \subset \widetilde{Q}_j$ (recall that we set $\widetilde{Q}_j = \widetilde{A}_{x_j}(P_j)$). Let $z \in E_k \cap aB_j$ be given, and use (14.13) to find $w \in P_j \cap aB_j$ such that $|z - w| \leq 3\varepsilon r_j$; then

$$\begin{aligned} \operatorname{dist}(\widetilde{f}(z), \widetilde{\mathcal{E}}) &\leq |\widetilde{f}(z) - \widetilde{f}(w)| + |\widetilde{f}(w) - \widetilde{A}_{x_j}(w)| + \operatorname{dist}(\widetilde{A}_{x_j}(w), \widetilde{\mathcal{E}}) \\ &= |\widetilde{f}(z) - \widetilde{f}(w)| + |\widetilde{f}(w) - \widetilde{A}_{x_j}(w)| \\ &\leq 3\varepsilon r_j |\widetilde{f}|_{lip} + \lambda\varepsilon |w - x_j| \leq (1 + 3\Lambda |f|_{lip}) \lambda \varepsilon r_j \end{aligned} \tag{14.34}$$

because $\widetilde{A}_{x_j}(w) \in \widetilde{A}_{x_j}(P_j \cap aB_j) = \widetilde{\mathcal{E}}$, by (12.52), by (12.75), and because $w \in P_j \cap aB_j$.

By the analogue of (12.78), $\widetilde{g}_j(z) = \widetilde{\pi}_j(\widetilde{f}(z))$, so it lies in $\widetilde{Q}_j$ (by definition of $\widetilde{\pi}_j$), and (by (14.34)) in a $(1 + 3\Lambda |f|_{lip}) \lambda \varepsilon r_j$-neighborhood $\widetilde{\mathcal{E}}'$ of $\widetilde{\mathcal{E}}$ in $\widetilde{Q}_j$. So we just checked that $\widetilde{g}_j(E_k \cap aB_j) \subset \widetilde{\mathcal{E}}'$ for $k$ large.

Now $\widetilde{\mathcal{E}}$ is an ellipsoid in $\widetilde{Q}_j$, its axes all have lengths smaller than $2\lambda\Lambda |f|_{lip} r_j$ by (12.37), and one of them is much shorter. Indeed, let $\nu$ be a unit vector in $P'_{x_j}$ such that

$|DA_{x_j}(\nu)| \leq \gamma$; such a vector exists because $x_j \in X_7$ (see (14.5) and (14.7)); notice that both $f$ and $\widetilde{f}$ are differentiable at $x_j$ in the direction of $\nu$, and recall that $\widetilde{f}(x) = \psi(\lambda f(x))$ near $x_j$. Then

$$\begin{aligned} |D\widetilde{A}_{x_j}(\nu)| &= \big| \lim_{t\to 0} t^{-1}[\widetilde{A}_{x_j}(x_j + t\nu) - \widetilde{A}_{x_j}(x_j)]\big| = \big| \lim_{t\to 0} t^{-1}[\widetilde{f}(x_j + t\nu) - \widetilde{f}(x_j)]\big| \\ &\leq \lambda\Lambda \big| \limsup_{t\to 0} t^{-1}[f(x_j + t\nu) - f(x_j)]\big| \\ &= \lambda\Lambda \big| \limsup_{t\to 0} t^{-1}[A_x(x_j + t\nu) - A_x(x_j)]\big| = \lambda\Lambda |DA_{x_j}(\nu)| \leq \lambda\Lambda\gamma \end{aligned} \tag{14.35}$$

by (12.39), because $\widetilde{A}_{x_j}(x_j) = \widetilde{f}(x_j)$, and by (11.40); hence the smallest axis of $\widetilde{\mathcal{E}}$ has length at most $2\lambda\Lambda\gamma r_j$. Thus

$$\mathcal{H}^d(\widetilde{g}_j(E_k \cap aB_j)) \leq \mathcal{H}^d(\widetilde{\mathcal{E}}') \leq C\lambda^d\Lambda^d(1 + \Lambda|f|_{lip})^d \gamma r_j^d \tag{14.36}$$

for $k$ large, and hence also

$$\mathcal{H}^d(g_j(E_k \cap aB_j)) \leq \Lambda^d\lambda^{-d}\mathcal{H}^d(\widetilde{g}_j(E_k \cap aB_j)) \leq C\Lambda^{2d}(1 + \Lambda|f|_{lip})^d \gamma r_j^d \tag{14.37}$$

by (14.29), and as in (14.15).

Finally the estimates (14.16)-(14.24) go through with only minor modifications. †

## 15. Step 4. The remaining main part of $X_0$

Return to the rigid assumption. We care about $X_9$ now (see near (14.22)). Set $Y_9 = f(X_9)$ and, for $y \in Y_9$,

$$Z(y) = X_9 \cap f^{-1}(y) = \big\{x \in X_9\,;\, f(x) = y\big\}. \tag{15.1}$$

Notice that $Z(y)$ has at most $N$ points, because $X_9 \subset X_6 \subset X_5 \setminus f^{-1}(Y_N)$ (see (14.21), the definition of $X_9$ just below (14.21), (14.3), and (12.1)). We claim that for each $y \in Y_9$, there is a positive radius $r(y)$ such that

$$X_9 \cap f^{-1}(B(y,r)) \subset \bigcup_{x \in Z(y)} B(x, 2\gamma^{-1}r) \ \text{ for } 0 < r \leq r(y). \tag{15.2}$$

Here $\gamma$ is the same as in the definition (14.5) of $X_7$. The proof uses a small compactness argument (to make sure that it is enough to control $f$ near the $x \in Z(y)$, (11.46) (to show that $A_x$ controls $f$ near $x \in Z(y)$), and the fact that we excluded $X_7$ in (14.21) (to exclude points near $x \in Z(y)$ that don't lie in $B(x, 2\gamma^{-1}r)$). We don't repeat it here because it is the same as in Lemma 4.69 in [D2].

Since $X_9$ is compact and disjoint from the finite collection of $B_j$, $j \in J_1 \cup J_2$ (see (14.21), (14.3), and (14.1)), the number

$$\delta_7 = \operatorname{dist}\big(X_9, \bigcup_{j \in J_1\cup J_2} \frac{1+a}{2}\, B_j\big) \tag{15.3}$$

is positive. Just by making $r(y)$ smaller if needed, we may assume that for each $y \in Y_9$,

$$0 < r(y) < \frac{\gamma}{4} \operatorname{Min}\big(\delta_6, \delta_7, \operatorname{Min}\{|x - x'| \,;\, x, x' \in Z(y), x \neq x'\}\big). \tag{15.4}$$

Then

$$B(x, 2\gamma^{-1} r(y)) \cap \frac{1+a}{2} B_j = \emptyset \text{ for } y \in Y_9,\, x \in Z(y), \text{ and } j \in J_1 \cup J_2, \tag{15.5}$$

and (for each $y \in Y_9$)

$$\text{the balls } B(x, 2\gamma^{-1} r(y)),\ x \in Z(y), \text{ are disjoint.} \tag{15.6}$$

We'll need some uniformity (i.e., to know that $r(y)$ is not too small), so let us choose a new small constant $\delta_8 \in (0, \delta_7] > 0$ such that if we set

$$Y_{10} = \big\{y \in Y_9 \,;\, r(y) > \delta_8\big\} \text{ and } X_{10} = X_9 \cap f^{-1}(Y_{10}), \tag{15.7}$$

then

$$H^d(X_9 \setminus X_{10}) \leq \eta. \tag{15.8}$$

As usual, such a $\delta_8$ exists, because the monotone union of the sets $X_{10}$, when $\delta_8$ tends to 0, is $X_9$. Next set

$$Y_{11} = \big\{y \in Y_{10} \,;\text{ all the affine planes } Q_x = A_x(P_x),\ x \in Z(y), \text{ coincide}\big\}. \tag{15.9}$$

Notice that the $Q_x$ are $d$-planes, because we excluded the case when $A_x$ has a very contracting direction in $P_x$. Also set

$$X_{11} = X_{10} \cap f^{-1}(Y_{11}). \tag{15.10}$$

As we check in (4.77) of [D2],

$$\mathcal{H}^d(X_{10} \setminus X_{11}) = 0. \tag{15.11}$$

The same proof is valid here; we sketch it to prevent the reader from worrying. We use the fact that $f(E)$ is rectifiable, and prove that it does not have any approximate tangent plane at points of $Y_{10} \setminus Y_{11}$ (too many tangent directions exist), so $\mathcal{H}^d(Y_{10} \setminus Y_{11}) = 0$. For this last, we use again the fact that we excluded contracting directions. For the accounting, we also use the fact that $f$ is at most $N$-to-1 on $X_{10}$, to return from $Y_{10}$ to $X_{10}$ and prove (15.11). Incidentally, (4.77) of in [D2] is wrongly referred to as (4.78) at the end of the proof in [D2] (sorry!).

We want to cover $X_{11}$, but it will be more efficient to cover $Y_{11}$ first. We choose a finite collection of balls $D_j = B(y_j, r_j)$, $j \in J_3$, so that

$$y_j \in Y_{11} \text{ and } 0 < r_j < \delta_8 \text{ for } j \in J_3, \tag{15.12}$$

$$(15.13) \qquad \text{the } \overline{D}_j,\ j \in J_3, \text{ are disjoint}$$

and

$$(15.14) \qquad \mathcal{H}^d\big(X_{11} \setminus f^{-1}\big(\bigcup_{j\in J_3} \overline{D}_j\big)\big) \leq \eta.$$

See Lemma 4.79 in [D2], where (15.14) is deduced form a similar estimate on $Y_9$, using again the fact that we excluded contracting directions and $f$ is at most $N$-to-1 on $X_{11}$.

Observe that for $j \in J_3$,

$$(15.15) \qquad r_j < \delta_8 \leq r(y_j) \leq \frac{\gamma}{4} \,\mathrm{Min}\,\big(\delta_6, \delta_7, \mathrm{Min}\{|x-x'|\,;\, x, x' \in Z(y_j), x \neq x'\}\big)$$

by (15.12), because $y_j \in Y_{10}$, and by (15.7) and (15.4).

We want to modify $f$ on the sets $f^{-1}(D_j)$, as we did in the balls $B_j$, $j \in J_1 \cup J_2$. We shall be able to proceed independently on each $f^{-1}(D_j)$, because the $D_j$ are disjoint by (15.13). In fact, for each $j$ we shall only modify $f$ on the $f^{-1}(D_j) \cap B(x, 2\gamma^{-1}r_j)$, $x \in Z(y_j)$, which are disjoint by (15.15) and contain the interesting part of $f^{-1}(D_j)$ by (15.2).

Fix $j \in J_3$. By definition, the $d$-planes $A_x(P_x)$, $x \in Z(y_j)$, are all equal; let us call $Q_j$ this common $d$-plane that we get. For each $x \in Z(y_j)$, set

$$(15.16) \qquad E(x) = P_x \cap A_x^{-1}(Q_j \cap D_j).$$

This is a $d$-dimensional ellipsoid in $P_j$, whose axes have lengths between $2|f|_{lip}^{-1} r_j$ and $2\gamma^{-1} r_j$, by (11.36) and the definition (14.5) of $X_7$ (which we excluded in (14.21)). The analogues of $aB_j$, $\frac{1+a}{2}B_j$, and $B_j$ in the previous sections will be

$$(15.17) \qquad B_{j,x}^- = \big\{z \in \mathbb{R}^n\,;\, \mathrm{dist}(z, aE(x)) < 20^{-1}(1+|f|_{lip})^{-1}(1-a)r_j\big\}$$

and

$$(15.18) \qquad B_{j,x}^+ = \big\{z \in \mathbb{R}^n\,;\, \mathrm{dist}(z, aE(x)) \leq 10^{-1}(1+|f|_{lip})^{-1}(1-a)r_j\big\},$$

and

$$(15.19) \qquad B_{j,x} = \big\{z \in \mathbb{R}^n\,;\, \mathrm{dist}(z, (2-a)E(x)) \leq (1-a)r_j\big\}.$$

Observe that

$$(15.20) \qquad B_{j,x}^- \subset B_{j,x}^+ \subset B_{j,x} \subset B(x, \frac{3}{2}\gamma^{-1}r_j)$$

if $(1-a)$ is small enough, depending on $|f|_{lip}$ and $\gamma$. Recall from Remark 11.17 that $a$ is allowed to depend on $\gamma$. By (15.20) and (15.15),

$$(15.21) \qquad \text{for each } j \in J_3, \text{ the } B_{j,x},\ x \in Z(y_j), \text{ are disjoint.}$$

Let us also check that

$$f(B_{j,x}^{+}) \subset \frac{1+a}{2} D_j. \tag{15.22}$$

Let $z \in B_{j,x}^{+}$ be given, and let $w$ be a point of $aE(x)$ such that

$$|z-w| \leq 10^{-1}(1+|f|_{lip})^{-1}(1-a)r_j. \tag{15.23}$$

Observe that

$$\begin{aligned} |w-x| &\leq |w-z|+|z-x| \leq 10^{-1}(1+|f|_{lip})^{-1}(1-a)r_j + \frac{3}{2}\gamma^{-1}r_j \\ &\leq 2\gamma^{-1}r_j \leq \delta_6 < \delta_3/10 \end{aligned} \tag{15.24}$$

by (15.23), (15.20), if $a$ is small enough, and by (15.15) and (12.7). In addition, $w \in P_x$, so (11.46) applies and says that

$$|f(w) - A_x(w)| \leq \varepsilon|w-x| \leq 2\varepsilon\gamma^{-1}r_j. \tag{15.25}$$

Now

$$\begin{aligned} |f(z)-y_j| &\leq |f(z)-f(w)| + |f(w)-A_x(w)| + |A_x(w)-y_j| \\ &\leq |z-w||f|_{lip} + 2\varepsilon\gamma^{-1}r_j + |A_x(w)-y_j| \\ &\leq 10^{-1}(1-a)r_j + 2\varepsilon\gamma^{-1}r_j + ar_j < \frac{1+a}{2}r_j \end{aligned} \tag{15.26}$$

by (15.25), (15.23), because $A_x(w) \in aD_j$ by definition of $E(x)$ (see (15.16)), and if $\varepsilon$ is small enough; (15.22) follows.

Because of (15.22), (15.21), and (15.13),

$$\text{the } B_{j,x}^{+},\ j \in J_3 \text{ and } x \in Z(y_j), \text{ are all disjoint,} \tag{15.27}$$

even for different values of $j$, because the $D_j$ are disjoint. Set

$$R_{j,x} = B_{j,x}^{+} \setminus B_{j,x}^{-} \tag{15.28}$$

by analogy with the previous constructions. Denote by $\pi_j$ the orthogonal projection onto $Q_j$, and define $g_{j,x}$ as follows: set

$$g_{j,x}(z) = \pi_j(f(z)) \ \text{ when } z \in B_{j,x}^{-}, \tag{15.29}$$

$$g_{j,x}(z) = f(z) \ \text{ when } z \in \mathbb{R}^n \setminus B_{j,x}^{+}, \tag{15.30}$$

and interpolate in the usual linear way in the remaining intermediate region $R_{j,x}$. That is, set

$$g_{j,x}(z) = (1-\beta(z))\pi_j(f(z)) + \beta(z)f(z) \quad \text{for } z \in R_{j,x}, \tag{15.31}$$

with

$$\beta(z) = \frac{20(1+|f|_{lip})\,\mathrm{dist}(z, aE(x))}{(1-a)r_j} - 1. \tag{15.32}$$

We also define a function $g_j$: we set $g_j = g_{j,x}$ on each $B^+_{j,x}$, $x \in Z(y_j)$, and $g_j(z) = f(z)$ on the rest of $\mathbb{R}^n$; the definition is coherent, by (15.30) and (15.21), and we even get a lipschitz mapping (possibly with very bad constants). Let us check that

$$||g_j - f||_\infty \leq r_j \leq \delta_6. \tag{15.33}$$

The second inequality comes from (15.15). By (15.31), it is enough to check that $|\pi_j(f(z)) - f(z)| \leq r_j$ for $z \in B^+_{j,x}$, and this is clear because $f(z) \in D_j$ by (15.22), and $D_j$ is centered on $Q_j$.

When $z \in B^+_{j,x} \cap E^{\varepsilon r_j} = \{z \in \mathbb{R}^n\,;\, \mathrm{dist}(z,E) \leq \varepsilon r_j\}$, the estimate improves: we can choose $w \in E$ such that $|w-z| \leq \varepsilon r_j$, and, by (11.45), $p \in P_x$ such that $|p-w| \leq \varepsilon|w-x| \leq 2\varepsilon r_j$ (with the same sort of justification as (15.24) for (15.25)). Thus $|p-z| \leq 3\varepsilon r_j$ and $|p-x| \leq |p-z| + |z-x| \leq 3\varepsilon r_j + \frac{3}{2}\gamma^{-1}r_j \leq 2\gamma^{-1}r_j$, by (15.20) and if $\varepsilon$ is small enough. Then

$$\begin{aligned}
|g_j(z) - f(z)| &\leq |\pi_j(f(z)) - f(z)| \leq \mathrm{dist}(f(z), Q_j) \\
&\leq \mathrm{dist}(A_x(p), Q_j) + |A_x(p) - f(p)| + |f(p) - f(z)| \\
&= |A_x(p) - f(p)| + |f(p) - f(z)| \\
&\leq \varepsilon|p-x| + |p-z||f|_{lip} \leq (2\gamma^{-1} + 3|f|_{lip})\varepsilon r_j
\end{aligned} \tag{15.34}$$

because $A_x(p) \in Q_j$ by definition of $Q_j$ and by (11.46). Then

$$g_j \ \text{ is } C(1+|f|_{lip})\text{-Lipschitz on } B^+_{j,x} \cap E^{\varepsilon r_j}. \tag{15.35}$$

by the same proof as for (12.66); here again, the small $\varepsilon$ wins against the large $\gamma^{-1}$, $|f|_{lip}$, and $(1-a)^{-1}$.

We shall also need to know that

$$X_9 \cap f^{-1}\Big(\bigcup_{j\in J_3} \overline{D}_j\Big) \subset \bigcup_{j\in J_3}\ \bigcup_{x\in Z(y_j)} B_{j,x}. \tag{15.36}$$

Indeed let $z \in X_9 \cap f^{-1}\big(\bigcup_{j\in J_3} \overline{D}_j\big)$ be given, and let $j \in J_3$ be such that $f(z) \in \overline{D}_j$. Notice that $|f(z) - y_j| \leq r_j < \delta_8 < r(y_j)$ by (15.12), because $y_j \in Y_{11} \subset Y_{10}$ (see (15.12) and (15.9)), and by (15.7), so (15.2) says that $z \in \overline{B}(x, 2\gamma^{-1}r_j)$ for some $x \in Z(y_j)$. Now

$|z-x| \leq 2\gamma^{-1}r_j \leq \delta_3/10$ by the last part of (15.24), so $\mathrm{dist}(z, P_x) \leq \varepsilon|z-x|$ by (11.45). Let $w \in P_x$ be such that $|z-w| \leq \varepsilon|z-x|$; then

$$
\begin{aligned}
|A_x(w) - y_j| &\leq |A_x(w) - f(w)| + |f(w) - f(z)| + |f(z) - y_j| \\
&\leq \varepsilon|w-x| + |f|_{lip}|z-w| + r_j < r_j + 3\varepsilon\gamma^{-1}(1+|f|_{lip})\, r_j
\end{aligned}
\tag{15.37}
$$

by (11.46) and because $|w-x| \leq 3\gamma^{-1}r_j < \delta_3/5$. Recall that $D_j$ is centered at $y_j = f(x) = A_x(x) \in A_x(P_x) = Q_j$, so (15.37) says that $A_x(w) \in Q_j \cap (1+3\varepsilon\gamma^{-1}(1+|f|_{lip}))D_j$. Then

$$
w \in (1+3\varepsilon\gamma^{-1}(1+|f|_{lip}))E(x) \subset (2-a)E(x)
\tag{15.38}
$$

by (15.16), because $a < 1$, and if $\varepsilon$ is small enough. Now

$$
\mathrm{dist}(z, (2-a)E(x)) \leq |z-w| \leq \varepsilon|z-x| \leq 2\varepsilon\gamma^{-1}r_j < (1-a)r_j
\tag{15.39}
$$

if $\varepsilon$ is small enough, and hence $z \in B_{x,j}$ (see (15.19)). This proves (15.36).

† When we work under the Lipschitz assumption, we need a few modifications to the definitions above. Surprisingly, we do not modify anything before (15.28). One could argue that it would be more natural to cover $\widetilde{Y}_{11} = \psi(\lambda Y_{11})$ instead of $Y_{11}$, but we prefer to keep the same definitions, and we will be able to handle the differences. In particular, we shall prove that

$$
\text{for each } y \in Y_{11}, \text{ all the affine planes } \widetilde{Q}_x = \widetilde{A}_x(P_x),\ x \in Z(y), \text{ coincide.}
\tag{15.40}
$$

But let us first check that if $y \in Y_{10}$, $x \in Z(y)$, and $Q_x = A_x(P_x)$, then the restriction of $\psi$ to $\lambda Q_x$ is differentiable at $\lambda y$, with a derivative $D_\psi$ such that

$$
\lambda D_\psi(DA_x(v)) = D\widetilde{A}_x(v) \quad \text{for } v \in P'_x,
\tag{15.41}
$$

where $P'_x$ denotes the vector space parallel to $P_x$ (and as we would expect from the chain rule). And indeed,

$$
\begin{aligned}
D\widetilde{A}_x(v) &= \lim_{t\to 0} t^{-1}[\widetilde{A}_x(x+tv) - \widetilde{A}_x(x)] = \lim_{t\to 0} t^{-1}[\widetilde{f}(x+tv) - \widetilde{f}(x)] \\
&= \lim_{t\to 0} t^{-1}[\psi(\lambda f(x+tv)) - \psi(\lambda f(x))],
\end{aligned}
\tag{15.42}
$$

by (12.39), and where the last line comes from the convention that we used for (12.38) and (12.39), that $\widetilde{f}$ is defined by the formula (12.36) near $x$. But

$$
\begin{aligned}
f(x+tv) - y &= f(x+tv) - f(x) = A_x(x+tv) - f(x) + o(t) \\
&= A_x(x+tv) - A_x(x) + o(t) = tDA_x(v) + o(t)
\end{aligned}
\tag{15.43}
$$

because $x \in Z(y)$, by (11.40), and because $A_x(x) = f(x)$ by (11.40) and $A_x$ is affine, so

$$
\psi(\lambda f(x+tv)) = \psi[\lambda(y + tDA_x(v) + o(t))] = \psi[\lambda y + \lambda t DA_x(v)] + o(t)
\tag{15.44}
$$

because $\psi$ is Lipschitz. So (15.42) says that

$$D\widetilde{A}_x(v) = \lim_{t\to 0} t^{-1}\big(\psi[\lambda y + \lambda t DA_x(v)] - \psi(\lambda f(x))\big)$$
$$= \lim_{t\to 0} t^{-1}\big(\psi[\lambda y + \lambda t DA_x(v)] - \psi(\lambda y)\big). \tag{15.45}$$

Now let $w$ be any vector in the vector space $Q'_x$ parallel to $Q_x$, write $w = DA_x(v)$ for some $v \in P'_x$, and observe that (15.45) says that $\psi$ is differentiable at $\lambda y$ in the direction $\lambda w$, with a derivative equal to $D\widetilde{A}_x(v)$ (and hence that satisfies (15.41)). We could easily get the differentiability (instead of the differentiability in each direction), because $\psi$ is Lipschitz, but let us not even bother, because we just need the formula (15.41) for the directional derivatives. Notice however that since $DA_x : P'_x \to Q'_x$ is a bijection (because $DA_x$ has no contracting direction because $x \in X_9$; see (14.5), (14.21) and the line below it, and (15.1)), (15.41) allows us to compute $D_\psi$ from $DA_x$ and $D\widetilde{A}_x$.

We are now ready to prove (15.40). Let $y \in Y_{11}$ be given; by (15.9), all the affine planes $A_x(P_x)$, $x \in Z(y)$, are equal to some affine space $Q_y$; in addition, we just checked that $\psi$ has directional derivatives at $ty$ along $Q_y$, given by a mapping $D_\psi$ that we can compute from the values of $DA_x$ and $D\widetilde{A}_x$ at some $x \in Z(y)$. Now (15.41) says that for each $x \in Z(y)$, the vector space $\widetilde{Q}'_x$ parallel to $\widetilde{A}_x(P_x)$ is given by

$$\widetilde{Q}'_x = D\widetilde{A}_x(P'_x) = D_\psi(DA_x(P'_x)) = D_\psi(Q'_y), \tag{15.46}$$

where $Q'_y$ is the vector plane parallel to $Q_y$. In particular, $\widetilde{Q}'_x$ does not depend on $x \in Z(y)$. Since all the $\widetilde{A}_x(P_x)$ go through $\widetilde{f}(x) = \psi(\lambda f(x)) = \psi(\lambda y)$ by construction, they are all equal, and (15.40) follows.

Return to the definition of the $g_{j,x}$ near (15.29). As before, we first define auxiliary functions $\widetilde{g}_{j,x}$ on the set $U_{int}$ defined by (12.72), and on which we extended $\widetilde{f}$ in (12.75). Notice that for $j \in J_3$ and $x \in Z(y_j)$,

$$B^+_{j,x} \subset B_{j,x} \subset B(x, \frac{3}{2}\gamma^{-1} r_j) \subset B(x, 3\gamma^{-1} r_j) \subset U_{int} \tag{15.47}$$

by (15.20), because $x \in X_9 \subset X_0$ (by (15.1)), since $r_j \leq \frac{\gamma}{4}\delta_6 \leq \frac{\gamma\delta_0}{40(1+|f|_{lip})}$ by (15.15) and (12.7), and by the definition (12.72).

For $j \in J_3$, we denote by $\widetilde{Q}_j$ the common value of the affine planes $\widetilde{A}_x(P_x)$, $x \in Z(y_j)$, and by $\widetilde{\pi}_j$ the orthogonal projection onto $\widetilde{\pi}_j$. Then we set

$$\widetilde{g}_{j,x}(z) = \widetilde{\pi}_j(\widetilde{f}(z)) \ \text{ when } z \in B^-_{j,x}, \tag{15.48}$$

$$\widetilde{g}_{j,x}(z) = \widetilde{f}(z) \ \text{ when } z \in U_{int} \setminus B^+_{j,x}, \tag{15.49}$$

and

$$\widetilde{g}_{j,x}(z) = (1-\beta(z))\widetilde{\pi}_j(\widetilde{f}(z)) + \beta(z)\widetilde{f}(z) \ \text{ for } z \in R_{j,x}, \tag{15.50}$$

with $\beta(z)$ as in (15.32). Also set $\widetilde{g}_j = \widetilde{g}_{j,x}$ on each $B^+_{j,x}$, $x \in Z(y_j)$, and $\widetilde{g}_j(z) = \widetilde{f}(z)$ on the rest of $E \cup U_{int}$; the definition is still coherent, for the same reasons as before, and $\widetilde{g}_j$ is Lipschitz. The analogue of (15.33) is

$$||\widetilde{g}_j - \widetilde{f}||_{L^\infty(U_{int})} \leq \lambda\Lambda r_j \leq \lambda\Lambda\delta_6, \tag{15.51}$$

which we prove as before: the second inequality follows from (15.15), and for the first one it is enough to observe that for $z \in B^+_{j,x}$,

$$\begin{aligned} |\widetilde{g}_{j,x}(z) - \widetilde{f}(z)| &\leq |\widetilde{\pi}_j(\widetilde{f}(z)) - \widetilde{f}(z)| = \mathrm{dist}(\widetilde{f}(z), \widetilde{Q}_j) \leq |\widetilde{f}(z) - \widetilde{f}(x)| \\ &\leq \lambda\Lambda|f(z) - f(x)| = \lambda\Lambda|f(z) - y| \leq \lambda\Lambda r_j \end{aligned} \tag{15.52}$$

because $\widetilde{Q}_j$ goes through $\widetilde{f}(x)$ and by (15.22).

Next we want to define $g_j$. We want to set

$$g_j(z) = \lambda^{-1}\psi^{-1}(\widetilde{g}_j(z)) \ \text{ for } z \in U_{int} \tag{15.53}$$

so let us check that $\widetilde{g}_j(z) \in B(0,1)$. When $z \in B^+_{j,x}$ for some $j \in J_3$ and $x \in Z(y_j)$,

$$\begin{aligned} \mathrm{dist}(\widetilde{g}_{j,x}(z), \mathbb{R}^n \setminus B(0,1)) &\geq \mathrm{dist}(\widetilde{f}(z), \mathbb{R}^n \setminus B(0,1)) - \lambda\Lambda r_j \\ &\geq \lambda\Lambda^{-1}\,\mathrm{dist}(f(z), \mathbb{R}^n \setminus U) - \lambda\Lambda r_j \end{aligned} \tag{15.54}$$

by (15.52) and because $\widetilde{f}(z) = \psi(\lambda f(z))$ and $\psi : \lambda U \to B(0,1)$ is bilipschitz; then

$$\begin{aligned} \mathrm{dist}(f(z), \mathbb{R}^n \setminus U) &\geq \mathrm{dist}(f(x), \mathbb{R}^n \setminus U) - |f(z) - f(x)| \\ &= \mathrm{dist}(f(x), \mathbb{R}^n \setminus U) - |f(z) - y_j| \\ &\geq \mathrm{dist}(f(x), \mathbb{R}^n \setminus U) - r_j \end{aligned} \tag{15.55}$$

because $f(z) \in D_j$ by (15.22), and

$$\mathrm{dist}(f(x), \mathbb{R}^n \setminus U) \geq \mathrm{dist}(\widehat{W}, \mathbb{R}^n \setminus U) = \delta_0 \geq 10\Lambda^2(1 + |f|_{lip})\delta_6 \tag{15.56}$$

because $x \in X_0$, hence $f(x) \in \widehat{W}$ by (11.20), (2.1), and (2.2), and by (12.6) and (12.7). Altogether,

$$\begin{aligned} \mathrm{dist}(\widetilde{g}_{j,x}(z), \mathbb{R}^n \setminus B(0,1)) &\geq \lambda\Lambda^{-1}\,\mathrm{dist}(f(z), \mathbb{R}^n \setminus U) - \lambda\Lambda r_j \\ &\geq \lambda\Lambda^{-1}[10\Lambda^2(1 + |f|_{lip})\delta_6 - r_j] - \lambda\Lambda r_j \\ &\geq 8\lambda\Lambda(1 + |f|_{lip})\delta_6 \end{aligned} \tag{15.57}$$

by (15.54), (15.55), (15.56), and because $r_j \leq \delta_6$ by (15.15). So $\widetilde{g}_{j,x}(z) \in B(0,1)$ and $g_j(z)$ is correctly defined in (15.53).

When $z \in U_{int} \setminus B^+_{j,x}$ for all $j \in J_3$ and $x \in Z(y_j)$, we defined $\widetilde{g}_j(z) = \widetilde{f}(z)$ below (15.50), and $\widetilde{f}(z) = \psi(\lambda f(z))$ by (12.75), so (15.53) makes sense, and even yields $g_j(z) =$

$f(z)$. We can thus extend the definition of $g_i$, and set $g_j(z) = f(z)$ for $z \in U \setminus U_{int}$, but in fact we won't even need that. Anyway, we get that

$$g_j(z) = f(z) \quad \text{for } z \in U_{int} \setminus \bigcup_{j\in J_3} \bigcup_{x\in Z(y_j)} B^+_{j,x}. \tag{15.58}$$

Notice that

$$||g_j - f||_{L^\infty(U_{int})} \leq \lambda^{-1}\Lambda||\widetilde{g}_j - \widetilde{f}||_\infty \leq \Lambda^2 r_j \leq \Lambda^2\delta_6, \tag{15.59}$$

by (15.58), (15.53) and (15.51). Let us also check that

$$|g_j(z) - f(z)| \leq \Lambda^2(2\gamma^{-1} + 3|f|_{lip})\,\varepsilon r_j \quad \text{for } z \in B^+_{j,x} \cap E^{\varepsilon r_j}. \tag{15.60}$$

We prove this as in (15.34). We can again choose $w \in E$ such that $|w - z| \leq \varepsilon r_j$, and (by (11.45)) $p \in P_x$ such that $|p - w| \leq \varepsilon|w - x| \leq 2\varepsilon r_j$; thus $|p - z| \leq 3\varepsilon r_j$ and $|p - x| \leq |p - z| + |z - x| \leq 3\varepsilon r_j + \frac{3}{2}\gamma^{-1}r_j \leq 2\gamma^{-1}r_j$ by (15.20), and

$$\begin{aligned}
|\widetilde{g}_j(z) - \widetilde{f}(z)| &\leq |\widetilde{\pi}_j(\widetilde{f}(z)) - \widetilde{f}(z)| \leq \mathrm{dist}(\widetilde{f}(z), \widetilde{Q}_j) \\
&\leq \mathrm{dist}(\widetilde{A}_x(p), \widetilde{Q}_j) + |\widetilde{A}_x(p) - \widetilde{f}(p)| + |\widetilde{f}(p) - \widetilde{f}(z)| \\
&= |\widetilde{A}_x(p) - \widetilde{f}(p)| + |\widetilde{f}(p) - \widetilde{f}(z)| \\
&\leq \lambda\varepsilon|p - x| + |p - z||\widetilde{f}|_{lip} \leq (2\gamma^{-1} + 3\Lambda|f|_{lip})\lambda\varepsilon r_j
\end{aligned} \tag{15.61}$$

because $\widetilde{A}_x(p) \in \widetilde{Q}_j = \widetilde{A}_x(P_x)$, and by (12.52) (with the same justification as for (15.25)); (15.60) follows.

We claim that now

$$g_j \quad \text{is } C\Lambda^2(1 + |f|_{lip})\text{-Lipschitz on } B^+_{j,x} \cap E^{\varepsilon r_j}, \tag{15.62}$$

with the same proof as for (12.96). Finally, (15.36) still holds in the Lipschitz context; its proof only involves $f$ and arguments anterior to (15.29) and the definition of the $g_{j,x}$, so we can keep it. †

## 16. The modified function $g$, and a deformation for $E$.

We are now ready to define a (new) function $U : \mathbb{R}^n \to \mathbb{R}^n$, which is a first competitor for the replacement of $f$. We already defined a function $g$ in Step 2.f, by (13.12) or (13.29) and (13.31), and we intend to keep it like this on

$$V_1 = \bigcup_{j\in J_1} \frac{1+a}{2} B_j = \bigcup_{j\in J_1} B(x_j, \frac{(1+a)t}{2}). \tag{16.1}$$

That is, we set

$$g(z) = f(z) + \sum_{j\in J_1} \psi_j(z)[g_j(z) - f(z)] \quad \text{for } z \in V_1 \tag{16.2}$$

in the rigid case, and

$$g(z) = \lambda^{-1}\psi^{-1}(\widetilde{g}(z)) \text{ for } z \in V_1, \tag{16.3}$$

with

$$\widetilde{g}(z) = \widetilde{f}(z) + \sum_{j\in J_1} \psi_j(z)[\widetilde{g}_j(z) - \widetilde{f}(z)] \tag{16.4}$$

under the Lipschitz assumption. We also set

$$g(z) = g_j(z) \text{ for } z \in \frac{1+a}{2}B_j = B(x_j, \frac{(1+a)r_j}{2}) \tag{16.5}$$

when $j \in J_2$, and

$$g(z) = g_j(z) = g_{j,x}(z) \text{ for } z \in B^+_{j,x} \tag{16.6}$$

when $j \in J_3$ and $x \in Z(y_j)$. Finally, set

$$V_1^+ = \Big[\bigcup_{j\in J_1\cup J_2} \frac{1+a}{2}B_j\Big] \cup \Big[\bigcup_{j\in J_3\,;\,x\in Z(y_j)} B^+_{j,x}\Big]; \tag{16.7}$$

we just defined $g$ on $V_1^+$, and we keep

$$g(z) = f(z) \text{ for } z \in \mathbb{R}^n \setminus V_1^+. \tag{16.8}$$

† Under the Lipschitz assumption, we also have a function $\widetilde{g}$, defined on $V_1^+$, and such that $\widetilde{g}(z) = \psi(\lambda g(z))$. On $V_1$, we wrote this explicitly in (16.3) and (16.4); on the balls $\frac{1+a}{2}B_j$, $j \in J_2$, this comes from the fact that $g_i$ was defined by (14.29) (also recall that $2B_j \subset U_{int}$ for $j \in J_2$); on the $B^+_{j,x}$, this comes from (15.47) and (15.53) (also see the line below (15.50)). †

Let us check that all these definitions are independent because the corresponding sets are disjoint. First, the $B_j$, $j \in J_2$, are disjoint from each other and from $\bigcup_{j\in J_1} \frac{1+a}{2}B_j$, by (14.8). The $B^+_{j,x}$ are disjoint from each other by (15.27). Finally, if $j \in J_3$ and $x \in Z(y_j)$,

$$B^+_{j,x} \subset B(x, \frac{3}{2}\gamma^{-1}r_j) \subset B(x, \frac{\delta_7}{2}) \tag{16.9}$$

(15.20) and (15.15). This last ball does not meet any $\frac{1+a}{2}B_j$, $j \in J_1 \cup J_2$, by the definition (15.3) of $\delta_7$ and because $x \in X_9$ (by (15.1)).

Next we check that

$$g \text{ is Lipschitz on } U \tag{16.10}$$

(but possibly with a very bad norm). Recall that (16.2) or (16.3) would also yield $g(z) = f(z)$ for $z \in \partial V_1$, by (13.18) or (13.31) (recall that our initial $g$ was Lipschitz). Similarly, $g_j(z) = f(x)$ for $j \in J_2$ and $z \in \partial B(x_j, \frac{(1+a)r_j}{2})$; in the rigid case, this is because we still use (12.59) (see above (14.10)), and in the Lipschitz case this comes from (12.77) and (14.29), or directly from (14.30). Finally, $g_{j,x}(z) = f(z)$ for $j \in J_3$, $x \in Z(y_j)$, and $z \in \partial B^+_{j,x}$, by (15.30) or (15.49) and (15.53). Thus (16.8) does not introduce any discontinuity, and (16.10) follows easily, because $g$ is Lipschitz on the closure of each piece.

Let us finally record that

$$||g - f||_\infty \leq 4\Lambda^2(1 + |f|_{lip})\delta_6 \tag{16.11}$$

in the rigid case by (13.13) and (12.8), (14.11), and (15.33), and in the Lipschitz case by (13.35) and (12.8), (14.31), and (15.59).

We would like to use $g$ to define new competitors, and a natural first step is to check that $g$ is the endpoint of a one-parameter family of functions $g_t$, that satisfies the conditions (1.4)-(1.8), and in particular the boundary conditions (1.7), relative to $E$.

This will not be entirely satisfactory, because we would like (1.7) to hold with respect to the $E_k$, but we shall take care about that in the next section.

Recall that $f$ itself is defined as $f(x) = \varphi_1(x)$, for some one-parameter family of functions $\varphi_t$, which we extended from $E$ to $\mathbb{R}^n$ at the beginning of Section 11, and for which (1.4)-(1.8) hold by assumption. We start under the rigid assumption and set

$$g_t(x) = \varphi_{2t}(x) \ \text{ for } 0 \leq t \leq 1/2 \tag{16.12}$$

and

$$g_t(x) = (2-2t)f(x) + (2t-1)g(x) \ \text{ for } 1/2 \leq t \leq 1. \tag{16.13}$$

Recall that (1.4)-(1.8) for the $\varphi_t$ holds with respect to the ball $B = B(X_0, R_0)$ of (11.1); here we shall find it convenient to use a slightly larger ball $B'$.

**Lemma 16.14.** *The functions $g_t$, $0 \leq t \leq 1$, satisfy (1.4)-(1.8), relative to $E$ and the ball $B' = \overline{B}(X_0, R_0 + 4\Lambda^2(1 + |f|_{lip})\delta_6)$.*

We shall need to know that

$$\mathrm{dist}(x, X_1) \leq \delta_6 \ \text{ and } \ \mathrm{dist}(x, \mathbb{R}^n \setminus W_f) > \delta_1/2 \ \text{ for } x \in V_1^+, \tag{16.15}$$

where we defined $V_1^+$ in (16.7) and $\delta_1 = \mathrm{dist}(X_1, \mathbb{R}^n \setminus W_f)$ in (11.22). The second part follows from the first part, because $\delta_6 < \delta_1/2$ by (12.7). For the first part there are three similar cases. When $x \in B_j$ for some $j \in J_1$, this is true because $x_j \in X_N(\delta_4) \subset X_1$ and $|x - x_j| \leq t < \delta_6$; see the line below (12.8), the various definitions of the $X_j$, and (12.8). When $x \in B_j$ for some $j \in J_2$, we use (14.7) instead. When $x \in B^+_{j,z}$ for some $j \in J_3$ and $z \in Z(y_j)$, we use the fact that $z \in X_9 \subset X_1$ by (15.1), and $B^+_{j,x} \subset B(x, \frac{3}{2}\gamma^{-1}r_j) \subset B(x, \delta_6)$ by (15.20) and (15.15). So (16.15) holds.

The properties (1.4) and (1.8) hold by construction. For (1.5), since we know that $g_0(x) = \varphi_0(x) = x$ for $x \in \mathbb{R}^n$, it is enough to check that

$$g_t(x) = x \ \text{ for } x \in E \setminus B \text{ and } 0 \le t \le 1. \tag{16.16}$$

Let $x \in E \setminus B$ be given. By (1.5) for the $\varphi_t$, $\varphi_t(x) = x$ for $0 \le t \le 1$, hence by (16.12) $g_t(x) = x$ for $0 \le t \le 1/2$. If $x \in V_1^+$, (16.15) says that $x \in W_f$ and, since $x \in E$, this forces $x \in B$ (because $\varphi_1(x) \neq x$ by the definition (11.19), and by (1.5)); this is impossible. So $x \in \mathbb{R}^n \setminus V_1^+$, and $g_t(x) = f(x) = x$ by (16.13) and (16.8); this proves (16.16) and (1.5).

For (1.6), we need to check that

$$g_t(x) \in B' \ \text{ when } x \in E \cap B' \text{ and } 0 \le t \le 1. \tag{16.17}$$

This is trivial when $x \in E \cap B' \setminus B$, because $g_t(x) = x \in B$. If $x \in B$ and $0 \le t \le 1/2$, $g_t(x) = \varphi_{2t}(x) \in B$ by (16.12) and (1.6) for the $\varphi_t$. Finally, if $x \in B$ and $t > 1/2$, $g_t(x)$ lies on the segment $[f(x), g(x)]$, which is contained in $B'$ because $f(x) \in B$ and $|g(x) - f(x)| \le 4\Lambda^2(1 + |f|_{lip})\delta_6$ by (16.11).

We still need to check (1.7), i.e., that for $0 \le k \le j_{max}$,

$$g_t(x) \in L_k \ \text{ when } x \in E \cap L_k \cap B' \text{ and } 0 \le t \le 1. \tag{16.18}$$

[We just used the letter $k$ to avoid a conflict with the notation for the $B_j$, but of course $k$ is not the index for the sequence $\{E_k\}$ here.] We may assume that $x \in B$, because otherwise $g_t(x) = x \in L_k$, and that $t \ge 1/2$, because otherwise $g_t(x) = \varphi_{2t}(x) \in L_k$ by (1.7) for the $\varphi_t$. By (16.13), $g_t(x)$ lies on the segment $[f(x), g(x)]$, so we just need to check that

$$[f(x), g(x)] \subset L_k \text{ for } x \in E \cap L_k \cap B. \tag{16.19}$$

Since this is trivial when $g(x) = f(x)$, we may assume that $x \in V_1^+$.

First suppose that $x \in V_1$, and let $j \in J_1$ be such that $x \in B_j$.

Return to the definition of $Q_j$ (Step 2.e, starting above (12.42)). Still denote by $x_j$ the the center of $B_j$; we chose $l \in \mathcal{L}$ such that $f(x_j) \in D_l$, and observed that we can find $x(l) \in X_5$ such that $y_l = f(x(l))$.

But $X_5 \subset X_2$, so by (11.26) there is an $m \in [0, n]$ such that $x(l) \in X_{1,\delta_2}(m)$. That is, by (11.23)-(11.24) $y_l = f(x(l)) \in \mathcal{S}_m \setminus \mathcal{S}_{m-1}$, and (if $m \ge 1$)

$$\mathrm{dist}(y_l, \mathcal{S}_{m-1}) \ge \delta_2. \tag{16.20}$$

Still denote by $F_l$ the smallest face of our grid that contains $y_l$, and by $W(y_l)$ the affine plane spanned by $F_l$; obviously $F_l$ and $W(y_l)$ are $m$-dimensional. Also notice that

$$|f(x) - y_l| \le |f(x) - f(x_j)| + |f(x_j) - y_l| \le t|f|_{lip} + t \le \delta_6(1 + |f|_{lip}) \tag{16.21}$$

because $f(x_j) \in D_l$ and by (12.8). Let us check that

$$\text{any face } F \text{ of our grid that contains } f(x) \text{ contains } F_l \text{ too.} \tag{16.22}$$

We use coordinates and the dyadic structure to prove this, but probably polyhedra would work as well. Also recall that we work under the rigid assumption for the moment. For $1 \leq i \leq n$, denote by $a_i$ and $b_i$ the $i$-th coordinate of $f(x)$ and $y_l$ respectively. Thus

$$|b_i - a_i| \leq \frac{1}{10} \min(\delta_2, r_0) \tag{16.23}$$

by (16.21) and (12.7). Set $I_0 = \big\{i \in [1,n]\,;\, b_i \notin r_0\mathbb{Z}\big\}$ (recall that $r_0$ is the scale of our dyadic grid). For $i \in I_0$, (16.20) says that $\mathrm{dist}(b_i, r_0\mathbb{Z}) > \delta_2$, so $[a_i, b_i]$ does not meet $r_0\mathbb{Z}$ (by (16.23)).

Denote by $w$ the point obtained from $f(x)$ by replacing each $a_i$, $i \in I_0$, with $b_i$. We get that $w \in F$ too. And if we want to go from $w$ to $z$, we just need to replace each coordinate $a_i$, $i \notin I_0$, with $b_i$, which by (16.23) and the definition of $I_0$ is the closest point of $r_0\mathbb{Z}$. Then $y_l$ lies in any face that may contain $w$, including $F$. Altogether, $y_l \in F$, and since $F_l$ is the smallest face that contains $y_l$, we get that $F_l \subset F$, as needed for (16.22).

Recall that $g_j(x) \in [f(x), \pi_j(f(x))]$ (by (12.59)-(12.61)), where $\pi_j$ is the orthogonal projection on the affine plane $\widehat{Q}_j$ spanned by $Q_j$, that $Q_j$ lies in $\mathcal{F}_l$ (see above (12.43)), and hence goes through $D_l$ and is contained in $W(y_l)$ (see above (12.18)). Let $\pi$ denote the orthogonal projection onto the affine plane through $y_l$ parallel to $Q_j$; then

$$\begin{aligned} |\pi_j(f(x)) - y_l| &\leq |\pi(f(x)) - y_l| + ||\pi - \pi_j||_\infty \leq |f(x) - y_l| + t \leq t|f|_{lip} + 2t \\ &\leq 2\delta_6(1 + |f|_{lip}) \leq \delta_2/5 < \mathrm{dist}(y_l, \partial F_l) \end{aligned} \tag{16.24}$$

by various parts of (16.21), (12.7), and (16.20). Also, $\pi_j(f(x)) \in \widehat{Q}_j \subset W(y_l)$, the affine space spanned by $F_l$; then (16.24) implies that $\pi_j(f(x)) \in F_l$ because the segment $[\pi_j(f(x)), y_l] \subset W(y_l)$ does not meet $\partial F_l$. Thus

$$\pi_j(f(x)) \in F_l \subset F \text{ for any face } F \text{ of our grid that contains } f(x), \tag{16.25}$$

by (16.22) for the second part.

By (1.7) for the $\varphi_t$, $f(x) = \varphi_1(x) \in L_k$. Let $F$ be a face of $L_k$ that contains $f(x)$. The proof of (16.25) shows that $\pi_i(f(x)) \in F$ for each $i \in J_1$ such that $x \in B_i$ (that is, not only for $i = j$), and then $g_i(x) \in [f(x), \pi_i(f(x))]$ lies in $F$ too (because every face is convex).

By (16.2), (13.8), and (13.9), $g(x)$ lies in the convex hull of $f(x)$ and the $g_i(x)$, where $i \in J_1$ is such that $\psi_i(x) \neq 0$. For such $i$, (13.5) and (13.6) imply that $x \in B_i$, so $g_i(x) \in F$. Altogether, $g(x) \in F$ and $[f(x), g(x)] \subset F \subset L_k$, as needed for (16.19).

Our second case for the proof of (16.19) is when $x \in \frac{1+a}{2}B_j$ for some $j \in J_2$. Set $y_j = f(x_j)$, denote by $F(y_j)$ the smallest face that contains $y_j$, by $W(y_j)$ the affine subspace spanned by $F(y_j)$, and by $m$ their dimension. This time $x_j \in X_7 \subset X_2$ by (14.7) and various definitions, so $y_j = f(x_j) \in \mathcal{S}_m \setminus \mathcal{S}_{m-1}$ and

$$\mathrm{dist}(y_l, \mathcal{S}_{m-1}) \geq \delta_2 \tag{16.26}$$

if $m \geq 1$, by the proof of (16.20). Since

$$|f(x) - y_j| = |f(x) - f(x_j)| \leq r_j |f|_{lip} \leq \delta_6 |f|_{lip} \leq \delta_2/10 \tag{16.27}$$

by (14.7), and (12.7), the same proof as for (16.22) shows that any face $F$ of our grid that contains $f(x)$ contains $F(y_j)$ too.

Here $g(x) = g_j(x) \in [f(x), \pi_j(f(x))]$ by (16.5) and because $g_j(x)$ is given by (12.59)-(12.61), and where $\pi_j$ now denotes the orthogonal projection on $Q_j = A_{x_j}(P_j)$ (see below (14.9)). But Lemma 12.27 says that $Q_j = A_{x_j}(P_j) \subset W(f(y_j))$, and since $|\pi_j(f(x)) - y_j| \leq |f(x) - y_j| \leq \delta_2/10$ because $Q_j$ goes through $y_j$ and by (16.27), the proof of (16.25) shows that $\pi_j(f(x)) \in F(y_j)$.

As before, the $\varphi_t(x)$ and $f(x) = \varphi_1(x)$ lie in $L_k$. Let $F$ be a face of $L_k$ that contains $f(x)$; then $\pi_j(f(x)) \in F(y_j) \subset F$, and $g(x) \in [f(x), \pi_j(f(x))]$ lies in $F$ too (by convexity). So $[f(x), g(x)] \subset F \subset L_k$, and (16.19) holds in this case too.

Our last case is when $x$ lies in $B^+_{j,z}$ for some $j \in J_3$ and $z \in Z(y_j)$ (recall that $x \in V_1^+$ and see the definition (16.7)). We proceed as in the second case, notice that $x_j \in X_9$ by (15.1), replace (16.27) with the fact that $f(x) \in D_j = B(y_j, r_j) \subset B(y_j, \delta_6)$ by (15.22) and (15.15) (see the definition of $D_j$ above (15.12)). Then $g(x) = g_{j,z}(x) \in [f(x), \pi_j(f(x))]$ by (16.6) and (15.29)-(15.32), and where $\pi_j$ denotes the orthogonal projection on $Q_j = A_z(P_z)$ (see above (15.29) and (15.16)), which is again contained in $W(y_j)$ by Lemma 12.27. The rest of the argument is the same. This completes our proof of (16.19) and, by the same token, of (16.18); this was our last verification; Lemma 16.14 follows. $\square$

† Under the Lipschitz assumption, we keep $g_t(z) = \varphi_{2t}(z)$ for $0 \leq t \leq 1/2$, as in (16.12), but for $t \geq 1/2$, we want to preserve the faces when this is possible, and this is easier to do after the usual change of variable, so we want to set

$$g_t(z) = \lambda^{-1}\psi^{-1}(\widetilde{g}_t(z)) \quad \text{for } z \in U_{int}, \tag{16.28}$$

where

$$\widetilde{g}_t(z) = (2 - 2t)\widetilde{f}(z) + (2t - 1)\widetilde{g}(z), \tag{16.29}$$

and $\widetilde{g}(z)$ is as in (16.4) when $z \in V_1$, $\widetilde{g}(z) = \widetilde{g}_j(z)$ when $z \in \frac{1+a}{2}B_j$ for some $j \in J_2$, $\widetilde{g}(z) = \widetilde{g}_j(z) = \widetilde{g}_{j,x}(z)$ when $z \in B^+_{j,x}$ for some $j \in J_3$ and $x \in Z(y_j)$, and $\widetilde{g}(z) = \widetilde{f}(z)$ when $z \in \mathbb{R}^n \setminus V_1^+$. On $V_1^+$, this definition is the same as in the remark below (16.8), which was also based on (16.3)-(16.4) (also see (13.28) and (13.29)), (14.29), and (15.53).

We need to check that

$$\widetilde{g}_t(z) \in B(0,1) \text{ for } t \geq 1/2, \tag{16.30}$$

so that (16.28) makes sense. This is clear when $z \in \mathbb{R}^n \setminus V_1^+$, because $\widetilde{g}_t(z) = \widetilde{f}(z) = \psi(\lambda f(z)$ by (12.75); otherwise, we already checked that $\widetilde{g}(z) \in B(0,1)$ (typically, when we wanted to define $g$ by $g(z) = \lambda^{-1}\psi^{-1}(\widetilde{g}(z)))$; see (12.82), above (14.29), and below (15.53). Then $\widetilde{g}_t(z)$, which lies on the segment between $\widetilde{g}(z)$ and $\widetilde{f}(z) = \psi(\lambda f(z)$, lies in $B(0,1)$ too. Thus (16.28) makes sense and $g_t(z) \in U$ for $z \in U_{int}$ and $1/2 \leq t \leq 1$.

When $t = 1/2$, (16.28) and (16.29) yield $\widetilde{g}_t = \widetilde{f}$ and $g_t = f = \varphi_1$, so $g_t$ is continuous across $t = 1/2$. When $t = 1$, we retrieve $\widetilde{g}_1 = \widetilde{g}$ and $g_1 = g$.

We only defined $\widetilde{g}_t(z)$ and $g_t(z)$ when $z \in U_{int}$; when $z \in U \setminus U_{int}$, we do not define $\widetilde{g}_t(z)$ and directly set $g_t(z) = f(z)$, as in (16.8). This does not create a discontinuity, because $V_1^+$ lies well inside $U_{int}$ (recall the definition (12.72) and the inclusions in (12.76), the lines above (14.25), and (15.47)), and because the definition above also gives $g_t(z) = f(z)$ when $z \in U_{int} \setminus V_1^+$.

Now we check that Lemma 16.14 is still valid in the present case. We do not need to change anything before the last line of the proof of (16.17), where we just need to observe that (again for $t \geq 1/2$ and $x \in B$) $|g_t(x) - f_t(x)| \leq \lambda^{-1}\Lambda|\widetilde{g}_t(x) - \widetilde{f}(x)| \leq 4\Lambda^2(1+|f|_{lip})\delta_6$ by (16.28) and the proof of (16.11) (more precisely, the line above (13.35), (12.8), (14.25), and (15.51), but if you are ready to loose an extra $\Lambda^2$, just use (16.11)); so $g_t(x) \in B'$ as before.

Thus we may turn to (1.7), or equivalently (16.18) or, after a change of variable, the fact that

$$\widetilde{g}_t(x) \in \widetilde{L}_k = \psi(\lambda L_k) \quad \text{when } x \in E \cap L_k \cap B' \text{ and } 0 \leq t \leq 1. \tag{16.31}$$

The verification for $0 \leq t \leq 1/2$ is the same as before, so we may assume that $t \geq 1/2$, and by (16.29) we just need to check that

$$[\widetilde{f}(x), \widetilde{g}(x)] \subset \widetilde{L}_k \text{ for } x \in E \cap L_k \cap B \tag{16.32}$$

(compare with (16.19)).

We continue the argument as below (16.19), starting with the case when $x \in V_1$ and so $x \in B_j$ for some $j \in J_1$. Let $l \in \mathcal{L}$ be as before; thus $f(x_j) \in D_l$ and $y_l = f(x(l))$ for some $x(l) \in X_5$. We shall also use $\widetilde{y}_l = \psi(\lambda y_l) \in B(0,1)$, and $m \in [0,n]$ such that $y_l \in \mathcal{S}_m \setminus \mathcal{S}_{m-1}$ (just $y_l \in \mathcal{S}_m$ if $m = 0$); then (16.20) holds as before. Still denote by $F_l$ the smallest face of the twisted grid that contains $y_l$, set $\widetilde{F}_l = \psi(\lambda F_l)$ (the smallest rigid face that contains $\widetilde{y}_l$), and call $\widetilde{W}(y_l)$ the affine space spanned by $\widetilde{F}_l$. Next we check that (16.22) holds, or equivalently that

$$\text{any face } \widetilde{F} \text{ of the true grid that contains } \widetilde{f}(x) \text{ contains } \widetilde{F}_l \text{ too.} \tag{16.33}$$

The proof needs to be modified slightly. From (16.20) we deduce that

$$\operatorname{dist}(\widetilde{y}_l, \widetilde{\mathcal{S}}_{m-1}) = \operatorname{dist}(\psi(\lambda y_l), \psi(\lambda \mathcal{S}_{m-1})) \geq \Lambda^{-1}\lambda\delta_2. \tag{16.34}$$

We still have (16.21), with the same proof, which yields

$$|\widetilde{f}(x) - \widetilde{y}_l| = |\psi(\lambda f(x)) - \psi(\lambda y_l)| \leq \lambda\Lambda|f(x) - y_l| \leq \lambda\Lambda\delta_6(1+|f|_{lip}). \tag{16.35}$$

Denote by $a_i$ and $b_i$ the coordinates of $\widetilde{f}(x)$ and $\widetilde{y}_l$; now

$$|b_i - a_i| \leq \lambda\Lambda\delta_6(1+|f|_{lip}) \leq \frac{\lambda}{10\Lambda}\min(\delta_2, \lambda^{-1}r_0) \leq \frac{1}{10}\min(\Lambda^{-1}\lambda\delta_2, r_0). \tag{16.36}$$

still by (12.7). From this and (16.34) we deduce the analogue of (16.22) as before, when we had (16.23) and (16.20).

Now we use the fact that $\widetilde{g}_j(x) \in [\widetilde{f}(x), \widetilde{\pi}_j(\widetilde{f}(x))]$, by (12.77)-(12.79), where $\widetilde{\pi}_j$ is the orthogonal projection onto the affine plane $\widetilde{P}_j$ that contains $\widetilde{Q}_j$; see the description above (12.77), and recall that $\widetilde{P}_j$ satisfies (12.23).

Let $\widetilde{\pi}$ be the projection onto the affine plane through $\widetilde{y}_l$ parallel to $\widetilde{P}_j$; the analogue of (16.24) is

$$
\begin{aligned}
|\widetilde{\pi}_j(\widetilde{f}(x)) - \widetilde{y}_l| &\leq |\widetilde{\pi}(\widetilde{f}(x)) - \widetilde{y}_l| + ||\widetilde{\pi} - \widetilde{\pi}_j||_\infty \\
&\leq |\widetilde{f}(x) - \widetilde{y}_l| + 2\lambda\Lambda(1 + |f|_{lip})t \\
&\leq \lambda\Lambda|f(x) - y_l| + 2\lambda\Lambda(1 + |f|_{lip})t \\
&\leq 3\lambda\Lambda(1 + |f|_{lip})t \leq 3\lambda\Lambda(1 + |f|_{lip})\delta_6 \\
&\leq \frac{3\lambda\delta_2}{10\Lambda} < \mathrm{dist}(\widetilde{y}_l, \partial\widetilde{F}_l)
\end{aligned} \tag{16.37}
$$

by (12.23), (16.21), (12.7), and (16.34).

As before, $\widetilde{\pi}_j(\widetilde{f}(x))$ lies on the affine plane $\widetilde{P}_j$ that contains $\widetilde{Q}_j$, which is contained in $\widetilde{W}(y_l)$ by (12.23); since $\widetilde{W}(y_l)$ is the affine space spanned by $\widetilde{F}_l$, and $\widetilde{y}_l \in \widetilde{F}_l$, we get that $\widetilde{\pi}_j(\widetilde{f}(x)) \in \widetilde{F}_l \subset \widetilde{F}$ for any (straight) face $\widetilde{F}$ that contains $\widetilde{f}(x)$ (by (16.37)). The rest of the proof of (16.33) (by convexity) goes as before.

The other cases are easier (see near (16.26)); we replace Lemma 12.27 with Lemma 12.40 when needed, and otherwise proceed as above. This completes our proof of Lemma 16.14 under the Lipschitz assumption. †

## 17. Magnetic projections onto skeletons, and a deformation for the $E_k$.

We just checked that $g$ and the $g_t$ define (a hopefully stabler) acceptable deformation for $E$, but we still want to modify them so that they work for the $E_k$, at least for $k$ large. For this we will need some way to push points back to the $L_j$ (when they are close to the $L_j$). The name magnetic for the projections below was used in [Fv1] in a similar context; it is nice because it conveys the idea of a strong attraction, but with a very short range.

### 17.a. Magnetic projections onto the faces.

We start with a projection on nearby faces of a given dimension, and then we shall see how to work in all dimensions at the same time. In what follows, $m \in [0, n)$ is an integer, and $s$ is a small number that plays the role of an attraction range, which will later depend on various parameters. Also recall that $\mathcal{S}_m$ denotes the $m$-dimensional skeleton of our usual dyadic grid.

**Lemma 17.1.** *Let a dimension $m \in [0, n[$ and $s \in (0, \frac{r_0}{10})$ be given. There is a mapping $\Pi = \Pi_{m,s} : \mathbb{R}^n \to \mathbb{R}^n$, with the following properties:*

$$
\Pi_{m,s}(x) = x \quad \text{when } x \in \mathcal{S}_m \text{ and when } \mathrm{dist}(x, \mathcal{S}_m) \geq 2s, \tag{17.2}
$$

$$\Pi_{m,s}(x) \in \mathcal{S}_m \quad \textit{when } \operatorname{dist}(x, \mathcal{S}_m) \leq s, \tag{17.3}$$

$$\Pi_{m,s}(x) \textit{ is a } C\textit{-Lipschitz function of } s \in (0, \tfrac{r_0}{10}) \textit{ and } x \in \mathbb{R}^n, \tag{17.4}$$

*where $C$ depends only on $n$, and*

$$\Pi_{m,s} \textit{ preserves all the faces of our usual grid,} \tag{17.5}$$

*which means that if $F$ is a face of any dimension, then $\Pi_{m,s}(x) \in F$ for $x \in F$.*

We start with the (rigid) case when $r_0 = 1$. Naturally we shall use Lemma 3.17, with $L = \mathcal{S}_m$ and $\eta = 1/3$; we get a mapping $\Pi_L : L^\eta \times [0,1] \to \mathbb{R}^n$, with the properties (3.18)-(3.22). Recall that $L^\eta$ is, as in (3.5), an $\eta$-neighborhood of $L$. For convenience, we extend $\Pi_L$ by setting $\Pi_L(x,0) = x$ for $x \in \mathbb{R}^n$; this is compatible with (3.18).

Let $\theta : [0,+\infty) \to [0,1]$ be a smooth cut-off function such that

$$\theta(t) = 1 \text{ for } 0 \leq t \leq 1,\ 0 \leq \theta(t) \leq 1 \text{ for } 1 \leq t \leq 2,\ \theta(t) = 0 \text{ for } t \geq 2, \tag{17.6}$$

and $|\theta'(t)| \leq 2$ everywhere. Set $d(x) = \operatorname{dist}(x, \mathcal{S}_m)$ for $x \in \mathbb{R}^n$, and then

$$\Pi_{m,s}(x) = \Pi_L(x, \theta(s^{-1}d(x))) \ \text{ for } x \in \mathbb{R}^n \text{ and } 0 < s \leq 10^{-1}. \tag{17.7}$$

First observe that if $x \in \mathbb{R}^n \setminus L^\eta$, then $d(x) \geq \eta = 1/3$, $s^{-1}d(x) \geq 10/3$, hence $\theta(s^{-1}d(x)) = 0$ and $\Pi_L(x, \theta(s^{-1}d(x)))$ is well defined (and is equal to $x$). So $\Pi_{m,s}$ is well defined on $\mathbb{R}^n$.

The second part of (17.2) holds for the same reason: if $\operatorname{dist}(x, \mathcal{S}_m) \geq 2s$, then $\theta(s^{-1}d(x)) = 0$ and $\Pi_{m,s}(x) = \Pi_L(x,0) = x$ by (3.18). Similarly, if $\operatorname{dist}(x, \mathcal{S}_m) \leq s$, then $\theta(s^{-1}d(x)) = 1$ and $\Pi_{m,s}(x) = \Pi_L(x,1) \in L = \mathcal{S}_m$, by (3.19) and the definition of $\pi_L$ in Lemma 3.4, so (17.3) holds. Finally, if $x \in \mathcal{S}_m$, $\Pi_L(x,t) = x$ for all $t$, by (3.18), so $\Pi_{m,s}(x) = x$ and the first part of (17.2) holds too.

Let us check that $\Pi_{m,s}(x)$ is Lipschitz in $x$. First consider $x, y \in L^\eta$; then

$$\begin{aligned}
|\Pi_{m,s}(x) - \Pi_{m,s}(y)| &= |\Pi_L(x, \theta(s^{-1}d(x))) - \Pi_L(y, \theta(s^{-1}d(y)))| \\
&\leq |\Pi_L(x, \theta(s^{-1}d(x))) - \Pi_L(x, \theta(s^{-1}d(y)))| \\
&\qquad + |\Pi_L(x, \theta(s^{-1}d(y))) - \Pi_L(y, \theta(s^{-1}d(y)))| \\
&\leq C d(x) |\theta(s^{-1}d(x)) - \theta(s^{-1}d(y))| + C|x-y|
\end{aligned} \tag{17.8}$$

by (3.20) and (3.21). Let us check that

$$d(x)|\theta(s^{-1}d(x)) - \theta(s^{-1}d(y))| \leq 6|x-y|. \tag{17.9}$$

If $d(x) \leq 3s$, simply say that $|\theta(s^{-1}d(x)) - \theta(s^{-1}d(y))| \leq 2s^{-1}|d(x) - d(y)| \leq 2s^{-1}|x-y|$, and (17.9) follows. If $d(x) \geq 3s$ and $d(y) \geq 2s$, then $\theta(s^{-1}d(x)) = \theta(s^{-1}d(y)) = 0$ and (17.9) is trivial. In the last case when $d(x) \geq 3s$ and $d(y) \leq 2s$,

$$d(x)|\theta(s^{-1}d(x)) - \theta(s^{-1}d(y))| = d(x)\theta(s^{-1}d(y)) \leq d(x) \leq 3|d(x) - d(y)| \leq 3|x-y|, \tag{17.10}$$

and (17.9) holds too. Then $|\Pi_{m,s}(x)-\Pi_{m,s}(y)| \leq C|x-y|$, by (17.8) and (17.9), and this takes care of our first case when $x, y \in L^\eta$.

Suppose $x \in L^\eta$ and $y \in \mathbb{R}^n \setminus L^\eta$, and let $z \in [x,y]$ lie on the boundary of $L^\eta$; then $\Pi_{m,s}(z) = z$ and $\Pi_{m,s}(y) = y$ by (17.2), and

$$
\begin{aligned}
|\Pi_{m,s}(x)-\Pi_{m,s}(y)| &\leq |\Pi_{m,s}(x)-\Pi_{m,s}(z)| + |\Pi_{m,s}(z)-\Pi_{m,s}(y)| \\
&= |\Pi_{m,s}(x)-\Pi_{m,s}(z)| + |z-y| \\
&\leq C|x-z| + |z-y| \leq C|x-y|
\end{aligned}
\tag{17.11}
$$

by the previous case. The case when $x \in \mathbb{R}^n \setminus L^\eta$ and $y \in L^\eta$ is similar, and when $x, y \in \mathbb{R}^n \setminus L^\eta$ we simply get that $|\Pi_{m,s}(x)-\Pi_{m,s}(y)| = |x-y|$ by (17.2). So $\Pi_{m,s}$ is $C$-Lipschitz.

For the Lipschitz dependence on $s$, first let $x \in L^\eta$ and $0 \leq s \leq t \leq 10^{-1}$ be given. Then

$$
\begin{aligned}
|\Pi_{m,s}(x)-\Pi_{m,t}(x)| &= |\Pi_L(x, \theta(s^{-1}d(x))) - \Pi_L(x, \theta(t^{-1}d(x)))| \\
&\leq C d(x) |\theta(s^{-1}d(x)) - \theta(t^{-1}d(x))|
\end{aligned}
\tag{17.12}
$$

by (3.20). If $d(x) \leq 3s \leq 3t$, then

$$
d(x)|\theta(s^{-1}d(x)) - \theta(t^{-1}d(y))| \leq 2d(x)\Big|\frac{d(x)}{s} - \frac{d(x)}{t}\Big| = 2d(x)^2\,\frac{|s-t|}{st} \leq 18|s-t|,
\tag{17.13}
$$

and we are happy. If $d(x) \geq 2t \geq 2s$, then $\theta(s^{-1}d(x)) = \theta(t^{-1}d(x)) = 0$ by (17.6), and we are happier. We are left with the case when $3s \leq d(x) \leq 2t$; then

$$
d(x)|\theta(s^{-1}d(x)) - \theta(t^{-1}d(y))| = d(x)\theta(t^{-1}d(y)) \leq d(x) \leq 2t \leq 6(t-s)
\tag{17.14}
$$

by (17.6) and because $3s \leq 2t$. This takes care of the case when $x \in L^\eta$. The other case is trivial, since $\Pi_{m,s}(x) = \Pi_{m,t}(x) = x$ when $x \in \mathbb{R}^n \setminus L^\eta$. So $\Pi_{m,s}(x)$ is Lipschitz in $s$ too, and (17.4) holds.

Finally, (17.5) is a direct consequence of the fact that $\Pi_L$ preserves the faces too, by (3.22).

We still need to prove the lemma when $r_0 < 1$; denote by $\Pi'_{m,s}$ the mapping that we just obtained for the unit grid; naturally, we set

$$
\Pi_{m,s}(x) = r_0 \Pi'_{m, r_0^{-1}s}(r_0^{-1}x) \quad \text{for } x \in \mathbb{R}^n \text{ and } 0 \leq s \leq 10^{-1}r_0;
\tag{17.15}
$$

the properties (17.2), (17.3), and (17.5) follow at once by conjugation, and for (17.4) a rapid inspection shows that the two Lipschitz constants for $\Pi_{m,s}(x)$ do not even depend on $r_0$. (We don't really need to know this, but it feels better.) Lemma 17.1 follows. $\square$

We shall need to know that

$$
|\Pi_{m,s}(x) - x| \leq C\,\mathrm{Min}\big(s, \mathrm{dist}(x, \mathcal{S}_m)\big) \text{ for } x \in \mathbb{R}^n \text{ and } 0 \leq s \leq 10^{-1}r_0.
\tag{17.16}
$$

And indeed, by (17.2) we may assume that $d(x) = \mathrm{dist}(x, \mathcal{S}_m) \leq 2s$, because otherwise $\Pi_{m,s}(x) = x$. Then pick $z \in \mathcal{S}_m$ such that $|z - x| = d(x)$, and observe that

$$\begin{aligned} |\Pi_{m,s}(x) - x| &\leq |\Pi_{m,s}(x) - \Pi_{m,s}(z)| + |\Pi_{m,s}(z) - x| \\ &= |\Pi_{m,s}(x) - \Pi_{m,s}(z)| + |z - x| \leq C|z - x| = Cd(x) \end{aligned} \tag{17.17}$$

because $\Pi_{m,s}(z) = z$ by (17.2); (17.16) follows.

Next we want a version of Lemma Lemma 17.1 that works for all the dimensions $m$ at the same time; naturally we shall obtain it by composing mappings $\Pi_{m,s}(x)$ provided by Lemma 17.1. We keep our usual dyadic grid of mesh $r_0$.

**Lemma 17.18.** *There is a mapping $\Pi : \mathbb{R}^n \times [0, 10^{-1}r_0] \to \mathbb{R}^n$, with the following properties:*

$$|\Pi(x, s) - x| \leq Cs \quad \text{for } x \in \mathbb{R}^n \text{ and } 0 \leq s \leq 10^{-1}r_0, \tag{17.19}$$

$$\Pi(x, s) \in F \quad \text{when } F \text{ is any face of the grid, } x \in \mathbb{R}^n, \text{ and } \mathrm{dist}(x, F) \leq C^{-1}s, \tag{17.20}$$

$$\Pi \text{ is } C\text{-Lipschitz on } \mathbb{R}^n \times [0, 10^{-1}r_0] \tag{17.21}$$

*and*

$$\text{every } \Pi(\cdot, s) \text{ preserves all the faces of our usual grid.} \tag{17.22}$$

For $s \in [0, 10^{-1}r_0]$, set

$$s_m = (6C)^{-m}s \quad \text{for } 0 \leq m \leq n-1, \tag{17.23}$$

where $C$ is the constant of (17.16) (chosen so that $C \geq 1$) and then

$$\Pi(x, s) = \Pi_{0,s_0} \circ \Pi_{1,s_1} \cdots \circ \Pi_{n-1,s_{n-1}}(x) \tag{17.24}$$

for $x \in \mathbb{R}^n$. Notice that $\Pi_{m,s_m}$ is well defined, because $0 \leq s_m \leq 10^{-1}r_0$, and that (17.19) holds (with a larger constant $C$) by successive applications of (17.16). Also, (17.21) follows from (17.4) and the chain rule, and (17.22) is a consequence of (17.5).

We are left with (17.20) to check. Let $F$ be a face and $x \in \mathbb{R}^n$ be such that

$$\mathrm{dist}(x, F) \leq s_{n-1} = (6C)^{1-n}s; \tag{17.25}$$

we want to check that $\Pi(x, s) \in F$. Set $x_{n+1} = x_n = x$, then $x_{n-1} = \Pi_{n-1,s_{n-1}}(x)$, and by induction

$$x_k = \Pi_{k,s_k}(x_{k+1}) \quad \text{for } 0 \leq k \leq n-1. \tag{17.26}$$

Thus $\Pi(x,s) = x_0$. Notice that

$$|x_k - x| \le C(s_{n-1} + \cdots s_k) \tag{17.27}$$

by successive applications of (17.16), and where for the few next lines $C$ will stay the same as in (17.16) and (17.23).

Let $m$ denote the dimension of $F$; observe that

$$\operatorname{dist}(x_{m+1}, F) \le \operatorname{dist}(x, F) + |x_{m+1} - x| \le s_{n-1} + C \sum_{k>m} s_k. \tag{17.28}$$

Next denote by $l$ the smallest nonnegative integer such that

$$\operatorname{dist}(x_{l+1}, F') \le s_{n-1} + 4C \sum_{k>l} s_k \tag{17.29}$$

for some face $F' \subset F$ of dimension $l$. Thus $l \le m$, by (17.28). Let us check that

$$s_{n-1} + 4C \sum_{k>l} s_k \le s_l. \tag{17.30}$$

If $l = n-1$, (17.30) holds because the left-hand side is $s_l$. If $l < n-1$,

$$s_{n-1} + 4C \sum_{k>l} s_k \le 5C \sum_{k>l} s_k \le 5C s_l \sum_{j\ge 1} (6C)^{-j} \le \frac{5s_l}{6} \sum_{j\ge 0} 6^{-j} = s_l \tag{17.31}$$

because we assumed that $C \ge 1$; so (17.30) holds. Now

$$\operatorname{dist}(x_{l+1}, \mathcal{S}_l) \le \operatorname{dist}(x_{l+1}, F') \le s_{n-1} + 4C \sum_{k>l} s_k \le s_l \tag{17.32}$$

by (17.29) and (17.30), so (17.3) says that $x_l = \Pi_{l,s_l}(x_{l+1})$ lies in $\mathcal{S}_l$.

If $x_l \in F$, we are happy because all the later $\Pi_{k,s_k}$ preserve the faces, so $\Pi(x,s) = x_0$ lies in $F$ too. So assume that $x_l \notin F$. Let $F''$ denote a face of dimension $l$ that contains $x_l$, and notice that $F'' \neq F'$ because $x_l \notin F'$ since $F' \subset F$. Use (17.29) to choose $z \in F'$ such that

$$|z - x_{l+1}| \le s_{n-1} + 4C \sum_{k>l} s_k. \tag{17.33}$$

If $l > 0$, (3.8) says that

$$\operatorname{dist}(z, \partial F') \le \operatorname{dist}(z, F'') \le |z - x_l| \le |z - x_{l+1}| + Cs_l \le s_{n-1} + 4C \sum_{k>l} s_k + Cs_l \tag{17.34}$$

by (17.26), (17.16), and (17.33), and so

$$\operatorname{dist}(x_l, \partial F') \le \operatorname{dist}(z, \partial F') + |z - x_l| \le 2s_{n-1} + 8C \sum_{k>l} s_k + 2Cs_l. \tag{17.35}$$

Since $2s_{n-1} + 8C \sum_{k>l} s_k \le 2s_l$ by (17.30), we get that

$$\operatorname{dist}(x_l, \partial F') \le 4Cs_l \le 4C \sum_{k>l-1} s_k, \tag{17.36}$$

which contradicts the minimality of $l$, because $\partial F' \subset F' \subset F$. So in fact $l = 0$, and $F'$ and $F''$ are just points of the grid. Then $F'' = \{x_l\}$ and $F' = \{z\}$, and these points are distinct. But the last part of (17.34) is still valid, and says that $|z - x_l|$ is very small. This contradiction shows that $x_l \in F$ was the only option, and completes our proof of (17.20); Lemma 17.18 follow. □

**17.b. A stable deformation for the $E_k$.**

Recall from Section 16 that we have defined a family of mappings $g_t$, $0 \le t \le 1$, that satisfy the constraints (1.4)-(1.8) with respect to our limit set $E$. We want to use the magnetic projection given by Lemma 17.18 to modify the $g_t$ and make them work for the $E_k$ as well. As usual, we start in the rigid case.

Let $\varepsilon_0$ be small, to be chosen below, and set

$$h_t(x) = \Pi(g_t(x), s_t(x)) \ \text{ for } x \in U \text{ and } 0 \le t \le 1, \tag{17.37}$$

where we set

$$s_t(x) = C \operatorname{Min}(\varepsilon_0, |g_t(x) - x|), \tag{17.38}$$

where $C$ is the constant of (17.20). We shall choose $\varepsilon_0$ much smaller than $(10C)^{-1} r_0$, so $h_t(x)$ is well defined. Observe that since $0 \le s_t(x) \le C\varepsilon_0$, (17.19) yields

$$|h_t(x) - g_t(x)| = |\Pi(g_t(x), s_t(x)) - g_t(x)| \le Cs_t(x) \le C\varepsilon_0 \tag{17.39}$$

for $x \in U$ and $0 \le t \le 1$ (and with a new constant $C$). We are interested in the following.

**Lemma 17.40.** *For $k$ large enough, the mappings $h_t$, $0 \le t \le 1$, satisfy the conditions (1.4)-(1.8), relative to $E_k$ and the ball $B'' = \overline{B}(X_0, R'')$, where*

$$R'' = R_0 + 4\Lambda^2(1 + |f|_{lip})\delta_6 + C\Lambda\varepsilon_0. \tag{17.41}$$

We give the statement with $\Lambda$ because it will be valid in the Lipschitz case, but for the moment we may take $\Lambda = 1$.

There is still no difficulty with (1.4) and (1.8), since we merely composed $g_t(x)$ with continuous functions of $x$ and $t$, which happen to be Lipschitz when $t = 1$. Notice that by (17.19),

$$h_t(x) = g_t(x) = x \ \text{ when } g_t(x) = x, \tag{17.42}$$

because then $s_t(x) = 0$. Because of this,

$$h_0(x) = x \quad \text{for } x \in U, \tag{17.43}$$

by (16.12) and because $\varphi_0(x) = x$ for $x \in U$, by (11.14). Let us also check that

$$h_t(x) = x \quad \text{for } x \in U_{ext} \text{ and } 0 \leq t \leq 1, \tag{17.44}$$

where $U_{ext} = \big\{x \in \mathbb{R}^n\,;\, \mathrm{dist}(x, \mathbb{R}^n \setminus U) \leq \delta_0/2\big\}$ as in (11.2). Let $x \in U_{ext}$ be given. By (11.3), $\varphi_t(x) = x$ for $0 \leq t \leq 1$, and in particular $f(x) = x$. We will be finished as soon as we check that $g(x) = f(x)$, because then $g_t(x) = x$ for all $t$, by (16.12) and (16.13), and we can apply (17.42).

But $\mathrm{dist}(x, \widehat{W}) \geq \delta_0/2$ because $\delta_0 = \mathrm{dist}(\widehat{W}, \mathbb{R}^n \setminus U)$ by (11.2), so $\mathrm{dist}(x, X_1) \geq \delta_0/2$ because $X_1 \subset X_0 \subset \widehat{W}$ (by (11.20)). Finally, $\delta_0/2 > \delta_6$ by (12.7), so the first part of (16.15) says that $x \in U \setminus V_1^+$, and then $g(x) = f(x)$ by (16.8); (17.44) follows.

For the next verifications, we shall often need to restrict to $E_k$ (we shall not have enough information on the values of the $\varphi_t$ far from $E$), and we shall find it more convenient to work on the set

$$H = \big\{x \in U\,;\, \mathrm{dist}(x, \mathbb{R}^n \setminus U) \geq \delta_0/2\big\} \supset \mathbb{R}^n \setminus U_{ext}. \tag{17.45}$$

because $H$ is a compact subset of $U$ and it will be easier to use our assumption (10.4) (i.e., the convergence of the $E_k$ to $E$) on that set. Indeed, set

$$d_k = \sup_{x \in E_k \cap H} \mathrm{dist}(x, E); \tag{17.46}$$

it is easy to deduce from (10.4)-(10.6) that $\lim_{k\to+\infty} d_k = 0$ (cover $H$ with a finite set of balls).

We claim that for $k$ large,

$$h_t(x) = x \quad \text{for } 0 \leq t \leq 1 \text{ when } x \in E_k \setminus B(X_0, R_0 + \varepsilon_0). \tag{17.47}$$

Because of (17.44), it is enough to prove this when $x \in E_k \cap H \setminus B(X_0, R_0 + \varepsilon_0)$. Then

$$\mathrm{dist}(x, E) \leq d_k < \frac{1}{2}\,\mathrm{Min}(\varepsilon_0, \delta_1) \tag{17.48}$$

for $k$ large.

For each $t \in [0, 1]$, the set $W_t$ of (11.13) is contained in $B = \overline{B}(X_0, R_0)$, by (1.5) and (11.1), so $\mathrm{dist}(x, E) < \varepsilon_0/2 \leq \frac{1}{2}\,\mathrm{dist}(x, W_t)$ by (17.48) and because $x \notin B(X_0, R_0 + \varepsilon_0)$. This is good, because (11.12) says that then $\varphi_t(x) = x$. [This is not a surprise; recall that we computed $\varphi_t(x) - x$ by Whitney-extending the values of $\varphi_t(\xi) - \xi$ on $E \cup E_{ext}$, which happen to vanish near $x$.]

If we also prove that $g(x) = f(x)$, (16.12) and (16.13) will say that $g_t(x) = x$ for all $t$, and the result will follow by (17.42). For this, it is enough to prove that $x \notin V_1^+$, by

(16.8). By (17.48), we can find $z \in E$ such that $|z-x| < \frac{1}{2}\,\mathrm{Min}(\varepsilon_0, \delta_1)$. In particular, $z \notin B$ (because $x \notin B(X_0, R_0 + \varepsilon_0)$), so (1.5) says that $f(z) = \varphi_1(z) = z$. Thus $z \in \mathbb{R}^n \setminus W_f$ (see (11.19)), and so $\mathrm{dist}(x, \mathbb{R}^n \setminus W_f) \leq |z - x| < \delta_1/2$, and indeed this makes $x \in V_1^+$ impossible, by the second part of (16.15). This proves our claim (17.47), and (1.5) (for $E_k$ and the $h_t$, and with a slightly larger ball) follows.

