SHOULD WE SOLVE PLATEAU'S PROBLEM AGAIN?

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Résumé. Après une courte description de plusieurs solutions classiques du problème de Plateau, on parle d'autres modélisations des films de savon, et de problèmes ouverts liés. On insiste un peu plus sur un modèle basé sur des déformations et des conditions glissantes à la frontière.

Abstract. After a short description of various classical solutions of Plateau's problem, we discuss other ways to model soap films, and some of the related questions that are left open. A little more attention is payed to a more specific model, with deformations and sliding boundary conditions.

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1. Introduction

The main goal of this text is to give a partial account of the situation of Plateau's problem, on the existence and regularity of soap films with a given boundary. We intend to convince the reader that there are many reasonable ways to state a Plateau problem, most of which give interesting questions that are still wide open. This is even more true when we want our models to stay close to Plateau's original motivation, which was to describe physical phenomena such as soap films.

Plateau problems led to lots of beautiful results; we shall start the paper with a rapid description of some of the most celebrated solutions of Plateau's problem (Section 2), followed by a description of a few easy examples (Section 3), mostly to explain more visibly some objections and differences between the models.

With these examples in mind, we shall shortly return to the modeling problem, and mention a few additional ways to state a Plateau problem and (in some cases) get solutions with a nice physical flavor (Section 4).

It turns out that the author has a preference for a specific way to state Plateau problems, coming from Almgren's notion of minimal sets, which we accommodate at the boundary with what we shall call sliding boundary conditions.

In Section 5, we shall describe briefly the known local regularity properties of the Almgren minimal sets (i.e., far from the boundary), and why we would like to extend some of these regularity results to sliding minimal sets, all the way to the boundary. We want to do this because little seems to be known on the boundary behavior of soap films and similar objects, but also because we hope that this may help us get existence results.

We try to explain this in Section 6, and at the same time why, even though beautiful compactness and stability results for various classes of objects (think about Almgren minimal sets, but also currents or varifold) yield relatively systematic partial solutions of

Plateau problems, these solutions are not always entirely satisfactory (we shall call these amnesic solutions).

We explain in Section 7 why the regularity results for sliding Almgren minimal sets also apply to solutions of the Reifenberg and size minimization problems described in Section 2.

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2. SOME CELEBRATED SOLUTIONS

In this section we describe some of the most celebrated ways to state a Plateau problem and solve it. To make things easier, we shall mostly think about the simplest situation where we give ourselves a smooth simple loop Γ in $\mathbb{R}^3$, and we look for a surface bounded by Γ, with minimal area. Even that way, a few different definitions of the terms "bounded by" and "area" will be used.

2.a. Parameterizations by disks, Garnier, Douglas, and Radó

Here we think of surfaces as being parameterized, and compute their area as the integral of a Jacobian determinant. For instance, let $\Gamma \subset \mathbb{R}^n$ be a simple curve, which we parameterize with a continuous function $g : \partial D \to \Gamma$, where D denotes the unit disk in $\mathbb{R}^2$; we decide to minimize the area

$$A(f) = \int_D J_f(x)dx, \tag{2.1}$$

where $J_f(x)$ denotes the positive Jacobian of f at x, among a suitable class of functions $f : \overline{D} \to \mathbb{R}^n$ such that $f_{|\partial B} = g$. In fact, it is probably a good idea to allow also functions f such that $f_{|\partial B}$ is another equivalent parameterization of Γ.

There are obvious complications with this problem, and the main one is probably the lack of compactness of the reasonable classes of acceptable parameterizations. That is, if $\{f_k\}$ is a minimizing sequence, i.e., if $A(f_k)$ tends to the infimum of the problem, and even if the sets $f_k(\mathbb{D})$ converge very nicely to a beautiful smooth surface, the parameterizations f_k themselves could have no limit. Even if Γ is the unit circle and each $f_k(\mathbb{D})$ is equal to D, it could be that we stupidly took a sequence of smooth diffeomorphisms $f_k : D \to D$ that behave more and more wildly.

For 2-dimensional surfaces, there is at least one standard way to deal with this problem: we can decide to use conformal parameterizations of the image, normalized in some way, and gain compactness this way. This is more or less the approach that was taken, for instance, by Garnier [Ga], and then Tibor Radó [Ra1] (1930) and J. Douglas [Do] (1931).

Let us say a few words about the existence theorem of Douglas, who was able to prove the existence of a function f that minimizes $A(f)$ under optimal regularity conditions for Γ.

Let us say a few words about the idea because it is beautiful. We decide that f will be the harmonic extension of its boundary values on ∂D, which we require to be a parameterization of Γ (but not given in advance). This is a reasonable thing to do, because we know that such harmonic parameterizations exist, at least in the smooth case. Then we can compute $A(f)$ in terms of $g = f_{|\partial B}$, and we get that $A(f) = B(g)$, where

$$B(g) = \int_0^{2\pi} \int_0^{2\pi} \frac{\sum_{j=1}^n |g_j(\theta) - g_j(\varphi)|^2}{\sin^2\left(\frac{\theta-\varphi}{2}\right)} \, d\theta d\varphi, \tag{2.2}$$

where the g_j are just the coordinates of g.

Now B is a much easier functional to minimize, in particular because there is one less variable, and Douglas obtains a solution rather easily. The paper [Do] seems very simple and pleasant to read.

Of course the mapping f is smooth, but there were still important regularity issues to be resolved, concerning the way $f(D)$ is embedded, or whether f may have critical points. See [Laws], [Ni], or [Os].

There are two or three important difficulties with this way of stating Plateau's problem. The minor one is that getting reasonably normalized parameterizations will be much harder for higher dimensional sets, thus making existence results in these dimensions much less likely.

Even when the boundary is a nice curve Γ, many of the physical solutions of Plateau's problem are not parameterized by a disk, but by a more complicated set, typically a Riemann surface with a boundary. So we should also allow more domains than just disks; this is not such a serious issue though.

But also, in many cases the solutions of Douglas do not really describe soap films (which were at the center of Plateau's initial motivation). For instance, if Γ is folded in the right way, the minimizing surface $f(D)$ of Douglas will cross itself, and because we just minimize the integral of the Jacobian, the various pieces that cross don't really interact with each other. In a soap film, two roughly perpendicular surfaces would merge and probably create a singularity like the ones that are described below. See Example 3.a and Figures 2, 3, 5 below.

If we replaced $A(f)$ with the surface measure (or the Hausdorff measure) of $f(D)$, which may be smaller if f is not one-to-one, we would probably get a much better description of soap films, but also a much harder existence theorem to prove. We shall return to similar ways of stating Plateau problems in Section 4.e, when we discuss sliding minimal sets.

2.b. Reifenberg's homology problem

The second approach that we want to describe is due to Reifenberg [R1] (1960). Let us state things for a d-dimensional surface (so you may take $d = 2$ for simplicity). Here we do not want to assume any a priori smoothness for the solution, it will just be a closed set E. Because of this, we shall define the area of E to be its d-dimensional Hausdorff measure. Recall that for a Borel-measurable set $E \subset \mathbb{R}^n$, the d-dimensional Hausdorff measure of E is

$$\mathcal{H}^d(E) = \lim_{\delta \to 0^+} \mathcal{H}^d_\delta(E), \tag{2.3}$$

where

$$\mathcal{H}^d_\delta(E) = c_d \inf \Big\{ \sum_{j\in\mathbb{N}} \mathrm{diam}(D_j)^d \Big\}, \tag{2.4}$$

c_d is a normalizing constant, and the infimum is taken over all coverings of E by a countable collection $\{D_j\}$ of sets, with $\mathrm{diam}(D_j) \leq \delta$ for all j. Let us choose c_d so that $\mathcal{H}^d$ coincides with the Lebesgue measure on subsets of $\mathbb{R}^d$.

The main point of $\mathcal{H}^d$ is that it is a measure defined for all Borel sets, and we don't lose anything anyway because $\mathcal{H}^d(E)$ coincides with the total surface measure of E when E is a smooth d-dimensional submanifold. Also, we don't assume E to be parameterized, which does not force us to worry about counting multiplicity when the parameterization is not one-to-one, and we allow E to take all sorts of shapes, even if our boundary is a nice curve; see Section 3 for examples of natural minimizers that are not topological disks.