Before we prove (1.6), let us check that for $x \in U$ and $0 \leq t \leq 1$, we can find $s \in [0,1]$ such that

$$|g_t(x) - \varphi_s(x)| \leq 4\Lambda^2(1 + |f|_{lip})\delta_6. \tag{17.49}$$

When $t \leq 1/2$, we just take $s = t/2$ and observe that $g_t(x) = \varphi_s(x)$ by (16.12). When $t \geq 1$, we take $s = 1$ and observe that $|g_t(x) - \varphi_1(x)| \leq |g(x) - f(x)| \leq 4\Lambda^2(1 + |f|_{lip})\delta_6$, by (16.13) and (16.11). So (17.49) holds. Notice also that it implies that

$$|h_t(x) - \varphi_s(x)| \leq 4\Lambda^2(1 + |f|_{lip})\delta_6 + C\varepsilon_0, \tag{17.50}$$

by (17.39).

We are ready to prove (1.6). In fact, we just need to prove that for $k$ large,

$$h_t(x) \in B'' \text{ when } x \in E_k \cap B(X_0, R_0 + \varepsilon_0), \tag{17.51}$$

where $B''$ is as in the statement of Lemma 17.40, since (17.47) says that $h_t(x) = x \in B''$ when $x \in E_k \cap B'' \setminus B(X_0, R_0 + \varepsilon_0)$. Pick $z \in E$ such that $|z - x| < \frac{4}{3}\,\mathrm{dist}(x, E) \leq \frac{4}{3} d_k$. For $k$ large enough, $z \in B(X_0, R_0 + 2\varepsilon_0)$, and by (1.5) and (1.6) for $E$ and the $\varphi_t$, $\varphi_t(z) \in B(X_0, R_0 + 2\varepsilon_0)$ for $0 \leq t \leq 1$.

Also let $H'$ denote a compact neighborhood of $H$ in $U$. The function $(y, t) \in H' \times [0,1] \to \varphi_t(x)$ is uniformly continuous; if $k$ is large enough, $z \in H'$ because $x \in H$, and $|z - x|$ is so small that $|\varphi_t(z) - \varphi_t(x)| \leq \varepsilon_0$. Then $\varphi_t(x) \in B(X_0, R_0 + 3\varepsilon_0)$ for $0 \leq t \leq 1$. We then use (17.50) and get that $h_t(x) \in B''$, if the constant $C$ in (17.41) is large enough). This proves (1.6) for the $h_t$.

Finally we need to prove that (1.7) holds, relative to $E_k$, and for $k$ large. As before, we shall restrict our attention to $H$ first. Set, for $0 \leq j \leq j_{max}$ and $k \geq 0$,

$$d_{j,k} = \sup_{x \in L_j \cap E_k \cap H} \mathrm{dist}(x, E \cap L_j); \tag{17.52}$$

we claim that for each $j$,

$$\lim_{k \to +\infty} d_{j,k} = 0. \tag{17.53}$$

Otherwise, we can find $j \leq j_{max}$ and a sequence of points $x_k \in E_k \cap L_j \cap H$, for which $t_k = \mathrm{dist}(x_k, E \cap L_j)$ does not tend to 0. Passing to a subsequence, we may even assume that $t_k \geq a$ for some $a > 0$, and that $x_k$ tends to a limit $x_\infty$. Then $x_\infty \in L_j \cap H$ because $L_j \cap H$ is closed, and $x_\infty \in E$ because $E$ is closed and $\mathrm{dist}(x_k, E)$ tends to 0 by (10.4) (see near (17.46)). Now the fact that $x_\infty \in L_j \cap E$ contradicts the fact that $t_k \geq a$; our claim (17.53) follows. Next set

$$\eta_{j,k} = \sup\big\{|g_t(x) - g_t(y)|\,;\, x \in H, |x-y| \leq 2d_{j,k}, \text{ and } 0 \leq t \leq 1\big\}; \tag{17.54}$$

then $\lim_{k\to+\infty} \eta_{j,k} = 0$, by (17.53) and because $(x,t) \to g_t(x)$ is uniformly continuous on $H' \times [0,1]$, where $H'$ a compact neighborhood of $H$ in $U$.

Let us check that (1.7) holds when $k$ is large enough. Let $j \leq j_{max}$ and $x \in E_k \cap L_j$ be given; we want to check that $h_t(x) \in L_j$ for $0 \leq t \leq 1$. We may assume that $x \in H$, because otherwise $h_t(x) = x$ by (17.44). Pick $y \in E \cap L_j$ such that $|y - x| \leq 2d_{j,k}$; then

$$\mathrm{dist}(g_t(x), L_j) \leq |g_t(x) - g_t(y)| \leq \eta_{j,k} \leq \varepsilon_0 \tag{17.55}$$

because $g_t(y) \in L_j$ by (1.7) for the $g_t$ relative to $E$, and if $k$ is so large that $\eta_{j,k} < \varepsilon_0$. Also, $\mathrm{dist}(g_t(x), L_j) \leq |g_t(x) - x|$ because $x \in L_j$; altogether,

$$\mathrm{dist}(g_t(x), L_j) \leq C^{-1} s_t(x), \tag{17.56}$$

by (17.38) and where $C$ is as in (17.38) and (17.20). Let $F$ be a face of $L_j$ such that $\mathrm{dist}(g_t(x), L_j) = \mathrm{dist}(g_t(x), F)$; then

$$h_t(x) = \Pi(g_t(x), s_t(x)) \in F \tag{17.57}$$

by (17.37), (17.20), and (17.56). Thus $h_t(x) \in F \subset L_j$; this completes our proof of (1.7), and Lemma 17.40 follows. □

We shall also need to know about the analogue, for the mappings $h_t$ and the set $E_k$, of the sets $W_t$ and $\widehat{W}$. We claim that for $k$ large and $x \in E_k$,

$$h_t(x) = x \ \text{ for } 0 \leq t \leq 1 \text{ when } \mathrm{dist}\big(x, \bigcup_{0\leq t\leq 1} W_t\big) \geq 2d_k \tag{17.58}$$

(where $W_t$ is as in (11.13) and $d_k$ as in and (17.46)) and

$$\begin{aligned}\mathrm{dist}(h_t(x), \widehat{W}) \leq 4\Lambda^2(1+|f|_{lip})\delta_6 + C\Lambda\varepsilon_0 \ &\text{ for } 0 \leq t \leq 1 \\ &\text{when } \mathrm{dist}\big(x, \bigcup_{0\leq t\leq 1} W_t\big) \leq 2d_k.\end{aligned} \tag{17.59}$$

The proof will be almost the same as for (17.47) and (17.51). First let $x \in E_k$ be such that $\mathrm{dist}\big(x, \bigcup_{0\leq t\leq 1} W_t\big) \geq 2d_k$. If $x \in U_{ext}$, (17.44) says $h_t(x) = x$, as needed. So we may assume that $x \in \mathbb{R}^n \setminus U_{ext} \subset H$ (by (17.45)). Then $\mathrm{dist}(x,E) \leq \frac{1}{2}\mathrm{dist}(x, W_t)$ for all $t$, by (17.46). Hence, by (11.12) $\varphi_t(x) = x$ for $0 \leq t \leq 1$. In particular, $f(x) = x$. Next let us check that $x \notin V_1^+$. Let $y \in E$ be such that $|y - x| < \frac{3}{2}\mathrm{dist}(x,E) \leq \frac{3}{2}d_k$. Then $\mathrm{dist}(y, W_1) > d_k/2$, so $f(y) = \varphi_1(y) = y$. That is, $y \in \mathbb{R}^n \setminus W_f$ (see the definition (11.19)) and $\mathrm{dist}(x, \mathbb{R}^n \setminus W_f) \leq |y - x| \leq 2d_k$. For $k$ large this forces $x \notin V_1^+$, by the second part of (16.15). Hence $g(x) = f(x) = x$, by (16.8), and then $g_t(x) = x$ for all $t$, by (16.12) and (16.13). Finally, $h_t(x) = x$ by (17.42), as needed for (17.58).

Now suppose that $\mathrm{dist}\big(x, \bigcup_{0\leq t\leq 1} W_t\big) \leq 2d_k$, and choose $y \in W_t$ be such that $|y-x| < \frac{3}{2}\,\mathrm{dist}(x,E) \leq 3d_k$. By (2.1) and (2.2), $\varphi_t(y) \in \widehat{W}_t$ for $0 \leq t \leq 1$. If $x \in U_{ext}$, $h_t(x) = x$ by (17.44), hence $\mathrm{dist}(h_t(x), \widehat{W}) \leq |x-y| \leq 3d_k \leq \varepsilon_0$ (for $k$ large). Otherwise, (17.45) says that $x \in H$, and we use the uniform continuity of $(y,t) \to \varphi_t(y)$ on $H' \times [0,1]$ to show that for $k$ large enough, $\mathrm{dist}(\varphi_t(x), \widehat{W}) \leq \varepsilon_0$ for $0 \leq t \leq 1$. Then we apply (17.50) and get that $\mathrm{dist}(g_t(x), \widehat{W}) \leq 4\Lambda^2(1+|f|_{lip})\delta_6 + C\Lambda\varepsilon_0$ for all $t$, as needed for (17.59).

† Let us now say how we modify all this when we work with the Lipschitz assumption. We don't need to change Lemmas 17.1 and 17.18, but we need to modify the definition of the $h_t$. We first set

$$h_t(x) = x \quad \text{when } x \in U_{ext} \text{ and } 0 \leq t \leq 1; \tag{17.60}$$

this is not too shocking, because of (17.44). Let us also define the $h_t$ on $E_k \cap H_1$, where

$$H_1 = \big\{x \in U\,;\, \mathrm{dist}(x, \mathbb{R}^n \setminus U) \geq \delta_0/3\big\} \tag{17.61}$$

We shall see soon that although the two sets overlap, our two definitions coincide on their intersection $E_k \cap H_1 \cap U_{ext}$. We modify our original definition by (17.37) and (17.38) and set, for $x \in E_k \cap H_1$,

$$\widetilde{s}_t(x) = C\,\mathrm{Min}(\lambda\varepsilon_0, |\widetilde{g}_t(x) - \psi(\lambda x)|), \tag{17.62}$$

where $C$ is still as in (17.20) and $\widetilde{g}$ was defined near (16.29), and then

$$\widetilde{h}_t(x) = \Pi(\widetilde{g}_t(x), \widetilde{s}_t(x)) \quad \text{for } 0 \leq t \leq 1. \tag{17.63}$$

We naturally intend to set

$$h_t(x) = \lambda^{-1}\psi^{-1}(\widetilde{h}_t(x)) \quad \text{for } x \in E_k \cap H_1 \text{ and } 0 \leq t \leq 1, \tag{17.64}$$

but as usual we first need to check that $\widetilde{h}_t(x) \in B(0,1)$, and this will be easier after some estimates on $\widetilde{h}_t(x)$. Let us first describe some situations where $\widetilde{h}_t(x) = \psi(\lambda x)$. The analogue of (17.42) is now that

$$\widetilde{h}_t(x) = \widetilde{g}_t(x) = \psi(\lambda x) \quad \text{when } x \in E_k \cap H_1 \text{ and } g_t(x) = x \tag{17.65}$$

(or equivalently when $\widetilde{g}_t(x) = \psi(\lambda x)$, see (16.28) and recall that $x \in H_1 \subset \mathbb{R}^n \setminus U_{ext}$ by (17.61) and (11.2)). The proof stays the same: just observe that $\widetilde{s}_t(x) = 0$ and apply (17.19). Because of this, we shall easily get that $\widetilde{h}_t(x) = \widetilde{g}_t(x) = \psi(\lambda x)$ in the situations where we proved that $h_t(x) = g_t(x) = x$.

First observe that we still get that $h_t(x) = g_t(x) = x$ when $x \in E_k \cap H_1 \cap U_{ext}$, by the proof of (17.44) (and where we replace (16.13) with (16.29)). Because of this, our two definitions on that set coincide, which is good because we also get that $h_t(x)$ is continuous in both variables and Lipschitz in $x$.

We also get that

$$\widetilde{h}_t(x) = \widetilde{g}_t(x) = \psi(\lambda x) \text{ when } t = 0, \text{ when } x \in E_k \setminus B(X_0, R_0 + \varepsilon_0), \\ \text{and when } \operatorname{dist}\big(x, \bigcup_{0 \leq t \leq 1} W_t\big) \geq 2d_k, \tag{17.66}$$

as in (17.43), in (17.47), and in (17.58). In all these cases, we need $k$ to be large enough (so that we can define $h_t(x)$, but not only), and then we follow the proofs above (except that we replace (6.13) with (16.29) and (17.42) with (17.65)). Of course in all these cases, the formula in (17.64) makes sense because $\widetilde{h}_t(x) \in B(0,1)$, and yields $h_t(x) = g_t(x) = x$.

Next we generalize the formulas (17.49) and (17.50). We shall restrict to $x \in E_k \cap H_1$, because for $x \in E_k \setminus H_1$, (17.60) will be enough.

Let us first check that for $x \in U_{int}$ (the set defined in (12.72) and where $\widetilde{f}$ is nicely defined) and $0 \leq t \leq 1$, we can find $s \in [0,1]$ such that

$$|\widetilde{g}_t(x) - \psi(\lambda\varphi_s(x))| \leq 4\lambda\Lambda(1 + |f|_{lip})\delta_6. \tag{17.67}$$

When $t \leq 1/2$, we take $s = 2t$, and (17.67) holds trivially because $\widetilde{g}_t(x) = \psi(\lambda\varphi_s(x))$ by (16.12) and (12.75). When $t \geq 1/2$, we take $s = 1$ (and hence $\psi(\lambda\varphi_s(x)) = \widetilde{f}(x)$) and use (16.29) to get that

$$|\widetilde{g}_t(x) - \widetilde{f}(x)| \leq |\widetilde{g}(x) - \widetilde{f}(x)| \leq 4\lambda\Lambda(1 + |f|_{lip})\delta_6, \tag{17.68}$$

by the line above (13.35), (12.8), (14.26), and (15.51) (or faster, if you are willing to lose a factor $\Lambda^2$, by (16.11) and (16.3)). This proves (17.67); since

$$|\widetilde{h}_t(x) - \widetilde{g}_t(x)| \leq C\widetilde{s}_t(x) \leq C\lambda\varepsilon_0 \tag{17.69}$$

by (17.63), (17.19), and (17.62), we deduce from (17.67) that (still for $x \in E_k \cap H_1$)

$$|\widetilde{h}_t(x) - \psi(\lambda\varphi_s(x))| \leq 4\lambda\Lambda(1 + |f|_{lip})\delta_6 + C\lambda\varepsilon_0. \tag{17.70}$$

When $x \in E_k \cap H_1 \setminus U_{int}$, we can even say a bit more. For $t \leq 1/2$, we still have that $\widetilde{g}_t(x) = \psi(\lambda\varphi_s(x))$ with $s = 2t$ by (16.12). For $t \geq 1/2$, we did not want to define $\widetilde{g}_t(x)$ directly in Section 16, and instead we set $g_t(x) = f(x)$ directly; see the definition about nine lines below (16.30). This is because if $x$ is any point of $U \setminus U_{int}$, we cannot be sure that $f(x) \in U$, and the we cannot define $\widetilde{f}(x) = \psi(\lambda f(x))$ or $\widetilde{g}_t(x)$. But here we only care about points $x \in E_k \cap H_1$. Let $H_1'$ be a compact neighborhood of $H_1$, with $H_1' \subset U$. Notice that for $x \in E \cap H_1'$ and $0 \leq t \leq 1$, $\varphi_t(x) \in \widehat{W} \cup H_1'$, either by (2.2) or because $\varphi_t(x) = x \in H_1'$. Then there is a compact neighborhood $H''$ of $E \cap H_1'$, such that $\varphi_t(H'') \subset U$ for $0 \leq t \leq 1$. Since for $k$ large, $E_k \cap H_1 \subset H''$, we get that $\varphi_t(E_k \cap H_1) \subset\subset W$ for $k$ large. That is, for $k$ large we can define $\psi(\lambda\varphi_t(x))$ for $x \in E_k \cap H_1$. Then we can set

$\widetilde{g}_t(x) = \psi(\lambda\varphi_1(x)) = \psi(\lambda f(x))$ for $t \geq 2$, because $g_t(x) = f(x) = \varphi_1(x)$. This is of course better than (17.68). And we still have (17.69) (with the same proof) and hence (17.70) in this case. So we get a good definition of $\widetilde{g}_t$ and $\widetilde{h}_t$ on $E_k \cap H_1$.

Recall that we shall not worry about $x \in E_k \setminus H_1$, because we have the simpler formula (17.60).

Now we want to generalize (17.59) (on the set $E_k \cap H_1$). Let us first prove that

$$\text{dist}(\widetilde{h}_t(x), \psi(\lambda\widehat{W})) \leq 4\lambda\Lambda(1 + |f|_{lip})\delta_6 + C\lambda\varepsilon_0 \quad \text{for } 0 \leq t \leq 1 \tag{17.71}$$

when $x \in E_k \cap H_1$ is such that $\text{dist}\big(x, \bigcup_{0\leq t\leq 1} W_t\big) \leq 2d_k$. For such $x$ and $k$, we can find $t \in [0,1]$ and $y \in W_t$ such that $|y - x| \leq 3d_k$; then $y \in \widehat{W}$ by (2.2), and also $\varphi_s(y) \in \widehat{W}$ for $0 \leq s \leq 1$, either because $y \in W_s$ and by (2.2), or because $y \in E \setminus W_s$ and $\varphi_x(y) = y \in \widehat{W}$. Thus

$$\psi(\lambda\varphi_s(y)) \in \psi(\lambda\widehat{W}) \quad \text{for } 0 \leq s \leq 1. \tag{17.72}$$

If $x \in E_k \cap H_1 \cap U_{ext}$, we checked that (17.60) is still valid; it says that $h_t(x) = x$, so $\widetilde{h}_t(x) = \psi(\lambda x)$, and (17.71) holds because

$$\text{dist}(\widetilde{h}_t(x), \psi(\lambda\widehat{W})) \leq |\psi(\lambda x) - \psi(\lambda y)| \leq \lambda\Lambda|x - y| \leq 3\lambda\Lambda d_k \tag{17.73}$$

(since $y \in \widehat{W}$) and for $k$ large. By (17.45) we are left with the case when $x \in H$, and (17.70) says that $|\widetilde{h}_t(x) - \psi(\lambda\varphi_s(x))| \leq 4\lambda\Lambda(1 + |f|_{lip})\delta_6 + C\lambda\varepsilon_0$ for some $s$. In addition, $|\psi(\lambda\varphi_s(x)) - \psi(\lambda\varphi_s(y))| \leq \lambda\varepsilon_0$ if $k$ is large enough, because $|y - x| \leq 3d_k$ and by the usual uniform continuity argument near $H$ (see below (17.51) for instance). We deduce (17.71) from this and (17.72).

Notice that

$$\text{dist}(\psi(\lambda\widehat{W}), \mathbb{R}^n \setminus B(0,1)) \geq \lambda\Lambda^{-1}\,\text{dist}(\widehat{W}, \mathbb{R}^n \setminus U) = \lambda\Lambda^{-1}\delta_0 \geq 10\lambda\Lambda(1+|f|_{lip})\delta_6 \tag{17.74}$$

by (12.6) and (12.7); thus (17.71) implies that $\widetilde{h}_t(x) \subset B(0,1)$ when $k$ is large and $x \in E_k \cap H_1$ is such that $\text{dist}\big(x, \bigcup_{0\leq t\leq 1} W_t\big) \leq 2d_k$.

If instead $x \in E_k \cap H_1$ but $\text{dist}\big(x, \bigcup_{0\leq t\leq 1} W_t\big) > 2d_k$, (17.66) says that $\widetilde{h}_t(x) = \psi(\lambda x) \in B(0,1)$ (as in 17.58). So we proved that for $k$ large,

$$\widetilde{h}_t(x) \subset B(0,1) \text{ for } x \in E_k \cap H_1 \text{ and } t \in [0,1]. \tag{17.75}$$

This is good to know, because now we may define $h_t$ on $E_k \cap H_1$ by (17.64), and complete our verifications with a free mind.

Notice for instance that (17.70) implies that for $x \in E_k \cap H_1$

$$|h_t(x) - \varphi_s(x)| \leq 4\Lambda^2(1 + |f|_{lip})\delta_6 + C\Lambda\varepsilon_0 \tag{17.76}$$

(for the same $s$ that we found for (17.67)); this is almost as good as (17.50).

We already checked that (1.4) and (1.8) hold (see above (17.66), when we checked that our two definitions coincide on $E_k \cap H_1 \cap E_{ext}$), and (1.5) follows from (17.66). We still need to check (1.6), but we may now repeat the proof given near (17.51), except that when we consider points of $E_k \setminus H_1$, we just use the simpler (16.60). A similar argument applies to the final comment (17.59). We even added the constant $\Lambda$ in advance, to take care of the extra $\Lambda$ in (17.76).

We are left with the proof of (1.7), which we only need to modify slightly. We keep the same definition for the $d_{j,k}$ in (17.52), but define the modulus of continuity $\eta_{j,k}$ of (17.54) as before, only with respect to the $\widetilde{g}_t$; they both tend to 0 for the same reasons as before.

Then suppose that $\eta_{j,k} \leq \lambda\varepsilon_0$, and let $x \in L_j \cap E_k$ be given. If $x \in E_{ext}$, (16.60) says that $h_t(x) = x \in L_j$ (as needed), so we may assume that $x \in H_1$, and then we shall use (17.62) and (17.63). Choose $y \in L_j \cap E$ such that $|y - x| \leq d_k$, and observe that

$$\begin{aligned} \operatorname{dist}(\widetilde{g}_t(x), \psi(\lambda L_j)) &\leq \operatorname{Min}\big(|\widetilde{g}_t(x) - \widetilde{g}_t(y)|, |\widetilde{g}_t(x) - \psi(\lambda x)|\big) \\ &\leq \operatorname{Min}\big(\lambda\varepsilon_0, |\widetilde{g}_t(x) - \psi(\lambda x)|\big) \leq C^{-1}\widetilde{s}_t(x) \end{aligned} \tag{17.77}$$

since $g_t(y)$ and $x$ both lie in $L_j$, because $|\widetilde{g}_t(x) - \widetilde{g}_t(y)| \leq \eta_{j,k} \leq \lambda\varepsilon_0$, and by (17.62). This allows us to use (17.20), get that $\widetilde{h}_t(x) = \Pi(\widetilde{g}_t(x), \widetilde{s}_t(x)) \in \widetilde{F}$, where $\widetilde{F}$ is a face of $\psi(\lambda L_j)$ that lies close to $\widetilde{g}_t(x)$, and conclude as before. So Lemma 17.40 also holds under the Lipschitz assumption.

Notice finally that our final estimates (17.58) and (17.59) still hold in the present context. For (17.58), this follows from (17.66) and (17.64), or directly (17.60); for (17.59) we use (17.71) and (17.64) if $x \in E_k \cap H$, and (17.60) and the fact that $\operatorname{dist}(h_t(x), \widehat{W}) = \operatorname{dist}(x, \widehat{W}) \leq 2d_k$ otherwise. †

**17.c. A last minute modification of our deformation.**

The family $\{h_t\}$ that we just constructed is almost perfect, but we would also like to make the set where $h_1(x) \neq x$ a little smaller, so that it stays away from the boundary of

$$W_f = \big\{x \in U\,;\, f(x) \neq x\big\}. \tag{17.78}$$

The point is that the sets $E_k$ could have a large piece in $W_f$ very near $\partial W_f$, while the corresponding piece of $E$ lies in $\partial W_f$ and is not accounted for in some estimates; this could be bad.

So we want to replace $g_t$ and $h_t$ by mappings that coincide with the identity very near $\partial W_f$. We want to continue the $g_t$, and then the $h_t$, with a deformation that only moves points near $\partial W_f$. First recall from (16.13) (or (16.28) and (16.29)) and (16.8) that

$$g_1(x) = g(x) = f(x) \quad \text{for } x \in \mathbb{R}^n \setminus V_1^+. \tag{17.79}$$

Thus, by the second part of (16.15),

$$g_1(x) = g(x) = f(x) \quad \text{when } \operatorname{dist}(x, \mathbb{R}^n \setminus W_f) \leq \delta_1/2. \tag{17.80}$$

As usual, we start with the rigid case. Let us define $g_t$ for $1 \le t \le 2$. Let $\varepsilon_*$ be (much) smaller than $\delta_1/4$, to be chosen below. First set

$$g_t(x) = g_1(x) \quad \text{for } 1 \le t \le 2 \text{ when } \operatorname{dist}(x, \mathbb{R}^n \setminus W_f) \ge 2\varepsilon_*. \tag{17.81}$$

When $\varepsilon_* \le \operatorname{dist}(x, \mathbb{R}^n \setminus W_f) \le 2\varepsilon_*$, first define

$$\begin{aligned} g_2(x) &= \frac{\operatorname{dist}(x, \mathbb{R}^n \setminus W_f) - \varepsilon_*}{\varepsilon_*}\, g_1(x) + \frac{2\varepsilon_* - \operatorname{dist}(x, \mathbb{R}^n \setminus W_f)}{\varepsilon_*}\, x \\ &= \frac{\operatorname{dist}(x, \mathbb{R}^n \setminus W_f) - \varepsilon_*}{\varepsilon_*}\, f(x) + \frac{2\varepsilon_* - \operatorname{dist}(x, \mathbb{R}^n \setminus W_f)}{\varepsilon_*}\, x, \end{aligned} \tag{17.82}$$

where the identity comes from (17.80); just set

$$g_2(x) = x \quad \text{when } \operatorname{dist}(x, \mathbb{R}^n \setminus W_f) < \varepsilon_*. \tag{17.83}$$

Notice that $g_2$ is continuous across the two obvious boundaries. Now set

$$g_t(x) = (2-t)g_1(x) + (t-1)g_2(x) = (2-t)f(x) + (t-1)g_2(x) \tag{17.84}$$

when $\operatorname{dist}(x, \mathbb{R}^n \setminus W_f) \le 2\varepsilon_*$ (i.e., in the two last cases) and for $1 \le t \le 2$.

Now we have a complete definition of the $g_t$, $0 \le t \le 2$. Again, $g_t(x)$ is a continuous function of $t$ and $x$, by construction. We still define the $h_t$, $1 \le t \le 2$, by the formulas (17.37) and (17.38); observe that

$$h_t(x) = h_1(x) \text{ for } 1 \le t \le 2 \text{ when } \operatorname{dist}(x, \mathbb{R}^n \setminus W_f) \ge 2\varepsilon_*, \tag{17.85}$$

by (17.81) and because $s_t(x)$, and then $h_t(x)$ depend only on the values $g_t(x)$. We are happier because

$$h_2(x) = g_2(x) = x \quad \text{when } \operatorname{dist}(x, \mathbb{R}^n \setminus W_f) < \varepsilon_*, \tag{17.86}$$

because $s_2(x) = 0$ by (17.83) and (17.38). Now we want to check that

$$\text{Lemma 17.40 also holds for the } h_{2t},\ 0 \le t \le 1. \tag{17.87}$$

Since $h_t(x)$, $1 \le t \le 2$ is continuous in both variables and Lipschitz in $x$, (1.4) and (1.8) still hold as before; (17.42) still holds for $t \ge 1$, for the same reasons. Then

$$h_t(x) = g_1(x) = f(x) = x \text{ for } 1 \le t \le 2 \text{ and } x \in \mathbb{R}^n \setminus W_f, \tag{17.88}$$

because $f(x) = x$ by definition of $W_f$, $g_2(x) = x$ by (17.83), hence $g_t(x) = x$ by (17.84), and finally $h_t(x) = x$ by the extended (17.42).

For (1.5), it is enough to check that for $k$ large, $h_t(x) = x$ for $x \in E_k \setminus B(X_0, R_0 + \varepsilon_0)$. By (17.47), this is true for $t \le 1$. Also, $f(x) = x$ (if $k$ is large enough): this was one of the intermediate steps in the proof of (17.47) (we proved that $\operatorname{dist}(x, W_t) > 0$ when

$\operatorname{dist}(x, E) \leq d_k$; otherwise use (17.44) and its proof). Then $x \in \mathbb{R}^n \setminus W_f$, and we can apply (17.88). So (1.5) holds.

For (1.6) we also need to check that $h_t(x)$ does not escape too far when $x \in E_k \cap B(X_0, R_0 + \varepsilon_0)$. We may restrict to $t > 1$, since Lemma 17.40 itself takes care of $0 \leq t \leq 1$.

If $\operatorname{dist}(x, \mathbb{R}^n \setminus W_f) \leq 2\varepsilon_*$,

$$\begin{aligned} |g_t(x) - x| &\leq \operatorname{Max}(|g_1(x) - x|, |g_2(x) - x|) = \operatorname{Max}(|f(x) - x|, |g_2(x) - x|) \\ &\leq |f(x) - x| \leq (1 + |f|_{lip}) \operatorname{dist}(x, \mathbb{R}^n \setminus W_f) \leq 2(1 + |f|_{lip})\varepsilon_* \end{aligned} \tag{17.89}$$

by the various definitions (17.80)-(17.84), and because $f(x) - x$ is a $(1 + |f|_{lip})$-Lipschitz mapping that vanishes on $\mathbb{R}^n \setminus W_f$. Then $|h_t(x) - x| \leq C(1 + |f|_{lip})\varepsilon_*$ too, by (17.37), (17.38), and (17.19). We get that $h_t(x) \in B''$ if $x \in B(X_0, R_0 + \varepsilon_0)$ (and if $\varepsilon_*$ is small enough compared to $\varepsilon_0$ or $\delta_6$; see Remark 11.17 to check that we do not cheat).

In the remaining case when $\operatorname{dist}(x, \mathbb{R}^n \setminus W_f) \geq 2\varepsilon_*$, (17.81) says that $g_t(x) = g_1(x)$ for $t \geq 1$, so $h_t(x) = h_1(x) \in B''$ because we use the same formula (17.37), and by (17.51). This proves (1.6).

The verification of (1.7) is a little easier than before. The only case when we do not already know that $h_t(x) \in L_j$ is when $\operatorname{dist}(x, \mathbb{R}^n \setminus W_f) \leq 2\varepsilon_*$, and then (17.89) implies that

$$\operatorname{dist}(g_t(x), L_j) \leq |g_t(x) - x| \leq 2(1 + |f|_{lip})\varepsilon_* \leq \varepsilon_0 \tag{17.90}$$

if $\varepsilon_*$ is small enough compared to $\varepsilon_0$. Then (17.55) holds and we can conclude as before (that is, $g_t(x)$ is so close to $L_j$ that the magnetic projection sends it back to $L_j$). This completes our proof of (17.87).

Let us record the fact that

$$h_t(x) = x \ \text{ for } x \in U_{ext} \text{ and } 0 \leq t \leq 2; \tag{17.91}$$

for $t \leq 1$, this comes from (17.44), and the proof of (17.44), which uses (17.42), also gives that $g(x) = f(x) = x$. Then (17.81)-(17.83) yield $g_t(x) = g_1(x) = x$ for $t \geq 1$, and hence $h_t(x) = x$ by (17.37) and (17.38) (because $s_t(x) = 0$).

Let us also say a few words about the analogue of $\widehat{W}$ for the $h_t$, $0 \leq t \leq 2$. Set

$$W_{k,t} = \big\{x \in E_k\,;\, h_t(x) \neq x\big\} \text{ for } 0 \leq t \leq 2 \ \text{ and } \ \widehat{W}_k = \bigcup_{0 \leq t \leq 2} W_{k,t} \cup h_t(W_{k,t}). \tag{17.92}$$

We claim that

$$\widehat{W}_k \subset\subset U \ \text{ for } k \text{ large}. \tag{17.93}$$

Because of (17.91), we know that the $W_{k,t}$ do not meet $U_{ext}$. Also, (17.58) and (17.59) give the desired control on the $W_{k,t}$, $0 \leq t \leq 1$ (recall that $4\Lambda^2(1 + |f|_{lip})\delta_6 < \delta_0/2 = \operatorname{dist}(\widehat{W}, \mathbb{R}^n \setminus U)/2$ by (12.6) and (12.7)). So it is enough to consider $t > 1$.

Let $x \in W_{k,t}$ be given. If $\operatorname{dist}(x, \mathbb{R}^n \setminus W_f) \geq 2\varepsilon_*$, (17.81) and the fact that we still use (17.37) say that $h_t(x) = h_1(x)$, so $x \in W_{k,1}$ and the desired control comes from (17.58) and (17.59). Otherwise, observe that $|g_t(x) - x| \leq 2(1 + |f|_{lip})\varepsilon_*$ by (17.89), and $|h_t(x) - g_t(x)| \leq C\varepsilon_0$ because we use the formulas (17.37) and (17.38) and by the proof of (17.39). On the other hand, $\operatorname{dist}(x, \mathbb{R}^n \setminus U) \geq \delta_0/2$ because $x \in U \setminus U_{ext}$; so $\operatorname{dist}(h_t(x), \mathbb{R}^n \setminus U) \geq \delta_0/4$ if $\varepsilon_0$ and $\varepsilon_*$ are small enough, and (17.93) follows.

† Let us do a similar construction under the Lipschitz assumption. We still have (17.79) (by (16.8), (16.28), and (16.29); also see eight lines below (16.30) for the definition on $U \setminus U_{int}$) and (17.80) (for the same reason). As we did in (17.60), we first set

$$h_t(x) = x \text{ when } x \in U_{ext} \text{ and } 1 \leq t \leq 2. \tag{17.94}$$

This way, it will be enough to define $h_t$ on the set $E_k \cap H_1$, where we shall find it convenient to first define mappings $\widetilde{g}_t$, $1 \leq t \leq 2$. We shall also check that the two formulas coincide on $E_k \cap H_1 \cap U_{ext}$.

Recall that when $x \in E_k \cap H_1$, the $h_t(x)$, $0 \leq t \leq 1$, were defined by (17.64), in terms of functions $\widetilde{h}_t(x)$, themselves defined by (17.62) and (17.63). In particular, we had first observed that the $\widetilde{g}_t(x)$, and in particular $\widetilde{f}(x) = \widetilde{g}_1(x)$ were well defined (in terms of $g_t(x)$ and $f(x)$). Here we proceed similarly. First we set

$$\widetilde{g}_2(x) = \widetilde{g}_1(x) = \widetilde{g}(x) \text{ when } \operatorname{dist}(x, \mathbb{R}^n \setminus W_f) \geq 2\varepsilon_* \tag{17.95}$$

where the fact that $\widetilde{g}_1(x) = \widetilde{g}(x)$ comes from (16.29), or (when $x \in \mathbb{R}^n \setminus V_1^+$) from the definition above (16.30). When $\varepsilon_* \leq \operatorname{dist}(x, \mathbb{R}^n \setminus W_f) \leq 2\varepsilon_*$ (and $x \in E_k \cap H_1$), set

$$\begin{aligned} \widetilde{g}_2(x) &= \frac{\operatorname{dist}(x, \mathbb{R}^n \setminus W_f) - \varepsilon_*}{\varepsilon_*}\, \widetilde{g}_1(x) + \frac{2\varepsilon_* - \operatorname{dist}(x, \mathbb{R}^n \setminus W_f)}{\varepsilon_*}\, \psi(\lambda x) \\ &= \frac{\operatorname{dist}(x, \mathbb{R}^n \setminus W_f) - \varepsilon_*}{\varepsilon_*}\, \widetilde{f}(x) + \frac{2\varepsilon_* - \operatorname{dist}(x, \mathbb{R}^n \setminus W_f)}{\varepsilon_*}\, \psi(\lambda x), \end{aligned} \tag{17.96}$$

where the identity comes from (17.80) through the usual change of variable. Finally set

$$\widetilde{g}_2(x) = \psi(\lambda x) \text{ when } \operatorname{dist}(x, \mathbb{R}^n \setminus W_f) < \varepsilon_*. \tag{17.97}$$

This defines the mapping $\widetilde{g}_2$ on $E_k \cap H_1$, and now we define the $\widetilde{g}_t(x)$, $1 \leq t \leq 2$, by

$$\widetilde{g}_t(x) = (2-t)\widetilde{g}_1(x) + (t-1)\widetilde{g}_2(x) = (2-t)\widetilde{f}(x) + (t-1)\widetilde{g}_2(x). \tag{17.98}$$

We proceed as near (17.62), define

$$\widetilde{s}_t(x) = C \operatorname{Min}(\lambda\varepsilon_0, |\widetilde{g}_t(x) - \psi(\lambda x)|) \tag{17.99}$$

also for $1 < t \leq 2$, and where $C$ is still as in (17.20), and then

$$\widetilde{h}_t(x) = \Pi(\widetilde{g}_t(x), \widetilde{s}_t(x)) \text{ for } x \in E_k \cap H_1 \text{ and } 1 < t \leq 2, \tag{17.100}$$

as in (17.63). As usual, we want to set

$$g_t(x) = \lambda^{-1}\psi^{-1}(\widetilde{g}_t(x)) \ \text{ and } \ h_t(x) = \lambda^{-1}\psi^{-1}(\widetilde{h}_t(x)) \tag{17.101}$$

for $x \in E_k \cap H_1$ and $1 \leq t \leq 2$, but we shall first need to check that this makes sense.

When $\mathrm{dist}(x, \mathbb{R}^n \setminus W_f) \geq 2\varepsilon_*$, (17.95) implies that $\widetilde{g}_2(x) \in B(0,1)$, so $g_t(x) = \lambda^{-1}\psi^{-1}(\widetilde{g}_t(x))$ is well defined, and $g_t(x) = g(x)$. Since we used the same formula (17.63) to define $\widetilde{h}_1(x)$, we also get that $\widetilde{h}_t(x) = \widetilde{h}_1(x)$, so $h_t(x) = \lambda^{-1}\psi^{-1}(\widetilde{h}_t(x))$ is well defined, and we also get that

$$h_t(x) = h_1(x) \ \text{ for } 1 \leq t \leq 2 \tag{17.102}$$

when $x \in E_k \cap H_1$ is such that $\mathrm{dist}(x, \mathbb{R}^n \setminus W_f) \geq 2\varepsilon_*$.

So suppose now that $\mathrm{dist}(x, \mathbb{R}^n \setminus W_f) \leq 2\varepsilon_*$. Let check that then

$$|\widetilde{h}_t(x) - \psi(\lambda x)| \leq C\lambda\Lambda(1 + |f|_{lip})\varepsilon_* \tag{17.103}$$

for $t \geq 1$. Notice that

$$\begin{aligned} |\widetilde{g}_t(x) - \psi(\lambda x)| &\leq \mathrm{Max}(|\widetilde{g}_1(x) - \psi(\lambda x)|, |\widetilde{g}_2(x) - \psi(\lambda x)|) \\ &= \mathrm{Max}(|\widetilde{f}(x) - \psi(\lambda x)|, |\widetilde{g}_2(x) - \psi(\lambda x)|) \\ &= |\widetilde{f}(x) - \psi(\lambda x)| \leq \lambda\Lambda|f(x) - x| \\ &\leq \lambda\Lambda(1 + |f|_{lip})\,\mathrm{dist}(x, \mathbb{R}^n \setminus W_f) \leq 2\lambda\Lambda(1 + |f|_{lip})\varepsilon_* \end{aligned} \tag{17.104}$$

by (17.98), (17.96) or (17.97), and the fact that $f(x) - x$ is $(1 + |f|_{lip})$-Lipschitz and vanishes on $\mathbb{R}^n \setminus W_f$. Also,

$$\widetilde{s}_t(x) \leq C|\widetilde{g}_t(x) - \psi(\lambda x)| \tag{17.105}$$

by (17.99), so

$$|\widetilde{h}_t(x) - \widetilde{g}_t(x)| \leq C\widetilde{s}_t(x) \leq C|\widetilde{g}_t(x) - \psi(\lambda x)| \leq C\lambda\Lambda(1 + |f|_{lip})\varepsilon_* \tag{17.106}$$

by (17.100), (17.19), and (17.104). Now (17.103) follows from (17.104) and (17.106).

We are now ready to prove that $\widetilde{g}_t(x)$ and $\widetilde{h}_t(x)$ lie in $B(0,1)$ when $x \in E_k \cap H_1$. We already treated the case when $\mathrm{dist}(x, \mathbb{R}^n \setminus W_f) \leq 2\varepsilon_*$ (see (17.102)); in the other case, notice that $\mathrm{dist}(\psi(\lambda x); \mathbb{R}^n \setminus B(0,1)) \geq \Lambda^{-1}\lambda\,\mathrm{dist}(x, \mathbb{R}^n \setminus U) \geq \Lambda^{-1}\lambda\delta_0/3$ by (17.61), hence $\widetilde{h}_t(x)$ lie in $B(0,1)$, by (17.103), and similarly for $\widetilde{g}_t(x)$, by (17.104). So the definitions in (17.101) make sense.

When $x \in E_k \cap H_1 \cap U_{ext}$, (17.101) yields the same result as (17.94), because (17.94) says that $h_t(x) = x$, while $h_1(x) = g_1(x) = f(x) = x$ by (17.44) and its proof by (17.42), then $\widetilde{g}_2(x) = \psi(\lambda x)$ by (17.95)-(17.97), then $\widetilde{g}_t(x) = \psi(\lambda x)$ by (17.98), and finally $g_t(x) = x$ by (17.101).

We continue with the verifications that follow the definition of the $h_t$. We still have (17.85), by (17.94) or (17.102). Instead of (17.86), let us just check that

$$h_2(x) = x \text{ when } x \in E_k \text{ and } \mathrm{dist}(x, \mathbb{R}^n \setminus W_f) < \varepsilon_*. \tag{17.107}$$

When $x \in U_{ext}$, $h_2(x) = x$ by (17.94). When $x \in E_k \cap H_1$, (17.97) yields $\widetilde{g}_2(x) = \psi(\lambda x)$; then (17.98) yields $\widetilde{g}_t(x) = \psi(\lambda x)$ for $t = 2$ (yes, there is a double definition of $\widetilde{g}_2(x)$, but the notation is acceptable because they give the same result), and finally $\widetilde{s}_2(x) = 0$ because $\widetilde{g}_2(x) - \psi(\lambda x) = 0$ and hence $\widetilde{h}_2(x) = \widetilde{g}_2(x) = \psi(\lambda x)$ by (17.100); (17.107) follows.

Our next verification is (17.87), the fact that Lemma 17.40 still holds for our extended family. As before, only (1.5)-(1.7) need to be checked, because (1.4) and (1.8) can be seen from the definition (this is why we made sure to have an overlap when we used two definitions, on $U_{ext}$ and on $E_k \cap H_1$). For these verifications, we already know the desired result for $0 \leq t \leq 1$, and we only need to worry when $h_t(x) \neq h_t(1)$ for some $t > 1$. Hence we may restrict to $x \in E_k \cap H_1$ such that $\mathrm{dist}(x, \mathbb{R}^n \setminus W_f) \leq 2\varepsilon_*$ (see (17.95) and (17.98), and compare (17.99)-(17.101) to (17.62)-(17.64)).

For (1.5), we suppose in addition $x \in E_k \setminus B(X_0, R_0 + \varepsilon_0)$ and we want to know that $h_t(x) = x$ for $t > 1$. But we already know that $g_1(x) = x$, so (17.96) yields $\widetilde{g}_2(x) = \psi(\lambda x)$, hence $\widetilde{g}_t(x) = \psi(\lambda x)$ by (17.98), $\widetilde{s}_t(x) = 0$ by (17.99), and finally $h_t(x) = g_t(x) = x$ for $t > 1$, by (17.100) and (17.101).

For (1.6) we suppose that $x \in E_k \cap B(X_0, R_0 + \varepsilon_0)$ and we want to prove that $h_t(x) \in B''$ for $t > 1$. Since we may assume that $x \in E_k \cap H_1$ and $\mathrm{dist}(x, \mathbb{R}^n \setminus W_f) \leq 2\varepsilon_*$ we can use (17.103), which by (17.101) implies that

$$|h_t(x) - x| \leq C\Lambda^2(1 + |f|_{lip})\varepsilon_*, \tag{17.108}$$

from which we easily deduce that $h_t(x) \in B''$, because $x \in B(X_0, R_0 + \varepsilon_0)$ and if $\varepsilon_*$ is small enough.

We are left with (1.7). We are given $x \in E_k \cap L_j$, and we want to check that $h_t(x) \in L_j$ for $t \geq 1$. Again we can assume that $x \in E_k \cap H_1$ and $\mathrm{dist}(x, \mathbb{R}^n \setminus W_f) \leq 2\varepsilon_*$, so (17.103) holds, and hence

$$\mathrm{dist}(\widetilde{g}_t(x), \widetilde{L}_j) \leq |\widetilde{g}_t(x) - \psi(\lambda x)| \leq C\lambda\Lambda(1 + |f|_{lip})\varepsilon_* \leq \lambda\varepsilon_0 \tag{17.109}$$

because $x \in L_j$ and if $\varepsilon_*$ is small enough compared to $\varepsilon_0$. Then $\widetilde{s}_t(x) = C|\widetilde{g}_t(x) - \psi(\lambda x)|$ in (17.99), and since there is a face $\widetilde{F}$ of $\widetilde{L}_j$ such that $\mathrm{dist}(\widetilde{g}_t(x), \widetilde{F}) \leq |\widetilde{g}_t(x) - \psi(\lambda x)| = C^{-1}\widetilde{s}_t(x)$ (by (17.109)), we get that $\Pi(\widetilde{g}_t(x), \widetilde{s}_t(x)) \in \widetilde{F}$ by (17.20) and our choice of $C$ in (17.99); Thus $\widetilde{h}_t(x) = \Pi(\widetilde{g}_t(x), \widetilde{s}_t(x)) \in \widetilde{F} \subset \widetilde{L}_j$ by (17.100) and $h_t(x) \in L_j$, as needed. This completes our verification of (17.87) in the Lipschitz context.

We also want to generalize (17.91)-(17.93). In fact, (17.91) still holds, by our definition (17.94), (17.92) is a definition, and the proof of (17.93) also goes the same way: we just need to consider points $x \in E_k \cap H_1$ such that $\mathrm{dist}(x, \mathbb{R}^n \setminus W_f) \leq 2\varepsilon_*$, hence for which (17.108) holds. But $\mathrm{dist}(x, \mathbb{R}^n \setminus U) \geq \delta_0/3$ because $x \in H_1$, so $\mathrm{dist}(h_t(x), \mathbb{R}^n \setminus U) \geq \delta_0/4$, by (17.108), and (17.93) follows.

## 18. The final accounting and the proof of Theorem 10.8 in the rigid case

In the previous sections, we managed to construct deformations of $E$ and the $E_k$; we are especially interested in the last one, the family $h_{2t}$, $0 \leq t \leq 1$. By (17.87), or the corresponding verification in the Lipschitz case (see above (17.108), Lemma 17.40 holds for the $h_{2t}$, $0 \leq t \leq 1$. Also, (17.93) says that the analogue of (2.4) for $E_k$ holds for $k$ large enough. Thus we can apply Definition 2.3 and the quasiminimality of $E_k$; we get that for $k$ large,

$$\mathcal{H}^d(E_k \cap W) \leq M\mathcal{H}^d(h_2(E_k \cap W)) + (R'')^d h, \tag{18.1}$$

where $R''$ is as in (17.41) and

$$W = \big\{y \in U\,; h_2(x) \neq x\big\}. \tag{18.2}$$

Most of this section will consist in estimating the two sides of (18.1) (and especially the right-hand side). We shall need to cut $U$ into small pieces, and we shall start with the least important ones. We shall also try to treat the rigid and Lipschitz assumptions simultaneously when this is possible, but some estimates for the Lipschitz cases will be done in the next section.

Return to the definition of $h_2$. First recall from (17.86) (or (17.107) in the Lipschitz case) that $h_2(x) = x$ when $x \in E_k$ is such that $\mathrm{dist}(x, \mathbb{R}^n \setminus W_f) < \varepsilon_*$; hence

$$E_k \cap W \subset \big\{x \in W_f\,; \mathrm{dist}(x, \mathbb{R}^n \setminus W_f) \geq \varepsilon_*\big\} \subset W_f. \tag{18.3}$$

**The exterior skin.** In the rigid case and on the set

$$A_* = \big\{x \in W\,; \varepsilon_* \leq \mathrm{dist}(x, \mathbb{R}^n \setminus W_f) \leq 2\varepsilon_*\big\}, \tag{18.4}$$

we defined $g_2$ by (17.82), and then decided to use (17.37) as before (see below (17.84)). That is

$$h_2(x) = \Pi(g_2(x), s_2(x)), \tag{18.5}$$

where $s_2(x)$ is still defined as in (17.38), and $g_2(x)$ is defined by (17.82). We shall need to know that

$$h_2 \text{ is } C\text{-Lipschitz on } A_*. \tag{18.6}$$

Since $\Pi$ is $C$-Lipschitz (by (17.21)), (15.8) and (17.38) say that it is enough to show that $g_2$ is $3(1+|f|_{lip})$-Lipschitz on $A_*$. And indeed, for $x, y \in A_*$,

$$\begin{aligned}
|g_2(x) - g_2(y)| &= \Big|\frac{\mathrm{dist}(x, \mathbb{R}^n \setminus W_f) - \varepsilon_*}{\varepsilon_*}\, f(x) + \frac{2\varepsilon_* - \mathrm{dist}(x, \mathbb{R}^n \setminus W_f)}{\varepsilon_*}\, x \\
&\qquad - \frac{\mathrm{dist}(y, \mathbb{R}^n \setminus W_f) - \varepsilon_*}{\varepsilon_*}\, f(y) - \frac{2\varepsilon_* - \mathrm{dist}(y, \mathbb{R}^n \setminus W_f)}{\varepsilon_*}\, y\Big| \\
&\leq \frac{\mathrm{dist}(x, \mathbb{R}^n \setminus W_f) - \varepsilon_*}{\varepsilon_*}\, |f(x) - f(y)| + \frac{2\varepsilon_* - \mathrm{dist}(x, \mathbb{R}^n \setminus W_f)}{\varepsilon_*}\, |x-y| \\
&\qquad + \frac{|\,\mathrm{dist}(x, \mathbb{R}^n \setminus W_f) - \mathrm{dist}(y, \mathbb{R}^n \setminus W_f)|}{\varepsilon_*}\, |f(y) - y| \\
&\leq |x-y| + |f(x) - f(y)| + |x-y|\, \frac{|f(y) - y|}{\varepsilon_*} \leq 3(1 + |f|_{lip})\, |x-y|
\end{aligned} \tag{18.7}$$

by (17.82), and because $y \in A_*$ and $f(y)-y$ is a $(1+|f|_{lip})$-Lipschitz function that vanishes on $\mathbb{R}^n \setminus W_f$. So (18.6) holds.

† Under the Lipschitz assumption, we start with an analogue of (18.6). We first work on $E_k \cap H_1$ (where $H_1$ is as in (17.61)) and check that

$$\widetilde{g}_2 \text{ is } C(\Lambda, f)\lambda\text{-Lipschitz on } A_* \cap E_k \cap H_1. \tag{18.8}$$

On $A_* \cap E_k \cap H_1$, we defined $\widetilde{g}_2$ by (17.96) (see (18.4)). Recall that we restricted to $E_k \cap H_1$ because we were able to define $\widetilde{f}(x) = \psi(\lambda f(x))$ for $x \in E_k \cap H_1$, and compute with it. Then we can follow the same proof of (18.7) and get (18.8).

Then we observe that $\widetilde{h}_2$ was defined by (17.99) and (17.100), so it is also $C(\Lambda, f)\lambda$-Lipschitz on $E_k \cap H_1 \cap A_*$. Finally,

$$h_2 \text{ is } C(\Lambda, f)\text{-Lipschitz on } A_* \cap E_k \cap H_1, \tag{18.9}$$

because of (17.101).

Recall from (17.94) that $h_2(x) = x$ on $U_{ext}$, and that $H_1$ and $U_{ext}$ cover $\mathbb{R}^n$ (even, with an overlap), by (17.61) and (11.2). So (18.9) will be good enough. For instance,

$$\begin{aligned} \mathcal{H}^d(h_2(E_k \cap A_*)) &\leq \mathcal{H}^d(h_2(E_k \cap A_* \cap H_1)) + \mathcal{H}^d(h_2(E_k \cap A_* \setminus H_1)) \\ &\leq C(\Lambda, f)\mathcal{H}^d(E_k \cap H_1 \cap A_*)) + \mathcal{H}^d(E_k \cap A_* \setminus H_1) \\ &\leq C(\Lambda, f)\mathcal{H}^d(E_k \cap A_*)) \end{aligned} \tag{18.10}$$

by (18.9) and the trivial estimate on $U_{ext}$. †

In the rigid case, we also have the conclusion of (18.10), directly by (18.6). Thus the contribution of $A_*$ to the right-hand side of (18.1) is easily controlled in both cases. Next return to the general case, and recall that

$$W_f \text{ is compactly contained in } U, \tag{18.11}$$

by (17.78) or (11.19), (11.3), and (11.2). Since $A_* \subset W_f$ by (18.4), it is a compact subset of $U$. Then we can apply (10.14) to $A_*$ and get that

$$\limsup_{k \to +\infty} \mathcal{H}^d(E_k \cap A_*)) \leq C_M \mathcal{H}^d(E \cap A_*)). \tag{18.12}$$

Finally

$$\limsup_{\varepsilon_* \to 0} \mathcal{H}^d(E \cap A_*) \leq \limsup_{\varepsilon_* \to 0} \mathcal{H}^d\big(\{x \in E\,;\, 0 < \operatorname{dist}(x, \mathbb{R}^n \setminus W_f) \leq 2\varepsilon_*\}\big) = 0 \tag{18.13}$$

(the monotone intersection of these sets is empty), so we deduce from (18.10)-(18.13) that

$$\mathcal{H}^d(h_2(E_k \cap A_*)) \leq \eta + C\mathcal{H}^d(E \cap A_*)) \leq 2\eta \tag{18.14}$$

for $k$ large, and provided that we choose $\varepsilon_*$ small enough (depending on the usual quantities).

Observe that by (18.3) and (18.4),

$$E_k \cap W \setminus A_* \subset W_* := \big\{x \in W_f\,;\mathrm{dist}(x, \mathbb{R}^n \setminus W_f) > 2\varepsilon_*\big\}, \tag{18.15}$$

and so we shall now concentrate on $E_k \cap W_*$. On this set, and in the rigid case, (17.85), (17.37), and (16.13) say that

$$h_2(x) = h_1(x) = \Pi(g_1(x), s_1(x)) = \Pi(g(x), s_1(x)), \tag{18.16}$$

where by (17.38) and (16.13)

$$s_1(x) = C\,\mathrm{Min}(\varepsilon_0, |g_1(x) - x|) = C\,\mathrm{Min}(\varepsilon_0, |g(x) - x|). \tag{18.17}$$

† In the Lipschitz case, either $x \in E_k \cap U_{ext}$ and (17.94) says that $h_2(x) = x$, or else $x \in E_k \cap H_1$ and (since $x \in W_*$) (17.101) says that $h_2(x) = \lambda^{-1}\psi^{-1}(\widetilde{h}_2(x))$, where $\widetilde{h}_2(x) = \Pi(\widetilde{g}_2(x), \widetilde{s}_2(x))$ by (17.100). Since (17.95) says that $\widetilde{g}_2(x) = \widetilde{g}_1(x) = \widetilde{g}(x)$, the definition (17.99) yields

$$\widetilde{s}_2(x) = C\,\mathrm{Min}(\lambda\varepsilon_0, |\widetilde{g}_2(x) - \psi(\lambda x)|) = \widetilde{s}_1(x) = C\,\mathrm{Min}(\lambda\varepsilon_0, |\widetilde{g}(x) - \psi(\lambda x)|) \tag{18.18}$$

and hence

$$\widetilde{h}_2(x) = \Pi(\widetilde{g}_2(x), \widetilde{s}_2(x)) = \Pi(\widetilde{g}_1(x), \widetilde{s}_1(x)) = \Pi(\widetilde{g}(x), \widetilde{s}_1(x)) \tag{18.19}$$

for $x \in E_k \cap H_1 \cap W_*$ (a good enough analogue of (18.16) and (18.17)). †

In both cases, (18.16)-(18.19), (16.10), and (17.21) yield

$$h_2 \text{ is } C\text{-Lipschitz on } E_k \cap W_* \tag{18.20}$$

with a constant $C$ that depends on $M$ and $|f|_{lip}$ in particular, and also on $\Lambda$ in the Lipschitz case.

**The part outside the balls.** Our next small set is

$$W_*^1 = W_* \setminus V_1^+, \tag{18.21}$$

where $V_1^+$ is as in (16.7). Then (18.20) yields

$$\mathcal{H}^d(h_2(E_k \cap W_*^1)) \leq C\mathcal{H}^d(E_k \cap W_*^1)), \tag{18.22}$$

where we don't mention the dependence on $\Lambda$ and $|f|_{lip}$. Again $\overline{W}_*^1$ is a compact subset of $U$, because $W_* \subset W_f$ by (18.15), and by (18.11). So (10.14) yields

$$\limsup_{k\to+\infty} \mathcal{H}^d(E_k \cap W_*^1)) \leq C_M\mathcal{H}^d(E \cap \overline{W}_*^1)) \tag{18.23}$$

and hence

$$\mathcal{H}^d(h_2(E_k \cap W^1_*)) \leq \eta + C\mathcal{H}^d(E \cap \overline{W}^1_*)) \tag{18.24}$$

for $k$ large. Of course it will be interesting to control $\mathcal{H}^d(E \cap \overline{W}^1_*))$, and we shall do this more easily after the next step.

**The small rings.** Next we want to control the contribution of the small rings.

**Lemma 18.25.** *Set*

$$R^1 = \bigcup_{j\in J_1} [\overline{B}_j \setminus aB_j]\,, \quad R^2 = \bigcup_{j\in J_2} [\overline{B}_j \setminus aB_j]\,, \text{ and } R^3 = \bigcup_{j\in J_3\,;\, x\in Z(y_j)} [B_{j,x} \setminus B^-_{j,x}]\,, \tag{18.26}$$

*where $B_{j,x}$ and $B^-_{j,x}$ are as in (15.19) and (15.17). Then*

$$\mathcal{H}^d(E \cap [R^1 \cup R^2 \cup R^3]) \leq C(f,\gamma)(1-a), \tag{18.27}$$

*where $C(f,\gamma)$ depends on $|f|_{lip}$, $\mathcal{H}^d(E \cap W_f)$, and $\gamma$.*

Indeed, we know from (13.24) and the definition (11.20) that

$$\mathcal{H}^d(E \cap R^1) \leq C(1-a)\mathcal{H}^d(E \cap W_f). \tag{18.28}$$

Similarly, (14.20) says that

$$\mathcal{H}^d(E \cap R^2) \leq \sum_{j\in J_2} \mathcal{H}^d(E \cap \overline{B}_j \setminus aB_j) \leq C(1-a)\mathcal{H}^d(E \cap W_f). \tag{18.29}$$

We are thus left with the $B_{j,x} \setminus B^-_{j,x}$. First fix $j \in J_3$ and $x \in Z(y_j)$, recall the definitions (15.16)-(15.19), and cover $(2-a)E(x) \setminus aE(x)$ by balls $A_k$ of radius $(1-a)r_j$. By (15.19) and (15.20) (but we could also use the definition (15.16) of $E(x)$), the ellipsoid $(2-a)E(x)$ is contained $B(x, \frac{3}{2}\gamma^{-1}r_j)$; by the definition (15.16), it is also contained in the $d$-plane $P_x$, and so we need less than $C\gamma^{-d}(1-a)^{1-d}$ balls $A_k$ to cover $(2-a)E(x) \setminus aE(x)$. The slightly larger balls $2A_k$ cover $B_{j,x} \setminus B^-_{j,x}$, so

$$\mathcal{H}^d(E \cap B_{j,x} \setminus B^-_{j,x}) \leq \sum_k \mathcal{H}^d(E \cap 2A_k). \tag{18.30}$$

Let us apply Proposition 4.1 (the local Ahlfors-regularity of $E$) to each $2A_k$. This is allowed, because $4A_k \subset W_f \subset U$, since

$$r_j \leq \frac{\gamma}{4}\delta_6 \leq \frac{\gamma}{40}\delta_1 \leq \frac{\gamma}{40}\,\mathrm{dist}(x, \mathbb{R}^n \setminus W_f) \tag{18.31}$$

by (15.15), (12.7), (11.22), and because $x \in X_9 \subset X_1$ (see (15.1)). We get that

$$\begin{aligned}\mathcal{H}^d(E \cap B_{j,x} \setminus B^-_{j,x}) &\leq \sum_k \mathcal{H}^d(E \cap 2A_k) \leq C\sum_k (1-a)^d r_j^d \leq C(1-a)\gamma^{-d}r_j^d \\ &\leq C(1-a)\gamma^{-d}|f|^d_{lip}\mathcal{H}^d(E \cap B(x, |f|^{-1}_{lip}r_j))\end{aligned} \tag{18.32}$$

by (18.30) and Proposition 4.1, applied a second time but in the other direction. The reader should not worry about $|f|_{lip}^{-1}$ being too large: we know that $|f|_{lip} \geq 1$ because $f(z) = z$ near $\infty$. We claim that

$$\text{the balls } B(x, |f|_{lip}^{-1} r_j),\ j \in J_3 \text{ and } x \in Z(y_j), \text{ are all disjoint.} \tag{18.33}$$

For different $j$, this is because $f(B(x, |f|_{lip}^{-1} r_j)) \subset B(f(x), r_j) = D_j$, and the $D_j$ are disjoint by (15.13). For the same $j$ and different $x \in Z(y_j)$, this follows from (15.6) and the fact that $y_j \in Y_9$ by (15.12), because we can safely assume that $\gamma \leq 1$. Now

$$\begin{aligned} H^d(E \cap R^3) &\leq \sum_{j \in J_3} \sum_{z \in Z(y_j)} \mathcal{H}^d(E \cap B_{j,z} \setminus B_{j,z}^-) \\ &\leq C(1-a)\gamma^{-d} |f|_{lip}^d \sum_{j \in J_3} \sum_{z \in Z(y_j)} \mathcal{H}^d(E \cap B(x, |f|_{lip}^{-1} r_j)) \\ &\leq C(1-a)\gamma^{-d} |f|_{lip}^d \mathcal{H}^d(E \cap W_f) \end{aligned} \tag{18.34}$$

by (18.32), (18.33), and because $B(x, |f|_{lip}^{-1} r_j) \subset W_f$ by (18.31). Lemma 18.25 follows. □

By (18.20),

$$\mathcal{H}^d(h_2(E_k \cap W_* \cap [R^1 \cup R^2 \cup R^3])) \leq C\mathcal{H}^d(E_k \cap [R^1 \cup R^2 \cup R^3])). \tag{18.35}$$

But $R^1 \cup R^2 \cup R^3$ is compact because the sets $J_1$, $J_2$, and $J_3$ are finite, and it is contained in $U$ by construction of the $B_j$ and $B_{j,x}$, so (10.14) and (18.27) say that

$$\mathcal{H}^d(E_k \cap [R^1 \cup R^2 \cup R^3])) \leq C\mathcal{H}^d(E \cap [R^1 \cup R^2 \cup R^3])) + \eta \leq C(f,\gamma)(1-a) + C\eta \tag{18.36}$$

for $k$ large. Altogether,

$$\mathcal{H}^d(h_2(E_k \cap W_* \cap [R^1 \cup R^2 \cup R^3])) \leq C(f,\gamma)(1-a) + C\eta. \tag{18.37}$$

**Return to the part outside the balls.** We still want to estimate $\mathcal{H}^d(E \cap \overline{W}_*^1))$, but we shall even consider the set $E \cap W^2$, where

$$W^2 = W_f \setminus \mathrm{int}(V_1^+). \tag{18.38}$$

Let us first check that $\overline{W}_*^1 \subset W^2$. Let $x \in \overline{W}_*^1$ be given; then $x$ is the limit of some sequence $\{x_k\}$ in $W_*^1 = W_* \setminus V_1^+$ (see (18.21)). By (18.15), $\mathrm{dist}(x_k, \mathbb{R}^n \setminus W_f) > 2\varepsilon_*$ for all $k$, so $x \in W_f$. But also $x_k$ lies out of $V_1^+$, so $x \notin \mathrm{int}(V_1^+)$. Thus $\overline{W}_*^1 \subset W^2$.

Now let $x \in E \cap W^2$ be given; we want to show that it lies in one of many small sets. Observe that $x$ lies in $X_0 = E \cap W_f$, by (11.20). First assume that $x \in \overline{B}_j$ for some $j \in J_1 \cup J_2$. Notice that $\frac{1+a}{2} B_j$ is contained in $\mathrm{int}(V_1^+)$ by (16.7), so it does not meet $W^2$. Hence $x \in \overline{B}_j \setminus \frac{1+a}{2} B_j \subset R^1 \cup R^2$, and the corresponding subset of $E \cap W^2$ is small, by (18.27).

So we may assume that $x \in X_0 \setminus \bigcup_{j\in J_1\cup J_2} \overline{B}_j$. The case when $x \notin X_9$ is controlled by (14.24), which says that $\mathcal{H}^d\big(X_0 \setminus \big[X_9 \cup \big(\bigcup_{j\in J_1\cup J_2} \overline{B}_j\big)\big]\big) \le 7\eta$, so we may even assume that $x \in X_9$.

The case when $x \in X_9 \setminus X_{11}$ is covered by (15.8) and (15.11), so we may assume that $x \in X_{11}$. In addition, (15.14) allows us to assume that $x \in f^{-1}\big(\bigcup_{j\in J_3} \overline{D}_j\big)$, and by (15.36) $x$ lies in $B_{j,z}$ for some $j \in J_3$ and $z \in Z(y_j)$. At the same time, $B^+_{j,z}$ is contained in $V_1^+$ by (16.7), so its interior does not meet $W^2$. The interior contains $B^-_{j,z}$ (see the definitions (15.17)-(15.19)), hence $x \in B_{j,z} \setminus B^-_{j,z} \subset R^3$ (compare with (18.26)). The corresponding set is again controlled by (18.27), and altogether

$$\mathcal{H}^d(E \cap W^2)) \le C(f,\gamma)(1-a) + C\eta. \tag{18.39}$$

Since $\overline{W}^1_* \subset W^2$, we get that

$$\mathcal{H}^d(h_2(E_k \cap W^1_*)) \le \eta + C\mathcal{H}^d(E \cap \overline{W}^1_*)) \le C(f,\gamma)(1-a) + C\eta \tag{18.40}$$

for $k$ large, by (18.24) and (18.39).

We are left with the set $V_1^+$. Recall from (16.15) that

$$\mathrm{dist}(x, X_1) \le \delta_6 \text{ and } \mathrm{dist}(x, \mathbb{R}^n \setminus W_f) > \delta_1/2 \;\text{ for } x \in V_1^+. \tag{18.41}$$

In addition, recall from (11.2) that $\delta_0 = \mathrm{dist}(\widehat{W}, \mathbb{R}^n \setminus U)$; since $X_1 \subset X_0 \subset \widehat{W}$ by (11.20), we get that $\mathrm{dist}(z, \mathbb{R}^n \setminus U) \ge \delta_0$ for $z \in X_1$, and hence

$$\mathrm{dist}(x, \mathbb{R}^n \setminus U) \ge \frac{2\delta_0}{3} \;\text{ for } x \in V_1^+, \tag{18.42}$$

because $\delta_6 < \delta_0/3$ by (12.7). The definition (17.61) then immediately yields

$$V_1^+ \subset H_1. \tag{18.43}$$

Next we check that

$$|g(x) - x| \ge \frac{\delta_5}{2} \;\text{ for } x \in V_1^+. \tag{18.44}$$

Use (18.41) to find $z \in X_1$ such that $|z - x| \le \delta_6$, and notice that $|f(z) - z| \ge \delta_5$ by the definition (12.5). Then

$$\begin{aligned}
|g(x) - x| &\ge |f(x) - x| - |g(x) - f(x)| \ge |f(x) - x| - ||f - g||_\infty \\
&\ge |f(x) - x| - 4\Lambda^2(1 + |f|_{lip})\delta_6 \\
&\ge |f(z) - z| - (1 + |f|_{lip})|z - x| - 4\Lambda^2(1 + |f|_{lip})\delta_6 \\
&\ge \delta_5 - (1 + 4\Lambda^2)(1 + |f|_{lip})\delta_6 \ge \frac{\delta_5}{2}
\end{aligned} \tag{18.45}$$

by (16.11) and (12.7). So (18.44) holds. We want to use this to show that for $k$ large,

$$|h_2(x) - x| \geq \frac{\delta_5}{4} > 0 \ \text{ for } x \in E_k \cap V_1^+, \tag{18.46}$$

so we want to estimate $|h_2(x) - g(x)|$.

We start in the rigid case. If $\varepsilon_0$ is chosen small enough (compared to $\delta_5$), we get that $s_1(x) = C\,\mathrm{Min}(\varepsilon_0, |g(x) - x|) = C\varepsilon_0$ (by (18.17) and (18.45)). In this case, (18.16) simplifies to

$$h_2(x) = \Pi(g(x), C\varepsilon_0) \ \text{ for } x \in V_1^+ \tag{18.47}$$

and hence, by (17.19),

$$|h_2(x) - g(x)| \leq C\varepsilon_0; \tag{18.48}$$

in this case (18.46) follows from (18.44).

† Similarly, under the Lipschitz assumption, let $x \in E_k \cap V_1^+$ be given. First observe that $x \in E_k \cap H_1$, by (18.43). In addition, $x \in W_*$ if $\varepsilon_* < \delta_1/4$ (compare the definition (18.15) with (18.41)), so (18.18) and (18.19) hold.

Recall from the remark below (16.8) that since $x \in V_1^+$, $g(x)$ was defined in terms of a function $\widetilde{g}$, through the relation $\widetilde{g}(x) = \psi(\lambda g(x))$; thus (18.18) yields

$$\widetilde{s}_1(x) = C\,\mathrm{Min}(\lambda\varepsilon_0, |\widetilde{g}(x) - \psi(\lambda x)|) = C\,\mathrm{Min}(\lambda\varepsilon_0, |\psi(\lambda g(x)) - \psi(\lambda x)|) = C\lambda\varepsilon_0 \tag{18.49}$$

because $|\psi(\lambda g(x)) - \psi(\lambda x)| \geq \lambda\Lambda^{-1}|g(x) - x| \geq \lambda\varepsilon_0$ if $\varepsilon_0$ is small enough, by (18.44), and (18.19) simplifies to

$$\widetilde{h}_2(x) = \Pi(\widetilde{g}(x), \widetilde{s}_1(x)) = \Pi(\widetilde{g}(x), C\lambda\varepsilon_0). \tag{18.50}$$

Since (17.101) says that

$$h_2(x) = \lambda^{-1}\psi^{-1}(\widetilde{h}_2(x)), \tag{18.51}$$

we deduce from (18.50) and (17.19) that

$$|h_2(x) - g(x)| = |\lambda^{-1}\psi^{-1}(\widetilde{h}_2(x)) - \lambda^{-1}\psi^{-1}(\widetilde{g}(x))| \leq \lambda^{-1}\Lambda|\widetilde{h}_2(x) - \widetilde{g}(x)| \leq C\varepsilon_0 \tag{18.52}$$

where we we don't care that $C$ depends on $\Lambda$. That is, (18.48) still holds under the Lipschitz assumption, and (18.46) follows from (18.44) as before. †

Return to the general case; (18.46) implies that

$$E_k \cap V_1^+ \subset W, \tag{18.53}$$

where $W = \big\{y \in U\,; h_2(x) \neq x\big\}$ is as in (18.2).

**The balls $B_j$, $j \in J_1 \cup J_2$.** Next we estimate the contribution of $E_k \cap V_1^+$ to the right-hand side of (18.1). We start with $\bigcup_{j\in J_1} \frac{1+a}{2} B_j \subset \bigcup_{j\in J_1} B_j$. Observe that the set $R$ in (13.3) and (13.23) is contained in the more recent $R^1$ of (18.26), so (13.23) and (11.20) say that

$$\mathcal{H}^d\Big(g\Big(\bigcup_{j\in J_1} B_j \setminus R^1\Big)\Big) \leq \mathcal{H}^d\Big(g\Big(\bigcup_{j\in J_1} B_j \setminus R\Big)\Big) \leq C(\alpha, f)N^{-1}\mathcal{H}^d(E \cap W_f). \tag{18.54}$$

In the rigid case, (18.47) and (17.21) imply that on $V_1^+$, $h_2(x)$ is a $C$-Lipschitz function of $g(x)$; hence (18.54) yields

$$\begin{aligned}\mathcal{H}^d\Big(h_2\Big(E_k \cap V_1^+ \cap \Big(\bigcup_{j\in J_1} B_j \setminus R^1\Big)\Big)\Big) &\leq C\mathcal{H}^d\Big(g\Big(\bigcup_{j\in J_1} B_j \setminus R^1\Big)\Big)\\ &\leq C(\alpha, f)N^{-1}\mathcal{H}^d(E \cap W_f).\end{aligned} \tag{18.55}$$

† Under the Lipschitz assumption, (18.50) says that on $E_k \cap V_1^+$, $\widetilde{h}_2(x)$ is a $C$-Lipschitz function of $\widetilde{g}(x)$; hence, since $\widetilde{g}(x) = \psi(\lambda g(x))$ is a $\lambda\Lambda$-Lipschitz function of $g(x)$ and $h_2(x) = \lambda^{-1}\psi^{-1}(\widetilde{h}_2(x))$ is a $\lambda^{-1}\Lambda$-Lipschitz function of $\widetilde{h}_2(x)$ (see (18.51)), $h_2(x)$ is a $C\Lambda^2$-Lipschitz function of $g(x)$. Thus (18.55) still holds, only with a larger constant that also depends on $\Lambda$. †.

We can treat $\bigcup_{j\in J_2} \frac{1+a}{2} B_j \subset \bigcup_{j\in J_2} B_j$ almost the same way as for $J_1$. Indeed,

$$\begin{aligned}\mathcal{H}^d\Big(g\Big(E_k \cap \Big(\bigcup_{j\in J_2} B_j \setminus R^2\Big)\Big)\Big) &\leq \sum_{j\in J_2} \mathcal{H}^d(g(E_k \cap B_j \setminus R^2))\\ &\leq \sum_{j\in J_2} \mathcal{H}^d(g(E_k \cap aB_j)) \leq C\Lambda^{2d}(1+\Lambda|f|_{lip})^d\gamma \sum_{j\in J_2} r_j^d\\ &\leq C(\Lambda, |f|_{lip})\gamma\,\mathcal{H}^d(E \cap W_f) = C\gamma\,\mathcal{H}^d(E \cap W_f)\end{aligned} \tag{18.56}$$

for $k$ large enough, by (18.26), (14.15) or (14.37), the fact that $g = g_j$ on $aB_j$ (see (16.5)), and (14.19). As for the case of $J_1$, $h_2(x)$ is a $C$-Lipschitz function of $g(x)$ on $E_k \cap V_1^+$ (by (18.47) or (18.50)), so

$$\begin{aligned}\mathcal{H}^d\Big(h_2\Big(E_k \cap V_1^+ \cap \Big(\bigcup_{j\in J_2} B_j \setminus R^2\Big)\Big)\Big) &\leq C\mathcal{H}^d\Big(g\Big(E_k \cap \Big(\bigcup_{j\in J_2} B_j \setminus R^2\Big)\Big)\Big)\\ &\leq C\gamma\,\mathcal{H}^d(E \cap W_f).\end{aligned} \tag{18.57}$$

**The main contribution from the $B_{j,x}^+$, $j \in J_3$ and $x \in Z(y_j)$.** Our last piece of $V_1^+$ is the union of the $B_{j,x}^+$, and more precisely, of the $B_{j,x}^-$, since the rest is contained in $R^3$ by (18.26). We start with the rigid case. Let us check that for $j \in J_3$, $x \in Z(y_j)$, and $z \in B_{j,x}^-$,

$$h_2(z) = g(z) \in Q_j \cap D_j. \tag{18.58}$$

Recall that $Q_j$ is the common value of the $d$-planes $A_x(P_x)$, $x \in Z(y_j)$; see above (15.16). Fix $j \in J_3$, $x \in Z(y_j)$, and $z \in B^-_{j,x}$. By (16.6) and (15.29), $g(z) = g_{j,z}(z) = \pi_j(f(z))$, where $\pi_j$ denotes the orthogonal projection onto $Q_j$. By (15.22), $f(z) \in \frac{1+a}{2} D_j \subset D_j$, so $g(z) \in Q_j \cap D_j$ (recall that $Q_j$ goes through $y_j = f(x) = A_x(x)$). Now $h_2(z) = \Pi(g(z), C\varepsilon_0)$ by (18.47), and we still need to check that $h_2(z) = g(z)$.

Denote by $F(y_j)$ the smallest face of our grid that contains $y_j = f(x)$, by $W(y_j)$ the affine subspace spanned by $F(y_j)$, and by $m$ the dimension of $F(y_j)$ and $W(y_j)$. By Lemma 12.27, $Q_j \subset W(y_j)$. Also recall from (15.1) and (11.26) that $x \in X_9 \subset X_2 = X_{1,\delta_2}$. So (11.23)-(11.25) say that $x \in X_{1,\delta_2}(m)$ and

$$\operatorname{dist}(y_j, \mathcal{S}_{m-1}) \geq \delta_2. \tag{18.59}$$

But

$$|g(z) - y_j| \leq r_j \leq \delta_6 \leq \frac{\delta_2}{10} \tag{18.60}$$

because $g(z) \in D_j = B(y_j, r_j)$ and by (15.15) and (12.7). We know that $g(z) \in Q_j \subset W(y_j)$ and of course $y_j \in W(y_j)$; hence the line segment $[y_j, g(z)]$ is contained in $W(y_j)$. In addition, by (18.59) and (18.60), it does not meet $\partial F(y_j) \subset \mathcal{S}_{m-1}$, and since $y \in F(y_j)$, we get that $g(z) \in F(y_j)$ too. We also deduce from (18.59) and (18.60) that

$$\operatorname{dist}(g(z), \mathcal{S}_{m-1}) \geq \frac{\delta_2}{2}. \tag{18.61}$$

We return to $h_2(z) = \Pi(g(z), C\varepsilon_0)$, and use the definition of $\Pi$ in (17.23) and (17.24). Thus

$$h_2(z) = \Pi_{0,s_0} \circ \Pi_{1,s_1} \cdots \circ \Pi_{n-1,s_{n-1}}(g(z)), \tag{18.62}$$

where $s_j = (6C)^j (C\varepsilon_0) \leq C'\varepsilon_0$ for $0 \leq j \leq n-1$. For $j \geq m$, $g(z) \in F(y_j) \subset \mathcal{S}_m$ and so $\Pi_{j,s_j}(g(z)) = g(z)$ by (17.2). For $j < m$, $\operatorname{dist}(g(z), \mathcal{S}_j) \geq \frac{\delta_2}{2} > 2s_j$ by (18.61), and now $\Pi_{j,s_j}(g(z)) = g(z)$ by the second part of (17.2). Altogether, $h_2(z) = g(z)$ and (18.58) follows.

We may now sum over $j$ and $x$:

$$\begin{aligned} \mathcal{H}^d\Big(h_2\Big(\bigcup_{j \in J_3} \bigcup_{x \in Z(y_j)} B^+_{j,x} \setminus R^3\Big)\Big) &\leq \mathcal{H}^d\Big(h_2\Big(\bigcup_{j \in J_3} \bigcup_{x \in Z(y_j)} B^-_{j,x})\Big)\Big) \\ &\leq \sum_{j \in J_3} \mathcal{H}^d(Q_j \cap D_j) = \omega_d \sum_{j \in J_3} r_j^d, \end{aligned} \tag{18.63}$$

by (18.26), (18.58), and where $\omega_d$ denotes the $\mathcal{H}^d$-measure of the unit ball in $\mathbb{R}^d$. We put everything together and get that

$$\mathcal{H}^d(h_2(E_k \cap W)) \leq C\eta + C(f,\gamma)(1-a) + C(\alpha, f)N^{-1} + C(f)\gamma + \omega_d \sum_{j \in J_3} r_j^d, \tag{18.64}$$

where we no longer write the dependence on $\mathcal{H}^d(E \cap W_f)$, by (18.14), (18.15), (18.21) and (18.40), (18.37) (which control everything except $E_k \cap V_1^+ \setminus [R^1 \cup R^2 \cup R^3]$), and then (18.55), (18.57), and (18.63) which take care of the rest of $E_k \cap V_1^+$.

† Let us say what we get easily under the Lipschitz assumption; additional information will be needed, but we shall only take care of that in the next section. We first check that for $j \in J_3$, $x \in Z(y_j)$, and $z \in E_k \cap B_{j,x}^-$,

$$\widetilde{h}_2(z) = \widetilde{g}(z) \in \widetilde{Q}_j, \tag{18.65}$$

where $\widetilde{Q}_j$ denotes, as above (15.48), the common image $\widetilde{A}_x(P_x)$, $x \in Z(y_j)$. By (16.6), $g(z) = g_j(z) = g_{j,z}(z)$; by the remark below (16.8), we can set $\widetilde{g}(z) = \psi(\lambda g(z))$; by (15.47), $z \in U_{int}$; by (15.53), $\widetilde{g}_j(z) = \psi(\lambda g_j(z))$, and hence $\widetilde{g}(z) = \widetilde{g}_j(z)$; by the line below (15.50), $\widetilde{g}(z) = \widetilde{g}_{j,x}(z)$; finally, by (15.48), $\widetilde{g}(z) = \widetilde{\pi}_j(f(z))$, where $\widetilde{\pi}_j$ denotes the orthogonal projection onto $\widetilde{Q}_j$. So $\widetilde{g}(z) \in \widetilde{Q}_j$. Also, $\widetilde{h}_2(z) = \Pi(\widetilde{g}(z), C\lambda\varepsilon_0)$ by (18.50).

We still need to check that $\widetilde{h}_2(z) = \widetilde{g}(z)$. Denote by $F(y_j)$ the smallest face of our (deformed) grid that contains $y_j = f(x)$; thus $\widetilde{F} = \psi(\lambda F(y_j))$ is a flat face of the usual dyadic grid, the smallest one that contains $\widetilde{y}_j = \psi(\lambda y_j)$. Denote by $\widetilde{W}$ the affine subspace spanned by $\widetilde{F}$, and by $m$ the dimension of $\widetilde{F}$ and $\widetilde{W}$. By Lemma 12.40, $\widetilde{Q}_j \subset \widetilde{W}$. As before, $x \in X_9 \subset X_2 = X_{1,\delta_2}$, so (11.23)-(11.25) say that $x \in X_{1,\delta_2}(m)$ and

$$\mathrm{dist}(y_j, \mathcal{S}_{m-1}) \geq \delta_2 \tag{18.66}$$

as in (18.59). Now

$$\begin{aligned} |g(z) - y_j| &= |\lambda^{-1}\psi^{-1}(\widetilde{g}(z)) - \lambda^{-1}\psi^{-1}(\widetilde{y}_j)| \leq \lambda^{-1}\Lambda|\widetilde{g}(z) - \widetilde{y}_j| \\ &\leq \lambda^{-1}\Lambda|\widetilde{f}(z) - \widetilde{y}_j| = \lambda^{-1}\Lambda|\psi(\lambda f(z)) - \psi(\lambda y_j)| \\ &\leq \Lambda^2|f(z) - y_j| \leq \Lambda^2 r_j \leq \Lambda^2\delta_6 \leq \frac{\delta_2}{10} \end{aligned} \tag{18.67}$$

because $\widetilde{g}(z) = \widetilde{\pi}_j(\widetilde{f}(z))$ and $\widetilde{Q}_j$ goes through $\widetilde{y}_j = \widetilde{f}(y_j) = \widetilde{A}_x(x)$, then by (12.75) and (15.47), then because $f(z) \in D_j$ by (15.22), and by (15.15) and (12.7). Since $\widetilde{g}(z) \in \widetilde{Q}_j \subset \widetilde{W}$ and $\widetilde{y}_j \in \widetilde{W}$, the line segment $L = [\widetilde{y}_j, \widetilde{g}(z)]$ is contained in $\widetilde{W}$. Since for $\xi \in L$,

$$\begin{aligned} |\lambda^{-1}\psi^{-1}(\xi) - y_j| &= |\lambda^{-1}\psi^{-1}(\xi) - \lambda^{-1}\psi^{-1}(\widetilde{y}_j)| \leq \lambda^{-1}\Lambda|\xi - \widetilde{y}_j| \\ &\leq \lambda^{-1}\Lambda|\widetilde{g}(z) - \widetilde{y}_j| \leq \frac{\delta_2}{10} \end{aligned} \tag{18.68}$$

by the end of (18.67), (18.66) says that $\lambda^{-1}\psi^{-1}(L)$ does not meet $\partial F(y_j) \subset \mathcal{S}_{m-1}$. Since $y \in F(y_j)$, we get that $g(z) \in F(y_j)$ too. We deduce from (18.66) and (18.67) that $\mathrm{dist}(g(z), \mathcal{S}_{m-1}) \geq \frac{\delta_2}{2}$, as in (18.61), which implies that

$$\mathrm{dist}(\widetilde{g}(z), \widetilde{\mathcal{S}}_{m-1}) \geq \frac{\lambda\Lambda^{-1}\delta_2}{2} \tag{18.69}$$

where we denote by $\widetilde{\mathcal{S}}_{m-1} = \psi(\lambda \mathcal{S}_{m-1})$ the $(m-1)$-dimensional skeleton in the standard dyadic grid.

Recall that $\widetilde{h}_2(z) = \Pi(\widetilde{g}(z), C\lambda\varepsilon_0)$ by (18.50); we may now use the definitions (17.23) and (17.24) as above, and the same argument based on (17.2) yields that $\widetilde{h}_2(z) = \widetilde{g}(z)$, if $\varepsilon_0$ is small enough compared to $\delta_2$; (18.65) follows.

We'll also need to know that for $j \in J_3$ and $x \in Z(y_j)$, and $k$ large enough

$$h_2(E_k \cap B^-_{j,x}) \subset D_j \cap \lambda^{-1}\psi^{-1}(\widetilde{Q}_j). \tag{18.70}$$

Let $z \in E_k \cap B^-_{j,x}$ be given. We already know from (18.65) that $h_2(z) = g(z) \in \lambda^{-1}\psi^{-1}(\widetilde{Q}_j)$, so we just need to check that $g(z) \in D_j$.

By (15.22), $f(z) \in \frac{1+a}{2} D_j$. If $k$ is large enough, then by (10.4) $z \in E^{\varepsilon r_j}$, and (16.6) and (15.60) say that $|g(z) - f(z)| = |g_j(z) - f(z)| \leq \Lambda^2(1 + 3|f|_{lip})\varepsilon r_j$; hence $g(z) \in D_j$ and (18.70) follows.

We may now follow the proof of (18.63) and (18.64); we get that for $k$ large

$$\mathcal{H}^d\Big(h_2\Big(E_k \cap \bigcup_{j\in J_3} \bigcup_{x\in Z(y_j)} B^+_{j,x} \setminus R^3\Big)\Big) \leq \sum_{j\in J_3} \mathcal{H}^d(D_j \cap \lambda^{-1}\psi^{-1}(\widetilde{Q}_j)) \tag{18.71}$$

and then

$$\begin{aligned}\mathcal{H}^d(h_2(E_k \cap W)) \leq C\eta + C(f,\gamma)(1-a) + C(\alpha, f)N^{-1} + C(f)\gamma \\ + \sum_{j\in J_3} \mathcal{H}^d(D_j \cap \lambda^{-1}\psi^{-1}(\widetilde{Q}_j)).\end{aligned} \tag{18.72}$$

We will only see in the next section how to use our assumption (10.7) or a weaker (but more complicated) one to control the last sum. †

**A lower bound for $H^d(f(E \cap W))$.** We found in (18.64) or (18.72), a first upper bound for the right-hand side of (18.1). The main term in (18.64) is $\omega_d \sum_{j\in J_3} r_j^d$, and we want to bound it by $H^d(f(E\cap W))$, plus small errors. The following lemma, which is our analogue of Lemma 4.111 in [D2], will be useful.

**Lemma 18.73.** *For each $j \in J_3$,*

$$\mathcal{H}^d(D_j \cap f(E \cap W_f)) \geq \big(1 - C(f,\gamma)\varepsilon\big)\, \omega_d r_j^d, \tag{18.74}$$

*where $C(f,\gamma)$ depends only on $|f|_{lip}$ and $\gamma$.*

We shall give a different proof here, so as not to have to construct a competitor again. Instead we shall take advantage of the fact that we could choose extremely small balls $D_j$, which we control by differentiability and density results. Our proof of Lemma 18.73 will also work, with no modification, in the Lipschitz context.

Let $j \in J_3$ be given and pick some $x \in Z(y_j)$; by (15.12), $y_j \in Y_{11} \subset Y_9 = f(X_9)$ (see above (15.1)), so $Z(y_j)$ is not empty and we can choose $x \in Z(y_j)$. Then $x \in X_9 \subset$

$X_5 = \cup_{s\in S} X_5(s)$ (see (15.1) and (11.47)), so there is an index $s$ such that the description in (11.42)-(11.46) is valid. In particular, the graph $\Gamma_s$ contains all the points $z + F_x(z)$, $z \in P_x \cap B(x, \delta_3)$, where $F_x : P_x \to P_x^\perp$ is the $C^1$ function of (11.42). Let $Q_j = A_x(P_x)$ be as above, denote by $\pi_j$ the orthogonal projection onto $Q_j$, and define $G : P_x \cap B(x, \frac{\delta_3}{2}) \to Q_j$ by

$$G(z) = \pi_j(f(z + F_x(z))) \text{ for } z \in P_x \cap B(x, \frac{\delta_3}{2}). \tag{18.75}$$

Even under the Lipschitz assumption, we really want to use $Q_j$ and the fact that $f$ itself (and not $\widetilde{f}$) is well approximated by the affine function $A_x$, as in (11.46). Notice that for $z \in P_x \cap B(x, \frac{\delta_3}{2})$ (as in (18.75)),

$$|F_x(z)| \le ||DF_x||_\infty |z - x| \le \varepsilon|z - x| \tag{18.76}$$

by (11.42), so $z + F_x(z) \in \Gamma_s \cap B(x, \delta_3)$ by (11.43), and

$$|f(z + F_x(z)) - A_x(z + F_x(z))| \le \varepsilon|z + F_x(z) - x| \le 2\varepsilon|z - x| \tag{18.77}$$

by (11.46). Then

$$\begin{aligned} |G(z) - A_x(z)| &= |\pi_j(f(z + F_x(z))) - A_x(z)| = |\pi_j(f(z + F_x(z))) - \pi_j(A_x(z))| \\ &\le |f(z + F_x(z)) - A_x(z)| \\ &\le |f(z + F_x(z)) - A_x(z + F_x(z))| + |A_x(z + F_x(z)) - A_x(z)| \\ &\le 2\varepsilon|z - x| + |f|_{lip}|F_x(z)| \le (2 + |f|_{lip})\varepsilon|z - x| \end{aligned} \tag{18.78}$$

by (18.75), because $\pi_j(A_x(z)) = A_x(z)$ (since $z \in P_x$ and so $A_x(z) \in Q_j$ by definition of $Q_j$), and by (18.77), (11.36), and (18.76).

We can apply this to $z \in 2\overline{E}(x)$, where $E(x) = P_x \cap A_x^{-1}(Q_j \cap D_j)$ is as in (15.16), because

$$2E(x) \subset 2B_{j,x} \subset B(x, 3\gamma^{-1}r_j) \subset B(x, \delta_6) \subset B(x, \frac{\delta_3}{10}) \tag{18.79}$$

by (15.19), (15.20), (15.15), and (12.7). [Again all those things hold in the Lipschitz case; see the remark below (15.39).] We get that

$$|G(z) - A_x(z)| \le (2 + |f|_{lip})\varepsilon|z - x| \le 3\gamma^{-1}(2 + |f|_{lip})\varepsilon r_j \text{ for } z \in 2\overline{E}(x). \tag{18.80}$$

Denote by $\partial$ the boundary of $2E(x)$ in $P_x$; because $x \in X_9 \subset X_8$ and by the definitions (14.21) and (14.5), $A_x$ is a bijective affine mapping from $\partial$ to $Q_j \cap \partial(2D_j)$. Call $\partial'$ the unit sphere in the vector $d$-space parallel to $Q_j$. For each $w \in Q_j \cap D_j$; the mapping $a_w : z \to \frac{A_x(z)-w}{|A_x(z)-w|}$, from $\partial$ to $\partial'$, is well-defined (because $A_x(z)$ does not take the value $w$). Its degree is the same for all $w \in Q_j \cap D_j$, and it is equal to $\pm 1$ because this is its value at $w = y_j$.

For $w \in Q_j \cap D_j$ and $z \in \partial$, $A_x(z) \in Q_j \cap \partial(2D_j)$, so $|A_x(z) - w| \geq r_j$, hence by (18.80) the segment $[A_x(z), G(z)]$ does not contain $w$. Then

$$(z,t) \to a_{w,t}(z) = \frac{(1-t)A_x(z) + tG(z) - w}{|(1-t)A_x(z) + tG(z) - w|} \tag{18.81}$$

is defined and continuous on $\partial \times [0,1]$. It defines a homotopy from $a_w$ to $a_{w,1}$, among mappings from $\partial$ to $\partial'$. Thus $a_{w,1}$ has the same degree as $a_w$, namely, $\pm 1$. Then $a_{w,1}$ does not extend continuously as a mapping from $2\overline{E}(x)$ to $\partial'$, and this forces $G(z) - w$ to vanish at some point $z \in 2\overline{E}(x)$ (otherwise, use $\frac{G(z)-w}{|G(z)-w|}$). We just proved that

$$G(2\overline{E}(x)) \text{ contains } Q_j \cap D_j. \tag{18.82}$$

Set $\lambda = 1 - 3\gamma^{-1}(2 + |f|_{lip})\varepsilon$. We want to estimate the size of

$$Y = Q_j \cap \lambda D_j \setminus \pi_j[D_j \cap f(E \cap W_f)]. \tag{18.83}$$

Let $w \in Y$ be given, and use (18.82) to find $z \in 2\overline{E}(x)$ such that $G(z) = w$. Set $y = z + F_x(z)$ and observe that

$$|y - x| = |z + F_x(z) - x| \leq |z - x| + \varepsilon|z - x| < 4\gamma^{-1} r_j < \frac{\delta_3}{5} \tag{18.84}$$

by (18.76) and (18.79), so $y \in \Gamma_s$ by (11.43). Notice that

$$w = G(z) = \pi_j(f(z + F_x(z))) = \pi_j(f(y)) \tag{18.85}$$

by (18.75) and other definitions.

If $y \in E \cap W_f$, we observe that

$$\begin{aligned} |w - f(y)| &= |\pi_j(f(y)) - f(y)| \leq \operatorname{dist}(f(y), Q_j) \leq |f(y) - A_x(z)| \\ &= |f(z + F_x(z)) - A_x(z)| \leq (2 + |f|_{lip})\varepsilon|z - x| \leq 3\gamma^{-1}(2 + |f|_{lip})\varepsilon r_j \end{aligned} \tag{18.86}$$

by (18.85), because $A_x(z) \in Q_j$, by the last lines of (18.78) and the end of (18.80). So $f(y) \in D_j$ because $w \in \lambda D_j$, and $w = \pi_j(f(y))$ lies in $\pi_j[D_j \cap f(E \cap W_f)]$, a contradiction with the definition of $Y$.

So $y \notin E \cap W_f$. But

$$y \in B(x, 4\gamma^{-1} r_j) \subset B(x, \delta_6) \subset B(x, \delta_1/10) \subset W_f, \tag{18.87}$$

by (18.84), (15.15), (12.7), (11.22), and the fact that $x \in X_9 \subset X_1$. So $y \notin E$. In addition, $y \in \Gamma_s$ (see below (18.84)). Set $\rho = 4\gamma^{-1} r_j$; we just proved that $y \in B(x,\rho) \cap \Gamma_s \setminus E$; hence $w = \pi_j(f(y)) \in \pi_j(f(B(x,\rho) \cap \Gamma_s \setminus E))$ and now

$$\mathcal{H}^d(Y) \leq \mathcal{H}^d(\pi_j(f(B(x,\rho) \cap \Gamma_s \setminus E))) \leq |f|_{lip}^d \mathcal{H}^d(B(x,\rho) \cap \Gamma_s \setminus E). \tag{18.88}$$

We want to apply (11.44) to $B(x,\rho)$; this is allowed because $\rho = 4\gamma^{-1} r_j \leq \delta_6 \leq \delta_3/10$ (again by (15.15) and (12.7)), and (11.44) yields

$$\mathcal{H}^d(B(x,\rho) \cap [\Gamma_s \cup E] \setminus X_3(s)) \leq \varepsilon \rho^d \leq C(\gamma)\varepsilon r_j^d \tag{18.89}$$

Now $\Gamma_s \setminus E \subset \Gamma_s \cup E \setminus X_3(s)$, just because $X_3(s) \subset X_0 \subset E$, so (18.88) yields

$$\mathcal{H}^d(Y) \leq C(\gamma)|f|_{lip}^d \varepsilon r_j^d. \tag{18.90}$$

Finally,

$$\begin{aligned} \mathcal{H}^d(D_j \cap f(E \cap W_f)) &\geq \mathcal{H}^d(\pi_j[D_j \cap f(E \cap W_f)]) \geq \mathcal{H}^d(Q_j \cap \lambda D_j) - \mathcal{H}^d(Y) \\ &= \lambda^d \omega_d r_j^d - \mathcal{H}^d(Y) \geq \omega_d r_j^d - C(f,\gamma)\varepsilon r_j^d \end{aligned} \tag{18.91}$$

by (18.83), (18.90), and because $\lambda = 1 - 3\gamma^{-1}(2 + |f|_{lip})\varepsilon$. Lemma 18.73 follows. □

**The final estimate.** We are now ready to conclude, at least under the rigid assumption. We sum (18.74) over $j \in J_3$ and get that

$$\begin{aligned} \sum_{j \in J_3} \omega_d r_j^d &\leq \big(1 - C(f,\gamma)\varepsilon\big)^{-1} \sum_{j \in J_3} \mathcal{H}^d(D_j \cap f(E \cap W_f)) \\ &\leq \big(1 - C(f,\gamma)\varepsilon\big)^{-1} \mathcal{H}^d(f(E \cap W_f)) \leq \mathcal{H}^d(f(E \cap W_f)) + C'(f,\gamma)\varepsilon \end{aligned} \tag{18.92}$$

because the $D_j$ are disjoint (see (15.13)), and if $\varepsilon$ is small enough. We compare this with (18.64) and get that

$$\mathcal{H}^d(h_2(E_k \cap W)) \leq \mathcal{H}^d(f(E \cap W_f)) + \mathcal{E}, \tag{18.93}$$

with

$$\mathcal{E} \leq C\eta + C(f,\gamma)(1-a) + C(\alpha, f)N^{-1} + C(f)\gamma + C'(f,\gamma)\varepsilon. \tag{18.94}$$

Recall from (18.46) that $|h_2(x) - x| \geq \frac{\delta_5}{4}$ for $x \in E_k \cap V_1^+$. Since $\{E_k\}$ converges to $E$ and $h_2$ is continuous, we also get that

$$|h_2(x) - x| \geq \frac{\delta_5}{4} \text{ for } x \in E \cap \operatorname{int}(V_1^+), \tag{18.95}$$

and hence $E \cap \operatorname{int}(V_1^+) \subset W$ (recall that $W = \big\{y \in U\,; h_2(x) \neq x\big\}$ by (18.2)). Hence $E \setminus W \subset E \setminus \operatorname{int}(V_1^+)$, and

$$\begin{aligned} \mathcal{H}^d(E \cap W_f) - \mathcal{H}^d(E \cap W) &\leq \mathcal{H}^d(E \cap W_f \setminus W) \leq \mathcal{H}^d(E \cap W_f \setminus \operatorname{int}(V_1^+)) \\ &= \mathcal{H}^d(E \cap W^2) \leq C(f,\gamma)(1-a) + C\eta, \end{aligned} \tag{18.96}$$

by (18.38) and (18.39). At the same time, $W$ is open (see the definition above), and Theorem 10.97 (our main lower semicontinuity result) says that for $k$ large,

$$\mathcal{H}^d(E \cap W) \leq \mathcal{H}^d(E_k \cap W) + \eta. \tag{18.97}$$

By (18.96), (18.97), and then (18.1) and (18.93),

$$\begin{aligned} \mathcal{H}^d(E \cap W_f) &\leq \mathcal{H}^d(E \cap W) + C(f,\gamma)(1-a) + C\eta \\ &\leq \mathcal{H}^d(E_k \cap W) + C(f,\gamma)(1-a) + C\eta \\ &\leq M\mathcal{H}^d(h_2(E_k \cap W)) + h(R'')^d + C(f,\gamma)(1-a) + C\eta \\ &\leq M\mathcal{H}^d(f(E \cap W_f)) + M\mathcal{E} + h(R'')^d + C(f,\gamma)(1-a) + C\eta \\ &\leq M\mathcal{H}^d(f(E \cap W_f)) + \mathcal{E}' + h(R'')^d \end{aligned} \tag{18.98}$$

for $k$ large, where $\mathcal{E}'$ is given by the same sort of formula (18.94) as $\mathcal{E}$, and where

$$R'' = R_0 + 4\Lambda^2(1 + |f|_{lip})\delta_6 + C\Lambda\varepsilon_0 \tag{18.99}$$

is still as in (17.41). Now we choose our various small constants $\gamma$, $a$, $\alpha$, $N$, $\eta$, $\varepsilon$, $\delta_6$, and $\varepsilon_0$ in this order (as announced in Remark 11.17), so as to make $\mathcal{E}'$ and $(R'')^d - R_0^d$ arbitrarily small. Since (18.98) holds for all these choices, we get that

$$\mathcal{H}^d(E \cap W_f) \leq M\mathcal{H}^d(f(E \cap W_f)) + hR_0^d. \tag{18.100}$$

This is the same as (2.5) in the circumstances that were described at the beginning of Section 11 : compare $E \cap W_f$ in (11.19) with $W_1$ in Definition 2.3, and recall that $f = \varphi_1$ (see above (11.18)). So we completed the verification of (10.9), and therefore proved Theorem 10.8 in the special case when we have the rigid assumption. □

## 19. Proof of Theorem 10.8, and variants, under the Lipschitz assumption.

In this section we work under the Lipschitz assumption, and try to prove (18.100) (almost) as in the previous section.

The only remaining difficulty is that we have (18.72) instead of (18.64), and the difference between the two right-hand sides is

$$\Delta = \sum_{j \in J_3} \mathcal{H}^d(D_j \cap \lambda^{-1}\psi^{-1}(\widetilde{Q}_j)) - \omega_d \sum_{j \in J_3} r_j^d. \tag{19.1}$$

If we follow the proof above and use (18.72) instead of (18.64), we do not need to modify Lemma 18.73 (which is still valid in the Lipschitz context), but we need to add $\Delta$ to $\mathcal{E}$ in (18.93) and (18.94), and then $M\Delta$ to $\mathcal{E}'$ in (18.98). So, if we could prove that $\Delta$ can be made arbitrarily small (with a choice of constants as above), then we could quietly follow the same proof above and get the conclusion. Before we do this, we need to modify a little our definition of our final mapping $h_2$ in the $B_{j,x}$, $x \in Z(y_j)$, for some $j \in J_3$.

**The indices $j \in J_4$ for which $y_j \in L_i$ only if it lies in its interior.** We define the set $J_4 \subset J_3$ to be the set of indices $j \in J_3$ such that, for all $i \in [0, j_{max}]$ such that $y_j \in L_i$, $y_j$ actually lies in the $n$-dimensional interior of $L_i$ (that is, for the ambient topology of $\mathbb{R}^n$).

For these $j$, we were a little too prudent with the definition of $g_j$ and $h_2$, because the boundary condition (1.7) is much easier to fulfill in these cases. The truth is that we should have defined the corresponding $g_j$ differently, but rather than modifying the construction above, we shall fix it by continuing our deformation $\{h_t\}$, $0 \leq t \leq 2$, a little further. Set

$$h_t(z) = h_2(z) \quad \text{when } 2 < t \leq 3 \text{ and } z \in U \setminus \bigcup_{j \in J_4} \bigcup_{x \in Z(y_j)} B^-_{j,x}. \tag{19.2}$$

In the remaining sets $B^-_{j,x}$, $j \in J_4$ and $x \in Z(y_j)$, we can proceed independently (because these sets are disjoint by (15.27) and (15.20)). Fix $j \in J_4$, denote by $Q_j$ the common value of the $A_x(P_x)$, $x \in Z(y_j)$, as we did before, and define the $h_t$, $2 < t \leq 3$, on $B^-_{j,x}$ by

$$h_t(z) = (1 - \beta_{j,x}(z,t))h_2(z) + \beta_{j,x}(z,t)\pi_j(h_2(z)), \tag{19.3}$$

where $\pi_j$ denotes the orthogonal projection on $Q_j$ and

$$\beta_{j,x}(z,t) = \text{Min}\Big(1, \frac{100(1 + |f|_{lip})}{(1-a)r_j}\,(t-2)\,\mathrm{dist}(z, \partial B^-_{j,x})\Big). \tag{19.4}$$

Notice that $\beta_{j,x}(z,t) = 0$, and hence $h_t(z) = h_2(z)$, when $t = 2$ and when $z \in \partial B^-_{j,x}$, so we glue things in a continuous way. The final mapping $h_3$ is even Lipschitz, because it is Lipschitz on each $B^-_{j,x}$ (there is a finite collection of them), on the rest of $\mathbb{R}^n$, and is continuous across the boundary.

We need to check that the $h_{3t}$, $0 \leq t \leq 3$ still satisfy the conditions (1.4)-(1.8), relative to the set $E_k$, as in Lemma 17.40 and (17.87) (proved below (17.107)).

The continuity condition (1.4) follows from its counterpart for the $h_{2t}$, and the Lipschitz property (1.8) was just discussed. For the other properties (1.5)-(1.7), and the constraint on the $\widehat{W}$-set, we just need to worry about the only places where we change something, i.e., the sets $B^-_{j,x}$.

So let $j \in J_4$, $x \in Z(y_j)$, and $z \in E_k \cap B^-_{j,x}$ be given, and let us derive some general information on $z$ and the $h_t(z)$. We know that $x \in X_0$, so $\varphi_1(x) \neq x$ by (11.19) and (11.20), which implies that $x \in B$ (by (1.5)) and that $x$ and $f(x)$ both lie in $\widehat{W}$ (by the definition (2.2)). Since in addition

$$|z - x| \leq 2\gamma^{-1} r_j \leq \delta_6 \tag{19.5}$$

by (15.20) and, say, (15.24), the first part implies that $z \in B(X_0, R_0 + \delta_6) \subset B(X_0, R'')$ (see (11.1) and (17.41)). Then (1.5) holds.

Let us check that

$$h_2(z) = g(z) \quad \text{for } z \in E_k \cap B^-_{j,x}. \tag{19.6}$$

By (18.65), $\widetilde{h}_2(z) = \widetilde{g}(z)$. In addition, $z \in V_1^+$ (see the definition (16.7) and (15.20)), hence $z \in H_1$ by (18.43). By (17.101), we can compute $h_2(z)$ and $g(z)$ in terms of $\widetilde{h}_2(z)$ and $\widetilde{g}(z)$; (19.6) follows.

Moreover, we get that for $z \in E_k \cap B^-_{j,x}$ and $t \geq 2$,

$$\text{(19.7)}\quad |h_t(z)-y_j| \leq \mathrm{Max}(|h_2(z)-y_j|, |\pi_j(h_2(z))-y_j|) = |h_2(z)-y_j| = |g(z)-y_j| \leq \Lambda^2\delta_6$$

by (19.3), because $y_j = f(x) = A_x(x) \in Q_j$ by definitions, and by (18.67) (which holds because $z \in E_k \cap B^-_{j,x}$).

Return to our verifications. For (1.6), we need to check that $h_t(z) \in B(X_0, R'')$ when $z \in E_k \cap B(X_0, R'')$, and since we already know about $h_t$, $0 \leq t \leq 2$, we can restrict to $z \in E_k \cap B^-_{j,x}$ as above. The desired estimate follows from (19.7) because $y_j = f(x) \in B$, by (1.5) for $f = \varphi_1$ and because $x \in X_9 \subset E_0 = E \cap W_f \subset B$ (see (15.1), (11.20), and (11.19)).

For the verification of (2.4), recall that $x$ and $y_j = f(x)$ both lie in (the old set) $\widehat{W}$, and so $z$ and the new $h_t(z)$ lie within $\Lambda^2\delta_6$ of $\widehat{W}$, by (19.7) in particular. This still puts them in a compact subset of $U$, by (12.6) and (12.7).

We are left with (1.7) to check. That is, we fix $0 \leq i \leq j_{max}$, and we want to check that for $k$ large, $h_t(z) \in L_i$ for $0 \leq t \leq 3$ as soon as $z \in E_k \cap L_i$. We know this when $0 \leq t \leq 2$, by (1.7) for the $h_{2t}$ (see (17.87)); so we may assume that $t > 2$. We also know this when $z \in E_k \setminus \bigcup_{j\in J_4} \bigcup_{x\in Z(y_j)} B^-_{j,x}$, because (19.2) says that $h_t(z) = h_2(z) \in L_i$. So we may fix $j \in J_4$, $x \in Z(y_j)$, and it is enough to check that

$$\text{(19.8)}\qquad h_t(z) \in L_i \text{ for } z \in E_k \cap L_i \cap B^-_{j,x} \text{ and } 2 \leq t \leq 3.$$

We first assume that $y_j \notin L_i$. Let us check that for $k$ large,

$$\text{(19.9)}\qquad E_k \cap L_i \cap B^-_{j,x} = \emptyset$$

for $x \in Z(y_j)$; (19.8) will follow trivially. Let $z \in B^-_{j,x}$ be given; we proved below (18.68) that $g(z) \in F(y_j)$, where $F(y_j)$ is the smallest face of our grid that contains $y_j$. In fact, since $g(z)$ is also far from the boundary of $F(y_j)$ (by (18.69)), we see that $g(z)$ lies in $I$, the interior of $F(y_j)$. Now suppose that $z \in L_i$ for some $i$. Since $g(z) = h_2(z)$, (1.7) for $h_2$ says that $g(z) \in L_i$. Then some face $F$ of $L_i$ meets $I$, and since $I$ is the interior of a face, $F$ contains $I$. But $y_j \in I$, by definition of the smallest face $F(y_j)$, so $y_j \in L_i$. Our current assumption says that this is impossible, so $z \notin L_i$, and (19.9) follows.

We are left with the case when $y_j \in L_i$. Since $j \in J_4$, this implies that $y_j$ is an interior point of $L_i$, and we want to deduce from this that

$$\text{(19.10)}\qquad \overline{B}(y_j, \Lambda^2\delta_6) \subset L_i.$$

Since by (19.7) $|h_t(z) - y_j| \leq \Lambda^2\delta_6$ for $z \in E_k \cap B^-_{j,x}$ and $t \geq 2$, (19.8) will follow at once.

Denote by $\delta(L_i)$ the (true $n$-dimensional) boundary of $L_i$, and set $D = \mathrm{dist}(y_j, \delta(L_i))$; we know that $D > 0$, and we want to show that $D \geq \Lambda^2\delta_6$. Take $\xi \in \delta(L_i)$ such that $|\xi - y_j| = D$, and let $F$ denote the smallest face of our grid that contains $\xi$. Since $\delta(L_i)$ is composed of full faces too, $F$ is contained in $\delta(L_i)$. Hence $y_j \notin F$, and $F$ does not contain the smallest face $F(y_j)$ that contains $y_j$. First assume that $F(y_j)$ is not reduced to $\{y_j\}$; then (3.8) (and a conjugation by $\psi$ that makes us lose a constant $\Lambda^2$) yields

$$D = |\xi - y_j| \geq \mathrm{dist}(y_j, F) \geq \Lambda^{-2}\,\mathrm{dist}(y_j, \partial F(y_j)). \tag{19.11}$$

In addition, $x \in Z(y_j)$, so (15.1) says that $x \in X_9 \subset X_2 = X_{1,\delta_2}$. By (11.23)-(11.25), this means that $x \in X_{1,\delta_2}(m)$, where $m$ is the dimension $F(y_j)$, and hence $\mathrm{dist}(y_j, \mathcal{S}_{m-1}) \geq \delta_2$. Then $D \geq \Lambda^{-2}\delta_2 \geq 10\delta_6$ by (12.7). The sad truth is that this is not enough, but this is easy to fix: we just need to require that $\delta_6 \leq 2\Lambda^{-4}\delta_2$ in (addition to) (12.7), or replace $\Lambda^2$ with $\Lambda^4$ in (12.7), and then we get the desired estimate which implies (19.10). In the remaining case when $F(y_j) = \{y_j\}$, $D$ is at least equal to the smallest distance between a vertex of the grid (namely $y_j$) and a face that does not contain it. This distance is at least $\lambda^{-1}\Lambda^{-1}r_0$; hence $D \geq \lambda^{-1}\Lambda^{-1}r_0 \geq 2\Lambda^2\delta_6$, if we put an extra power of $\Lambda$ in (12.7).

This completes our verification of (1.7) for our extended family of $h_t$, and also the verification of (1.4)-(1.8). We also checked (2.4) for the extended family $\{h_t\}$ along the way, so we have the analogue of (18.1) for $h_3$.

Now we want to see why $h_3$ is possibly better than $h_2$. We do not need to modify any of our estimates, except for the ones relative to the $B^-_{j,x}$, $x \in Z(y_j)$ and $j \in J_4$. Fix such $j$ and $x$, and let us first check that for $k$ large,

$$h_3 \text{ is } C\Lambda^2(1+|f|_{lip})\text{-Lipschitz on } E_k \cap B^-_{j,x}. \tag{19.12}$$

We already know that $h_2(z) = g(z) = g_j(z)$ on $E_k \cap B^-_{j,x}$ (also see (16.6)). But (15.62) says that $g_j$ is $C\Lambda^2(1+|f|_{lip})$-Lipschitz on $E^{\varepsilon r_j} \cap B^-_{j,x}$, hence also on $E_k \cap B^-_{j,x}$ (just because $E_k \cap B^-_{j,x} \subset E^{\varepsilon r_j}$ for $k$ large). But we need to worry a little about the rapid fluctuations of $\beta_{j,x}(z,3)$. As usual, pick $z, w \in E_k \cap B^-_{j,x}$ and write

$$\begin{aligned} h_3(z) - h_3(w) &= (1-\beta_{j,x}(z,3))h_2(z) + \beta_{j,x}(z,3)\pi_j(h_2(z)) \\ &\qquad\qquad - (1-\beta_{j,x}(w,3))h_2(w) - \beta_{j,x}(w,3)\pi_j(h_2(w)) \\ &= -[\beta_{j,x}(z,3) - \beta_{j,x}(w,3)]\,[h_2(z) - \pi_j(h_2(z))] \\ &\qquad + (1-\beta_{j,x}(w,3))[h_2(z) - h_2(w)] + \beta_{j,x}(w,3)[\pi_j(h_2(z)) - \pi_j(h_2(w))] \end{aligned} \tag{19.13}$$

so that

$$\begin{aligned} |h_3(z) - h_3(w)| &\leq |\beta_{j,x}(z,3) - \beta_{j,x}(w,3)|\,|h_2(z) - \pi_j(h_2(z))| + C\Lambda^2(1+|f|_{lip})|z-w| \\ &\leq \frac{100\Lambda(1+|f|_{lip})}{(1-a)r_j}|z-w|\,|h_2(z) - \pi_j(h_2(z))| + C\Lambda^2(1+|f|_{lip})|z-w| \end{aligned} \tag{19.14}$$

because $h_2$ is $C\Lambda^2(1+|f|_{lip})$-Lipschitz and by (19.4). Then we want to estimate $|h_2(z) - \pi_j(h_2(z))|$, i.e., prove that $h_2(z)$ is close to $Q_j$. Recall from (18.65) and the line below that

$$\widetilde{h}_2(z) \in \widetilde{Q}_j = \widetilde{A}_x(P_x); \tag{19.15}$$

also, we checked below (19.6) that $z \in E_k \cap H_1$, so we may apply (17.101) and we get that

$$\widetilde{h}_2(z) = \psi(\lambda h_2(z)). \tag{19.16}$$

Then

$$\begin{aligned} |\widetilde{h}_2(z) - \widetilde{f}(x)| &= |\psi(\lambda h_2(z)) - \psi(\lambda f(x))| \leq \lambda\Lambda |h_2(z) - f(x)| \\ &= \lambda\Lambda |h_2(z) - y_j| \leq \lambda\Lambda r_j \end{aligned} \tag{19.17}$$

because $h_2(z) \in D_j$ by (18.70).

Use (19.15) to write $\widetilde{h}_2(z) = \widetilde{A}_x(\xi)$, with $\xi \in P_x$; we want to evaluate $|\xi - x|$. Recall from just above (19.3) that $Q_j = A_x(P_x)$. The discussion near (15.41) says that that since $y_j \in Y_{10}$ (by (15.12)), the restriction of $\psi$ to $Q_j$ is differentiable at $\lambda y_j$, with a derivative $D_\psi$ such that

$$\lambda D_\psi(DA_x(v)) = D\widetilde{A}_x(v) \text{ for } v \in P'_x\,, \tag{19.18}$$

where $P'_x$ denotes the vector space parallel to $P_x$. Since $\widetilde{A}_x$ is affine and $\widetilde{A}_x(x) = \widetilde{f}(x)$ (see (12.38)), we get that

$$\begin{aligned} |\widetilde{h}_2(z) - \widetilde{f}(x)| &= |\widetilde{A}_x(\xi) - \widetilde{A}_x(x)| = |D\widetilde{A}_x(\xi - x)| = |\lambda D_\psi(DA_x(\xi - x))| \\ &\geq \lambda\Lambda^{-1} |DA_x(\xi - x)| \geq \lambda\Lambda^{-1}\gamma|\xi - x| \end{aligned} \tag{19.19}$$

because $D_\psi$, just like $\psi$ itself, is $\Lambda$-biLipschitz, and because $DA_x$ has no contracting direction since $x \in Z(y_j) \subset X_9 \subset X_8 \subset X_6 \setminus X_7$; see (15.1), (14.21), and (14.5). We compare (19.19) to (19.17) and get that

$$|\xi - x| \leq \Lambda^2\gamma^{-1} r_j \leq \Lambda^2\delta_6 \leq \frac{\delta_3}{10} \tag{19.20}$$

by (15.15) and (12.7). Now

$$\begin{aligned} |h_2(z) - \pi_j(h_2(z))| &= \mathrm{dist}(h_2(z), Q_j) \leq |h_2(z) - A_x(\xi)| \\ &\leq |h_2(z) - f(\xi)| + |f(\xi) - A_x(\xi)| \end{aligned} \tag{19.21}$$

because $A_x(\xi) \in Q_j$ (since $\xi \in P_x$ and $Q_j = A_x(P_x)$). By (19.20), we can apply (11.46) and get that

$$|f(\xi) - A_x(\xi)| \leq \varepsilon|\xi - x|. \tag{19.22}$$

Also, (19.20) allows us to apply (12.52), which says that $\widetilde{f}(\xi) = \psi(\lambda f(\xi))$ is well defined, and also that $|\widetilde{f}(\xi) - \widetilde{A}_x(\xi)| \leq \lambda\varepsilon|\xi - x|$. Then

$$\begin{aligned} |h_2(z) - f(\xi)| &= |\lambda^{-1}\psi^{-1}(\widetilde{h}_2(z)) - \lambda^{-1}\psi^{-1}(\widetilde{f}(\xi))| \leq \lambda^{-1}\Lambda|\widetilde{h}_2(z) - \widetilde{f}(\xi)| \\ &= \lambda^{-1}\Lambda|\widetilde{A}_x(\xi) - \widetilde{f}(\xi)| \leq \Lambda\varepsilon|\xi - x| \end{aligned} \tag{19.23}$$

by (19.16), because $\widetilde{h}_2(z) = \widetilde{A}_x(\xi)$ by definition of $\xi$, and by (12.52). Altogether,

$$|h_2(z) - \pi_j(h_2(z))| \leq (1+\Lambda)\varepsilon|\xi - x| \leq (\Lambda^2 + \Lambda^4)\gamma^{-1}\varepsilon r_j \tag{19.24}$$

by (19.21), (19.22), (19.23), and (19.20).

For the record, notice that (19.3) and (19.24) imply that for $z \in E_k \cap B^-_{j,x}$,

$$|h_3(z) - h_2(z)| \leq |h_2(z) - \pi_j(h_2(z))| \leq \varepsilon(\Lambda^4 + \Lambda^2)\gamma^{-1} r_j \leq C\varepsilon\Lambda^4\delta_6 \leq \delta_5/10 \tag{19.25}$$

by (15.15), (12.7), and if $\varepsilon$ is small enough (depending on $\Lambda$). We used $\varepsilon$ here just so that we don't have to put an extra power of $\Lambda$ in the definition of $\delta_6$, but we could have done that too. Since we simply have that $h_3(z) = h_2(z)$ when $z$ lies in no $B^-_{j,x}$, we get that for $k$ large,

$$|h_3(z) - h_2(z)| \leq \delta_5/10 \quad \text{for } z \in E_k. \tag{19.26}$$

Let us return to our $z$ and $w$, plug (19.24) into (19.14), and get that

$$\begin{aligned}
|h_3(z) - h_3(w)| &\leq \frac{100\Lambda(1+|f|_{lip})}{(1-a)r_j}|z-w|\,|h_2(z) - \pi_j(h_2(z))| + C\Lambda^2(1+|f|_{lip})|z-w| \\
&\leq C(\Lambda^4+\Lambda^2)\gamma^{-1}\varepsilon r_j \frac{\Lambda(1+|f|_{lip})}{(1-a)r_j}\,|z-w| + C\Lambda^2(1+|f|_{lip})|z-w| \\
&\leq C\Lambda^2(1+|f|_{lip})|z-w|
\end{aligned} \tag{19.27}$$

if $\varepsilon$ is small enough, depending on $\Lambda$, $\gamma$, and $a$. This proves (19.13).

Next we take care of little rims. Set

$$R(j,x) = \Big\{ z \in B^-_{j,x}\,;\, \mathrm{dist}(z, \partial B^-_{j,x}) \leq \frac{(1-a)r_j}{100(1+|f|_{lip})} \Big\} \tag{19.28}$$

for $j \in J_4$ and $x \in Z(y_j)$. By (15.17), (15.20), and the proof of (18.32), we get that

$$\mathcal{H}^d(E \cap \overline{R}(j,x)) \leq C(f)(1-a)\gamma^{1-d} r_j^d \leq C(f,\gamma)(1-a)\mathcal{H}^d(E \cap B(x, |f|^{-1}_{lip} r_j)). \tag{19.29}$$

The total contribution of these annuli to the right-hand side of (18.1) is still small, because (as happened near (18.34))

$$\begin{aligned}
\sum_{j\in J_4}\sum_{x\in Z(y_j)} \mathcal{H}^d(h_3(E_k \cap R(j,x))) &\leq C \sum_{j\in J_4}\sum_{x\in Z(y_j)} \mathcal{H}^d(E_k \cap R(j,x)) \\
&\leq \eta + C\sum_{j\in J_4}\sum_{x\in Z(y_j)} \mathcal{H}^d(E \cap \overline{R}(j,x)) \\
&\leq \eta + C(f,\gamma)(1-a)\sum_{j\in J_4}\sum_{x\in Z(y_j)} \mathcal{H}^d(E \cap B(x, |f|_{lip} r_j)) \\
&\leq \eta + C(f,\gamma)(1-a) H^d(E \cap W_f) = \eta + C(f,\gamma)(1-a)
\end{aligned} \tag{19.30}$$

by (19.12), for $k$ large and by (10.14), by (19.29), and because the $B(x,|f|_{lip}^{-1}r_j)$ are disjoint by (18.33) and contained in $W_f$ by (18.31).

We are left with $B_{j,x}^- \setminus R(j,x)$. Observe that $\beta_{j,x}(z,3) = 1$ for $z \in B_{j,x}^- \setminus R(j,x)$, by (19.4), so (19.3) and the first part of (18.70) yield

$$h_3(z) = \pi_j(h_2(z)) \in Q_j \cap D_j \text{ for } z \in E_k \cap B_{j,x}^- \setminus R(j,x), \tag{19.31}$$

at least for $k$ large. Again all the sets $Q_j \cap D_j = A_x(P_x) \cap D_j$, $x \in Z(y_j)$, coincide, and now

$$\begin{aligned} &\mathcal{H}^d\Big(h_3\Big(E_k \cap \bigcup_{j\in J_4} \bigcup_{x\in Z(y_j)} B_{j,x}^-\Big)\Big) \\ &\qquad \leq \sum_{j\in J_4} \sum_{x\in Z(y_j)} \mathcal{H}^d(h_3(E_k \cap R(j,x))) + \sum_{j\in J_4} \mathcal{H}^d(Q_j \cap D_j) \\ &\qquad \leq \omega_d \sum_{j\in J_4} r_j + \eta + C(f,\gamma)(1-a). \end{aligned} \tag{19.32}$$

Thus the contribution of all the sets $B_{j,x}^-$ where we modified $h_2$ is just as good as in the rigid case, and we shall only need to worry about the contribution of the indices $j \in J_3 \setminus J_4$.

**We get rid of some small set in $Y_{11}$.** Let us introduce a small bad set $Z_0 \subset Y_{11}$. For each $y \in U$, denote by $F(y)$ the smallest face of our grid on $U$ that contains $y$. Also set $\widetilde{y} = \psi(\lambda y)$ and call $\widetilde{F}(y) = \psi(\lambda F(y))$ the smallest face of the usual dyadic grid that contains $\widetilde{y}$. Finally call $\widetilde{W}(y)$ the smallest affine space that contains $\widetilde{F}(y)$. Then set

$$\begin{aligned} A_r(y) = r^{-d} \sup \big\{ &\mathcal{H}^d(B(y,r) \cap \lambda^{-1}\psi^{-1}(\widetilde{Q}))\,;\, \widetilde{Q} \text{ is a } d\text{-dimensional} \\ &\text{affine subspace of } \widetilde{W}(y) \text{ that contains } \widetilde{y} \big\}; \end{aligned} \tag{19.33}$$

when the dimension of $\widetilde{W}(y)$ is less than $d$, just set $A_r(y) = 0$. For $0 \leq i \leq j_{max}$, set $L_i' = L_i \setminus \mathrm{int}(L_i)$, where $\mathrm{int}(L_i)$ is really the interior of $L_i$, taken in $\mathbb{R}^n$ and regardless of the dimension of the faces that compose it, and then set

$$\widehat{L} = \bigcup_{0\leq i\leq j_{max}} L_i \text{ and } \widehat{L}' = \bigcup_{0\leq i\leq j_{max}} L_i'. \tag{19.34}$$

Still denote by $\omega_d$ the $d$-dimensional Hausdorff measure of the unit ball in $\mathbb{R}^d$. Set

$$Z = \big\{ y \in \widehat{L}'\,;\, \limsup_{r\to 0} A_r(y) > \omega_d \big\}. \tag{19.35}$$

We chose this definition because it will be easy to use, and we chose to use the condition (10.7) because it is not too complicated, and because it implies that

$$H^d(Z) = 0. \tag{19.36}$$

Let us check this. Let us rather use the translation of (10.7) that is given below (10.7) itself. This condition gives an exceptional set $Z_0$ such that $\mathcal{H}^d(Z_0) = 0$ and, if $y \in U \setminus Z_0$ lies in $L'_i = L_i \setminus \mathrm{int}(L_i)$ and is such that dimension$(F(y)) > d$, then we can find $t = t(y) > 0$ such that the restriction of $\psi$ to $\lambda F(y) \cap B(\lambda y, t)$ is $C^1$.

We want to show that $\mathcal{H}^d$-almost every $y \in Z$ lies in $Z_0$. Let $y \in Z$ be given; then $y \in L'_i$ for some $i$, there is a face of $L_i$ that contains $y$, and this face contains $F(y)$ by definition of $F(y)$ as a smallest face.

If dimension$(F(y)) > d$ and $y \in Z \setminus Z_0$, we can find $t = t(y) > 0$ such that the restriction of $\psi$ to $\lambda F(y) \cap B(\lambda y, t)$ is $C^1$. Since $\psi$ is Lipschitz, this also means that the restriction of $\psi^{-1}$ to the face $\widetilde{F}(y) = \psi(\lambda F(y))$ is $C^1$ in a neighborhood of $\widetilde{y} = \psi(\lambda y)$. Recall that $y$ is an interior point of $F(y)$, so $\widetilde{F}(y)$ coincides with $\widetilde{W}(y)$ (the affine affine space spanned by $\widetilde{F}(y)$) near $\widetilde{y}$.

With the notation above, if $\widetilde{Q}$ is a $d$-dimensional affine subspace of $\widetilde{W}(y)$, the restriction of $\psi^{-1}$ to $\widetilde{Q}$ is also $C^1$ near $\widetilde{y}$, with uniform estimates with respect to $\widetilde{Q}$. Then $\lambda^{-1}\psi^{-1}(\widetilde{Q})$ is a $C^1$ surface near $y$, and $\lim_{r\to 0} r^{-d}\mathcal{H}^d(B(y,r) \cap \lambda^{-1}\psi^{-1}(\widetilde{Q})) = \omega_d$, uniformly in $\widetilde{Q}$. Thus $\limsup_{r\to 0} A_r(y) \leq \omega_d$, which contradicts the fact that $y \in Z$ and takes care of the case when dimension$(F(y)) > d$.

If dimension$(F(y)) < d$, then by definition $A_r(y) = 0$ for $r > 0$ small, and $y \notin Z$ (a contradiction).

If dimension$(F(y)) = d$, there is only one possible choice of $\widetilde{Q}$ in the definition (19.33) of $A_r(y)$, namely $\widetilde{W}(y)$, and

$$\begin{aligned} A_r(y) &= r^{-d}\mathcal{H}^d(B(y,r) \cap \lambda^{-1}\psi^{-1}(\widetilde{W}(y)) = r^{-d}\mathcal{H}^d(B(y,r) \cap \lambda^{-1}\psi^{-1}(\widetilde{F}(y)) \\ &= r^{-d}\mathcal{H}^d(B(y,r) \cap F(y)) \end{aligned} \tag{19.37}$$

for $r$ small, because $y$ is an interior point of $F(y)$ (and hence $\widetilde{y}$ is an interior point of $\widetilde{F}(y)$). But for each face $F$ of dimension $d$ and $\mathcal{H}^d$-almost-every interior point $y \in F$, $\lim_{r\to 0} r^{-d}\mathcal{H}^d(B(y,r) \cap F(y)) = \omega_d$ (because $F$ is rectifiable), so $\mathcal{H}^d(\mathrm{int(F)} \cap Z) = 0$. This takes care of the case when dimension$(F(y)) = d$. This was our last case, and (19.36) follows.

We now assume (19.36) (and the other assumptions of Theorem 10.8, except perhaps (10.7)) and show that $E$ is quasiminimal as in (10.9). We proceed as in the last sections, with only two modifications. The first one occurs in Step 4 (in Section 15), and we shall explain it now. The second one is the one that was described earlier in this section, and concerns the indices $j \in J_4$.

So we do not change anything up to Section 15; we also define $Y_9$, $Y_{10}$, and $Y_{11}$ as before, but before we cover $Y_{11}$ by disks $D_j$ (near (15.12)), we remove some small pieces.

First set $Y_{12} = Y_{11} \setminus Z$. Then $\mathcal{H}^d(Y_{11} \setminus Y_{12}) = 0$ by (19.36). Set $X_{12} = X_{11} \cap f^{-1}(Y_{12})$; the same proof as for (15.11) (or, more precisely, for (4.77) in [D2]) yields that

$$\mathcal{H}^d(X_{11} \setminus X_{12}) = 0. \tag{19.38}$$

We shall remember that by (19.35),

$$\limsup_{r\to 0} A_r(y) \leq \omega_d \quad \text{when } y \in Y_{12} \cap \widehat{L}'. \tag{19.39}$$

Let $\delta_9 > 0$ be small, set

$$Y_{13} = Y_{13}(\delta_9) = [Y_{12} \setminus \widehat{L}'] \cup \big\{y \in Y_{12} \cap \widehat{L}' \,;\, A_r(y) \leq \omega_d + \varepsilon \text{ for } 0 < r \leq \delta_9\big\} \tag{19.40}$$

and then

$$X_{13} = X_{13}(\delta_9) = X_{12} \cap f^{-1}(Y_{13}). \tag{19.41}$$

Notice that $Y_{12}$ is, by (19.39), the monotone union of the sets $Y_{13}(\delta_9)$, so $X_{12}$ is the monotone union of the sets $X_{13}(\delta_9)$. Thus we can choose $\delta_9 > 0$ so small that

$$\mathcal{H}^d(X_{12} \setminus X_{13}) \leq \eta. \tag{19.42}$$

We choose $\delta_9 > 0$ like this, and then cover $Y_{13}$ as we did before (for $Y_{11}$) by balls $D_j = B(y_j, r_j)$, $j \in J_3$, so that

$$y_j \in Y_{13} \ \text{ and } \ 0 < r_j < \mathrm{Min}(\delta_8, \delta_9) \ \text{ for } j \in J_3, \tag{19.43}$$

$$\text{the } \overline{D}_j,\ j \in J_3, \text{ are disjoint} \tag{19.44}$$

and

$$\mathcal{H}^d\big(X_{13} \setminus f^{-1}\big(\bigcup_{j \in J_3} \overline{D}_j\big)\big) \leq \eta. \tag{19.45}$$

Then we continue the construction as before, all the way through Section 18, and arrive to the second modification. We define $J_4 \subset J_3$ as we did near (19.2) and we continue the deformation all the way to $h_3$, just as was explained at the beginning of this section.

We follow the proof of (18.71), but restrict to the indices $j \in J_3 \setminus J_4$; we get that

$$\mathcal{H}^d\Big(h_3\Big(E_k \cap \bigcup_{j \in J_3 \setminus J_4} \ \bigcup_{x \in Z(y_j)} B_{j,x}^+ \setminus R^3\Big)\Big) \leq \sum_{j \in J_3 \setminus J_4} \mathcal{H}^d(D_j \cap \lambda^{-1}\psi^{-1}(\widetilde{Q}_j)) \tag{19.46}$$

(recall that $h_3 = h_2$ on these sets). Now let $j \in J_3 \setminus J_4$ be given. By definition of $J_4$, we can find $i \in [0, j_{max}]$ such that $y_j \in L_i$, without lying on the interior of $L_i$. That is, $y_j \in L_i' \subset \widehat{L}'$ (see (19.34) and the definition above it). Notice that $y_j \in Y_{13}$ because of our first modification. Since $y_j \in \widehat{L}'$, (19.40) implies that

$$A_r(y_j) \leq \omega_d + \varepsilon \ \text{ for } 0 < r \leq \delta_9 \tag{19.47}$$

But $D_j = B(y_j, r_j)$ and $r_j < \delta_9$ by (19.43), so

$$r_j^{-d}\mathcal{H}^d(D_j \cap \lambda^{-1}\psi^{-1}(\widetilde{Q}_j)) \leq \omega_d + \varepsilon \tag{19.48}$$

by (19.47), the definition (19.33), and because $\widetilde{Q}_j$ contains $y_j$ and is a $d$-dimensional subspace of the affine space spanned by $\widetilde{F}(y_j)$ (see the discussion below (18.65)). We replace in (19.46) and get that

$$\mathcal{H}^d\Big(h_3\Big(E_k \cap \bigcup_{j\in J_3\setminus J_4}\ \bigcup_{x\in Z(y_j)} B^+_{j,x} \setminus R^3\Big)\Big) \leq \sum_{j\in J_3\setminus J_4} (\omega_d+\varepsilon) r_j^d. \tag{19.49}$$

Then we add this to (19.32) and get that

$$\mathcal{H}^d\Big(h_3\Big(E_k \cap \bigcup_{j\in J_3}\ \bigcup_{x\in Z(y_j)} B^+_{j,x} \setminus R^3\Big)\Big) \leq \sum_{j\in J_3} (\omega_d+\varepsilon) r_j^d + \eta + C(f,\gamma)(1-a) \tag{19.50}$$

because $B^+_{j,x} \setminus R^3 \subset B^-_{j,x}$ (see (18.26)). The last part is an error term smaller than $\mathcal{E}$ in (18.93) and (18.94). We also have the small term

$$\varepsilon \sum_{j\in J_3} r_j^d \leq C(f)\varepsilon \sum_{j\in J_3} \mathcal{H}^d(E\cap B(x,|f|_{lip}^{-1} r_j)) \leq C(f)\,\varepsilon\,\mathcal{H}^d(E\cap W_f) \tag{19.51}$$

by Proposition 4.1 (which we can apply because of (18.31) and $W_f \subset U$, as for the proof of (18.32)), and then the disjointness (18.33) and the fact that $B(x,|f|_{lip}^{-1} r_j) \subset W_f$ by (18.31). This term too is dominated by $\mathcal{E}$, so (19.50) is essentially as good as (18.63) (the difference is controlled by $\mathcal{E}$).

We may now continue the proof as before. There is a last place where we need to be careful, when we use (18.95) to prove set inclusions in (18.97). Previously the sets $W$ and $W^2$ were defined in terms of $h_2$, and now we need the same inclusions with the sets $W_3$ and $W_3^2$ defined in terms of $h_3$. Fortunately, (19.26) says that $|h_3(z)-h_2(z)| \leq \delta_5/10$ for $z\in E_k$; this stays true for $z\in E$ (because $h_3-h_2$ is continuous and $E$ is the limit of $\{E_k\}$); then (18.95) also holds for $h_3$, with the smaller constant $\delta_5/10$, and we can complete the argument as before (i.e., $E\setminus W_3 \subset E\setminus \mathrm{int}(V_1^+)$, and (18.96)-(18.100) are valid).

This finally completes our proof of Theorem 10.8 in the remaining Lipschitz case. □

**Remark 19.52.** Our proof shows that in Theorem 10.8 (and under the Lipschitz assumption), we can replace the assumption (10.7) with the slightly weaker (but more complicated) (19.36).

It is a little sad that the author was not able to get rid of (10.7) or (19.36) altogether. We seem to be close to that, but not quite close enough. It would seem natural to try the following modification of what we do for $d$-dimensional faces. Notice that we just need to apply the definition of $A_r(y)$ at points $y_j$, $j\in J_3$, and to the specific $d$-dimensional set $\widetilde{Q}_j = \widetilde{A}_x(P_x)$, $x\in Z(y_j)$. Modulo some additional cutting, we could restrict to a subset of $Y_{11}$ where $\psi(\lambda\cdot)$ coincides with a $C^1$ mapping. Then we are supposed to go from $y_j$ to $\widetilde{y}_j = \psi(\lambda y_j)$, get $\widetilde{Q}_j$, which is also the image by $D\psi(\lambda\cdot)$ of the tangent place to $Y_{11}$ of $f(E)$ at $y_j$, and show that it has density 1. For instance, we would know this for $\widetilde{f}(E) = \psi(\lambda f(E))$, which is tangent to $\widetilde{Q}_j$ at $\widetilde{y}_j$. But could it be that by bad luck, $\lambda^{-1}\psi^{-1}(\widetilde{Q}_j)$ has more little wrinkles than $f(E)$, even though they are tangent.

One option to try to overcome this could be to try to project points back from $\lambda^{-1}\psi^{-1}(\widetilde{Q}_j)$ to $f(E)$, or a flatter set, but there are difficulties because we need to do this in a Lipschitz way, and more importantly along the faces (because of (1.7)), and for instance $f(E)$ may have little holes (although probably small because $E$ has surjective projections at places where it is flat). Because all this seems complicated, the author decided to leave Theorem 10.8 as it is for the moment.

## PART V : ALMOST MINIMAL SETS AND OTHER THEOREMS ABOUT LIMITS

In this part we apply the limiting results of the previous part to sequences of almost minimal, or even minimal sets. The proofs will usually not be very hard, but this part should be useful because it is likely that the results of this paper will more often be applied in the almost minimal context.

In Section 20 we give three slightly different definitions of sliding almost minimal sets (Definition 20.2), and then show that the two last ones are equivalent (Proposition 20.9). The definitions and proof are inspired of [D5]; the point is to unify some of the definitions, and to make it easier to check some assumptions.

In Section 21 we use Theorem 10.8 (our main result about limits) to show that limits of coral sliding almost minimal sets (of a given type and with a given gauge function) are also coral sliding almost minimal sets, of the same type and with the same gauge function. See Theorem 21.3.

In Section 22 we prove an upper semicontinuity result for $\mathcal{H}^d$, which says that if $\{E_k\}$ is a convergent sequence of coral sliding almost minimal sets in $U$ (as in Theorem 21.3), then for each compact set $H \subset U$, $\limsup_{k\to+\infty} \mathcal{H}^d(E_k \cap H) \leq \mathcal{H}^d(E \cap H)$. See Theorem 22.1. We also prove Lemma 22.3, where we only assume that the $E_k$ lie in $GSAQ(U, M, \delta, h)$ and merely get that $\limsup_{k\to+\infty} \mathcal{H}^d(E_k \cap H) \leq (1 + Ch)M\mathcal{H}^d(E \cap H)$.

In Section 23 we consider sequences of almost minimal sets $E_k$ that live in domains $U_k$ and with boundary sets $L_{j,k}$ that depend slightly on $k$. We get an analogue of Theorems 10.8 and 21.3 that works when $U_k$ and the $L_{j,k}$ are small bilipschitz variations of the limits $U$ and the $L_j$. See Theorem 23.8, which is proved brutally with a change of variables.

We apply this result in Section 24, to the special case of blow-up limits. We find two sets of flatness conditions on the sets $L_j$ (see Definitions 24.8 and 24.29) under which the blow-up limits at the origin of a sliding almost minimal set are sliding minimal sets in $\mathbb{R}^n$, associated to boundary sets $L_j^0$ obtained from the $L_j$ by the same blow-up. See Theorem 24.13 and Proposition 24.35.

### 20. Three notions of almost minimal sets.

We shall more often apply the regularity results above, and in particular Theorem 10.8 about limits, in the simpler context of almost minimal sets.

In this section we adopt the same point of view as in [D5], and introduce three types of almost minimal sets; we shall mostly restrict to the two last ones, which are slightly weaker, turn out to be equivalent to each other under mild assumptions, and for which the desired limiting theorem will easily follow from Theorem 10.8. The main point of this section will be the equivalence between our second and third definitions. It is perhaps not vital because we can hope to work with a single definition at a time, but the author will feel much better for not hiding a little secret under the rug. Also, the regularity results of the previous sections translate a little better in terms of our second definition, while the third one seems a little simpler.

So we shall give three different definitions of almost minimal sets, for which we keep the same setting as in Definition 2.3. That is, we are given an open set $U$ (equal to the unit ball when we work under the rigid assumption, and to a bilipschitz image of the unit ball when we work under the Lipschitz assumption), and boundaries $L_j$, $0 \leq j \leq j_{max}$. We give a special name to $\Omega = L_0$, and require that all our sets be contained in $\Omega$ (but we can take $\Omega = U$).

Now we also give ourselves a gauge function, i.e., a function $h : (0, +\infty) \to [0, +\infty]$ such that

$$h \text{ is continuous from the right and } \lim_{t \to 0} h(t) = 0; \tag{20.1}$$

let us not assume that $h$ is nondecreasing for the moment, because we don't need this. It would also make sense, in view of the definition below, to assume that the product $h(r)r^d$ is nondecreasing, but let us not do that either.

**Definition 20.2.** *Let $E \subset \Omega \cap U$ be a relatively closed in $U$ and such that, as in (1.2),*

$$\mathcal{H}^d(E \cap B) < +\infty \quad \textit{for every compact ball } B \textit{ such that } B \subset U. \tag{20.3}$$

*We say that $E$ is an $A_+$-almost minimal set (of dimension $d$) in $U$, with the sliding conditions given by the closed sets $L_j$, $0 \leq j \leq j_{max}$, and the gauge function $h$, if for every choice of one-parameter family $\{\varphi_t\}$, $0 \leq t \leq 1$, of continuous functions with the properties (1.4)-(1.8) relative to a ball $B = \overline{B}(x, r)$, and also such that $\widehat{W} \subset\subset U$ as in (2.4), we have that*

$$\mathcal{H}^d(W_1) \leq (1 + h(r))\mathcal{H}^d(\varphi_1(W_1)), \tag{20.4}$$

*where as usual $W_1 = \{y \in E\,;\, \varphi_1(y) \neq y\}$.*
*We say that $E$ is an $A$-almost minimal set (with the sliding conditions given by the closed sets $L_j$, $0 \leq j \leq j_{max}$, and the gauge function $h$) if under the same circumstances,*

$$\mathcal{H}^d(W_1) \leq \mathcal{H}^d(\varphi_1(W_1)) + h(r)r^d. \tag{20.5}$$

*Finally, we say that $E$ is an $A'$-almost minimal set (with the sliding conditions given by the closed sets $L_j$, $0 \leq j \leq j_{max}$, and the gauge function $h$) if under the same circumstances,*

$$\mathcal{H}^d(E \setminus \varphi_1(E)) \leq \mathcal{H}^d(\varphi_1(E) \setminus E) + h(r)r^d. \tag{20.6}$$

So the accounting in the three cases is slightly different, but the competitors are the same (and are the same as for the generalized quasiminimal sets in Definition 2.3). We could also have forced the competitors to be such that $\widehat{W}$, instead of being merely compactly contained in $U$, is contained in a ball of radius $r$ which itself is compactly contained in $U$, and this would probably not have made a big difference in practice, but we decided to keep the same competitors as above.

The last two definitions look slightly easier to use. Let us also check that in Definition 20.2, we could replace (20.6) with

$$\mathcal{H}^d(E \cap \widehat{W}) \leq \mathcal{H}^d(\varphi_1(E) \cap \widehat{W}) + h(r)r^d \tag{20.7}$$

and get an equivalent definition. Notice that $\varphi_1(E)$ coincides with $E$ out of $\widehat{W}$ (by the definition (2.4) of $\widehat{W}$), and $\mathcal{H}^d(E \cap \widehat{W}) < +\infty$ (by (20.3) and because $\widehat{W} \subset\subset U$); then (20.7) is obtained from (20.6) by adding $\mathcal{H}^d(E \cap \varphi_1(E) \cap \widehat{W})$ to both sides.

We shall now worry about the inclusion relations between our three classes of almost minimal sets.

It is fairly easy to see that if $E$ is $A_+$-almost minimal in $U$, then $E$ is also $A$-almost minimal in every smaller open set $U_\tau = \big\{x \in U\,;\, \overline{B}(x,\tau) \subset U\big\}$, with the same boundaries $L_j$, but a slightly larger gauge function $\widetilde{h}$ (that depends only on $h$, $\tau$, and $n$ through local Ahlfors-regularity constants). The proof is the same as in Remark 4.5 of [D5], and it is fairly easy once you notice that $E$ is quasiminimal, hence locally Ahlfors-regular. This is also the reason why we restrict to a smaller set $U_\tau$. The converse looks like it could be wrong, but the author does not know for sure, even in the case without boundary.

It is also easy to see that if $E$ is $A'$-almost minimal in $U$, then it is $A$-almost minimal in $U$, with the same $L_j$ and the same gauge function $h$. To see this, let $E$ be $A'$-almost minimal, let the $\varphi_t$ be as in the definition, and let us deduce (20.5) from (20.7). Set $Z = E \cap \widehat{W} \setminus W_1$ and observe that

$$\begin{aligned} \mathcal{H}^d(W_1) &= \mathcal{H}^d(E \cap \widehat{W}) - \mathcal{H}^d(E \cap \widehat{W} \setminus W_1) = \mathcal{H}^d(E \cap \widehat{W}) - \mathcal{H}^d(Z) \\ &\leq \mathcal{H}^d(\varphi_1(E) \cap \widehat{W}) - \mathcal{H}^d(Z) + h(r)r^d \\ &\leq \mathcal{H}^d(\varphi_1(E) \cap \widehat{W} \setminus Z) + h(r)r^d \end{aligned} \tag{20.8}$$

because $W_1 \subset E \cap \widehat{W}$, by (2.7), and because $Z \subset \varphi_1(E) \cap \widehat{W}$ since $\varphi_1(z) = z$ for $z \in Z$ (by definition of $W_1$). For (2.5) is is enough to check that $\varphi_1(E) \cap \widehat{W} \setminus Z \subset \varphi_1(W_1)$. So let $y \in \varphi_1(E) \cap \widehat{W} \setminus Z$ be given, and write $y = \varphi_1(x)$. If $x \in W_1$, we are happy. Otherwise, $\varphi_1(x) = x$, hence $y = x \in E \cap \widehat{W} \setminus W_1 = Z$, which is impossible. The $A$-minimality of $E$ follows.

Notice that if $E$ is $A$-almost minimal (and hence also if $E$ is $A'$-almost minimal), then it is quasiminimal in every ball $B(x,r) \subset U$, with $M = 1$ and $h = h(r)$. So we shall be able to apply the regularity results of the previous parts to almost minimal sets.

The fact that $A$-minimality implies $A'$-minimality will be a little more complicated to prove, and in fact, under the Lipschitz assumption we shall only be able to do it under the same additional assumption (10.7) as for Theorem 10.8. The following is a generalization of Proposition 4.10 in [D5].

**Proposition 20.9.** *Suppose that the rigid assumption holds, or that the Lipschitz assumption, plus one of the two technical conditions (10.7) or (19.36), hold. Let $E$ be* *an $A$-almost minimal set in $U$, with the sliding conditions given by the closed sets $L_j$, $0 \leq j \leq j_{max}$, and the gauge function $h$. Then $E$ is also $A'$-almost minimal in $U$, with the same $L_j$ and the same $h$. The converse also holds (see above).*

Without of our extra assumption (10.7), we do not know whether, under the Lipschitz assumption alone, $A_+$ minimality and $A$-minimality always imply $A'$-minimality. But we do not have good reasons to think that it fails either.

Our assumption (20.1) should not bother much, but if it fails we can still do something. The fact that $h(r)$ tends to 0 when $r$ tends to 0 will be used only once, at the beginning of the proof in the Lipschitz case, to show that $E^*$ is rectifiable and Ahlfors-regular. If we do not suppose this, we can suppose instead that $E \in QSAQ(U, M, \delta, h)$ for a number $h > 0$ that is small enough (depending on $n$, $M$, and $\Lambda$) for Theorem 5.16 and Propositions 4.1 and 4.74 to apply. If $h$ is not continuous from the right, our proof will only show that $E$ is $A'$-almost minimal with the larger gauge function $h'(r) = \liminf_{\varepsilon \to 0^+} h(r + \varepsilon)$.

We shall need to revise the proof of [D5], because it involves a modification of a family $\{\varphi_t\}$, $0 \leq t \leq 1$, and we want to make sure that we do not destroy the boundary conditions (1.7). Also, the proof in the Lipschitz case will be a little more complicated, and will use the rectifiability of $E$, so we shall give two different arguments, one for the rigid case and one for the Lipschitz case. Of course the second argument also works in the rigid case.

**Proof of Proposition 20.9 under the rigid assumption.** Let $E$ be $A$-almost minimal; we want to prove that $E$ is $A'$-almost minimal, so we give ourselves mappings $\{\varphi_t\}$, $0 \leq t \leq 1$, that satisfy (1.4)-(1.8) relative to a ball $B = \overline{B}(x_0, r_0)$, and are also such that $\widehat{W} \subset\subset U$. If $\varphi_1(W_1)$ were disjoint from $E \setminus W_1$, we could easily deduce (20.7) from (20.5): we would say that

$$
\begin{aligned}
\mathcal{H}^d(E \cap \widehat{W}) &= \mathcal{H}^d(E \cap \widehat{W} \setminus W_1) + \mathcal{H}^d(W_1) \\
&\leq \mathcal{H}^d(E \cap \widehat{W} \setminus W_1) + \mathcal{H}^d(\varphi_1(W_1)) + h(r_0) r_0^d \\
&= \mathcal{H}^d(\varphi_1(E \cap \widehat{W} \setminus W_1)) + \mathcal{H}^d(\varphi_1(W_1)) + h(r_0) r_0^d \\
&= \mathcal{H}^d(\varphi_1(E \cap \widehat{W})) + h(r_0) r_0^d,
\end{aligned}
\tag{20.10}
$$

as needed. In general, we want to modify $\varphi_1$ slightly, so as to be able to almost apply the argument above. And rather than move $\varphi_1(W_1)$, it will be more convenient to make $W_1$ artificially larger, by a minor modification that will not change (20.7) significantly, but will make (20.5) more useful.

We first want to construct a vector-valued function $v$, defined on $U$, which we shall see as a direction in which we are allowed to move the points. Denote by $\mathcal{F}$ the set of faces of dimension at least $d$ of our usual dyadic grid. For each $F \in \mathcal{F}$, set

$$
\begin{aligned}
F_\tau &= \big\{x \in F\,;\, \mathrm{dist}(x, \partial F) \geq \tau\big\} \ \text{ and} \\
F_\tau^+ &= \big\{x \in \mathbb{R}^n\,;\, \mathrm{dist}(x, F_\tau) \leq \frac{\tau}{10}\big\},
\end{aligned}
\tag{20.11}
$$

where the very small $\tau > 0$ will be chosen later, and then set

$$
\begin{aligned}
h_F(x) &= 1 - 10\tau^{-1}\,\mathrm{dist}(x, F_\tau) \quad \text{for } x \in F_\tau^+, \\
h_F(x) &= 0 \qquad\qquad\qquad\qquad\quad\ \text{for } x \in \mathbb{R}^n \setminus F_\tau^+.
\end{aligned}
\tag{20.12}
$$

Finally choose for each $F \in \mathcal{F}$ a vector $v_F$ in the vector space $\mathrm{Vect}(F)$ parallel to $F$, and such that $\frac{1}{2} \leq |v_F| \leq 1$, and set

$$v(x) = \sum_{F \in \mathcal{F}} h_F(x)\, v_F \quad \text{for } x \in \mathbb{R}^n. \tag{20.13}$$

Recall from (3.8) that if $F, G \in \mathcal{F}$ are different faces, with $\dim(F) \geq \dim(G)$, then

$$\mathrm{dist}(y, G) \geq \mathrm{dist}(y, \partial F) \quad \text{for } y \in F. \tag{20.14}$$

In particular, $\mathrm{dist}(y, G) \geq \tau$ if $y \in F_\tau$, and hence

$$\mathrm{dist}(F_\tau^+, G_\tau^+) \geq \frac{8\tau}{10}. \tag{20.15}$$

Thus the sum in (20.13) has at most one term, and when we compute the differential of $v$ term by term, we get that

$$v \text{ is } 10\tau^{-1}\text{-Lipschitz}. \tag{20.16}$$

Also, (20.12), (20.13), and (20.15) yield

$$v(x) = v_F \quad \text{for } x \in F_\tau. \tag{20.17}$$

We also need a cut-off function $\chi$. First select compact subsets $S$ and $S'$ of $U$, such that

$$\widehat{W} \subset S \subset \mathrm{int}(S') \subset S' \subset U \tag{20.18}$$

and $S' \subset B(x_0, r_0 + \tau)$, where $B = \overline{B}(x_0, r_0)$ is as in (1.4)-(1.8). Let us also make sure, for instance by replacing $S'$ with a smaller compact set, that

$$\mathcal{H}^d(E \cap S' \setminus S) \leq \varepsilon, \tag{20.19}$$

where the small number $\varepsilon > 0$ is chosen in advance. Then choose a Lipschitz function $\chi$ on $U$, so that

$$0 \leq \chi(x) \leq 1 \;\text{ everywhere, } \chi(x) = 1 \text{ on } S, \text{ and } \chi(x) = 0 \text{ on } U \setminus S'. \tag{20.20}$$

We shall select an extremely small $t_0 > 0$ and continue the one parameter family $\{\varphi_t\}$ with mappings $\varphi_t$, $1 \leq t \leq 1 + t_0$, defined by

$$\varphi_t(x) = \psi_t(\varphi_1(x)), \quad \text{where } \; \psi_t(y) = y + (t-1)\chi(y)v(y). \tag{20.21}$$

Our constant $t_0$ will be chosen last, depending on $\varphi_1$, $\tau$, $\varepsilon$, $S$, $S'$, and even $\chi$ if needed, so small that $\psi_t$ is 2-Lipschitz for $1 \leq t \leq 1 + t_0$, and hence

$$\varphi_{1+t_0} \text{ is } 2\,||\varphi_1||_{lip}\text{-Lipschitz on } E. \tag{20.22}$$

Next we want to check that the $\varphi_{(1+t_0)t}$, $0 \leq t \leq 1$, satisfy the required conditions (1.4)-(1.8), but this time with respect to the slightly larger ball $B' = \overline{B}(x_0, r_0+\tau+t_0)$. We still have (1.4) and (1.8) because $v$ and $\chi$ are Lipschitz. For (1.5), we just need to worry about $t > 1$. Observe that when $x \in E \setminus B'$, $\chi(\varphi_1(x)) = \chi(x) = 0$ and hence $\varphi_t(x) = \varphi_1(x)$ for $t \geq 1$; thus (1.5) holds. For (1.6), let $x \in E \cap B'$ be given. If $x \in B$, (1.6) for our initial $\varphi_t$ says that $\varphi_1(x) \in B$, and then $\varphi_t(x) \in B'$ for $t \geq 1$, by (20.21). If $x \in B(x_0, r_0+\tau) \setminus B$, then $\varphi_1(x) = x \in B(x_0, r_0 + \tau) \setminus B$ by (1.5) for our initial $\varphi_t$, so $\varphi_t(x) \in B'$ for $t \geq 1$, again by (20.21) and because $t - 1 \leq t_0$. Finally, if $x \in B' \setminus B(x_0, r_0 + \tau)$, we still have that $\varphi_1(x) = x$, and now $\chi(x) = 0$ and so $\varphi_t(x) = x$ for $t > 1$. So (1.6) holds.

We are left with (1.7) to check. Let $j \leq j_{max}$ and $x \in E \cap L_j \cap B'$ be given; we want to check that $\varphi_t(x) \in L_j$ for $t \geq 1$ (we already know about $t \leq 1$, by assumption). Set $y = \varphi_1(x)$; thus

$$\varphi_t(x) = \psi_t(y) = y + (t-1)\chi(y)v(y) \tag{20.23}$$

by (20.21). If $v(y) = 0$, then $\varphi_t(x) = \varphi_1(x) \in L_j$. Otherwise, $y \in F_\tau^+$ for some $F \in \mathcal{F}$, and

$$\varphi_t(x) = y + (t-1)\chi(y)h_F(y)v_F \tag{20.24}$$

by (20.12) and (20.13) (also recall that the $F_\tau^+$ are disjoint by (20.15)). Let $G$ denote the smallest face of our grid that contains $y$. We claim that $G$ contains $F$. Let $z \in F_\tau$ be such that $|z - y| \leq \tau/10$; if $G$ does not contain $F$, (3.8) applies and says that $\mathrm{dist}(z, G) \geq \mathrm{dist}(z, \partial F) \geq \tau$, a contradiction since $y \in G$. So $F \subset G$.

Next we claim that $\varphi_t(x) \in G$, at least if we take $t_0 < \tau/10$. Denote by $z'$ the orthogonal projection of $y$ onto the smallest affine space $W$ that contains $F$; then $|z' - y| \leq |z - y| \leq \tau/10$ (because $z \in W$), so $\mathrm{dist}([z, z'], \partial F) \geq \mathrm{dist}(z, \partial F) - |z' - y| \geq \mathrm{dist}(z, \partial F) - 2\tau/10 \geq 8\tau/10$ (because $z \in F_\tau$), and so $z'$ lies in the interior of $F$.

By (20.24), $\varphi_t(x) = y + \lambda v_F$, with $|\lambda| \leq t - 1 \leq t_0 < \tau/10$ and $v_F \in \mathrm{Vect}(F)$. Let us compute with coordinates. The face $F$ is given by some equations $z_i = a_i$, where the $z_i$ are coordinates of the current point $z$, and the $a_i \in 2^{-m}\mathbb{Z}$ are constants, and some inequalities $z_j \in I_j$, where each $I_j$ is a dyadic interval of size $2^{-m}$. When we replace $y$ with $\varphi_t(x) = y + \lambda v_F$, we only modify some of the $z_j$, but since $\mathrm{dist}(z', \partial F) \geq 8\tau/10$, the corresponding coordinates stay in the interior of corresponding $I_j$. The other coordinates $z_i$ stay whatever they were, and altogether $\varphi_t(x)$ lies in exactly the same faces that contain $y$. Since $y \in G$, we get that $\varphi_t(x) \in G$. By definition of $G$ as the the smallest face that contains $y$, $G \subset L_j$ because $y = \varphi_1(x) \in L_j$ (by (1.7) for $\varphi_1$). Hence $\varphi_t(x) \in L_j$, as needed for (1.7).

We also need to check the assumption (2.4) for our extended family. Let $x \in E$ and $t \in [0, 1 + t_0]$ be such that $\varphi_t(x) \neq x$. If $t \leq 1$, we know (by assumption) that $x \in W_t$ and $\varphi_t(x) \in \widehat{W}$, a compact subset of $U$. So we may assume that $t > 1$, and also that $\varphi_t(x) \neq \varphi_1(x)$. Set $y = \varphi_1(x)$; since $\varphi_t(x) = y + (t-1)\chi(y)v(y)$ by (20.23), we get that $\chi(y) \neq 0$, hence $y \in S'$ by (20.20). Thus $\mathrm{dist}(\varphi_t(x), S') \leq t - 1 \leq t_0$ and, if $t_0$ is small enough, this forces $\varphi_t(x)$ to stay in a (fixed) compact subset of $U$. Also, either $x = y$, and then $x \in S'$, or else $x \neq y = \varphi_1(x)$, hence $x \in W_1 \subset S'$ too, so $W_t \subset S'$ for $t \geq 1$. Thus (2.4) holds.

We may now use our assumption that $E$ is $A$-minimal. Set

$$\varphi = \varphi_{1+t_0} \text{ and } W = W_{1+t_0} = \big\{x \in E\,;\, \varphi(x) \neq x\big\}; \tag{20.25}$$

then

$$\mathcal{H}^d(W) \leq \mathcal{H}^d(\varphi(W)) + h(r_0+\tau+t_0)(r_0+\tau+t_0)^d = H^d(\varphi(W)) + h(r_1)r_1^d \tag{20.26}$$

by (20.5) for the extended family, and with $r_1 = r_0 + \tau + t_0$.

We want to say that $W$ is large. First observe that

$$\big\{x \in E\,;\, |\varphi_1(x) - x| > t_0\big\} \subset W, \tag{20.27}$$

just because $|\varphi(x) - \varphi_1(x)| = |\psi_{1+t_0}(\varphi_1(x)) - \varphi_1(x)| \leq t_0$ by (20.21). Set

$$A_\tau = \bigcup_{F\in\mathcal{F}} F_\tau, \tag{20.28}$$

and notice that by (20.17), $v(x) \neq 0$ on $A_\tau$; then

$$S \cap A_\tau \cap E \setminus W_1 \subset W \tag{20.29}$$

because if $x \in S \cap A_\tau \cap E \setminus W_1$, then $\varphi_1(x) = x \in S \cap A_\tau$ and hence

$$\varphi(x) = \varphi_{1+t_0}(x) = \psi_{1+t_0}(x) = x + t_0\chi(x)v(x) \neq x \tag{20.30}$$

by (20.25), (20.21), because $\xi(x) = 1$ by (20.20), and because $v(x) \neq 0$.

Recall that we want to prove that $E$ is $A'$-almost minimal, so we want to establish (20.7), i.e., estimate $\mathcal{H}^d(E \cap \widehat{W})$, where

$$\widehat{W} = \bigcup_{0<t\leq 1} W_t \cup \varphi_t(W_t) \tag{20.31}$$

is as in (2.2). But it will be more convenient to work with the compact set $S$ of (20.18), and estimate $\mathcal{H}^d(E \cap S)$; we shall see that it makes no difference for (2.7). We write $S = (S \cap W) \cup (S \setminus W)$, and so

$$\mathcal{H}^d(E\cap S) \leq \mathcal{H}^d(W) + \mathcal{H}^d(E\cap S\setminus W) \leq \mathcal{H}^d(\varphi(W)) + h(r_1)r_1^d + \mathcal{H}^d(E\cap S\setminus W) \tag{20.32}$$

by (20.26). Next we estimate $\mathcal{H}^d(E \cap S \setminus W)$. Set

$$Z_\tau = \mathbb{R}^n \setminus A_\tau = \mathbb{R}^n \setminus \bigcup_{F\in\mathcal{F}} F_\tau \tag{20.33}$$

(by (20.28)). By the definition (20.11), every interior point of a face of dimension $\geq d$ lies in $F_\tau$ for $\tau$ small, so

$$\bigcap_{\tau>0} Z_\tau = \mathcal{S}_{d-1}, \tag{20.34}$$

where $\mathcal{S}_{d-1}$ still denotes the union of the faces of dimension $d-1$ of our net; since the intersection is decreasing and $\mathcal{H}^d(E \cap S) < +\infty$ (because $S \subset\subset U$), we get that

$$\mathcal{H}^d(E \cap S \setminus A_\tau) = \mathcal{H}^d(E \cap S \cap Z_\tau) \leq \varepsilon \tag{20.35}$$

if $\tau$ is chosen small enough, and where $\varepsilon > 0$ is the same small number given in advance as in (20.19).

Similarly, $W_1 = \{x \in E\,;\, \varphi_1(x) \neq x\}$ is the monotone union of the sets $\{x \in E\,;\, |\varphi_1(x) - x| > t_0\}$ that show up in (20.27), so (20.27) says that if $t_0$ is small enough,

$$\mathcal{H}^d(W_1 \setminus W) \leq \varepsilon \tag{20.36}$$

(again, this holds because $W_1 \subset E \cap \widehat{W} \subset E \cap S$ and hence $\mathcal{H}^d(W_1) < +\infty$). Then

$$\begin{aligned} \mathcal{H}^d(E \cap S \setminus W) &\leq \mathcal{H}^d(W_1 \setminus W) + \mathcal{H}^d(E \cap S \setminus (W \cup W_1)) \\ &\leq \varepsilon + \mathcal{H}^d(E \cap S \setminus (W \cup W_1)) \\ &= \varepsilon + \mathcal{H}^d(E \cap S \setminus (W \cup W_1 \cup A_\tau)) \\ &\leq \varepsilon + \mathcal{H}^d(E \cap S \setminus A_\tau) \leq 2\varepsilon \end{aligned} \tag{20.37}$$

by (20.36), because $E \cap S \cap A_\tau \setminus (W \cup W_1) = \emptyset$ by (20.29), and by (20.35). So (20.32) yields

$$\mathcal{H}^d(E \cap S) \leq \mathcal{H}^d(\varphi(W)) + h(r_1)r_1^d + 2\varepsilon \tag{20.38}$$

and our next step is to estimate $\mathcal{H}^d(\varphi(W))$. Recall from (20.25) and (20.21) that $\varphi = \psi_{1+t_0} \circ \varphi_1$, so

$$\varphi(W) = \psi_{1+t_0}(\varphi_1(W)). \tag{20.39}$$

We first consider $\psi_{1+t_0}(\varphi_1(W) \setminus S)$. Let $x \in W$ be such that $\varphi_1(x)$ lies outside of $S$; then $\varphi_1(x) = x$, because otherwise $x \in W_1$ and $\varphi_1(x) \in \varphi_1(W_1) \subset \widehat{W} \subset S$ by (20.31) and (20.18). In addition, $x \in S'$ because otherwise $\chi(x) = 0$ by (20.20) and $\varphi(x) = \psi_{1+t_0}(x) = x$ by (20.21); this is impossible because $x \in W$. So $x \in E \cap S' \setminus W_1$, and even $x \in E \cap S' \setminus S$ because $\varphi_1(x) = x$ and we assumed that $\varphi_1(x)$ lies outside of $S$. Hence $\psi_{1+t_0}(\varphi(x)) = \psi_{1+t_0}(x) \in \psi_{1+t_0}(E \cap S' \setminus S)$. We just checked that $\psi_{1+t_0}(\varphi_1(W) \setminus S) \subset \psi_{1+t_0}(E \cap S' \setminus S)$, and so

$$\mathcal{H}^d(\psi_{1+t_0}(\varphi_1(W) \setminus S)) \leq \mathcal{H}^d(\psi_{1+t_0}(E \cap S' \setminus S)) \leq 2^d \mathcal{H}^d(E \cap S' \setminus S) \leq 2^d \varepsilon \tag{20.40}$$

because $\psi_{1+t_0}$ is 2-Lipschitz (see above (20.22)), and by (20.19).