So we want to minimize $\mathcal{H}^d(E)$ among all closed sets E that are bounded by a given $(d-1)$-dimensional set Γ.

We still need to say what we mean when we say that E bounded by Γ, and Reifenberg proposes to define this in terms of homology in E. He says that E bounded by Γ if the following holds. First, $\Gamma \subset E$. This is not too shocking, even though we shall see later soap films E bounded by a smooth curve, but that seem to leave it at some singular point of E. See Figures 4, 5, and 17. And even in this case, we are just saying that we see Γ itself as (a lower dimensional) part of the film. The main condition concerns the Čech homology of E and Γ on some commutative group G. Since $\Gamma \subset E$, the inclusion induces a natural homomorphism from $\check{H}_{d-1}(\Gamma; G)$ to $\check{H}_{d-1}(E; G)$, and Reifenberg demands this homomorphism to be trivial. Or, we could take a subgroup of $\check{H}_{d-1}(\Gamma; G)$ and just demand that each element of this subgroup be mapped to zero in $\check{H}_{d-1}(E; G)$.

In the simple case when $d = 2$ and Γ is a simple curve, $\check{H}_{d-1}(E; G)$ is generated by a loop (run along Γ once), and when we require this loop to be a boundary in E, we are saying that there is a way to fill that loop (by a 2-dimensional chain whose support lies) in E.

Reifenberg proved the existence of minimizers in all dimensions, but only when $G = \mathbb{Z}_2$ or $G = \mathbb{R}/\mathbb{Z}$. This is a beautiful (although apparently very technical) proof by "hands“", where haircuts are performed on minimizing sequences to make them look nicer and allow limiting arguments. Unfortunately, difficulties with limits force him to use Čech homology (instead of singular homology, for instance) and compact groups.

Reifenberg also obtained some regularity results for the solutions; see [Re1], [Re2].

Later, F. Almgren [Al3] proposed another proof of existence, which works for more general elliptic integrands and uses integral varifolds. But I personally find the argument very sketchy and hard to read.

Recently, De Pauw [Dp] obtained the 2-dimensional case when Γ is a finite union of curves and now $G = \mathbb{Z}$, with a proof that uses currents; he also proved that in that case the infimum for this problem is the same number as for the size-minimizing currents of the next subsection. But even then it is not known whether one can build size-minimizing currents supported on the sets that De Pauw gets.

Reifenberg's solutions are nice and seem to give a good description of many soap films. Using finite groups G like $\mathbb{Z}_2$, one can even get non-orientable sets E. But there are some "real-life" soap spanned by a curve that cannot be obtained as Reifenberg solutions. See Example 3.c, for instance.

Anyway, many interesting problems (existence for other homologies and groups like $\mathbb{Z}$, equivalence with other problems) remain unsolved in the Reifenberg framework of this subsection. The author does not know whether too much was done after [Re2], concerning the regularity of the Reifenberg minimizers. But at least the regularity results proved for the Almgren minimal sets (see Section 5) are also valid for the Reifenberg minimizers; we check this in Section 7 for the convenience of the reader.

2.c. Integral currents

Currents provide a very nice way to solve two problems at the same time. First, they will allow us to work with a much more general class of objects, possibly with better compactness properties. That is, suppose we want to work with surfaces in a certain class $\mathcal{S}$, typically defined by some level of regularity, and we want to prove the existence of some $S \in \mathcal{S}$ that minimizes (some notion of) the area $A(S)$ under some boundary constraints. Then let $\{S_k\}$ be a minimizing sequence, which means that each S_k lies in $\mathcal{S}$ and satisfies the boundary constraints, and that $A(S_k)$ tends to the infimum of the problem. We would like to extract a subsequence of $\{S_k\}$ that converges, but typically the $\mathcal{S}$-norms of S_k will tend to $+\infty$, and we will not be able to produce a limit set $S \in \mathcal{S}$. Of course even if S exists, we shall not be finished, because we also need to check that S satisfies the boundary constraints, and that $A(S) \leq \lim_{k\to+\infty} A(S_k)$, but this is a different story.

So, in the same spirit as for weak (or more recently viscosity) solutions to PDE, we want to define Plateau problems in a rough setting, and of course hope that as soon as we get a minimizer, we shall be able to prove that it is so regular that in fact it deserves to be called a surface.

The second positive point of using currents is that even with these rough objects, we will be able to define a notion of boundary, inherited from differential geometry, and thus state a Plateau problem. We need a few definitions.

A d-dimensional <u>current</u> is nothing but a continuous linear form on the space of smooth d-forms. This is thus the same as a d-vector valued distribution. In fact, most of the distributions that will be used here are (d-vector valued) finite measures, which are thus not too wild.

There are two main examples of currents that we want to mention here. The first one is the <u>current</u> S <u>of integration on a smooth, oriented surface</u> Σ of dimension d, which is simply defined by

$$\langle S, \omega \rangle = \int_\Sigma \omega \quad \text{for every } d\text{-form } \omega. \tag{2.5}$$

But of course we are interested in more general objects. The second example is the <u>rectifiable current</u> T defined on a d-dimensional rectifiable set E such that $\mathcal{H}^d(E) < +\infty$, on which we choose a measurable orientation τ and an integer-valued multiplicity m. Recall that a rectifiable set of dimension d is a set E such that $E \subset N \cup \bigcup_{j\in\mathbb{N}} G_j$, where

$\mathcal{H}^d(N) = 0$ and each G_j is a C^1 embedded submanifold of dimension d. But we could have said that G_j is the Lipschitz image of a subset of $\mathbb{R}^d$, and obtained an equivalent definition. We shall only consider Borel sets E, and such that $\mathcal{H}^d(E) < +\infty$. For such a set E and $\mathcal{H}^d$-almost every $x \in E$, E has what is called an approximate tangent d-plane $x + V(x)$ at x, which of course coincides with the usual tangent plane in the smooth case. A measurable orientation can be defined as the choice of a simple d-vector $\tau(x)$ that spans $V(x)$, which is defined $\mathcal{H}^d$-almost everywhere on E and measurable. We set

$$\langle T, \omega \rangle = \int_E m(x)\, \omega(x) \cdot \tau(x)\, d\mathcal{H}^d(x) \tag{2.6}$$

when ω is a (smooth) d-form, and where the number $\omega(x) \cdot \tau(x)$ is defined in a natural way that won't be detailed here. We assume that the multiplicity m is integrable against $\mathbf{1}_E d\mathcal{H}^d$, and then the integral in (2.6) converges.

Return to general currents. The boundary of any d-dimensional current T is defined by duality with the exterior derivative d on forms, by

$$\langle \partial T, \omega \rangle = \langle T, d\omega \rangle \text{ for every } (d-1)\text{-form } \omega. \tag{2.7}$$

When Σ is a smooth oriented surface with boundary Γ, S is the current of integration on Σ, and G is the current of integration on Γ, Green's theorem says that $\partial S = G$. This allows us to define Plateau boundary conditions for currents. We start with a $(d-1)$-dimensional current S, for instance the current of integration on a smooth oriented $(d-1)$-dimensional surface without boundary, and simply look for currents T such that

$$\partial T = S. \tag{2.8}$$

Notice that $\partial\partial = 0$ among currents, just because $dd = 0$ among forms. So, if we want (2.8) to have solutions, we need $\partial S = 0$; this is all right with loops (when $d = 2$), and this is why we want the smooth surface above to have no boundary. But other choices of currents S would be possible.

We often prefer to restrict to integral currents. An integral current is a rectifiable current T as above (hence such that the multiplicity m is integer-valued and integrable against $\mathbf{1}_E d\mathcal{H}^d$), and such that ∂T is such a rectifiable current too. If T solves (2.8), the condition on ∂T will be automatically satisfied, just because we shall only consider rectifiable boundaries S. The fact that we only allow integer multiplicities should make our solutions more realistic, and make regularity theorem easier to prove. Otherwise, we would expect to obtain solutions with very low density, or (at best) foliated and obtained by integrating other solutions.

So here is how we want to define a Plateau problem in the context of integral currents: we start from a given integral current S, with $\partial S = 0$ (for instance, the current of integration on a smooth loop) and we minimize the area of T among integral currents T such that $\partial T = S$.

The most widely used notion of area for currents is the mass $Mass(T)$, which is just the norm of T, seen as acting on the vector space of smooth forms equipped with the supremum norm (the norm of uniform convergence). In the case of the rectifiable current of (2.6),

$$Mass(T) = \int_E |m(x)| d\mathcal{H}^d(x). \tag{2.9}$$

The corresponding Plateau problem works like a geometric measure theorist's dream. First, the problem of finding an integral current T such that $Mass(T)$ is minimal among all the solutions of (2.8) has solutions in all dimensions, and as soon as S is an integral current with compact support and such that $\partial S = 0$. This was proved a long time ago by Federer and Fleming [FF], [Fe1]; De Giorgi also had existence results in the codimension 1 case, in the framework of BV functions and Caccippoli sets.

What helps us here is the the lowersemicontinuity of M (not so surprising, it is a norm), and the existence of a beautiful compactness theorem that says that under reasonable circumstances (the masses of the currents T_k and ∂T_k stay bounded, their supports lie in a fixed compact set), the weak limit of a sequence $\{T_k\}$ of integral currents is itself an integral current.

Moreover, the solutions of this Plateau problem (we shall call them mass minimizers) are automatically very regular away from the boundary. Let T be a mass minimizer of dimension d in $\mathbb{R}^n$, and denote by F its support (the closure of the subset of E above where $m(x)$ is nonzero), and by H the support of $S = \partial T$. If $d = n - 1$ and $n \leq 7$, F is a C^∞ embedded submanifold of $\mathbb{R}^n$ away from H, and if $d = n - 1$ but $n \leq 8$, E may have a singularity set of dimension $n - 8$ away from H, but no more. See [Fe2]. And in larger codimensions, the dimension of the singularity set is as most $d-2$ [Al5]. There were also important partial results by W. Fleming [Fl], Simons, and Almgren [Al2]; see [Mo5], Chapter 8 for details.

With this amount of smoothness, mass minimizers cannot give a good description of all soap films in 3-space, because some of these have obvious interior singularities. Also, the fact that the notion of current naturally comes with an orientation is a drag for some examples (like Möbius films). We shall discuss this a little more in the next section.

For the author, the main reason why mass minimizers do not seem to be a good model for soap films (regardless of their obvious mathematical interest), is because the mass is probably not the right notion of area for soap films. So we may want to consider the size $Size(T)$ which, is the case of the rectifiable current of (2.6), is defined by

$$Size(T) = \mathcal{H}^d\big(\{x \in E\,;\, m(x) \neq 0\big)\big\}. \tag{2.10}$$

That is, we no longer count the multiplicity as in (2.9), we just compute the Hausdorff measure of the Borel support. This setting allows one to recover more examples of soap films, and to eliminate some sets that are obviously not good soap films; see the next section. But the price to pay is that the existence and regularity results are much harder to get.

The Plateau problem that you get when you pick a rectifiable current S with $\partial S = 0$, and try to minimize $Size(T)$ among all integral currents T such that $\partial T = S$, is very interesting, but as far as the author knows, far from being solved. There are existence results, where one use intermediate notions of area like $\int_E |m(x)|^\alpha d\mathcal{H}^d(x)$ (instead of (2.9) or (2.10)) for some $\alpha \in (0,1)$; see [DpH]. But for instance we do not have an existence result when $d = 2$ and S is the current of integration over a (general) smooth closed curved in $\mathbb{R}^3$. The compactness theorem above does not help as much here, because if $\{T_k\}$ is a minimizing sequence, we control $Size(T_k)$ but we don't know that $Mass(T_k)$ stays bounded, so we may not even be able to define a limit which is a current. See [Mo1], though, for the special case when Γ is contained in the boundary of a convex body.

The size-minimizing problem is not so different from the Reifenberg Plateau problem of Section 2.b, and in some cases the infimum for the two problems was even proved to be the same [Dp].

Let us not comment much about the (mostly interior) regularity results for size minimizers, and just observe that the regularity results for Almgren minimal sets apply to the support of T when T is a size minimizer. See Section 7.

3. SIMPLE CLASSICAL EXAMPLES

It is probably time to rest a little, and try the various definitions above on a few simple examples. Most of the examples below are very well known; hopefully they will be convincing, even though we can almost never prove that a given set, current, or surface, is minimal. (Sadly, the main way to prove minimality is by exhibiting a calibration for the given object, which would often implies an algebraic knowledge that don't have). We will try to give a more detailed account than usual of what happens, to make it easier for the reader.

3.a. Crossing surfaces: the disk with a tongue

The next example is essentially the same as in [Al1]. Construct a (smooth) boundary curve Γ, as in Figure 1, which contains a main circular part, and two roughly parallel lines that cross the disk, near the center; then solve a Plateau problem or drop the wire into a soap solution and pull it back.

Most probably (but in fact the author can't compute!), the parameterized solution of Douglas is a smooth, immersed surface E_d, that crosses itself along a curve I, which lies near the center of the disk and connects the two parallel lines; the two pieces have no reason to really interact (the functional A in (2.1) does not see that the two pieces get close to each other). See Figure 1.

Next orient Γ, let S be the current of integration on Γ, and minimize $Mass(T)$ among integral currents T such that $\partial T = S$; then let E_m denote the closed support of T. By Fleming's regularity theorem [Fl], E_m is smooth away from Γ, so we can be sure that $E_m \neq E_d$ (also see Subsection 3.d for a simpler argument). Quite probably, E_m looks like the surface suggested in Figure 2, which we could obtain from E_d by splitting E_d along I into two surfaces, and letting them go away from each other and evolve into something minimal. This creates a hole near I, where one could pass without meeting E_m. This also changes the topology of the surface: now E_m is (away from Γ) a smooth surface with a hole, not the continuous image of a disk. Also, E_m is oriented, because it is smooth and comes from the current T, whose boundary lives in Γ. That is, locally and away from Γ, we can write T as in (2.6), where $\tau(x)$ is smooth; then the fact that $\partial T = 0$ locally implies that the multiplicity m is locally constant, and this gives an orientation on E_m (if E_m connected, as in the picture).

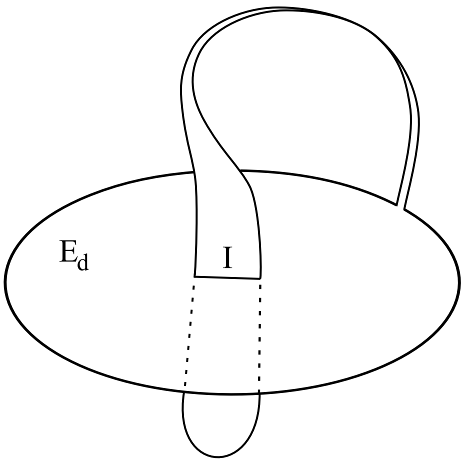


Figure 1. The Douglas solution.