We are left with $\psi_{1+t_0}(\varphi_1(W) \cap S)$. By (20.34), the monotone intersection of the sets $\varphi_1(E) \cap S \cap Z_\tau$, when $\tau$ tends to 0, is contained in $\mathcal{S}_{d-1}$. Since $\mathcal{H}^d(\varphi_1(E) \cap S) < +\infty$, we get that

$$\mathcal{H}^d(\varphi_1(E) \cap S \cap Z_\tau) \leq \varepsilon \tag{20.41}$$

if $\tau$ is chosen small enough (depending on $\varphi_1$). And then

$$\mathcal{H}^d(\psi_{1+t_0}(\varphi_1(W) \cap S \cap Z_\tau)) \leq 2^d \mathcal{H}^d(\varphi_1(W) \cap S \cap Z_\tau) \leq 2^d \varepsilon \tag{20.42}$$

because $W \subset E$ and $\psi_{1+t_0}$ is 2-Lipschitz. We are now left with $\psi_{1+t_0}(\varphi_1(W) \cap S \cap A_\tau)$. Write

$$\varphi_1(W) \cap S \cap A_\tau = \bigcup_{F \in \mathcal{F}} G_F, \text{ with } G_F = \varphi_1(W) \cap S \cap F_\tau, \tag{20.43}$$

and observe that for $y \in S \cap F_\tau$, $\chi(y) = 1$ by (20.20), and

$$\psi_{1+t_0}(y) = y + t_0 v(y) = y + t_0 v_F \tag{20.44}$$

by (20.21) and (20.17). Hence

$$\psi_{1+t_0}(\varphi_1(W) \cap S \cap A_\tau) = \bigcup_{F \in \mathcal{F}} \psi_{1+t_0}(G_F) = \bigcup_{F \in \mathcal{F}} [G_F + t_0 v_F] \tag{20.45}$$

by (20.43), and

$$\begin{aligned} \mathcal{H}^d(\psi_{1+t_0}(\varphi_1(W) \cap S \cap A_\tau)) &\leq \sum_{F \in \mathcal{F}} \mathcal{H}^d(G_F + t_0 v_F) \\ &= \sum_{F \in \mathcal{F}} \mathcal{H}^d(G_F) \leq \mathcal{H}^d(\varphi_1(W))) \end{aligned} \tag{20.46}$$

because the $G_F$ are disjoint (by (20.15)) and contained in $\varphi_1(W)$. Altogether,

$$\mathcal{H}^d(\varphi(W)) = \mathcal{H}^d(\psi_{1+t_0}(\varphi_1(W))) \leq \mathcal{H}^d(\varphi_1(W)) + 2^{d+1}\varepsilon \tag{20.47}$$

by (20.39), (20.40), (20.42), and (20.46). Also recall that if $x \in W \setminus S$, then $\varphi_1(x) = x$ (because (20.18) says that $W_1 \subset \widehat{W} \subset S$; also see the definition of $W_1$ below (20.4)) and $x \in S'$ (because otherwise $\varphi(x) = \psi_{1+t_0} \circ \varphi_1(x) = \psi_{1+t_0}(x) = x$ by (20.21) and because $\chi(x) = 0$ by (20.20)); hence

$$\begin{aligned} \mathcal{H}^d(\varphi_1(W)) &\leq \mathcal{H}^d(\varphi_1(W \cap S)) + \mathcal{H}^d(\varphi_1(W \setminus S)) \\ &\leq \mathcal{H}^d(\varphi_1(E \cap S)) + \mathcal{H}^d(W \setminus S) \\ &\leq \mathcal{H}^d(\varphi_1(E \cap S)) + \mathcal{H}^d(E \cap S' \setminus S) \\ &\leq \mathcal{H}^d(\varphi_1(E \cap S)) + \varepsilon \end{aligned} \tag{20.48}$$

because we just saw that $\varphi_1(x) = x$ on $W \setminus S$, then because $W \subset E \cap S'$ (see the definition (20.25) and recall that on $U \setminus S'$, $\varphi_1(x) = x$ by (20.18) and (20.31) and hence $\varphi_{1+t_0}(x) = x$ by (20.21) and (20.20)), and finally by (20.19). Hence

$$\mathcal{H}^d(E \cap S) \leq \mathcal{H}^d(\varphi(W)) + h(r_1)r_1^d + 2\varepsilon \leq \mathcal{H}^d(\varphi_1(E \cap S)) + h(r_1)r_1^d + (2^{d+1} + 3)\varepsilon \tag{20.49}$$

by (20.38), (20.47), and (20.48).

Recall that $\varphi_1(x) = x$ for $x \in E \setminus S$ and $\varphi_1(E \cap S) \subset S$ (because $\widehat{W} \subset S$); then $E$ and $\varphi_1(E)$ coincide out of $S$, and so

$$E \setminus \varphi_1(E) = S \cap E \setminus \varphi_1(E) \text{ and } \varphi_1(E) \setminus E = S \cap \varphi_1(E) \setminus E. \tag{20.50}$$

Since both sets have a finite measure and contain $E \cap \varphi_1(E) \cap S$, we get that

$$\begin{aligned} \mathcal{H}^d(E \setminus \varphi_1(E)) - \mathcal{H}^d(\varphi_1(E) \setminus E) &= \mathcal{H}^d(S \cap E \setminus \varphi_1(E)) - \mathcal{H}^d(S \cap \varphi_1(E) \setminus E) \\ &= \mathcal{H}^d(S \cap E) - \mathcal{H}^d(S \cap \varphi_1(E)) \end{aligned} \tag{20.51}$$

by subtracting $\mathcal{H}^d(E \cap \varphi_1(E) \cap S)$ from both terms. In addition, $S \cap \varphi_1(E) = \varphi_1(E \cap S)$ because $\varphi_1(E \cap S) \subset S$ and $\varphi_1(x) = x \notin S$ for $x \notin S$. Now (20.51) and (20.49) yield

$$\mathcal{H}^d(E \setminus \varphi_1(E)) - \mathcal{H}^d(\varphi_1(E) \setminus E) \leq h(r_1) r_1^d + (2^{d+1} + 3)\varepsilon. \tag{20.52}$$

Recall from the line below (20.26) that $r_1 = r_0 + \tau + t_0$, which is as close to $r_0$ as we want; since $h$ is continuous from the right, $h(r_1)$ is as close to $h(r_0)$ as we want. Also, $\varepsilon$ is as small as we want too, and since (20.52) holds with all these choices, we get (20.6). This completes our proof of Proposition 20.9 in the rigid case. □

**Proof of Proposition 20.9 in the general case.** The proof that we give below will use the same strategy as in the rigid case, but will be more complicated because we have a technical problem. When we modify $\varphi_1$, we move the points a little bit along the faces of our grid, and we do this because we want to preserve the boundary conditions (1.7). If we try to do this with our bilipschitz faces, this small translation along the faces may well multiply $\mathcal{H}^d(\varphi_1(E))$ by a factor of 2, even if our translation is very small, and this would of course be bad for our estimates. So we will have to find flatter parts of our faces where we can translate things without increasing the measure too much, and for this an almost-covering argument with disjoint small balls where $\varphi_1(E)$ looks nice will be helpful. The proof below also works in the rigid case, with some simplifications.

Let $E$ be $A$-almost minimal, and let the $\{\varphi_t\}$, $0 \leq t \leq 1$, satisfy (1.4)-(1.8) relative to a ball $B = \overline{B}(x_0, r_0)$ and be such that $\widehat{W} \subset\subset U$.

Observe that $E$ is rectifiable because it is quasiminimal. More precisely, choose $h > 0$ small, as in Theorem 5.16 with $M = 1$, and then use (20.1) to find $\delta > 0$ such that $h(r) \leq \delta$ for $0 < r \leq \delta$. Then $E \in QMAQ(U, M, \delta, h)$, with $M = 1$ (compare the definitions 20.2 and 2.3). Now Theorem 5.16 says that $E$ is rectifiable, as needed. Similarly, Propositions 4.1 and 4.74 say that $E^*$ (the core of $E$, defined in (3.2)), is locally Ahlfors-regular.

Let $\varepsilon > 0$ be small, and let $S$ be a compact set such that

$$\widehat{W} \subset \operatorname{int}(S) \subset S \subset U,\ S \subset B(x_0, r_0 + \varepsilon), \text{ and } \mathcal{H}^d(E \cap S \setminus \widehat{W}) < \varepsilon \tag{20.53}$$

(recall from (1.5), (1.6), and (2.2) that $\widehat{W} \subset \overline{B}(x_0, r_0)$). Denote by $\mu$ the restriction of $\mathcal{H}^d$ to the set $\varphi_1(E \cap S) = \varphi_1(E) \cap S$ (recall that $\varphi_1(E \cap \widehat{W}) \subset \widehat{W}$ and $\varphi_1(x) = x$ for $x \in E \setminus \widehat{W}$), and by $\nu$ the image by $\varphi_1$ of the restriction of $\mathcal{H}^d$ to $E \cap S$, defined by

$\nu(A) = \mathcal{H}^d(E \cap S \cap \varphi_1^{-1}(A))$ for $A \subset R^n$ (a Borel set). By (20.3), $\mathcal{H}^d(E \cap S) < +\infty$, hence $\mu$ and $\nu$ are finite measures (recall that $\varphi_1$ is Lipschitz).

We want to cover a substantial part of

$$G_0 = \varphi_1(E^* \cap \widehat{W}) \subset \widehat{W} \tag{20.54}$$

(as before, the inclusion comes from the fact that $\varphi_1(E \cap W_1) \subset \widehat{W}$; see (2.1) and (2.2)) by a collection of disjoint balls $B_j$, and then we will continue our mapping $\varphi_1$ by composing $\varphi_1$ by deformations defined on the $B_j$. Our first task is to eliminate various pieces of $G_0$. We defined $G_0$ in terms of the core $E^*$, because it costs nothing in terms of measure (recall from (3.29) or (8.26) on page 58 of [D4] that $\mathcal{H}^d(E \setminus E^*) = 0$) and $E^*$, being locally Ahlfors-regular, is a little easier to control. For instance let us check that

$$\liminf_{r \to 0} r^{-d} \nu(B(y,r)) > 0 \ \text{ for } y \in G_0. \tag{20.55}$$

Let $y \in G_0$ be given, pick $x \in E^* \cap \widehat{W}$ such that $\varphi_1(x) = y$, and for $r > 0$ small, set $\rho = (1 + |\varphi_1|_{lip})^{-1} r$; then $\varphi_1(B(x,\rho)) \subset B(y,r)$. By (20.53), $B(x,\rho) \subset S$ for $\rho$ small, so $\nu(B(y,r) \geq \mathcal{H}^d(E \cap B(x,\rho) \cap S) \geq C^{-1}\rho^d \geq C^{-1} r^d$, where we don't even want to know what $C$ depends on. This is enough for (20.55), which we just mention because it simplifies the discussion below.

First we remove the points where $\nu$ is much larger than $\mu$. Let $M > 1$ be very large, and set

$$Y_0 = \big\{ y \in G_0 \,;\, \limsup_{r \to 0} \frac{\nu(B(y,r))}{\mu(B(y,r))} \geq M \big\}. \tag{20.56}$$

We don't need to worry about the value of $0/0$ here, since $\nu(B(y,r)) > 0$ by (20.53). Also set

$$X_0 = E \cap \varphi_1^{-1}(Y_0) = E \cap \widehat{W} \cap \varphi_1^{-1}(Y_0) \tag{20.57}$$

(recall that $Y_0 \subset G_0 \subset \widehat{W}$ and $\varphi_1(x) = x \notin \widehat{W}$ when $x \in E \setminus \widehat{W}$). By (2) in Lemma 2.13 of [Ma],

$$\nu(A) \geq M\mu(A) \ \text{ for every Borel set } A \subset Y_0, \tag{20.58}$$

and in particular

$$\mathcal{H}^d(X_0) = \nu(Y_0) \geq M\mu(Y_0) \tag{20.59}$$

because $X_0 \subset S$ (since $\widehat{W} \subset S$ by (20.53)). We will not need to worry too much about $Y_0$ because

$$\mathcal{H}^d(Y_0) \leq M^{-1}\mathcal{H}^d(X_0) \leq M^{-1}\mathcal{H}^d(E \cap S) \leq M^{-1}(1 + \mathcal{H}^d(E \cap \widehat{W})), \tag{20.60}$$

by (20.53). On the other hand, (1) in Lemma 2.13 of [Ma], says that

$$\nu(A) \leq M\mu(A) \quad \text{for every Borel set } A \subset G_0 \setminus Y_0; \tag{20.61}$$

we don't intend to use the huge constant $M$, but merely the fact that $\nu$ is absolutely continuous with respect to $\mu$ on the set $G_1 = G_0 \setminus Y_0$. This will help, because we can now remove some small sets in $G_1$ without fear of losing a large mass in the source space.

Denote by $G_2$ the set of points $y \in G_1$ with the following good properties. First,

$$\varphi_1(E^*) \text{ has an approximate tangent plane } P(y) \text{ at } y \tag{20.62}$$

and

$$\lim_{r \to 0} r^{-d} \mathcal{H}^d(\varphi_1(E^*) \cap B(y,r)) = \omega_d, \tag{20.63}$$

where as usual $\omega_d$ is the $\mathcal{H}^d$-measure of the unit ball in $\mathbb{R}^d$. These properties are true for $H^d$-almost every $y \in G_0$, because $E^*$ is rectifiable (with finite measure in a neighborhood of $S$) and $\varphi_1$ is Lipschitz. We can also replace $E^*$ with $E$ in (20.62) and (20.63), since none of these properties are sensitive to adding a set of vanishing $H^d$-measure.

Next, if we are in the Lipschitz case, set $\widetilde{\varphi}_1(x) = \psi(\lambda\varphi_1(x))$ for $x \in U$ (and where $\lambda$ and $\psi$ are as in Definition 2.7), and also set $\widetilde{y} = \psi(\lambda y)$; we require that

$$\widetilde{\varphi}_1(E^*) \text{ has an approximate tangent plane } \widetilde{P}(y) \text{ at } \widetilde{y}. \tag{20.64}$$

In addition, denote by $F(y)$ the smallest face of our (twisted) net that contains $y$ and by $\dim(F(y))$ its dimension. We demand that

$$\dim(F(y)) \geq d, \tag{20.65}$$

and also that if $\widetilde{W}(y)$ denotes the smallest affine space that contains $\widetilde{F}(y) = \psi(\lambda F(y))$,

$$\widetilde{P}(y) \subset \widetilde{W}(y). \tag{20.66}$$

Finally we exclude the exceptional set $Z$ of (19.35). In other words, we demand that if $y$ lies in some boundary piece $L_i$, $0 \leq i \leq j_{max}$, but does not lie in its $n$-dimensional interior (see the definition of $L'_i$ and $\widehat{L}'$ near (19.34)), then

$$\limsup_{r \to 0} A_r(y) \leq \omega_d \tag{20.67}$$

where $A_r(y)$ is given by (19.33). Let us check that all these properties are true for $\mathcal{H}^d$-almost every $y \in G_1$, i.e., that

$$\mathcal{H}^d(G_1 \setminus G_2) = 0. \tag{20.68}$$

We know that (20.62) and (20.63) hold almost everywhere, and so does (20.64), because $\widetilde{\varphi}_1(E^*)$ is rectifiable and (for the invariance of negligible sets) $\psi$ is bilipschitz. For (20.65) we remove a set of dimension $d-1$, and (20.67) holds almost everywhere because we assumed (19.36) or the stronger (10.7). The fact that (10.7) implies (19.36) is proved below (19.36). Finally, let us check that we can arrange (20.66) almost everywhere. Let $F$ be any face, and let us say how we can get (20.66) for almost every $y \in G_1$ such that $F(y) = y$. Set $A = F \cap G_1$ and $\widetilde{A} = \psi(\lambda A)$. This last set is rectifiable (it is also a subset of $\widetilde{\varphi}_1(E^*)$), so for almost every $y \in A$, we can find an approximate tangent $d$-plane to $\widetilde{A}$ at $\widetilde{y} = \psi(\lambda y)$. Call it $Q(y)$, and observe that by definitions it is contained in the smallest affine subspace that contains $\psi(\lambda F)$. By the almost-everywhere uniqueness of the approximate tangent plane to $\widetilde{\varphi}_1(E^*)$, we just have to show that for almost every $y \in A$, $Q(y)$ is also an approximate tangent plane to $\widetilde{\varphi}_1(E^*)$ (and not merely $\widetilde{A}$) at $\widetilde{y}$. But by Theorem 6.2 on page 89 of [Ma], $\lim_{r\to 0} r^{-d}\mathcal{H}^d(\widetilde{\varphi}_1(E^*) \cap B(\widetilde{y}, r) \setminus \widetilde{A}) = 0$ for $\mathcal{H}^d$-almost every $\widetilde{y} \in \widetilde{A}$. For such $\widetilde{y}$, any approximate tangent plane to $\widetilde{\varphi}_1(E^*)$ at $\widetilde{y}$ also works for $\widetilde{\varphi}_1(E^*)$, as needed. This completes the proof of (20.68).

Let us now select, for each point $y \in G_2$, a small radius $r(y)$ with the following good properties. First,

$$r(y) \leq \frac{1}{4\Lambda^2}\,\mathrm{dist}(y, U \setminus S); \tag{20.69}$$

this true for $r(y)$ small enough, because (20.54) and (20.53) say that $y \in G_0 \subset \widehat{W} \subset \mathrm{int}(S)$. We also choose $r(y)$ so small that

$$r(y) \leq \frac{1}{4\Lambda^4}\,\mathrm{dist}(y, \partial F(y)) \tag{20.70}$$

(where $\mathrm{dist}(y, \partial F(y))$, the distance to the boundary of $F(y)$, is positive because $y$ lies in the interior of $F(y)$)),

$$\omega_d - \varepsilon \leq r^{-d}\mathcal{H}^d(\varphi_1(E^*) \cap B(y, r)) \leq \omega_d + \varepsilon \quad \text{for } 0 < r \leq r(y) \tag{20.71}$$

(we use the same small $\varepsilon > 0$ as before to save notation). We shall not need a uniform variant for the existence of a tangent plane to $\varphi_1(E)$, because in the delicate part of the argument, we shall work with $\widetilde{\varphi}_1(E)$. So we use (20.64) to require that for $0 < r \leq r(y)$,

$$\mathcal{H}^d\big(\big\{z \in \widetilde{\varphi}_1(E) \cap B(\widetilde{y}, \lambda\Lambda r)\,;\, \mathrm{dist}(z, \widetilde{P}(y)) \geq \varepsilon\lambda r\big\}\big) \leq \varepsilon\lambda^d r^d. \tag{20.72}$$

Finally we require a uniform version of (20.67), i.e., that if $y$ lies in some boundary piece $L_i$, but not in the (true) interior of $L_i$,

$$A_r(y) \leq \omega_d + \varepsilon \quad \text{for } 0 \leq r \leq 2r(y). \tag{20.73}$$

This completes our definition of $r(y)$ when $y \in G_2$.

We now use a consequence of Besicovitch's covering lemma. Consider, for each $y \in G_2$, the balls $\overline{B}(y, r)$, $0 < r < \min(\varepsilon, r(y))$ (we are again using the same $\varepsilon$ in a different role), and for which $\mu(\partial B(y,r)) = 0$ (almost every $r$ satisfies this, since the $\partial B(y,r)$ are disjoint). By Theorem 2.8 in [Ma] (applied to all these balls) we get a collection of disjoint balls $B_j = B(y_j, r_j)$, with the following properties:

$$0 < r_j < \min(\varepsilon, r(y_j)) \tag{20.74}$$

and $\mu(\partial B_j) = 0$ for all $j$, and

$$\mu(G_2 \setminus \bigcup_j B_j) = \mu(G_2 \setminus \bigcup_j \overline{B}_j) = 0. \tag{20.75}$$

Now we can define a continuation for our family $\{\varphi_t\}$, with which we shall eventually apply the definition of $A$-minimality. We want to define $\varphi_t$ for $1 \leq t \leq 2$, by

$$\varphi_t(x) = g_t(\varphi_1(x)) \ \text{ for } x \in E \text{ and } 1 \leq t \leq 2, \tag{20.76}$$

where the functions $g_t : U \to U$ are such that

$$g_t(y) = y \ \text{ for } y \in U \setminus \bigcup_j B_j \text{ and for } t = 1 \tag{20.77}$$

and will now be defined separately on the $B_j$. We shall use cut-off functions $\xi_j$, defined by $\xi_j(y) = 0$ for $y \in U \setminus B_j$, and

$$\xi_j(y) = \min\left\{1, (\tau r_j)^{-1} \operatorname{dist}(y, \partial B_j)\right\} \ \text{ for } y \in B_j. \tag{20.78}$$

Here $\tau > 0$ is another small constant that will be chosen soon.

We start with the simpler case when $y_j$ does not lie in any $L_i' = L_i \setminus \operatorname{int}(L_i)$, where $\operatorname{int}(L_i)$ is the $n$-dimensional interior of $L_i$. In this case we can pick any unit vector $v_j$, and set

$$g_t(y) = y + (t-1)\xi_j(y)\eta r_j v_j \tag{20.79}$$

for $y \in B_j$ and $1 \leq t \leq 2$, where $\eta > 0$ is a minuscule constant, to be chosen later.

When the rigid assumption holds, we define the $g_t$ by the same formula (20.79), but we make sure to choose $v_j$ in the vector space parallel to the smallest face $F(y_j)$ that contains $y_j$. This precaution will only help if $y_j$ lies in some $L_i'$.

In the remaining case when the Lipschitz assumption holds and $y_j$ lies in some $L_i'$, we need to be more careful and use the mapping $\psi$ of Definition 2.7. Still denote by $\widetilde{P}(y_j)$ the approximate tangent plane to $\widetilde{\varphi}_1(E^*)$ at $\widetilde{y}_j = \psi(\lambda y_j)$, as in (20.64), and denote by $\widetilde{\pi}_j$ the orthogonal projection onto $\widetilde{P}(y_j)$. Also choose a unit vector $\widetilde{v}_j$ in the vector space parallel to $\widetilde{P}(y_j)$, and then set

$$\widetilde{g}_t(y) = \psi(\lambda y) + (t-1)\xi_j(y)[\widetilde{\pi}_j(\psi(\lambda y)) - \psi(\lambda y)] + (t-1)\xi_j(y)\eta\lambda r_j \widetilde{v}_j \tag{20.80}$$

for $y \in B_j$ and $1 \leq t \leq 2$. Notice that

$$\begin{aligned}
|\widetilde{g}_t(y) - \psi(\lambda y_j)| &\leq |\psi(\lambda y) - \psi(\lambda y_j)| + (t-1)|\widetilde{\pi}_j(\psi(\lambda y)) - \psi(\lambda y)| + (t-1)\eta\lambda r_j \\
&\leq 2|\psi(\lambda y) - \psi(\lambda y_j)| + \eta\lambda r_j \leq 2\lambda\Lambda r_j + \eta\lambda r_j \leq 3\lambda\Lambda r_j
\end{aligned} \tag{20.81}$$

because $\psi(\lambda y_j)$ lies in $\widetilde{P}(y_j)$ and if $\eta$ is small enough. Recall from (20.69) and (20.74) that

$$\mathrm{dist}(y_j, U \setminus S) \geq 4\Lambda^2 r(y_j) \geq 4\Lambda^2 r_j \tag{20.82}$$

Since $\psi$ maps $\lambda U$ to $B(0,1)$, this implies that $\psi(\lambda y_j)$ is at distance at least $4\lambda\Lambda r_j$ from $B(0,1) \setminus \psi(\lambda S)$, hence also from $\mathbb{R}^n \setminus \psi(\lambda S)$ (recall that $y_j \subset S$, which is compactly contained in $U$). Now (20.81) yields $\mathrm{dist}(\widetilde{g}_t(y), \mathbb{R}^n \setminus \psi(\lambda S)) \geq \lambda\Lambda r_j$. In particular $\widetilde{g}_t(y) \in \psi(\lambda S)$, so we can define

$$g_t(y) = \lambda^{-1}\psi^{-1}(\widetilde{g}_t(y)) \in S. \tag{20.83}$$

Observe that by (20.81)

$$|g_t(y) - y_j| \leq \lambda^{-1}\Lambda|\widetilde{g}_t(y) - \psi(\lambda y_j)| \leq 3\Lambda^2 r_j. \tag{20.84}$$

This completes our definition of the $g_t$ and the $\varphi_t$. Our next task is to show that the $\varphi_{2t}$, $0 \leq t \leq 1$, define an acceptable competitor for $E$. There is no problem with (1.4) and (1.8); our mappings $\varphi_t(x)$, $t \geq 1$ are clearly continuous in $x$ and $t$, and Lipschitz in $x$ (notice in particular that all our definitions yield $g_t(y) = y$ on the $\partial B_j$). Also,

$$\varphi_t(x) = x \quad \text{for } 0 \leq t \leq 2 \text{ when } x \in E \setminus S, \tag{20.85}$$

just because $S$ contains $\widehat{W}$ and the $B_j$ (see (20.53), (20.69), and (20.77)). In addition, we claim that

$$\varphi_t(x) \in S \text{ when } x \in E \cap S. \tag{20.86}$$

When $t \leq 1$, this comes from the fact that $\varphi_t(W_t) \subset \widehat{W} \subset S$. When $t \geq 1$, we know that $\varphi_1(x) \in S$, and then we just need to use (20.76) and (20.77) or (20.83). This proves (20.86), and (1.5) and (1.6), relative to $B(x_0, r_0 + \varepsilon)$, follow (by (20.53)). We also get the analogue of (2.4), where we use the compact set $S \subset U$ in lieu of $\widehat{W}$.

Next we check the boundary condition (1.7). We do this under the Lipschitz assumption; the rigid case is just simpler. Let $i \leq j_{max}$ and $x \in E \cap L_i \cap \overline{B}(x_0, r_0 + \varepsilon)$ be given; recall that we want to check that $\varphi_t(x) \in L_i$ for all $t$. We already know this for $t \leq 1$, by (1.7) for the initial $\varphi_t$ (and (1.5) if $x \notin B(x_0, r_0)$), so we can assume that $t > 1$. Set $y = \varphi_1(x)$; by (20.76) $\varphi_t(x) = g_t(y)$, and we can assume that $y \in B_j$ for some $j$, because otherwise $\varphi_t(x) = g_t(y) = y = \varphi_1(x) \in L_i$ by (20.76). Let us record that (1.7) will follow as soon as we show that

$$g_t(y) \in L_i \quad \text{for } 1 < t \leq 2 \tag{20.87}$$

when $i$, $j$, $x \in E \cap L_i$, and $y = \varphi_1(x) \in B_j$ are as above.

By (1.7) for $\varphi_1$, $y = \varphi_1(x)$ lies in $L_i$. Let $F$ be a face of $L_i$ that contains $y$, and let us check that

$$F(y_j) \subset F. \tag{20.88}$$

Set $\widetilde{F} = \psi(\lambda F)$ and $\widetilde{F}_j = \psi(\lambda F(y_j))$, and observe that $\widetilde{F}_j$ is the smallest rigid face that contains $\widetilde{y}_j = \psi(\lambda y_j)$. Suppose that (20.88) fails; then $\widetilde{F}_j$ is not contained in $\widetilde{F}$. If in addition $\widetilde{F}_j$ is not reduced to the point $\widetilde{y}_j$, (3.8) yields

$$\operatorname{dist}(\widetilde{y}_j, \widetilde{F}) \geq \operatorname{dist}(\widetilde{y}_j, \partial(\widetilde{F}_j)) \geq \lambda\Lambda^{-1} \operatorname{dist}(y_j, \partial F(y_j)) \geq 4\lambda\Lambda^3 r(y_j) \geq 4\lambda\Lambda r_j \tag{20.89}$$

by (20.70) and (20.74). If instead $\widetilde{F}_j$ is reduced to the point $\widetilde{y}_j$, then $\widetilde{F}$ is a rigid face that does not contain the vertex $\widetilde{y}_j$, hence $\operatorname{dist}(\widetilde{y}_j, \widetilde{F}) \geq 2^{-m} \geq 4\lambda\Lambda r_j$, by (20.74) and if $\varepsilon$ is small enough; so the conclusion of (20.89) holds in both cases. But $\psi(\lambda y) \in \widetilde{F}$ because $y \in F$, so

$$\operatorname{dist}(\widetilde{y}_j, \widetilde{F}) \leq |\psi(\lambda y_j) - \psi(\lambda y)| \leq \lambda\Lambda|y - y_j| \leq \lambda\Lambda r_j, \tag{20.90}$$

a contradiction which proves (20.88).

Recall that we want to check (20.87). We start with the most interesting case when $y_j \in L_i'$. Then $g_t(y)$ was defined by (20.80) and (20.83), and (by (20.83)) it is enough to check that $\widetilde{g}_t(y) \in \widetilde{F}$.

Recall that $\widetilde{\pi}_j$ is the orthogonal projection onto the approximate tangent plane $\widetilde{P}(y_j)$ of (20.64), which itself is contained in the affine plane $\widetilde{W}_j$ spanned by $\widetilde{F}_j$, by (20.66). Denote by $\widetilde{W}$ the affine span of $F$; by (20.88), $\widetilde{F}_j \subset \widetilde{F}$ and hence $\widetilde{W}_j \subset \widetilde{W}$. Thus the points $\widetilde{y} = \psi(\lambda y)$, $\widetilde{\pi}_j(\widetilde{y})$, and even $\widetilde{w} = \widetilde{y} + (t-1)\xi_j(y)[\widetilde{\pi}_j(\widetilde{y}) - \widetilde{y}]$ all lie in $\widetilde{W}$. Observe that by (20.80), $\widetilde{g}_t(y) = \widetilde{w} + (t-1)\xi_j(y)\eta\lambda r_j \widetilde{v}_j$, and since we chose $\widetilde{v}_j$ in the vector space parallel to $\widetilde{P}(y_j) \subset \widetilde{W}$, we see that $\widetilde{g}_t(y) \subset \widetilde{W}$.

We want to show that $\widetilde{g}_t(y)$ even lies in $\widetilde{F}$. We start from the fact that $\widetilde{y}_j \in \widetilde{F}_j \subset \widetilde{F}$ (by definition of $\widetilde{F}_j = \psi(\lambda F(y_j))$ and by (20.88)), with

$$\operatorname{dist}(\widetilde{y}_j, \partial\widetilde{F}) \geq \operatorname{dist}(\widetilde{y}_j, \partial\widetilde{F}_j) \geq \lambda\Lambda^{-1} \operatorname{dist}(y_j, \partial F(y_j)) \geq 4\lambda\Lambda^3 r(y_j) \geq 4\lambda\Lambda r(y_j) \tag{20.91}$$

by (20.70) and (20.74) (that is, as in (20.89)). But $|\widetilde{g}_t(y) - \widetilde{y}_j| \leq 3\lambda\Lambda r_j$ by (20.81), so the line segment $[\widetilde{y}_j, \widetilde{g}_t(y)] \subset \widetilde{F}$ does not meet $\partial\widetilde{F}$, and $\widetilde{g}_t(y) \in \widetilde{F}$; (20.87) follows, because $F \subset L_i$ by definition, and this takes care of our first case.

We are left with the case when $y_j \notin L_i'$. Since $y_j \in F(y_j) \subset F \subset L_i$ by (20.88), this implies that $y_j$ lies in the interior of $L_i$. We want to show that in fact

$$\overline{B}(y_j, 4\Lambda^2 r_j) \subset L_i\,, \tag{20.92}$$

and for this we shall proceed as for (19.10). Denote by $\delta(L_i)$ the boundary of $L_i$, set $D = \operatorname{dist}(y_j, \delta(L_i)) > 0$, and pick $\xi \in \delta(L_i)$ such that $|\xi - y_j| = D$. Denote by $G$ the smallest face of our grid that contains $\xi$; since $\delta(L_i)$ is itself an union of faces, $G$ is contained in $\delta(L_i)$. Since $D > 0$, $G$ does not contain $y_j$, and even less $F(y_j)$.

First assume that $F(y_j)$ is not reduced to $\{y_j\}$; then (3.8) (applied to the rigid faces $\psi(\lambda G)$ and $\widetilde{F}_j$) yields

$$\begin{aligned} D = |y_j - \xi| &\geq \operatorname{dist}(y_j, G) \geq \lambda^{-1}\Lambda^{-1} \operatorname{dist}(\psi(\lambda y_j), \psi(\lambda G)) \\ &\geq \lambda^{-1}\Lambda^{-1} \operatorname{dist}(\psi(\lambda y_j), \widetilde{F}_j) \geq \Lambda^{-2} \operatorname{dist}(y_j, \partial F(y_j)) \geq 4\Lambda^2 r(y_j) \geq 4\Lambda^2 r_j \end{aligned} \tag{20.93}$$

by (20.70) and (20.74). If instead $F(y_j) = \{y_j\}$, and since $D$ is the distance from the vertex $y_j$ to a face that does not contain it, we get that $D \geq \lambda^{-1}\Lambda^{-1}2^{-m} \geq 4\Lambda^2 r_j$, by (20.74) and if $\varepsilon$ is small enough. Thus $D \geq 4\Lambda^2 r_j$ in both cases, and (20.92) follows. In this case the fact that $\varphi_t(x)$ lies in $L_i$ is trivial because $\varphi_t(x) = g_t(y) \in \overline{B}(y_j, 4\Lambda^2 r_j)$, by (20.84).

This completes our proof of (1.7), and the series of verifications for the extended family $\{\varphi_t\}$, and now we can use the $A$-minimality of $E$. This yields

$$\mathcal{H}^d(W_2) \leq \mathcal{H}^d(\varphi_2(W_2)) + h(r_0 + \varepsilon)(r_0 + \varepsilon)^d, \tag{20.94}$$

by (20.5) and where $W_2 = \big\{x \in E\,;\, \varphi_2(x) \neq x\big\}$.

Recall that $W_2 \cup \varphi_2(W_2) \subset S$, by (20.85) and (20.86). We start with an estimate of $\mathcal{H}^d(\varphi_2(W_2))$. Set $A = S \cap \varphi_1(E) \setminus \bigcup_j [B_j \cap \varphi_1(E)]$. Then

$$\begin{aligned} \mathcal{H}^d(\varphi_2(W_2)) &\leq \mathcal{H}^d(S \cap \varphi_2(E)) = \mathcal{H}^d(g_2(S \cap \varphi_1(E))) \\ &\leq \mathcal{H}^d(A) + \sum_j \mathcal{H}^d(g_2(B_j \cap \varphi_1(E))) \end{aligned} \tag{20.95}$$

by (20.76), because $S \cap \varphi_1(E) \subset\subset A \cup \big(\bigcup_j [B_j \cap \varphi_1(E)]\big)$, and because $g_2(y) = y$ on $A$ (by (20.77)). This will be compared to the fact that

$$\mathcal{H}^d(S \cap \varphi_1(E)) = \mathcal{H}^d(A) + \sum_j \mathcal{H}^d(B_j \cap \varphi_1(E)) \tag{20.96}$$

by definition of $A$ and because the $B_j$ are disjoint. Next we estimate the sum in (20.96). There are two types of indices $j$; we start with the simple case when $g_t$ was defined by (20.79). That is, $g_2(y) = y + \xi_j(y)\eta r_j v_j$ for $y \in B_j$. Write $B_j = B_{j,int} \cup B_{j,ext}$, where $B_{j,int} = \big\{y \in B_j; \operatorname{dist}(y, \partial B_j) \geq \tau r_j\big\}$ and $B_{j,ext} = B_j \setminus B_{j,int}$. On $B_{j,int}$, (20.78) yields $\xi_j(y) = 1$ and $g_2(y) = y + \eta r_j v_j$, and hence

$$\mathcal{H}^d(g_2(B_{j,int} \cap \varphi_1(E))) = \mathcal{H}^d(B_{j,int} \cap \varphi_1(E)) \leq \mathcal{H}^d(B_j \cap \varphi_1(E)). \tag{20.97}$$

On $B_{j,ext}$, although $\xi_j$ is only $(\tau r_j)^{-1}$-Lipschitz, we can choose $\eta$ so small that $g_2$ is 2-Lipschitz on $B_{j,ext}$, and we get that

$$\mathcal{H}^d(g_2(B_{j,ext} \cap \varphi_1(E))) \leq 2^d \mathcal{H}^d(B_{j,ext} \cap \varphi_1(E)) \tag{20.98}$$

But two applications of (20.71) yield

$$
\begin{aligned}
\mathcal{H}^d(B_{j,int} \cap \varphi_1(E)) = \mathcal{H}^d(B_{j,int} \cap \varphi_1(E^*)) &\geq [(1-\tau)r_j]^d(\omega_d - \varepsilon) \\
&\geq (1-\tau)^d \, \frac{\omega_d - \varepsilon}{\omega_d + \varepsilon} \, \mathcal{H}^d(B_j \cap \varphi_1(E))
\end{aligned}
\tag{20.99}
$$

because $\mathcal{H}^d(E \setminus E^*) = 0$. Thus, if $\varepsilon$ is small enough, depending on $\tau$, we get that

$$
\mathcal{H}^d(B_{j,ext} \cap \varphi_1(E)) = \mathcal{H}^d(B_j \cap \varphi_1(E)) - \mathcal{H}^d(B_{j,int} \cap \varphi_1(E)) \leq C\tau \mathcal{H}^d(B_j \cap \varphi_1(E))
\tag{20.100}
$$

and, by (20.97) and (20.98),

$$
\mathcal{H}^d(g_2(B_j \cap \varphi_1(E))) \leq (1 + C\tau)\mathcal{H}^d(B_j \cap \varphi_1(E)).
\tag{20.101}
$$

Now we consider the more complicated case when we used (20.80)-(20.83) to define $g_2$. By (20.80),

$$
\widetilde{g}_2(y) = \psi(\lambda y) + \xi_j(y)[\widetilde{\pi}_j(\psi(\lambda y)) - \psi(\lambda y)] + \xi_j(y)\eta\lambda r_j \widetilde{v}_j
\tag{20.102}
$$

for $y \in B_j$. We start with the good set

$$
G(j) = \big\{ y \in B_{j,int} \cap \varphi_1(E) \, ; \, \mathrm{dist}(\psi(\lambda y), \widetilde{P}(y_j)) \leq \varepsilon \lambda r_j \big\},
\tag{20.103}
$$

where $\widetilde{P}(y_j)$ is the tangent plane that shows up in (20.64) and (20.72), for instance. If $y \in G(j)$, (20.78) yields $\xi_j(y) = 1$, then (20.102) says that $\widetilde{g}_2(y) = \widetilde{\pi}_j(\psi(\lambda y)) + \eta\lambda r_j \widetilde{v}_j$, hence

$$
|\widetilde{g}_2(y) - \psi(\lambda y)| \leq |\widetilde{\pi}_j(\psi(\lambda y)) - \psi(\lambda y)| + \eta\lambda r_j \leq (\eta + \varepsilon)\lambda r_j
\tag{20.104}
$$

(by (20.103) and because $\widetilde{\pi}_j$ is the orthogonal projection on $\widetilde{P}(y_j)$). Then $|g_2(y) - y| \leq (\eta + \varepsilon)\Lambda r_j$ by (20.83), and hence $g_2(y) \in B(y_j, (1 + \eta\Lambda + \varepsilon\Lambda) r_j)$. Also, $\widetilde{\pi}_j(\psi(\lambda y)) \in \widetilde{P}(y_j)$, and since we chose $\widetilde{v}_j$ in the vector space parallel to $\widetilde{P}(y_j)$ (see above (20.80)), we see that $\widetilde{g}_2(y) \in \widetilde{P}(y_j)$. Thus

$$
g_2(y) \in \lambda^{-1}\psi^{-1}(\widetilde{P}(y_j)) \cap B(y_j, (1 + \eta\Lambda + \varepsilon\Lambda)\, r_j).
\tag{20.105}
$$

by (20.83). By definition of $\widetilde{P}(y_j)$ (see (20.64)), $\psi(\lambda y_j) \in \widetilde{P}(y_j)$. By (20.66), $\widetilde{P}(y_j) \subset \widetilde{W}(y_j)$, the affine span of $\widetilde{F}_j = \psi(\lambda F(y_j))$. We now deduce from (20.105) and the definition (19.33) of $A_r(y_j)$ that

$$
\begin{aligned}
\mathcal{H}^d(g_2(G(j))) &\leq \mathcal{H}^d(\lambda^{-1}\psi^{-1}(\widetilde{P}(y_j)) \cap B(y_j, (1 + \eta\Lambda + \varepsilon\Lambda) r_j)) \\
&\leq (1 + \eta\Lambda + \varepsilon\Lambda)^d r_j^d A_{(1+\eta\Lambda+\varepsilon\Lambda) r_j}(y_j) \\
&\leq (\omega_d + \varepsilon)(1 + \eta\Lambda + \varepsilon\Lambda)^d r_j^d \\
&\leq (1 + C\varepsilon\Lambda + C\eta\Lambda)\, \mathcal{H}^d(B_j \cap \varphi_1(E))
\end{aligned}
\tag{20.106}
$$

by (20.73) and (20.71). Next we consider the less good set

$$G'(j) = \big\{y \in B_{j,ext} \cap \varphi_1(E)\,;\, \mathrm{dist}(\psi(\lambda y), \widetilde{P}(y_j)) \leq \varepsilon\lambda r_j\big\}. \tag{20.107}$$

We claim that $\widetilde{g}_2$ is $C\lambda\Lambda$-Lipschitz on $G'(j)$. The first term in the definition (20.102) is $\psi(\lambda y)$, which is $\lambda\Lambda$-Lipschitz; the third one, $\xi_j(y)\eta\lambda r_j\widetilde{v}_j$, is $C\tau^{-1}\eta\lambda$-Lipschitz, which is much better if $\eta$ is small enough. Notice that $|\widetilde{\pi}_j(\psi(\lambda y)) - \psi(\lambda y)| \leq \varepsilon\lambda r_j$ on $G'(j)$, hence the second term $\xi_j(y)[\widetilde{\pi}_j(\psi(\lambda y)) - \psi(\lambda y)]$ is $C\tau^{-1}\varepsilon\lambda + C\Lambda\lambda$-Lipschitz, our claim follows, and $g_2$ is $C\Lambda^2$-Lipschitz on $G'(j)$. Then

$$\begin{aligned}\mathcal{H}^d(g_2(G'(j))) &\leq C\Lambda^{2d}\mathcal{H}^d(G'(j)) \leq C\Lambda^{2d}\mathcal{H}^d(B_{j,ext} \cap \varphi_1(E))\\ &\leq C\tau\mathcal{H}^d(B_j \cap \varphi_1(E))\end{aligned} \tag{20.108}$$

by (20.100), and where we no longer write the dependence on $\Lambda$ in the last line. We are left with

$$G''(j) = \big\{y \in B_j \cap \varphi_1(E)\,;\, \mathrm{dist}(\psi(\lambda y), \widetilde{P}(y_j)) > \varepsilon\lambda r_j\big\}. \tag{20.109}$$

On this set (20.102) only yields that $\widetilde{g}_2$ is $C\lambda\Lambda\tau^{-1}$-Lipschitz, hence by (20.83) $g_2$ is $C\Lambda^2\tau^{-1}$-Lipschitz. Fortunately $G''(j)$ is small. Indeed if $y \in G''(j)$, then $\psi(\lambda y)$ lies in the bad set of (20.72), whose measure is at most $\varepsilon\lambda^d r_j^d$ hence $\mathcal{H}^d(G''(j)) \leq \varepsilon\Lambda^d r^d$ and (dropping soon the dependence on $\Lambda$ and by (20.71) again),

$$\mathcal{H}^d(g_2(G''(j))) \leq C\Lambda^{2d}\tau^{-d}\mathcal{H}^d(G''(j)) \leq C\tau^{-d}\varepsilon r_j^d \leq C\tau^{-d}\varepsilon\mathcal{H}^d(B_j \cap \varphi_1(E)). \tag{20.110}$$

We add (20.106), (20.108), and (20.110) and get that

$$\mathcal{H}^d(g_2(B_j \cap \varphi_1(E))) \leq (1 + C\eta + C\tau + C\varepsilon\tau^{-d})\mathcal{H}^d(B_j \cap \varphi_1(E)). \tag{20.111}$$

We had a slightly better estimate (20.101) in the first case, so (20.111) holds in all cases, and when we compare (20.95) to (20.96), we now get that

$$\begin{aligned}\mathcal{H}^d(\varphi_2(W_2)) &\leq \mathcal{H}^d(S \cap \varphi_1(E)) + C(\eta + \tau + \varepsilon\tau^{-d})\sum_i \mathcal{H}^d(B_j \cap \varphi_1(E))\\ &\leq \mathcal{H}^d(S \cap \varphi_1(E)) + C(\eta + \tau + \varepsilon\tau^{-d})\mathcal{H}^d(S \cap \varphi_1(E))\end{aligned} \tag{20.112}$$

(recall that the $B_j$ are disjoint, and (by (20.69) and (20.74)) contained in $S$). Now we want to check that

$$\mathcal{H}^d(E \cap S \setminus W_2) \leq C(M^{-1} + \tau), \tag{20.113}$$

where $C$ is allowed to depend on $\mathcal{H}^d(E \cap \widehat{W})$, and $M$ is as in (20.56) and (20.60).

Let $E \cap S \setminus W_2$ be given. Let us remove a few small sets. A first possibility is that $x \in W = \big\{x \in U\,;\, \varphi_1(x) \neq x\big\}$. Set $y = \varphi_1(x)$; then $y \neq x$ because $x \in W$. Since $\varphi_2(x) = x$ (because $x \notin W_2$) and $\varphi_2(x) = g_t(y)$ by (20.76) we get that $g_2(y) \neq y$, and even $|g_2(y) - y| = |y - x| = |\varphi_1(x) - x|$. Then $y$ lies in some $B_j$. If $g_2(y)$ was computed by (20.79), this implies that $|\varphi_1(x) - x| \leq \eta r_j$. Otherwise, (20.84) says that $|g_2(y) - y| \leq |g_2(y) - y_j| + r_j \leq 4\Lambda^2 r_j$. In both cases, $|\varphi_1(x) - x| \leq 4\Lambda^2 r_j \leq 4\Lambda^2\varepsilon$ by our precaution (10.74). If $\varepsilon$ is small enough, depending on $\tau$, we deduce from this that

$$\mathcal{H}^d(E \cap W \setminus W_2) \leq \mathcal{H}^d\big(\{x \in E \cap W\,;\, |\varphi_1(x) - x| \leq 4\Lambda^2\varepsilon\}\big) \leq \tau \tag{20.114}$$

because the monotone intersection, when $\varepsilon$ tends to 0, of the sets in (20.114) is empty, and all these sets are contained in $E \cap W$ for which $\mathcal{H}^d(E \cap W) < +\infty$. So we may restrict to $x \in E \cap S \setminus [W_2 \cap W]$. Since $\mathcal{H}^d(E \cap S \setminus \widehat{W}) \leq \varepsilon$ by (20.53), this set contributes little to (20.113), and we may assume that $x \in \widehat{W}$. Since $x \in E \setminus W$, we get that $\varphi_1(x) = x$, and so $x \in \varphi_1(E \cap \widehat{W})$. Thus (20.54) says that $x$ almost always lies in $G_0$. Next we take care of $Y_0$, which by (20.60) is such that

$$\mathcal{H}^d(Y_0) \leq M^{-1}(1 + \mathcal{H}^d(E \cap \widehat{W})); \tag{20.115}$$

this is less than the right-hand side of (20.113), so we may now assume that $x \in G_1 = G_0 \setminus Y_0$ (see below (20.61)), or even that $x$ lies in some $B_j$, because (20.68) says that $\mathcal{H}^d(G_1 \setminus G_2) = 0$ and then (20.75) says that the $B_j$ almost cover $G_2$ (recall from the line below (20.53) that $\mu$ is the restriction of $\mathcal{H}^d$ to $\varphi_1(E \cap S)$). By (20.100),

$$\sum_j \mathcal{H}^d(B_{j,ext} \cap \varphi_1(E)) \leq C\tau \sum_j \mathcal{H}^d(B_j \cap \varphi_1(E)) \leq C\tau H^d(S \cap \varphi_1(E)) \tag{20.116}$$

(recall again that the $B_j$ are disjoint and contained in $S$ (by (20.69) and (20.74)). This bound is also compatible with (20.113), so we are left with the case when $x \in B_{j,int}$. In this case, $\xi_j(x) = 1$, and we claim that $\varphi_2(x) = g_2(x) \neq x$. The first part follows from the (20.76) because $\varphi_1(x) = x$. When $g_2(x)$ is given by (20.79), the second part is obvious. When we use (20.80), projecting on $\widetilde{P}_j$ yields

$$\widetilde{\pi}_j(\widetilde{g}_2(x)) = \widetilde{\pi}_j(\psi(\lambda x)) + \eta\lambda r_j \widetilde{v}_j \neq \widetilde{\pi}_j(\psi(\lambda x)) \tag{20.117}$$

because $\widetilde{v}_j$ was chosen to be a unit vector in the direction of $\widetilde{P}(y_j)$. Then $\widetilde{g}_2(x) \neq \psi(\lambda x)$ and, by (20.83), $g_2(x) \neq x$, as needed. But this is impossible, because we assumed that $x \in E \cap S \setminus W_2$. Then (20.113) holds, and we may now put all our estimates together:

$$\begin{aligned}\mathcal{H}^d(E \cap \widehat{W}) &\leq \mathcal{H}^d(E \cap S) \leq \mathcal{H}^d(W_2) + C(M^{-1} + \tau)\\ &\leq \mathcal{H}^d(\varphi_2(W_2)) + h(r_0 + \varepsilon)(r_0 + \varepsilon)^d + C(M^{-1} + \tau)\\ &\leq \mathcal{H}^d(S \cap \varphi_1(E)) + C(\eta + \tau + \varepsilon\tau^{-d}) + h(r_0 + \varepsilon)(r_0 + \varepsilon)^d + C(M^{-1} + \tau)\end{aligned} \tag{20.118}$$

by (20.53), (20.113), (20.94), and (20.112) (where we now see $\mathcal{H}^d(S \cap \varphi_1(E))$ as a constant). Let us check that

$$\mathcal{H}^d(S \cap \varphi_1(E)) \leq \mathcal{H}^d(\widehat{W} \cap \varphi_1(E)) + \varepsilon. \tag{20.119}$$

Suppose $y \in S \cap \varphi_1(E) \setminus \widehat{W}$, and let $x \in E$ such that $\varphi_1(x) = y$. If $y \neq x$, (2.2) says that $y \in \widehat{W}$, which is impossible. So $y = x$, and now $y \in E \cap S \setminus \widehat{W}$; (20.119) then follows from (20.53).

When we add (20.118) and (20.119), we get that $\mathcal{H}^d(E \cap \widehat{W}) \leq \mathcal{H}^d(\widehat{W} \cap \varphi_1(E)) + e$, with

$$e = C(\eta + \tau + \varepsilon\tau^{-d}) + h(r_0 + \varepsilon)(r_0 + \varepsilon)^d + C(M^{-1} + \tau). \tag{20.120}$$

Of course, $C$ depends on $E$ and $\varphi_1$ in various ways, but we can choose $\tau$, then $\varepsilon$ and $M$ (recall that we never used $M$ in the estimates, so we can choose it as large as we want), then $\eta$ so small that $e$ is as close to $h(r_0)r_0^d$ as we want. This proves (20.7), the $A'$-almost minimality of $E$ follows, and so does Proposition 20.9 (in the general case). $\square$

## 21. Limits of almost minimal sets and of minimizing sequences.

In this section we just rewrite Theorem 10.8 in the context of almost minimal sets. For our first statement, we consider a gauge function $h : (0, +\infty) \to [0, +\infty]$ which is right-continuous, i.e., such that

$$h(r) = \lim_{\rho \to r\,;\, \rho > r} h(\rho) \quad \text{for } r > 0, \tag{21.1}$$

and for which

$$\lim_{r \to 0} h(r) = 0. \tag{21.2}$$

**Theorem 21.3.** *Let an open set $U$ and boundary pieces $L_j$, $0 \leq j \leq j_{max}$, be given, and suppose that the Lipschitz assumption holds (see Definition 2.7). Also suppose that the technical assumption (10.7), or the weaker (19.36) holds (but this is not needed under the rigid assumption (2.6)). Let $\{E_k\}$ be a sequence of coral (see Definition 3.1) and relatively closed sets in $U$, that converges locally in $U$ to the closed set $E$ (as in (10.4)-(10.6)).*
**1.** *If each $E_k$ is an $A_+$-almost minimal set in $U$, with the sliding conditions given by the sets $L_j$, and the gauge function $h$ (see Definition 20.2), then $E$ is coral, and it is an $A_+$-almost minimal set in $U$, with the sliding conditions given by the same sets $L_j$ and the same gauge function $h$.*
**2.** *If each $E_k$ is an $A$-almost minimal set in $U$, with the sliding conditions given by the sets $L_j$, and the gauge function $h$, then $E$ is coral, and it is an $A$-almost minimal set in $U$, with the sliding conditions given by the same sets $L_j$ and the gauge function $h$.*
**3.** *If each $E_k$ is an $A'$-almost minimal set in $U$, with the sliding conditions given by the sets $L_j$, and the gauge function $h$, then $E$ is coral, and it is an $A'$-almost minimal set in $U$, with the sliding conditions given by the same sets $L_j$ and the gauge function $h$.*

**Proof.** We start with limits of $A_+$-almost minimal sets. Since we want to apply Theorem 10.8, we compare Definition 20.2 with the definition 2.3 of quasiminimality. If $E_k$ is $A_+$-almost minimal as above, then for each $\delta > 0$, $E_k \in GSAQ(U, M(\delta), \delta, 0)$, with $M(\delta) = 1 + h(\delta)$.

By Theorem 10.8, or its variant in Remark 19.52 where we assume (19.36) instead of (10.7), $E$ also satisfies this property. Notice in particular that since here the last constant $h$ in the definition of $GSAQ$ is zero, the additional constraint above (10.2) that requires $h$ to be small is automatically satisfied. That is, for each $\delta > 0$, $E \in GSAQ(U, M(\delta), \delta, 0)$. But then $E$ is $A_+$-almost minimal with the gauge function $h'$ defined by

$$h'(r) = \liminf_{\delta \to r^+} h(\delta). \tag{21.4}$$

By (21.1), $h' = h$ and Part 1 of our result follows.

Next consider a sequence of $A$-almost minimal sets. If the $E_k$ are as in Part 2, then for each $\delta > 0$, $E_k \in GSAQ(U, 1, \delta, h(\delta))$ for all $k$.

We have a minor additional difficulty here, because in order to apply Theorem 10.8, we have to assume that $E_k \in GSAQ(U, M, \delta, h)$ with $h$ sufficiently small, depending on $n$, $M$ and $\Lambda$. Here this is true for $\delta$ small, by (21.2), but maybe not for $\delta$ large.

Fortunately, as was noted below the statement of Theorem 10.8, this assumption that $h$ be small enough is only needed to get the right regularity and lower semicontinuity properties, but as soon as it is satisfied for some acceptable combination of $M, \delta, h$ (here with $M = 1$ and $\delta$ so small that $h = h(\delta)$ works), we get the limiting theorem for the other combinations. Thus $E \in GSAQ(U, 1, \delta, h(\delta))$ for $\delta > 0$.

Then we return to Definition 20.2 and get that $E$ is $A$-almost minimal with the gauge function $h'$ of (21.4). Since $h' = h$ by (21.1), Part 2 follows.

For Part 3, we just need to observe that because of Proposition 21.9, we do not need to distinguish between $A$-almost minimal and $A'$-almost minimal (notice that the additional sufficient condition for the equivalence, (10.7) or (19.36), is satisfied). Then Part 3 follows from Part 2. □

**Remark 21.5.** Probably we could modify our proof of Theorem 10.8 to make it work also for $A'$-almost minimal sets (and even with the variant of quasiminimal sets defined with the same accounting as in (20.6)). We should not expect a huge simplification, and in particular we cannot content ourselves with applying the almost minimality of $E_k$ with any extension of our initial mapping $\varphi_1$, because it still could be that $\varphi_1(E_k)$ is a very bad competitor because it contains may parallel sheets, that could easily be merged to produce a better competitor, while these sheets are already merged for $E$.

Also, we would have to take into account the possibility that the set $\varphi_1(E \cap W_1)$ meets $E \setminus W_1$ (where as usual $W_1 = \big\{x \in \mathbb{R}^n\,;\, \varphi_1(x) \neq x\big\}$), while this does not happen with $E_k$. Then $\varphi_1$ defines a better competitor for $E$ than for $E_k$, which is also bad for our proof. We did not pay attention to this case in the proof of Theorem 10.8, because it did not matter with the accounting for quasiminimal sets, but of course we could try to fix it, for instance by allowing a larger piece of $\widehat{W}$ in the definition of $X_0$ in (11.20). But this becomes similar to our proof of Proposition 21.9, so the author does not expect to win much by trying a direct proof.

**Remark 21.6.** If we did not assume (21.1), we would still have that the limit $E$ is almost minimal, but this would be with the gauge function $h'$ defined in (21.4); this is easy to see from the proof, and (for Part 3) the similar comment below Proposition 20.9.

**Remark 21.7.** Similarly, we do not really need to assume (21.2), but instead we can assume that (10.2) holds, i.e., that there are constants $M$, $\delta$, and $\hbar$, with $\hbar$ small enough (depending on $n$, $\Lambda$, and $M$) such that $E_k \in GSAQ(U, M, \delta, \hbar)$ for all $k$. Then we can use the remark below Theorem 10.8 (as we did for Part 2) and proceed as above, because (10.2) is enough for the regularity results of Section 10. We shall apply this now, in the context of local minimizing sequences.

Here is the notation for the next corollary. We are given, as in Theorem 21.3, an open set $U$ and boundary pieces $L_j$, $0 \leq j \leq j_{max}$, and we suppose that

$$\text{the Lipschitz assumption holds, as well as (10.7) or (19.36)} \tag{21.8}$$

(again see Definition 2.7 and observe that (10.7) is automatic under the rigid assumption). We are also given a sequence $\{E_k\}$ of coral relatively closed sets in $U$, and we assume that

$$\text{the } E_k \text{ converge locally in } U \text{ to the relatively closed set } E \subset U. \tag{21.9}$$

In addition, we assume that there are constants $M$, $\delta$, and $h$, with $h$ small enough (depending on $n$, $\Lambda$, and $M$) such that (10.2) holds, i.e.,

$$E_k \in GSAQ(U, M, \delta, h) \text{ for all } k. \tag{21.10}$$

Finally, we assume that $\{E_k\}$ is a locally minimizing sequence, in the following sense. Given $\delta > 0$, we say that one-parameter family $\{\varphi_t\}$ of functions is $\delta$-admissible for $E_k$ if it satisfies the conditions (1.4)-(1.8), relative to $E_k$ and some ball $B$ of radius $r < \delta$, and in addition the compactness condition (2.4) holds (relative to $E_k$). Recall that (2.4) says that $\widehat{W}(E_k) \subset\subset U$, where we set

$$\widehat{W}(E_k) = \bigcup_{0<t\leq 1} W_t(E_k) \cup \varphi_t(W_t(E_k)), \tag{21.11}$$

with

$$W_t(E_k) = \big\{ y \in E_k \,; \varphi_t(y) \neq y \big\} \ \text{ for } 0 < t \leq 1. \tag{21.12}$$

We shall assume that there exists $\delta > 0$ such that, for each $\varepsilon > 0$ we can find $k_0 \geq 0$ such that

$$\mathcal{H}^d(W_1(E_k)) \leq \mathcal{H}^d(\varphi_1(W_1(E_k))) + \varepsilon \tag{21.13}$$

for every $k \geq k_0$ and every one-parameter family $\{\varphi_t\}$ which is $\delta$-admissible for $E_k$.

Or we shall assume that, with the same quantifiers,

$$\mathcal{H}^d(E_k \setminus \varphi_1(E_k)) \leq \mathcal{H}^d(\varphi_1(E_k) \setminus E_k) + \varepsilon \tag{21.14}$$

for every $k \geq k_0$ and every one-parameter family $\{\varphi_t\}$ which is $\delta$-admissible for $E_k$. This second assumption, which is more in the mode of $A'$-almost minimal set, is more natural in some contexts.

**Corollary 21.15.** *Let $U$, the boundary pieces $L_j$, and the sequence $\{E_k\}$ of coral quasiminimal sets satisfy the conditions (21.8)-(21.14). Then $E$ is a coral local minimizer in $U$, in the sense that*

$$\mathcal{H}^d(E \setminus \varphi_1(E)) \leq \mathcal{H}^d(\varphi_1(E) \setminus E) \tag{21.16}$$

*for every one-parameter family $\{\varphi_t\}$ which is $\delta$-admissible for $E$.*

To prove the corollary, observe that for $k \geq k_0$, (21.13) or (21.14) says that $E_k$ is $A$-almost minimal or $A'$-almost minimal, with the strange gauge function $h_\varepsilon$ defined by $h_\varepsilon(r) = r^{-d}\varepsilon$ for $0 < r < \delta$ and $h_\varepsilon(r) = +\infty$ for $r \geq \delta$.

This function does not satisfy (21.2), but Remark 21.7 and our assumption (21.10) allow us to dispense with this condition. Then by Theorem 21.3, $E$ is coral, and almost minimal with the same gauge function $h_\varepsilon$. Since this is true for all $\varepsilon > 0$, we also get that $E$ is almost minimal with the gauge function $h_0$. If we were dealing with (21.14) and $A'$-almost minimal sets, we directly get (21.16) from this. If we were dealing with (21.13) and $A$-almost minimal sets, we get

$$\mathcal{H}^d(W_1(E)) \leq \mathcal{H}^d(\varphi_1(W_1(E))) \tag{21.17}$$

instead of (21.16), but by the easy part of Proposition 20.9, (21.17) implies (21.16); Corollary 21.15 follows. □

**Remark 21.18.** In the conclusion of Corollary 21.15, we may also replace (21.16) with (21.17), since the two conditions are equivalent (by Proposition 20.9, applied with $h_0$).

## 22. Upper semicontinuity of $H^d$ along sequences of almost minimal sets.

The main result of this section is the following upper semicontinuity result.

**Theorem 22.1.** *Let $U$, the $L_i$, the sequence $\{E_k\}$, and the set $E$ satisfy the assumptions of Theorem 21.3 (any part) or Corollary 21.15. Then for every compact set $H \subset U$,*

$$\mathcal{H}^d(E \cap H) \geq \limsup_{k \to +\infty} \mathcal{H}^d(E_k \cap H). \tag{22.2}$$

Notice that if the $E_k$ are only supposed to be quasiminimal, the conclusion may fail, even when there is no boundary condition. For instance, $E_k$ may coincide locally with the graph of the function $x \to 2^{-k}\sin(2^k x)$, which converges to a line; then (22.2) fails. So, for the sequences of Theorem 21.3, the condition (21.2) is really needed this time.

The proof will only use the rectifiability of $E^*$, a covering argument, and an application of the quasiminimality (or almost minimality) of the $E_k$ in balls where $E$ is flat. It is essentially a special case of the following lemma, which is a generalization of Lemma 3.12 on page 85 of [D5], and which we shall prove first.

**Lemma 22.3.** *Let $U$, the $L_i$, the sequence $\{E_k\}$, and the set $E$ satisfy the assumptions of Theorem 10.8. Then for every compact set $H \subset U$,*

$$(1 + Ch)M\mathcal{H}^d(E \cap H) \geq \limsup_{k \to +\infty} \mathcal{H}^d(E_k \cap H), \tag{22.4}$$

*with a constant $C$ that depends only on $n$, $M$, and $\Lambda$.*

Our proof of Lemma 22.3 will be similar to the proof of Proposition 21.9 in the Lipschitz case. Let $\{E_k\}$, $E$, and $H$ be as in the statement. We first try to cover a big piece of $E \cap H$ by small balls.

Our assumptions allow us apply the results of Section 10. In particular, the $E_k$ are uniformly locally Ahlfors-regular (by (10.10)), and $E$ is locally Ahlfors-regular (by (10.11)) and rectifiable (by Proposition 10.15).

Let $\varepsilon > 0$ be given, and use the fact that $\mathcal{H}^d(E)$ is locally finite in $U$ (for instance, because $E^*$ is locally Ahlfors regular) to choose an open set $V$ such that

$$H \subset V \subset\subset U \text{ and } \mathcal{H}^d(E \cap V \setminus H) \leq \varepsilon. \tag{22.5}$$

Next, the fact that $E$ is rectifiable implies that for $\mathcal{H}^d$-almost every $x \in E \cap H$,

$$\lim_{r \to 0} r^{-d} \mathcal{H}^d(E \cap B(x,r)) = \omega_d, \tag{22.6}$$

(see Theorem 17.6 on page 240 in [Ma]), and

$$E \text{ has a tangent plane } P(x) \text{ at } x. \tag{22.7}$$

Recall that the fact that an approximate tangent plane to $E$ is a true tangent plane comes from the local Ahlfors-regularity of $E$; see for instance Exercise 41.21 on page 277 of [D4].

We shall assume that the Lipschitz assumption holds; the rigid case is easier, and we could also obtain it the complicated way, by pretending that $U = B(0,1)$ and $\psi$ is the identity. For $x \in E \cap H$, denote by $F(x)$ the smallest (twisted) face of our grid that contains $x$. We also set $\widetilde{x} = \psi(\lambda x)$ and $\widetilde{F}(x) = \psi(\lambda F(x))$ (a true dyadic face). For almost every $x \in E \cap H$ such that (22.7) holds, we also have that

$$\widetilde{E} = \psi(\lambda E) \text{ has a tangent plane } \widetilde{P}(x) \text{ at } \widetilde{x}, \tag{22.8}$$

because $\widetilde{E}$ is also rectifiable and locally Ahlfors-regular (recall that $\psi$ is bilipschitz). We also want to show that $\mathcal{H}^d$-almost everywhere on $E \cap H$,

$$\widetilde{P}(x) \text{ is contained in the smallest affine space that contains } \widetilde{F}(x). \tag{22.9}$$

We proceed roughly as for (20.66). Fix a face $F$ of our twisted grid, and first observe that by Theorem 6.2 on page 89 of [Ma],

$$\lim_{r \to 0} r^{-d} \mathcal{H}^d(E \cap B(x,r) \setminus F) = 0. \tag{22.10}$$

for $\mathcal{H}^d$-almost every $x \in F \cap E$. Then

$$\lim_{\rho \to 0} \rho^{-d} \mathcal{H}^d(\widetilde{E} \cap B(\widetilde{x}, \rho) \setminus \widetilde{F}) = 0, \quad \text{with } \widetilde{F} = \psi(\lambda F) \tag{22.11}$$

(because $\psi$ is bilipschitz). Next notice that $\widetilde{E} \cap \widetilde{F}$ is rectifiable; hence for $\mathcal{H}^d$-almost every $x \in F \cap E$, $\widetilde{E} \cap \widetilde{F}$ has an approximate tangent $\widetilde{P}'(x)$ at $\widetilde{x}$, which of course can be chosen inside the affine span of $\widetilde{F}$. When (22.11) holds, $\widetilde{P}'(x)$ is also an approximate tangent plane to the whole $\widetilde{E}$ (the additional part has vanishing density). By local Ahlfors-regularity of $\widetilde{E}$, $\widetilde{P}'(x)$ is even a true tangent plane to $\widetilde{E}$. It is easy to see that for local Ahlfors-regular sets, the tangent plane is unique, so $\widetilde{P}'(x) = \widetilde{P}(x)$ almost everywhere on $F$. Since there is only a finite number of faces to try, we get that $\widetilde{P}(x)$ is contained in the affine span of $\widetilde{F}$ for $\mathcal{H}^d$-almost every $x \in E$ and all the faces $F$ that contain $x$; we apply this to $F = F(x)$ and get (22.9).

We don't even need to know that the tangent plane is unique to make the argument work, because we just need to find, for almost every $x \in E \cap H$, a tangent plane that satisfies (22.9); so we could use the plane $\widetilde{P}'(x)$ associated to $F(x)$, for instance.

Observe also that the set of points $x \in E \cap H$ for which the dimension of $F(x)$ is less than $d$ is excluded by (22.9); this is all right, because this set is $\mathcal{H}^d$-negligible.

We also exclude the exceptional set $Z$ of (19.35). That is, let us denote by $X$ the set of points $x \in E \cap H$ that satisfy the conditions (22.6)-(22.9) above, and in addition, if $x$ is contained in one of the sets $L_i' = L_i \setminus \mathrm{int}(L_i)$, where $\mathrm{int}(L_i)$ denotes the true ($n$-dimensional) interior of $L_i$, and

$$\limsup_{r \to 0} A_r(x) \leq \omega_d, \tag{22.12}$$

where $A_r(x)$ is given by (19.33). Thus, by the discussion above,

$$\mathcal{H}^d(E \cap H \setminus X) = 0. \tag{22.13}$$

For the next stage of the proof, we select a small radius $r(x)$ for every $x \in X$. we choose $r(x)$ so that

$$r(x) \leq \frac{1}{4\Lambda^2} \min(\mathrm{dist}(x, U \setminus V), \mathrm{dist}(x, \partial F(x))) \tag{22.14}$$

(which is positive because $x \in E \cap H \subset V = \mathrm{int}(V)$ and $x$ lies in the (face) interior of $F(x)$),

$$\omega_d - \varepsilon \leq r^{-d}\mathcal{H}^d(E \cap B(x,r)) \leq \omega_d + \varepsilon \quad \text{for } 0 < r \leq r(x) \tag{22.15}$$

(possible by (22.6)),

$$\mathrm{dist}(z, P(x)) \leq \varepsilon r \quad \text{for } z \in E \cap B(x, 2r) \text{ and } 0 < r \leq r(x), \tag{22.16}$$

$$\mathrm{dist}(\widetilde{z}, \widetilde{P}(x)) \leq \varepsilon\lambda r \quad \text{for } \widetilde{z} \in \widetilde{E} \cap B(\widetilde{x}, 2\lambda\Lambda r) \text{ and } 0 < r \leq r(x), \tag{22.17}$$

and, when $x$ lies in some $L_i' = L_i \setminus \mathrm{int}(L_i)$,

$$A_r(x) \leq \omega_d + \varepsilon \quad \text{for } 0 \leq r \leq r(x). \tag{22.18}$$

We add two constraints that will simplify our life when we check the boundary condition (1.7). We require that for each $i \in [0, j_{max}]$,

$$r(x) < \frac{1}{4\Lambda^2} \operatorname{dist}(x, \mathbb{R}^n \setminus L_i) \tag{22.19}$$

when $x$ lies in the ($n$-dimensional) interior of $L_i$, and on the opposite

$$r(x) < \frac{1}{2} \operatorname{dist}(x, L_i) \tag{22.20}$$

when $x \in U \setminus L_i$.

Let us apply Theorem 2.8 in [Ma] to the family of balls $\overline{B}(x, r)$, $x \in X$ and $0 < r < \min(r(x), \rho_0)$, where $\rho_0$ will be chosen later, and such that $\mathcal{H}^d(E \cap \partial B(x, r)) = 0$. We get a collection of disjoint $B_j = B(x_j, r_j)$, $j \in J_1$, such that

$$0 < r_j < \min(r(y_j), \rho_0), \tag{22.21}$$

$\mathcal{H}^d(E \cap \partial B_j) = 0$ for all $j$, and

$$\mathcal{H}^d(X \setminus \bigcup_{j \in J_1} B_j) = \mathcal{H}^d(X \setminus \bigcup_{j \in J_1} \overline{B}_j) = 0. \tag{22.22}$$

Let us choose a finite subset $J$ of $J_1$, so that

$$\mathcal{H}^d(X \setminus \bigcup_{j \in J} B_j) \leq \varepsilon. \tag{22.23}$$

Set $X_1 = E \cap H \setminus \bigcup_{j \in J} B_j$; then by (22.13),

$$\mathcal{H}^d(X_1) = \mathcal{H}^d(X \setminus \bigcup_{j \in J} B_j) \leq \varepsilon \tag{22.24}$$

and we can use the definition of $\mathcal{H}^d$ to cover $X_1$ by balls $B_i = B(x_i, r_i)$, $i \in I$, so that

$$r_i \leq \rho_0 \text{ for } i \in I \ \text{ and } \ \sum_{i \in I} r_i^d \leq C\varepsilon. \tag{22.25}$$

Because $X_1$ is compact, we can replace $I$ with a finite subset for which the $B_i$ still cover $X_1$ and (removing the useless balls) each $B_i$ meets $X_1$. By definition,

$$E \cap H \subset \bigcup_{j \in I \cup J} B_j. \tag{22.26}$$

Since $E \cap H$ is compact, $I \cup J$ is finite, and $\{E_k\}$ converges to $E$, we also get that

$$E_k \cap H \subset \bigcup_{j \in I \cup J} B_j \ \text{ for } k \text{ large enough.} \tag{22.27}$$

For each $i \in I$ pick $y_i \in E \cap B_i$. Then for $k$ large, we can find $y_{i,k} \in E_k \cap B_i$ for every $i \in I$, and of course $B_i \subset B(y_{i,k}, 2r_i)$. We shall choose

$$\rho_0 < \frac{1}{10\Lambda^2} \min(\mathrm{dist}(H, U \setminus V), \lambda^{-1} r_0, \delta), \tag{22.28}$$

where the constants $\lambda$ and $r_0$ come from Definition 2.7 (the Lipschitz assumption), and $\delta$ comes from our $GSAQ(U, M, \delta, h)$ assumption. We don't care how small they are, the main point is that they depend only on $E$ and the sequence $\{E_k\}$. Then $B(y_{i,k}, 4r_i) \subset V \subset U$, and by (10.11) (the uniform local Ahlfors-regularity of the $E_k$),

$$\mathcal{H}^d(E_k \cap B_i) \leq \mathcal{H}^d(E_k \cap B(y_{i,k}, 2r_i)) \leq C r_i^d. \tag{22.29}$$

This holds for $k$ large enough (and all $i \in I$), with a constant that depends only on $E$ and $\{E_k\}$. By (22.25), this yields

$$\mathcal{H}^d(E_k \cap \bigcup_{i \in I} B_i) \leq C\varepsilon \tag{22.30}$$

for $k$ large, and we are left with the contributions of the balls $B_j$, $j \in J$.

We need to use the quasiminimality of $E_k$, and for this we construct a one parameter family of mappings $\{\varphi_{j,t}\}$, $0 \leq t \leq 1$, for each $j \in J$.

Fix $j \in J$ for the moment, and define the cut-off function $\xi_j$ by $\xi_j(y) = 0$ for $y \in U \setminus B_j$, and

$$\xi_j(y) = \min\left\{1, (\tau r_j)^{-1} \mathrm{dist}(y, \partial B_j)\right\} \text{ for } y \in B_j, \tag{22.31}$$

where the small constant $\tau > 0$ will be chosen later (before $\varepsilon$ and $\rho_0$).

We start with the easier case when when $x_j$ does not lie in any $L'_i$. Then we pick a unit vector $v_j$ parallel to $P(x_j)$, and set

$$\varphi_{j,t}(x) = x + t\xi_j(x)[\pi_j(x) - x + \eta r_j v_j] \tag{22.32}$$

for $y \in B_j$ and $0 \leq t \leq 1$, where $\pi_j$ denotes the orthogonal projection onto $P(x_j)$ and $\eta > 0$ is a minuscule constant, to be chosen later (depending on $\tau$ and $\varepsilon$). We do nothing on $U \setminus B_j$, i.e., set

$$\varphi_{j,t}(x) = x \text{ for } x \in U \setminus B_j \text{ and } 0 \leq t \leq 1. \tag{22.33}$$

This is also the formula that we would use under the rigid assumption, but because of (1.7) (and under the Lipschitz assumption) we shall need to be more careful in our second case.

If $x_j \in L'_i$ for some $i$, we proceed as we did near (20.80). Denote by $F_j = F(x_j)$ the smallest face of our twisted grid that contains $x_j$, set $\widetilde{F}_j = \psi(\lambda F_j)$ (a rigid face), call $\widetilde{W}_j$ the affine space spanned by $\widetilde{F}_j$, choose a unit vector $\widetilde{v}_j$ in the vector space parallel to $\widetilde{P}(x_j)$ (the approximate tangent plane to $\widetilde{E} = \psi(\lambda E)$ at $\widetilde{x}_j = \psi(\lambda x_j)$). Notice that by

(22.9), $\widetilde{v}_j$ also lies in the vector space parallel to $\widetilde{W}_j$ and $\widetilde{F}_j$. Denote by $\widetilde{\pi}_j$ the orthogonal projection onto $\widetilde{P}_j$, and finally set

$$\widetilde{\varphi}_{j,t}(x) = \psi(\lambda x) + t\xi_j(x)\big[\widetilde{\pi}_j(\psi(\lambda x)) - \psi(\lambda x) + \eta\lambda r_j \widetilde{v}_j\big] \tag{22.34}$$

for $x \in B_j$ and $0 \leq t \leq 1$. Notice that

$$\begin{aligned} |\widetilde{\varphi}_{j,t}(x) - \psi(\lambda x_j)| &\leq |\psi(\lambda x) - \psi(\lambda x_j)| + t|\widetilde{\pi}_j(\psi(\lambda x)) - \psi(\lambda x)| + t\eta\lambda r_j \\ &\leq 2|\psi(\lambda x) - \psi(\lambda x_j)| + \eta\lambda r_j \leq 2\lambda\Lambda r_j + \eta\lambda r_j \leq 3\lambda\Lambda r_j \end{aligned} \tag{22.35}$$

because $\psi(\lambda x_j)$ lies in $\widetilde{P}_j$ and if $\eta$ is small enough. Then by (22.14) and (22.21),

$$\begin{aligned} \operatorname{dist}(\widetilde{\varphi}_{j,t}(x), \mathbb{R}^n \setminus \psi(\lambda V)) &\geq \operatorname{dist}(\psi(\lambda x_j), \mathbb{R}^n \setminus \psi(\lambda V)) - 3\lambda\Lambda r_j \\ &\geq \lambda\Lambda^{-1} \operatorname{dist}(x_j, U \setminus V) - 3\lambda\Lambda r_j \\ &\geq 4\lambda\Lambda r(x_j) - 3\lambda\Lambda r_j \geq \lambda\Lambda r_j. \end{aligned} \tag{22.36}$$

In particular $\widetilde{\varphi}_{j,t}(x) \in \psi(\lambda V) \subset B(0,1)$ and we can define

$$\varphi_{j,t}(x) = \lambda^{-1}\psi^{-1}(\widetilde{\varphi}_{j,t}(x)) \in V. \tag{22.37}$$

Observe that by (22.35)

$$|\varphi_{j,t}(x) - x_j| \leq \lambda^{-1}\Lambda|\widetilde{\varphi}_{j,t}(x) - \psi(\lambda x_j)| \leq 3\Lambda^2 r_j. \tag{22.38}$$

This completes our definition of the $\varphi_{j,t}$ on $B_j$, and naturally we keep the trivial definition (22.33) on $U \setminus B_j$. Our next task is to show that the $\varphi_{j,t}$, $0 \leq t \leq 1$, define an acceptable competitor for $E_k$. There is no problem with (1.4) and (1.8); our mappings $\varphi_{j,t}(x)$, $t \leq 1$, are clearly continuous in $x$ and $t$ and Lipschitz in $x$.

Notice that $\varphi_{j,t}(x) = x$ when $t = 0$. By (22.33) $\varphi_{j,t}(x) = x$ when $x \in U \setminus B_j$. When $x \in B_j$ and we use (22.32), notice that $x + t\xi_j(x)[\pi_j(x) - x] \in B_j$ (because $\pi_j(x) \in B_j$ and $B_j$ is convex), so $\varphi_{j,t}(x) \in B(x_j, (1+\eta)r_j)$. When $x \in B_j$ but we use (22.34) and (22.37), we only get that $\varphi_{j,t}(x) \in B(x_j, 3\Lambda^2 r_j)$, by (22.38). In both cases,

$$\varphi_{j,t}(B_j) \subset B(x_j, 3\Lambda^2 r_j), \tag{22.39}$$

and the $\varphi_{j,t}$ satisfy (1.5) and (1.6), relative to the ball $B = \overline{B}(x_j, 3\Lambda^2 r_j)$. They also satisfy (2.4) with $\widehat{W} \subset B$, which is compact and contained in $V \subset U$ by (22.14) and (22.21).

Finally let us check (1.7). We do this under the Lipschitz assumption; the rigid case is just simpler. Let $i \leq j_{max}$ and $x \in E_k \cap L_i$ be given. Recall that we want to check that $\varphi_{j,t}(x) \in L_i$ for all $t$, so we may assume that $x \in B_j$, because otherwise $\varphi_{j,t}(x) = x \in L_i$.

A first case is when $x_i \in \operatorname{int}(L_i)$. In this case, (22.21) and (22.19) imply that $B(x_j, 4\Lambda^2 r_j) \subset L_i$, and then (22.39) implies that $\varphi_{j,t}(x) \in \varphi_{j,t}(B_j) \subset L_i$, as needed. The case when $x_j \notin L_i$ is impossible, because (22.21) and (22.20) would imply that $B_j$

does not meet $L_i$. So we are left with the case when $x_j \in L_i'$. Recall that in this case $\varphi_{j,t}(x)$ was defined by (22.34) and (22.37).

Let $F$ be a face of $L_i$ that contains $x$, and set $\widetilde{F} = \psi(\lambda F)$ and $\widetilde{F}(x_j) = \psi(\lambda F(x_j))$. We first check that

$$\widetilde{F}(x_j) \subset \widetilde{F}. \tag{22.40}$$

Suppose not, first assume that $\widetilde{F}(x_j)$ is not reduced to $\{\psi(\lambda x_j)\}$, and apply (3.8), (22.14), and (22.21), to get that

$$\begin{aligned} \operatorname{dist}(\psi(\lambda x_j), \widetilde{F}) &\geq \operatorname{dist}(\psi(\lambda x_j), \partial(\widetilde{F}(x_j))) \geq \lambda\Lambda^{-1} \operatorname{dist}(x_j, \partial F(x_j)) \\ &\geq 4\lambda\Lambda r(x_j) \geq 4\lambda\Lambda r_j. \end{aligned} \tag{22.41}$$

If instead $\widetilde{F}(x_j) = \{\psi(\lambda x_j)\}$, $\psi(\lambda x_j)$ is a vertex, so $\operatorname{dist}(\psi(\lambda x_j), \widetilde{F}) \geq r_0$ because $\widetilde{F}$ is a face that does not contain it, and the conclusion of (22.41) still holds, by (22.21) and (22.28). But $\psi(\lambda x) \in \widetilde{F}$ because $x \in F$, so

$$\operatorname{dist}(\psi(\lambda x_j), \widetilde{F}) \leq |\psi(\lambda x_j) - \psi(\lambda x)| \leq \lambda\Lambda |x - x_j| \leq \lambda\Lambda r_j, \tag{22.42}$$

a contradiction which proves (22.40).

Return to $\varphi_{j,t}(x)$, which was defined by (22.34) and (22.37); we want to show that $\varphi_{j,t}(x) \subset L_i$ , and (by definition of $F$) it is enough to show that $\varphi_{j,t}(x) \in F$, or equivalently that $\widetilde{\varphi}_{j,t}(x) \in \widetilde{F}$. Recall that $\widetilde{\pi}_j$ is the orthogonal projection onto $\widetilde{P}_j$, which by (22.9) is contained in the affine span $\widetilde{W}_j$ of $\widetilde{F}(x_j)$, and (by (22.40)) in the affine span $\widetilde{W}$ of $\widetilde{F}$.