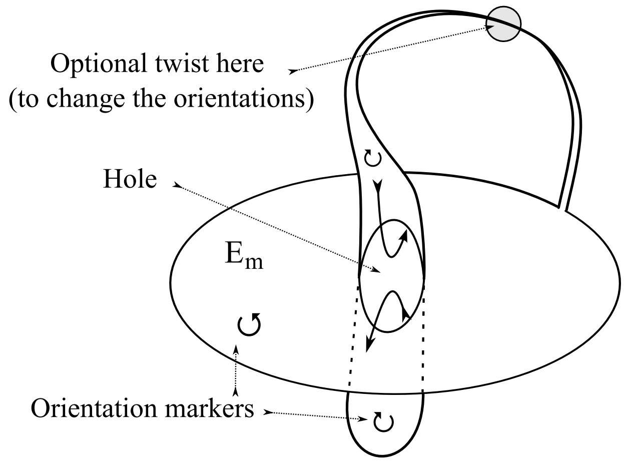
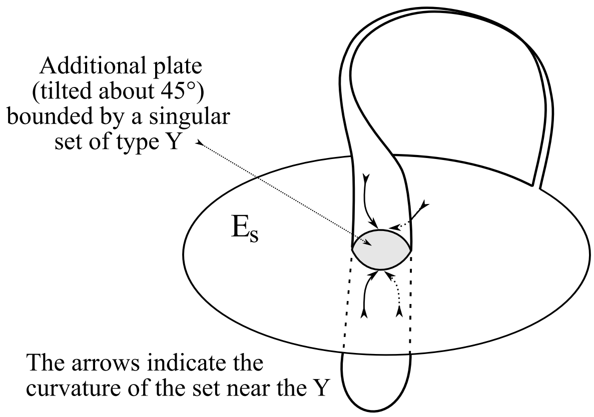


Figure 2. The mass minimizer E_m. The circular little arrows mark the orientation, and the longer oriented arrows try to show how E_m turns in the two curved transition zones.

Notice that the surface in Figure 2 is orientable, but if we had chosen to use the symmetric way to split E_d along I, we would have drawn a surface that goes in the direction of the front when we go down along the upper part of the tongue. This would have created a surface $\widetilde{E}_m$ which, near I, is roughly no one the image of E_m by the symmetry with respect to a vertical plane. But $\widetilde{E}_m$ is not orientable, hence the mass minimizing Plateau problem is not allowed to chose $\widetilde{E}_m$. The situation is reversed when we twist the thin part of Γ to change the orientation, which physically should not matter much though.

Now let us build Γ and plunge it into a film solution. Based on a few rough experiments, the author claims the following (which should surprise no one anyway).



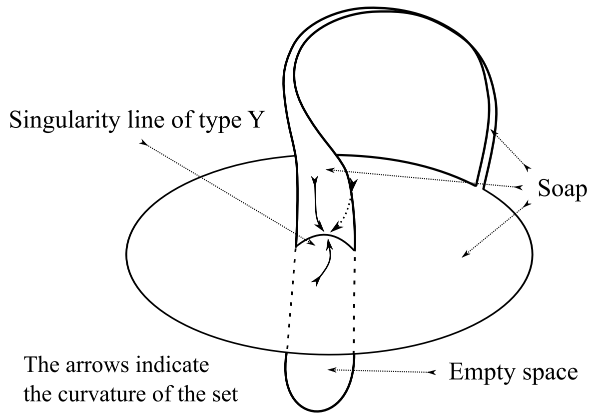


Figure 3 (left). The size minimizer E_s.
Figure 4 (right). A soap film which does not touch the whole boundary.

We should not even try to obtain E_d, if we believe that Almgren almost minimal sets give a good description of soap films, and indeed the author managed not to see E_d. Films that look like E_m are not so hard to get; we also get a third set E_s which we could roughly obtain from E_m as follows: add a small (topological) disk to fill the hole, and let again evolve into something minimal (so that the boundary of the disk will be a set of singularities of type Y); see Figure 3, or the left part of Figure 5. Also see the double disk example in Figure 7 for a more obvious singularity set of type Y). Or we could obtain E_s from E_d by pinching it along I. In practice, we often obtain E_s first, especially if our tongue is far from perpendicular to the horizontal disk (otherwise, it seems a little less stable), and one can obtain E_m from E_s by killing the small connecting disk.

We can also kill the lower part of the tongue by touching it, and get a singular curve of type Y where the upper tongue connects to the large disk-like piece. See Figure 4, or the right-hand part of Figure 5. When this happens, we get an example of a soap film whose apparent boundary is a proper subset of the curve Γ. This phenomenon is known, and has been described in the case of a trefoil knot in [Br4], for instance. See Figure 17 for another, more beautiful example. For all these physical observations, the fact that the two main pieces of the boundary are connected and make a single curve is not useful, and we get a similar description when Γ is composed of two disjoint non parallel ellipses (a large one and a small one) with the same center.

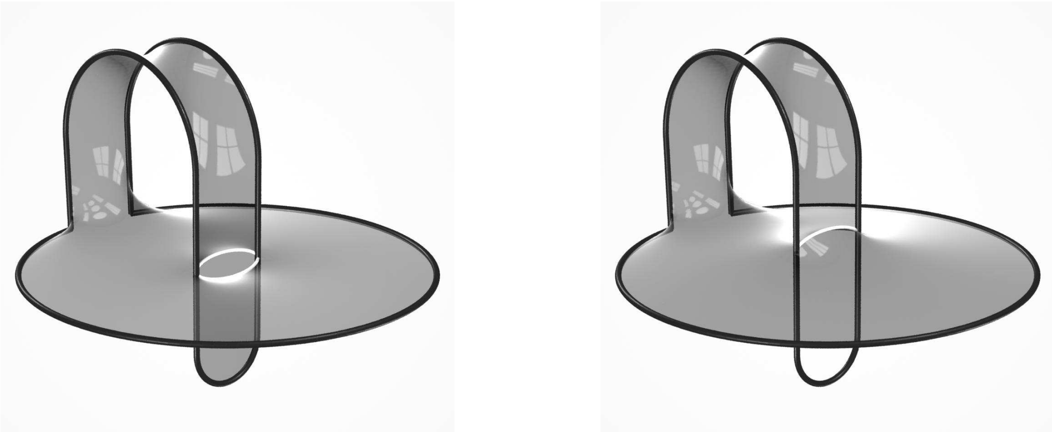


Figure 5. The sets of Figure 3 and 4. Images by John M. Sullivan, Technische Universität Berlin, used by permission.

Finally (returning to the case of a single curve), if the part that connects the tongue to the circular part of Γ is not too long or complicated, we obtain an often more stable, more complicated soap film with a singularity of type T, like the set depicted Figure 6. In fact, this one is often easier to get, and we then retrieve E_s and E_m by removing faces. See Section 3.e, and in particular Figures 9 and 10, for explanations and pictures of type T singularities.

We leave it as an exercise for the reader to determine whether E_s (or its vaguely symmetric variant $\widetilde{E}_s$) is probably the support of a size minimizer with the boundary S, and whether E_s and $\widetilde{E}_s$ are probable solutions of Reifenberg's problem. We shall treat the simpler example of two circles instead.

Let us nonetheless say that (not even based on real experiments) the author believes that the fact that soap films will choose E_m rather than $\widetilde{E}_m$, mostly depends on the angle of the tongue with the horizontal disk, or rather the way we pull Γ out of the soap solution, and is not based on orientation. Often we get E_m or $\widetilde{E}_m$ by passing through E_s and $\widetilde{E}_s$ first, which themselves should not depend much on orientation.

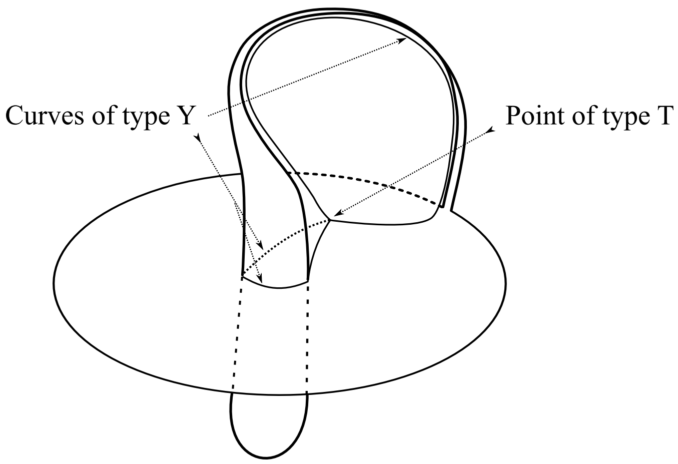


Figure 6. Another soap film bounded by the same curve. The picture only shows the singular set composed of curves where E looks like a Y, and which meet at a point where E looks like a T.

3.b. Two parallel circles

Let Γ be the union of two parallel circles C_1 and C_2, as in Figure 7.

There are three obvious soap film solutions: a catenoid H (that here looks a lot like a piece of vertical cylinder), the union $D_1 \cup D_2$ of two disks, and a set E composed of a slightly smaller disk D in the center, connected along the circle ∂D to C_1 and C_2 by two piece of catenoids that make angles of 120 degrees along ∂D. See Figure 7.

For the analogue of the Douglas problem, we would probably decide first to parameterize with two disks or with the cylinder, and then get $D_1 \cup D_2$ or H.

For the mass minimizing problem, we first choose an orientation on both circles C_1 and C_2, and for instance, use the sum $S = S_1 + S_2$, where S_i is the current of integration on C_i. If we choose parallel orientations, H is not allowed because the orientations do not fit, and, with a fairly easy proof by calibration (use the definition of ∂T to compute $\langle T, dx_1 dx_2 \rangle$) we can show that the current of integration on $D_1 \cup D_2$ is the unique mass minimizer.

If we choose opposite orientations on the C_i (or equivalently set $S = S_1 - S_2$), then both $D_1 \cup D_2$ and H will provide acceptable competitors, and we should pick the one with the smallest mass. Here too, we should be able to pick a vector field (pick one for each of the two competitors) and use it to prove minimality by a calibration argument.

We could also try another choice of S, like $S = S_1 + 2S_2$, but we would get the set $D_1 \cup D_2$ again.

The size-minimizing problem has a different solution when we take parallel orientations. Let D be a slightly smaller disk, parallel to D_1 and D_2, and that we put right in the middle. Complete D with two pieces of catenoids H_1 and H_2, connecting ∂D to the C_i (see Figure 7). The best choice will be when the H_i make an angle of 120 degrees with D along ∂D.

It is easy to construct a current T, with $\partial T = S$, and which is supported on E: take twice the integration on D, plus once the integration on each catenoid, all oriented the same way. The fact that $\partial T = S$ is even easier to see when one notices that $T = T_1 + T_2$, where T_i is the current of integration on $D \cup H_i$ and $\partial T_i = S_i$. If the two circles are close enough, it is easy to see that $Size(T)$ is smaller (almost twice smaller) than the size of the mass minimizer above. It should not be too hard to show that in this case, T is the unique size minimizer, but the author did not try. Again, H is not allowed, because the orientations do not fit.

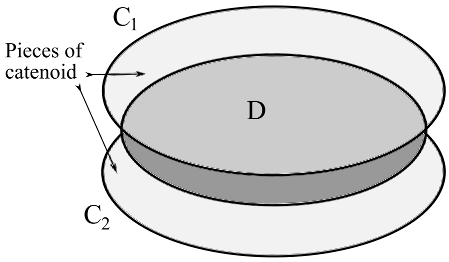


Figure 7. The set E.

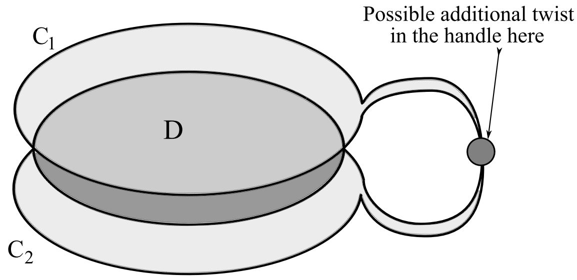


Figure 8. Same thing with a handle.

When we use opposite orientations on the C_i, H is allowed, and E never shows up: if T is supported on E and $\partial T = S_1 - S_2$, the multiplicity on each H_i should be constant, equal to $(-1)^i$ (check ∂T); the multiplicity on D should also be constant, and in fact equal to 0 if we want ∂T to vanish near ∂D. Then the support of T is $H_1 \cup H_2$, and in fact H was doing even better than both $H_1 \cup H_2$ and E, so we should have no regret.

Returning to the case of parallel orientations, notice that when $H^2(E) \leq \mathcal{H}^2(D_1 \cup D_2)$, T is a size minimizer but $Size(T) = \mathcal{H}^2(E) > \mathcal{H}^2(H)$, which is possible because H is not a competitor.

The situation is simple enough for us to comment on the Reifenberg problem. But even then, let us just think about simplicial homology on the group $\mathbb{Z}$. Here $\Gamma = C_1 \cup C_2$, and there are two obvious generators γ_1 and γ_2 for the homology group $H_1(\Gamma, \mathbb{Z})$. Let us choose the γ_i so that they correspond to parallel orientations of the C_i.

One way to state a Plateau problem is to minimize $\mathcal{H}^2(F)$ among sets F such that $\gamma_1 + \gamma_2$ represents a null element in the homology group $H_2(F, \mathbb{Z})$. Notice then that $F = D_1 \cup D_2$ is allowed (because $D_1 \cup D_2$ gives a simplicial chain supported in F and whose boundary is $\gamma_1 + \gamma_2$), but H is not. To check this we would have to check that $\gamma_1 + \gamma_2$ does not vanish in $H_2(H, \mathbb{Z})$, which we kindly leave as an exercise because we do not want to offend any reader that would know anything about homology. Here $E = D \cup H_1 \cup H_2$ is allowed too, because each $D \cup H_i$ contains a simplicial chain whose boundary is γ_i. And, if D_1 and D_2 are close enough, the Reifenberg minimizer will be E (but the verification will be more painful than before).

The situation would be the same if we required that $\gamma_1 + 2\gamma_2$ represents a null element in $H_2(F, \mathbb{Z})$, or if we required the whole group $H_1(\Gamma, \mathbb{Z})$ to be mapped to 0 in $H_2(F, \mathbb{Z})$.

On the other hand, if we just require $\gamma_1 - \gamma_2$ to be mapped to 0, then both H, $D_1 \cup D_1$, and E are allowed, and the minimizer will be H if D_1 is close to D_2.

As far as the author knows, the verifications would be more painful, but at the end similar to the verifications for size-minimizing currents. The point should be that if F is a competitor in the Reifenberg problem, we should be able to construct chains inside F (whose boundary is a representative for $\gamma_1 + \gamma_2$, for instance), possibly approximate them with polyhedral chains, and integrate a differential form on them to complete a calibration argument. This is an easy exercise, but slightly above the author's competence.

We may also want to work on the group $\mathbb{Z}/2\mathbb{Z}$; then $\gamma_1 + \gamma_2 = \gamma_1 - \gamma_2$, H is allowed in all cases, and is the unique minimizer when the disks are close to each other.

Very easy soap experiments show that the three sets E, $D_1 \cup D_2$, and H are constructible soap films, with a noticeable preference for E when D_1 and D_2 are close to each other.

3.c. Two disks but a single curve

In the previous example, we managed to represent all the soap films as reasonable competitors in one of the two standard problems about currents, even though in some cases, stable soap films are not absolute minimizers. But we expect difficulties in general, again because of orientation issues

In fact, is easy to produce a Möbius soap film, which is not orientable and hence is not the support of a mass- or size-minimizing current. Unless we use some tricks (like work modulo 2, or use covering spaces), as in Sections 4.a and 4.b. For instance, the author believes that a Möbius soap film may be a solution of the Reifenberg problem above, where we decide to compute homology over the group $\mathbb{Z}_2$

Let us sketch another example, which is just a minor variant of the previous one. Cut two very small arcs out of $C_1 \cup C_2$, one above the other, and replace them with two parallel curves g_1 and g_2 that go from C_1 to C_2; this gives a single simple curve Γ_1, as in Figure 8. This is also almost the same curve that is represented in K. Brakke's site under the name "double catenoid" soap films. There are many constructible soap films bounded by Γ_1 (see Brakke's site) but let us concentrate on the one that looks like the set E above, plus a thin surface bounded by g_1 and g_2 (call this set E_1). The difference with the previous example is that now there is only one curve, the orientations on the two circular main parts of Γ_1 are opposite, and (as in the case of opposite orientations above) E_1 is not the support of an integral current T such that ∂T is the current of integration on Γ_1. Similarly, E_1 cannot solve the Reifenberg problem either, just because a strictly smaller subset is just as good (we can check that we can remove the central disk from E_1, and that Γ_1 is the boundary of the remaining surface, which is orientable).