Set $\widetilde{x} = \psi(\lambda x) \in \widetilde{F}$; we know that $\widetilde{x} \in \widetilde{F}$, and then, by (22.34),

$$\widetilde{\varphi}_{j,t}(x) = \widetilde{x} + t\xi_j(x)\big[\widetilde{\pi}_j(\widetilde{x}) - \widetilde{x} + \eta\lambda r_j \widetilde{v}_j\big] \in \widetilde{W} \tag{22.43}$$

because $\widetilde{v}_j$ was chosen in the vector space parallel to $\widetilde{P}_j \subset \widetilde{W}$. But $\psi(\lambda x_j) \in \widetilde{F}(x_j)$ and $\widetilde{F}(x_j) \subset \widetilde{F}$, so

$$\operatorname{dist}(\psi(\lambda x_j), \partial\widetilde{F}) \geq \operatorname{dist}(\psi(\lambda x_j), \partial\widetilde{F}(x_j)) \geq 4\lambda\Lambda r(x_j) \geq 4\lambda\Lambda r_j \tag{22.44}$$

by (22.14) and (22.21). Set $I = [\psi(\lambda x_j), \widetilde{\varphi}_{j,t}(x)]$. By (22.35), its length is at most $3\lambda\Lambda r_j$, so (22.44) says that $\operatorname{dist}(I, \partial\widetilde{F}) \geq \lambda\Lambda r_j$. In particular, $I$ does not cross $\partial\widetilde{F}$; since its initial point is $\psi(\lambda x_j) \in \widetilde{F}(x_j) \subset \widetilde{F}$ (by (22.40)), and $I \subset \widetilde{W}$ (by (22.43)), we get that $I \subset \widetilde{F}$. Hence $\widetilde{\varphi}_{j,t}(x) \in \widetilde{F}$, as desired, and (1.7) follows.

We are now allowed to test the quasiminimality of $E_k$ on the $\varphi_{j,t}$; notice in particular that the radius of our ball $B = \overline{B}(x_j, 3\Lambda^2 r_j)$ is smaller than the threshold $\delta > 0$ in our quasiminimality assumption (10.2), by (22.21) and (22.28). We get that

$$\mathcal{H}^d(W_1(E_k)) \leq M\mathcal{H}^d(\varphi_{j,1} W_1(E_k)) + 3h\Lambda^2 r_j^d, \tag{22.45}$$

where

$$W_1(E_k) = \{x \in E_k\,;\, \varphi_{j,1}(x) \neq x\} \subset E_k \cap B_j \tag{22.46}$$

(by (22.33)). Next we estimate $\mathcal{H}^d(\varphi_{j,1}(E_k \cap B_j))$. Let us first check that for $k$ large,

$$\varphi_{j,1} \text{ is } 2\Lambda^2\text{-Lipschitz on } E_k \cap B_j. \tag{22.47}$$

We start in the easier case when $\varphi_{j,1}$ was defined by (22.32). Observe that for $x, y \in B_j$,

$$\begin{aligned}
\varphi_{j,1}(x) - \varphi_{j,1}(y) &= x + \xi_j(x)[\pi_j(x) - x + \eta r_j v_j] - y - \xi_j(y)[\pi_j(y) - y + \eta r_j v_j] \\
&= (x-y) + \xi_j(x)[\pi_j(x) - \pi_j(y) - (x-y)] \\
&\qquad + [\xi_j(x) - \xi_j(y)][\pi_j(y) - y + \eta r_j v_j] \\
&= (1 - \xi_j(x))(x-y) + \xi_j(x)[\pi_j(x) - \pi_j(y)] \\
&\qquad + [\xi_j(x) - \xi_j(y)][\pi_j(y) - y + \eta r_j v_j]
\end{aligned} \tag{22.48}$$

and hence

$$\begin{aligned}
|\varphi_{j,1}(x) - \varphi_{j,1}(y)| &\leq |x-y| + |\xi_j(x) - \xi_j(y)|[|\pi_j(y) - y| + \eta r_j] \\
&\leq |x-y| + (\tau r_j)^{-1}|x-y|[|\pi_j(y) - y| + \eta r_j]
\end{aligned} \tag{22.49}$$

by (22.31). Since $\{E_k\}$ converges to $E$, for $k$ large enough we have that for each $y \in E_k \cap B_j$, there is a $z \in E$ such that $|z - y| \leq \varepsilon r_j$. Then $z \in B(x_j, 2r_j)$, and (since $r_j \leq r(x_j)$ by (22.21)) (22.16) says that $\operatorname{dist}(z, P(x)) \leq \varepsilon r_j$. Hence $|\pi_j(y) - y| \leq |\pi_j(z) - z| + \varepsilon r_j \leq 2\varepsilon r_j$; then (22.47) easily follows from (22.49), if $\varepsilon$ and $\eta$ are small enough compared to $\tau$.

When $\varphi_{j,1}$ is given by (22.34) and (22.37), it is enough to check that $\widetilde{\varphi}_{j,1}$ is $2\lambda\Lambda$-Lipschitz on $E_k \cap B_j$. Let $x, y \in E_k \cap B_j$ be given; by (22.34) and the same computation as for (22.49),

$$\begin{aligned}
|\widetilde{\varphi}_{j,1}(x) - \widetilde{\varphi}_{j,1}(y)| &\leq |\psi(\lambda x) - \psi(\lambda y)| + |\xi_j(x) - \xi_j(y)|\big[|\widetilde{\pi}_j(\psi(\lambda y)) - \psi(\lambda y)| + \eta\lambda r_j\big] \\
&\leq \lambda\Lambda|x-y| + (\tau r_j)^{-1}|x-y|\big[|\widetilde{\pi}_j(\psi(\lambda y)) - \psi(\lambda y)| + \eta\lambda r_j\big].
\end{aligned} \tag{22.50}$$

Let $z$ be, as before, a point of $E$ such that $|z - y| \leq \varepsilon r_j$. Set $\widetilde{z} = \psi(\lambda z)$, and notice that $|\widetilde{z} - \psi(\lambda y)| \leq \lambda\Lambda|z - y| \leq \lambda\Lambda\varepsilon r_j$, and also $|\widetilde{z} - \psi(\lambda x_j)| \leq \lambda\Lambda|z - x_j| < 2\lambda\Lambda r_j$. Since $\widetilde{z} \in \widetilde{E} = \psi(\lambda E)$, (22.17) yields $\operatorname{dist}(\widetilde{z}, \widetilde{P}(x_j)) \leq \varepsilon\lambda r_j$. Thus

$$\operatorname{dist}(\psi(\lambda y), \widetilde{P}(x_j)) \leq \operatorname{dist}(\widetilde{z}, \widetilde{P}(x_j)) + |\widetilde{z} - \psi(\lambda y)| \leq \varepsilon\lambda r_j + \lambda\Lambda\varepsilon r_j \leq 2\lambda\Lambda\varepsilon r_j \tag{22.51}$$

for $y \in E_k \cap B_j$ (and $k$ large). Then (22.50) yields

$$|\widetilde{\varphi}_{j,1}(x) - \widetilde{\varphi}_{j,1}(y)| \leq \lambda\Lambda|x-y| + (\tau r_j)^{-1}|x-y|\big[2\lambda\Lambda\varepsilon r_j + \eta\lambda r_j\big], \tag{22.52}$$

$\widetilde{\varphi}_{j,1}$ is $2\lambda\Lambda$-Lipschitz on $E_k \cap B_j$ (for $k$ large and if $\varepsilon$ and $\eta$ are small enough), and we get (22.47) in our last case.

Write $B_j = B_{j,int} \cup B_{j,ext}$, where $B_{j,int} = \{y \in B_j; \mathrm{dist}(y, \partial B_j) < \tau r_j\}$, $B_{j,ext} = B_j \setminus B_{j,int}$, and $\tau > 0$ is still the small constant in the definition (22.31) of $\xi_j$. Observe that by (22.15),

$$\begin{aligned}\mathcal{H}^d(E \cap B_{j,ext}) &= \mathcal{H}^d(E \cap B_j) - \mathcal{H}^d(E \cap B_{j,int}) \\ &\leq (\omega_d + \varepsilon) r_j^d - (\omega_d - \varepsilon)(1-\tau)^d r_j^d \leq C\tau r_j^d\end{aligned} \tag{22.53}$$

if $\varepsilon$ is small enough compared to $\tau$. Since $B_{j,ext}$ is closed, we can apply the weak lower semicontinuity result of (10.14), and we get that for $k$ large

$$\mathcal{H}^d(E_k \cap B_{j,ext}) \leq C_M \mathcal{H}^d(E \cap B_{j,ext}) + \tau r_j^d \leq C\tau r_j^d; \tag{22.54}$$

of course if the reader does not feel like using (10.14), he may also use the flatness of $E \cap 2B_j$ (see (22.16)), the fact that the $E_k$ converge to $E$, and the uniform Ahlfors regularity of the $E_k^*$ near $B_j$ to get (22.54). Next, by (22.47),

$$\mathcal{H}^d(\varphi_{j,1}(E_k \cap B_{j,ext})) \leq C\tau r_j^d, \tag{22.55}$$

where we no longer write the dependence on $\Lambda$.

We are left with the contribution of $B_{j,int}$. If we used (22.32) to define $\varphi_{j,1}$, we get that $\varphi_{j,1}(x) = \pi_j(x) + \eta r_j v_j$ for $x \in B_{j,int}$, because $\xi_j(x) = 1$ on $B_{j,int}$. Then

$$\mathcal{H}^d(\varphi_{j,1}(E_k \cap B_{j,int})) = \mathcal{H}^d(\pi_j(E_k \cap B_{j,int})) \leq \mathcal{H}^d(P(x_j) \cap B_j) \leq \omega_d r_j^d. \tag{22.56}$$

If instead we used (22.34) and (22.37), observe that by (22.34), $\widetilde{\varphi}_{j,1}(x) = \widetilde{\pi}_j(\psi(\lambda x)) + \eta\lambda r_j \widetilde{v}_j$ for $x \in E_k \cap B_{j,int}$. We chose $\widetilde{v}_j$ parallel to $\widetilde{P}(x_j)$, so $\widetilde{\varphi}_{j,1}(x) \in \widetilde{P}(x_j)$, and hence

$$\varphi_{j,1}(x) \in \lambda^{-1}\psi^{-1}(\widetilde{P}(x_j)), \tag{22.57}$$

by (22.37). In addition,

$$\begin{aligned}|\varphi_{j,1}(x) - x| &\leq \lambda^{-1}\Lambda|\widetilde{\varphi}_{j,1}(x) - \psi(\lambda x)| \leq \lambda^{-1}\Lambda\big[|\widetilde{\pi}_j(\psi(\lambda x)) - \psi(\lambda x)| + \eta\lambda r_j\big] \\ &\leq \lambda^{-1}\Lambda\,\mathrm{dist}(\psi(\lambda x), \widetilde{P}(x_j)) + \eta\Lambda^2 r_j \leq 2\Lambda^2\varepsilon r_j + \Lambda^2\eta r_j\end{aligned} \tag{22.58}$$

by (22.37) again and (22.51).

Since $x \in B_{j,int}$, we get that $|x - x_j| \leq (1-\tau)r_j$, and then (22.58) implies that $\varphi_{j,1}(x) \in B_j$, if $\varepsilon$ and $\eta$ are small enough compared to $\tau$. Thus $\varphi_{j,1}(x) \in B_j \cap \lambda^{-1}\psi^{-1}(\widetilde{P}(x_j))$.

But $\widetilde{P}(x_j)$ is a $d$-plane which is contained in the affine span of the face $\widetilde{F}_j = \psi(\lambda F(x_j)$ (see (22.9) or the description above (22.34)), so the definition (19.33) yields

$$\mathcal{H}^d(\varphi_{j,1}(E_k \cap B_{j,int})) \leq \mathcal{H}^d(B_j \cap \lambda^{-1}\psi^{-1}(\widetilde{P}(x_j))) \leq r_j^d A_{r_j}(x_j). \tag{22.59}$$

In addition, we only used (22.34) and (22.37) when $x \in L_i'$ for some $i$, and then (22.18) (together with (22.21)) says that $A_{r_j}(x_j) \leq \omega_d + \varepsilon$. Thus

$$\mathcal{H}^d(\varphi_{j,1}(E_k \cap B_{j,int})) \leq r_j^d A_{r_j}(x_j) \leq (\omega_d + \varepsilon) r_j^d \leq (1 + C\varepsilon)\,\omega_d r_j^d. \tag{22.60}$$

Since (22.56) was a better estimate, we shall remember that (22.60) holds in all cases. We group this with (22.55) and get that

$$\mathcal{H}^d(\varphi_{j,1}(E_k \cap B_j)) \leq C\tau r_j^d + (1 + C\varepsilon)\,\omega_d r_j^d \leq (1 + C\tau)\omega_d r_j^d \tag{22.61}$$

(if $\varepsilon$ is small enough).

We now return to (22.45) and give a lower bound for $\mathcal{H}^d(W_1(E_k))$. We claim that $E_k \cap B_{j,int} \subset W_1(E_k)$. Let $x \in E_k \cap B_{j,int}$ be given. If we used (22.32), $\varphi_{j,1}(x) = \pi_j(x) + \eta r_j v_j$ and $\pi_j(\varphi_{j,1}(x)) = \pi_j(x) + \eta r_j v_j \neq \pi_j(x)$ because we chose $v_j$ parallel to $P(x_j)$. Then $\varphi_{j,1}(x) \neq x$ and $x \in W_1(E_k)$. If instead we used (22.34) and (22.37), (22.34) yields $\widetilde{\varphi}_{j,1}(x) = \widetilde{\pi}_j(\psi(\lambda x)) + \eta\lambda r_j \widetilde{v}_j$, and now $\widetilde{\pi}_j(\widetilde{\varphi}_{j,1}(x)) = \widetilde{\pi}_j(\psi(\lambda x)) + \eta\lambda r_j \widetilde{v}_j \neq \widetilde{\pi}_j(\psi(\lambda x))$ because $\widetilde{v}_j$ is parallel to $\widetilde{P}_j(x)$; in this case $\widetilde{\varphi}_{j,1}(x) \neq \psi(\lambda x)$, hence $\varphi_{j,1}(x) \neq x$ and $x \in W_1(E_k)$. So $E_k \cap B_{j,int} \subset W_1(E_k)$, and (22.45) yields

$$\begin{aligned} \mathcal{H}^d(E_k \cap B_{j,int}) &\leq \mathcal{H}^d(W_1(E_k)) \leq M\mathcal{H}^d(\varphi_{j,1} W_1(E_k)) + 3h\Lambda^2 r_j^d \\ &\leq M\mathcal{H}^d(\varphi_{j,1}(E_k \cap B_j)) + 3h\Lambda^2 r_j^d \end{aligned} \tag{22.62}$$

by (22.46). By (22.54), (22.62), and (22.61),

$$\begin{aligned} \mathcal{H}^d(E_k \cap B_j) &\leq \mathcal{H}^d(E_k \cap B_{j,int}) + C\tau r_j^d \\ &\leq M\mathcal{H}^d(\varphi_{j,1}(E_k \cap B_j)) + 3h\Lambda^2 r_j^d + C\tau r_j^d \\ &\leq \big[M\omega_d + 3h\Lambda^2 + C\tau\big] r_j^d \\ &\leq \big[M\omega_d + 3h\Lambda^2 + C\tau\big](\omega_d - \varepsilon)^{-1}\mathcal{H}^d(E \cap B_j) \end{aligned} \tag{22.63}$$

by (22.15). We sum over $j$ and get that

$$\begin{aligned} \mathcal{H}^d(E_k \cap H) &\leq \mathcal{H}^d(E_k \cap \bigcup_{j \in I \cup J} B_j) \leq C\varepsilon + \sum_{j \in J} \mathcal{H}^d(E_k \cap B_j) \\ &\leq C\varepsilon + \big[M\omega_d + 3h\Lambda^2 + C\tau\big](\omega_d - \varepsilon)^{-1} \sum_{j \in J} \mathcal{H}^d(E \cap B_j) \\ &\leq C\varepsilon + \big[M\omega_d + 3h\Lambda^2 + C\tau\big](\omega_d - \varepsilon)^{-1}\mathcal{H}^d(E \cap V) \\ &\leq C\varepsilon + \big[M\omega_d + 3h\Lambda^2 + C\tau\big](\omega_d - \varepsilon)^{-1}[\mathcal{H}^d(E \cap H) + \varepsilon] \end{aligned} \tag{22.64}$$

by (22.27), (22.30), (22.63), the fact that the $B_j$, $j \in J$ are disjoint and contained in $V$ (by (22.14) and (22.21)), and finally (22.5).

This holds for all choices of small constants $\tau$ and $\varepsilon$, with $\varepsilon$ small enough (depending on $\tau$) and $k$ large enough (depending on $\tau$ and $\varepsilon$). We let $k$ tend to $+\infty$ in (22.64), notice that we can take $\tau$ and $\varepsilon$ as small as we want (depending on $E$, the $E_k$, and $H$), and get (22.4). This completes our proof of Lemma 22.3. $\square$

**Proof of Theorem 22.1.** We start with the case when the $E_k$ satisfy the assumptions of Theorem 21.3. In both of our three cases, the sets $E_k$ lie in some fixed $GSAQ(U, M, \delta, h)$, and we shall be able to apply Theorem 10.8 and Lemma 22.3.

In our first case when the $E_k$ are $A_+$-almost minimal, we can take $h = 0$ and $M = 1 + h(\delta)$ as close to 1 as we want (because (21.2) holds), only at the price of choosing $\delta$ small. We do that, get (22.4) with $h = 0$ and $M$ close to 1, let $M$ tend to 0, and get (22.2).

In the second case when the $E_k$ are $A$-almost minimal, we take $M = 1$ and $h = h(\delta)$, which is also as small as we want by (21.2), apply Lemma 22.3, let $\delta$ tend to 0, and get (22.2) as above.

In the third case, we just need to observe that the $E_k$ are also $A$-almost minimal, by the easy part of Proposition 20.9, and use the second case.

So we may assume now that the $E_k$ satisfy the assumptions of Corollary 21.15. Because of (21.10) we can still apply the results of Section 10, with some acceptable choice of $M$ and $h > 0$, but we do not want to use Lemma 22.3 with these constants, because we want to get rid of $M$ and $h$.

Instead we want to use the proof of Lemma 22.3, change a little bit the definitions and accounting at the end, and use our asymptotic minimality assumption (21.13) or (21.14).

The main difference will be that we want to group some of the families $\varphi_{j,t}$. Recall that (21.13) and (21.14) come with some $\delta > 0$. Let us pick a maximal family $z_l$, $1 \leq l \leq L$, of points of $H$, so that the $y_l$ lie at mutual distances $\delta/10$ from each other. Then do the construction of Lemma 22.3, with $\rho_0 < 10^{-2}\Lambda^{-2}\delta$ (in addition to the constraint in (22.28)). This gives, in particular, a family of balls $B_j$, $j \in J$, and for $j \in J$ a family $\varphi_{j,t}$.

For each $l$, denote by $J_0(l)$ the set of indices $j \in J$ such that $|x_j - y_l| \leq \delta/10$. Each $j \in J$ lies in some $J_0(l)$, because otherwise we could add $x_j \in H$ to our collection of points $y_l$ and that collection would not be maximal. We prefer to have disjoint sets, so we set

$$J(l) = J_0(l) \setminus \bigcup_{m<l} J(m) \ \text{ for } 1 \leq l \leq L. \tag{22.65}$$

Then we fix $l$, and define a new family $\{\varphi_t\}$, $0 \leq t \leq 1$, by

$$\varphi_t(x) = x \text{ for } 0 \leq t \leq 1 \text{ and } x \in U \setminus \bigcup_{j \in J(l)} B_j \tag{22.66}$$

and

$$\varphi_t(x) = \varphi_{j,t}(x) \text{ for } 0 \leq t \leq 1 \text{ and } x \in B_j. \tag{22.67}$$

The $B_j$ are disjoint, so there in no ambiguity in (22.67). Also, the $\varphi_t$ are continuous across the natural boundaries $\partial B_j$, $j \in J(l)$, because the $\varphi_{j,t}$ are, and by (22.33). Because of this, the $\varphi_t$ satisfy the conditions (1.4) and (1.8).

We first need to check that the family $\{\varphi_t\}$ is $\delta$-admissible for $E_k$ (see the definition below (21.10)). Let $W_t(E_k)$ and $\widehat{W}(E_k)$ be as in (21.11) and (21.12). If $x \in W_t(E_k)$ for some $t \in [0,1]$, then $x \in B_j$ for some $j \in J(l)$ (by (22.66)), and then $\varphi_t(x) = \varphi_{j,t}(x) \in B(x_j, 3\Lambda^2 r_j)$, by (22.39). This means that

$$\widehat{W}(E_k) \subset \bigcup_{j \in J(l)} B(x_j, 3\Lambda^2 r_j) \subset B(y_l, \delta/3) \tag{22.68}$$

because $r_j \leq \rho_0 \leq 10^{-2}\Lambda^{-2}\delta$ and $|x_j - y_l| \leq \delta/10$. Thus (1.4) and (1.5) hold for the $E_k$, with respect to the ball $B = \overline{B}(y_l, \delta/3)$. Also, the $\overline{B}(x_j, 3\Lambda^2 r_j)$ are still contained in $V \subset U$ (by (22.14) and (22.21)), so $\widehat{W}(E_k) \subset\subset U$ and $\{\varphi_t\}$ is $\delta$-admissible for $E_k$. We thus get that for each $\varepsilon > 0$ (we can keep the same one as before), we can find $k_0 \geq 0$ such that for $k \geq k_0$,

$$\mathcal{H}^d(W_1(E_k)) \leq \mathcal{H}^d(\varphi_1(W_1(E_k))) + \varepsilon \tag{22.69}$$

(as in (21.13) or

$$\mathcal{H}^d(E_k \setminus \varphi_1(E_k)) \leq \mathcal{H}^d(\varphi_1(E_k) \setminus E_k) + \varepsilon. \tag{22.70}$$

(as in (21.14). We first assume that (22.69) holds for $k \geq k_0$, because it is closer to what we had for Lemma 22.3. Then

$$\begin{aligned} \mathcal{H}^d(\varphi_1(W_1(E_k))) &\leq \mathcal{H}^d(\varphi_1(E_k \cap \bigcup_{j\in J(l)} B_j)) \leq \sum_{j\in J(l)} \mathcal{H}^d(\varphi_1(E_k \cap B_j)) \\ &= \sum_{j\in J(l)} \mathcal{H}^d(\varphi_{1,j}(E_k \cap B_j)) \leq \sum_{j\in J(l)} (1+C\tau)\omega_d r_j^d \end{aligned} \tag{22.71}$$

because $W_1(E_k) \subset \bigcup_{j\in J(l)} B_j$ by (22.66), and by (22.67) and (22.61).

On the other hand, $W_1(E_k)$ contains $E_k \cap B_{j,int}$ for $k$ large, by the proof above (22.62); hence

$$\begin{aligned} \sum_{j\in J(l)} \mathcal{H}^d(E_k \cap B_j) &\leq \sum_{j\in J(l)} \left[\mathcal{H}^d(E_k \cap B_{j,int}) + C\tau r_j^d\right] \\ &\leq \mathcal{H}^d(E_k \cap \bigcup_{j\in J(l)} B_{j,int}) + C\sum_{j\in J(l)} \tau r_j^d \leq \mathcal{H}^d(W_1(E_k)) + C\sum_{j\in J(l)} \tau r_j^d \\ &\leq \mathcal{H}^d(\varphi_1(W_1(E_k))) + \varepsilon + C\sum_{j\in J(l)} \tau r_j^d \leq \varepsilon + \sum_{j\in J(l)} (1+C\tau)\omega_d r_j^d \end{aligned} \tag{22.72}$$

by (22.54), because the $B_j$ are disjoint, and by (22.69) and (22.71). We sum this over $l$, use the fact that $J$ is the disjoint union of the $J(l)$, and get that

$$\sum_{j\in J} \mathcal{H}^d(E_k \cap B_j) \leq L\varepsilon + \sum_{j\in J} (1+C\tau)\omega_d r_j^d\,; \tag{22.73}$$

the extra $L$ will not disturb, because it is fixed as soon as we know $\delta$, and we can choose $\varepsilon$ later. We proceed as in (22.64):

$$\begin{aligned} \mathcal{H}^d(E_k \cap H) &\leq \mathcal{H}^d(E_k \cap \bigcup_{j\in I\cup J} B_j) \leq C\varepsilon + \sum_{j\in J} \mathcal{H}^d(E_k \cap B_j) \\ &\leq C\varepsilon + L\varepsilon + \sum_{j\in J} (1+C\tau)\omega_d r_j^d \\ &\leq C\varepsilon + L\varepsilon + \sum_{j\in J} (1+C\tau)\omega_d(\omega_d - \varepsilon)^{-1}\mathcal{H}^d(E \cap B_j) \\ &\leq C\varepsilon + L\varepsilon + (1+C\tau)\frac{\omega_d}{\omega_d - \varepsilon}\mathcal{H}^d(E \cap V) \\ &\leq C\varepsilon + L\varepsilon + (1+C\tau)\frac{\omega_d}{\omega_d - \varepsilon}[\mathcal{H}^d(E \cap H) + \varepsilon] \end{aligned} \tag{22.74}$$

by (22.27), (22.30), (22.73), (22.15), the fact that the $B_j$, $j \in J$ are disjoint and contained in $V$ (by (22.14) and (22.21)), and (22.5).

This is true for $\tau$ small enough, $\varepsilon$ small enough (depending on $\tau$ as well), and $k$ large (depending on both). We let $k$ tend to$+\infty$, and then let $\varepsilon$ and $\tau$ tend to 0, and we get (22.2).

Now assume that (22.70) holds. The simplest is to show that (22.69) holds as well. Set $W = W_1(E_k)$ and observe that

$$W \setminus \varphi_1(W) \subset E_k \setminus \varphi_1(E_k) \tag{22.75}$$

because if $x \in W \setminus \varphi_1(W)$, it lies in $E_k$, does not lie in $\varphi_1(W)$, and does not lie in $\varphi_1(E_k \setminus W)$ either, because $\varphi_1(y) = y \notin W$ when $y \in E_k \setminus W$. Similarly,

$$\varphi_1(E_k) \setminus E_k \subset \varphi_1(W) \setminus W \tag{22.76}$$

because if $x \in \varphi_1(E_k) \setminus E_k$ and $y \in E_k$ is such that $\varphi_1(y) = x$, then $y \in W$ (otherwise, $x = \varphi_1(y) = y \in E_k$) and of course $x \notin W$ (because $W \subset E_k$ and $x \notin E_k$). Then

$$\begin{aligned}
\mathcal{H}^d(W) &= \mathcal{H}^d(W \cap \varphi_1(W)) + \mathcal{H}^d(W \setminus \varphi_1(W)) \\
&\leq \mathcal{H}^d(W \cap \varphi_1(W)) + \mathcal{H}^d(E_k \setminus \varphi_1(E_k)) \\
&\leq \mathcal{H}^d(W \cap \varphi_1(W)) + \mathcal{H}^d(\varphi_1(E_k) \setminus E_k) + \varepsilon \\
&\leq \mathcal{H}^d(W \cap \varphi_1(W)) + \mathcal{H}^d(\varphi_1(W) \setminus W) + \varepsilon = \mathcal{H}^d(\varphi_1(W)) + \varepsilon
\end{aligned} \tag{22.77}$$

by (22.75), (22.70), and (22.76), as needed for (22.69). This completes our proof of (22.2) when (22.70) holds; Theorem 22.1 follows. □

## 23. Limits of quasiminimal and almost minimal sets in variable domains.

The main result of this section is a variant of Theorem 10.8 where we allow the domain $U$ and the boundary pieces $L_i$ to vary slightly along the sequence. We shall not try to obtain an optimal result here (this would probably involve following the long proof carefully), and instead we shall state a result (Theorem 23.8) that we can easily deduce from Theorem 10.8 by a change of variable.

Let us explain the notation for Theorem 23.8. We are given an open set $U$ (the limit) and boundary pieces $L_j$, $0 \leq j \leq j_{max}$, and we assume as in (10.1) that

(23.1) the Lipschitz assumption are satisfied in $U$.

But we also give ourselves a sequence $\{U_k\}$ of open sets, and for each $k \geq 0$ a collection of boundary pieces $L_{j,k}$, $0 \leq j \leq j_{max}$. We shall make our life simpler and assume that $U_k$ and the $L_{j,k}$ are parameterized by $U$ and the $L_j$, using a single bilipschitz mapping $\xi_k : U \to U_k$. That is,

$$U_k = \xi_k(U) \text{ and } L_{j,k} = \xi_k(L_j) \text{ for } 0 \leq j \leq j_{max}. \tag{23.2}$$

In addition, we assume that the $\xi_k$ become optimally bilipschitz, in the sense that there exist constants $\eta_k \geq 1$ such that

$$\xi_k \text{ is } \eta_k\text{-bilipschitz on } U \text{ and } \lim_{k\to+\infty} \eta_k = 1. \tag{23.3}$$

We also assume that

$$\lim_{k\to+\infty} \xi_k(x) = x \text{ for } x \in U. \tag{23.4}$$

These are quite strong assumptions on our sequence, but our main example will be a blow-up sequence at a point of an initial domain where each $L_j$ has a tangent cone, in which case the $\xi_k$ can be constructed by hand. See Section 24.

Now we give ourselves a sequence $\{E_k\}$ of quasiminimal sets. We assume that the following properties hold for some choice of constants $M \geq 1$, $\delta \in (0, +\infty]$, and $h > 0$. First, each $E_k$ is a relatively closed subset of the corresponding set $U_k$, $E_k$ is coral (as in (10.3) and Definition 3.1), and

$$E_k \in GSAQ(U_k, M, \delta, h), \tag{23.5}$$

where of course we define $GSAQ(U_k, M, \delta, h)$ relative to the sets $L_{j,k}$, $0 \leq j \leq j_{max}$. We assume, as in (10.4), that for some relatively closed set $E \subset U$,

$$\lim_{k\to+\infty} E_k = E \text{ locally in } U. \tag{23.6}$$

Since the $E_k$ are contained in slightly different domains $U_k$, let us say what this means: for each compact set $H \subset U$ and each $\varepsilon > 0$, we can find $k_0 \geq 0$ such that for $k \geq k_0$,

$$\operatorname{dist}(x, E_k) \leq \varepsilon \text{ for every } x \in E \cap H \text{ and } \operatorname{dist}(x, E) \leq \varepsilon \text{ for every } x \in E_k \cap H. \tag{23.7}$$

When the rigid assumption does not hold, we also assume that (10.7) or (19.36) holds (in $U$, for the $L_j$).

**Theorem 23.8.** *Let $U$, $\{U_k\}$, the $L_j$, the $L_{j,k}$, and $\{E_k\}$ satisfy all the conditions above, including (10.7) or (19.36) when the $L_j$ don't satisfy the rigid assumption. Also assume that $h$ is small enough, depending on $M$, $n$, and the constant $\Lambda$ that comes from (23.1). Then $E$ is coral, and*

$$E \in GSAQ(U, M, \delta, h), \tag{23.9}$$

*with the same constants $M$, $\delta$, and $h$ as in (23.5).*

We first prove the theorem, and then comment later. Since we want to reduce to a fixed domain, we consider the sets

$$\widetilde{E}_k = \xi_k^{-1}(E_k) \subset U, \tag{23.10}$$

and we want to apply Theorem 10.8 to the sequence $\{\widetilde{E}_k\}$. Since $\xi_k$ is bilipschitz, $\widetilde{E}_k$ is closed in $U$ and coral. We claim that

$$\widetilde{E}_k \in GSAQ(U, \eta_k^{2d} M, \eta_k^{-1}\delta, \eta_k^{2d} h), \tag{23.11}$$

where $GSAQ$ is defined in terms of the boundaries $L_j$. The proof is the same as for Proposition 2.8, so we won't repeat all the verifications, but just give the outline. For each one parameter family of mappings $\widetilde{\varphi}_t : \widetilde{E}_k \to \mathbb{R}^n$, $0 \leq t \leq 1$, that satisfies (1.4)-(1.8) with a radius $\widetilde{r} < \eta_k^{-1}\delta$ and (2.4), we observe that $\widetilde{\varphi}_t(\widetilde{E}_k) \subset U$ by (2.4), and so we can define mappings $\varphi_t : E_k \to U_k$ by

$$\varphi_t = \xi_k \circ \widetilde{\varphi}_t \circ \xi_k^{-1}. \tag{23.12}$$

It is easy to see that the properties (1.4)-(1.8) and (2.4) for the $\varphi_t$ (relative to $U_k$ and the $L_{j,k}$, and with a radius $r \leq \eta_k \widetilde{r}$ smaller than $\delta$) follow from their counterpart for the $\widetilde{\varphi}_t$, so we can apply (2.5) to $\varphi_1$; this yields

$$\mathcal{H}^d(W_1) \leq M\mathcal{H}^d(\varphi_1(W_1)) + hr^d, \tag{23.13}$$

with $W_1 = \big\{x \in E_k\,;\, \varphi_1(x) \neq x\big\}$. The analogue of $W_1$ for the $\widetilde{\varphi}_t$ is

$$\widetilde{W}_1 = \big\{x \in \widetilde{E}_k\,;\, \widetilde{\varphi}_1(x) \neq x\big\} = \xi_k^{-1}(W_1), \tag{23.14}$$

so (23.13) yields

$$\mathcal{H}^d(\widetilde{W}_1) \leq \eta_k^d \mathcal{H}^d(W_1) \leq \eta_k^d M\mathcal{H}^d(\varphi_1(W_1)) + \eta_k^d h r^d \leq \eta_k^{2d} M\mathcal{H}^d(\widetilde{\varphi}_1(\widetilde{W}_1)) + \eta_k^{2d} h \widetilde{r}^d, \tag{23.15}$$

as needed for (23.11).

We shall want to check that

$$\lim_{k\to+\infty} \widetilde{E}_k = E \ \text{ locally in } U, \tag{23.16}$$

but we start with simple consequences of (23.3) and (23.4). First notice that (23.4) implies the apparently stronger fact that

$$\lim_{k\to+\infty} \xi_k(x) = x \text{ uniformly on every compact subset of } U, \tag{23.17}$$

simply because (23.3) says that the $\xi_k$ are Lipschitz on $U$, with uniform bounds. Next we check that for every compact set $H \subset U$, we can find a compact set $K \subset U$ and an integer $k_0$ such that

$$H \subset \xi_k(K) \ \text{ for } k \geq k_0. \tag{23.18}$$

Set $r = \frac{1}{4}\operatorname{dist}(H, \mathbb{R}^n \setminus U) > 0$ and cover $H$ by a finite number of balls $B(x_i, r)$. For each $i$, we (23.17) says that for $k$ large, $|\xi_k(y) - y| \leq r$ for $y \in \partial B(x_i, 3r)$. For such $k$, and each

$x \in B(x_i, r)$, the mapping from $\partial B(x_i, 3r)$ to the unit sphere which maps $z \in \partial B(x_i, 3r)$ to $\frac{\xi_k(z)-x}{|\xi_k(z)-x|}$ is well defined (because $|\xi_k(z) - x| \geq |z - x| - r \geq r$), continuous, and of degree 1 because we can easily find a homotopy from this map to to the map $z \to \frac{|z-x|}{|z-x|}$ (among continuous mappings : $\partial B(x_i, 3r) \to \partial B(0, 1)$). Then there is no homotopy from it to a constant, which implies that $x \in \xi_k(B(x_i, 3r))$ (because otherwise we could use the mappings $z \to \frac{\xi_k(z_t)-x}{|\xi_k(z_t)-x|}$, with $z_t = tx_i + (1-t)(z - x_i)$. Set $K = \cup_i \overline{B}(x_i, 3r)$; we just proved that $\xi_k(K)$ contains all the $B(x_i, r)$, and (23.8) follows.

We are now ready to check that $\{\widetilde{E}_k\}$ converges to $E$, as in (23.16). Let $H \subset U$ be compact, set $d_H = \mathrm{dist}(H, \mathbb{R}^n \setminus U)$, and let $\varepsilon \in (0, d_H/2)$ be given. First we want to show that if $k$ large enough, then

$$\text{for every } x \in E \cap H \text{ we can find } \widetilde{y}_k \in \widetilde{E}_k \text{ such that } |x - y_k| \leq 2\varepsilon. \tag{23.19}$$

By (23.7), we can find $y_k \in E_k$ such that $|x - y_k| \leq \varepsilon$. Notice that $y_k$ lies in the compact set $H_1 = \{y \in \mathbb{R}^n\,;\, \mathrm{dist}(y, H) \leq d_H/2\}$. By (23.18), there is a compact set $K$ such that $\xi_k(K)$ contains $H_1$ for $k$ large.

Set $\widetilde{y}_k = \xi_k^{-1}(y_k)$. We already knew that $\xi_k^{-1}(y_k)$ is defined and lies in $\widetilde{E}_k$, because $E_k \subset U_k = \xi_k(U)$ and $\widetilde{E}_k = \xi_k^{-1}(E_k)$; now we also know that $\widetilde{y}_k \in K$ for $k$ large, and (23.17) says that for $k$ large, $|\widetilde{y}_k - y_k| = |\widetilde{y}_k - \xi_k(\widetilde{y}_k)| \leq \varepsilon$. Thus $\widetilde{y}_k \in \widetilde{E}_k$ and $|x - y_k| \leq 2\varepsilon$; this proves (23.19).

Conversely, if $k$ is large enough, then for each $x_k \in \widetilde{E}_k \cap H$ (23.17) says that $|\xi_k(x_k) - x_k| \leq \varepsilon$; but $\xi_k(x_k) \in E_k \in H_1$ (by (23.10) and the definition of $H_1$), and (23.7) gives (again if $k$ is large enough) a point $x \in E$ such that $|x - \xi_k(x_k)| \leq \varepsilon$. Thus $|x - x_k| \leq 2\varepsilon$. This completes our proof of (23.16).

We may now return to Theorem 23.8. If we prove that

$$E \in GSAQ(U, M', \delta', h') \tag{23.20}$$

for every choice of $M' > M$, $\delta' < \delta$, and $h' > h$, the desired conclusion (23.9) will follow; this comes from the way we defined $GSAQ(U, M, \delta, h)$ in Definition 2.3. So it is enough to show that the $\widetilde{E}_k$ satisfy the assumptions of Theorem 10.8, with any such choice of $M', \delta', h'$, and where the limit in (10.4) is still $E$.

But (10.1) is the same as (23.1), (10.2) follows from (23.11) (if we use the fact that $\eta_k$ tends to 1 by (23.3), and restrict to $k$ large), and (10.4) follows from (23.16). Since we also assumed (10.7) or (19.36) if the rigid assumption does not hold, we can apply Theorem 10.8 or Remark 19.52 and get (23.20). Theorem 23.8 follows. $\square$

We end this section with a few comments and extensions of Theorem 23.8.

**Remark 23.21.** Our bilipschitz assumption (23.3) could easily be made more local. That is, if instead of (23.3) we assume that each $\xi_k : U \to U_k$ is bilipschitz, and that for each compact set $K \subset U$ and each $\eta > 1$ we can find $k_0 \geq 0$ such that

$$\eta^{-1}|x - y| \leq |\xi_k(x) - \xi_k(y)| \leq \eta|x - y| \quad \text{for } k \geq k_0 \text{ and } x, y \in K, \tag{23.22}$$

we still get the same conclusion. This is easy, because given a competitor (i.e., a one parameter family $\{\varphi_t\}$ with the usual conditions), we can restrict our attention to a relatively compact open set $V$ such that $\widehat{W} \subset V \subset\subset U$, and then apply Theorem 23.8 with the open set $V$.

To be honest, even though we claim that the proof still works, the situation is a little more complicated than this, because $V$, together with the boundary sets $L_j \cap V$, may not satisfy the Lipschitz assumption as in (23.1). This gets better if we choose $V$, possibly a little larger, so that $\psi(\lambda V)$ is a ball centered at the origin (and with a radius close to 1). Without saying more, we can observe that our proof of Theorem 10.8 still works in that case (we never used the fact that the radius of $B(0,1)$ is a dyadic number). Or, on a more formal level, we can also make sure that $\psi(\lambda V) = B(0, 1-2^{-m})$ for some integer $m$, and observe that the sets $(1-2^{-m})^{-1}\psi(L_j \cap V)$ satisfy the rigid assumption, although perhaps with a grid with a $2^m$ times smaller mesh. This has a small incidence on the statement of Theorem 10.8, so we can get an acceptable formal proof this way.

**Remark 23.23.** The lower semicontinuity of $\mathcal{H}^d$, as in Theorem 10.97, is still true under the assumptions of Theorem 23.8. That is, under the assumptions of Theorem 23.8, we also have that

$$\mathcal{H}^d(E \cap V) \leq \liminf_{k \to +\infty} \mathcal{H}^d(E_k \cap V) \ \text{ for every open set } V \subset U. \tag{23.24}$$

(as in (10.98)). Indeed, for this it is enough to show that if $V \subset U$ is open, and if $H$ is any compact subset of $V$,

$$\mathcal{H}^d(E \cap H) \leq \liminf_{k \to +\infty} \mathcal{H}^d(E_k \cap V), \tag{23.25}$$

just because we can write $\mathcal{H}^d(E \cap V) = \lim_{l \to +\infty} \mathcal{H}^d(E \cap H_l)$, where $\{H_l\}$ is an increasing sequence of compact subsets of $V$.

Let $V$ and $H$ be given, and let $W$ be an open set that contains $H$ and is relatively compact in $V$. We have seen in the proof of Theorem 23.8 that the sets $\widetilde{E}_k \cap V = \xi_k^{-1}(E_k) \cap V$ satisfy the assumptions of Theorem 10.8 and Theorem 10.97 for some acceptable choices of constants (i.e., with $h$ small enough), and with the same limit $E$, so Theorem 10.97 yields

$$\mathcal{H}^d(E \cap H) \leq \liminf_{k \to +\infty} \mathcal{H}^d(\widetilde{E}_k \cap W). \tag{23.26}$$

For $k$ large, $\xi_k(\widetilde{E}_k \cap W) = \xi_k(\widetilde{E}_k) \cap \xi_k(W) \subset E_k \cap V$ (by (23.10) and because (23.17) says that the $\xi_k$ converge to the identity uniformly on $\overline{W}$), and

$$\mathcal{H}^d(\widetilde{E}_k \cap W) \leq \eta_k^d \mathcal{H}^d(\xi_k(\widetilde{E}_k \cap W)) \leq \eta_k^d \mathcal{H}^d(E_k \cap V) \tag{23.27}$$

by (23.3); now (23.25) and then (23.24) follow by letting $k$ tend to $+\infty$.

**Remark 23.28 (limits of almost minimal sets).** Our statements about limits of almost minimal sets, namely Theorem 21.3 and Corollary 21.15, also hold in the present setting.

We start with Theorem 21.3. If, in Theorem 23.8 above, we replace the quasiminimality assumption (23.5) with the assumption that

$$\text{(23.29)}\qquad \begin{gathered} E_k \text{ is an almost minimal set in } U_k, \text{ with the sliding} \\ \text{conditions given by the } L_{j,k} \text{ and the gauge function } h, \end{gathered}$$

where $h$ is a gauge function such that (21.1) and (21.2) hold (otherwise, see Remarks 21.6 and 21.7), and one type of almost minimal set ($A_+$, $A$, or $A'$) is chosen. The conclusion is then, as in Theorem 21.3, that the limit set $E$ is an almost minimal set in $U$, with the sliding conditions given by the $L_j$, the gauge function $h$, and the same type ($A_+$, $A$, or $A'$). The proof consists in following our short proof of Theorem 21.3, and applying Theorem 23.8 instead of Theorem 10.8.

Similarly, if we replace the assumption (23.5) with the assumptions (21.10)-(21.14) (and taken, for $E_k$, in the domain $U_k$ and relative to the boundary sets $L_{j,k}$), we get the same conclusion as in Corollary 21.15, namely that $E$ is a coral local minimal set in $U$, that satisfies (21.16). Once again, we just follow the proof of Corollary 21.15.

**Remark 23.30 (upper semicontinuity of $\mathcal{H}^d$).** The upper semicontinuity results of Section 22 can also be generalized in the context of slowly changing domains. We start with an extension of Lemma 22.3.

**Lemma 23.31.** *Let $U$, the $L_i$, the $U_k$ the $L_{j,k}$, the $\{E_k\}$, and the limit $E$ satisfy the assumptions of Theorem 10.8. Then for every compact set $H \subset U$,*

$$\text{(23.32)}\qquad (1+Ch)M\mathcal{H}^d(E\cap H) \geq \limsup_{k\to+\infty} \mathcal{H}^d(E_k\cap H),$$

*with a constant $C$ that depends only on $n$, $M$, and $\Lambda$.*

We proceed and as in Theorem 23.8, to deduce Lemma 23.31 from Lemma 22.3 and a change of variable. We already proved near (23.20) that for each choice of $M' > M$, $\delta' < \delta$, and $h' > h$, the end of the sequence $\{\widetilde{E}_k\}$ satisfies the hypotheses of Theorem 10.8 or Remark 19.52. Then (if $h$ is small enough) they also satisfy the conclusion of Lemma 22.3: for each compact set $K \subset U$,

$$\text{(23.33)}\qquad (1+Ch)M'\mathcal{H}^d(E\cap K) \geq \limsup_{k\to+\infty} \mathcal{H}^d(\widetilde{E}_k\cap K),$$

with a constant $C$ which we can still bound in terms of $n$, $M$, and $\Lambda$ (the constant $C$ that is associated to $2M$ also works for all $M' \leq 2M$).

Return to (23.32), let $H \subset U$ be compact, and let $\varepsilon > 0$ be given. Pick a compact set $K \subset U$ such that $H$ is contained in the interior of $K$ and $\mathcal{H}^d(E\cap K) \leq \mathcal{H}^d(E\cap H)+\varepsilon$. Observe that for $k$ large,

$$\text{(23.34)}\qquad \xi_k(E_k\cap H) = \xi_k(E_k)\cap\xi_k(H) = \widetilde{E}_k\cap\xi_k(H) \subset \widetilde{E}_k\cap K$$

by (23.10) and because by (23.17) the $\xi_k$ converge to the identity uniformly on $H$. Thus for $k$ large,

$$\begin{aligned} \mathcal{H}^d(E_k \cap H) &\leq \eta_k^d \mathcal{H}^d(\xi_k(E_k \cap H)) \leq \eta_k^d \mathcal{H}^d(\widetilde{E}_k \cap K) \\ &\leq \eta_k^d[(1+Ch)M'\mathcal{H}^d(E \cap K) + \varepsilon] \\ &\leq \eta_k^d[(1+Ch)M'(\mathcal{H}^d(E \cap H) + \varepsilon) + \varepsilon] \end{aligned} \tag{23.35}$$

by (23.34). We let $k$ tend to $+\infty$, use (23.3), let $\varepsilon$ tend to 0, and get (23.29). We take the *limsup* of this, recall that $\eta_k$ tends to 1 by (23.3), then observe that $\varepsilon$ and $M' - M$ are arbitrarily small, and get (23.32) □

Here is a generalization of Theorem 22.1.

**Lemma 23.36.** *Let $U$, the $L_i$, the $U_k$ the $L_{j,k}$, the $\{E_k\}$, and $E$ satisfy the same assumptions as in Theorem 21.3 and Corollary 21.15, but modified as in Remark 23.28. Then for every compact set $H \subset U$,*

$$\limsup_{k \to +\infty} \mathcal{H}^d(E_k \cap H) \leq \mathcal{H}^d(E \cap H). \tag{23.37}$$

In the case of Theorem 21.3, we can just follow the proof of Theorem 21.3, and replace Lemma 22.3 with Lemma 23.31 when needed. In the case of Corollary 21.15, the simplest seems to show that the sets $\widetilde{E}_k$ of (23.10) satisfy the assumption of Corollary 21.15, and then compute as in Lemma 23.31. We claim that the verifications are quite similar to what we did for Theorem 23.8 and Lemma 23.31, and we skip them. □

## 24. Blow-up limits.

In this section we apply the results of Section 23 to the case of blow-up limits of an almost minimal set $E$, at a point $x_0 \in E$ near which the boundary pieces $L_j$ behave in a roughly $C^1$ way.

We fix an open set $U \subset \mathbb{R}^n$, boundary pieces $L_j$, $0 \leq j \leq j_{max}$, a point $x_0 \in U$, and a quasiminimal (or almost minimal) set $E \subset U$. We shall systematically assume that

$$U \text{ and the } L_j \text{ satisfy the Lipschitz assumption} \tag{24.1}$$

(see Definition 2.7), and things will be more interesting when $E$ is assumed to be coral and we take $x_0 \in E$.

We are also given a sequence $\{r_k\}$, with

$$\lim_{k \to +\infty} r_k = 0, \tag{24.2}$$

along which we want to define a blow up. We define the sets $E_k$ by

$$E_k = r_k^{-1}(E - x_0) = \big\{z \in \mathbb{R}^n\, ;\, x_0 + r_k z \in E\big\}. \tag{24.3}$$

A simple computation shows that

$$E_k \in QSAQ(U_k, M, r_k^{-1}\delta, h) \text{ if } E \in QSAQ(U, M, \delta, h), \tag{24.4}$$

with $U_k = r_k^{-1}(U - x_0)$, and where on $U_k$ we use the boundary pieces

$$L_{j,k} = r_k^{-1}(L_j - x_0),\ 0 \leq j \leq j_{max}. \tag{24.5}$$

Similarly, if $E$ is almost minimal in $U$, with the sliding conditions coming from the $L_j$, and the gauge function $h$, then $E_k$ is almost minimal in $U_k$, with the sliding conditions coming from the $L_{j,k}$, the gauge function $h_k$ defined by $h_k(t) = h(tr_k)$, and the same type of almost minimality ($A_+$, $A$, or $A'$) as $E$; see Definition 20.2. Notice that, by (24.2), the sets $U_k$ converge to $\mathbb{R}^n$.

A blow-up sequence for $E$ at the point $x_0$ is any sequence $\{E_k\}$ defined by (24.3), under the condition (24.2). In some cases, the sequence converges, i.e., there is closed set $E_\infty \subset \mathbb{R}^n$ such that

$$E_\infty = \lim_{k \to +\infty} E_k \ \text{ locally in } \mathbb{R}^n. \tag{24.6}$$

When this is the case, i.e., when (24.6) holds for some sequence $\{r_k\}$ that tends to 0, we say that $E_\infty$ is a blow-up limit of $E$ at $x_0$. Of course, different sequences may yield different blow-up limits $E_\infty$, even though in some cases one can prove that $E$ has only one blow-up limit at $x_0$.

By general compactness arguments, we can always extract, from any blow-up sequence for $E$ at the point $x_0$, a convergent subsequence. Thus $E$ always has at least one blow-up limit at $x_0$. We shall only consider the case when $x_0 \in E$ (because otherwise $E_\infty = \emptyset$) and $E$ is coral (because otherwise the fuzzy sets $E_k \setminus E_k^*$ could just tend to anything). When in addition, $E \in QSAQ(U, M, \delta, h)$ with a small enough $\delta$, we know that $E$ is locally Ahlfors-regular, and so $E_\infty$ is Ahlfors-regular too, so it cannot be too bad.

Our intention is to prove that under reasonable assumptions on the $L_j$ near $x_0$, $E_\infty$ is quasiminimal and sometimes minimal, with sliding boundary conditions coming from limit sets $L_j^0$ that we want to define now.

Let us forget about $E$ for a moment, and restrict our attention to the sets $L_j$. We assume that for each $0 \leq j \leq j_{max}$ such that $x_0 \in L_j$, there is a closed set $L_j^0$ such that

$$\text{the sets } L_{j,k} = r_k^{-1}(L_j - x_0) \text{ converge to } L_j^0, \text{ locally in } \mathbb{R}^n. \tag{24.7}$$

This is a weak way of asking for a tangent set at $x_0$, at least along the sequence $\{r_k\}$. When $x_0 \notin L_j$, it is easy to see that the $L_{j,k}$ go away to infinity, and we can always set $L_j^0 = \emptyset$.

This condition will not be enough for us, we want to say more uniform about the way the $L_{j,k}$ converge to the $L_j^0$. The main point of the following definition is that it is just what we need to apply Theorem 23.8, and its conditions should still be easy to check in simple concrete situations. Otherwise it is a little too complicated, and we shall partially address this point later.

**Definition 24.8.** *Suppose the Lipschitz assumption (24.1) holds, and let $x_0 \in U$ and a sequence $\{r_k\}$ that tends to $0$ (as in (24.2)) be given. We say that the* configuration *of $L_j$* is flat *at $x_0$,* along the sequence *$\{r_k\}$, if we can find closed sets $L_j^0$, $0 \le j \le j_{max}$, such that (24.7) holds when $x_0 \in L_j$, and for each radius $R > 0$, numbers $\eta_k \ge 1$ such that $\lim_{k\to+\infty} \eta_k = 1$, and bilipschitz mappings $\xi_k : B(0,R) \to \xi_k(B(0,R)$ such that $\xi_k(0) = 0$,*

$$\rho_k^{-1}|x-y| \le |\xi_k(x)-\xi_k(y)| \le \rho_k|x-y| \text{ for } x,y \in B(0,R), \tag{24.9}$$

$$L_{j,k} \cap B(0,\rho_k^{-1}R) \subset \xi_k(L_j^0 \cap B(0,R)) \subset L_{j,k} \cap B(0,\rho_k R) \tag{24.10}$$

*for $0 \le j \le j_{max}$ and $k$ large enough, and*

$$\lim_{k\to+\infty} \xi_k(x) = x \text{ for } x \in B(0,R). \tag{24.11}$$

Probably a more reasonable notion would be that the configuration of $L_j$ is flat at $x_0$, meaning along any sequence $\{r_k\}$ that tends to 0. Then it would not be too hard to check that the $L_j^0$ are cones, and don't depend on the sequence $\{r_k\}$. This corresponds more to the usual notion of being $C^1$ at $x_0$.

The main defect of Definition 24.8 is that it concerns the whole configuration of the sets $L_j$, ie., both the faces that compose them and their relative positions. For the existence of limits $L_j^0$ as in (24.7), this is not a problem and we can check it face by face. That is, if for each of the faces $F$ that compose an $L_j$, we can find a closed set $F_0$ such that

$$\text{the } F^k = r_k^{-1}(F - x_0) \text{ converge to } F^0\text{, locally in } \mathbb{R}^n, \tag{24.12}$$

(with $F^0 = \emptyset$ if $x_0 \notin F$), then (24.7) holds, with a set $L_j^0$ which is just the union of the $F^0$, where $F$ runs along the faces of $L_j$ that contain $x_0$.

To compensate this defect, we shall give in Proposition 24.35 a sufficient condition for the existence of the $\xi_k$ in Definition 24.8, that can be checked face by face, so that we don't have to worry about how the different faces (if they are nice) are glued to each other.

Also notice that the flatness condition above is satisfied trivially under the Lipschitz assumption; most of the work in this section will consist in dealing with the other case.

We are ready for the main statement of this section.

**Theorem 24.13.** *Let $E$ be a coral closed set in $U$, $x_0 \in E$, and let the sequence $\{r_k\}$ tend to $0$ as in (24.2). Assume that (24.1) holds, that the configuration of $L_j$ is flat at $x_0$, along the sequence $\{r_k\}$, and that the limit sets $L_j^0$ defined by (24.7) satisfy (10.7) or (19.36). Finally assume that $E_\infty$ is a closed subset of $\mathbb{R}^n$ such that (24.6) holds.*
*If $E \in QSAQ(U,M,\delta,h)$, with a constant $h \ge 0$ which is small enough (depending on $n$, $M$, and $\Lambda$), then*

$$E_\infty \in QSAQ(\mathbb{R}^n, M, +\infty, h), \tag{24.14}$$

*with respect to the sliding boundary conditions associated to the $L_j^0$, $0 \le j \le j_{max}$, defined by (24.5).*

*If $E$ is an almost minimal set in $U$, with the sliding conditions coming from the $L_j$, and a gauge function $h$ such that $\lim_{r\to 0} h(r) = 0$, then*

$$E_\infty \text{ is locally minimal in } \mathbb{R}^n\text{, with the sliding boundary conditions defined by the } L_j^0,\ 0 \le j \le j_{max}. \tag{24.15}$$

Some comments about the definitions are in order. Concerning (10.7) or (19.36), we will see that the $L_j^0$ have natural decompositions into faces, so the definitions make sense, and since we expect the $L_j^0$ to be at least as nice as the $L_j$, assuming (10.7) or (19.36) for them does not hurt more than usual.

For (24.15), we accept the three types ($A_+$, $A$, or $A'$) of almost minimality, and our conclusion (24.15) means that for each one parameter family of mappings $\varphi_t : E_\infty \to \mathbb{R}^n$ that satisfy (1.4)-(1.8) for some closed ball $B \subset \mathbb{R}^n$, we have the two minimality properties

$$\mathcal{H}^d(E_\infty \setminus \varphi_1(E_\infty)) \le \mathcal{H}^d(\varphi_1(E_\infty) \setminus E_\infty) \tag{24.16}$$

as in (20.6) with $h(r) = 0$, and

$$\mathcal{H}^d(\varphi_1(W_1)) \le \mathcal{H}^d(W_1),\ \text{ with } W_1 = \big\{x \in E_\infty\,;\, \varphi_1(x) \ne x\big\} \tag{24.17}$$

as in (20.4) or (20.5), which coincide when $h(r) = 0$. Notice that our conditions for the competitors (i.e, the family $\{\varphi_t\}$) simplify here, because (2.4) is automatic, and (1.5)-(1.6) reduce to the fact that $\varphi_t(x) = x$ for $0 \le t \le 1$ when $|x|$ is large enough.

The class $QSAQ(\mathbb{R}^n, M, +\infty, h)$ is as in Definition 2.3, and it makes sense even without the Lipschitz assumption.

Let us now prove Theorem 24.13. Let $R \ge 1$ be a (large) number; we want to apply Theorem 23.8 in a ball comparable to $B(0, R)$, and our first task will be to check the Lipschitz assumption for the boundary pieces $L_j^0$, with a grid that we need to construct. Of course we shall use the grid that is provided by the Lipschitz assumption (24.1) for the $L_j$. In the special case of the rigid assumption, we don't even need to worry about this, because we can directly apply Theorem 10.7 or Corollary 21.15 (once we look at them in a small enough ball, the $L_j$ coincide with simple cones and we don't eve have a variable domain).

Recall that the Lipschitz assumption in Definition 2.7 comes with constants $\lambda > 0$, $\Lambda \ge 1$, and a $\Lambda$-bilipschitz mapping $\psi : \lambda U \to B(0,1)$. For each large enough $k$, we define a mapping $\psi_k : B(0, 3\Lambda) \to \mathbb{R}^n$ by

$$\psi_k(z) = \rho_k^{-1}\psi(\lambda(x_0 + Rr_k z)) - \rho_k^{-1}\psi(\lambda x_0), \tag{24.18}$$

where $\rho_k$ is a power of 2 that we choose so that

$$\lambda R r_k \le \rho_k \le 2\lambda R r_k. \tag{24.19}$$

Thus $\psi_k$ is defined on $B(0, 3\Lambda)$ as soon as $B(x_0, 3\Lambda R r_k) \subset U$. By the normalization (24.19), the $\psi_k$ are $2\Lambda$-bilipschitz, and since $\psi_k(0) = 0$ there is a subsequence of $\{r_k\}$ for which $\psi_k$ converges uniformly on $B(0,3\Lambda)$ to some $2\Lambda$-bilipschitz mapping $\psi_0 : B(0,3\Lambda) \to \psi_0(B(0,3\Lambda))$. We replace $\{r_k\}$ with such a subsequence; this will not alter the construction of a grid for the limit set. Of course $\psi_0(B(0,2\Lambda))$ contains $B(0,1)$. Set

$$U_R = R\psi_0^{-1}(B(0,1)) \text{ and } \lambda_R = R^{-1}; \tag{24.20}$$

then we have a $2\Lambda$-bilipschitz mapping $\psi_0 : \lambda_R U_R \to B(0,1)$, that we can use to define a grid on $U_R$ and then check the rigid assumption. Before we do this, record the fact that by (24.20) and because $\psi_0$ is $2\Lambda$-Lipschitz,

$$B(0,(2\Lambda)^{-1}R) \subset U_R \subset B(0,2\Lambda R). \tag{24.21}$$

So we want to construct the grid. To each face $F$ of our initial grid $\mathcal{G}$, we associate the face $\widetilde{F} = \psi(\lambda F)$, then for each $k$ the larger face $\widetilde{F}^k = \rho_k^{-1}(\widetilde{F} - \psi(\lambda x_0))$. We took $\rho_k$ dyadic, so it is a translation of a dyadic cube, and it contains the origin if $x_0 \in F$.

We claim that for each $F$ such that $x_0 \in F$, there is a finite union $\widetilde{F}^\infty$ of faces of the standard unit dyadic grid $\mathcal{G}_0$, such that

$$\widetilde{F}^k \cap B(0,2) = \widetilde{F}^\infty \cap B(0,2) \ \text{ for } k \text{ large.} \tag{24.22}$$

The simplest will be to check this with coordinates. The face $\widetilde{F}$ is given by some equations $z_\ell = a_\ell 2^{-m}$, $\ell \in I_1$, and some inequalities $a_\ell 2^{-m} \leq z_\ell \leq (a_\ell+1)2^{-m}$, $\ell \in I_2$, where the $a_\ell$ are integers, $I_1$ and $I_2$ form a partition of $\{1,\dots,n\}$, and $2^{-m}$ is the scale of our initial dyadic grid. Notice that if $\widetilde{F}$ was a face of a larger size, we could just adapt the argument below (and work with a smaller $m$).

Denote by $b_\ell$ the $\ell$-th coordinate of $\psi(\lambda x_0)$; then $\widetilde{F}^k = \rho_k^{-1}(\widetilde{F} - \psi(\lambda x_0))$ is given by the equations $x_\ell = \rho_k^{-1}[a_\ell 2^{-m} - b_\ell]$, $\ell \in I_1$, and the inequalities

$$\rho_k^{-1}[a_\ell 2^{-m} - b_\ell] \leq x_\ell \leq \rho_k^{-1}[(a_\ell+1)2^{-m} - b_\ell], \tag{24.23}$$

$\ell \in I_2$. When $b_\ell \in 2^{-m}\mathbb{Z}$ (and $k$ is so large that $\rho_k^{-1}$ is a multiple of $2^m$), the corresponding equation or inequality has integer coefficients. We shall now check that when $b_\ell \notin 2^{-m}\mathbb{Z}$, we get an inequality which is automatically satisfied when $|x_\ell| \leq 2$; then $\widetilde{F}^k$ coincides, in $B(0,2)$, with a union of faces of $\mathcal{G}_0$, as needed.

So let us check that we get no condition when $b_\ell \notin 2^{-m}\mathbb{Z}$. Since $x_0$ lies in $F$, we get that $\psi(\lambda x_0) \in \widetilde{F}$. The equation $b_\ell = a_\ell 2^{-m}$ is not satisfied (because $b_\ell \notin 2^{-m}\mathbb{Z}$), so $\ell \in I_2$ and $a_\ell 2^{-m} \leq b_\ell \leq (a_\ell+1)2^{-m}$. In addition, both inequalities are strict, so there is an $\varepsilon > 0$ such that $a_\ell 2^{-m} + \varepsilon \leq b_\ell \leq (a_\ell+1)2^{-m} - \varepsilon$. Then, as soon as $k$ is so large that $\rho_k^{-1}\varepsilon > 2$, we get that $\rho_k^{-1}[a_\ell 2^{-m} - b_\ell] \leq -2$ and $\rho_k^{-1}[(a_\ell+1)2^{-m} - b_\ell] \geq 2$, and (24.23) is a tautology on $[-2,2]$, as needed for our claim (24.22).

This gives us an idea of a grid on $U_R$: simply use the cubes $\lambda_R^{-1}\psi_0(Q \cap B(0,1)) \subset U_R$, where $Q$ runs in the (rather small) family of unit dyadic cubes that meet $B(0,1)$. We still need to check that our family of boundaries $L_j^0$ satisfy the Lipschitz assumption, as in Definition 2.7, and this means that each set $A_j = \psi_0(\lambda_R L_j^0 \cap \lambda_R U_R))$ is (the intersection with $B(0,1)$ of) a finite union of faces of $\mathcal{G}_0$.

So let $0 \leq j \leq j_{max}$ be given, and assume that $x_0 \in L_j$; otherwise, $L_j^0 = \emptyset$ and the result is trivial. We claim that for $k$ large,

$$A_j = B(0,1) \cap \Big( \bigcup_{F \in \mathcal{F}} \widetilde{F}^\infty \Big), \tag{24.24}$$

where $\mathcal{F}$ is the set of faces of $L_j$ that touch $x_0$ and $\widetilde{F}^\infty$ is associated to $F$ as in (24.22). First pick $w \in A_j$, and write $w = \psi_0(\lambda_R z)$ for some $z \in L_j^0 \cap U_R$. By (24.7), we can write $z = \lim_{k \to +\infty} z_k$, with $z_k \in L_{j,k}$. Set $x_k = x_0 + r_k z_k$; thus $x_k \in L_j$, and since there is only a finite number of faces, we can assume (after taking a subsequence) that all the $x_k$ lie in a same face $F$. In addition, $F$ contains $x_0$, because otherwise the $z_k$ would go away to infinity. Recall from (24.20) and (24.21) that $\lambda_R U_R = R^{-1} U_R \subset B(0, 2\Lambda)$, so the $z_k$ lie in $B(0, 2\Lambda)$ for $k$ large (because $z \in B(0, 2\Lambda)$), and hence

$$w = \psi_0(\lambda_R z) = \lim_{k \to +\infty} \psi_0(\lambda_R z_k) = \lim_{k \to +\infty} \psi_k(\lambda_R z_k) \tag{24.25}$$

because the $\psi_k$ converge to $\psi_0$ uniformly on $B(0, 3\Lambda)$. Now

$$\begin{aligned} \psi_k(\lambda_R z_k) = \psi_k(R^{-1} z_k) &= \rho_k^{-1} \psi(\lambda(x_0 + R r_k R^{-1} z_k)) - \rho_k^{-1} \psi(\lambda x_0) \\ &= \rho_k^{-1} \psi(\lambda x_k) - \rho_k^{-1} \psi(\lambda x_0) \end{aligned} \tag{24.26}$$

by (24.20) and (24.18). But $x_k \in F$, so $\psi(\lambda x_k) \in \widetilde{F}$ and hence (by (24.26)) $\psi_k(\lambda_R z_k) \in \rho_k^{-1}(\widetilde{F} - \psi(\lambda x_0)) = \widetilde{F}^k$. By (24.22) (and because $\lambda_R z_k \in B(0,2)$ for $k$ large; see above (24.25)), $\psi_k(\lambda_R z_k) \in \widetilde{F}^\infty$. But $\widetilde{F}^\infty$ is closed (and no longer depends on $k$), so (24.25) implies that $w \in \widetilde{F}^\infty$, as needed for the first inclusion.

For the other inclusion, let $w$ lie in $B(0,1) \cap \widetilde{F}^\infty$ for some face $F$ of $L_j$ such that $x_0 \in F$. By (24.22), $w \in \widetilde{F}^k$ for $k$ large, so we can write $w = \rho_k^{-1}(\psi(\lambda x_k) - \psi(\lambda x_0))$ for some $x_k \in F$. Then set $z_k = r_k^{-1}(x_k - x_0) \in L_{j,k}$ (by (24.5)); notice that

$$|z_k| = r_k^{-1}|x_k - x_0| \leq r_k^{-1} \lambda^{-1} \Lambda |\psi(\lambda x_k) - \psi(\lambda x_0)| = r_k^{-1} \lambda^{-1} \Lambda \rho_k |w| \leq 2\Lambda R |w| \tag{24.27}$$

by (24.19), so the $z_k$ lie in $B(0, 2\Lambda R)$, and there is a subsequence for which they converge to a limit $z$. By (24.26) (or rather its proof), $\psi_k(\lambda_R z_k) = w$. By the uniform convergence of the $\psi_k$ on $B(0, 3\Lambda)$, $w = \lim_{k \to +\infty} \psi_0(\lambda_R z_k) = \psi_0(\lambda_R z)$. But $z \in L_i^0$ because $z_k \in L_{j,k}$ and by (24.7), and $z \in U_R$ by (24.20) and because $\psi_0(\lambda_R z) = w \in B(0,1)$. Thus $w \in A_j$, and the converse inclusion holds.

This completes our proof of (24.24); we now know that each $A_j$ is a union of faces of $\mathcal{G}_0$, hence the $L_j^0$ satisfy the Lipschitz assumption in the domain $U_R$.

Now we are ready to apply Theorem 23.8 in the domain $U_R$ and with the sequence $\{E_k\}$. We apply Definition 24.8, with the radius $3\Lambda R$; this gives, for $k$ large, a $\rho_k$-bilipschitz mapping $\xi_k$ defined on $B(0, 3\Lambda R)$. We are interested in the restriction of $\xi_k$ to $U_R$ (recall from (24.21) that $U_R \subset B(0, 3\Lambda R)$), and the domain $U_{R,k} = \xi_k(U_R)$. We want to apply Theorem 23.8 to the sets $E_k \cap U_{R,k}$ and the boundaries $L_{j,k} \cap U_{R,k}$, so let us check the assumptions.

We checked (23.1) (the fact that the $L_j^0$ satisfy the Lipschitz assumption on $U_R$), and $U_{R,k} = \xi_k(U_R)$ by definition. For (23.2), we also need to check that

$$L_{j,k} \cap U_{R,k} = \xi_k(L_j^0 \cap U_R). \tag{24.28}$$

First let $x \in L_{j,k} \cap U_{R,k}$ be given. Since $U_{R,k} = \xi_k(U_R) \subset B(0, 2\rho_k \Lambda R)$ by (24.9) and because $\xi_k(0) = 0$, (24.10) (with $R$ replaced with $3\Lambda R$) says that we can write $x = \xi_k(y)$ for some $y \in L_j^0 \cap B(0, 3\Lambda R)$. But then $y = \xi_k^{-1}(x) \in U_R$ because $x \in U_{R,k}$ and hence $x \in \xi_k(L_j^0 \cap U_R)$. Conversely, if $x \in \xi_k(L_j^0 \cap U_R)$ and $y \in L_j^0 \cap U_R$ is such that $\xi_k(y) = x$, (24.10) says that $x \in L_{j,k}$, and obviously $x \in U_{R,k}$ because $y \in U_R$. So (24.28) holds.

The bilipschitz condition (23.3) comes from (24.9), and (23.4) follows from (24.11). Also, we assumed that the $L_j^0$ satisfy the unpleasant additional condition (10.7) or (19.36), so their restriction to $U_R$ does too.

We start under the first assumption that $E \in QSAQ(U, M, \delta, h)$, and by (24.4) we get that $E_k \in QSAQ(U_k, M, r_k^{-1}\delta, h)$. For $k$ large, $U_{R,k} \subset U_k = r_k^{-1}(U - x_0)$, and $r_k^{-1}\delta$ becomes much larger than $4\Lambda R$ and the diameter of $U_{R,k}$, so when we restrict to $U_{R,k} \subset U_k$, we get that $E_k \in QSAQ(U_k, M, +\infty, h)$. That is, (23.5) holds with $\delta = +\infty$. Since (23.6) (the limit in $U_R$) follows from (24.6), and if $h$ is small enough, Theorem 23.8 says that $E_\infty \cap U_R \in QSAQ(U_R, M, +\infty, h)$.

Now this holds for every $R > 0$, and since (by (24.21)) $U_R$ tends to the whole $\mathbb{R}^n$ (when $R \to +\infty$), we get that $E_\infty \in QSAQ(\mathbb{R}^n, M, +\infty, h)$, as promised in (24.14).

Now suppose that $E$ is $A_+$-almost minimal, with a gauge function $h$ that tends to 0. Then for each $M > 1$, we can find $\delta > 0$ such that $E \in QSAQ(U, M, \delta, 0)$ (just compare with Definition 20.2); by our first case, we get that $E_\infty \in QSAQ(\mathbb{R}^n, M, +\infty, 0)$, and since this is true for each $M > 1$, we even get that $E$ is locally minimal in $\mathbb{R}^n$, as needed.

If $E$ is $A$-almost minimal, still with a gauge function $h$ that tends to 0, Definition 20.2 says that for each small number $h' > 0$, we can find $\delta > 0$ such that $E \in QSAQ(U, 1, \delta, h')$. Then by our first case, $E_\infty \in QSAQ(\mathbb{R}^n, 1, +\infty, h')$, and again $E$ is locally minimal in $\mathbb{R}^n$.

If $E$ is $A'$-almost minimal, we can still conclude as above, except that we now use the generalization of Theorem 21.3 that goes like Theorem 23.8 (but with almost minimal sets), as explained in Remark 23.28. Observe that we can always apply this statement with a gauge function $\widetilde{h}$ which is larger than $h$, continuous from the right, and still tending to 0. Or we could use Remark 21.6.

This concludes our proof of Theorem 24.13. □

Let us now give a slightly simpler sufficient condition for the flatness of the configuration of the $L_i$, which depends only on the regularity at $x_0$ of the faces of the $L_j$ (and not, apparently, on the way they are arranged in space). As we shall see, the construction of the bilipschitz mapping $\xi_k$ will be simpler than expected, because the bilipschitz property will come from the fact that the differential stays close to the identity.

We keep the notation of the beginning of this section; that is, $U$ is an open set, $x_0 \in U$, and the boundary pieces $L_j$ satisfy the Lipschitz assumption (as in (24.1)). Denote by $\mathcal{G}$ the associated grid on $U$, and by $\mathcal{F}$ the set of faces of $\mathcal{G}$ that contain $x_0$ and are contained in some set $L_j$. Also call $\mathcal{F}_\ell$ the set of faces $F \in \mathcal{F}$ that are $\ell$-dimensional.

Let a sequence $\{r_k\}$, that tends to 0, be given too. For $F \in \mathcal{F}$ and $k \geq 0$, set $F^k = r_k^{-1}(F - x_0)$.

**Definition 24.29.** *We say that* <u>*the faces*</u> *of the* $L_j$ <u>*are flat*</u> *at* $x_0$, <u>*along the sequence*</u> $\{r_k\}$, *when for each* $1 \leq \ell \leq n$ *and each face* $F \in \mathcal{F}_\ell$, *there is a set* $F^0$ *that contains* $0$, *such that*

$$F^0 \text{ is a closed convex } \ell\text{-dimensional polyhedron in some } \ell\text{-dimensional vector space } V_F, \tag{24.30}$$

*and with the following connection with the* $F^k$. *For* $1 \leq R < +\infty$, *there is a sequence* $\{\varepsilon_k\}$ *such that*

$$\lim_{k \to +\infty} \varepsilon_k = 0, \tag{24.31}$$

*and for each large enough* $k$, *a Lipschitz mapping* $\psi_{F,k} : F^0 \cap B(0,R) \to F^k$, *such that*

$$\psi_{F,k}(0) = 0, \tag{24.32}$$

$$|D\psi_{F,k} - I| \leq \varepsilon_k \quad \mathcal{H}^\ell\text{-almost everywhere on } F^0 \cap B(0,R), \tag{24.33}$$

*and*

$$\psi_{F,k}(F^0 \cap B(0,R)) \supset F^k \cap B(0,(1-\varepsilon_k)R). \tag{24.34}$$

Let us comment on the slightly strange aspects of this definition. We shall see soon (in (24.37)) that $F^0$ is the limit of the $F^k$, and this is also why we require that $0 \in F^0$ (recall that $x_0 \in F$ for $F \in \mathcal{F}$, hence $0 \in F^k$). We could also check that when $F$ is of dimension $\ell > 0$, our polyhedron $F^0$ is unbounded and has a nonempty interior in $V_F$. It can even fill the whole space $V_F$. We decided not to let the $F^0$ depend on $R$ (this would have at least complicated the proof, maybe with no true additional generality), even though the relation with $F^k$ is only stated in each ball $B(0,R)$. Similarly, requiring that $F^0$ is a convex polyhedron will simplify our life, and will probably not hurt in applications. Finally, we shall not try to see whether (24.34) could, or could not, be deduced from the other assumptions.

**Proposition 24.35.** *Let* $U$, $x_0$, $\{r_k\}$, *and the* $L_j$ *be as above. If the the faces of the* $L_j$ *are flat at* $x_0$ *along the sequence* $\{r_k\}$, *then the configuration of* $L_j$ *is flat at* $x_0$ *along the sequence* $\{r_k\}$.

Before we start the construction of mappings $\xi_k : B(0,R) \to \mathbb{R}^n$, as in Definition 24.8, let us use the $\psi_{F,k}$ to control some of the geometry of the $F^0$. Let $\ell \geq 1$ and $F \in \mathcal{F}_\ell$ be given.

First observe that if $\psi_{F,k}$ is as in Definition 24.29, then for $x, y \in F^0 \cap B(0,R)$,

$$|\psi_{F,k}(x) - \psi_{F,k}(y) - x + y| \leq \varepsilon_k |x - y|. \tag{24.36}$$

Indeed, for almost all choices of $x$ and $y$ (for the $2\ell$-dimensional product of Lebesgue measures), we can compute $\psi_{F,k}(x) - \psi_{F,k}(y)$ as the integral of $D\psi_{F,k}$ on the segment

$[x, y]$; this comes from the fact that $\psi_{F,k}$ is Lipschitz, hence absolutely integrable on almost every line. Notice that $[x, y] \subset F^0$, because $F^0$ is convex by (24.30). Thus (24.36) holds for almost all choices of $x$ and $y$; the general case follows because $\psi_{F,k}$ is continuous.

Next we check that

$$F^0 = \lim_{k \to +\infty} F^k \quad \text{in } \mathbb{R}^n, \tag{24.37}$$

(with the same definition as in (10.4)-(10.6)). Pick $r > 0$, and let us first check that $d_k = \sup\left\{ \operatorname{dist}(x, F^k)\,;\, x \in F^0 \cap B(0, r)\right\}$ tends to 0. Let the $\psi_{F,k}$ be, for $k$ large, as in Definition 24.29, with $R = 2r$, and simply observe that for $x \in F^0 \cap B(0, r)$, $\operatorname{dist}(x, F^k) \le |\psi_{F,k}(x) - x| \le \varepsilon_k |x|$ by (24.33) and (24.32). Thus $d_k \le \varepsilon_k r$ for $k$ large.

Then we control $d_k' = \sup\left\{ \operatorname{dist}(x, F^0)\,;\, x \in F^k \cap B(0, r)\right\}$. We keep the same choice of $R = 2r$ and $\psi_{F,k}$, and observe that for $x \in F^k \cap B(0, r)$, (24.34) allows us to write $x = \psi_{F,k}(z)$ for some $z \in F^0 \cap B(0, R)$. Then $\operatorname{dist}(x, F^0) \le |x - z| = |\psi_{F,k}(z) - z| \le \varepsilon_k |z| \le \varepsilon_k R$, so $d_k' \le \varepsilon_k R$ for $k$ large, and (24.37) follows.

Next we check that the mappings $\psi_{F,k}$ essentially preserve the boundaries. That is, denote by $\partial F^0$ the boundary of $F^0$, seen as a subset of the vector space $V_F$. We claim that for $k$ large,

$$\text{for } x \in F^0 \cap B(0, R), \psi_{F,k}(x) \in \partial F^k \text{ if and only if } x \in \partial F^0. \tag{24.38}$$

For this we shall use a little bit of topology. Notice that by (24.36), $\psi_{F,k}$ is a bilipschitz mapping from $F^0 \cap B(0, R)$ to its image. We compose with the affine mapping $\rho_k : x \to x_0 + r_k x$, and get an image $\rho_k \circ \psi_{F,k}(F^0 \cap B(0, R)) \subset F$, which is contained in $U$ for $k$ large. We further compose with the usual bilipschitz mapping $z \to \psi(\lambda z)$, and get a bilipschitz mapping $h_k : F^0 \cap B(0, R) \to h_k(F^0 \cap B(0, R)) \subset \widetilde{F}$, where now $\widetilde{F}$ is a (straight) dyadic cube of dimension $\ell$.

Let $x \in F^0 \cap B(0, R)$ be an interior point of $F^0$, suppose that $\psi_{F,k}(x) \in \partial F^l$, and derive a contradiction. Recall that $\partial F$ was in fact defined as the bilipschitz image of $\partial\widetilde{F}$, so we are assuming that $h_k(x) \in \partial\widetilde{F}$. Let $S$ be the unit sphere in $V_F$, and let us see what happens to the mapping $f : S \to V_F$, defined by $f(\xi) = x + t\xi$, where we choose $t > 0$ so small that $f(S) \subset F^0 \cap B(0, R)$. This map cannot be homotoped to a constant, through mappings from $S$ to $V_F \setminus \{x\}$, yet, for $t$ small we shall use $h_k$ to find such a homotopy. Let $L$ denote the bilipschitz constant for $h_k$; then $h_k \circ f(S) \subset A$, where $A = \left\{z \in \widetilde{F}\,;\, L^{-1}t \le |z - h_k(x)| \le Lt\right\}$. Now, because $h_k(x) \in \partial\widetilde{F}$ and $\widetilde{F}$ is a dyadic cube, and if $t$ is small enough, $A$ can be contracted (to a point) inside the slightly larger annular region $A_C = \left\{z \in \widetilde{F}\,;\, (CL)^{-1}t \le |z - h_k(x)| \le CLt\right\}$. That is, there is a continuous function $\varphi : A \times [0, 1] \to A_C$ such that $\varphi(z, 0) = z$ and $\varphi(z, 1) = c$ for $z \in A$, and where $c \in A_C$ is a constant. The desired deformation is the mapping $\widetilde{\varphi} : S \times [0, 1] \to V_F$ defined by $\widetilde{\varphi}(\xi, t) = h_k^{-1}(\varphi_t(h_k \circ f(\xi), t))$; it is easy to see that for $t$ small, it is defined (because $A_C \subset h_k(F^0 \cap B(0, R))$) and continuous, that it avoids the value $x$, and that $\widetilde{\varphi}(\xi, 1) = h_k^{-1}(c)$ is constant. This contradiction shows that $\psi_{F,k}(x)$ is an interior point of $F^l$.

The same argument, applied with the mapping $h_k^{-1}$, shows that if $x \in F^0 \cap B(0, R)$ and $z = \psi_{F,k}(x)$ is an interior point of $F^l$ (which means that $h_k(x)$ is an interior point of $\widetilde{F}$), then $x$ lies in the interior of $F^0$. This time, we use the fact that since $F^0$ is a convex polyhedron, there is a constant $C$ such that, for $x \in \partial F^0$ and $t$ small enough, $\{w \in F^0\,;\, L^{-1}t \leq |w-x| \leq Lt\}$ can be contracted inside $\{w \in F^0\,;\, C^{-1}L^{-1}t \leq |w-x| \leq CLt\}$. This proves (24.38).

It will also be good to know that $\partial F^0$ can also be seen as the combinatorial boundary of $F^0$, i.e., that

$$\partial F^0 = \bigcup_{H \in \mathcal{F}(F)} H^0, \tag{24.39}$$

where $\mathcal{F}(F)$ denotes the set of strict subfaces of $F$ that meet $x_0$. Indeed, let $x \in \partial F^0$ be given, choose $R > |x|$, and notice that by (24.38), $\psi_{F,k}(x) \in \partial F^k$ for $k$ large. Then there is a strict subface $H$ of $F$ such that $\psi_{F,k}(x) \in H^k$ for infinitely many $k$. But $x = \lim_{k\to+\infty} \psi_{F,k}(x)$ by (24.36) with $y = 0$, so by (24.37) $x$ lies in $H^0$. Conversely, let $x \in H^0$ for some $H \in \mathcal{F}(F)$, and choose $\psi_{F,k}$ as above. By (24.37), $x$ is the limit of some sequence $\{x_k\}$, with $x_k \in H^k$. By (24.34) and since $H^k \subset F^k$, we can write $x_k = \psi_{F,k}(z_k)$ for some $z_k \in F^0 \cap B(0,R)$, and by (24.36) $|z_k - x_k|$ tends to 0. Also, $z_k \in \partial F^0$ by (24.38), so $x = \lim_{k\to+\infty} x_k = \lim_{k\to+\infty} z_k$ lies in $\partial F^0$ too. This completes the proof of (24.39).