If instead of taking g_1 and g_2 parallel, we make them twist and exchange their upper extremities, we get parallel orientations again, and the corresponding set $\widetilde{E}_1$ is a competitor in the Reifenberg and size-minimizing problems.

Two last comments for this type of examples: if we have only one Plateau problem, as is the case in any of the categories above if the boundary is just one curve, then in generic situations we can only get one solution. This is usually not enough to cover the variety of different soap films. But this alone would not be so bad if we could cover all the example as stable local minima. Also, orientation seems to be the main source of trouble here, which suggests that we use varifolds. But boundary conditions are harder to define for varifolds.

3.d. The minimal cones Y and T, and why mass minimizers do not cross

There are three types of tangent cones that can easily be seen in soap films: planes (that we see at all the points where the film is smooth), the sets Y composed of three half planes that meet along a line with 120 degrees angles, and the sets T that are obtained as the cone over the union of the edges of a regular tetrahedron (seen from its center). See Figure 9. Also see Figure 10 for a soap film with a singularity of type T.

These are the three cones of dimension 2 in $\mathbb{R}^3$ that are minimal sets in the sense of Almgren (see Section 5 below), but let us consider currents for the moment.

The plane is the only cone that can be seen as a blow-up limit of the support of a 2-dimensional mass minimizer in $\mathbb{R}^3$; this comes from Fleming's regularity theorem, but we want to discuss this a little more.

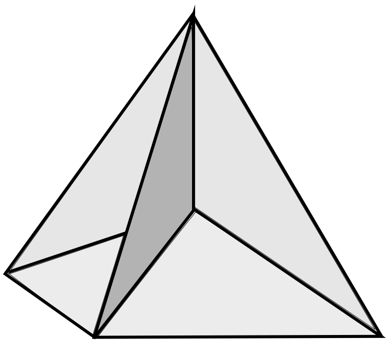


Figure 9. The set T.

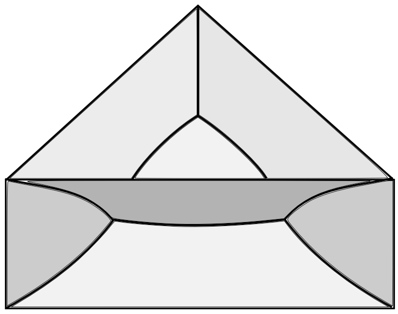


Figure 10. A film with three T points.

Let us first say why the support of a mass minimizer T never looks like two smooth surfaces that cross neatly. Denote by E this support, and suppose that in a small box, E looks like the union of the two planes (or smooth surfaces) suggested in Figure 11. First assume that T has multiplicity 1 on these planes, with the orientation suggested by the arrows. Replace E, near the center, with two smoother surfaces (as suggested in Figure 12). This gives a new current T_1, and it is easy to see that $\partial T_1 = \partial T$ (both ∂T and ∂T_1 vanish near the center, and the contributions away from the center are the same too). It is also easy to choose the two smoother surfaces so that $Mass(T_1) < Mass(T)$.

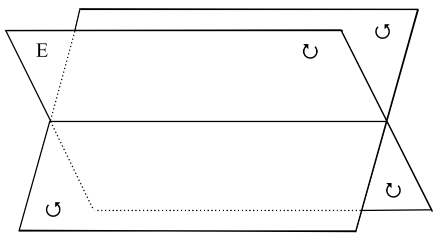
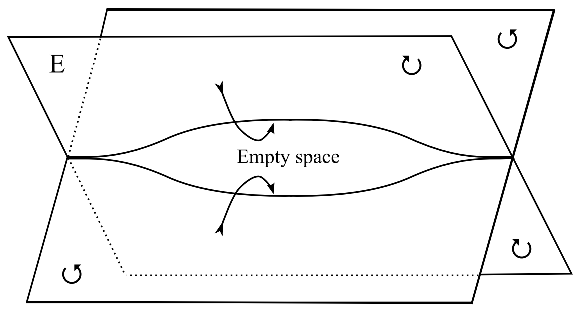


Figure 11. Two oriented planes

Notice that the modification suggested in Figure 13) is not allowed (how would we orient the surfaces?), and that the modifications above are the same as what we suggested in the tongue example of Subsection 3.a.



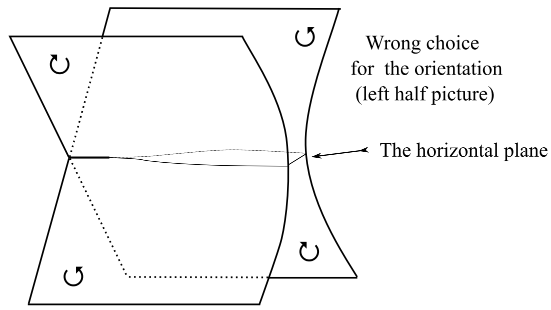


Figure 12. A better competitor. **Figure 13.** Unauthorized competitor (half picture).

We assumed above that T has multiplicity 1 on the two planes. If the multiplicities are different, say, $1 \leq m_1 < m_2$, denote by T_0 the current on E with the constant multiplicity m_1, replace T_0 with m_1 times the current of integration on the two smooth surfaces, and keep $T - T_0$ as it is. We still get a better competitor. This looks strange, and the author sees this as a hint that the mass is too linear a functional to be completely honest.

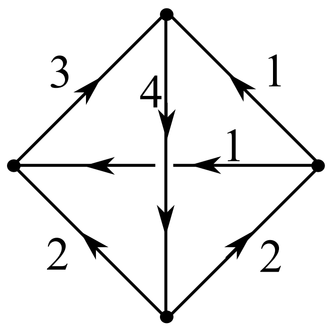
Return to the cone Y, and let us first say why it is the support of a current with no boundary. Put an orientation on the three faces F_j that compose Y. The orientation of each F_j gives an orientation of the common boundary L, and we can safely assume that the three orientations of L that we get coincide. Denote by T_j the current of integration on F_j, and set $T = T_1 + T_2 - 2T_3$. It is easy to see that $\partial T = 0$ because the three contributions cancel.

One can show (and the best argument uses a calibration; see [Mo1]) that T is locally size minimizing. More precisely, for each choice of $R > 0$, set $T_R = \mathbf{1}_{B(0,R)} T$; the computation shows that ∂T_R is of the form $S_1 + S_2 - 2S_3$, where S_j denotes the current of integration on the (correctly oriented) half circle $F_j \cap \partial B(0, R)$, and one can show that for each $R > 0$, T_R is the unique size minimizer W under the Plateau condition $\partial W = \partial T_R$.

Now Y is not the support of a local mass minimizer T: the multiplicity would need to be constant on the three faces F_j, the sum of these multiplicities should be zero because $\partial T = 0$ near the line, and then we could split T into two pieces (each with two faces) that we could improve independently, as in the previous case.

The third minimal cone T is also the support of a current W with vanishing boundary; again we have to put (nonzero integer) multiplicities m_j on the six faces F_j, so that their contribution to each of the four edges cancel. Let us give an example of multiplicities that work. Let H denote the closed convex hull of the tetrahedron that was used to define T. Then $T \cap \partial H$ is composed of six edges $\Gamma_j = F_j \cap \partial H$; we orient Γ_j so that $\partial(\mathbf{1}_H T_j) = S_j$ along Γ_j, and then we set $S = \sum_j m_j S_j$. Figure 14 describes a choice of multiplicities and orientations of the Γ_j for which $\partial S = O$ and so (this is the same condition) $\partial W = 0$.

With such multiplicities, $\mathbf{1}_H W$ is the only size minimizer for the Plateau condition $\partial T = S$. See [Mo1]. Of course it is not a mass minimizer, because it contains a lot of singularities of type Y, which are not allowed.



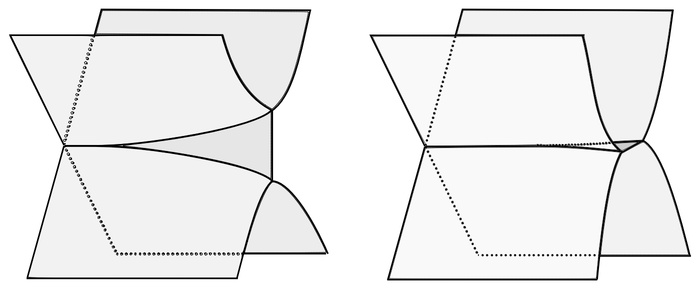


Figure 14 (left). Orientations and multiplicities to make a size minimizer supported on T. The dots represent vertices of a tetrahedron, and the arrows sit on the edges Γ_j.

Figure 15 (right). Two size-competitors for the union of two planes (half pictures).

Return to the union of two planes in Figure 11 and let T be obtained by putting nonzero integer multiplicities on the four half faces; suppose that $\partial T = 0$ in some large ball and, for convenience, that the two planes are nearly orthogonal. Then the two sets suggested in Figure 15 support currents that have a smaller size than T. (It could be that we can take a vanishing multiplicity on the middle triangular surface, and then one of the competitors of Figures 12 and 13 is allowed and does even better.)

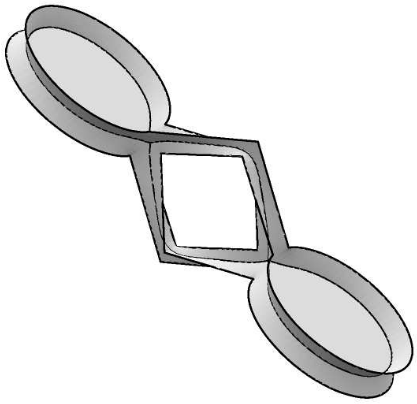
3.e. Mere local minima can make soap films

We have seen that (real-life) soap films can exist, even if they do not minimize mass, size, or Hausdorff measure in one of the Plateau problems quoted above. If Γ is a curve, each Plateau problem usually comes with one solution, and many soap films exist, often with different topologies (but maybe the same homology constraints). We shall see in the next section a few attempts to multiply the number of Plateau problems, in order to accommodate more examples.

It is expected that soap films are not necessarily global minima, even for a given topology, i.e., that stable local minima work as well. Things like this are unavoidable. But the situation is even worse: Figure 16 shows a soap film which can be retracted, inside itself, into its boundary which is a curve. But the retraction is long, and the soap will not see it and stay at the local minimum. The author admits he does not see the retraction either, and was unable to construct a soap example.

This example is due to J. Frank Adams (in the appendix of [Re1]), and the picture comes from K. Brakke's web page.

We add in Figure 17 one of the many pictures of soap films bounded by a union of three circles in a Borromean position. This one leaves one of the three circles for some time, like the film of Figures 4 and 5. See the web page of K. Brakke for many more.



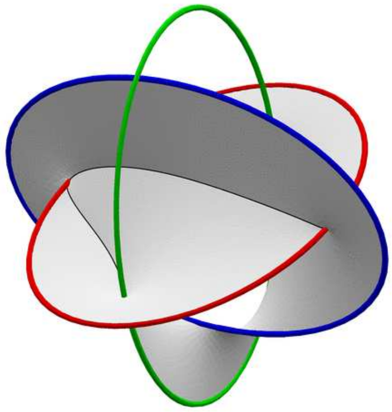


Figure 16 (left). A soap film that retracts onto itself (J. Frank Adams).
Figure 17 (right). One of the many soap films bounded by a Borromean ring. Both pictures are courtesy of K. Brakke.

4. OTHER MODELS FOR PLATEAU PROBLEMS

Various tricks, often clever, have been invented to increase the number of Plateau problems, for instance associated to a given curve, and thus accommodate the various examples. As we have seen, one of the unpleasant things that we often have to deal with is orientation. We briefly report some of these tricks.

4.a. Compute modulo p

One can define integral currents modulo k, by saying that two integral currents are equivalent when their difference is k times an integral current. Then Möbius strips, for instance, can can support currents modulo 2 without being orientable, and even solve a mass minimizing Plateau problem as in Section 2, but with integral currents modulo 2. Similarly, the double disk set E of Figure 2 is the support of a a current modulo 3 with multiplicity ± 1.

4.b. Use covering spaces

In [Br4], K. Brakke manages to treat many of the simple soap film examples above in the general formalism of mass (not even size!) minimizers. His construction works for sets of codimension 1. He starts from the base manifold $M = \mathbb{R}^n \setminus \Gamma$, where Γ is our boundary set, constructs a covering space over M, which branches along Γ, and eventually gets the desired soap film E as the projection of boundaries of domains in the covering space, that minimizes the mass under some constraints that we don't want to describe here. The boundaries are orientable even if E is not, and almost every point of E comes from two boundary points in the covering (as if, a little as what happens in real soap film, E were locally composed of two layers, coming from different levels in the covering).

The construction is very beautiful, but apparently limited to codimension 1. Also, the fact that a whole new construction seems to be needed for each example is a little unsettling.

4.c. Use varifolds

Almgren [Al3] used varifolds to present another proof of existence for the solutions of the Reifenberg problem of Section 2.b. Varifolds look like a very nice concept here, because they don't need to be orientable (which was an unpleasant feature of currents), and we can compute variations of the area functional on them. This gives a notion of stationary integral varifold, which englobes the minimal cones above, and Allard and Almgren prove compactness theorems on these classes that are almost as pleasant as for the integral currents above. See [Al1], where the author expresses very high hopes that varifolds are the ultimate tool for the study of minimal sets and Plateau problems. But (possibly because varifolds are not oriented), it seems hard to state and solve a Plateau problem, as we did in Section 2.c for currents with the boundary operator ∂. We shall return briefly to this issue in Section 6.

4.d. Differential chains

J. Harrison [Ha3] proposes yet another way to model and solve a Plateau problem. Again the point is to get rid of the difficulties with orientation, and at the same time to keep a boundary operator. To make things easier, let us consider 2-dimensional sets in $\mathbb{R}^3$. We are looking for a representation of soap films as slight generalizations of "dipole surfaces", where a dipole surface associated to a smooth oriented surface Σ is defined as

$$\Delta S = \lim_{t \to 0} \frac{1}{t}(S^+ - S), \tag{4.1}$$

where S is the current of integration on Σ and S^+ is the current of integration on the surface Σ^+ obtained from Σ by a translation of t in the normal direction. Thus dipole surfaces are currents of dimension 2, but they act on forms by taking an additional normal derivative (which means that they have one less degree of smoothness).

One of the ideas is that the orientation disappears (the dipole surface obtained from Σ with the opposite orientation is the same), which allows one to represent the branching examples of Section 3 as (limits of) dipole surfaces. The boundary $\partial(\Delta S)$ is a dipole version of ∂S, taken in the direction of the unit normal to Σ (along $\partial\Sigma$).

J. Harrison works with the closure $\mathcal{F}$ of all finite sums of ΔS as above, with the norm coming from the duality with Lipschitz 2-forms (with the L^∞ norm on the form and its derivative), starts from a nice curve Γ, chooses a smooth normal vector field on Γ, uses it to define a dipole curve G based as above on the current of integration on Γ, and finally looks for a current $T \in \mathcal{F}$ that solves $\partial T = G$, and for which the correct analogue of the mass (called the volume form) is minimal. In the case of a small ΔS above, the volume form is equal to $\mathcal{H}^2(\Sigma)$; for general elements of $\mathcal{F}$, it is defined by density.

It turns out that when we restrict to dipoles (that is, limits of dipole polyhedral chains, for the norm of duality with the forms with one Lipschitz derivative), there is a construction (based on filling the holes with a Poincaré lemma) that allows one to invert ∂ and construct competitors in the problem above. This construction may be a main difference with the situation of varifolds.

But at the same time the elements of $\mathcal{F}$ are not so smooth, and one still needs to check some details, not only about the existence of minimizer for the problem above, but also to get some control on the solutions and show that they are more than extremely weak solutions. For instance, it would be nice to prove that their support is a rectifiable set, maybe locally Almgren-minimal, as in the discussion below.

4.e. Sliding deformations and sliding Almgren minimal sets

We now describe the author's favorite model, which we describe with some generality, but for which very little is known, even for 2-dimensional sets in $\mathbb{R}^3$.