Finally, we shall need the following consequence of (3.8), to control the geometry of the faces $F^0$. Let $F$ and $G$ be two faces of $\mathcal{F}$, and suppose that $F$ is neither reduced to a point nor contained in $G$; we claim that

$$\operatorname{dist}(y, \partial F^0) \leq \Lambda^2 \operatorname{dist}(y, G^0) \quad \text{for } y \in F^0. \tag{24.40}$$

Let $y \in F^0$ be given, and use (24.37) to choose points $y_k \in F^k$ such that $\lim_{k\to+\infty} y_k = y$. Set $x_k = x_0 + r_k y_k \in F$, $\widetilde{x}_k = \psi(\lambda x_k)$, and as usual $\widetilde{F} = \psi(\lambda F)$ and $\widetilde{G} = \psi(\lambda G)$. Then

$$\begin{aligned}
\operatorname{dist}(y, G^0) &= \lim_{k\to+\infty} \operatorname{dist}(y, G^k) = \lim_{k\to+\infty} \operatorname{dist}(y_k, G^k) = \lim_{k\to+\infty} r_k^{-1} \operatorname{dist}(x_k, G) \\
&\geq \Lambda^{-1}\lambda^{-1} \limsup_{k\to+\infty} r_k^{-1} \operatorname{dist}(\widetilde{x}_k, \widetilde{G}) \\
&\geq \Lambda^{-1}\lambda^{-1} \limsup_{k\to+\infty} r_k^{-1} \operatorname{dist}(\widetilde{x}_k, \partial(\widetilde{F})) \\
&\geq \Lambda^{-2} \limsup_{k\to+\infty} r_k^{-1} \operatorname{dist}(x_k, \partial F) = \Lambda^{-2} \limsup_{k\to+\infty} \operatorname{dist}(y_k, \partial F^k)
\end{aligned} \tag{24.41}$$

by (24.37), various definitions, and (3.8). Use (24.41) to choose a sequence of points $z_k \in \partial F^k$, such that

$$\limsup_{k\to+\infty} \operatorname{dist}(y_k, z_k) \leq \Lambda^2 \operatorname{dist}(y, G^0). \tag{24.42}$$

Now $\partial F$ is the union of a certain number of strict subfaces $H$, and each $\partial F^k$ is the union of the corresponding subfaces. So we can choose a subsequence for which all the $z_k$ lie in $H^k$ for the same $H$.

Of course $\operatorname{dist}(y, G^0) < +\infty$, and by (24.42) $\{z_k\}$ is bounded. So we can choose a new subsequence so that $\{z_k\}$ tends to a limit $z_\infty$. Since $x_0 + r_k z_k \in H$, $\{z_k\}$ is bounded, and $r_k$ tends to 0, we get that $x_0 \in H$, hence $H \in \mathcal{F}$. Also, $z_\infty \in H^0$, since $z_k \in H^k$ and by (24.37) for $H$. Hence $z_\infty \in \partial F^0$, by the representation formula (24.39). Finally, (24.42) implies that $\operatorname{dist}(y, z_\infty) \leq \Lambda^2 \operatorname{dist}(y, G^0)$, and (24.40) follows.

We are now ready to start the construction of our mappings $\xi_k$. Let $R > 0$ be given; we apply our flatness assumption to every face $F \in \mathcal{F}$, in the larger ball $B(0, 4^{n+1}R)$, to get mappings $\psi_{F,k} : F^0 \cap B(0, 4^{n+1}R) \to F^k$ with the properties (24.31)-(24.34).

We shall obtain $\xi_k$ after building successive extensions, defined on the following collection of skeletons. Set $\mathcal{S}_0 = \{0\}$ and for $1 \leq \ell \leq n$,

$$B_\ell = B(0, 4^{n-\ell+1}R),\ \mathcal{S}_\ell = \bigcup_{F \in \mathcal{F}_\ell} F^0 \cap B_\ell\,, \quad \text{and } \mathcal{S}_\ell^+ = \bigcup_{m \leq \ell} \mathcal{S}_m. \tag{24.43}$$

We gave a special definition to $\mathcal{S}_0$ just because, even when $x_0$ is not a point of the grid, we find it more pleasant to take $\mathcal{S}_0 = \{0\}$.

Set $\xi_k^0(0) = 0$ to start the process. We want to define successive extensions $\xi_k^\ell$ of $\xi_k^0$, with the following properties. First, $\xi_k^\ell$ is defined on $\mathcal{S}_\ell^+$, and is an extension of $\xi_k^{\ell-1}$ if $\ell \geq 1$. Next, for each $F \in \mathcal{F}_\ell$ and $k$ large enough,

$$F^k \cap \frac{1}{2} B_\ell \subset \xi_k^\ell(F^0 \cap B_\ell) \subset F^k \cap 2B_\ell, \tag{24.44}$$

and $\xi_k^\ell$ is locally Lipschitz on $F^0 \cap B_\ell$, with

$$|D\xi_k^\ell - I| \leq C_\ell \varepsilon_k \quad \mathcal{H}^\ell\text{-almost everywhere on } F^0 \cap B_\ell, \tag{24.45}$$

where $I$ denotes the identity on $\mathbb{R}^n$ and $C_\ell \geq 1$ is a geometric constant that will be computed by induction. Finally, we require that

$$|\xi_k^\ell(x) - \xi_k^\ell(y) - x + y| \leq 2(1 + 2\Lambda^2)^2 C_\ell \varepsilon_k |x - y| \ \text{ for } x, y \in \mathcal{S}_\ell^+. \tag{24.46}$$

We already have $\xi_k^0$, and let us construct $\xi_k^1$ to warm up. We choose it so that

$$\xi_k^1 = \psi_{F,k} \ \text{ on } F^0 \cap B_1 \tag{24.47}$$

for every $F \in \mathcal{F}_1$. This definition is coherent: there is no conflict with the fact that $\xi_k^0(0) = 0$, by (24.32), and similarly if $F$ and $G$ are different faces of $\mathcal{F}_1$, then $F^0 \cap G^0 = \{0\}$ (for instance by (24.40) and because $\partial F^0$ and $\partial G^0$ are contained in $\{0\}$ by (24.39)), and $\psi_{F,k}(0) = \psi_{G,k}(0) = 0$. Notice that it could happen that $\mathcal{S}_1$ is empty because $\mathcal{F}_1$ is empty, but this does not disturb. Next, (24.44) (i.e., the fact that $F^k \cap \frac{1}{2}B_1 \subset \psi_{F,k}(F^0 \cap B_1) \subset F^k \cap 2B_1$ holds by definition of $\psi_{F,k}$, and in particular (24.34) (for the surjectivity) and (24.36) with $y = 0$ (for the management of radii). Also, (24.45) holds with $C_1 = 1$, by (24.33). We don't need to check (24.46) for $\ell = 1$, because we shall do it in a more general case now.

Now we check that (24.46) (for some $\ell \geq 1$) follow from (24.45) for $\ell$ and (24.46) for $\ell - 1$ (notice that (24.46) is obvious for $\ell = 0$). First we claim that because of (24.45) for $\ell$, we have that

$$|\xi_k^l(x) - \xi_k^l(y) - x + y| \leq C_\ell \varepsilon_k |x - y| \quad \text{for } x, y \in F^0 \cap B_\ell \tag{24.48}$$

for each face $F \in \mathcal{F}_\ell$. The verification is the same as for (24.36): we first check this for $x, y$ in a dense subset, using the convexity of $F^0 \cap B_\ell$ and the absolute continuity of $\xi_k^l$ on almost all lines, and the general case follows because $\xi_k^l$ is also continuous.

Then we check (24.46); we intend to use (24.40) to control the position of the different faces. Let $x, y \in \mathcal{S}_\ell^+$ be given. By the definition (24.43), we can find $m_x, m_y \leq \ell$ so that $x \in \mathcal{S}_{m_x}$ and $y \in \mathcal{S}_{m_y}$; choose $m_x$ and $m_y$ are as small as possible, i.e., consider the first occurrence. Then use (24.43) again to choose $F \in \mathcal{F}_{m_x}$ and $G \in \mathcal{F}_{m_y}$ such that $x \in F^0 \cap B_{m_x}$ and $y \in G^0 \cap B_{m_y}$.

First assume that $m_x = \ell$. If $x \in G^0$, then in fact it lies in $G^0 \cap B_{m_y}$ (because $m_y \leq m_x$), and (24.46) follows from (24.48) for $G$. So we may assume that $x \notin G^0$, and then $F$ is neither reduced to a point (it would be $x_0$, and then $x = 0 \in G^0$) nor contained in $G$ (because if $F \subset G$, then (24.37) shows that $F^0 \subset G^0$), so we may apply (24.40). We get that

$$\operatorname{dist}(x, \partial F^0) \leq \Lambda^2 \operatorname{dist}(x, G^0) \leq \Lambda^2 |x - y| < +\infty. \tag{24.49}$$

In particular, $\partial F^0$ is not empty, and we can pick $x' \in \partial F^0$ such that $|x' - x| = \operatorname{dist}(x, \partial F^0) \leq \Lambda^2 |x - y|$. By the representation formula (24.39) for $\partial F^0$, and the fact that each $H^0$ is convex and contains the origin, we see that $tx' \in \partial F^0$ for $0 \leq t \leq 1$; we use this to see that $|x'| \leq |x|$ (otherwise, some $tx'$ is closer to $x$). Hence $x' \in B_\ell$. By (24.39) again, $x' \in H^0$ for some $H \in \mathcal{F}$ of dimension $m \leq \ell - 1$, so

$$x' \in H^0 \cap B_\ell \subset \mathcal{S}_m \subset \mathcal{S}_{\ell-1}^+. \tag{24.50}$$

Since $x' \in \partial F^0 \cap B_\ell \subset F^0 \cap B_\ell$, we can apply (24.48) to get that

$$|\xi_k^\ell(x') - \xi_k^\ell(x) - x' + x| \leq C_\ell \varepsilon_k |x' - x| \leq C_\ell \Lambda^2 \varepsilon_k |x - y|. \tag{24.51}$$

We will continue with the proof in a moment, but let us record some cases first.

If $m_x < \ell$, we simply keep $x' = x \in \mathcal{S}_{\ell-1}^+$, and some estimates will be simpler.

If $m_y = \ell$, we can assume (as above) that $y \notin F^0$, and then we choose $y' \in \mathcal{S}_{\ell-1}^+$ as we did for $x$; if $m_y < \ell$, we just keep $y' = y$. Now

$$\begin{aligned} |\xi_k^\ell(x') - \xi_k^\ell(y') - x' + y'| &= |\xi_k^{\ell-1}(x') - \xi_k^{\ell-1}(y') - x' + y'| \\ &\leq 2(1 + 2\Lambda^2) C_{\ell-1} \varepsilon_k |x' - y'| \\ &\leq 2(1 + 2\Lambda^2)^2 C_{\ell-1} \varepsilon_k |x - y| \end{aligned} \tag{24.52}$$

because $x', y' \in \mathcal{S}_{\ell-1}^+$, by (24.46) for $\ell - 1$, and because $|x' - y'| \leq |x - y| + |x' - x| + |y' - y| \leq (1 + 2\Lambda^2)|x - y|$. If $x' \neq x$ or $y' \neq y$, we add (24.51) or its analogue for $y$, and get that

$$|\xi_k^\ell(x) - \xi_k^\ell(y) - x + y| \leq 2(1 + 2\Lambda^2)^2 C_{\ell-1} \varepsilon_k |x - y| + 2 C_\ell \Lambda^2 \varepsilon_k |x - y| \tag{24.53}$$

which implies (24.46) if we make sure to choose $C_\ell \geq 2(1+2\Lambda^2)C_{\ell-1}$.

Next we define $\xi_k^{\ell+1}$ when $1 \leq \ell < n$, assuming that we already have $\xi_k^\ell$. We shall construct our extension $\xi_k^{l+1}$ independently on each $F^0 \cap B_{\ell+1}$, $F \in \mathcal{F}_{\ell+1}$, and of course we shall make sure to keep the same values as $\xi_k^\ell$ on $F^0 \cap B_{\ell+1} \cap \mathcal{S}_\ell^+$. We claim that if we proceed this way there will be no conflict of definition between faces. More precisely, if $F, G \in \mathcal{F}_{\ell+1}$ are different faces, we claim that $F^0 \cap G^0 \cap B_{\ell+1} \subset \mathcal{S}_\ell^+$, so $\xi_k^l = \xi_k^{l+1}$ was in fact already defined on the intersection.

To prove the claim, let $y \in F^0 \cap G^0 \cap B_{\ell+1}$ be given. Since $F$ is neither reduced to a point nor contained in $G$, (24.40) says that $\operatorname{dist}(y, \partial F^0) \leq \Lambda^2 \operatorname{dist}(y, G^0) = 0$. Recall from (24.39) that $\partial F^0$ is the finite union of the closed sets $H^0$, where $H$ is a strict subface of $F$ that contains $x_0$. Then $y$ lies in such an $H^0$, and by the definition (24.43), $y \in \mathcal{S}_\ell^+$; our claim follows.

So we now fix $F \in \mathcal{F}_{l+1}$ and proceed to define $\xi_k^{l+1}$ on $F^0 \cap B_{\ell+1}$. By (24.39), $\partial F^0 = \bigcup_{H \in \mathcal{F}(F)} H^0$, and by induction assumption $\xi_k^\ell$ is defined on $\partial F^0 \cap B_\ell$, with values in $\partial F^k$. We want to extend this mapping to $F^0 \cap B_{\ell+1}$, with values in $F^k$, and for this it will be easier to use our bilipschitz mapping $\psi_{F,k}$ to return to the vector space $V_F$ and work there.

By (24.48) (on the faces $H^0$, and with $y=0$),

$$|\xi_k^\ell(x) - x| \leq C_l \varepsilon_k |x| \quad \text{for } x \in \partial F^0 \cap B_\ell\,; \tag{24.54}$$

in particular, $\xi_k^\ell(x) \in 2B_\ell$ and (24.34) (which we can apply in the larger ball $B(0, 4^{n+1}R)$; compare with (24.43) and recall that $\ell > 1$) allows us to set

$$h(x) = \psi_{F,k}^{-1} \circ \xi_k^\ell(x) \in F^0 \quad \text{for } x \in \partial F^0 \cap B_\ell. \tag{24.55}$$

In fact, (24.38) (with the same radius $4^{n+1}R$ that we used to define $\psi_{F,k}$, and applied to $h(x)$) says that $h(x) \in \partial F^0$. And of course, by (24.36) with $y=0$ and because $\xi_k^\ell(x) \in 2B_\ell \subset B(0, 4^{n+1}R)$, we get that $h(x) \in 3B_\ell$. That is,

$$h(x) \in \partial F^0 \cap 3B_\ell \quad \text{for } x \in \partial F^0 \cap B_\ell. \tag{24.56}$$

We also have good estimates on the Lipschitz constant for $h_1 = h - I$. Indeed, for $x, y \in \partial F^0 \cap B_\ell$,

$$|h_1(x) - h_1(y)| = |\psi_{F,k}^{-1} \circ \xi_k^\ell(x) - \psi_{F,k}^{-1} \circ \xi_k^\ell(y) - x + y| \leq a + b, \tag{24.57}$$

with

$$a = |\psi_{F,k}^{-1} \circ \xi_k^\ell(x) - \xi_k^\ell(x) - \psi_{F,k}^{-1} \circ \xi_k^\ell(y) + \xi_k^\ell(y)| \tag{24.58}$$

and $b = |\xi_k^\ell(x) - \xi_k^\ell(y) - x + y|$. By (24.36) applied to $\psi_{F,k}^{-1} \circ \xi_k^\ell(x)$ and $\psi_{F,k}^{-1} \circ \xi_k^\ell(y)$, we get that

$$a \leq \varepsilon_k |\psi_{F,k}^{-1} \circ \xi_k^\ell(x) - \psi_{F,k}^{-1} \circ \xi_k^\ell(y)| \leq 2\varepsilon_k |\xi_k^\ell(x) - \xi_k^\ell(y)| \leq 4\varepsilon_k |x - y| \tag{24.59}$$

(because (24.36) and (24.46) also imply that $\psi_{F,k}$ and $\xi_k^\ell$ are 2-bilipschitz on the domains where we work). Also, (24.46) implies that $b \leq 2(1+2\Lambda^2)^2 C_\ell \varepsilon_k |x-y|$; altogether,

$$|h_1(x) - h_1(y)| \leq \big(4 + 2(1+2\Lambda^2)^2 C_\ell\big)\varepsilon_k |x-y|. \tag{24.60}$$

Let us take advantage that $F^0$ and the values of $h_1$ lie in the space $V_F$ to apply the Whitney extension theorem to $h_1$; we get an extension of $h_1$ to the whole $F^0 \cap B_\ell$, so that

$$h_1 : F^0 \cap B_\ell \to V_F \text{ is } C'_\ell \varepsilon_k\text{-Lipschitz}, \tag{24.61}$$

with $C'_\ell \leq C(4 + 2(1+2\Lambda^2)^2 C_\ell$. Now set

$$h(x) = h_1(x) + x \text{ for } x \in F^0 \cap B_\ell; \tag{24.62}$$

This function is an extension of the mapping in (24.55), it still a Lipschitz function (like $h_1$), and by (24.61)

$$|Dh - I| \leq C'_\ell \varepsilon_k \quad \mathcal{H}^{\ell+1}\text{-almost everywhere on } F^0 \cap B_\ell. \tag{24.63}$$

We now need some topological information, which will lead to (24.44) for $\ell+1$. We start with a control on the restriction of $h$ to $\partial F^0$. We claim that

$$\partial F^0 \cap \frac{1}{3} B_\ell \subset h(\partial F^0 \cap B_\ell). \tag{24.64}$$

Let $y \in \partial F^0 \cap \frac{1}{3} B_\ell$, and set $y_k = \psi_{F,k}(y)$. Notice that $y_k \in F^k \cap \frac{1}{2} B_\ell$ by (24.36), and $y_k \in \partial F^k$ by (24.38). This means that $y_k \in H^k$ for some $\ell$-dimensional subface $H$ of $F$. Observe that then $x_0 + y_k r_k \in H$, hence $\mathrm{dist}(H, x_0) \leq r_k 4^{n-\ell+1} R$. If $k$ is large enough (depending on the finite list of faces $H$ that get close to $x_0$), this can only happen if $x_0 \in H$, i.e., if $H \in \mathcal{F}(F)$. By induction assumption, (24.44) holds for $H$, and since $y_k \in H^k \cap \frac{1}{2} B_\ell$, we get that $y_k = \xi_k^\ell(x)$ for some $x \in H^0 \cap B_\ell$. Now $h(x) = \psi_{F,k}^{-1}(y_k) = y$ by (24.55), and this proves (24.64).

Next we use (24.64) and a connectedness argument to show that

$$F^0 \cap \frac{1}{3} B_\ell \subset h(F^0 \cap B_\ell). \tag{24.65}$$

Denote by $W = F^0 \setminus \partial F^0$ the interior of $F^0$ in the space $V_F$; we just need to show that $h(W \cap B_\ell)$ contains $W \cap \frac{1}{3} B_\ell$, because (24.64) takes care of $\partial F^0 \cap \frac{1}{3} B_\ell$.

Of course we may assume that $W \cap \frac{1}{3} B_\ell$ is not empty. Set $Y = W \cap \frac{1}{3} B_\ell \cap h(W \cap B_\ell)$; we want to show that $Y = W \cap \frac{1}{3} B_\ell$.

First we check that if $k$ is large enough, $Y$ is not empty. Notice that

$$\lim_{k \to +\infty} h(z) = z \text{ for } z \in F^0 \cap B_\ell, \tag{24.66}$$

by (24.62) and because $\lim_{k \to +\infty} h_1(z) = 0$ by (24.61) and because $h_1(0) = 0$; thus we pick $z \in W \cap \frac{1}{3} B_\ell$ and observe that $h(z) \in W \cap \frac{1}{3} B_\ell$ for $k$ large, hence $Y$ is not empty.

Next, $Y$ is open. This is because $h : W \cap B_\ell \to V_F$ is bilipschitz (by (24.61) and (24.62)), hence open (for instance by degree theory, or a fixed point theorem as in the implicit function theorem).

Finally, $Y$ is closed in $W \cap \frac{1}{3}B_\ell$: if $\{y_j\}$ is a sequence in $Y$, with a limit $y \in W \cap \frac{1}{3}B_\ell$, then $y \in Y$ because we can find $x_j \in W \cap B_\ell$ such that $h(x_j) = y_j$, the $x_j$ converge to a limit $x$ because $h$ is bilipschitz on $F^0 \cap B_\ell$, $h(x) = y$ because $h$ is continuous, $x \in B_\ell$ because all the $x_j$ lie in $\frac{5}{6}B_\ell$ (again because $h$ is bilipschitz and $h(0) = 0$), and $x \in W$ (because otherwise $x \in \partial F^0$ and $h(x) = y \in \partial F^0$ by (24.56), a contradiction).

Since $W \cap \frac{1}{3}B_\ell$ is connected (and even convex), we get that $Y = W \cap \frac{1}{3}B_\ell$, as needed for (24.65).

Let us now deduce from (24.65) that

$$h(F^0 \cap \frac{1}{4}B_\ell) \subset F^0. \tag{24.67}$$

If $W \cap \frac{1}{4}B_\ell = \emptyset$, i.e., if $F^0 \cap \frac{1}{4}B_\ell \subset \partial F^0$, this is a direct consequence of (24.55). Otherwise, first observe that for $k$ large, $h(W \cap \frac{1}{4}B_\ell)$ meets $F^0$ (pick $x \in W \cap \frac{1}{4}B_\ell$, and observe that $h(x) \in W$ for $k$ large, by (24.66)). Next we claim that

$$h(W \cap \frac{1}{4}B_\ell) \text{ does not meet } \partial F^0. \tag{24.68}$$

Indeed, suppose that $x \in W \cap \frac{1}{4}B_\ell$ is such that $h(x) \in \partial F^0$. Notice that $h(x) \in \frac{1}{3}B_\ell$, so by (24.64) we can find $z \in \partial F^0 \cap B_\ell$ such that $h(z) = h(x)$. This is impossible, because $x \in W = F^0 \setminus \partial F^0$ and $h$ is bilipschitz (hence injective) on $F^0 \cap B$. So (24.68) holds.

Since $W \cap \frac{1}{4}B_\ell$ is convex and $h(W \cap \frac{1}{4}B_\ell)$ meets $F^0$, (24.68) says that $h(W \cap \frac{1}{4}B_\ell) \subset W \subset F^0$. But $h(\partial F^0 \cap \frac{1}{4}B_\ell) \subset F^0$ by (24.55), so (24.67) holds.

We are now ready to define $\xi_k^{\ell+1}$ on $F^0$, and check our induction assumptions. By (24.67) and the bilipschitz property of $h$ (see (24.61) and (24.62)), $h(F^0 \cap \frac{1}{4}B_\ell) \subset F^0 \cap \frac{1}{3}B_\ell$, and we can we set

$$\xi_k^{\ell+1} = \psi_{F,k} \circ h \ \text{ on } F^0 \cap \frac{1}{4}B_\ell = F^0 \cap B_{\ell+1}. \tag{24.69}$$

This gives the desired definition of $\xi_k^{\ell+1}$ (recall that we can proceed face by face). Notice that for $x \in F^0 \cap B_{\ell+1}$, $\xi_k^{\ell+1}(x) \in F^k$ (by definition of $\psi_{F,k}$), and $\xi_k^{\ell+1}(x) \in 2B_{\ell+1}$, by (24.36). This proves the second inclusion in (24.44) (for our $F \in \mathcal{F}_{\ell+1}$).

For the first inclusion we pick $y \in F^k \cap \frac{1}{2}B_{\ell+1}$ and we need to find $x \in F^0 \cap B_{\ell+1}$ such that $\xi_k^{\ell+1}(x) = y$. First apply (24.34) to find $w \in F^0$ such that $\psi_{F,k}(w) = y$. By (24.36), $w \in \frac{2}{3}B_{\ell+1} \subset \frac{1}{3}B_\ell$. By (24.65), we can find $x \in F^0 \cap B_\ell$ such that $h(x) = w$. By (24.61) and (24.62), and because $w \in \frac{2}{3}B_{\ell+1}$, $x \in B_{\ell+1}$. Then $\xi_k^{\ell+1}(x) = \psi_{F,k} \circ h(x) = y$ by (24.69) and our definitions, which proves the second inclusion in (24.44).

The estimate (24.45) on $D\xi_k^{\ell+1}$ follows from (24.69), (24.33), (24.63), and the fact that all our mappings are Lipschitz. We just need to pick $C_{\ell+1}$ somewhat larger that $C_\ell$.

We already checked (near (24.48)) that (24.46) follows from (24.45) and the induction assumption, so we completed our induction step, and we get mappings $\xi_k^\ell$, $1 \leq \ell \leq n$, with the properties (24.44)-(24.46).

For the verification of flatness, we need to construct a $\xi_k$, as in Definition 24.8, which is defined on the whole $B(0, R)$, so we need to extend $\xi_k^n$ a last time. We proceed as before, set $g_k = \xi_k^n - I$ on $\mathcal{S}_n^+$, observe that by (24.46) $g_k$ is $2(1+2\Lambda^2)^2 C_n \varepsilon_k$-Lipschitz on $\mathcal{S}_n^+$, use the Whitney extension theorem (on $\mathbb{R}^n$) to extend $g_k$, and set $\xi_k = g_k + I$ on $B(0, R)$ (we shall not need the values further away). Let us now check that $\xi_k$ satisfies the properties required in Definition 24.8.

First, we need to find sets $L_j^0$, $0 \leq j \leq j_{max}$, such that (24.7) holds when $x_0 \in L_j$ (otherwise, we don't need to find $L_j^0$, or we can take the empty set). Fix $j$, and let us try $L_j^0 = \bigcup_{F \in \mathcal{F}; F \subset L_j} F^0$. From (24.37) we deduce that

$$L_j^0 = \lim_{k \to +\infty} \bigcup_{F \in \mathcal{F}; F \subset L_j} F^k. \tag{24.70}$$

Recall that $F^k = r_k^{-1}(F - x_0)$ (see before Definition 24.29), and set $L'_j = \bigcup_{F \in \mathcal{F}; F \subset L_j} F$ and $L'_{j,k} = r_k^{-1}(L'_j - x_0)$; then (24.70) says that $L_j^0 = \lim_{k \to +\infty} L'_{j,k}$, while (24.7) requires $L_j^0 = \lim_{k \to +\infty} L_{j,k}$, i.e., for the possibly larger set $L_j$. That is, $L_j$ is the union of all the faces $F$ that are contained in $L_j$, and not only those that contain $x_0$, as in the definition of $\mathcal{F}$ (above Definition 24.29). But the difference does not hurt: there is only a finite number of faces $F$ in $L_j$ that do not contain $x_0$, and removing them does not change the limit of the $L_{j,k}$ because the corresponding sets $F^k$ go away to infinity. So our $L_j^0$ satisfy (24.7).

We know that $\xi_k$ is bilipschitz, that $\xi_k(0) = \xi_k^0(0) = 0$, the bilipschitz condition (24.9) holds with any $\rho_k \geq 1 + C(1+2\Lambda^2)^2 C_n \varepsilon_k$ (because of the small Lipschitz constant for $g_k = \xi_k - I$), and (24.11) holds because $\lim_{k \to +\infty} g_k(x) = 0$, so we are left with (24.10) to check.

First let $y \in L_{j,k} \cap B(0, \rho_k^{-1} R)$. If $k$ is large enough, and by the discussion above, $y \in L'_{j,k}$, which means that $y \in F^k$ for some $F \in \mathcal{F}$ such that $F \subset L_j$. Let $\ell$ be such that $F \in \mathcal{F}_\ell$; by (24.44) and because $\frac{1}{2}B_\ell \supset \frac{1}{2}B_n \supset B(0, R)$, we can find $x \in F^0 \cap B_\ell$ such that $\xi_k^\ell(x) = y$. Then $x \in L_j^0$, $\xi_k(x) = \xi_k^\ell(x) = y$, and $x \in B(0, R)$ by (24.48), and if $\rho_k \geq 1 + CC_\ell \varepsilon_k$. This yields the first part of (24.10). For the second part, we take $x \in L_j^0 \cap B(0, R)$, choose a face $F \subset \mathcal{F}$ such that $F \subset L_j$ and $x \in F^0$, and notice that $\xi_k(x) = \xi_k^\ell(x) \in F^k$, by (24.44), and that $|\xi_k(x)| \leq (1 + C_\ell \varepsilon_k)|x| \leq (1 + C_\ell \varepsilon_k) R \leq \rho_k R$ by (24.48) and if $\rho_k \geq 1 + 2C_\ell \varepsilon_k$. This proves the second inclusion in (24.11), the $\xi_k$ satisfy the requirements for Definition 24.8, and this concludes our proof of Proposition 24.35. □

## PART VI : OTHER NOTIONS OF QUASIMINIMALITY

### 25. Elliptic integrands; the main lower semicontinuity result.

Up to now we used the Hausdorff measure $\mathcal{H}^d(E)$ to measure the size of our sets $E$, but it is natural to consider other measure like $\int_E f(x) d\mathcal{H}^d(x)$ (if our space is not homogeneous), and even with functions $h$ that depend not only on the position of $x$ in space, but also on the tangent plane to $E$ at $x$, to model nonisotropic spaces. To the author's knowledge, the question of nonisotropic integrands like $h$, in the context of the Plateau problem, was raised by F. Almgren. In [A1], he states his generalization of Reifenberg's theorem on the homological Plateau problem in terms of elliptic integrands, and even adds, probably to explain the use of currents and varifolds in [A1]: "It does not seem possible to extend the arguments of De Giorgi or of Reifenberg to general elliptic integrands. In particular, the orthogonal invariance of the $m$ area integrand $F = 1$ is essential for the applicability of Reifenberg's methods". The author of these notes does not know whether this sentence is taken too seriously by the specialists, but just to make sure we shall explain in this section why many of the results of the previous sections still hold when $\mathcal{H}^d$ is integrated against a reasonable elliptic integrand. A good part of it is based on an adaptation of Dal Maso, Morel, and Solimini's uniform concentration lemma, which Y. Fang's proved to make his extension of Reifenberg's existence result for the homological Plateau problem work also in the context of elliptic integrands; see [Fa]. As usual, we shall need to change the lemmas because of the boundary conditions, but not the general scheme of the proof.

Let us first say what sort of elliptic integrands we shall consider, and how we integrate them on (rectifiable) sets. Our integrands will be Borel-measurable positive functions $f : U \times G(n,d) \to (0,+\infty)$, where $U$ is an open set in $\mathbb{R}^n$ and $G(n,d)$ denotes the Grasssman manifold of vector $d$-planes in $\mathbb{R}^n$, and their integral on rectifiable sets $E \subset U$ will be defined by

$$J_f(E) = \int_E f(x, T_x E)\, d\mathcal{H}^d(x), \tag{25.1}$$

where $T_x E$ denotes the non oriented vector $d$-plane which gives the approximate tangent plane to $E$ at $x$; thus $T_x E$ is defined $\mathcal{H}^d$-almost everywhere on $E$ because $E$ is rectifiable.

We shall not really need to define $J_f(E)$ when $E$ is not rectifiable, because we shall concentrate on quasiminimal sets, but let us mention here that we could do so easily with a trick: we could define an auxiliary function $\widetilde{f} : U \to (0,+\infty)$ (possibly using the values of $f$, but not necessarily), and then set

$$J_{f,\widetilde{f}}(E) = J_f(E_{rec}) + \int_{E \setminus E_{rec}} \widetilde{f}(x)\, d\mathcal{H}^d(x) \tag{25.2}$$

for Borel sets $E$ with $\mathcal{H}^d(E) < +\infty$, and where $E_{rec}$ denotes the rectifiable part of $E$. This may sound a little artificial, but the issue typically shows up when we want to state a result connected to the Plateau problem, want to define a functional $J$ even for sets that are not rectifiable, but know anyway that the minimizers (or even the very good competitors) will be rectifiable.

We will work with the following class of integrands, which is the same as in Fang's paper [Fa]; Almgren [A1] and [A3] mentions slightly more restricted classes, but the spirit is the same.

**Definition 25.3.** *Let $U \in \mathbb{R}^n$ be open. For $0 < a \leq b < +\infty$, we denote by $\mathcal{I}(U,a,b)$ (or just $\mathcal{I}(a,b)$) the set of continuous functions $f : U \times G(n,d) \to (0,+\infty)$, such that*

$$a \leq f(x,T) \leq b \quad \text{for } x \in U \text{ and } T \in G(n,d), \tag{25.4}$$

*and, for each $x \in U$, there is a radius $r(x) > 0$ and function $\varepsilon_x : (0, r(x)] \to [0,1]$, with*

$$\lim_{r \to 0} \varepsilon_x(r) = 0 \tag{25.5}$$

*(that will measure the near optimality of planes near $x$), such that*

$$J_f(P \cap B(x,r)) \leq J_f(S \cap B(x,r)) + \varepsilon_x(r) r^d \tag{25.6}$$

*when $0 < r \leq r(x)$, $P$ is a $d$-plane through $x$, and $S \subset \overline{B}(x,r)$ is a compact rectifiable set which cannot be mapped into $P \cap \partial B(x,r)$ by any Lipschitz mapping $\psi : \overline{B}(x,r) \to \overline{B}(x,r)$ such that $\psi(y) = y$ for $y \in P \cap \partial B(x,r)$.*

This definition is probably not optimal, but something like (25.6) is needed if we want to have existence results for the (local) minimization of $J_f$. Let us just explain what may go wrong, without computing a precise example. Take $n = 2$, $d = 1$, parameterize $G(2,1)$ by the angle $\theta$ of a line of $G(2,1)$ with the horizontal direction, and consider functions of the form $f(x,\theta) = f_1(x_1) f_2(\theta)$, where $f_1(x)$ is a function of the horizontal variable of $\mathbb{R}^2$ which is minimal on the $x_1$-axis, and $f_2(\theta)$ is an nice function, but such that $f_2(\pm\pi/4) < f_2(0)/2$. A good minimizing sequence will be composed of zig-zag curves that stay close to the $x_1$-axis, with for instance slopes that stay close to $\pm 1$; it will converge to the axis itself, which is not a minimizer because $f_2(0)$ is too large. Then we can easily cook up some some problems for which there is no minimizer because, if one existed, it would have to be the $x_1$-axis.

Our definition is a little unpleasant because it is hard to control the list of sets $S$ that satisfy the non-retractability condition above, but there are convexity conditions that imply that $f \in \mathcal{I}(a,b)$ (assuming (25.4)). At least we have one example : the constant 1 lies in $\mathcal{I}(1,1)$ because the orthogonal projection of any $S$ as above contains $P \cap B(x,r)$. Similarly, if $f$ is a continuous function of $x$ alone such that $a \leq f(x) \leq b$ everywhere, then $f \in \mathcal{I}(a,b)$. We are a little sorry because we do not allow functions $f$ that are merely lower semicontinuous; see Remark 25.87, Claim 25.89, and (25.96) below for slightly more general conditions that work.

The main result of this section is the following generalization of Theorem 10.97.

**Theorem 25.7.** *Let $U$, $\{E_k\}$, and $E$ satisfy the hypotheses (10.1), (10.2), (10.3), and (10.4). Also suppose that $h$ is small enough, depending only on $n$, $M$, and $\Lambda$. Then*

$$J_f(E \cap V) \leq \liminf_{k \to +\infty} J_f(E_k \cap V) \tag{25.8}$$

*for $0 < a \leq b < +\infty$, every open set $V \subset U$, and every $f \in \mathcal{I}(V,a,b)$.*

Notice that the sets $E_k$ and $E$ are rectifiable, by Theorem 5.16 and Proposition 10.15, so $J_f(E_k \cap V)$ and $J_f(E \cap V)$ are well defined by (25.1).

We shall first prove this for sets $V$ that are relatively compact in $U$. As in [Fa], our argument will be inspired by Dal Maso, Morel, and Solimini's [DMS], but with some pleasant simplifications in the covering argument. The present argument will rely directly on the rectifiability of the limit $E$ (which gives flatness almost everywhere), rather than the concentration lemma (which gives flatness with a more quantitative control, even though on balls that are not centered at the original point); this will allow us to apply a more standard version of the Vitali covering lemma, at the (small) price of a less constructive argument.

Let $\{E_k\}$, $E$, and $V \subset\subset U$ be as in the statement. Our first task is to find a large set $E^1$ of $E \cap V$, and lots of nice small balls centered on $E^1$. First observe that $E$ is rectifiable, by Proposition 10.15, hence for $\mathcal{H}^d$-almost every $x \in E \cap V$,

$$\lim_{r \to 0} r^{-d} \mathcal{H}^d(E \cap B(x,r)) = \omega_d \tag{25.9}$$

(see Theorem 17.6 on page 240 in [Ma]), and

$$E \text{ has a tangent plane } P(x) \text{ at } x \tag{25.10}$$

(here Theorem 15.19 (3) on page 212 of [Ma] only gives an approximate tangent plane, but since $E$ is locally Ahlfors-regular near $x$ (by (10.11)) Exercise 41.21 on page 277 of [D4] says that this plane is a true tangent plane.

Set $E^0$ denote the set of points $x \in E \cap V$ such that (25.9) and (25.10) hold. In fact, other constraints will appear later, which will lead us to removing other negligible pieces of $E^0$, but this will not matter. Let $\varepsilon > 0$ be small; we shall let it tend to 0 at the end of the estimate. For each $x \in E^0$, we select $r(x) > 0$ with various properties, which are all true for $r$ small enough. First,

$$r(x) \text{ is small enough, depending on } r_0,\ \lambda,\ \Lambda, \text{ and } \operatorname{dist}(x, \mathbb{R}^n \setminus U), \tag{25.11}$$

where $\lambda$, $\Lambda$, and $r_0$ (the scale of the the dyadic cube that we use in the unit ball) are as in the Lipschitz assumption; how small will depend on simple geometric constraints that will arise in our construction, and we don't need to know this precisely. Next,

$$\left|\omega_d - r^{-d}\mathcal{H}^d(E \cap B(x,r))\right| \leq \varepsilon \ \text{ for } 0 < r < 2r(x), \tag{25.12}$$

and

$$\operatorname{dist}(y, P(x)) \leq \varepsilon |y - x| \ \text{ for } y \in E \cap B(x, 3r(x)). \tag{25.13}$$

We also need to control the variations of our integrand near $x$. The function $x \to T_x E$ (where with our new notation, $T_x E$ is the vector space parallel to $P(x)$) is Borel-measurable on $E^0$; this is unpleasant, but not hard to check, especially because $E$ is locally Ahlfors-regular, but anyway we leave the proof to the reader. Then by Lusin's theorem, we can find a Borel set $E^1 \subset E^0$, such that

$$\mathcal{H}^d(E \cap V \setminus E^1) \leq \varepsilon, \tag{25.14}$$

and on which $T_x E$ is a continuous function of $x$. We use this, and the uniform continuity of $f$ on $W \times G(n,d)$ for some neighborhood $W$ of $x$, to say that if $r(x) > 0$ small enough (depending on $x$),

$$|f(y, T_y E) - f(x, T_x E)| \leq \varepsilon \ \text{ for } y \in E \cap B(x, 2r(x)) \tag{25.15}$$

and

$$|f(y,T_xE) - f(x,T_xE)| \le \varepsilon \ \text{ for } y \in B(x, 2r(x)). \tag{25.16}$$

We certainly require this from $r(x)$, but other similar constraints concerning the set $E^1$ and the radius $r(x)$ will show up, in relation with our boundary constraints; we shall find it more pleasant to mention them later, as we use them.

For the moment, we fix $x \in E^1$ and $0 < r \le r(x)$, set $B = \overline{B}(x,r)$, and try to evaluate the contribution of $B$ to the two sides of (25.8). Set $D = P(x) \cap B$; we first compare $J_f(E \cap B)$ with $J_f(D)$. We just observe that

$$\begin{aligned} J_f(E \cap B) &= J_f(E^1 \cap B) + J_f((E \setminus E^1) \cap B) \\ &\le \int_{E^1 \cap B} f(y, T_yE)\, d\mathcal{H}^d(y) + b\mathcal{H}^d((E \setminus E^1) \cap B) \\ &\le (f(x,T_xE) + \varepsilon)\, \mathcal{H}^d(E^1 \cap B) + b\mathcal{H}^d((E \setminus E^1) \cap B) \end{aligned} \tag{25.17}$$

by (25.1), because $f \in \mathcal{I}(a,b)$, and by (25.15). Then

$$\mathcal{H}^d(E^1 \cap B) \le \mathcal{H}^d(E \cap B) \le (\omega_d + \varepsilon) r^d \tag{25.18}$$

by (25.12), and

$$f(x,T_xE)\omega_d r^d = \int_D f(x,T_xE)\, d\mathcal{H}^d(y) \le J_f(D) + \varepsilon\omega_d r^d = J_f(P(x) \cap B) + \varepsilon\omega_d r^d \tag{25.19}$$

by (25.16) and because $D = P(x) \cap B$. Thus

$$\begin{aligned} (f(x,T_xE) + \varepsilon)\, \mathcal{H}^d(E^1 \cap B) &\le (\omega_d^{-1} r^{-d} J_f(P(x) \cap B) + 2\varepsilon)(\omega_d + \varepsilon) r^d \\ &\le J_f(P(x) \cap B) + r^d(\varepsilon\omega_d^{-1} r^{-d} J_f(P(x) \cap B) + 2\varepsilon\omega_d + 2\varepsilon^2) \\ &\le J_f(P(x) \cap B) + \varepsilon r^d(b + 2\omega_d + \varepsilon) \end{aligned} \tag{25.20}$$

by (25.18), (25.19), and (25.1), and now (25.17) yields

$$J_f(E \cap B) \le J_f(P(x) \cap B) + \varepsilon(b + 2\omega_d + \varepsilon) r^d + b\mathcal{H}^d((E \setminus E^1) \cap B). \tag{25.21}$$

We also need a lower bound for $J_f(E_k \cap B)$, and for this we shall need to introduce a set $S$ as in Definition 25.3 and use the quasiminimality of $E_k$ to show that $S$ cannot be retracted.

First observe that for $k$ large enough (depending on $x$, but this will not matter)

$$E_k \cap \frac{3}{2}B \subset H, \quad \text{where } H = \big\{y \in \mathbb{R}^n\,;\, \mathrm{dist}(y, P(x)) \le 3\varepsilon r\big\}, \tag{25.22}$$

just by (25.13) and because the $E_k$ converge to $E$ in $2B$. We want to modify $E_k$ a first time, in the set

$$A_{20} = \big\{y \in \mathbb{R}^n\,;\, (1 - 20\varepsilon) r \le |y - x| \le (1 + 20\varepsilon) r\big\} \tag{25.23}$$

because we want a set $S$ such that $S \cap \partial B(x,r) \subset P(x)$. Denote by $\pi$ the orthogonal projection on $P(x)$, and set

$$g(y) = \alpha(|y-x|)\pi(y) + (1-\alpha(|y-x|))y \tag{25.24}$$

for $y \in \mathbb{R}^n$, where $\alpha$ is the continuous, piecewise affine mapping defined by $\alpha(t) = 0$ for $t \in [0,(1-20\varepsilon)r] \cup [(1+20\varepsilon)r,+\infty)$, $\alpha(t) = 1$ for $t \in [(1-10\varepsilon)r,(1+10\varepsilon)r]$, and $\alpha$ is affine on each of the two remaining intervals $[(1-20\varepsilon)r,(1-10\varepsilon)r]$ and $[(1+10\varepsilon)r,(1+20\varepsilon)r]$. Notice that

$$|g(y) - y| \le |\pi(y) - y| \le 3\varepsilon r \quad \text{for } y \in H, \tag{25.25}$$

and hence, if we set $A_5 = \big\{y \in \mathbb{R}^n\,;\, (1-5\varepsilon)r \le |y-x| \le (1+5\varepsilon)r\big\}$,

$$g(H) \cap A_5 \subset P(x) \tag{25.26}$$

because if $y \in H$ is such that $g(y) \in A_5$, then $||y-x|-r| \le 8\varepsilon r$, and hence $g(y) = \pi(y)$. We set $S = g(E_k) \cap B$; the next lemma is probably the key step of the proof.

**Lemma 25.27.** *There is no Lipschitz mapping $\psi : \overline{B} \to \overline{B}$ such that (as in Definition 25.3) $\psi(y) = y$ for $y \in P(x) \cap \partial B$ and $\psi(S) \subset P(x) \cap \partial B$.*

This will allow us to apply Definition 25.3 and get (25.6). We want to prove the lemma by contradiction, suppose there exists such a $\psi$, and use it to construct a new mapping $\varphi$ and an impossible competitor for $E_k$. First observe that $\pi \circ \psi$ has the same properties as $\psi$, so, at the price of replacing $\psi$ with $\pi \circ \psi$, we may assume that

$$\psi(z) \in P(x) \cap B \quad \text{for } z \in B. \tag{25.28}$$

We want to extend $\psi$ to $\mathbb{R}^n$, and we do this in two steps. First we set

$$\psi(z) = z \quad \text{for } z \in (P(x) \setminus B) \cup [\mathbb{R}^n \setminus (1+5\varepsilon)B]. \tag{25.29}$$

The first extension that we get this way is still Lipschitz; we can easily check this by hand, using the fact that $\psi(z) = z$ for $z \in P(x) \cap \partial B$ (connect a point of $\overline{B}$ to any point of the rest of a domain through a point of $P(x) \cap \partial B$). Then we extend $\psi$ to the whole $\mathbb{R}^n$, using for instance the Whitney extension theorem; we can even make sure that $\psi((1+5\varepsilon)B) \subset (1+5\varepsilon)B$, because otherwise we can compose the restriction to $(1+5\varepsilon)B$ with the radial projection from $\mathbb{R}^n$ onto $(1+5\varepsilon)B$.

If we were dealing with quasiminimal sets with no boundary constraints, we would use the mapping $\varphi = \psi \circ g$ as the endpoint of a one parameter family (as in Definition 1.3), to test the quasiminimality of $E_k$ and get a contradiction. The point is that $E_k \cap B$ is sent to $S$ by $g$, and then to $P(x) \cap \partial B$ by $\psi$, which means that all its measure disappears. We will see that $\mathcal{H}^d(E_k \cap A)$ is small, and so is $\mathcal{H}^d(\varphi(E_k \cap A))$, so we would get that $\mathcal{H}^d(\varphi(E_k \cap (A \cup B)))$ is small, while $\mathcal{H}^d(E_k \cap B)$ is reasonably large, because $E_k$ is locally Ahlfors-regular. The ensuing contradiction would prove Lemma 25.27.

But we have boundary constraints coming from the $L_j$, and so we will need to modify $\varphi$ before we use it to test the quasiminimality of $E_k$. This will be easier if we first add a few constraints on the set $E^1$ and the radius $r(x)$.

We first replace the sets $E^0$ and $E^1$ above by the slightly smaller sets where we add the requirement that every point $x \in E^0$, is a Lebesgue density point of $E \cap F$ for every face $F$ of our dyadic grid on $U$ that contains $x$. This means that

$$\lim_{r \to 0} r^{-d} \mathcal{H}^d(E \cap B(x,r) \setminus F) = 0 \tag{25.30}$$

for every such face. For each face $F$, (25.30) is true for $\mathcal{H}^d$-almost every $x \in E \cap F$, so our new constraint only removes a $\mathcal{H}^d$-negligible set from $E^0$ and $E^1$. Notice that for each $x \in U$, there is a smallest face $F_x$ of our grid that contains $x$ (because the intersection of two faces that contain $x$ is a face that contains $x$), and (25.30) for $F_x$ is stronger than for the other faces.

We also put additional conditions on the radius $r(x)$, $x \in E^1$. Namely, we require that

$$\mathrm{dist}(x, F) \geq 2r(x) \text{ for every face } F \text{ of our grid that does not contain } x, \tag{25.31}$$

we recall our constraint (25.11) that $r(x)$ be small enough (constraints of that type will arise soon), and we also demand that

$$\mathcal{H}^d(E \cap B(x,r) \setminus F_x) \leq \varepsilon^d r^d \quad \text{for } 0 < r < 3r(x). \tag{25.32}$$

Because of the boundary constraints, we shall need a retraction on the smallest face $F = F_x$ that contains $x$. Recall from Lemma 3.14 and Remark 3.25 that when $F$ is a standard dyadic cube (i.e., under the rigid assumption), there is a natural projection $\pi = \pi^F$, defined on a $(r_0/3)$-neighborhood of $F$ (where $r_0$ is the scale of our smallest cubes). Under the Lipschitz assumption, we use the rigid face $\widetilde{F} = \psi(\lambda F)$, and the projection $\pi_F$ defined by

$$\pi_F(y) = \lambda^{-1} \psi^{-1}(\pi^{\widetilde{F}}(\psi(\lambda y)), \tag{25.33}$$

which is now defined on a neighborhood of $F$ whose width near $x$ could easily be computed in terms of $r_0$, $\lambda$, $\Lambda$, and $\mathrm{dist}(x, \mathbb{R}^n \setminus U)$. With this projection comes a retraction, defined by

$$\Pi_F(y,s) = \lambda^{-1} \psi^{-1}\big(s(\pi^{\widetilde{F}}(\psi(\lambda y)) + (1-s)\psi(\lambda y)\big), \tag{25.34}$$

for $y$ the same neighborhood of $F$ and $0 \leq s \leq 1$. We shall use a different time for different points; that is, for $y \in \mathbb{R}^n$ and $0 \leq t \leq 1/2$, we set

$$s(y,t) = 0 \text{ when } |y - x| \geq (1 + 30\varepsilon)r, \tag{25.35}$$

and

$$s(y,t) = 2t\ \min\big(1, \frac{(1+30\varepsilon)r - |y-x|}{10\varepsilon r}\big) \ \text{ when } |y-x| \leq (1+30\varepsilon)r \tag{25.36}$$

(so that in particular $s(y,t) = 2t$ when $y \in \overline{B}(x,(1+20\varepsilon)r)$. Then we set

$$s(y,t) = s(y,1/2) \text{ when } 1/2 \leq t \leq 1. \tag{25.37}$$

We also want to interpolate between $y$ and $\varphi(y) = \psi(g(y))$, so we set

$$\varphi_t(y) = y \ \text{ for } 0 \leq t \leq 1/2 \tag{25.38}$$

and

$$\varphi_t(y) = (2t-1)\varphi(y) + (2-2t)y \ \text{ for } 1/2 \leq t \leq 1. \tag{25.39}$$

Finally, we want to use the family $\{h_t\}$ defined by

$$h_t(y) = \Pi_F(\varphi_t(y), s(y,t)), \tag{25.40}$$

but a few verifications will be needed.

First of all, $\pi_F(z)$ and $\Pi_F(z,s)$ are well defined when $z \in 2B$ and $0 \leq s \leq 1$. Indeed, $x \in F$ and hence $\mathrm{dist}(z,F) \leq |z-x| \leq 2r$, and (25.11) allows us to choose $r(x)$ so small that $2B$ is contained in the neighborhood of $F$ that was mentioned below (25.33). Next observe that

$$\varphi_t(y) = y \text{ for } 0 \leq t \leq 1 \text{ when } |y-x| \geq (1+20\varepsilon)r, \tag{25.41}$$

since $g(y) = y$ by (25.24) and because $\alpha(|y-x|) = 0$, hence $\varphi(y) = \psi(y) = y$ by (25.29), and finally $\varphi_t(y) = y$ by (25.38) and (25.39).

If moreover $(1+30\varepsilon)r \leq |y-x| \leq 2r$, then $s(y,t) = 0$ by (25.35) and (25.37), so $h_t(y) = \varphi_t(y) = y$ by (25.40) and (25.34).

If instead $y \in \mathbb{R}^n \setminus 2B$, and even though $\Pi_F(y,s)$ is not formally defined above, we can safely extend the definitions (for instance, set $\Pi_F(z,0) = z$) and keep $h_t(y) = y$ there. So

$$h_t(y) = y \text{ for } 0 \leq t \leq 1 \text{ when } |y-x| \geq (1+30\varepsilon)r, \tag{25.42}$$

and nothing happens there.

Before we continue with other regions, it will be useful to know that for some constant $C_\Lambda$, that depends on the local Ahlfors regularity constant for $E$ near $x$,

$$\mathrm{dist}(y,F) \leq C_\Lambda \varepsilon r \ \text{ for } y \in E \cap \frac{29}{10}B. \tag{25.43}$$

Indeed otherwise, $E \cap B(y, C_\Lambda \varepsilon r)$ does not meet $F$, and then $\mathcal{H}^d(E \cap B(x,3r) \setminus F) \geq \mathcal{H}^d(E \cap B(y, C_\Lambda \varepsilon r) \setminus F) \geq C^{-1}(C_\Lambda \varepsilon r)^d$ by local Ahlfors regularity of $E$ (see Propositions 4.1 and 4.74). This contradicts (25.32) if $C_\Lambda$ is chosen large enough; (25.43) follows. Since the $E_k$ converge to $E$, we also get that for $k$ large,

$$\operatorname{dist}(y, F) \leq 2C_\Lambda \varepsilon r \quad \text{for } y \in E_k \cap \frac{28}{10}B. \tag{25.44}$$

For the moment, we only care about $y \in E_k \cap 2B$. Set $\widetilde{y} = \psi(\lambda y)$, and notice that for $0 \leq s, s' \leq 1$,

$$\begin{aligned} |\Pi_F(y,s) - \Pi_F(y,s')| &= \lambda^{-1}\Big[\psi^{-1}\big(s(\pi^{\widetilde{F}}(\widetilde{y}) + (1-s)\widetilde{y}\big) - \psi^{-1}\big(s'(\pi^{\widetilde{F}}(\widetilde{y}) + (1-s')\widetilde{y}\big)\Big] \\ &\leq \lambda^{-1}\Lambda|s-s'||\pi^{\widetilde{F}}(\widetilde{y}) - \widetilde{y}| \leq C\lambda^{-1}\Lambda|s-s'| \operatorname{dist}(\widetilde{y}, \widetilde{F}) \\ &\leq C\Lambda^2|s-s'| \operatorname{dist}(y,F) \leq C(\Lambda)|s-s'|\varepsilon r \end{aligned} \tag{25.45}$$

by (25.34), because $\pi^{\widetilde{F}}$ is Lipschitz and $\pi^{\widetilde{F}}(z) = z$ on $\widetilde{F}$ (see (3.6)), and by (25.43); here and below, $C(\Lambda)$ is our notation for a constant that depends on $\Lambda$ (but also on $M$ and the other usual constants). Since $\Pi_F(y,0) = y$, we get that

$$|\Pi_F(y,s) - y| \leq C(\Lambda)\varepsilon r \quad \text{for } y \in E_k \cap \tfrac{18}{10}B \text{ and } 0 \leq s \leq 1. \tag{25.46}$$

Notice also that for $y, z \in 2B$ and $0 \leq s \leq 1$,

$$|\Pi_F(y,s) - \Pi_F(z,s)| \leq C\Lambda^2|y-z|, \tag{25.47}$$

by (25.34) and because $\pi^{\widetilde{F}}$ is $C$-Lipschitz.

We continue our study of the $h_t$ with what happens in the region

$$R_1 = \big\{y \in E_k\,;\, (1+10\varepsilon)r \leq |y-x| \leq (1+30\varepsilon)r\big\}. \tag{25.48}$$

Let $y \in R_1$ be given; first observe that $|g(y) - y| \leq 3\varepsilon r$ by (25.25) and (25.22), hence $|g(y) - x| \geq 7\varepsilon r$ (by definition of $R_1$), and (25.29) says that

$$\varphi(y) = \psi(g(y)) = g(y) \quad \text{for } y \in R_1. \tag{25.49}$$

We'll need to know that

$$|g(y) - g(z)| \leq 2|y-z| \quad \text{for } y, z \in E_k \cap 2B \tag{25.50}$$

so we return to the definition (25.24), write $g(y) = \alpha\pi(y) + (1-\alpha)y$, with $\alpha = \alpha(|y-x|)$, and similarly $g(z) = \beta\pi(y) + (1-\beta)y$ with $\beta = \alpha(|z-x|)$, and write that

$$\begin{aligned} |g(y) - g(z)| &= |\alpha\pi(y) + (1-\alpha)y - \beta\pi(y) - (1-\beta)y| \\ &\leq |(\alpha-\beta)(\pi(y) - y)| + \beta|\pi(y) - \pi(z)| + (1-\beta)|y-z| \\ &\leq 3\varepsilon r|\alpha-\beta| + |y-z| \leq 2|y-z| \end{aligned} \tag{25.51}$$

by (25.22), and because $|\alpha-\beta| \leq (10\varepsilon)^{-1}|y-z|$ by the definition below (25.24). So (25.50) holds.

By (25.50), the definitions (25.38) and (25.39), and the fact that $\varphi = g$ on $R_1$, each $\varphi_t$ is also 2-Lipschitz on $R_1$. In addition, notice that for $y \in R_1$,

$$|\varphi_t(y) - y| \leq |\varphi(y) - y| = |g(y) - y| \leq 3\varepsilon r \tag{25.52}$$

by (25.38) and (25.39), (25.49), and (25.25) and (25.22). Thus $\operatorname{dist}(\varphi_t(y), F) \leq C(\Lambda)\varepsilon r$ by (25.44), and the proof of (25.45) also yields

$$|\Pi_F(\varphi_t(y), s) - \Pi_F(\varphi_t(y), s')| \leq C(\Lambda)|s - s'|\varepsilon r \tag{25.53}$$

for $0 \leq s, s' \leq 1$ (just replace $\operatorname{dist}(y, F)$ with $\operatorname{dist}(\varphi_t(y), F)$). Recall that the $\varphi_t(y)$ stay in $2B$, where $\Pi_F$ is well defined and all the formulas that we use make sense (because $r \leq r(x)$ and if $r(x)$ is chosen small enough). Similarly, we still have (as in (25.47)) that

$$|\Pi_F(\varphi_t(y), s) - \Pi_F(\varphi_t(z), s)| \leq C(\Lambda)|\varphi_t(y) - \varphi_t(z)| \leq C(\Lambda)|y - z|, \tag{25.54}$$

for $y, z \in R_1$ and $0 \leq s \leq 1$ by (25.34), and because $\pi^{\widetilde{F}}$ is $C$-Lipschitz and $\varphi_t$ is 2-Lipschitz on $R_1$. Hence,

$$\begin{aligned} |h_t(y) - h_t(z)| &= |\Pi_F(\varphi_t(y), s(y,t)) - \Pi_F(\varphi_t(z), s(y,t))| \\ &\leq |\Pi_F(\varphi_t(y), s(y,t)) - \Pi_F(\varphi_t(y), s(z,t))| \\ &\qquad\qquad + |\Pi_F(\varphi_t(y), s(z,t)) - \Pi_F(\varphi_t(z), s(z,t))| \\ &\leq C(\Lambda)\varepsilon r|s(y,t) - s(z,t)| + C(\Lambda)|y - z| \leq C(\Lambda)|y - z| \end{aligned} \tag{25.55}$$

by (25.40), (25.53), (25.54), and our definition of $s(y,t)$ in (25.36). Thus $h_t$ is $C(\Lambda)$-Lipschitz on $R_1$, and in particular

$$\mathcal{H}^d(h_1(R_1)) \leq C(\Lambda)\mathcal{H}^d(R_1). \tag{25.56}$$

Next we consider

$$R_2 = \big\{y \in E_k\,;\, |y - x| < (1 + 10\varepsilon)r \text{ and } g(y) \notin B\big\}. \tag{25.57}$$

Let us first check that

$$h_t(y) \in B(x, 2\Lambda^2 r) \text{ when } y \in E_k \cap B(x, (1 + 10\varepsilon)r) \text{ and } 0 \leq t \leq 1. \tag{25.58}$$

When $t \leq 1/2$, $\varphi_t(y) = y$ by (25.38), hence $h_t(y) = \Pi_F(y, s(y,t))$ by (25.40). Then $|h_t(y) - y| \leq C(\Lambda)\varepsilon r < r/2$ by (25.46) and if $\varepsilon$ is small enough, and $h_t(y) \in 2B$ (which is better than promised).

When $t \geq 1/2$, $s(y,t) = s(y, 1/2) = 1$ by (25.37) and (the sentence below) (25.36), hence (25.40) yields

$$h_t(y) = \Pi_F(\varphi_t(y), 1) = \pi_F \circ \varphi_t(y) \tag{25.59}$$

(compare (25.34) with (25.33)). Notice that $\pi_F(x) = x$ (by (25.33), because $\psi(\lambda x) \in \widetilde{F}$ (since $x \in F$), and by (3.6)); then

$$|h_t(y) - x| = |\pi_F \circ \varphi_t(y) - \pi_F(x)| \leq \Lambda^2 |\varphi_t(y) - x| \tag{25.60}$$

because $\pi_F$ is $\Lambda^2$-Lipschitz by (25.33). If $g(y) \in B$, we also get that $\varphi(y) = \psi \circ g(y) \in B$, by (25.28), hence $\varphi_t(y) \in 2B$ by (25.38) and (25.39); then (25.60) says that $|h_t(y) - x| < 2\Lambda^2 r$, as needed for (25.58).

We are left with the case when $g(y) \notin B$, i.e., when $y \in R_2$. We claim that

$$\varphi(y) = \pi(y) \text{ for } y \in R_2. \tag{25.61}$$

As soon as we prove this, we will get that $\varphi_t(y) \in 2B$ (by (25.38) and (25.39)), and (25.58) will follow from (25.60).

Now we prove the claim. Let $y \in R_2$ be given. By definition of $R_2$, $g(y) \notin B$; hence by (25.25) and (25.22), $|y - x| \geq (1 - 3\varepsilon) r$. Since $|y - x| \leq (1 + 10\varepsilon) r$ by definition of $R_2$, we get that $\alpha(|y - x|) = 1$ (see below (25.24)), and $g(y) = \pi(y) \in P(x)$ by (25.24). Since $g(y) \notin B$, (25.29) yields $\varphi(y) = \psi \circ g(y) = g(y) = \pi(y)$, as needed for (25.61).

We are a little more interested in what happens for $t = 1$. Then (25.59), (25.39), and (25.61) say that

$$h_1(y) = \pi_F \circ \varphi_1(y) = \pi_F \circ \varphi(y) = \pi_F \circ \pi(y). \tag{25.62}$$

This is good, because it means that $h_1$ is Lipschitz on $R_2$, with a constant that depends on $\Lambda$, but not on $\varepsilon$, for instance. Then

$$\mathcal{H}^d(h_1(R_2)) \leq C(\Lambda) \mathcal{H}^d(R_2). \tag{25.63}$$

We are left with the region

$$R_3 = \big\{ y \in E_k\,;\, |y - x| \leq (1 + 10\varepsilon) r \text{ and } g(y) \in B \big\}. \tag{25.64}$$

On this last region, we do not control the Lipschitz norm of $h_1$ (because we do not control the Lipschitz norm of $\psi$), but fortunately (25.59) and (25.39) yield $h_1(y) = \pi_F \circ \varphi(y) = \pi_F \circ \psi(g(y))$, so

$$h_1(R_3) \subset \pi_F \circ \psi(g(R_3)) \subset \pi_F \circ \psi(g(E_k) \cap B) = \pi_F \circ \psi(S) \subset \pi_F(P(x) \cap \partial B) \tag{25.65}$$

because we set $S = g(E_k) \cap B$ (above Lemma 25.27) and by definition of $\psi$ (below that lemma). Since $\pi_F$ is Lipschitz, we get that

$$\mathcal{H}^d(h_1(R_3)) = 0. \tag{25.66}$$

We want to apply the definition of a quasiminimal set, so we check that the $h_t$ satisfy the conditions (1.4)-(1.8), relative to the ball $2\Lambda^2 B$. The continuity and Lipschitz conditions (1.4) and (1.8) are satisfied (all our maps are Lipschitz, even though with possibly huge constants), and (1.5) follows from (25.42).

For (1.6), we just need to check that $h_t(y) \in B(x, 2\Lambda^2 r)$ when $y \in E_k \cap B(x, (1+30\varepsilon)r$, because otherwise (25.42) says that $h_t(y) = y$. When $y \in B(x, (1+10\varepsilon)r$, this follows from (25.58), so we may assume that $y \in R_1$ (see (25.48)). By (25.52), $|\varphi_t(y) - y| \leq 3\varepsilon r$, and so (25.40) yields

$$
\begin{aligned}
|h_t(y) - x| = |\Pi_F(\varphi_t(y), s(y,t)) - x| &\leq |\Pi_F(\varphi_t(y), 0) - x| + C(\Lambda)\varepsilon r \\
&\leq |\varphi_t(y) - x| + C(\Lambda)\varepsilon r \leq 2r
\end{aligned}
\tag{25.67}
$$

by (25.53) with $s' = 0$, because $\Pi_F(\varphi_t(y), 0) = \varphi_t(y)$ and if $\varepsilon$ is small enough; (1.6) follows.

As usual, we end the verification with the boundary condition (1.7). Let $y \in E_k$ be given, suppose $y \in L_j$ for some $j$, and let us check that $h_t(y) \in L_j$ for $0 \leq t \leq 1$. There is nothing to check if $|y - x| \geq (1 + 30\varepsilon)r$, because $h_t(y) = y$ by (25.42). Otherwise, let $G$ be a face of our grid that contains $y$.

First assume that $(1 + 20\varepsilon)r \leq |y - x| \leq (1 + 30\varepsilon)r$. Then $\varphi_t(y) = y$ by (25.41), and $h_t(y) = \Pi_F(y, s(y,t)$ by (25.40). Both $\pi_F$ and $\Pi_F$ were designed to preserve all the faces of our grid: see Lemma 3.4 for $\pi^{\widetilde{F}}$, observe that $s\pi^{\widetilde{F}} + (1-s)I$ preserves the faces of the usual dyadic grid too (by convexity of the faces), and then $\pi_F$ and $\Pi_F$ preserve the face $G$, because we conjugate with $\psi(\lambda\cdot)$ (see (25.33) and (25.34)). Thus $h_t(y) \in G$, as needed.

So we may assume that $|y - x| \leq (1 + 20\varepsilon)r$. For $t \leq 1/2$, $\varphi_t(y) = y$ by (25.38), so $h_t(y) = \Pi_F(y, s(y,t))$, and we get that $h_t(y) \in G$ by the same argument as above. So we restrict to $t \geq 1/2$. Then $s(y,t) = s(y, 1/2) = 1$, by (25.37) and (the line below) (25.36). Then (25.40), together with (25.33) and (25.34), yields

$$
h_t(y) = \Pi_F(\varphi_t(y), 1) = \pi_F(\varphi_t(y)). \tag{25.68}
$$

Our next case is when $(1+10\varepsilon)r \leq |y-x| \leq (1+20\varepsilon)r$; then $y \in R_1$ (see (25.48)) and (25.52) says that $|\varphi_t(y) - y| \leq 3\varepsilon r$. Notice that $\mathrm{dist}(\varphi_t(y), F) \leq \mathrm{dist}(y, F) + 3\varepsilon r \leq C(\Lambda)\varepsilon r$ by (25.44), and hence, if $\varepsilon$ is small enough, $h_t(y) = \pi_F(\varphi_t(y)) \in F$ by definition of $\pi_F$ (see near (25.33), and then Lemma 3.4 and Remark 3.25). But by (25.31), $G$ contains $x$; since $F$ was chosen (below (25.32)) as the smallest face that contains $x$, we get that $F \subset G \subset L_j$, as needed.

Next we assume that $y \in R_2$. In this case we still have that $h_t(y) = \pi_F(\varphi_t(y))$, by (25.59), and in addition $\varphi(y) = \pi(y)$ by (25.61). In this case, $|\varphi_t(y) - y| \leq |\varphi(y) - y| = \mathrm{dist}(y, P(x)) \leq 3\varepsilon r$ by (25.39), the definition of $\pi$ above (25.24), and (25.22). Thus $\mathrm{dist}(\varphi_t(y), F) \leq C(\Lambda)\varepsilon r$ again, and we may conclude as before.

We are left with the case when $y \in R_3$. Then $g(y) \in B$, and by (25.29), $\varphi(y) = \psi(g(y)) \in P(x) \cap \partial B$. Recall that $\varphi_t(y) \in [y, \varphi(y)]$ by (25.39); since $\mathrm{dist}(y, P(x)) \leq 3\varepsilon r$ by (25.22) and $\mathrm{dist}(y, B) \leq |y - g(y)| \leq 3\varepsilon r$ by (25.25) and (25.22), this yields

$$
\mathrm{dist}(\varphi_t(y), P(x) \cap B) \leq 6\varepsilon r \tag{25.69}
$$

But we want to show that $\varphi_t(y)$ lies close to $F$, and since $F$ is a distorted face which may not be flat, we shall need to show that $\varphi_t(y)$ lies close to $E_k$, and then use (25.44).

Unfortunately, we shall need to use Lemma 9.14. Recall that $x \in E^1 \subset E$ (see the definitions above (25.17) and (25.9)), but since we want to apply the lemma to the set $E_k$, we restrict to $k$ large, choose $x_k \in E_k \cap B(x, \varepsilon r)$, and apply the lemma with $y = x_k$, $t = r$ (so that $B(x_k, 2t) \subset B(x, 3r)$), and $P = P(x)$. Recall that $E_k = E_k^*$ because we assumed (10.3). The size condition (9.15) is satisfied because $r \leq r(x)$ and if $r(x)$ is small enough (this is allowed by (25.11)). If some $L_i$ meets $B(x, 2r)$ and $G$ is a face of $L_i$ that meets $B(x, 2r)$, (25.31) says that $G$ contains $x$; since $F$ is the smallest face that contains $x$, we get that $F \subset G \subset L_i$. Thus the set $L$ of (9.16) contains $F$, and our assumption (9.17) holds with $\eta = C(\Lambda)\varepsilon$, by (25.44). Finally the assumption (9.18) is satisfied for $k$ large (and with the constant $2\varepsilon$), by (25.13) and because the $E_k$ converge to $E$ (recall that we apply the lemma to $E_k$, which is why we only get $2\varepsilon$). Thus, if $\varepsilon$ is small enough, the lemma applies, and we get (9.19). That is,

$$\operatorname{dist}(p, E_k) \leq 2\varepsilon r \quad \text{for } p \in P(x) \cap B(x_k, 3r/2). \tag{25.70}$$

Return to $y \in R_3$. For each $t \in [0,1]$, (25.69) gives $p \in P(x) \cap \overline{B}$ such that $|p - \varphi_t(y)| \leq 6\varepsilon r$. Then (25.70) gives $z \in E_k$ such that $|z - p| \leq 2\varepsilon r$. In turn $z \in 2B$, so (25.44) says that $\operatorname{dist}(z, F) \leq C(\Lambda)\varepsilon r$. Altogether $\operatorname{dist}(\varphi_t(y), F) \leq C(\Lambda)\varepsilon r$, and (if $\varepsilon$ is small enough), (25.68) implies that $h_t(y) \in F \subset L_j$, as needed.

This completes our proof of (1.7). Notice that it was surprisingly easy to get, by requiring $r(x)$ to be small, the only difficulty was to compose with $\pi_F$ in a way that would not destroy good Lipschitz bounds on $R_1 \cup R_2$ (because we need (25.56) and (25.63)); this is where we used our good control on $E_k \cap 2B$. This completes also the verification of (1.4)-(1.8). We also have (2.4), because by our proof of (1.6), the analogue of $\widehat{W}$ is contained in $B(x, 2\Lambda^2 r)$, which is compactly supported in $U$ if $r(x)$ was chosen small enough.

Anyway, the quasiminimality of $E_k$ now yields

$$\mathcal{H}^d(W_1) \leq M\mathcal{H}^d(h_1(W_1)) + hr^d, \tag{25.71}$$

as in (2.5), and where as usual $W_1 = \big\{y \in E_k \cap 2\Lambda^2 B\,;\, h_1(y) \neq y\big\}$. For each $y \in E_k \cap B(x, r)$, (25.24) says that $g(y) \in B(x, r)$; hence $y \in R_3$ (see (25.64)). If in addition $y \notin h_1(R_3)$, then $h_1(y) \neq y$ and $y \in W_1$. Thus

$$\mathcal{H}^d(W_1) \geq \mathcal{H}^d(E_k \cap B(x, r) \setminus h_1(R_3)) = \mathcal{H}^d(E_k \cap B(x, r)) \geq C^{-1} r \tag{25.72}$$

by (25.66), the local Ahlfors-regularity of $E_k$, and the fact that $E_k$ meets $B(x, r/10)$ because $x \in E$. As usual, this holds for $k$ large (depending on $x$), and with a constant $C$ that may depend on $\Lambda$, for instance, but not on $k$ or $x$.

On the other hand, $W_1 \subset R_1 \cup R_2 \cup R_3$, by (25.42), hence

$$\mathcal{H}^d(h_1(W_1)) \leq C\mathcal{H}^d(R_1 \cup R_2) \tag{25.73}$$

by (25.56), (25.63), and (25.66). Notice also that $|y - x| \geq (1 - 3\varepsilon)r$ for $y \in R_2$, because $g(y) \notin B$ and by (25.25) and (25.22). Thus

$$R_1 \cup R_2 \subset A, \text{ where } A = \big\{x \in E_k\,;\, (1 - 30\varepsilon)r \leq |y - x| \leq (1 + 30\varepsilon)r\big\}. \tag{25.74}$$

By (25.22) again, $A$ is contained in the thin strip $H$ around $P(x)$, and we can cover $A$ by less than $C\varepsilon^{-d+1}$ balls $D_l$ of radius $\varepsilon r$, which we may even choose centered on $A$. By the local Ahlfors regularity of $E_k$ (and because these balls stay far from $\mathbb{R}^n \setminus U$ if $r(x)$ is small enough), $\mathcal{H}^d(E_k \cap D_l) \leq C\varepsilon^d r^d$. We sum and get that

$$\mathcal{H}^d(A) \leq C\varepsilon r^d \tag{25.75}$$

and hence, by (25.73) and (25.74), $\mathcal{H}^d(h_1(W_1)) \leq C\varepsilon r^d$. If $h$ is small enough (depending on $n$, $M$, and $\Lambda$ through the constants $C$ of (25.72)), and $\varepsilon$ is small enough (depending on our various constants, but not $x$ or $r$), this contradicts (25.71) or (25.72). This contradiction proves that $\psi$ does not exist and finishes our proof of Lemma 25.27. □

We may now return to our initial construction, with $x \in E^1$, $r \leq r(x)$, and $S = g(E_k) \cap B$. By Lemma 25.27 and Definition 25.3, we get that (25.6) holds, i.e.,

$$J_f(P(x) \cap B(x,r)) \leq J_f(S \cap B(x,r)) + \varepsilon_x(r) r^d \tag{25.76}$$

with $\varepsilon_x(r)$ coming from (25.5). But $J_f(P(x) \cap B) = J_f(P(x) \cap B(x,r))$ because $\mathcal{H}^d(P(x) \cap \partial B) = 0$, and by (25.5) $\varepsilon_x(r) \leq \varepsilon$ if $r(x)$ was chosen small enough. Thus (25.76) implies that

$$J_f(P(x) \cap B) \leq J_f(S) + \varepsilon r^d. \tag{25.77}$$

Let $z \in S$ be given, and choose $y \in E_k$ such that $g(y) = z$. Notice that $|y - x| \leq (1 + 30\varepsilon)r$, because otherwise (25.24) would yield $g(y) = y \notin B$. Then $y \in H$ by (25.22), and (25.25) says that $|z - y| = |g(y) - y| \leq 3\varepsilon r$.

A first option is that $|z - x| \leq (1 - 23\varepsilon)r$; then $|y - x| \leq (1 - 20\varepsilon)r$, and (25.24) yields $g(y) = y$ (because $\alpha(|y - x|) = 0$). Then $z \in E_k$, and we get that

$$J_f(S \cap B(x, (1 - 23\varepsilon)r)) \leq J_f(E_k \cap B(x, (1 - 23\varepsilon)r)) \leq J_f(E_k \cap B). \tag{25.78}$$

If $|z - x| \geq (1 - 23\varepsilon)r$, then $|y - x| \geq (1 - 26\varepsilon)r$ and $y \in A$, the annulus in (25.74). Thus $S \setminus B(x, (1 - 23\varepsilon)r) \subset g(A)$. Now (25.50) says that $g$ is 2-Lipschitz on $A$ and hence

$$J_f(S \setminus B(x, (1 - 23\varepsilon)r)) \leq J_f(g(A)) \leq b\mathcal{H}^d(g(A)) \leq 2^d b \mathcal{H}^d(A) \leq Cb\varepsilon r^d \tag{25.79}$$

by (25.4) (observe also that $g(A)$ is rectifiable) and (25.75). Hence

$$J_f(P(x) \cap B) \leq J_f(S) + \varepsilon r^d \leq J_f(E_k \cap B) + Cb\varepsilon r^d + \varepsilon r^d \tag{25.80}$$

by (25.77), (25.78), and (25.79). We compare with (25.21) and get that

$$\begin{aligned} J_f(E \cap B) &\leq J_f(P(x) \cap B) + \varepsilon(b + 2\omega_d + \varepsilon) r^d + b\mathcal{H}^d((E \setminus E^1) \cap B) \\ &\leq J_f(E_k \cap B) + C\varepsilon r^d + b\mathcal{H}^d((E \setminus E^1) \cap B), \end{aligned} \tag{25.81}$$

where in the last line $C$ is allowed to depend on $b$ too.

This estimate is essentially what we wanted; note that for each $x \in E^1$, it holds for $k$ large (how large depends on $x$), and for a constant $C$ that does not depend on $x$ or $k$. We need a covering argument to complete our estimate.

We have a set $E^1 \subset E \cap V$, and for each $x \in E^1$ we have a family of closed balls $B = B(x,r)$, $0 < r \leq r(x)$, which forms a Vitali covering of $E^1$. Note also that $\mathcal{H}^d(E^1) \leq \mathcal{H}^d(E \cap V) < +\infty$ (because $V \subset\subset U$ for the moment). By Theorem 2.8 on page 34 in [Ma], we can extract from this large family of balls a disjoint family $\{B_i\}$, $i \in I$, so that $\mathcal{H}^d(E^1 \setminus \bigcup_{i \in I} B_i) = 0$. Then we can choose a finite set $I_0 \subset I$, such that

$$\mathcal{H}^d(E^1 \setminus \bigcup_{i \in I_0} B_i) \leq \varepsilon. \tag{25.82}$$

For $k$ large enough, (25.81) holds for every $B_i$, $i \in I_0$, and now

$$\begin{aligned}
J_f(E \cap V) &\leq \sum_{i \in I_0} J_f(E \cap B_i) + J_f(E \cap V \setminus \bigcup_{i \in I_0} B_i) \\
&\leq \sum_{i \in I_0} J_f(E \cap B_i) + b\mathcal{H}^d(E \cap V \setminus \bigcup_{i \in I_0} B_i) \leq \sum_{i \in I_0} J_f(E \cap B_i) + 2b\varepsilon \\
&\leq 2b\varepsilon + \sum_{i \in I_0} \left[ J_f(E_k \cap B_i) + C\varepsilon r_j^d + b\mathcal{H}^d((E \setminus E^1) \cap B_j) \right] \\
&\leq 2b\varepsilon + J_f(E_k \cap V) + b\mathcal{H}^d(E \cap V \setminus E^1) + C\varepsilon \sum_{i \in I_0} r_j^d \\
&\leq J_f(E_k \cap V) + C\varepsilon + C\varepsilon \sum_{i \in I_0} r_j^d
\end{aligned} \tag{25.83}$$

by (25.4), (25.14), (25.82), then by (25.81), where we set $B_j = \overline{B}(x_j, r_j)$, because the $B_j$ are disjoint and contained in $V$ (if each $r(x)$ was chosen small enough, according to (25.11)), and by (25.14) again. Since $E$ is locally Ahlfors-regular and each $B_j$ is centered on $E$ and such that $10B_j \subset V \subset U$, we get that

$$\sum_{i \in I_0} r_j^d \leq C \sum_{i \in I_0} \mathcal{H}^d(E \cap B_i) \leq C\mathcal{H}^d(E \cap V) \tag{25.84}$$

(because the $B_i$ are disjoint). Then (25.83) says that

$$J_f(E \cap V) \leq J_f(E_k \cap V) + C(1 + \mathcal{H}^d(E \cap V))\varepsilon \tag{25.85}$$

for $k$ large. Thus $J_f(E \cap V) \leq \liminf_{k \to +\infty} J_f(E_k \cap V) + C(1 + \mathcal{H}^d(E \cap V))\varepsilon$ and, since this estimate holds for every small $\varepsilon$, we get (25.8).

This takes care of the special case when $V$ is compactly contained in $U$. In the general case, we write $V$ as the increasing union of open sets $V_m \subset\subset U$, notice that

$$J_f(E \cap V_m) \leq \liminf_{k \to +\infty} J_f(E_k \cap V_m) \leq \liminf_{k \to +\infty} J_f(E_k \cap V) \tag{25.86}$$

for each $m$, then take the limit in $m$ and get (2.8) for $V$. This completes our proof of Theorem 25.7. $\square$

**Remark 25.87.** The author feels that it is a pity that we do not allow $f$ to be merely lower semicontinuous, but was not able to come up with a clean statement, so we will just give two possible substitutes here.

First observe that we used the continuity of $f$ only twice, in (25.15), and (25.16) (but where lower semicontinuity would have been enough), to be used in the last line of (25.17), then in (25.21), to prove that for our nice balls $B$, $J_f(E \cap B)$ is almost as small as $J_f(P(x) \cap B)$ (the measure of a nearby disk).

We want to replace our continuity assumption with the following one: for each $x \in U$, each $d$-plane $P$ through $x$, and each $C^1$, embedded, submanifold $\Gamma$ of dimension $d$ through $x$, which admits $P$ as a tangent plane at $x$,

$$\limsup_{r \to 0} \frac{1}{r^d} \big[ J_f(\Gamma \cap B(x,r)) - J_f(P \cap B(x,r)) \big] \leq 0. \tag{25.88}$$

Notice that this goes in the direction opposite to (25.6); this can be seen as a form of continuity is some direction, possibly much weaker than the full continuity asked above, but hard to think about as a lower semicontinuity property. We could have given the same definition, where instead $\Gamma$ is the graph of some $C^1$ mapping $F : P \to P^\perp$, with $DF(x) = 0$, and the two definitions would have been equivalent.

**Claim 25.89.** Theorem 25.7 also holds when we replace $\mathcal{I}(U,a,b)$ with the class $\mathcal{I}_l(U,a,b)$ of functions $f : U \times G(n,d) \to [a,b]$ that satisfy (25.5), (25.6), and (25.88).

That is, for the sake of Theorem 25.7, we can replace the continuity of $f$ by the condition (25.88) in the definition of $\mathcal{I}(U,a,b)$.

Our claim will follow as soon as we show that, with a suitable modification of the set $E^1$ and, for $x \in E^1$, of the radius $r(x)$, we still have (25.21) for $B = \overline{B}(x,r)$, when $x \in E^1$ and $r > 0$ is small enough. Recall that $E$ is rectifiable; thus we can write $E$ as null set, plus a countable collection of sets $F_i$, where $F_i$ is contained in a $C^1$, embedded, submanifold $\Gamma_i$ of dimension $d$. We may even assume that the $F_i$ are disjoint. Then almost every point $x \in E$ lies in some $\Gamma_i$, and is even a point of vanishing density for $E \setminus \Gamma_i$, i.e.,

$$\lim_{r \to 0} r^{-d} \mathcal{H}^d(E \cap B(x,r) \setminus \Gamma_i) = 0; \tag{25.90}$$

see [Ma], Theorem 6.2 (2) on page 89. We leave the definition of $E^0$ and $E^1$ as it was, except that we forget about the conditions (25.14)-(25.16) (which concerned the continuity of $f$, and replace them by the constraints that for $x \in E^1$, $x$ lies in some $\Gamma_i$, $i = i(x)$, and (25.90) holds. When we choose $r(x)$, we require that for $0 < r \leq 2r(x)$,

$$\mathcal{H}^d(E \cap \overline{B}(x,r) \setminus \Gamma_i) < \varepsilon r^d \tag{25.91}$$

and

$$J_f(\Gamma_{i(x)} \cap \overline{B}(x,r)) \leq J_f(P \cap B(x,r)) + \varepsilon r^d, \tag{25.92}$$

which we obtain as a limit of (25.88) for $r' > r$, because $\mathcal{H}^d(P \cap \partial B(x,r)) = 0$. Then for $x \in E^1$ and $0 < r \leq r(x)$, and if we set $B = \overline{B}(x,r)$ as before, we get that

$$
\begin{aligned}
J_f(E \cap B) &\leq J_f(E \cap \Gamma_{i(x)} \cap B) + b\mathcal{H}^d(E \cap B \setminus \Gamma_{i(x)}) \\
&\leq J_f(P(x) \cap B(x,r)) + (1+b)\varepsilon r^d,
\end{aligned}
\tag{25.93}
$$

which is even better than (25.21). Our claim follows.

The reader is probably worried about the *limsup* in (25.88), because the most logical statement would use a *liminf*. The proof above accommodates a liminf too, if we are more careful. Instead of having (25.92) for all the radii $r \leq 2r(x)$, we would only get it for a sequence of radii that tends to 0. Then, in the application of the Vitali covering lemma near (25.82), we would only choose balls $B_i$ with a radius that satisfies (25.92). We decided not to bother.

We can also try to take care of our semicontinuity issue by extending the class $\mathcal{I}(U,a,b)$ after the fact. That is, denote by $\mathcal{I}^+(U,a,b)$ the class of functions $f : U \times G(n,d)$ such that, for each compact set $H \subset U$, there is a sequence $\{f_m\}$ in $\mathcal{I}(U,a,b)$, with $a \leq f_m \leq f$ everywhere, and $\lim_{m\to+\infty} f_m(x,T) = f(x,T)$ for $x \in H$ and $T \in G(n,d)$. We claim that

(25.94) Theorem 25.7 also holds with $\mathcal{I}(U,a,b)$ replaced by $\mathcal{I}^+(U,a,b)$.

This would be nice if we could characterize easily $\mathcal{I}^+(U,a,b)$ (for instance, by lower semicontinuity and the conditions (25.4)-(25.6)), but the truth is that the author does not know how to manipulate (25.6) concretely.

Let us prove the claim anyway. Let $f \in \mathcal{I}^+(U,a,b)$, the sequence $\{E_k\}$ and its limit $E$, and $V \subset U$ be given. As for Theorem 25.7 itself, it is enough to prove the conclusion (25.8) when $V \subset\subset U$ (otherwise, write $V$ as an increasing union of open sets that are compactly contained in $U$). Then let $\{f_m\}$ be as in the definition of $\mathcal{I}^+(U,a,b)$, relative to $H = \overline{V}$; notice that for $m \geq 0$

$$
J_{f_m}(E \cap V) \leq \liminf_{k\to+\infty} J_{f_m}(E_k \cap V) \leq \liminf_{k\to+\infty} J_f(E_k \cap V)
\tag{25.95}
$$

because (25.8) holds for $f_m$ and $f_m \leq f$, and that

$$
J_f(E \cap V) = \lim_{m\to+\infty} J_{f_m}(E \cap V)
\tag{25.96}
$$

by the dominated convergence theorem; (25.8) and our claim (25.94) follow.

The next lemma will help with the extension of Theorem 10.8 to $f$-quasiminimal sets.

**Lemma 25.97.** *Let $f : U \times G(n,d) \to [a,b]$ satisfy the conditions (25.4)-(25.6). Then*

$$
\liminf_{r\to 0} \frac{1}{r^d}\big[J_f(\Gamma \cap B(x,r)) - J_f(P \cap B(x,r))\big] \geq 0
\tag{25.98}
$$

*for each $x \in U$, each $d$-plane $P$ through $x$, and each $C^1$, embedded, submanifold $\Gamma$ of dimension $d$ through $x$, which admits $P$ as a tangent plane at $x$.*

Notice that the conclusion is the opposite of (25.88), so we could replace (25.88) into the simpler (but apparently stronger)

$$\lim_{r\to 0} \frac{1}{r^d}\big[J_f(\Gamma \cap B(x,r)) - J_f(P\cap B(x,r))\big] = 0 \tag{25.99}$$

in Claim 25.89, without changing the result.

The proof of Lemma 25.97 goes a little as for Lemma 25.27. Let $x, P$, and $\Gamma$ be as in the statement, and let $r > 0$ be small. We want to apply (25.6) to a suitable set $S \subset \overline{B}(x,r)$, and since we want $S \cap \partial B(x,r)$ to be contained in $P$, we use $S = B \cap g(\Gamma)$, where $B = \overline{B}(x,r)$, $g$ is defined by (25.24), $\pi$ denotes the orthogonal projection onto $P$, $\alpha$ is defined as below (25.24), and $\varepsilon > 0$ is a small positive number, which will tend to 0 at the end of the argument.

As before, we want to show that there is no Lipschitz mapping $\psi : B \to B$ such that $\psi(y) = y$ for $y \in P \cap \partial B$ and $\psi(S) \subset P \cap \partial B$. Let us suppose that $\psi$ exists and use it to define an impossible mapping $h : P \cap B \to P \cap \partial B$. First extend $\psi$ to $P$, by setting $\psi(y) = \rho(y)$ for $y \in P \setminus B$, where $\rho(y) = x + r\frac{y-x}{|y-x|}$ is the radial projection of $y$ on $\partial B$. The extension is still Lipschitz because $\psi(y) = y$ on $P \cap \partial B$.

By definition of $\Gamma$, there is a $C^1$ function $F : P \to P^\perp$, with $DF(x) = 0$, such that for $r$ small, $\Gamma$ coincides with the graph of $F$ in $B(x, 2r)$. Also, for $r$ small enough, $|F(y)| \leq \varepsilon r$ for $y \in P \cap 2B$. Set $\lambda = 1 + 2\varepsilon$ and, for $y \in P \cap B$, set $\widetilde{y} = x + \lambda(y - x) \in \lambda B$, and then $z = \widetilde{y} + F(\widetilde{y})$ Thus $z \in B(x, (1 + 3\varepsilon)r)$ (because $|F(\widetilde{y})| \leq \varepsilon r$). If $g(z) \in B$, then $g(z) \in B \cap g(\Gamma) = S$, and $\psi(g(z))$ is defined and lies in $P \cap \partial B$. Otherwise, notice that

$$|g(z) - z| \leq |\pi(z) - z| = |F(\widetilde{y})| \leq \varepsilon r \tag{25.100}$$

by (25.24) and because $\widetilde{y} \in \lambda B$, and

$$|z - y| \leq |F(\widetilde{y})| + |\widetilde{y} - y| \leq 3\varepsilon r. \tag{25.101}$$

Hence $|z-x| \geq |g(z)-x| - \varepsilon r \geq (1-\varepsilon)r$ (because $g(z) \notin B$), and $|z-x| \leq |y-x| + 3\varepsilon r \leq (1+3)\varepsilon r$, so $\alpha(|x - z|) = 1$, hence $g(z) = \pi(z) \in P$, and again $\psi(g(z))$ is defined and lies in $P \cap \partial B$ by definition of our extension $\psi$ on $P \setminus B$. So we can define $h : P \cap B \to P \cap \partial B$ by $h(y) = \psi(g(z))$, and obviously $h$ is continuous. Also, if $y \in \partial B$, (25.101) still holds and yields $|z - y| \leq 3\varepsilon r$, then $\alpha(|x - z|) = 1$, and hence $g(z) = \pi(z)$ by (25.24). In addition, $|g(z) - z| = |\pi(z) - z| = |F(\widetilde{y})| \leq \varepsilon r$ as in (25.100), so $|g(z)| \geq |z| - \varepsilon r > r$ (by definition of $\lambda$), which means that $g(z) \in P \setminus B$ and $\psi(g(z)) = \rho(g(z))$. Thus $|h(y)-y| = |\psi(g(z))-y| = |\rho(g(z)) - y| \leq |g(z) - y| \leq 4\varepsilon r$; this implies that the restriction of $h$ to $P \cap \partial B$ is of degree 1, which is impossible because it has a continuous extension from $P \cap B$ to $P \cap \partial B$.

This contradiction shows that $\psi$ does not exist, and this allows us to apply (25.6). That is,

$$J_f(P \cap B(x,r)) \leq J_f(S \cap B(x,r)) + \varepsilon_x(r)r^d = J_f(g(\Gamma) \cap B(x,r)) + \varepsilon_x(r)r^d. \tag{25.102}$$

We claim that for $r$ small,

$$J_f(g(\Gamma) \cap B(x,r)) \leq J_f(\Gamma \cap B(x,r)) + C\varepsilon r^d. \tag{25.103}$$

Let $z \in g(\Gamma) \cap B(x,r)$ be given, and let $y \in \Gamma$ be such that $g(y) = z$; if $|y-x| \leq (1-20\varepsilon)r$, $\alpha(|y-x|) = 0$, hence $g(y) = y$. The corresponding subset of $g(\Gamma)$ is controlled by the first term in the second hand of (25.103). The case when $|y-x| \geq (1+20\varepsilon)r$ is impossible, because we would have that $g(y) = y$ for the same reasons. We are left with $g(A)$, where $A = \big\{y \in \Gamma\,;\,(1-20\varepsilon)r \leq |y-x| \leq (1-20\varepsilon)r\big\}$. We observe that $\mathcal{H}^d(A) \leq C\varepsilon r^d$, and that $g$ is $C$-Lipschitz on $A$ (recall that $|\pi(y) - y| \leq \varepsilon r$ for $y \in A$, and use the usual argument). This proves (25.103), and because of (25.102) we get that the *liminf* in (25.98) is larger than $-C\varepsilon$; since $\varepsilon > 0$ is arbitrarily small, (25.98) and Lemma 25.97 follow. □

**26. Limits of $f$-quasiminimal sets associated to elliptic integrands.**

We shall now describe a few implications of Theorem 25.7, in a context of quasiminimal and almost minimal sets relative to an integrand in the class $\mathcal{I}(U,a,b)$. We could also use the slightly larger classes $\mathcal{I}_l(U,a,b)$ and $\mathcal{I}^+(U,a,b)$ defined for Claim 25.89 and (25.94), but we shall stick to $\mathcal{I}(U,a,b)$ for simplicity.

We start with some simple observations on quasiminimal sets. Let $f \in \mathcal{I}(U,a,b)$ be given. Since we want to define $J_f(E)$ also for sets $E$ that are not necessarily rectifiable, define an auxiliary function $\widetilde{f} : U \to (0,+\infty)$, also with $a \leq \widetilde{f}(x) \leq b$; this way we can define $J_{f,\widetilde{f}}(E)$ as in (25.2). Of course $J_{f,\widetilde{f}}(E) = J_f(E)$, as defined in (25.1), when $E$ is rectifiable (which will be our main case).

Then, we can define the class $GSAQ_f(U,M,\delta,h)$, as we did in Definition 2.3, except that we replace (2.5) with the corresponding inequality

$$J_{f,\widetilde{f}}(W_1) \leq M J_{f,\widetilde{f}}(\varphi_1(W_1)) + hr^d. \tag{26.1}$$

With our assumptions, notice that

$$a\mathcal{H}^d(A) \leq J_{f,\widetilde{f}}(A) \leq b\mathcal{H}^d(A) \tag{26.2}$$

when $A$ is a Borel set such that $\mathcal{H}^d(A) < +\infty$. Then it is easy to see that

$$E \in GSAQ(U, a^{-1}bM, \delta, a \equiv^{-1} h) \text{ as soon as } E \in GSAQ_f(U,M,\delta,h). \tag{26.3}$$

That is, quasiminimal sets relative to $f$ are also quasiminimal relative to 1, and if $h$ is small enough (now depending on $a$ and $b$ as well), Theorem 5.16 says that $E$ is rectifiable. Then we can forget about $\widetilde{f}$ altogether (since $E$ and also its competitors $\varphi_1(E)$, where $\varphi_1$ satisfies (1.8), are rectifiable), and concentrate on $f$ and the formula (25.1). In particular, our class $GSAQ_f$ does not depend on $\widetilde{f}$.

We don't need to worry about the regularity results for $E \in GSAQ_f(U,M,\delta,h)$, since we can apply the results that we proved for plain quasiminimal sets.

Also, Theorem 25.7 applies to quasiminimal sets $E \in GSAQ_f(U,M,\delta,h)$, $h$ small enough, since they are plain quasiminimal sets.

**Claim 26.4.** Theorem 10.8 is still valid when we take $g \in \mathcal{I}_l(U,a,b)$, and replace $GSAQ(U,M,\delta,h)$ with $GSAQ_g(U,M,\delta,h)$ both in the assumption (10.2) and the conclusion (10.9).

We decided to call our integrand $g$ because the letter $f$ is used for the Lipschitz map of Sections 11-19. We work with the slightly larger class $\mathcal{I}_l(U,a,b)$ of Claim 25.89 because the proof for $g \in \mathcal{I}(U,a,b)$ works as well with $\mathcal{I}_l(U,a,b)$; we shall not attempt to see what happens when $g \in \mathcal{I}^+(U,a,b)$.

Also notice that under the Lipschitz assumption, this time we restrict to the additional condition (10.7), which is easy to use, and do not attempt to use the weaker (19.36).

Because of the length of the proof, we shall not check every detail, so the reader is invited to use a little more caution than usual before applying this result.

Most of the construction of stable competitors, as in Sections 10-17, does not need to be changed (we shall just modify the definition of the radii $r(y)$ defined in (15.4), before we define the balls $B_j$, $j \in J_3$). In particular, the estimates for all the small perturbation pieces will give equivalent results when we estimate sets with $J_g$ rather than $\mathcal{H}^d$, because of (25.4). Even in Section 18, nothing much happens before the estimates near (18.58), when we study the main contribution from the $B^+_{j,x}$, $j \in J_3$ and $x \in Z(y_j)$.

Under the rigid assumption, we still have (18.58), for the same reason as before, but instead of (18.63) and (18.64), we use this to prove that

$$\begin{aligned} J_g\Big(h_2\Big(\bigcup_{j\in J_3}\bigcup_{x\in Z(y_j)} B^+_{j,x}\setminus R^3\Big)\Big) &\leq J_g\Big(h_2\Big(\bigcup_{j\in J_3}\bigcup_{x\in Z(y_j)} B^-_{j,x})\Big)\Big) \\ &\leq \sum_{j\in J_3} J_g(Q_j\cap D_j), \end{aligned} \tag{26.5}$$

where, as we recall, $D_j = B(y_j, r_j)$ is a ball and $Q_j$ is a $d$-plane through $y_j$; we then deduce from this and previous estimates that (as in (18.64))

(26.6)
$$J_g(h_2(E_k\cap W)) \leq C\eta + C(f,\gamma)(1-a) + C(\alpha,f)N^{-1} + C(f)\gamma + \sum_{j\in J_3} J_g(Q_j\cap D_j).$$

Under the Lipschitz assumption, the same estimates as before lead to the following analogue of (18.72):

$$\begin{aligned} J_g(h_2(E_k\cap W)) &\leq C\eta + C(f,\gamma)(1-a) + C(\alpha,f)N^{-1} + C(f)\gamma \\ &\quad + \sum_{j\in J_3} J_g(D_j\cap\lambda^{-1}\psi^{-1}(\widetilde{Q}_j)). \end{aligned} \tag{26.7}$$

Now we need to change the argument a little, because we want lower bounds for $J_g(D_j \cap f(E\cap W_f))$ that fit with (26.6) or (26.7).

We begin with the rigid case, and explain how to modify the definition of the $D_j$ in Section 15, so as to have additional useful properties. We start with the sets $X_9 \subset E$ and $Y_9 = f(X_9)$, and of course $Y_9$ is rectifiable because $f$ is Lipschitz. This means that we can find a countable collection $\{\Gamma_m\}$, $m \geq 0$, of $C^1$ submanifolds of dimension $d$, such that $\mathcal{H}^d(Y_9 \setminus \bigcup_{m\geq 0}\Gamma_m) = 0$.

Set $\Gamma'_m = \Gamma_m \setminus \bigcup_{l<m} \Gamma_l$, and then $Y_9(m) = Y_9 \cap \Gamma'_m$; thus the $Y_9(m)$ are disjoint, and almost cover $Y_9$. Then denote by $Y'_9(m)$ the set of $y \in Y_9(m)$ such that

$$\lim_{r \to 0} r^{-d} \mathcal{H}^d(B(y,r) \cap \Gamma_m \setminus Y_9(m)) = 0; \tag{26.8}$$

we know from [Ma], Theorem 6.2 (2) on page 89 that (26.8) holds for $\mathcal{H}^d$-almost every $y \in Y_9(m)$ (because $Y_9(m) \subset \Gamma_m$), so $\mathcal{H}^d(Y_9 \setminus \bigcup_{m \geq 0} Y'_9(m)) = 0$. Set $X'_9(m) = X_9 \cap f^{-1}(Y'_9(m))$ for $m \geq 0$; we claim that

$$\mathcal{H}^d\big(X_9 \setminus \bigcup_{m \geq 0} X'_9(m)\big) = 0. \tag{26.9}$$

The justification is the same as for (15.11), relies on the fact that $f : X_9 \to Y_9$ is at most $N$-to-1, and is done for (4.77) in [D2].

For each $m \geq 0$ and $y \in Y'_9(m)$, there is a radius $r_1(y)$ such that

$$\mathcal{H}^d(B(y,r) \cap \Gamma_m \setminus Y_9(m)) \leq \eta r^d \ \text{ for } 0 < r \leq r_1(y) \tag{26.10}$$

(where $\eta > 0$ is the usual small number in Sections 11-18) and, because of Lemma 25.97,

$$J_g(Q_m(y) \cap B(y,r)) \leq J_g(\Gamma \cap B(y,r)) + \eta r^d \ \text{ for } 0 < r \leq r_1(y), \tag{26.11}$$

where $Q_m(y)$ denotes the tangent plane to $\Gamma_m$ at $y$. We now modify our definition of $Y_{10}$ in Section 15. We replace $Y_9$ by $Y'_9 = \bigcup_{m \geq 0} Y'_9(m)$ and $X_9$ by $X'_9 = \bigcup_{m \geq 0} X'_9(m)$ (we know from (26.9) that we don't lose any mass), and in addition to the defining condition (15.2) on $r(y)$, we require that $r(y) \leq r_1(y)$ for $y \in Y'_9$. Then we define $\delta_7$, $\delta_8$, and the sets $X_{10}$ and $Y_{10}$ as before, except that in (15.7) and (15.8) we replace $X_9$ and $Y_9$ with $X'_9$ and $Y'_9$. This way we get the additional property that $r_1(y) > \delta_8$ when $y \in Y_{10}$, and in particular, once we choose the balls $D_j = B(y_j, r_j)$, $j \in J_3$, that $r_j < r_1(y_j)$ for $j \in J_3$ (by (15.12)).

Now fix $j \in J_3$, and let $m$ be such that $y_j \in Y'_9(m)$. We have a $d$-plane $Q_j$, which is the common value of the $A_x(P_x)$, $x \in Z(y_j)$ (see above (15.16), and we claim that it is also equal to the tangent plane $Q_m(y_j)$ to $\Gamma_m$ at $y_j$. Since both are $d$-dimensional, it is enough to check that $Q_m(y_j) \subset Q_j$. Let $v$ be a unit vector in the direction of $Q_m(y_j)$, and let $\varepsilon > 0$ be given. For $r > 0$, (26.8) says that $B(y_j + rv/2, \varepsilon r)$ meets $Y_9(m)$. Since $Y_9(m) \subset Y_9 = f(X_9)$, we can find $z \in X_9$ such that $f(z) \in B(y_j + rv/2, \varepsilon r)$. By (15.2), $z \in B(x, 2\gamma^{-1} r)$ for some $x \in Z(y_j)$. By (11.40), $|f(z) - A_x(z)| \leq \varepsilon |z - x| \leq 2\gamma^{-1} \varepsilon r$ if $r$ is small enough (recall that $Z(y_j)$ is finite). At the same time, $\operatorname{dist}(z, P_x) \leq \varepsilon r$ for $r$ small, because $P_x$ is tangent to $E$ at $x$. Let $\widetilde{z}$ denote the projection of $z$ on $P_x$; then

$$\begin{aligned}
\operatorname{dist}(y_j + rv/2, Q_j) &\leq \varepsilon r + \operatorname{dist}(f(z), Q_j) = \varepsilon r + \operatorname{dist}(f(z), A_x(P_x)) \\
&\leq \varepsilon r + \operatorname{dist}(f(z), A_x(\widetilde{z})) \leq \varepsilon r + |f(z) - A_x(z)| + |A_x|_{lip} |z - \widetilde{z}| \\
&\leq \varepsilon r + 2\gamma^{-1} \varepsilon r + |f|_{lip} \varepsilon r \leq C \varepsilon r
\end{aligned} \tag{26.12}$$

by (11.36). For each $\varepsilon > 0$, this holds for $r$ small enough; since $y_j \in Q_j$, it follows that $v$ lies in the vector space parallel to $Q_j$, and $Q_m(y_j) = Q_j$, as needed.

Since $r_j < r_1(y_j)$, we can apply (26.11) and then (26.10) to get that

$$
\begin{aligned}
J_g(Q_j \cap D_j) &= J_g(Q_m(y_j) \cap B(y_j, r_j)) \leq J_g(\Gamma_m \cap B(y_j, r_j)) + \eta r_j^d \\
&\leq J_g(Y_9(m) \cap B(y_j, r_j)) + b\mathcal{H}^d(B(y_j, r_j) \cap \Gamma_m \setminus Y_9(m)) + \eta r_j^d \\
&\leq J_g(Y_9(m) \cap B(y_j, r_j)) + (1+b)\eta r_j^d.
\end{aligned} \tag{26.13}
$$

Now $Y_9(m) \subset f(E \cap W_f))$, because $Y_9(m) \subset Y_9 = f(X_9)$, and $X_9 \subset X_0 = E \cap W_f$ by (11.20), so (26.13) says that

$$
J_g(D_j \cap Q_j) \leq J_g(D_j \cap f(E \cap W_f)) + (1+b)\eta r_j^d. \tag{26.14}
$$

We sum over $j \in J_3$, and get that

$$
\begin{aligned}
\sum_{j \in J_3} J_g(D_j \cap Q_j) &\leq \sum_{j \in J_3} J_g(D_j \cap f(E \cap W_f)) + (1+b)\eta \sum_{j \in J_3} r_j^d \\
&\leq J_g(f(E \cap W_f)) + (1+b)\eta \sum_{j \in J_3} r_j^d
\end{aligned} \tag{26.15}
$$

because the $D_j$ are disjoint. Since $J_g(D_j \cap Q_j) \geq a\mathcal{H}^d(D_j \cap Q_j) = a\omega_d r_j^d$ for $j \in J_3$, we deduce from this that

$$
\sum_{j \in J_3} r_j^d \leq a^{-1}\omega_d^{-1} \sum_{j \in J_3} J_g(D_j \cap Q_j) \tag{26.16}
$$

and so, by (26.15) and if $\eta$ is small enough,

$$
\sum_{j \in J_3} r_j^d \leq 2a^{-1}\omega_d^{-1} J_g(f(E \cap W_f)) \tag{26.17}
$$

because the $D_j$ are disjoint. We now compare (26.15) with (26.6), and get that

$$
J_g(h_2(E_k \cap W)) \leq J_g(f(E \cap W_f)) + \mathcal{E}, \tag{26.18}
$$

where

$$
\mathcal{E} = C\eta + C(f,\gamma)(1-a) + C(\alpha, f)N^{-1} + C(f)\gamma + (1+b)\eta \sum_{j \in J_3} r_j^d \tag{26.19}
$$

is a small error term (observe that $\sum_{j \in J_3} r_j^d \leq C(f)$, by (26.17)). This is a good substitute for (18.93); from there, we estimate the difference between $W_f$ and $W$ as in (18.96), replace the lower semicontinuity estimate (18.97) by (25.8), and end the proof as before, with $\mathcal{H}^d$ replaced with $J_g$. This completes the proof under the rigid assumption.

Now suppose that we only have the Lipschitz assumption; thus we only have the estimate (26.7), and as before in Section 19, we need to estimate the quantity

$$
\Delta = \sum_{j \in J_3} J_g(D_j \cap \lambda^{-1}\psi^{-1}(\widetilde{Q}_j)) - \sum_{j \in J_3} J_g(Q_j \cap D_j) \tag{26.20}
$$

(compare with (19.1)). The first part of Section 19, where for $i \in J_4$, we extend our one parameter family to get a final set which in $D_j$ is almost contained in $Q_j$, does not need to be modified. We get an estimate like (19.32), with $\mathcal{H}^d$ replaced by $J_g$, which gives a contribution like the one we had in (26.6), and the effect is that we can remove from $\Delta$ the contribution from the indices $j \in J_4$.

For the second part of the argument, where we get rid of some small set $Z$, we need to change a few definitions. For $y \in U$, denote by $F(y)$ the smallest face of our grid that contains $y$.

Our first set $Z_1$ is the set of points $y \in U$ such that dimension$(F(y)) > d$, that lie in $L_i' = L_i \setminus \mathrm{int}(L_i)$ for some $i \in [0, j_{max}]$, but for which we cannot find $t > 0$ such that the restriction of $\psi$ to $\lambda F(y) \cap B(\lambda y, t)$ is of class $C^1$. By (10.7) (or rather the translation that was given below its statement), $\mathcal{H}^d(Z_1) = 0$.

Next denote by $Z_2$ the union of all the faces $F$ such that dimension$(F) < d$; again $\mathcal{H}^d(Z_2) = 0$.

Our third small set $Z_3$ is a subset of $Y_{11}$ (defined by (15.9)). Consider the set $Y$ of points $y \in Y_{11}$ such that dimension$(F(y)) = d$. This set is rectifiable, so we can find a countable collection of $C^1$ submanifolds $\Gamma_m$, $m \geq 0$, of dimension $d$, such that if we set $Y' = Y \cap \big(\bigcup_m \Gamma_m\big)$, then $\mathcal{H}^d(Y \setminus Y') = 0$. The $\Gamma_m' = \Gamma_m \setminus \bigcup_{l<m} \Gamma_l$ are disjoint, and still cover $Y'$. Now for each face $F$ of dimension $d$ and each $m$, we can apply [Ma], Theorem 6.2 (2) on page 89 to show that for $\mathcal{H}^d$-almost every $y \in Y' \cap F \cap \Gamma_m'$,

$$\lim_{r \to 0} r^{-d} \mathcal{H}^d(B(y,r) \cap \Gamma_m \setminus Y') = 0 \tag{26.21}$$

(say that $\Gamma_m \setminus Y' \subset [\Gamma_m \setminus \Gamma_m'] \cup [\Gamma_m' \setminus Y']$ and observe that $\mathcal{H}^d(\Gamma_m)$ is locally finite) and

$$\lim_{r \to 0} r^{-d} \mathcal{H}^d(B(y,r) \cap F \setminus \Gamma_m') = 0. \tag{26.22}$$

We remove from $Y_{11}$ the set $Z_3$ of $y \in F_{11}$ such that $y \in Y \setminus Y'$, or $y \in Y'$ but (26.21) or (26.22) fails for $F = F(y)$ and $m = m(y)$, the index such that $y \in G_m'$. That is, we set $Y_{12} = Y_{11} \setminus (Z_1 \cup Z_2 \cup Z_3)$. Of course $\mathcal{H}^d(Y_{11} \setminus Y_{12}) = 0$.

Now let $y \in Y_{12}$ be given. We want to define a radius $r_1(y)$ under which some good things happen. We start in the special case when dimension$(F(y)) = d$, set $F = F(y)$, and let $m$ be such that $y \in G_m'$; thus (26.21) and (26.22) hold, because we excluded $Z_3$.

First, we shall take $r_1(y)$ so small that

$$\mathcal{H}^d(B(y,r) \cap F \setminus \Gamma_m) \leq \eta r^d \tag{26.23}$$

for $0 < r \leq r_1(y)$; this is easy because of (26.22). For the next condition, pick any point $x \in Z(y)$; such a point exists because $y \in Y_9 = f(X_9)$ (see (15.1) and the line before), and in addition $y = f(x)$ and $x \in X_{11}$ (see (15.7) and (15.10)). Set $\widetilde{y} = \psi(\lambda y) = \widetilde{f}(x)$ (see (12.36)) and $\widetilde{F} = \psi(\lambda F)$; this last is the smallest face of the true dyadic grid that contains $\widetilde{y}$. By Lemma 12.40, $\widetilde{A}_x(P_x) \subset W(\widetilde{f}(x)) = W(\widetilde{y})$, where $W(\widetilde{y})$ denotes the smallest affine subspace that contains $\widetilde{F}$; since $F$, and hence $W(\widetilde{y})$, are $d$-dimensional, we will immediately get that

$$\widetilde{A}_x(P_x) = W(\widetilde{y}) \tag{26.24}$$

as soon as we check that $\widetilde{A}_x(P_x)$ is $d$-dimensional. We know that $A_x(P_x)$ is $d$-dimensional (compare the definition (14.5) with (14.21) and the line below); then our proof of (15.41) shows that shows that the restriction of $\psi$ to $\lambda A_x(P_x)$ is differentiable in every direction, and (15.41) gives a relation between the directional derivatives of $A_x$ and $\widetilde{A}_x$ on $P_x$ (and at $x$), which proves that $D\widetilde{A}_x$ is injective (because $D\psi$ is injective since $\psi$ is bilipschitz). So $\widetilde{A}_x(P_x)$ is $d$-dimensional and (26.24) holds. No $y$ lies in the interior of $F$ (by definition of $F$ as the smallest face that contains $y$, hence $\widetilde{y}$ lies in the interior of $\widetilde{F}$, the sets $\widetilde{A}_x(P_x)$, $W(\widetilde{y})$, and $\widetilde{F}$ coincide near $y$, and (by applying the bilipschitz map $\lambda^{-1}\psi^{-1}$) the sets $F$ and $\lambda^{-1}\psi^{-1}(\widetilde{A}_x(P_x))$ coincide near $y$. Now set $\widetilde{Q} = \widetilde{A}_x(P_x)$; we even know from (15.40) that all the $x \in Z(y)$ give the same $\widetilde{Q}$. If $r_1(y)$ is small enough, then for $0 < r < r_1(y)$,

$$\begin{aligned} J_g(B(y,r) \cap \lambda^{-1}\psi^{-1}(\widetilde{Q})) &= J_g(B(y,r) \cap F) \\ &\leq J_g(B(y,r) \cap \Gamma_m) + b\mathcal{H}^d(B(y,r) \cap F \setminus \Gamma_m) \\ &\leq J_g(B(y,r) \cap \Gamma_m) + b\eta r^d \end{aligned} \tag{26.25}$$

by (26.23). Denote by $P(\Gamma_m)$ the tangent $d$-plane to $\Gamma_m$ at $y$. We claim that $P(\Gamma_m) = A_x(P_x)$. We know from (15.9) that $A_x(P_x)$ does not depend on $x$; since both sets are $d$-dimensional, it is enough to check that $P(\Gamma_m) \subset A_x(P_x)$. We then proceed as we did near (26.12). Let $v$ be a unit vector in the direction of $P(\Gamma_m)$, and let $\varepsilon > 0$ be given. For $\rho > 0$, (26.21) says that $B(y + \rho v/2, \varepsilon\rho)$ meets $Y' \subset Y \subset Y_{11}$. Since $Y_{11} = f(X_{11})$, we can find $z \in X_{11}$ such that $f(z) \in B(y + \rho v/2, \varepsilon\rho)$. By (15.2), $z \in B(x, 2\gamma^{-1}\rho)$ for some $x \in Z(y)$. By (11.40), $|f(z) - A_x(z)| \leq \varepsilon|z - x| \leq 2\gamma^{-1}\varepsilon\rho$ if $\rho$ is small enough (don't worry, $Z(y)$ is finite). Also $\mathrm{dist}(z, P_x) \leq \varepsilon\rho$ for $\rho$ small, because $P_x$ is tangent to $E$ at $x$. Let $\widetilde{z}$ denote the projection of $z$ on $P_x$; then

$$\begin{aligned} \mathrm{dist}(y + \rho v/2, A_x(P_x)) &\leq \varepsilon\rho + \mathrm{dist}(f(z), A_x(P_x)) \leq \varepsilon\rho + \mathrm{dist}(f(z), A_x(\widetilde{z})) \\ &\leq \varepsilon\rho + |f(z) - A_x(z)| + |A_x|_{lip}|z - \widetilde{z}| \\ &\leq \varepsilon\rho + 2\gamma^{-1}\varepsilon\rho + |f|_{lip}\varepsilon\rho \leq C\gamma^{-1}\varepsilon\rho \end{aligned} \tag{26.26}$$

by (11.36). For each $\varepsilon > 0$, this holds for $\rho$ small; since $y \in A_x(P_x)$, it follows that $v$ lies in the vector space parallel to $A_x(P_x)$, and $P(\Gamma_m) = A_x(P_x)$, as needed.

We add one more constraint to the choice of $r_1(y)$ above: we apply the definition (25.88)-(25.89) of $\mathcal{I}_l(U, a, b)$, and require that

$$J_g(B(y,r) \cap \Gamma_m) \leq J_g(B(y,r) \cap P(\Gamma_m)) + \eta r^n = J_g(B(y,r) \cap A_x(P_x)) + \eta r^n \tag{26.27}$$

for $0 < r \leq r_1(y)$; then by (26.25)

$$J_g(B(y,r) \cap \lambda^{-1}\psi^{-1}(\widetilde{Q})) \leq J_g(B(y,r) \cap A_x(P_x)) + (1+b)\eta r^d. \tag{26.28}$$

We like this because if we ever pick $y_j = y$ and $r_j \leq r_1(y)$ for some $j \in J_3$, we will immediately deduce from (26.28) that

$$J_g(D_j \cap \lambda^{-1}\psi^{-1}(\widetilde{Q}_j)) \leq J_g(Q_j \cap D_j) + (1+b)\eta r_j^d, \tag{26.29}$$

just because $D_j = B(y_j, r_j)$, $Q_j$ is the common value of the $A_x(P_x)$, $x \in Z(y_j)$, and $\widetilde{Q}_j$ is the common value of the $\widetilde{A}_x(P_x)$.

This takes care of the definition of $r_1(y)$ in our first case when dimension$(F(y)) = d$. Notice that since we removed $Z_2$, dimension$(F(y)) < d$ is impossible. We are left with the case when dimension$(F(y)) > d$. If $y$ lies in no set $L'_i = L_i \setminus \mathrm{int}(L_i)$, we won't need $r_1(y)$, and we can set $r_1(y) = +\infty$. Finally, if $y \in L'_i$ for some $i \leq j_{max}$, the fact that we removed the set $Z_1$ implies that the restriction of $\psi$ to $\lambda F \cap B(\lambda y, t(y))$ is of class $C^1$.

Since $y \in Y_{11}$, and as in our first case, Lemma 12.40 says that for $x \in Z(y)$ the $d$-plane $\widetilde{A}_x(P_x)$ is contained in $W(\widetilde{y})$, where $W(\widetilde{y})$ is still the smallest affine subspace $W(\widetilde{y})$ that contains $\widetilde{F}$ (and $\widetilde{F}$ is the smallest rigid dyadic face that contains $\widetilde{y} = \widetilde{f}(x)$); but the difference is that now the dimension of $W(\widetilde{y})$ is larger than $d$. However, there is a neighborhood of $\widetilde{y}$ in $W(\widetilde{y})$ where the restriction of $\lambda^{-1}\psi^{-1}$ is of class $C^1$ (in fact, the $C^1$-regularity of this inverse map is the best definition of the $C^1$-regularity of $\psi$ on $\lambda F$). Then, if we set $\Gamma(y) = \lambda^{-1}\psi^{-1}(\widetilde{A}_x(P_x))$, there is a neighborhood of $y$ in $U$ where $\Gamma(y)$ is is a $C^1$ submanifold of $U$. By (25.88) again,

$$\limsup_{r \to 0} r^{-d}\big[J_g(B(y,r) \cap \Gamma(y)) - J_g(B(y,r) \cap P(y))\big] \leq 0, \tag{26.30}$$

where $P(y)$ denotes the tangent to $\Gamma(y)$ at $y$. Then we need to check that

$$P(y) = A_x(P_x) \text{ for } x \in Z(y). \tag{26.31}$$

The fact that all the sets $A_x(P_x)$ coincide comes from (15.9), and for the equality with $P(y)$ we shall be able to compute. Let $R$ denote the differential of $\lambda^{-1}\psi^{-1}$ at $\widetilde{y}$; this map is only defined on the vector space parallel to $\widetilde{F}$, but this will be enough. Also denote by $P'$ the vector space parallel to $P_x$; we know from (11.40) that the restriction of $DA_x$ to $P'$ is the differential of the restriction of $f$ to $P_x$. Similarly, (12.39) says that the restriction of $D\widetilde{A}_x$ to $P'$ is the differential of the restriction of $\widetilde{f}$ to $P_x$. We have seen that, because of Lemma 12.40, $\widetilde{A}_x(P_x)$ is contained in the vector space parallel to $\widetilde{F}$. Then the composition $R \circ D\widetilde{A}_x : P' \to \mathbb{R}^n$ makes sense, and we claim that it is also the differential of the mapping $f : P_x \to \mathbb{R}^n$. Indeed, for $v \in P'$, set $z = \widetilde{f}(x + tv)$ and denote by $w$ the projection of $\widetilde{f}(x+tv)$ on $\widetilde{A}_x(P_x)$. Then $z = \widetilde{f}(x+tv) = \widetilde{y} + tD\widetilde{A}_x(v) + o(t)$, so $w = \widetilde{y} + tD\widetilde{A}_x(v) + o(t)$ too (because $\widetilde{y} + tD\widetilde{A}_x(v) \in \widetilde{A}_x(P_x)$), and finally

$$\begin{aligned} f(x+tv) &= \lambda^{-1}\psi^{-1}(z) = \lambda^{-1}\psi^{-1}(w) + O(|z-w|) = \lambda^{-1}\psi^{-1}(w) + o(t) \\ &= \lambda^{-1}\psi^{-1}(\widetilde{y}) + R(w - \widetilde{y}) + o(|w - \widetilde{y}|) + o(t) \\ &= y + tR \circ D\widetilde{A}_x(v) + o(t) \end{aligned} \tag{26.32}$$

because $\psi$ is Lipschitz, and as needed. Since we also have the differential $DA_x$, we see that $DA_x = R \circ D\widetilde{A}_x$ on $P'$ and the direction of $A_x(P_x)$ is $R \circ D\widetilde{A}_x(P')$, which is indeed the direction of $P(y)$ (naturally obtained as the image by $R$ of the direction of the tangent plane to $\widetilde{A}_x(P_x)$). This proves (26.31).

We use (26.30) and (26.31) to choose $r_1(y)$ so small that for $0 < r \leq r_1(y)$,

$$J_g(B(y,r) \cap \Gamma(y)) \leq J_g(B(y,r) \cap A_x(P_x)) + \eta r^d \tag{26.33}$$

for $x \in Z(y)$. This way, if we ever pick $y_j = y$ and $r_j \leq r_1(y)$ for some $j \in J_3$, we will automatically get that (for $x \in Z(y_j)$)

$$\begin{aligned} J_g(D_j \cap \lambda^{-1}\psi^{-1}(\widetilde{Q}_j)) &= J_g(B(y_j,r_j) \cap \Gamma(y_j)) \\ &\leq J_g(B(y_j,r_j) \cap A_x(P_x)) + \eta r_j^d = J_g(Q_j \cap D_j) + \eta r_j^d, \end{aligned} \tag{26.34}$$

which is as good as (26.29).

We may now continue the construction as suggested in Section 19; recall that we set $Y_{12} = Y_{11} \setminus Z$, and $\mathcal{H}^d(Y_{11} \setminus Y_{12}) = 0$; then we set $X_{12} = X_{11} \cap f^{-1}(Y_{12})$, and use (4.77) in [D2] to get that

$$\mathcal{H}^d(X_{11} \setminus X_{12}) = 0. \tag{26.35}$$

Then we choose $\delta_9 > 0$, and set

$$Y_{13} = Y_{13}(\delta_9) = \big\{ y \in Y_{11} \,;\, r_1(y) < \delta_9 \big\} \ \text{ and } X_{13} = X_{13}(\delta_9) = X_{12} \cap f^{-1}(Y_{13}); \tag{26.36}$$

since the decreasing intersection of the $X_{13}(\delta_9)$ is $X_{12}$, we can choose $\delta_9$ so small that

$$\mathcal{H}^d(X_{11} \setminus X_{13}) = \mathcal{H}^d(X_{12} \setminus X_{13}) \leq \eta/2. \tag{26.37}$$

Then we proceed as before, choose the $D_j$ as we did near (15.12) but with the stronger constraint that (instead of (15.12))

$$r_j < \min(\delta_8, \delta_9) \ \text{ for } j \in J_3. \tag{26.38}$$

We continue our construction as before, except that we also define the modification $h_3$, which concerns the indices $j \in J_4$, as described near (26.20). This way, we only have to estimate the numbers

$$\Delta_j = J_g(D_j \cap \lambda^{-1}\psi^{-1}(\widetilde{Q}_j)) - J_g(Q_j \cap D_j) \tag{26.39}$$

for $j \in J_3 \setminus J_4$. For such $j$, we have defined a radius $r_1(y_j) > 0$, and made sure that since $y_j \in Y_{13}$, $0 < r_j < r(y_j)$. But then $\Delta_j \leq (1+b)\eta r_j^d$, by (26.29) or (26.34). We sum this over $j \in J_3 \setminus J_4$, and get an additional error term which is dominated by $\sum_{j \in J_3 \setminus J_4} \Delta_j \leq (1+b)\eta \sum_{j \in J_3 \setminus J_4} r_j^d \leq C(f)\eta$ by (26.17). This is small enough for us to complete our proof of the extension of Claim 26.4. □

We may also generalize many results of Part V to $f$-almost minimal sets, where $f$ is an elliptic integrand. Most of the time, the proof is the same once we have Theorem 25.7 and Claim 26.4, but we prefer to omit the details.

## 27. Smooth competitors.

In our definition of quasiminimal sets, we used competitors for $E$ that come from one-parameter families $\{\varphi_t\}$, $0 \leq t \leq 1$, for which the final mapping $\varphi_1$ is required to be Lipschitz, as in (1.8). We added this requirement because Almgren did it, and because this would not disturb in the proofs. The main advantage of this definition is probably that it makes it possible to show that some types of minimal currents (typically, size minimizers) have supports that are almost minimal sets. The author suspects this has been known for ages by specialists, but wrote a short proof for this in Section 7 of [D8] anyway. For other classes, such as Reifenberg homological solutions of Plateau's problem, we would not need (1.8).

To make the verification of quasiminimality easier for some other classes of sets, we may want to restrict the class of one-parameter families $\{\varphi_t\}$ (but without changing the main defining inequality (2.5)), typically by requiring the final mapping $\varphi_1$ (or maybe even the whole family of mappings $\varphi_t$) to be smoother. The issue appeared with some of the classes of differential chains introduced by J. Harrison, and at some point we even produced, with J. Harrison and H. Pugh, a sketch of proof for the some of the results in the present section (in the special case with no boundary). The details were never written down, essentially because Harrison and Pugh managed to verify the almost minimality of their supports in a different way.

The author thinks this is a reasonably interesting issue to mention, especially because we did not find a trivial way to deal with it directly with density arguments, hence the present section. He wishes to thank J. Harrison and H. Pugh for discussions about this issue and letting him write down this section.

We start with some definitions. Since all our sets (like $E$) may be thin, and we don't want to worry about about Whitney jets, let us agree that a function $f$ is of class $C^\alpha$ on the set $F$ when $f$ has a $C^\alpha$ extension to a neighborhood of $F$.

Let us define modified classes of quasiminimal sets. We keep most of the notation in Definition 2.3 as it was, and say that the closed set $E \subset \Omega$ is quasiminimal for competitors of class $C^\alpha$, with $\alpha \in \{1, 2, \dots, \infty\}$, if (2.5) holds for every one-parameter family $\{\varphi_t\}$, $0 \leq t \leq 1$, which satisfy (1.4)-(1.8) and (2.4), and for which, in addition, $\varphi_1$ is of class $C^\alpha$ on $E$. The corresponding classes are denoted by $GSAQ(U, M, \delta, h, \varphi_1 \in C^\alpha)$.

We shall also discuss the following intermediate notion of quasiminimal set for piecewise $C^\alpha$ competitors, denoted by $GSAQ(U, M, \delta, h, \varphi_1 \in PC^\alpha)$, where we only require the competitor to piecewise $C^\alpha$. This last means that the closure of $\{x \in E\,;\, \varphi_1(x) \neq x\}$, which by (2.4) is required to be a compact subset of $U$, can be covered by a finite number of compact sets $K_l$, and $\varphi_1(x)$ is $C^\alpha$ on each $K_l$, with the definition above. Thus we do not care whether the various pieces $K_l$ are smooth or not.

Let us mention yet another variant of these definitions. We say that $E$ is quasiminimal for families of class $C^\alpha$, and we write $E \in GSAQ(U, M, \delta, h, \varphi_t \in C^\alpha)$, if we only require (2.5) for families $\{\varphi_t\}$ that satisfy (1.4)-(1.8) and (2.4), and in addition define a $C^\alpha$ function on $V \times [0,1]$, where $V$ is some neighborhood of the set $E$. We define $GSAQ(U, M, \delta, h, \varphi_t \in PC^\alpha)$ likewise, with piecewise $C^\alpha$ functions defined on $E \times [0,1]$

We shall not worry too much about the difference between the $GSAQ(U, M, \delta, h, \varphi_t \in PC^\alpha)$ and the corresponding $GSAQ(U, M, \delta, h, \varphi_1 \in PC^\alpha)$, or directly $GSAQ(U, M, \delta, h, \varphi_t \in$

$C^\alpha$) and $GSAQ(U, M, \delta, h, \varphi_1 \in C^\alpha)$. We shall say a few words about this in Remark 27.47 though. We mention $GSAQ(U, M, \delta, h, \varphi_t \in PC^\alpha)$ now because our first result applies easily to that class with no special effort.

The main result of this section is that if our boundary pieces $L_j$ are smooth enough, the classes $GSAQ(U, M, \delta, h)$ and $GSAQ(U, M, \delta, h, \varphi_t \in C^1)$ are the same. See the remarks at the end of the section concerning possible further results, in particular concerning the case of $\alpha > 1$.

We start our discussion with a basic regularity result for the class $GSAQ(U, M, \delta, h, \varphi_t \in PC^\alpha)$. We mention it now because it seems hard to compare our different classes before we get the rectifiability of $E$. For this first result, we do not try to compare directly $GSAQ(U, M, \delta, h, \varphi_t \in PC^\alpha)$ with $GSAQ(U, M, \delta, h)$, but just observe that our initial proofs go through. We start with the Lipschitz assumption.

**Proposition 27.1.** *For each $M \geq 1$ we can find $h > 0$ and $C_M \geq 1$, depending on the dimensions $n$ and $d$, such that if $E \in GSAQ(B_0, M, \delta, h, \varphi_t \in PC^\infty)$, where $B_0 = B(0,1) \subset \mathbb{R}^n$, and if the rigid assumption holds, then $E$ is rectifiable and $E^*$ is locally $C_M$-Ahlfors regular in $B_0$.*

The local Ahlfors regularity condition means, as for Proposition 4.1, that

$$C_M^{-1} r^d \leq \mathcal{H}^d(E \cap B(x,r)) \leq C_M r^d \tag{27.2}$$

when $x \in E^*$ and $0 < r < \mathrm{Min}(r_0, \delta)$ are such that $B(x, 2r) \subset B_0$; we decided not to check that Proposition 3.3, which says that the closed support $E^*$ of $\mathcal{H}^d_{|E}$ is also a quasiminimal set, also holds in $GSAQ(B_0, M, \delta, h, \varphi_t \in PC^\infty)$, and this is why we have to deal with $E^*$.

Our proof will consist in checking that modulo a few minor modifications, all the competitors that we build to prove Proposition 4.1 and Theorem 5.16, under the rigid assumption, are obtained with piecewise $C^\infty$ functions. And indeed, these mappings are compositions of Federer-Fleming projections that can be described as follows. We fix a face $F$ of a dyadic cube, of some dimension $m \in [d, n]$, a point $\xi$ in the interior of $F$, and which lies outside of the image of the previous mapping, and then compose with the mapping $p_\xi$ which sends a point $y \in F \setminus \{\xi\}$ to its radial projection on $\partial F$ (centered at $\xi$). On the other faces of the same dimension, we set $p_\xi(y) = y$ (but then we compose with mappings coming from other faces). On the face $F$, we don't need to know $\pi_\xi$ near $\xi$, which is not in the current image of $E$, and away from $\xi$, $F$ is decomposed into a finite collection of closed pieces $F^l$, the inverse images of the faces of dimension $m-1$ that compose $\partial F$, where after a change of coordinates (so that $\xi = 0$ and the face of $\partial F$ is contained in the $(m-1)$-plane with equations $x_1 = a$, and $x_{m+1} = \ldots = x_n = 0$, the mapping $\pi_\xi$ is just given analytically by $\pi_{\xi,1}(y) = a$, $\pi_{\xi,m+1}(y) = \ldots = \pi_{\xi,n}(y) = 0$, and $\pi_{\xi,j}(y) = \frac{a y_j}{y_1}$ for $2 \leq j \leq m$. So it is easy to extend our definition of $\pi_\xi$ so that it is defined and Lipschitz on $\mathbb{R}^n$, and piecewise $C^\infty$. As before, the values of other $\pi_j$ outside of the faces don't matter. Recall also that for this result, we were not disturbed by the boundary condition (1.7), because we chose to project on cubes parallel to our grid, so that the Federer-Fleming projections, which preserve the faces, automatically preserve the $L_j$. □

The special case of Proposition 27.1 where we use the stronger assumption that $E \in GSAQ(B_0, M, \delta, h, \varphi_t \in PC^\infty)$ easily extends to the case when the Lipschitz assumption holds, but the bilipschitz function $\psi : \lambda U \to B(0,1)$ of Definition 2.7 is $C^\alpha$; we just conjugate the projections onto faces of dyadic cubes with $\psi$, and get projections on faces in $U$ that we can use as in the argument above. It seems hard to adapt the argument to make it work when $\psi$ is merely bilipschitz, and also the author is not sure that the notion of piecewise smooth competitors is interesting then.

Our next result will say that if the boundary pieces $L_j$ are sufficiently smooth, the adverb "piecewise" in the definition does not add anything. It will rely on the following simple extension lemma.

**Lemma 27.3.** *Let $F_1, F_2, \dots, F_m$ be closed subsets of $\mathbb{R}^n$, with $\operatorname{dist}(F_i, F_j) > 0$ for $i \neq j$, set $F = \cup_i F_i$, and let $f : F \to \mathbb{R}$ be a Lipschitz mapping. Suppose that for $1 \leq i \leq m$, the restriction of $f$ to $F_i$ has an extension $g_i$ to an open neighborhood $V_i$ of $F_i$, which is both $C^\alpha$ and Lipschitz. Then there $f$ has an extension to $\mathbb{R}^n$, which is of class $C^\alpha$, and which is Lipschitz with*

$$|f|_{Lip(\mathbb{R}^n)} \leq C|f|_{Lip(F)} + C\sum_i |g_i|_{Lip(V_i)}. \tag{27.4}$$

We had to mention that $g_i$ is Lipschitz on $V_i$, because when $F_i$ is not compact this may not follow from the fact that it is $C^\alpha$.

Let $\varepsilon > 0$ be so small that the sets $W_i = \{y \in \mathbb{R}^n\,;\, \operatorname{dist}(y, F_i) \leq 3\varepsilon\}$ are disjoint and contained in the corresponding $V_i$. Denote by $h$ the usual Lipschitz extension of $f$ to $\mathbb{R}^n$, obtained from the values of $f$ on $F$ with Whitney cubes, as in the first pages of [St]. Thus $h$ is $C|f|_{Lip(F)}$-Lipschitz, and it is also $C^\infty$ on $\mathbb{R}^n \setminus F$.

For each $i$, let $\xi_i$ denote a smooth function such that $\xi_i(y) = 1$ when $\operatorname{dist}(y, F_i) \leq \varepsilon$, $\xi_i(y) = 0$ when $\operatorname{dist}(y, F_i) \geq 2\varepsilon$, and $0 \leq \xi_i(y) \leq 1$ everywhere. We can choose $\xi_i$ so that it is $2\varepsilon^{-1}$-Lipschitz (and we leave the verification as an exercise). Also set $\xi_\infty = 1 - \sum_i \xi_i$; notice that $0 \leq \xi_\infty \leq 1$ by definition of $\varepsilon$. Finally we set

$$f = \xi_\infty h + \sum_i \xi_i g_i; \tag{27.5}$$

it is clear that it is as smooth as the $g_i$, so we just need to check the Lipschitz bound, and since $f$ is smooth we just need to bound $Df$. By (27.5), $Df = D\xi_\infty h + \sum_i D\xi_i g_i + \xi_\infty Dh + \sum_i \xi_i Dg_i$. The last two terms are bounded, so we are left with $A = D\xi_\infty h + \sum_i D\xi_i g_i$. Let $y \in \mathbb{R}^n$ be given. Notice that $D\xi_\infty(y) + \sum_i D\xi_i(y) = 0$ because $\xi_\infty + \sum \xi_i$ is constant. We may assume that $D\xi_i(y) \neq 0$ for some $i$, because otherwise $A(y) = 0$. Then $\operatorname{dist}(y, F_i) \leq 2\varepsilon$, which implies that $D\xi_j(y) = 0$ for $j \neq i$, and $A(y) = D\xi_j(y)(g_i(y) - h(y))$. Since $g_i$ and $h$ are Lipschitz and coincide on $F_i$, we get that $|A(y)| \leq |D\xi_j(y)|(|h|_{Lip(F)} + |g_i|_{Lip(V_i)}) \operatorname{dist}(y, F_i) \leq C(|f|_{Lip(F)} + |g_i|_{Lip(V_i)})$, and the lemma follows. □

For the next result, we allow the Lipschitz assumption, but require the boundary pieces $L_j$ to be sufficiently smooth and transverse. Of course, when we work with no boundary pieces (or just the unique $L_0 = \mathbb{R}^n$), thing are much simpler and we don't need the assumptions below, whose main goal is to allow a re-projection on the face when we leave them.

Rather than giving simple natural conditions that works well (the author tried to do this and did not manage), let us say what we will use; hopefully our assumptions will not be too brutal and will be easy to check in potential applications. We want local retractions on the $L_j$, which work for all the $L_j$ at the same time. More precisely, we shall assume that for each compact set $K \subset U$, we can find constants $\tau_0 > 0$ and $C_0 \geq 1$, so that the following holds. For $0 < \tau \leq \tau_0$, we can find a $C^\alpha$ mapping $\pi_\tau$, defined on

$$K^\tau = \big\{x \in U\,;\, \mathrm{dist}(x, K) < \tau\big\}, \tag{27.6}$$

such that

$$|\pi_\tau(x) - x| \leq C_0 \tau \quad \text{for } x \in K^\tau, \tag{27.7}$$

$$|\pi_\tau(x) - \pi_\tau(y)| \leq C_0 |x - y| \quad \text{for } x, y \in K^\tau, \tag{27.8}$$

$$\pi_\tau(x) \in L_j \quad \text{for } 0 \leq j \leq j_{max} \text{ and } x \in K^\tau \text{ such that } \mathrm{dist}(x, L_j) \leq C_0^{-1} \tau, \tag{27.9}$$

but also, setting

$$Z_j(\rho) = \big\{x \in U\,;\, 0 < \mathrm{dist}(x, L_j) < \rho\big\} \tag{27.10}$$

for $0 \leq j \leq j_{max}$ and $\rho > 0$ and

$$Z(\rho) = \bigcup_{0 \leq j \leq j_{max}} Z_j(\rho), \tag{27.11}$$

such that

$$\pi_\tau(x) = x \quad \text{for } x \in K^\tau \setminus Z(\tau). \tag{27.12}$$

Finally, we require that $\pi_\tau$ is the endpoint of a one parameter family $\{\pi_{\tau,t}\}$, such that $\pi_{\tau,t}(x)$ is a function of $x \in K^\tau$ and $t \in [0, 1]$ which is both $C^\alpha$ (with no precise bound needed), but also $C_0$-Lipschitz (we shall use this near (27.43)) and such that $\pi_{\tau,0}(x) = x$ and $\pi_{\tau,1}(x) = \pi_\tau(x)$ for $x \in K^\tau$,

$$|\pi_{\tau,t}(x) - x| \leq C_0 \tau \quad \text{for } x \in K^\tau \text{ and } 0 \leq t \leq 1, \tag{27.13}$$

and

$$\pi_{\tau,t}(x) \in L_j \quad \text{for } 0 \leq t \leq 1 \text{ when } x \in L_j. \tag{27.14}$$

This looks like a long list, but the reader may check that is easy to construct such retraction, say, when the $L_j$ are two transverse smooth submanifolds (first project on the first one parallel the second one, and continue with a projection on the second one along the first one), or when the $L_j$ are contained in each other (retract on the largest, then on the second largest inside the first one, etc.).

**Proposition 27.15.** *If the $L_j$ satisfy the assumption above, then the two classes $GSAQ(B_0, M, \delta, h, \varphi_1 \in PC^\alpha)$ and $GSAQ(B_0, M, \delta, h, \varphi_1 \in C^\alpha)$ are equal.*

The following example shows that this result will at least be harder to prove if we do not assume that the $L_j$ are smooth. Consider, in the unit disk $D \subset \mathbb{R}^2 \simeq \mathbb{C}$, a single boundary $L = [0, 1) \cup [0, i)$ (two orthogonal intervals), and the set $E = L \cup J$, where $J = D \cap [0, 1 + i]$ is a piece of the first diagonal. It is easy to produce better Lipschitz, or piecewise $C^1$ competitors, by replacing $J$ with a shorter curve $\Gamma$ that ends somewhere else on the positive first axis, for instance (push part of the first quadrant down and to the left). The obvious map $\varphi_1$ that does this (i.e., maps $L$ to itself and $J$ to $\Gamma$) is not $C^1$, and it looks like there is an obstruction because we changed the angles at the origin. But this is not a counterexample, because we can find a smoother mapping, with a vanishing derivative at the origin, and which does the job even though it destroys some angles. For instance, precompose the function $\varphi_1$ above with the mapping $x \to |x|^2 x$.

We shall not try to extend Proposition 27.15 to such situations; this may be hard, and the benefit is not clear, because the class $GSAQ(B_0, M, \delta, h, \varphi_1 \in C^\alpha)$ is not too natural in that case. Similarly, our assumptions are probably much too strong, but we prefer the proof to be short.

Let us prove the proposition. We only need to show that $GSAQ(B_0, M, \delta, h, \varphi_1 \in C^\alpha) \subset GSAQ(B_0, M, \delta, h, \varphi_1 \in PC^\alpha)$, since the other inclusion is trivial. Thus we are given $E \in GSAQ(B_0, M, \delta, h, \varphi_1 \in C^\alpha)$ and a family $\{\varphi_t\}$ for which $\varphi_1$ is piecewise $C^\alpha$, and we want to construct a modified family with a final map of class $C^\alpha$, apply the definition of $GSAQ(B_0, M, \delta, h, \varphi_1 \in C^\alpha)$, and get (2.5) for the initial $\varphi_1$.

Set $W_1 = \big\{x \in E\,;\, \varphi_1(x) \neq x\big\}$ and

$$K = \overline{W}_1 \cup \overline{h_1(W_1)}; \tag{27.16}$$

by (2.4), $K$ is a relatively compact subset of $U$. We use this $K$ to apply our assumption on the $L_j$, with a very small constant $\tau$ that will be chosen later; we get mappings $\pi_\tau$ and $\pi_{\tau,s}$, $0 \leq s \leq 1$, defined on $K^\tau$. Since we do not want to modify the $\varphi_t$ too far from $W_1$, we shall use a smooth cut-off function $\chi$ such that $0 \leq \chi(x) \leq 1$ everywhere,

$$\chi(x) = 1 \text{ when } \operatorname{dist}(x, W_1) \leq \tau/4, \ \chi(x) = 0 \text{ when } \operatorname{dist}(x, W_1) \geq \tau/2, \tag{27.17}$$

and $|\nabla\chi| \leq C\tau^{-1}$ everywhere.

We continue our family $\{\varphi_t\}$ a first time. Set

$$\varphi_t(x) = \pi_{\tau,(t-1)\chi(x)}(\varphi_1(x)) \ \text{ for } x \in E \cap K^\tau \text{ and } 1 \leq t \leq 2; \tag{27.18}$$

notice that when $x \in W_1$, $\varphi_1(x) \in K$ and $\pi_{\tau,(t-1)\chi(x)}(\varphi_1(x))$ is well defined. When $x \in E \cap K^\tau \setminus W_1$, $\varphi_1(x) = x$ and $\pi_{\tau,(t-1)\chi(x)}(\varphi_1(x))$ is well defined too. When in addition

$\mathrm{dist}(x, W_1) \geq \tau/2$, $\chi(x) = 0$ and so $\varphi_t(x) = \varphi_1(x) = x$ (see above (27.13). Thus we can safely set

$$\varphi_t(x) = x \quad \text{for } x \in E \setminus K^\tau \text{ and } 1 \leq t \leq 2, \tag{27.19}$$

and $\varphi_t(x)$ is a continuous function of $x$ and $t$. Because of what we just said, we even have that

$$\varphi_t(x) = x \quad \text{for } 1 \leq t \leq 2 \text{ when } x \in E \text{ is such that } \mathrm{dist}(x, W_1) \geq \tau/2. \tag{27.20}$$

We want to continue with mappings $\varphi_t$, $2 \leq t \leq 3$, so that the final mapping $\varphi_3$ is smooth. Recall that $\varphi_1$ is piecewise smooth; we shall single out one piece, $F_0 = E \setminus W_1$, on which we know that $\varphi_1$ is smooth because $\varphi_1(x) = x$ there. Then, since $\varphi_1$ is piecewise smooth, we can cover $\overline{W}_1$ with a finite collection of compact sets $H_l$, $1 \leq l \leq m$, such that $\varphi_1$ is $C^\alpha$ on some open neighborhood of $H_l$. Of course we may assume that $H_l \subset E$.

We want to replace the $H_l$, $1 \leq l \leq m$, with slightly smaller compact sets $F_l \subset H_l$, so that

$$\text{the } F_l,\ 0 \leq l \leq m, \text{ are disjoint,} \tag{27.21}$$

and, if we set

$$F = \bigcup_{0 \leq l \leq m} F_l, \tag{27.22}$$

such that

$$\mathcal{H}^d(E \setminus F) \leq \eta \tag{27.23}$$

and

$$\mathrm{dist}(x, F) \leq \eta \quad \text{for } x \in E, \tag{27.24}$$

where $\eta > 0$ is a very small constant that will be chosen later.

This is easy: we choose the $F_l$ one by one; if the $F_k$, $k < l$ have been chosen, we try $F_l = \big\{x \in H_l\,;\, \mathrm{dist}(x, F_k) \geq a_l \text{ for } 0 \leq k < l\big\}$, where $a_l > 0$ will be chosen soon. Set $H_l' = H_l \setminus \big(\bigcup_{0 \leq k < l} F_k\big)$, and observe that

$$E \setminus F \subset \bigcup_{l \geq 1} (H_l' \setminus F_l). \tag{27.25}$$

Also, for each $l$, $H_l' \setminus F_l$ decreases to the empty set when $a_l$ tends to 0, so $\mathcal{H}^d(H_l' \setminus F_l) \leq \eta/m$ if $a_l$ is chosen small enough. Then (27.23) follows from (27.25). In addition, if $a_l < \eta$ for $l \geq 1$ and $x \in E \setminus F$, then by (27.25) $x \in H_l' \setminus F_l$ for some $l \geq 1$, and this forces $\mathrm{dist}(x, F_k) \leq a_l \leq \eta$, as needed for (27.24).

Apply Lemma 27.3 to the function $\varphi_1$ and the disjoint sets $F_k$. We get a smooth extension of the restriction of $\varphi_1$ to $F$, which we call $f$. Thus

$$f(x) = \varphi_1(x) \quad \text{for } x \in F. \tag{27.26}$$

Notice that in $f$ is Lipschitz with a norm that does not depend on $\eta$ or $\tau$ (because for $l \geq 1$, the $C^\alpha$ extension of $\varphi_1$ in a neighborhood of $H_l$ that we used can be assumed to be Lipschitz too, since $H_l$ is compact). Of course $f$ may differ from $\varphi_1$ on the very small set $E \setminus F$, but nonetheless

$$|f(x) - \varphi_1(x)| \leq C|f|_{lip} \operatorname{dist}(x, F) \leq C|f|_{lip}\eta \leq \frac{\tau}{C_0 + 2} \tag{27.27}$$

for $x \in E$, by (27.24), our Lipschitz control on $f$, and if $\eta$ is small enough compared to $\tau$.

We go from $\varphi_1$ to $f$ by the usual linear interpolation, i.e., set

$$z(x,t) = (t-2)f(x) + (3-t)\varphi_1(x) \tag{27.28}$$

for $x \in E$ and $2 \leq t \leq 3$, and then compose with $\pi_{\tau\chi(x)}$ as we did for $\varphi_2$; that is, we want to set

$$\varphi_t(x) = \pi_{\tau,\chi(x)}(z(x,t)) \quad \text{for } x \in E \cap K^\tau \text{ and } 2 \leq t \leq 3. \tag{27.29}$$

We just need to check that

$$z(x,t) \in K^\tau \quad \text{when } x \in E \cap K^\tau \tag{27.30}$$

If $x \in W_1$, then $\varphi_1(x) \in K$ (by (27.16)), and the result follows because (27.27) says that $|f(x) - \varphi_1(x)| \leq \tau/2$. Otherwise, $x \in F_0 \subset F$, so $f(x) = \varphi_1(x) = x \in K^\tau$, and the result holds too. So (27.30) holds, and (27.29) makes sense.

On the rest of $E$, we set, as in (27.18),

$$\varphi_t(x) = x \quad \text{for } x \in E \setminus K^\tau \text{ and } 2 \leq t \leq 3. \tag{27.31}$$

When $x \in E \cap K^\tau$ but $\operatorname{dist}(x, W_1) \geq \tau/2$, observe that $x \in F_0 \subset F$, hence $f(x) = \varphi_1 = x$, and since $\chi(x) = 0$ by (27.17), we get that $\varphi_t(x) = \pi_{\tau,0}(x) = x$. So

$$\varphi_t(x) = x \quad \text{for } 2 \leq t \leq 3 \text{ when } x \in E \text{ is such that } \operatorname{dist}(x, W_1) \geq \tau/2, \tag{27.32}$$

as for (27.20).

This completes our definition of the extended family $\{\varphi_t\}$, $0 \leq t \leq 3$, modulo some choices of constants that we still need to make. We want to apply our assumption that $E \in GSAQ(B_0, M, \delta, h, \varphi_1 \in C^\alpha)$, so let us check the usual requirements for the $\varphi_{3t}$, $0 \leq t \leq 1$.

The continuity condition (1.4) is satisfied; in particular, for $t = 2$ and $x \in E \cap K^\tau$, (27.29) yields $\varphi_t(x) = \pi_{\tau,\chi(x)}(\varphi_1(x))$, just like (27.18). Of course $\varphi_0(x) = x$. Also,

$$\varphi_t(x) = x \quad \text{for } 0 \leq t \leq 3 \quad \text{when } x \in E \text{ is such that } \operatorname{dist}(x, W_1) \geq \tau/2, \tag{27.33}$$

by (2.1), (27.19), and (27.32). If $B = \overline{B}(x_0, r_0)$ was the ball for which (1.5) and (1.6) hold for the initial $\varphi_t$, we get (1.5) for the extended family, with any ball $B'$ that contains $B(x_0, r_0\tau)$.

Let us now check (1.6), with the ball $B' = \overline{B}(x_0, r_0 + (C_0 + 2)\tau)$. We are given $x \in E \cap B'$, and we want to check that $\varphi_t(x) \in B'$ for all $t$. We may assume that $\mathrm{dist}(x, W_1) < \tau/2$, because otherwise the result follows from (27.33), and also that $t > 1$, because we know (1.6) for the $\varphi_t$, $0 \le t \le 1$. Let us check that

$$|\varphi_t(x) - \varphi_1(x)| \le (C_0 + 1)\tau. \tag{27.34}$$

If $1 \le t \le 2$, $\varphi_t(x)$ is given by (27.18), and $|\varphi_t(x) - \varphi_1(x)| \le C_0\tau$ by (27.13). Otherwise, $\varphi_t(x)$ is given by (27.29); thus $\varphi_t(x) = \pi_{\tau,\chi(x)}(z(x,t))$, where $z(x,t)$ is defined by (27.28) and lies in $[f(x), \varphi_1(x)] \cap K^\tau$ by (27.30). Then

$$\begin{aligned}|\varphi_t(x) - \varphi_1(x)| &= |\pi_{\tau,\chi(x)}(z(x,t)) - \varphi_1(x)| \\ &\le |\pi_{\tau,\chi(x)}(z(x,t)) - z(x,t)| + |z(x,t) - \varphi_1(x)| \\ &\le |\pi_{\tau,\chi(x)}(z(x,t)) - z(x,t)| + |f(x) - \varphi_1(x)| \le (C_0 + 1)\tau\end{aligned} \tag{27.35}$$

by (27.13) and (27.27). So (27.34) holds. But now

$$\mathrm{dist}(\varphi_t(x), W_1) \le \mathrm{dist}(x, W_1) + (C_0 + 1)\tau \le (C_0 + 2)\tau, \tag{27.36}$$

which proves that $\varphi_t(x) \in B'$, because $W_1 \subset B = \overline{B}(x_0, r_0)$.

The compactness condition (2.4) also holds, because the analogue of $\widehat{W}$ for the extended family lies in a $(C_0 + 2)\tau$-neighborhood of $\widehat{W}$, by (27.33) and (27.36) in particular. Of course this neighborhood is compactly contained on $U$ if $\tau$ is small enough.

We managed to end our family with a mapping $\varphi_3$ which is $C^\alpha$. Indeed, (27.29) yields $\varphi_3(x) = \pi_{\tau,\chi(x)}(f(x))$ for $x \in E \cap K^\tau$, $f$ was constructed to be $C^\alpha$, $\pi_{\tau,s}(x)$ is a $C^\alpha$ function of $s$ and $x$, and as usual there is an overlap between the definitions by (27.29) and (27.31), where both definitions yield $\varphi_3(x) = x$. This takes care of the improved constraint (1.8) with $C^\alpha$.

Finally we check (1.7). We are given $x \in E \cap L_j$, and we want to check that $\varphi_t(x) \in L_j$ for all $t$. We can assume that $t > 1$ (otherwise, use the old (1.7)), and that $\mathrm{dist}(x, W_1) < \tau/2$ (by (27.33)). By the old (1.7), $\varphi_1(x) \in L_j$ and now (27.18) yields $\varphi_t(x) \in L_j$ for $1 \le t \le 2$, by (27.14). So we assume that $t \ge 2$, and $\varphi_t(x)$ is given by (27.29). If $x \in F$, then $f(x) = \varphi_1(x)$ and (27.29) yields $\varphi_t(x) = \varphi_2(x) \in L_j$ for $t \ge 2$, as needed. So can assume that $x \in E \backslash F$. Since $x \notin F_0 = E \backslash W_1$, we get that $x \in W_1$, and then $\chi(x) = 1$ (see (27.17)), so $\varphi_t(x) = \pi_{\tau,1}(z(x,t)) = \pi_\tau(z(x,t))$, with $z(x,t) = (t-2)f(x) + (3-t)\varphi_1(x) \in [f(x), \varphi_1(x)]$ (see above (27.13)). By (27.27),

$$\mathrm{dist}(z, L_j) \le |z(x,t) - \varphi_1(x)| \le |f(x) - \varphi_1(x)| \le C_0^{-1}\tau, \tag{27.37}$$

and now (27.9) says that $\varphi_t(x) = \pi_\tau(z(x,t)) \in L_j$.

This completes our list of verifications, and we may now apply the quasiminimality of $E$. Set $W_3 = \{x \in E\,;\, \varphi_3(x) \neq x\}$; then the analogue of (2.5) for $\varphi_3$ says that

$$\mathcal{H}^d(W_3) \leq M\mathcal{H}^d(\varphi_3(W_3)) + h\, r_1^d, \tag{27.38}$$

where $r_1 = r_0 + (C_0+2)\tau$ is the radius of our ball $B'$, and $r_0$ is the radius of our initial ball $B$.

We want to use $W_1$, so let us estimate the size of the symmetric difference $W_1 \Delta W_3$. By definition, $W_1 \Delta W_3 \subset \Xi$, where

$$\Xi = \{x \in E\,;\, \varphi_3(x) \neq \varphi_1(x)\}. \tag{27.39}$$

We claim that

$$\Xi \subset (E \setminus F) \cup X(\tau) \cup Y(\tau), \tag{27.40}$$

where

$$X(\tau) = \{x \in E\,;\, \tau/4 \leq \mathrm{dist}(x, W_1) \leq \tau/2\} \tag{27.41}$$

and

$$Y(\tau) = \{x \in E\,;\, \varphi_1(x) \in Z(\tau)\}, \tag{27.42}$$

where $Z(\tau)$ is defined by (27.10) and (27.11). Indeed, let $x \in \Xi$ be given. If $x \in E \setminus F$ we are happy, so we may assume that $x \in F$. Then $f(x) = \varphi_1(x)$. Also, $\mathrm{dist}(x, W_1) < \tau/2$, because otherwise (27.30) and (2.1) say that $\varphi_3(x) = x = \varphi_1(x)$. Thus (27.26) applies, and $\varphi_3(x) = \pi_{\tau,\chi(x)}(f(x)) = \pi_{\tau,\chi(x)}(\varphi_1(x))$. Since $\varphi_3(x) \neq \varphi_1(x)$, we get that $\chi(x) \neq 0$. If $x \in X(\tau)$, we are happy; otherwise, $\xi(x) = 1$ (see above (27.17)) and $\varphi_3(x) = \pi_{\tau,1}(\varphi_1(x)) = \pi_\tau(\varphi_1(x))$.

If $\varphi_1(x) \in Z(\tau)$, we are happy. Otherwise, (27.12) says that $\varphi_3(x) = \pi_\tau(\varphi_1(x)) = \varphi_1(x)$ (recall that we checked that $\varphi_1(x) \in K^\tau$ below (27.18)); this contradiction completes the proof of (27.40).

Our function $\varphi_3$ is $C$-Lipschitz, with a (possibly huge) constant $C$ that does not depend on $\tau$; thus

$$\begin{aligned}\mathcal{H}^d(\varphi_3(W_3)) &\leq \mathcal{H}^d(\varphi_3(W_3 \setminus \Xi)) + CH^d(\Xi)\\ &\leq \mathcal{H}^d(\varphi_1(W_3 \setminus \Xi)) + CH^d(\Xi)\\ &\leq \mathcal{H}^d(\varphi_1(W_1)) + CH^d(\Xi)\end{aligned} \tag{27.43}$$

because $\varphi_3 = \varphi_1$ on $W_3 \setminus \Xi$, and $W_1 \Delta W_3 \subset \Xi$. Also, $\mathcal{H}^d(W_1) \leq \mathcal{H}^d(W_3) + H^d(\Xi)$ because $W_1 \Delta W_3 \subset \Xi$, so (27.38) yields

$$\begin{aligned}\mathcal{H}^d(W_1) \leq \mathcal{H}^d(W_3) + H^d(\Xi) &\leq M\mathcal{H}^d(\varphi_3(W_3)) + h\, r_1^d + H^d(\Xi)\\ &\leq M\mathcal{H}^d(\varphi_1(W_1)) + C(1+M)H^d(\Xi) + h\, r_1^d.\end{aligned} \tag{27.44}$$

We shall now use (27.40) to estimate $H^d(\Xi)$. Recall that we may choose $\tau$ as small as we want, and then $\eta$ even smaller. By (27.23), $\mathcal{H}^d(E \setminus F) \leq \eta$ can be made as small as we want; similarly, $X(\tau) \subset X'(\tau) = \{x \in E\,;\, 0 < \mathrm{dist}(x, W_1) < \tau/2\}$, and since all the $X'(\tau)$ have a finite $\mathcal{H}^d$-measure and their monotone intersection is empty, $\mathcal{H}^d(X(\tau))$ can be made as small as we want too. The same argument applies to $Y(\tau)$ (recall from (27.10) and (27.11) that the monotone limit of $Z(\tau)$ is empty.

Thus (27.44) holds for arbitrarily small values of $\tau$ and hence $H^d(\Xi)$. In addition, $r_1 = r_0 + (C_0 + 2)\tau$ is as close to $r_0$ as we want, we get (2.5) for the initial $\varphi_1$, and Proposition 27.15 follows. $\square$

When $\alpha = 1$ we can use the fact that Lipschitz functions are not far from $C^1$ to obtain easily the main result of this section.

**Corollary 27.45.** *Suppose the $L_j$ satisfy the same assumption as for Proposition 27.15, with $\alpha = 1$. Then the two classes $GSAQ(B_0, M, \delta, h)$ and $GSAQ(B_0, M, \delta, h, \varphi_1 \in C^1)$ are equal.*

Recall that $GSAQ(B_0, M, \delta, h)$ is the usual class of quasiminimal sets that we studied in the rest of this text.

As before, one of the inclusions is trivial, and we just need to check that if $E \in GSAQ(B_0, M, \delta, h, \varphi_1 \in C^1)$, then $E \in GSAQ(B_0, M, \delta, h)$. We first apply Proposition 27.15 and get that $E \in GSAQ(B_0, M, \delta, h, \varphi_1 \in PC^1)$. This is good, because now we can apply Proposition 27.1 to show that $E$ is rectifiable.

Let $\{\varphi_t\}$ satisfy (1.4)-(1.8) and (2.4); we want to copy the proof of Proposition 27.15, but we need $C^1$ mappings, so we first pick a compact set $K$ that contains a neighborhood of $W_1 \cup \varphi_1(W_1)$, and then use the rectifiability of $E$ and Theorem 3.2.29 in [Fe] or Theorem 15.21 in [Ma] to find a countable collection of $C^1$ submanifolds $\Gamma_j \subset \mathbb{R}^n$, and disjoint Borel sets $F_j \subset \Gamma_j$, so that $\mathcal{H}^d(E \cap K \setminus \bigcup_i F_i) \leq \eta$, where the very small $\eta > 0$ will be chosen at the end of the argument. In fact, at the price of replacing $\eta$ with $2\eta$, we can suppose that the family is finite, and that each $F_i$ is compact. Even more, Theorem 3.1.16 in [Fe] allows us to (make $F_j$ a tiny bit smaller and) assume that $\varphi_1$ coincides on $F_i$ with a $C^1$ function on $\Gamma_j$. Notice that this function can be extended into a $C^1$ function $g_i$ defined on a neighborhood of $\Gamma_i$ (and hence $F_i$). We may now proceed as before; the only difference is that the small neighborhoods where we have $C^1$ extensions only cover $F = \cup_j F_j$ (and not $E$), but we did not use this to apply Lemma 27.3 and define our extension $f$. So we conclude as in Proposition 27.15. $\square$

We end this section with further results that may well be true, but which the author was too lazy to check. Thus the point of the following remarks is mostly to record what the author believes, just for the case when the potential results may become useful. The situation for $GSAQ(B_0, M, \delta, h, \varphi_1 \in PC^1)$ is correct, perhaps modulo our transversality assumption for the $L_j$. But we may feel bad about the very small difference between Lipschitz (as in (1.8)) and $C^\alpha$.

**Remark 27.46.** It is probably true that for $\alpha > 1$, the two classes $GSAQ(B_0, M, \delta, h)$ and $GSAQ(B_0, M, \delta, h, \varphi_1 \in C^\alpha)$ coincide under the same regularity condition for the $L_j$

as in Proposition 27.15. But even if there is no boundary piece $L_j$, it seems that some nontrivial argument is needed.

As before, and because of Proposition 27.15, it is enough to show that every $E \in GSAQ(B_0, M, \delta, h, \varphi_1 \in PC^\alpha)$ lies in $GSAQ(B_0, M, \delta, h)$. But this time we really need to modify our family $\{\varphi_t\}$ on a large set, because $\varphi_1$ may not be smooth anywhere.

We encountered this sort of problem before, when we were dealing with limits; we wanted to construct good competitors for sets $E_k$ that lie close to $E$, and were led to constructing stable competitors first. Here we probably want to do something similar, and proceed roughly as follows. We are given our set $E \in GSAQ(B_0, M, \delta, h, \varphi_1 \in PC^\alpha)$, and a family $\{\varphi_t\}$ that only satisfies the usual Lipschitz condition (1.8), and we want to construct a smoother family. We first build the stabler family of Sections 11-17, and because it is stable, we should be able to make it smoother without making is much worse. That is, the places where we expect the largest contributions are the $B_{j,x}$, $j \in J_3$ and $x \in Z(y_j)$, and on these places we composed the initial mapping $\varphi_1$ by a projection onto a $d$-plane and compared the measure of the image with the measure of a disk. We claim that replacing $\varphi_1$ with a smoother mapping before we project will not change the final estimates much.

On the other balls, or the intermediate regions (thin annuli, bad sets), we typically used no more than the fact that we project onto planes (which we still intend to do after we smooth out $\varphi_1$), and that our final mapping is Lipschitz with uniform bounds (which will not be disturbed by smoothing).

This description is probably enough if there is no boundary piece $L_J$, but in the general case we would also need to compose with retractions on the faces, as we did in Part IV and later, and for this the assumptions of Proposition 27.15 will probably be needed again. The fact that we are allowed competitors which are merely piecewise $C^\alpha$ may not be really needed (we can probably glue our pieces smoothly), but is at least psychologically comforting. At this point the reader probably guessed why we do not want to do all this here.

**Remark 27.47.** Let us say a few words about the difference between the two classes $GSAQ(U, M, \delta, h, \varphi_t \in C^\alpha)$ and $GSAQ(U, M, \delta, h, \varphi_1 \in C^\alpha)$.

The author believes that under the assumptions of Proposition 27.15, these two classes are probably equal, and that the same thing holds for their piecewise counterparts $GSAQ(U, M, \delta, h, \varphi_t \in PC^\alpha)$ and $GSAQ(U, M, \delta, h, \varphi_1 \in PC^\alpha)$.

In fact, the issue may also arise with our definition of the standard $GSAQ(U, M, \delta, h)$, where we only require the final mapping $\varphi_1$ to be Lipschitz, and we could have required instead that the whole map $(x, t) \to \varphi_t(x)$ be Lipschitz. We expect that this yields the same class of quasiminimal sets, but never checked.

This time, given a family $\{\varphi_t\}$ such that $\varphi_1$ is (piecewise) smooth, we want to change the $\varphi_t$, $0 < t < 1$, to make the family (piecewise) smooth. If there is no boundary piece, the point is merely to find a (continuous and piecewise) smooth mapping on $E \times [0, 1]$, with the given boundary value for $t = 0$ and $t = 1$. For this, the simplest is to use the definition of smoothness for $\varphi_1$, which gives a smooth extension to an open neighborhood of $E$, then decide brutally that $\varphi_t = \varphi_1$ for $1 - \varepsilon \leq t \leq 1$ (and similarly for $0 \leq t \leq \varepsilon$), and in the middle use partitions of unity and the values of $\varphi_t(x)$ on a discrete, but rather dense set of $E \times [0, 1]$ to interpolate. Maybe a small smooth gluing will be needed near

$t = \varepsilon$ and $t = 1 - \varepsilon$ too (as in Lemma 27.3). We proceeded a little like this in Section 11 when we extended $f$, and the advantage is that the construction is rather explicit, and in particular we can get that the new family $\{\widetilde{\varphi}_t\}$ is such that $||\widetilde{\varphi}_t - \varphi_t||_\infty$ is as small as we want.

When there are boundary pieces $L_j$, the new $\widetilde{\varphi}_t$ may not respect the $L_j$, and we need to use the smooth universal retractions $\pi_{\tau,t}$ of Proposition 27.15 (defined near (27.6)-(27.14)) to send points back to the $L_j$. That is, first observe that we can choose the $\widetilde{\varphi}_t$ so that the analogues for them of the sets $W_1$, $\varphi_1(W_1)$, and $\widehat{W}$ all stay within $\tau/100$ of the original $W_1$, $\varphi_1(W_1)$, and $\widehat{W}$, with $\tau$ as small as we want. Choose for $K$ the closure of $\widehat{W}$, $\tau$ very small and in particular such that $K^\tau \subset\subset U$, and use the assumptions for Proposition 27.15 to find the retractions $\pi_{\tau,t}$. Then set

$$\varphi_t^\sharp(x) = \pi_{\tau,\chi(x,t)}(\widetilde{\varphi}_t(x)) \tag{27.48}$$

for $x \in E$ and $0 \leq t \leq 1$, with a function $\chi$ that we still need to define.

The point of keeping $\varphi_t = \varphi_1$ for $1-\varepsilon \leq t \leq 1$ and similarly for $0 \leq t \leq \varepsilon$ is that we did not destroy anything on these intervals, and so we can take $\chi(x,t) = 0$ for $1 - \varepsilon/2 \leq t \leq 1$ and for $0 \leq t \leq \varepsilon/2$. We also want to take $\chi(x,t) = 0$ when $x \in E \setminus K^{\tau/2}$. We take $\chi(x,t) = 1$ when $x \in E \cap K^{\tau/3}$ and $\varepsilon \leq t \leq 1 - \varepsilon$. We also make $\chi$ smooth, with values in $[0,1]$. We claim (but will not check) that if $||\widetilde{\varphi}_t - \varphi_t||_\infty < C_0^{-1}\tau$ for all $t$, (27.48) gives a family $\{\varphi_t^\sharp\}$ which is smooth, for which (1.7) holds, and which still satisfies (2.2) and (1.4)-(1.6), although perhaps with a slightly larger ball $B$ because, to be safe, we want the new one to contain $K^\tau$. Thus the only effect in the verification of the quasiminimality property (2.5) is that, even though we did not change $\varphi_1$ and $W_1$, we have to replace $r^d$ in the right-hand side with a slightly larger $r_1^d$, which does not harm much. This completes our sketch of a potential proof of equivalence between the classes with $\varphi_1 \in C^\alpha$ and $\varphi_t \in C^\alpha$.

## PART VII : MONOTONE DENSITY

Monotonicity results for minimal sets or surfaces are very useful, for instance because they usually give a good control on the blow-up limits of these objects.

The starting point of this part is the following simple result (Theorem 28.4 below). Suppose that $E$ is a (locally) minimal set, with boundary pieces $L_j$ that are cones centered at $x$; then the density $\theta(x,r) = r^{-d}\mathcal{H}^d(E \cap B(x,r))$ is a nondecreasing function of $r$ (small).

We also show (in Theorem 29.1) that when in addition ($E$ is coral and) $\theta(x,\cdot)$ is constant, $E$ coincides with a minimal cone centered at $x$. This, with our result of Section 24, is our way to show that bow-up limits of almost minimal sets are minimal cones (Corollary 29.52 below).

We shall establish (with essentially the same proof as for Theorem 28.4) that $\theta(x,\cdot)$ is nearly monotone when $E$ is almost minimal with a sufficiently small gauge function $h$ (and the $L_j$ are still cones centered at $x$); see Theorem 28.7. This result is extended, with a slightly different density function to make the computations easier, to the case when the $L_j$ are not exactly cones. See Remark 28.11 and Theorem 28.15.

The equality case proved in Section 29 will allow us to show, by compactness, that (under suitable assumptions) if for the almost minimal set $E$, the density $\theta(x,\cdot)$ is almost constant, then $E$ is close to a minimal cone, both in Hausdorff distance and in measure. See Proposition 30.3 for a statement of approximation in an annulus, and Proposition 30.19 for a simpler statement of approximation in a ball.

The results of this part clearly have some interest, but we should observe that it would be much better to have monotonicity results for some quantity like $\theta(x,r)$, which would also hold when $x$ is not the center of the $L_j$. We shall not try to prove such formulae here.

**28. Monotone density for minimal sets; almost monotone density in some cases**

We start our study with the monotonicity of density for a minimal set. We consider a coral minimal set $E$, more precisely such that

$$E \in GSAQ(U, 1, \delta, 0) \tag{28.1}$$

for some open set $U$, and where the boundary pieces $L_j$, $0 \leq j \leq j_{max}$, satisfy the Lipschitz assumption. See Definitions 2.3 and 2.7. We are also given a ball $B(x_0, r_0) \subset U$, and we assume that $r_0 \leq \delta$, and also that for $0 \leq j \leq j_{max}$,

$$L_j \text{ coincides, in } B(x_0, r_0), \text{ with a closed cone centered at } x_0. \tag{28.2}$$

We allow $L_j \cap B(x_0, r_0) = \emptyset$ (even though in this case there is not much point in keeping $L_j$), and we allowed the Lipschitz assumption because we do not want to restrict to plane sectors that make square or flat angles. In fact our proof will only use the Lipschitz assumption to make sure that $E$ is rectifiable, and otherwise (28.2) will be enough. Next set

$$\theta(r) = r^{-d}\mathcal{H}^d(E \cap B(x_0, r)) \text{ for } 0 < r \leq r_0. \tag{28.3}$$

**Theorem 28.4.** *Let $U$, the $L_j$, the minimal set $E$, and $B(x_0, r_0) \subset U$ satisfy the assumptions above. Then $\theta : (0, r_0) \to \mathbb{R}_+$ is nondecreasing.*

This should not shock the reader. The result for minimal sets far from the boundary is classical, and relies on comparisons of $E$ with cones, which can be obtained as limits of radial deformations of $E$. These deformations will preserve the boundary pieces $L_j$, by (28.2), and we will be able to conclude from there.

We shall follow the proof of Proposition 5.16 in [D5], which was conveniently done in a similar context. We start with the integrated version of monotonicity which is stated as Lemma 5.1 in [D5]. In the statement of that lemma, the author required that $E$ be coral and that $x_0 \in E$, but this is not used in the proof (only later in the section). The proof then used the rectifiability of $E$ and a radial deformation, defined near (5.3), and it is clear that such deformations satisfy our boundary constraint (1.7) because of (28.2). So Lemma 5.1 in [D5] goes through. Once we have that lemma, the proof is a simple manipulation of measures and integrals, that does not use the minimality of $E$, and it goes through as it is. Theorem 28.4 follows. □

Let us now record a version of Theorem 28.4 for almost minimal sets. We give ourselves a gauge function $h : (0, +\infty) \to [0, +\infty]$, which we assume to be nondecreasing and continuous on the right; these are probably not exactly needed, but won't disturb much and the assumption was made in Section 4 of [D5] and the beginning of our Section 20. We also assume the Dini condition

$$\int_0^{r_0} h(2t)\frac{dt}{t} < +\infty \tag{28.5}$$

(which is really used in the proof; don't mind the fact that we have $h(2t)$, which we repeat from [D5] and is jut due to the fact that we wanted to estimate $h(t)$ with an integral). We replace our minimality condition (28.1) with the new one that

(28.6) $E$ is an $A$-almost, or an $A'$-almost minimal set in $U$, with gauge function $h$

(and with the sliding conditions given by the closed sets $L_j$). See Definition 20.2; we don't care whether the $A$-almost or $A'$-almost minimality is used, both are equivalently easy to use in the proof. We now copy the analogue in the present context of Proposition 5.24 in [D5].

**Theorem 28.7.** *There exist constants $\alpha > 1$ and $\varepsilon_n > 0$, that depend only on the dimension $n$ and on the constant $\Lambda$ in the Lipschitz assumption, such that the following holds. Let $U$, the $L_j$, the gauge function $h$, the almost minimal set $E$, and the ball $B(x_0, r_0) \subset U$ satisfy the assumptions above. Suppose in addition that $E$ is coral, that $x_0 \in E$, and that $h(r_0) \leq \varepsilon_n$. Then*

(28.8) $\theta(r) \exp \alpha\Big(\int_0^r h(2t)\frac{dt}{t}\Big)$ *is a nondecreasing function of* $r \in (0, r_0)$.

Notice that the exponential tends to 1 as $r$ tends to 0, so we can see it as a nice (increasing) extra term that we multiply with $\theta(r)$ to get a nondecreasing function. The fact that we use $h(2t)$ is just an artifact of the statement; the reader should not worry about the case when $t$ is close to $r_0$ and $h(2t)$ may not be defined naturally: just set $h(r) = h(r_0)$ for $r \geq r_0$.

It looks strange that now we require $E$ to be coral and $x_0 \in E$; this is because in the proof, we use a lower bound for $\theta(r)$, that comes from the local Ahlfors-regularity of $E$, to simplify a differential inequality. See Remark 28.9 though. For the proof we proceed as we did in [D5]; our almost minimality assumption is only used twice, once in Lemma 5.1 as before, and once, through the local Ahlfors regularity, in the computation of differential inequalities; so the proof goes through. We have to let $\varepsilon_n$ depend on $\Lambda$ because we use our regularity theorems to prove that $E$ is rectifiable and locally Ahlfors-regular, and $\alpha$ depends on $\Lambda$ too, through the local Ahlfors-regularity bounds that we use to modify a differential inequality. □

**Remark 28.9.** We may even drop our assumption that $E$ is coral and $x_0 \in E$ if we replace (28.6) with the stronger

(28.10) $E$ is an $A_+$-almost minimal set in $U$, with gauge function $h$.

This is Proposition 5.30 in [D5], and as before the proof just goes through. Then we don't need the local Ahlfors-regular bound and $\alpha$ does not depend on $\Lambda$.

**Remark 28.11.** Theorem 28.7 can also be generalized slightly to situations where the $L_j$ are not exactly cones.

Let us assume that, instead of (28.2), we have a bilipschitz mapping $\xi = B(x_0, r_0) \to \xi(B(x_0, r_0)) \subset \mathbb{R}^n$, with the following properties. First of all,

(28.12) $\xi(B(x_0, r_0) \cap L_j)$ coincides, in $\xi(B(x_0, r_0))$, with a closed cone centered at $\xi(x_0)$;

this will be our replacement for (28.2). We want a better control (typically, of $C^1$ type) in the smaller balls centered at $x_0$, so we assume that for $r \in (0, r_0]$, there is a constant $\rho(r))$ such that

(28.13) $$\text{the restriction of } \xi \text{ to } B(x_0, r) \text{ is } (1 + \rho(r)) \text{ bilipschitz}$$

and

(28.14) $$\int_0^{r_0} \rho(t) \frac{dt}{t} < +\infty.$$

Then we have the following extension of Theorem 28.7.

**Theorem 28.15.** *There exist constants $\alpha_1 > 1$ and $\varepsilon_n > 0$, that depend only on the dimension $n$ and on the constant $\Lambda$ in the Lipschitz assumption, such that the following holds. Let $U$, the $L_j$, the gauge function $h$, the coral almost minimal set $E$, and the ball $B(x_0, r_0) \subset U$ be such that $h$ is nondecreasing and continuous on the right, (28.5) and (28.6) hold, $x_0 \in E$, $h(r_0) \leq \varepsilon_n$, and there exists $\xi$ and $\rho$ as above, such that $\rho(r_0) \leq \varepsilon_n$ and (28.12), (28.13), and (28.14) hold. Set $\widetilde{B}(r) = \xi^{-1}(B(\xi(x_0), r))$ and*

(28.16) $$\Psi(r) = \mathcal{H}^d(E \cap \widetilde{B}(r)) \exp\Big(\alpha_1 \Big( \int_0^r (h(9t/4) + \rho(9t/4)) \frac{dt}{t} \Big)\Big)$$

*for $0 < r < r_0/2$; then $\Psi$ is nondecreasing on $(0, r_0/2]$.*

We were a little lazy here, because we measured the density in terms of the slightly distorted balls $\widetilde{B}(r)$. This way we will be able to reduce to Theorem 28.7 via a change of variable. Probably the more reasonable statement with the the same function $\theta$ as above also holds, but for this it seems that we would have to follow the proof above, and in due time modify the proof of Lemma 5.1 in [D5]. That is, we would obtain some estimate on the measure of thin annuli by constructing directly a competitor that expands the annulus and contracts the inside disk. Since our initial radial competitor probably does not satisfy (1.7) (because the $L_j$ are no longer cones), we could conjugate by $\xi$ and apply a radial transformation in the new variables. We decided to use the function $\Psi$ above and avoid the computations.

So we try to deduce Theorem 28.15 from Theorem 28.7 and a change of variable. Without loss of generality, we may assume that $\xi(x_0) = x_0 = 0$. Set $B_0 = B(x_0, r_0)$ and

$\widetilde{E} = \xi(E \cap B_0)$; we would like to say that $\widetilde{E}$ is almost minimal in $\xi(B_0)$, but this is probably wrong, because we did not assume $\xi$ to be asymptotically conformal near each point of $B_0$ (this would be a very strong assumption to make!), but only at the point $x_0$. So we need to be a little careful with our assertions.

Let us first prove that $\widetilde{E}$ is quasiminimal in $B_1 = \xi(B_0)$, with $M = 1$, $\delta = 2r_0$, and $h = C\varepsilon_n$. That is, with the notation of Definition 2.3, that

$$\widetilde{E} \in GSAQ(B_1, 1, 2r_0, C\varepsilon_n), \tag{28.17}$$

where on $B_1$, we use the boundary pieces $\widetilde{L}_j = B_1 \cap \xi(L_j)$. We put $\delta = 2r_0$ as a way to imply that we put no constraint on the size of the analogue of $\widehat{W}$ for competitors of $\widetilde{E}$.

The proof will be easy. Let the $\widetilde{\varphi}_t$, $0 \leq t \leq 1$, define a competitor for $\widetilde{E}$ in $B_1$; that is, assume that they satisfy the analogue of (1.4)-(1.8), in a ball $\widetilde{B}$ of radius $\widetilde{r}$, and (2.4). Then define $\varphi_t$, $0 \leq t \leq 1$, by $\varphi_t(x) = \xi^{-1}(\widetilde{\varphi}_t(\xi(x)))$ for $x \in \xi^{-1}(B_1)$ and $\varphi_t(x) = x$ otherwise. It is easy to see that the $\varphi_t$ define a competitor for $E$ (i.e, satisfy the usual constraints (1.4)-(1.8) and (2.4)), in a ball $B$ that contains $\xi^{-1}(\widetilde{B})$; we can choose $B$ or radius $r = \min(r_0, (1 + \rho(r_0))\widetilde{r})$. In addition, $\widetilde{W} = \big\{x \in \widetilde{E} \cap B_1\,;\, \widetilde{\varphi}_1(x) \neq x\big\}$ is equal to $\xi(W)$, where $W = \big\{x \in E \cap B_0\,;\, \varphi_1(x) \neq x\big\}$. We may assume that $E$ is $A$-almost minimal (because $A'$-almost minimality implies $A$-almost minimality with the same gauge function $h$; see near (20.8)), and the defining property (20.5) yields

$$\begin{aligned} \mathcal{H}^d(\widetilde{W}) = \mathcal{H}^d(\xi(W)) &\leq (1 + \rho(r_0))^d \mathcal{H}^d(W)) \\ &\leq (1 + \rho(r_0))^d \big[\mathcal{H}^d(\varphi_1(W)) + h(r) r^d\big] \\ &\leq (1 + \rho(r_0))^d \big[(1 + \rho(r_0))^d \mathcal{H}^d(\widetilde{\varphi}_1(\widetilde{W})) + h(r) r^d\big]. \end{aligned} \tag{28.18}$$

If $\mathcal{H}^d(\widetilde{W}) \leq \mathcal{H}^d(\widetilde{\varphi}_1(\widetilde{W}))$, we are happy. Otherwise,

$$\mathcal{H}^d(\widetilde{\varphi}_1(\widetilde{W})) \leq \mathcal{H}^d(\widetilde{W}) \leq \mathcal{H}^d(\widetilde{E} \cap \widetilde{B}) \leq (1 + \rho(r_0))^d \mathcal{H}^d(E \cap B) \leq C r^d \leq C \widetilde{r}^d \tag{28.19}$$

because $E$ is locally Ahlfors-regular in $B_0$ (if $h(r_0) < \varepsilon_n$ is small enough). Then (28.18) yields

$$\begin{aligned} \mathcal{H}^d(\widetilde{W}) &\leq (1 + \rho(r_0))^{2d} \mathcal{H}^d(\widetilde{\varphi}_1(\widetilde{W})) + (1 + \rho(r_0))^d h(r) r^d \\ &\leq \mathcal{H}^d(\widetilde{\varphi}_1(\widetilde{W})) + [C\rho(r_0) + h(r)]\, \widetilde{r}^d \end{aligned} \tag{28.20}$$

because $r = (1 + \rho(r_0))\widetilde{r}$; (28.17) follows because $h(r) \leq h(r_0) < \varepsilon_n$ and $\rho(r_0) \leq \varepsilon_n$.

When the ball $B$ is contained in $B(0, t)$ for some $t \leq 8r_0/9$, we can use the better bilipschitz control provided by (28.13), and the proof of (28.20) yields

$$\mathcal{H}^d(\widetilde{W}) \leq \mathcal{H}^d(\widetilde{\varphi}_1(\widetilde{W})) + [C\rho(9t/8) + h(9t/8)]\, \widetilde{r}^d. \tag{28.21}$$

Consequently,

$$\widetilde{E} \cap B(0, t) \in GSAQ(B(0, t), 1, 2r_0, C\rho(9t/8) + h(9t/8)). \tag{28.22}$$

We may now follow our proof of Theorem 28.7, which we want to apply to $\widetilde{E}$. For this proof we need to know that $\widetilde{E}$ is rectifiable and locally Ahlfors regular in $B_1$. Since $\xi$ is bilipschitz, we just need to check that $E$ is rectifiable and locally Ahlfors regular in $B_0$, and this follows from our assumption (28.10), together with the fact that $h(r_0) \leq \varepsilon_n$, and as usual Propositions 4.1 and 4.74 and Theorem 5.16. We even get bounds on $r^{-d}\mathcal{H}^d(\widetilde{E} \cap B(x,r))$, for $x \in E$ and $B(x, 2r) \subset B_1$, that depend only on $n$ and $\Lambda$, as long as $\varepsilon_n$ is chosen small enough. Incidentally, it is a little easier to proceed this way here, rather than trying to use (28.17) directly, because this way we don't need to worry about the fact that on $B_1$, the boundary pieces $\widetilde{L}_j = B_1 \cap \xi(L_j)$ may not satisfy the Lipschitz assumption exactly as it was stated.

Then we turn to the main ingredient of the proof, which is the comparison argument in [D5], Lemma 5.1, that we already used for Theorem 28.4. This lemma uses a radial deformation of the set (here, this means $\widetilde{E}$). The fact that the boundary sets $\xi(L_j)$ are conical allows us to use the same deformation, and then apply (28.20) or (28.21). That is, we do not need to use the full almost minimality of $\widetilde{E}$, because we just need to compare with a single competitor that lives in a small ball $B$ centered at the origin. Now this is the only place where we use the almost minimality of $E$ in [D5] (see (5.4) there); after this, the same argument as in Theorem 28.7 applies, and again does not use the full almost minimality (or the Lipschitz assumption), but just the rectifiability and local Ahlfors regularity of $\widetilde{E}$, plus measure theory. We get that $H(r)$ is a nondecreasing function of $r \in (0, 8r_0/3)$, where

$$H(r) = r^{-d}\mathcal{H}^d(\widetilde{E} \cap B(0,r)) \exp \alpha\Big(\int_0^r (C\rho(9t/4) + h(9t/4))\frac{dt}{t}\Big) \tag{28.23}$$

is the analogue for $\widetilde{E}$ of the function in (28.8). Of course there is a small difference between $\Psi$ and $H$, because $\mathcal{H}^d(E \cap \xi^{-1}(B(0,r))$ is not the same as $\mathcal{H}^d(\widetilde{E} \cap (B(0,r))$ (one set is the image of the other one by $\xi$), but this will not disturb much for the monotonicity. Once again, we want to avoid a computation, so let us recall how (28.23) is obtained in [D5]. We write $H(r) = l(r)g(r)$, with $l(r) = \mathcal{H}^d(\widetilde{E} \cap B(0,r))$ and $g(r) = r^{-d}e^{\alpha A(r)}$, with $A(r) = \int_0^r (C\rho(9t/4) + h(9t/4))\frac{dt}{t}$. Then we find out that those functions have a derivative almost everywhere, and that it is enough to show that $H'(r) \geq 0$ almost everywhere. A computation shows that

$$H'(r) = l'(r)g(r) + l(r)g'(r) = g(r)\big\{l'(r) - \frac{l(r)}{r}\,\big[d - \alpha(C\rho(9r/4) + h(9r/4))\big]\big\} \tag{28.24}$$

(see (5.27) in [D5]), and it tuns out that because of Lemma 5.1, the right-hand side of (28.24) is nonnegative almost everywhere. That is, setting $a = C\rho(9r/4) + h(9r/4)$,

$$l'(r) \geq \frac{l(r)}{r}(d - \alpha a) \tag{28.25}$$

Now we want to replace $l(r)$ with $l_1(r) = \mathcal{H}^d(E \cap \widetilde{B}(r)) = \mathcal{H}^d(\xi^{-1}(\widetilde{E} \cap B(0,r))$, and prove the same inequality, but with $\alpha$ replaced by a larger $\alpha_1$. We observe that, by a change of variable, $l_1(r) \leq (1+\rho(9r/4))^d l(r)$ and $l_1'(r) \geq (1+\rho(9r/4))^{-d} l'(r)$. Therefore

$$\begin{aligned} l_1'(r) &\geq (1+\rho(9r/4))^{-d} l'(r) \geq (1+\rho(9r/4))^{-d} \frac{l(r)}{r}(d-\alpha a) \\ &\geq (1+\rho(9r/4))^{-2d} \,\frac{l_1(r)}{r}\,(d-\alpha a). \end{aligned} \tag{28.26}$$

Now $\rho(9r/4) < a$ and $a$ is as small as we want, so $(1+\rho(9r/4))^{-2d}(d-\alpha a) \geq d - \alpha_1 a$ for some new constant $\alpha_1 > \alpha$, and (28.26) gives an analogue of (28.25) for $l_1$ and $\alpha_1$, which implies that $\Psi'(r) \geq 0$ almost everywhere, and then that $\Psi$ is monotone, by the same argument as in [D5]. $\square$

**Remark 28.27.** When $E$ is an $A_+$-almost minimal set (as in (28.10)), we do not need to assume that $E$ is coral and that $x_0 \in E$. The reason is the same as for Remark 28.9.

## 29. Minimal sets with constant density are cones

Our goal for this section is to prove that under the assumptions of Theorem 28.4, if in addition the density function $\theta$ is constant on some interval, the set $E$ (almost) coincides with a minimal cone on the corresponding annulus. As a corollary we will get that under mild assumptions, blow-up limits of coral almost minimal sets are minimal cones. See Corollary 29.53.

**Theorem 29.1.** *Let the open set $U$ and the boundary pieces $L_j$, $0 \leq j \leq j_{max}$ satisfy the Lipschitz assumption, let $E$ be a coral local minimal set in $U$, with $E \in GSAQ(U,1,\delta,0)$ for some $\delta > 0$, and let $B(x_0,r_0) \subset U$, with $0 < r_0 \leq \delta$ be given. Assume (as in (28.2)) that each $L_j$ coincides in $B(x_0,r_0)$ with a cone centered at $x_0$ and that we can find constants $a$, $b$, and $\theta$ such that $0 \leq a < b \leq r_0$ and*

$$\mathcal{H}^d(E \cap B(x_0,r)) = \theta r^d \quad \text{for } a < r < b. \tag{29.2}$$

*Then there is a closed coral minimal cone $\mathcal{C}$ centered at $x_0$, such that $E \cap B(x_0,b) \setminus B(x_0,a) = \mathcal{C} \cap B(x_0,b) \setminus B(x_0,a)$.*

Let $\widehat{L}_j$ denote the cone that coincides with $L_j$ in $B(x_0,r_0)$; the fact that $\mathcal{C}$ is a minimal cone in $\mathbb{R}^n$, associated to the boundary pieces $\widehat{L}_j$, is a fairly easy consequence of local minimality of $E$. The argument is given in more detail in [D5], pages 125-126, but we sketch it here for the convenience of the reader. In the ball $B(x_0,r_0)$, the cone $\mathcal{C}$ is a competitor for $E$, i.e., can be obtained as $\varphi_1(E)$ for some family $\{\varphi_t\}$ of mappings that contract part of $E$ along the rays through $x_0$. In particular, the constraint (1.7) is satisfied by (28.2) and because we move points along rays. The cone also has the same measure as $E$ in $B(x_0,r_0)$, by our assumption of constant density. Then every competitor for the cone gives rise to a competitor for $E$ (by scale invariance, we may assume that the modifications only occur in $B(x_0,r_0/2)$, and then we just compose our two deformations); the minimality of $E$ then implies the minimality of $\mathcal{C}$. Similarly, $\mathcal{C}$ is coral because $E$ is Ahlfors-regular.

We required $E$ to be coral to have a cleaner statement; if we don't, we just get that conclusion that the difference between $E \cap B(x_0, b) \setminus B(x_0, a)$ and $\mathcal{C} \cap B(x_0, b) \setminus B(x_0, a)$ is $\mathcal{H}^d$-negligible. Recall that the core of a minimal set is a coral minimal set; we would use this to replace an initial minimal cone $\mathcal{C}$ with a coral one.

Let us now prove the theorem. Just for convenience and by translation invariance, let us assume that $x_0 = 0$; notice that we work with the Lipschitz assumption, so the origin does not have a special position in our grid, except for the fact that it is the center of our cones.

Set $A = B(0, b) \setminus \overline{B}(0, a)$. We first follow carefully our proof of monotonicity for $\theta$ (Theorem 28.4), use the fact that all the inequalities are identities almost everywhere, and get that for $\mathcal{H}^d$-almost every $x \in E \cap A$, $E$ has a tangent plane $P(x)$ at $x$, which goes through the origin. The existence of a tangent plane follows from the rectifiability and local Ahlfors regularity of $E$; only the fact that $0 \in P(x)$ is new, and we get it essentially because otherwise the measure of $E$ in a thin annulus that contains $x$ would be too large, compared to $\mathcal{H}^{d-1}(\partial B(0, |x|)$. See (6.5) in [D5] (and its translation one page later); the same proof (of measure theory only) applies here.

Our next stage is to show that for $\mathcal{H}^d$-almost every $y \in E \cap A$,

$$E \text{ contains the line interval } A \cap L(y), \tag{29.3}$$

where $L(y) = \big\{\lambda y \,;\, \lambda > 0\big\}$ is the open half line through $y$. Once we prove this, the conclusion will follow, because (29.3) then also holds for all $y \in E \cap A$; see [D5] (below (6.12)) for the easy verification.

We can thus restrict our attention to the points $y \in E \cap A$ for which the tangent plane $P(y)$ exists and contains the origin, but we also add the following density constraint, which is valid $\mathcal{H}^d$-almost everywhere (see [Ma], Theorem 6.2 (2) on page 89). For $y \in U$, denote by $\mathcal{F}(y)$ the collection of all the faces $F$ of our grid that contain $y$. We require that for every face $F \in \mathcal{F}(y)$, $y$ be a density point of $E \cap F$ in $F$, i.e., that

$$\lim_{r \to 0} r^{-d} \mathcal{H}^d(B(y, r) \cap E \setminus F) = 0. \tag{29.4}$$

Let $y \in E$ be such a point, and assume that we can find $x \in A \cap L(y) \setminus E$; we want to apply the proof of Proposition 6.11 in [D5], with a few modifications, to get a contradiction.

The construction will use two radii $r_y$ and $r_x$, and will work as soon as $r_y$ is small enough (depending on $y$) and then $r_x$ is small enough (depending on $r_y$ and the position of $x$, $y$, and in particular on the ratio $|y|/|x|$, which may be large). Various smallness conditions will arise along the proof, but let us mention the first ones. First set

$$B_y = B(y, r_y),\ B_x = B(x, r_x), \quad \text{and } P = P(y) \tag{29.5}$$

to simplify the notation. As in (6.13) in [D5], we require that for some small $\varepsilon_0$ (that will be chosen near the end),

$$\operatorname{dist}(z, P) \le \varepsilon_0 r_y \quad \text{for } z \in E \cap B(y, 3r_y); \tag{29.6}$$

this is true for $r_y$ small because $P = P(y)$ is a true tangent plane, by Ahlfors-regularity (see Exercise 41.21 on page 277 of [D4]). We also demand that for each face $F$ that contains $y$, and in particular the smallest one,

$$\mathcal{H}^d(B(y, 3r_y) \cap E \setminus F) \leq \varepsilon_1^d r_y^d, \tag{29.7}$$

where $\varepsilon_1$ is another, even smaller, positive constant. Again small $r_y$ satisfy this, by (29.4). Let us deduce from this that

$$\operatorname{dist}(z, L_j) \leq C\varepsilon_0 r_y \text{ when } L_j \text{ contains } y \text{ and } z \in P \cap B_y. \tag{29.8}$$

We want to apply Lemma 9.14 to $E$ and the ball $B_y$. If $r_y$ is small enough, the first condition (9.15) on the size of $B_y$ is satisfied. Also, every $L_i$ that meets $3B_y$ contains $y$, so the set $L$ of (9.16) is the intersection of all the $L_i$ that contain $y$. The smallest face $F$ that contains $y$ is contained in all these $L_i$, hence $F \subset L$ and (29.7) says that $\mathcal{H}^d(B(y, 3r_y) \cap E \setminus L) \leq \varepsilon_1^d r_y^d$. Since $E$ is locally Ahlfors-regular, this implies that

$$\operatorname{dist}(z, L) \leq C\varepsilon_1 r_y \text{ for } z \in E \cap 2B_y. \tag{29.9}$$

That is, the constraint (9.17) is satisfied if $\varepsilon_1$ is small enough.

Next, the flatness condition (9.18) holds (for the same $P$ and with $\varepsilon = \varepsilon_0$); if $\varepsilon_0$ is small enough, we can apply Lemma 9.14 and we also get that

$$\operatorname{dist}(p, E) \leq \varepsilon_0 r_y \text{ for } p \in P \cap B(y, 3r_y/2), \tag{29.10}$$

as in (9.19). Now (29.8) follows from this and (29.9), if $\varepsilon_1$ is smaller than $\varepsilon_0$.

Let us try to describe the construction of [D5], without entering into too much detail. For simplicity, all the references of the type (6.x) will refer to the corresponding number in [D5].

The construction of [D5] starts with the choice of a set $T \subset B_x$, which is defined near (6.19); the fact that $T \subset B_x$ follows from Lemma 6.15 (of [D5]). Then we consider the cone $\widehat{T}$ over $T$, and more precisely the part that lives near $[x, y]$. That is, if $x_1$ and $x_2$ lie in the half line $L(y)$ through $y$, we denote by $V(x_1, x_2)$ the set of points of $\mathbb{R}^n$ whose orthogonal projection on the line that contains $L(y)$ lies between $x_1$ and $x_2$. We shall use a lot the piece of tube

$$T_0 = \widehat{T} \cap V(x_1, y) \tag{29.11}$$

where $x_1$ is a point of $L(y)$ that lies quite close to $x$ (on the other side of $x$ as $y$, so that $V(x, y) \subset V(x_1, y)$); see (6.22) for the definition of $x_1$, which was called $x_0$ in [D5] (but we want to avoid a conflict with the center of our main ball). Notice that

$$\operatorname{dist}(z, [x, y]) \leq (1 + |x|^{-1}|y|)\, r_x \text{ for } z \in T_0, \tag{29.12}$$

essentially because $T \subset B_x$. We shall use $T_0$ to connect $B_x$ to $B_y$; the point of the specific choice in Lemma 6.15 is to find a tube, inside $T_0$, that does not meet $T_0$, but we shall not need to know this here. Now set

$$Z = T_0 \cup \overline{B}_y, \tag{29.13}$$

as in (6.43); this is the place where most of the transformations will take place. That is, a few successive mappings $f_1, \dots, f_5$ are constructed in [D5], and the composition $f = f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1$ is used to define a competitor for $E$, and eventually get a contradiction. The point of these mappings is to use the hole that we have in $B_x$ (that is, there is no $E$ there) to push $E \cap B_y$ onto a set of much smaller measure, essentially contained in $\partial B_y$. We will not need to know here how these mappings are constructed, but for the information of the reader, let us rapidly say how it works. Our first mapping $f_1$ uses the hole in $\widehat{T}$ to send points of $E$ (vertically, i.e., in hyperplanes perpendicular to $L(y)$) to something that looks like a thin double tunnel (say, when $d = 2$ and $n = 3$), with a common flat floor that is contained in $P$, and which we shall use to communicate between $B_x$ (which does not meet $E$) and $P \cap B_y$ (which we want to kill). Then $f_2$ acts in $B_y$, where it pushes points of $E_1 = f_1(E)$ to $B_y \cap P$ (a large disk, with an entrance that was the floor of the tunnel), plus some small part of $\partial B_y$ near $P$. Then we compose with a mapping $f_3$ that pushes the points of the floor, starting from the empty part $P \cap B_x$, and eventually sends every point of the floor to a point of $P \cap \partial B_y$. This is good, because this is how we get rid of most of the measure of $E \cap B_y$. In codimension larger than 1, since $\partial B_y$ has an infinite measure, we need to compose with two additional Federer-Fleming projections on $d$-dimensional skeletons, one near the boundary of our tunnel that is not the floor, and one near $\partial B_y$, to get some good control on the image of the part of $E_3 = f_3 \circ f_2 \circ f_1(E)$ that lives there.

We return to the final mapping $f$, and record some of its properties. There is a slight enlargement $Z^*$ of $Z$, that will be described in a minute, such that

$$f(z) = z \text{ for } z \in E \setminus Z^*, \text{ and } f(Z^*) \subset Z^*. \tag{29.14}$$

This set is defined just above (6.90), and we just need to know that

$$Z^* = Z \cup Z_1^* \cap Z_2^* \tag{29.15}$$

(see just above (6.90)) with sets $Z_1^*$ and $Z_2^*$ with the following properties. First

$$\operatorname{dist}(z, T_0) \leq C r_x \quad \text{for } z \in Z_1^*; \tag{29.16}$$

see above (6.79). Next $Z_2^*$ is defined as a neighborhood, roughly of width $Cr_x$ (see (6.58)) of a set $Z_2$, which itself is defined by (6.51) and contained in $T_0$ (because $V(x, y) \subset V(x_1, y)$). And similarly

$$\operatorname{dist}(z, \partial B_y \cap P) \leq 3\varepsilon_0 r_y + C r_x \quad \text{for } z \in Z_2^*; \tag{29.17}$$

this time see the line above (6.88) for the definition of $Z_2^*$ in terms of $Z_2$, (6.51) for the definition of $Z_2$, and (6.39) for $H$.

The main nice thing about $f$ is that, as in (6.92),

$$\mathcal{H}^d(E_5 \cap Z^*) \leq C\varepsilon_0 r_y^d + C_{x,y} r_x^{d-1}, \tag{29.18}$$

where $E_5 = f(E)$, and the constant $C_{x,y}$ does not depend on $r_x$, so that we can choose $r_x$ very small to make $\mathcal{H}^d(E_5 \cap Z^*)$ as small as we want compared to $r_y^d$.

It is easy to find a continuous one parameter family of functions $\varphi_t : E \to \mathbb{R}^n$, such that $\varphi_0(z) = z$ and $\varphi_1(z) = f(z)$ for $z \in E$, $\varphi_t(z) = z$ for $z \in E \setminus Z^*$, and $\varphi_t(Z^*) \subset Z^*$. This is not exactly what was done in [D5], where a brutal linear interpolation was enough, but did not yield $\varphi_t(Z^*) \subset Z^*$ because $Z^*$ is not convex. Here we want to proceed with just a little more care, and the most natural thing to do is construct a family that goes from the identity to $f_1$, then from $f_1$ to $f_2 \circ f_1$, then to $f_3 \circ f_2 \circ f_1$, and so on. For the first two times, we proceed by brutal linear interpolation. When we go from $f_2 \circ f_1$ to $f_3 \circ f_2 \circ f_1$, we proceed slightly differently.

The main part of the definition of $f_3$ is (6.47), which is its definition on the floor $F = P \cap Z \cap V(x, b)$, where $b$ is a point of $L(y)$ a little further from $x$ than $B_y$; see the bottom of page 114 in [D5], and the statement of Theorem 6.2 for the definition of $b$ (which essentially lies on $\partial B(0, r_0)$ here). We don't care about the definition of $f_3$ on the rest of the interior of $Z$, because the intersection of $E_2 = f_2 \circ f_1(E)$ with the interior of $Z$ is reduced to $F$; see (6.46), and compare with (6.43) or see the comment three lines above (6.52) for a hint. And on $\overline{\mathbb{R}^n \setminus Z}$, we set $f_3(z) = z$ (see (6.48)). So for the interior of $Z$, the only interesting piece is the definition of $f_3$ on the floor $F$, and it is obtained by conjugating with a bilipschitz mapping $\psi$ that goes from $F$ to a cylinder a radial mapping on the cylinder that maps it to its boundary. To define the intermediate mappings on $F$, we just interpolate linearly the radial projection, and conjugate. Of course we keep the identity on $\overline{\mathbb{R}^n \setminus Z}$, and the way we extend to the rest of $Z$ does not matter anyway.

Even though this is not important (and a brutal interpolation would do), for the last two segments where we compose with $f_4$ and $f_5$, it is more natural to decompose the deformation into successive Federer-Fleming (radial) projections, and for each one interpolate linearly. This way we are sure not to leave the sets $Z_1^*$ and $Z_2^*$ introduced in (29.15).

Anyway, this allows us to define a one parameter family $\{\varphi_t\}$ that goes from the identity to $f$, but because of the boundary condition (1.7) we cannot use this family directly, and we shall compose it with retractions.

We shall use the mapping $\Pi$ that was constructed for Lemma 17.18, except that here we can work under the Lipschitz assumption, in which case we conjugate it with our usual mapping $\psi(\lambda\cdot)$ to make it work on a different grid. This only makes the constant $C$ in (17.19)-(17.21) larger, and also forces us to define $\Pi(z, s)$ only when $0 \leq s \leq C^{-1}$ (instead of $0 \leq s \leq 10^{-1}$), but this will not matter. We define a first part of our path by

$$g_t(z) = \Pi(z, t\chi(z)) \quad \text{for } z \in E \text{ and } 0 \leq t \leq 1, \tag{29.19}$$

where the cut-off function $\chi = \chi_1 + \chi_2$ is defined as follows. We start with $\chi_1$, for which we use the small scale $\tau_1 = C_1 r_x$, where the geometric $C_1$ will be chosen later and may also depend on $|y|/|x|$, and set

$$\chi_1(z) = \Big[2\tau_1 - \operatorname{dist}(z, T_0)\Big]_+ \quad \text{for } z \in \mathbb{R}^n. \tag{29.20}$$

Here $T_0$ is the truncated cone of (29.11) and $a_+$ denotes the positive part of $a \in \mathbb{R}$. Similarly, we choose $\tau_2 = C_2 \varepsilon_0 r_y$, where the large $C_2$ will also be chosen later, and we set

$$\chi_2(z) = \Big[2\tau_2 - \operatorname{dist}(z, B_y)\Big]_+ \quad \text{for } z \in \mathbb{R}^n. \tag{29.21}$$

Thus, by (17.19), $g_t(z) = z$ for $0 \leq t \leq 1$ unless $z$ lies in a $2\tau_1$-neighborhood of $T_0$, or a $2\tau_2$-neighborhood of $B_y$. At the end of this first stage, we are left with $g_1(z) = \Pi(z, \chi(z))$. For our next stage, we use the $\varphi_t$ above and set

$$g_t(z) = \Pi(\varphi_{t-1}(z), \chi(\varphi_{t-1}(z))) \text{ for } z \in E \text{ and } 1 \leq t \leq 2. \tag{29.22}$$

We want to check now that the $g_{2t}$, $0 \leq t \leq 1$, define an acceptable competitor for $E$. We start with (2.4). Set

$$Z_+ = \big\{z \in E\,;\, \chi(z) > 0\big\} = \big\{z \in E\,;\, \mathrm{dist}(z, T_0) < 2\tau_1 \text{ or } \mathrm{dist}(z, B_y) < 2\tau_2\big\}. \tag{29.23}$$

If $C_1$ and $C_2$ are large enough, $Z_+$ contains $Z^*$, (by (19.13), (29.16), and (29.17)). If $z \in E \setminus Z_+$, (29.14) and the definition of the intermediate mappings say that $\varphi_s(z) = z$ for all $s$, then $\chi(\varphi_s(z)) = \chi(z) = 0$, and by (17.19) $\Pi(\varphi_s(z), \chi(\varphi_s(z)) = \varphi_s(z) = z$ for all $s$; thus (29.19) and (29.22) yield

$$g_t(z) = z \text{ when } z \in E \setminus Z_+ \text{ and } 0 \leq t \leq 2. \tag{29.24}$$

Next suppose that $z \in Z_+ \setminus Z^*$. Since $\varphi_s(z) = z$ for $0 \leq s \leq 1$ (again by (29.14) and the definition of the intermediate mappings), we get that $g_t(z) = \Pi(z, s)$ for some $s \in [0, \chi(z)]$. Since $\chi(z) \leq 2\tau_1 + 2\tau_2$, (17.19) yields

$$\mathrm{dist}(g_t(z), Z_+) \leq |g_t(z) - z| \leq C(\tau_1 + \tau_2) \leq C(r_x + \varepsilon_0 r_y). \tag{29.25}$$

Finally, assume that $z \in Z^*$; then all the $\varphi_s(z)$ lie in $Z^*$ (by (29.14)), and in this case

$$\mathrm{dist}(g_t(z), Z^*) \leq |g_t(z) - \varphi_s(z)| \leq C(\tau_1 + \tau_2) \leq C(r_x + \varepsilon_0 r_y) \tag{29.26}$$

for some $s$, by the proof of (29.25).

Thus, if $r_y$ and then $r_x$ are chosen small enough, $z$ and the $g_t(z)$ lie in a compact subset of $B(x_0, r_0)$ when $z \in Z_+$; (2.4) follows, and also (1.5) and (1.6). Here we can take for $B$ a compact ball that is almost as wide as $B(x_0, r_0)$, and we do not care if its radius is quite large (provided that it stays smaller than $r_0$), because there is no price to pay in (2.5) when $B$ is large, since $E$ is minimal. The constraints (1.4) and (1.8) (continuity and Lipschitzness) hold by construction, so we are left with (1.7) to check.

Let $z \in E \cap L_j$ be given. We may assume that $z \in Z_+$, because otherwise $g_t(z) = z \in L_j$ for all $t$. If $z \in Z_+ \setminus Z^*$, (29.14) says that $\varphi_s(z) = z$ for all $s$, and then (29.19) or (22.22) says that every $g_t(z)$ is of the form $\Pi(z, s)$, which by (17.20) lies in any face of $L_j$ that contains $z$. We are left with the case when $z \in Z^*$.

Even in this case, (29.19) and (17.20) say that $g_t(z) = \Pi(z, t\chi(z)) \in L_j$ for $0 \leq t \leq 1$, so it is enough to show that

$$g_t(z) \in L_j \text{ when } z \in L_j \cap Z^* \text{ and } t > 1. \tag{29.27}$$

Let $t > 1$ be given, and set $w = \varphi_{t-1}(z)$; thus

$$g_t(z) = \Pi(w, \chi(w)) \tag{29.28}$$

by (29.22). As we shall see, the mapping $\Pi$ tends to send to $L_j$ points that lie sufficiently close to $L_j$, so we want to show that $w$ lies close to $L_j$. Let us first check that

$$\mathrm{dist}(\xi, L_j) \leq C(1+|y|/||x|)r_x \quad \text{for } \xi \in T_0. \tag{29.29}$$

Set $z' = z|y|/|z|$; first observe that $z' \in L_j$ because $z \in L_j$ and by the cone property (28.2). Also, $|z'-y| \leq Cr_y + C(1+|y|/||x|)r_x$, because $z \in Z^* \subset Z_+$, and by (29.15), (29.13), (29.12) and, for $Z_1^*$ and $Z_2^*$, (29.16) and (29.17). If $r_x$ and $r_y$ are chosen small enough, this forces $y \in L_j$. That is, we choose $r_x$ and $r_y$ so small that the ball centered at $y$ and with radius $Cr_y + C(1+|y|/||x|)r_x$ does not meet any $L_j$ that does not already contain $y$. Then $[x,y] \subset L_j$ too, by the cone property (28.2), and now (29.29) follows from (29.12). Also recall from (29.8) that

$$\mathrm{dist}(\xi, L_j) \leq C\varepsilon_0 r_y \quad \text{for } \xi \in P \cap B_y. \tag{29.30}$$

Let us return to $w = \varphi_{t-1}(z)$ and use this to evaluate its distance to $L_j$. We will unfortunately need to distinguish between cases. First assume that

$$\mathrm{dist}(w, T_1) \leq C(1+|y|/||x|)r_x, \tag{29.31}$$

where $T_1 = \widehat{T} \cap V(x_1, y_0)$ is defined like $T_0$ in (20.11), but with a point $y_0$ that lies on $L(y)$, but at distance $r_y/5$ from $y$, in the direction opposite to $x$. Thus $T_1$ is a little larger than $T_0$ (in the direction of $y$), but not much. By (29.29), $\mathrm{dist}(w, L_j) \leq C(1+|y|/||x|)r_x$ too. If $\tau_1$ is large enough (compared to $(1+|y|/||x|)r_x$), this implies that $\chi(w) \geq \chi_1(w) \geq \tau_1$ (by (29.20)), and that $\tau_1$ is much larger than $\mathrm{dist}(w, L_j)$. Then (17.20) (applied to any face of $L_j$ that lies near $w$) implies that $g_t(z) = \pi(w, \chi(w)) \in L_j$, by (29.28), and as needed. Similarly, if

$$\mathrm{dist}(w, P \cap B_y) \leq C\varepsilon_0 r_y, \tag{29.32}$$

(29.30) says that a similar estimate holds for $\mathrm{dist}(w, L_j)$; then, if $\tau_2$ is large enough compared to $\varepsilon_0 r_y$, and by (29.21), $\chi(z) \geq \chi_2(z) \geq \tau_2$ and (17.20) implies that $g_t(z) = \pi(w, \chi(w)) \in L_j$.

Finally, if $w = z$, we know that $g_t(z) = \pi(z, \chi(z)) \in L_j$, because $z \in L_j$ and by (17.20) again.

We now want to check that we always fall in one of these three cases. We start with the case when $t$ comes from the first part of the construction of $f$ in [D5], when we go from the identity to the first map $f_1$ that is defined on pages 107-110 of [D5]. Let us say that we define these intermediate functions $\varphi_t$ by linear interpolation, i.e., set $\varphi_t(z) = 2tf_1(z) + (1-2t)z$, for $0 \leq t \leq 1/2$. Also recall that $f_1$ was obtained by moving points in vertical hyperplanes (i.e., in directions perpendicular to the line $L(y)$ through $x$ and $y$), and inside the tube $T_1$. That is, $f_1(z) = z$ for $z \in E \setminus T_0$ (by (6.25) in [D5]), and $f_1(T_0) \subset T_0$ by (6.26) (also compare our definition of $y_0$ with (6.22)). The same thing holds for $\varphi_t(z)$, $0 \leq t \leq 1/2$, and we are happy because $w = z$ or (29.31) holds.

At the end of this first stage, all the points $z \in E$ are sent to $E_1 = f_1(E)$, and we now apply to them our second mapping $f_2$, defined on pages 111-114 of [D5]. This map only moves points of $B_y$ (see (6.37)), and moves them like a radial projection, centered on a $(n-d-1)$-dimensional sphere $S_y$, onto a part of $P \cap \partial B_y$ (recall that $Q$ on pages 106 and 111 of [D5] is $(n-d)$-dimensional). We do not care about the details here, we just need to know that if $\xi \in E_1 \subset \overline{B}_y$ (the only place where we may move something) then by (6.30),

$$\mathrm{dist}(\xi, P) \leq 2\varepsilon_0|\xi - y| \leq 2\varepsilon_0 r_y. \tag{29.33}$$

The function $f_2$ maps $\overline{B}_y$ to itself (see above (6.38)) and maps $E_1 \cap \overline{B}_y$ to a $2\varepsilon_0 r_y$-neighborhood of $P$, by (6.41) and the definition (6.39). Again we interpolate linearly, i.e. set $\varphi_t(z) = (4t-2)f_2 \circ f_1(z) + (3-4t)f_1(z)$ for $1/2 \leq t \leq 3/4$, and are happy because (29.32) holds as soon as $w \neq f_1(z)$. (In the other case, we already new that $w = f_1(z)$ satisfies (29.31)).

We now consider the case when $\varphi_t$ comes between $f_2 \circ f_1$ and $f_3 \circ f_2 \circ f_1$, where $f_3$ is described on pages 114-116 of [D5]. This is the place where we said below (29.18) that we do not interpolate linearly, but only after a conjugation with a biLipschitz mapping. There is only one part of $E_2 = f_2 \circ f_1(E)$ where $f_3$ moves points, which is the floor $F \subset P$. See our discussion below (29.32), or directly (6.47), (6.48), and (6.46) in [D5]. But we choose the $\varphi_t$ so that they move points of $F$ along $F$, which means that (29.31) or (29.32) holds. Thus we are happy, up to the stage of $f_3 \circ f_2 \circ f_1$. As was explained below (6.52) in [D5], in codimension 1 we could stop here, but in higher codimensions we have to complete our construction with Federer-Fleming projections that act near $Z$. For these we can interpolate linearly between each elementary projection and the (composition with the) next one. By construction we never leave the sets $Z_1^* \cup Z_2^*$ of (29.15) (see (6.58), the proof of (6.78) (the intermediate projection also stay close), (6.81), and the proof of (6.96)); then the desired conclusion follows from (29.16) and (29.17), which show that (29.31) or (29.32) holds whenever we move a point.

This completes the proof of the boundary constraint (1.7) for our family $\{g_t\}$. Thus we are allowed to use (2.5), which says that

$$\mathcal{H}^d(W) \leq \mathcal{H}^d(g_2(W)) \tag{29.34}$$

(recall that we work with minimizers here), with $W = \big\{z \in E\,;\, g_2(z) \neq z\big\}$. Recall that (29.22) yields $g_2(z) = \Pi(f(z), \chi(f(z)))$, because $\varphi_1 = f$. Also, $\Pi(w, \chi(w))$ is a $C$-Lipschitz function of $w$, by (29.20), (29.21), and (17.21), and with a constant $C$ that may now depend on the bilipschitz constant $\Lambda$ when we work under the Lipschitz assumption, but not on $\varepsilon_0$ or $r_x$, for instance. Because of this, we can easily take care of $g_2(E \cap Z^*)$, since (6.95) and (6.97) in [D5] yield

$$\mathcal{H}^d(g_2(E \cap Z^*)) \leq C\mathcal{H}^d(f(E \cap Z^*)) \leq C\mathcal{H}^d(E_5 \cap Z^*) \leq C\varepsilon_0 r_y^d + C_{x,y} r_x^{d-1}; \tag{29.35}$$

note that the $r_y^2$ in (6.97) is a misprint, but that we really get $r_x^{d-1}$ because the points of $T_0$ stay $C_{x,y}$-close to a line.

Because of (29.24), $W$ is contained in $Z_+$. So we still need to worry about the set $W_0 = W \setminus Z^* \subset Z_+ \setminus Z^*$. On this set $f(z) = z$ by (29.14), so $g_2(z) = \Pi(z, \chi(z)) \neq z$, where the last part holds by definition of $W$. Now we follow the construction of $\Pi(z,\chi(z))$, and notice that by (17.24)

$$\Pi(z,\chi(z)) = \Pi_{0,s_0} \circ \Pi_{1,s_1} \cdots \circ \Pi_{n-1,s_{n-1}}(z) \tag{29.36}$$

where the $\Pi_{k,s_k}$ come from Lemma 17.1 and we set

$$s_m = (6C)^{-m}\chi(z) \ \text{ for } 0 \leq m \leq n-1, \tag{29.37}$$

as in (17.23). Also denote by $x_k$, $n+1 \geq k \geq 0$, the successive images of $z = x_{n+1}$, defined as (above (17.26) and) in (17.26) by $x_k = \Pi_{k,s_k}(x_{k+1})$. By the property (17.2) of $\Pi_{k,s_k}$, we see that $x_k = x_{k+1}$ unless $0 < \mathrm{dist}(x_{k+1}, \mathcal{S}_k) < 2s_k$. That is, unless $x_k$ lies very close to some face of the grid, without actually lying on that face. By an easy induction, we see that $\Pi(z,\chi(z)) = z$ unless

$$0 < \mathrm{dist}(z, F) \leq 2\chi(z) \ \text{ for some face } F \text{ of our grid.} \tag{29.38}$$

First assume that $z \in 2B_y$. By choosing $r_y$ small enough, we can ensure that $2B_y$ only meets the faces $F$ that already contain $y$. But (29.38) means that $z \in E \cap 2B_y \setminus F$. Then of course $z \in E \cap 2B_y \setminus F_0$, where $F_0$ is the smallest face that contains $y$, and by (29.7)

$$\mathcal{H}^d(W_0 \cap 2B_y) \leq \mathcal{H}^d(E \cap 2B_y \setminus F_0) \leq \varepsilon_1^d r_y^d. \tag{29.39}$$

By (29.23), we are left with

$$W_0 \setminus 2B_y \subset E \cap Z_+ \setminus 2B_y \subset \big\{z \in E\,;\, \mathrm{dist}(z, T_0) < 2\tau_1\big\}. \tag{29.40}$$

Because $T_0$ stays so close to $[x,y]$ (see (29.12)), we can cover $W_0 \setminus 2B_y$ by less than $C_{x,y} r_x^{-1}$ balls centered on $E$ and with radius $Cr_x$. Then by local Ahlfors regularity (Propositions 4.1 and 4.74), $\mathcal{H}^d(W_0 \cap 2B_y) \leq C_{x,y} r_x^{d-1}$. Again $g_2(z) = \Pi(z,\chi(z))$ is a $C$-Lipschitz function of $z \in W_0$, so

$$\mathcal{H}^d(g_2(W_0)) \leq C\mathcal{H}^d(W_0) \leq C\varepsilon_1^d r_y^d + C_{x,y} r_x^{d-1}. \tag{29.41}$$

We add this to (29.35) and get that

$$\mathcal{H}^d(g_2(W)) \leq C\varepsilon_0 r_y^d + C\varepsilon_1^d r_y^d + C_{x,y} r_x^{d-1}. \tag{29.42}$$

On the other hand, we claim that $W$ is large because it contains most of $E \cap B_y$. More precisely, if $z \in E \cap B_y \setminus W$, then $g_2(z) = z$ and hence $z \in g_2(E \cap B_y) \subset g_2(E \cap Z^*)$ (by (29.13) and because $Z \subset Z^*$). Thus

$$\mathcal{H}^d(E \cap B_y \setminus W) \leq \mathcal{H}^d(g_2(E \cap Z^*)) \leq C\varepsilon_0 r_y^d + C_{x,y} r_x^{d-1} \tag{29.43}$$

by (29.35). But $\mathcal{H}^d(E \cap B_y) \geq C^{-1} r_y^d$ because $E$ is locally Ahlfors regular (by Propositions 4.1 and 4.74), hence

$$\mathcal{H}^d(W) \geq \mathcal{H}^d(E \cap B_y) - \mathcal{H}^d(E \cap B_y \setminus W) \geq C^{-1} r_y^d - C\varepsilon_0 r_y^d - C_{x,y} r_x^{d-1}. \tag{29.44}$$

If $\varepsilon_0$ and $\varepsilon_1$ are small enough and $r_x$ is small enough (depending also on the position of $x$ and $y$ through $|y|/|x|$), (29.42) and (29.44) contradict (29.34), and this concludes our proof of Theorem 29.1. $\square$

We complete this section with a simple consequence of Theorem 29.1 and the results of Section 10 on limits, revised in Section 24 so that they apply to blow-up limits.

Let us list the assumptions for the next theorem. Most of them are the same as for our Theorem 24.13 on blow-up limits, which we intend to combine with Theorem 29.1. We are given a coral almost minimal set $E$ in the open set $U$, an origin $x_0 \in E$, and a sequence $\{r_k\}$, with

$$\lim_{k\to+\infty} r_k = 0. \tag{29.45}$$

We assume that

$$U \text{ and the } L_j \text{ satisfy the Lipschitz assumption,} \tag{29.46}$$

as in Definition 2.27, and that

$$\text{the configuration of } L_j \text{ is flat at } x_0, \text{ along the sequence } \{r_k\}. \tag{29.47}$$

See Definition 24.8, but also recall that we have a simpler condition, the flatness of the faces of the $L_j$ along the sequence, which is introduced in Definition 24.29 and implies it; see Proposition 24.35. Recall that (29.47) comes with a collection of limit sets $L_j^0$, the natural blow-up limits of the $L_j$, that are defined by (24.7). We assume that

$$\text{the } L_j \text{ satisfy (10.7) or (19.36),} \tag{29.48}$$

the additional assumptions that we used for our theorems on limits, and that

$$\begin{gathered} E \text{ is a coral almost minimal set in } U, \text{ with} \\ \text{sliding conditions coming from the } L_j, \end{gathered} \tag{29.49}$$

and with a gauge function $h$ such that

$$\lim_{r\to 0} h(r) = 0. \tag{29.50}$$

For this, we accept the three types ($A_+$, $A$, or $A'$) of almost minimality; see Definition 20.2. Also see Definition 3.1 for corality.

We also give ourselves a closed set $E_\infty \subset U$, and we assume that

$$E_\infty = \lim_{k\to+\infty} r_k^{-1}(E - x_0) \ \text{ locally in } \mathbb{R}^n \tag{29.51}$$

(see near (10.5) for the definition). Finally, we suppose that the following limit exists:

$$\theta(x_0) = \lim_{\rho\to 0} \rho^{-d}\mathcal{H}^d(E \cap B(x_0,\rho)). \tag{29.52}$$

Notice that the existence of $\theta(x_0)$ follows from Theorem 28.7 when $h$ satisfies the Dini condition (28.5) and the $L_j$ are cones. One may also use Remark 28.11 to prove this when the $L_j$ are almost cones (but with conditions stronger than (29.47)).

**Corollary 29.53.** *Let the coral almost minimal set $E$ in $U$, the point $x_0 \in E$, the sequence $\{r_k\}$, and the set $E_\infty$ satisfy the conditions above. Then $E_\infty$ is a coral minimal cone, with the sliding boundary conditions defined by the sets $L_j^0$, $0 \leq j \leq j_{max}$, defined by (24.7), and with the constant density*

$$r^{-d}\mathcal{H}^d(E_\infty \cap B(x_0, r)) = \theta(x_0). \tag{29.54}$$

The first assumptions allow us to apply Theorem 24.13, which says that $E_\infty$ is a coral minimal set, with the sliding boundary conditions defined by the sets $L_j^0$. Recall that this is defined by (24.16) or (24.17), as the reader prefers.

We still need to check that $E_\infty$ is a cone and that (29.54) holds, and naturally we start with (29.54). For this shall need to say more about how Theorem 24.13 is proved. We consider the same sets $E_k$ (compare with (24.3)), and the main point of the proof is to show that Theorem 23.8 can be applied. A long first part consists in showing that for any fixed large radius $R \geq 1$, the $L_j^0$ satisfy the Lipschitz assumption on some appropriate domain $U_R$ (defined by (24.18)-(24.20)). Once this is done, we apply Theorem 23.8 to the domains $U_{R,k} = \xi_k(U_R)$ and the sets $E_k \cap U_{R,k}$. In turn Theorem 23.8 consists in applying Theorem 10.8 to a single domain (with single boundary sets), but the different sets $\widetilde{E}_k = \xi_k^{-1}(E_k)$, where the bilipschitz mappings $\xi_k$ come from the condition (29.47), and satisfy the asymptotic conditions (23.3) and (23.4). Eventually, one proves that these sets satisfy the assumptions of Theorem 10.8 (see (23.16)-(23.20)). Their limit is still $E_\infty$ (by (23.3) and mostly (23.4)), and Theorem 10.97, which has the same assumptions as Theorem 10.8, shows that for $0 < \rho < \rho_1 < R/2$,

$$\begin{aligned}\mathcal{H}^d(E_\infty \cap B(x_0, \rho)) &\leq \liminf_{k\to+\infty} \mathcal{H}^d(\widetilde{E}_k \cap B(x_0, \rho)) \leq \liminf_{k\to+\infty} \mathcal{H}^d(E_k \cap B(x_0, \rho_1)) \\ &= \liminf_{k\to+\infty} r_k^{-d}\mathcal{H}^d(E \cap B(x_0, r_k\rho_1)) = \rho_1^d\theta(x_0)\end{aligned} \tag{29.55}$$

where we used the asymptotic bilipschitz property (23.3) for the change of variable to control the measures, and then the scale invariance and (29.56). Since we may take $\rho_1$ as close to $\rho$ as we want, this gives the upper bound in (29.54).

Similarly, we can apply Lemma 22.2 (whose assumptions are the same as for Theorem 10.8) and with any choice of $M > 1$ and $h > 0$; this yields, for $0 < \rho_1 < \rho_2 < \rho < R/2$,

$$\begin{aligned}(1+Ch)M\mathcal{H}^d(E_\infty \cap B(x_0, \rho)) &\geq (1+Ch)M\mathcal{H}^d(E_\infty \cap \overline{B}(x_0, \rho_2)) \\ &\geq \limsup_{k\to+\infty} \mathcal{H}^d(\widetilde{E}_k \cap \overline{B}(x_0, \rho_2)) \\ &\geq \limsup_{k\to+\infty} \mathcal{H}^d(E_k \cap B(x_0, \rho_1)) \\ &= \limsup_{k\to+\infty} r_k^{-d}\mathcal{H}^d(E \cap B(x_0, r_k\rho_1)) = \rho_1^d\theta(x_0)\end{aligned} \tag{29.56}$$

by (22.4) and the same sort of computation as above. This yields the rest of (29.54).

Once we know (29.54), we can apply Theorem 29.1, we get that $E_\infty$ is a cone, and this completes the proof of Corollary 29.53. □

## 30. Nearly constant density and approximation by minimal cones

In this section we use Theorem 29.1 and the results of Section 10 on limits to give sufficient conditions, in terms of density, for an almost minimal set to be very close to a minimal cone.

For the main statement, we give ourselves a fixed ball $B_0 = B(x_0, r_0)$, boundary pieces $L_j^0$, $0 \leq j \leq j_{max}$, and we suppose that the Lipschitz assumption of Definition 2.7 holds for $B_0$ and the $L_j^0$ and (in the non rigid case), that the $L_j^0$ satisfy the technical assumption (10.7), or the weaker (19.36). We also suppose that $0 \in B_0$ and that, for $0 \leq j \leq j_{max}$,

$$L_j^0 \text{ coincides with a cone in } B_0. \tag{30.1}$$

We see $B_0$ and the collection of $L_j^0$ as a model for domains $U$, endowed with boundary pieces $L_j$, and such that there is a bilipschitz mapping $\xi$ such that

$$\xi(B_0) = U \quad \text{and} \quad L_j = \xi(L_j^0) \text{ for } 0 \leq j \leq j_{max}. \tag{30.2}$$

We want to say that when the bilipschitz constant of $\xi$ is small, $E$ is a coral quasiminimal set in $U$ with constants $M$ close enough to 1 and $h$ small enough, and the density ratios of $E$ in two different balls centered at $\xi(0)$ are close enough, then $E$ looks a lot like a minimal cone in the corresponding annulus. The statement will be a little complicated, but later on, in Proposition 30.19, we shall consider the simpler case when the annulus is just a ball centered at the origin.

**Proposition 30.3.** *Let $B_0$ and the $L_j^0$, $0 \leq j \leq j_{max}$, be as above. In particular, assume that we have (30.1), the Lipschitz assumption, and (10.7) or (19.36). For each $\tau > 0$, we can find $\varepsilon > 0$ such that the following holds. Let $U$, the $L_j$, and $\xi$ satisfy (30.2), and assume that*

$$\xi \text{ is } (1+\varepsilon)\text{-bilipschitz}. \tag{30.4}$$

*Let $E$ be a coral quasiminimal set in $U$, with*

$$E \in GSAQ(U, 1+\varepsilon, 2r_0, \varepsilon), \tag{30.5}$$

*set $x_0 = \xi(0) \in U$, and assume that for some choice of radii $0 < r_1 < r_2 \leq r_0$,*

$$r_2^{-d}\mathcal{H}^d(E \cap B(x_0, r_2)) \leq r_1^{-d}\mathcal{H}^d(E \cap B(x_0, r_1)) + \varepsilon. \tag{30.6}$$

*Then there is a minimal cone $T$ centered at $x_0$ such that*

$$\mathrm{dist}(y, T) \leq \tau r_0 \text{ for } y \in E \cap B(x_0, r_2 - \tau) \setminus B(x_0, r_1 + \tau) \tag{30.7}$$

$$\text{(30.8)} \qquad \mathrm{dist}(z,E) \leq \tau r_0 \text{ for } z \in T \cap B(x_0, r_2 - \tau) \setminus B(x_0, r_1 + \tau),$$

$$\text{(30.9)} \qquad \begin{gathered} \left|\mathcal{H}^d(T \cap B(y,t)) - \mathcal{H}^d(E \cap B(y,t))\right| \leq \tau r_0^d \\ \text{for } y \text{ and } t \text{ such that } B(y,t) \subset B(x_0, r_2 - \tau) \setminus B(x_0, r_1 + \tau), \end{gathered}$$

*and*

$$\text{(30.10)} \qquad \left|H^d(E \cap B(x_0,r)) - H^d(T \cap B(x_0,r))\right| \leq \tau r_0^d \text{ for } r_1 + \tau \leq r \leq r_2 - \tau.$$

Let us comment this statement before we prove it.

Proposition 30.3 is a generalization of Proposition 7.1 in [D5].

In (30.6), it could happen that $B(x_0,r_0))$ goes slightly out of $U$, so we could have written $\mathcal{H}^d(E \cap U \cap B(x_0,r_2))$ instead of $\mathcal{H}^d(E \cap B(x_0,r_2))$ to be more explicit (but the result is the same since $E \subset U$).

Of course (30.6) is only meaningful if $r_1$ is not too close to $r_2$, but otherwise the conclusion is empty anyway.

We can always apply the result to domains $U' \supset U$ and quasiminimal sets $E'$ in $U'$, since it is easy to check that $E' \cap U \in GSAQ(U, 1+\varepsilon, 2r_0, \varepsilon)$ as soon as $E' \in GSAQ(U', 1+\varepsilon, 2r_0, \varepsilon)$.

We are lucky because we don't need the Dini condition in (28.5), or even the fact that $E$ is almost minimal. Thus we may not be in the situation where we know for sure that the density $r^{-d}\mathcal{H}^d(E \cap B(x,r))$ is almost nondecreasing (as in Theorems 28.7 and 28.15). On the other hand, we only state a result at a fixed scale, not an asymptotic result, and we will not compute $\varepsilon$ in terms of $\tau$, but just apply a compactness argument.

The author is not too happy about the statement of Proposition 30.3, because we let $\varepsilon$ depend on the $L_j$. This is because the proof below, just like the limiting result of Section 23, for instance, uses the very fine bilipschitz convergence on domains, for which it seems too hard to extract converging subsequences. Both for Section 23 and here, there are probably ways to improve the statement, but the author is not really sure of what would be needed, and hopes that in practice Proposition 30.3 will often be enough. Anyway, we still put the scale invariant factors $r_0$ and $r_0^d$ in (30.7)-(30.10), even though our statement allows $\varepsilon$ to depend on $r_0$ through $B_0$ and the $L_j$.

We shall prove the proposition by compactness. It will be easier to take limits of sets like $\widetilde{E} = \xi^{-1}(E)$, because they live in the fixed domain $B_0$. A simple computation, using (30.4) and (30.5), shows that if $E$ is as in the statement,

$$\text{(30.11)} \qquad \widetilde{E} \in GSAQ(B_0, 1 + C\varepsilon, r_0, C\varepsilon),$$

where the associated boundary pieces on $B_0$ are still the $L_j^0$.

Now let us fix $B_0$, the $L_j^0$, and $\tau$, and assume that we cannot find $\varepsilon > 0$ as in the statement. Let $\xi_k$, $U_k = \xi_k(B_0)$, the sets $L_j^k$, $E_k$, and the radii $r_{1,k}$ and $r_{2,k}$ provide a counterexample, associated with $\varepsilon_k = 2^{-k}$. By translation invariance, we may assume that $\xi_k(0) = 0$. Also set $\widetilde{E}_k = \xi_k^{-1}(E_k)$.

Recall that $\xi_k$ is defined on $B_0$ and $(1+2^{-k})$-bilipschitz (by (30.4)). Since $\xi_k(0) = 0$ for all $k$, we see that modulo extracting a subsequence, we may assume that the $\xi_k$ converge, uniformly on $B_0$, to a mapping $\eta$ which in addition is 1-bilipschitz. That is, $\eta$ is the restriction of a linear isometry of $\mathbb{R}^n$ that fixes 0. Let us replace $\xi_k$ with $\eta^{-1} \circ \xi_k$, $E_k$ with $\eta^{-1}(E_k)$, and so on; we get a new counterexample for which $\eta$ is the identity. So we may assume that the $\xi_k$ converge, uniformly on $B_0$, to the identity. Because of our bilipschitz property (30.4), the $\xi_k^{-1}$ also converge, uniformly on compact subsets of $B_0$, to the identity.

Modulo extracting a new subsequence, we may assume that the sets $\widetilde{E}_k$ converge in $B_0$ to a limit $F$; then the sets $E_k = \xi_k(\widetilde{E}_k)$ converge, locally in $B_0$, to the same limit $F$. By (30.11), $\widetilde{E}_k \in GSAQ(B_0, 1+C2^{-k}, r_0, C2^{-k})$. Then for each small $\delta > 0$, we can apply Theorem 10.8 to the (end of the) sequence $\{\widetilde{E}_k\}$, in the domain $B_0$. We get that $F$ is a coral quasiminimal set (associated to the boundary pieces $L_j^0$), with $F \in GSAQ(B_0, 1+\delta, r_0, \delta)$. Since this holds for every $\delta > 0$, $F$ is a minimal set in $B_0$.

Now we want to take care of the measures. Let $B$ be an open ball, with $B \subset B(0, r_0 - \tau/3)$ (we may assume that $\tau << r_0$, so we don't lose much). By the lower semicontinuity property (10.98),

$$\mathcal{H}^d(F \cap B) \leq \liminf_{k \to +\infty} \mathcal{H}^d(\widetilde{E}_k \cap B). \tag{30.12}$$

By Lemma 22.3 (applied with $h$ and $M - 1$ as small as we want), we also get that

$$\mathcal{H}^d(F \cap \overline{B}) \geq \limsup_{k \to +\infty} \mathcal{H}^d(\widetilde{E}_k \cap \overline{B}). \tag{30.13}$$

We want to compare this with our density assumption (30.6). Let us replace our sequence with a subsequence for which $r_{1,k}$ tends to a limit $r_1$ and $r_{2,k}$ tends to a limit $r_2$. Notice that $r_{2,k} \geq r_{1,k} + 2\tau$ for all $k$ (otherwise the conclusion (30.7)-(30.10) would be trivially true, which is impossible for a counterexample), so $r_2 \geq r_1 + 2\tau$. Then take $\delta > 0$ very small, and notice that for $k$ large,

$$\widetilde{E}_k \cap B(0, r_2 - \delta) = \xi_k^{-1}(E_k) \cap B(0, r_2 - \delta)) \subset \xi_k^{-1}(E_k \cap B(0, r_2)) \tag{30.14}$$

(because (30.4) says that $\xi_k$ is $(1+2^{-k})$-bilipschitz). Then apply (30.12) to $B = B(0, r_2 - \delta)$, and get that

$$\begin{aligned}
\mathcal{H}^d(F \cap B(0, r_2 - \delta)) &\leq \liminf_{k \to +\infty} \mathcal{H}^d(\widetilde{E}_k \cap B(0, r_2 - \delta)) \\
&\leq \liminf_{k \to +\infty} \mathcal{H}^d(\xi_k^{-1}(E_k \cap B(0, r_2))) \\
&\leq \liminf_{k \to +\infty} (1 + 2^{-k})^d \mathcal{H}^d(E_k \cap B(0, r_2)) \\
&= \liminf_{k \to +\infty} \mathcal{H}^d(E_k \cap B(0, r_2)) \\
&\leq (r_1/r_2)^{-d} \liminf_{k \to +\infty} \mathcal{H}^d(E_k \cap B(0, r_1))
\end{aligned} \tag{30.15}$$

by (30.4) and because (30.6) holds for $E_k$, with $\varepsilon = 2^{-k}$. Similarly,

$$\widetilde{E}_k \cap \overline{B}(0, r_1 + \delta) = \xi_k^{-1}(E_k) \cap \overline{B}(0, r_1 + \delta)) \supset \xi_k^{-1}(E_k \cap B(0, r_1)) \tag{30.16}$$

for $k$ large, so when we apply (30.13) to $B = B(0, r_1 + \delta)$, we get that

$$\begin{aligned}
\mathcal{H}^d(F \cap \overline{B}(0, r_1 + \delta)) &\geq \limsup_{k\to+\infty} \mathcal{H}^d(\widetilde{E}_k \cap \overline{B}(0, r_1 + \delta)) \\
&\geq \limsup_{k\to+\infty} \mathcal{H}^d(\xi_k^{-1}(E_k \cap B(0, r_1))) \\
&\geq \limsup_{k\to+\infty} (1 + 2^{-k})^{-d}\, \mathcal{H}^d(E_k \cap B(0, r_1)) \\
&= \limsup_{k\to+\infty} \mathcal{H}^d(E_k \cap B(0, r_1)).
\end{aligned} \tag{30.17}$$

We compare (30.15) and (30.17), let $\delta$ tend to 0, and get that

$$\mathcal{H}^d(F \cap B(0, r_2)) \leq (r_1/r_2)^{-d}\mathcal{H}^d(F \cap \overline{B}(0, r_1)). \tag{30.18}$$

By Theorem 28.4 (and because $F$ is minimal in the full $B_0$) $\theta(r) = r^{-d}\mathcal{H}^d(F \cap B(0, r))$ is nondecreasing on $(0, r_0)$. But (30.18) says that $\theta(r_2) \leq \lim_{r\to r_1^+} \theta(r)$, so $\theta$ is constant on $(r_1, r_0)$. We apply Theorem 29.1 and get that $F$ coincides, in the annulus $B(0, r_2)\backslash\overline{B}(0, r_1)$, with a coral minimal cone $T$.

We shall now prove that the approximation properties (30.7)-(30.10) are satisfied for $k$ large (and the cone $T$ that we just found), and this will give the desired contradiction with the definition of $E_k$.

First notice that (30.7) and (30.8) hold, because we observed earlier that $F$ is also the limit of the $E_k$ in compact subsets of $B_0$. For (30.9) and (30.10), we deduce them from (30.12) and (30.13). The details of the verification were done in [D5], pages 128-129, so we refer to that and merely mention the two minor difficulties that may worry the reader. For (30.10), it is easy to deduce it, for a single radius, from (30.12), (30.13), and the fact that for the cone $T$, $\mathcal{H}^d(T \cap \partial B(0, r)) = 0$. But it is enough to check (30.10) for a finite collection of radii, because $\mathcal{H}^d(T \cap B(0, r))$ is a continuous function of $r$, while each $\mathcal{H}^d(E_k \cap B(0, r)) = 0$ is nondecreasing. For (30.9), we can proceed similarly, but we also need the less obvious fact that $\mathcal{H}^d(T \cap \partial B(y, t)) = 0$ for every ball $B(y, t)$. This is (7.14) in [D5], and the verification, done as Lemma 7.34 [D5], only uses the fact that $\mathcal{H}^d(T \cap B(0, 1)) < +\infty$ and a little bit of geometric measure theory, but not the fact that $T$ is minimal (which is good, because here minimal is merely meant with additional boundary constraints, so our cone $T$ is probably not minimal as in [D5]). This concludes our proof of Proposition 30.3 by contradiction and compactness. $\square$

Let us now state the analogue of Proposition 30.3 for the density in a ball (i.e., with $r_1 = 0$).

**Proposition 30.19.** *Let $0 < r_0$ be given, and let $B_0$ and the $L_j^0$, $0 \leq j \leq j_{max}$, be as in the statement of Proposition 30.3. In particular, assume that we have (30.1), the*

*Lipschitz assumption, and (10.7) or (19.36). For each $\tau > 0$, we can find $\varepsilon > 0$ such that the following holds. Let $\xi$, $U$, and the $L_j$ satisfy (30.2), and assume that*

$$\xi \text{ is } (1+\varepsilon)\text{-bilipschitz.} \tag{30.20}$$

*Let $E \subset U$ be a coral quasiminimal set, with*

$$E \in GSAQ(U, 1+\varepsilon, 2r_0, \varepsilon), \tag{30.21}$$

*set $x_0 = \xi(0) \in U$, and assume that for some $r_2 \in (0, r_0]$,*

$$r_2^{-d}\mathcal{H}^d(E \cap B(x_0, r_2)) \leq \varepsilon + \inf_{0<r<10^{-3}r_0} r^{-d}\mathcal{H}^d(E \cap B(x_0, r)). \tag{30.22}$$

*Then there is a minimal cone $T$ centered at $x_0$ such that*

$$\mathrm{dist}(y, T) \leq \tau r_0 \text{ for } y \in E \cap B(x_0, r_2 - \tau) \tag{30.23}$$

$$\mathrm{dist}(z, E) \leq \tau r_0 \text{ for } z \in T \cap B(x_0, r_2 - \tau), \tag{30.24}$$

$$\begin{aligned} &\left|\mathcal{H}^d(T \cap B(y,t)) - \mathcal{H}^d(E \cap B(y,t))\right| \leq \tau r_0^d \\ &\qquad\qquad \text{for } y \text{ and } t \text{ such that } B(y,t) \subset B(x_0, r_2 - \tau), \end{aligned} \tag{30.25}$$

*and in particular*

$$\left|H^d(E \cap B(x_0, r)) - H^d(T \cap B(x_0, r))\right| \leq \tau r_0^d \text{ for } 0 < r \leq r_2 - \tau. \tag{30.26}$$

This is now the generalization of Proposition 7.24 in [D5]. We write (30.22) in this strange way because, since we do not assume $E$ to be almost minimal with a small gauge function, we do not know that $\lim_{r\to 0} r^{-d}\mathcal{H}^d(E \cap B(x_0, r))$ exists and gives a good lower bound on $r^{-d}\mathcal{H}^d(E\cap B(x_0, r))$ for $r$ small. Of course, with suitable additional assumptions on the almost minimality of $E$, we could use Theorem 28.15 and replace the infimum in (30.22) with the density $\lim_{r\to 0} r^{-d}\mathcal{H}^d(E \cap B(x_0, r))$.

We repeat the proof of Proposition 30.3 because we don't want to worry about the way $\varepsilon$ depends on $r_1$, and also because in (30.25) we allow balls $B(y, t)$ that contain $x_0$. So we suppose that for $k \geq 0$, we have a counterexample $E_k$ to the statement with $\varepsilon = 2^{-k}$, extract suitable subsequences, and find a minimal set $F$, which is the joint limit in $B_0$ of the sequences $\{E_k\}$ and $\{\widetilde{E}_k\}$.

For each small $r_1 \in (0, 10^{-3}r_0)$, we can repeat the argument near (3.12)-(3.18), and get (3.18). This is why we required an infimum in (30.22). Then $\theta(r) = r^{-d}\mathcal{H}^d(F\cap B(0, r))$ is constant on $(r_1, r_2)$, and $F$ coincides with a minimal cone $T$ on $B(0, r_2) \setminus \overline{B}(0, r_1)$. We let $r_1$ tend to 0 and get that $F \cap B_0 = T \cap B_0$.

The desired contradiction, i.e., the fact that (30.23)-(30.26) actually hold for $k$ large, is then proved as before. In particular, since now $F$ coincides with $T$ near the origin, we are allowed balls $B(y,t)$ that contain it. Proposition 30.19 follows. $\square$

## 31. Where should we go now?

There were two main reasons for the present paper. The first one was to obtain some boundary regularity results for quasiminimal sets and almost minimal sets.

As far as quasiminimal sets are concerned, the author believes that the results in Parts I-III are not so far from being optimal, because of the bilipschitz invariance. That is, Lipschitz graphs are quasiminimal, and uniformly rectifiable sets are not so far the Lipschitz regularity. Of course it would be nice to know that the strange dimensional condition (6.2) can be removed, especially because this would mean that we found another proof of uniform rectifiability than the very complicated stopping time argument coming from [D1]. Also, the uniform rectifiability result of Theorem 6.1 barely contains more information than the fact that $E$ is locally uniformly rectifiable away from the boundary pieces $L_j$, plus the uniform rectifiability of the pieces themselves.

The situation is quite different for the almost minimal sets (typically, minimizers of a functional like $\int_E f(x) d\mathcal{H}^d(x)$, maybe plus some lower order terms, and where $f : \mathbb{R}^n \to [1, M]$ is continuous). For these sets, we expect much more regularity than what we obtained so far. Sufficiently flat sets are a little easier to control, because of Allard's theorem [All], but we should not expect precise general results, because we know that a general description is already hard away from the boundary. Recall that J. Taylor [Ta] gave a very good local description of the 2-dimensional almost minimal sets in $\mathbb{R}^3$. A similar, but already much less precise description is available for 2-dimensional almost minimal sets in $\mathbb{R}^n$ (see [D5,6]), and there are even some first descriptions of 3-dimensional almost minimal sets in $\mathbb{R}^4$ near special (but non flat) points ([Lu1,2]), but we expect a lot of trouble except in very small dimensions. Maybe see [D7] for a rapid description.

Because this is always a good way to start, a first step consists in studying the blow-up limits of our minimal sets at a point of the boundary, and Corollary 29.53 says that in the reasonable situations, these are sliding minimal cones associated to conical boundary pieces. So it seems interesting to study (find a list of) the minimal cones in some simple situations. Even for 2-dimensional minimal cones in $\mathbb{R}^4$, with no boundary piece, the list of minimal cones is not known. It was recently shown [Li2] that the almost orthogonal union of two planes in $\mathbb{R}^4$ is minimal (partially answering a conjecture of F. Morgan [Mo2]), and that the orthogonal product of two one dimensional sets $Y$ in $\mathbb{R}^4$ is minimal too [Li3], but there may be lots of other 2-dimensional minimal cones in $\mathbb{R}^4$ that we did not guess. To the author's best knowledge, the list of 2-dimensional minimal cones in $\mathbb{R}^3$, with a unique boundary $L_1$ which is a line, is not known either, and this would be a very good start for some versions of the most classical Plateau problem in 3-space.

Once we have an almost minimal set $E$, with a known blow-up limit at some point $x$ of the boundary, we can try to give a good description of $E$ near $x$, for instance a nice parameterization by the blow-up limit, like for the J. Taylor theorem [Ta]. This should not be too hard, in a very limited number of situations.

Of course it would be good to have a substitute for the monotonicity of the density $\theta(x,r) = r^{-d}\mathcal{H}^d(E \cap B(x,r))$ when $x \in E$ lies close to a boundary piece, but not on it. The monotonicity fails stupidly when $E$ is a half plane (bounded by the line $L_1$) and $x \in E \setminus L_1$, but one may dream to use more clever functions of $r$, that would probably need to depend on the approximate shape of $E$ and the distance to $L_j$.

A second motivation is that a good local knowledge of the sliding almost minimal sets near the boundary should help attacking some existence problems. A typical one is the version of Plateau's problem that was described in the introduction: take a simple curve $\Gamma$ in $\mathbb{R}^3$ and an initial set $E_0$ bounded by $\Gamma$. You do not have to know what this means, but if you pick a wrong $E_0$, the problem will probably have a trivial solution (like a point). Then try to minimize $\mathcal{H}^2(E)$ (or a variant) among all the competitors for $E$ (as defined in Definition 1.3). In [D3], the author proposed to use sequences of quasiminimal sets, together with the concentration lemma of [DMS] and a construction of adapted polyhedral networks by V. Feuvrier [Fv1], [Fv2] to find existence results for problem of this type. Some existence results were indeed found (see [Fv3], [Li1], [Fa]), but often avoiding complicated problems at the boundary. For the problem above, for instance, it would be good to know that for the limit $E$ of the minimizing sequence that we construct, and which is a sliding minimal set by Theorem 10.8, there is a Lipschitz retraction of a neighborhood of $E$ onto $E$. Also see [D8] for variants of this problem, probably not all easy to solve. As was mentioned in the introduction, since other categories (such as size minimizing currents) also yield sliding minimal sets, boundary regularity results could be useful there too.

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