We give ourselves a finite collection of boundary pieces Γ_j, $0 \leq j \leq j_{max}$, (those are just closed subsets of $\mathbb{R}^n$ for the moment), and an initial competitor E_0 (a closed set). We simply want to minimize $\mathcal{H}^d(E)$ in the class $\mathcal{F}(E_0)$ of sliding deformations of E, which we define as follows. A closed set E lies in $\mathcal{F}(E_0)$ if $E = \varphi_1(E_0)$, where $\{\varphi_t\}$, $0 \leq t \leq 1$, is a one-parameter family of mappings such that

$$(4.2) \qquad (t,x) \to \varphi_t(x) : [0,1] \times E_0 \to \mathbb{R}^n \text{ is continuous,}$$

$$(4.3) \qquad \varphi_0(x) = x \text{ for } x \in E_0,$$

$$(4.4) \qquad \varphi_t(x) \in \Gamma_j \text{ when } 0 \leq j \leq j_{max} \text{ and } x \in E_0 \cap \Gamma_j,$$

and

$$(4.5) \qquad \varphi_1 \text{ is Lipschitz.}$$

We decided to require (4.5), mostly by tradition and to accommodate size minimizers below, but this is negotiable. Notice that we do not require any quantitative Lipschitz bound for φ_1.

Of course we should check that

$$(4.6) \qquad 0 < \inf_{E \in \mathcal{F}(E_0)} H^d(E) < +\infty,$$

because otherwise the minimization problem below is not interesting. And then we want to minimize $H^d(E)$ in the class $\mathcal{F}(E_0)$.

Let us just give a few comments here, starting with the good news.

This definition seems natural for soap films. We allow the soap film to move continuously, and we impose the constraint that points that lie on a boundary piece Γ_j stay on Γ_j (but may move along Γ_j). We also allow deformations φ that are not injective (including on the Γ_j); so we are allowed to pinch and merge different parts of E_0. This seems to be all right with real soap films.

Some boundary pieces may play the role of the curve Γ in the Plateau problems of Section 2, but we may also consider the case when some Γ_j are two-dimensional, and the surface boundary is allowed to slide along Γ_j, like a soap film that would be attached to a wall. Or we could use a set Γ_0 that contains E_0 to force our films to stay in the given region Γ_0.

The fact that we choose an initial E_0 also gives us some extra flexibility; this looks like cheating, because given a soap film, we can always try it as our initial set E_0 and see what happens. But at the same time, it is probable that real soap films do something like this. And we don't have to think too hard about how to model each given soap film, or to wonder about which precise topological property (for instance, belonging to some homology class, for some group that we would need to choose) defines the correct class of competitors.

The definition above is not really new, even though the author did not find more than allusions to this way of stating Plateau problems in the literature. But Brakke's software "Surface Evolver" allows this as one of the main options. Maybe people just did not want to state a problem that looked too hard to solve.

Our definition looks like Almgren's definition of "restricted", or (M, ε, δ)-minimal sets, but in the present situation, Almgren would tend to work on the open set $U = \mathbb{R}^n \setminus \big(\cup_j \Gamma_j\big)$, and would use the (much too strong) condition that $\varphi_t(x) = x$ for x in a neighborhood of the Γ_j. A minor difference is that Almgren typically considers only φ_1, regardless of the existence of a one-parameter family of mappings that connects it to the identity; this usually makes no difference, because in most results everything happens in a small ball contained in U, and the one-parameter family can easily be obtained by convexity. He can also play with the small parameter δ, which limits the diameter of the authorized modifications and can be used to forbid mappings φ_1 that are not homotopic to the identity inside U.

We could have taken a more restrictive approach and just required that $\varphi_t(x) = x$ when $x \in \Gamma_j$, but the author thinks that soap does not really act like this. For instance, for 2-dimensional films in $\mathbb{R}^3$, with a boundary Γ_j which is a plane, we expect solutions that look like two half planes with a common boundary (a line) in Γ_j, and that lie on the same side of Γ_j. With (4.4), they should make equal angles with Γ_j; with the stronger "sticking boundary" condition, any two planes should work (provided that they make an angle of at least 120 degrees), and even more complicated sets bounded by a curve in Γ_j. The definition with (4.4) also has the advantage, since it allows more competitors, to make it slightly easier to prove regularity results for the minimizers.

Also, we decided to define the φ_t only on E, because we do not want to force any deformation $\{\varphi_t\}$ that the soap may choose, to extend continuously the the whole space and respect boundary constraints where the soap is not even present.

Of course we expect that some soap films in nature will exist just because they are stable local minima of $\mathcal{H}^d$ without really minimizing, even in the same class $\mathcal{F}(E_0)$. Even worse, the set depicted in Figure 6 (Section 3.e) can be deformed into a set of vanishing measure, without even going through a set of larger Hausdorff measure, so the only way to exclude this deformation would be to disqualify it because it is too long.

At the same time, we should observe that solutions of the mass-minimizing problem of Section 2.c (where the boundary is a nice curve) provide local solutions. Indeed let T minimize $Mass(T)$ among solution of $\partial T = S$, where S is the current of integration on a smooth curve Γ, and let E_0 denote its support. By [HS] and if Γ is C^2, the set E_0 is regular, including along the boundary, and then it is easy to see that E_0 is locally minimal in the following sense: there is a small $\delta > 0$ such that $\mathcal{H}^2(E_0) \leq \mathcal{H}^2(E)$ whenever E is a deformation of E_0 in a ball B of radius δ. This last means that $E \in \mathcal{F}(E_0)$ is obtained as in (4.2)-(4.5), but with mappings φ_t such that $\varphi_t(x) = x$ for $x \in B$ and $\varphi_t(B) \subset B$. Of course we only get one local sliding minimizer E_0 this way, and it is always smooth, so the main defect of this point of view, in the author's opinion, is that it never gives a soap film with a singularity.

Now the worse news. We are again trying to play with parameterizations (here, with the initial set E_0 as a source set), and we know that it will be hard to find optimal ones, and that we will lack compactness at the crucial moment if we are not careful. That is, if we select a minimizing sequence in $\mathcal{F}(E_0)$, the limits of convergent subsequences will in general not lie in $\mathcal{F}(E_0)$.

And indeed no general existence result is known so far, even when $d = 2$, $n = 3$, there is a single piece of boundary Γ, which is a loop, and E_0 is the continuous image of a disk that closes the loop. Notice that the Douglas solution does not help here, because we look at a different functional which takes care of interactions between pieces. Thus the situation is as bad (and probably a tiny bit worse) than for size-minimizing currents.

It still looks interesting to study sliding minimizers. First because there is still a small chance that this approach will work is some cases, by selecting carefully a nice minimizing sequence before we take any limit. See [Da6], where a short description of two recent results of this type is given, but for simpler problems where there the class $\mathcal{F}(E_0)$ is not given in terms boundary pieces Γ_j as above, but of softer topological conditions.

Also, the chances of proving existence results will probably increase if we understand better the regularity results for minimizers, all the way to the boundary. We shall see in Section 7 that such results could also be used for the solutions of the Reifenberg and size minimization problems (when they exist).

4.f. Variants of the Plateau problem

Let us just mention here that there are many other interesting problems where one tries to minimize $\mathcal{H}^d(E)$ (or some variant) under topological conditions on E (separations conditions in codimension 1, homology conditions in higher codimensions, etc.) We refer to [Da3] and [Da6] for some examples, but we do not develop here.

5. REGULARITY RESULTS FOR ALMGREN ALMOST MINIMAL SETS

5.a. Local regularity

The following notion was introduced by Almgren, as a very good model for studying the local regularity properties of soap films and bubbles (among other objects). We simplify the definition a bit, but not in a significant way.

We give ourselves an open set $U \subset \mathbb{R}^n$, a dimension $d \leq n$, and a nondecreasing gauge function $h : (0, +\infty) \to [0, +\infty)$, such that

$$\lim_{r \to 0} h(r) = 0, \tag{5.1}$$

which will measure how close we are from minimality. Taking $h(r) = 0$ corresponds to minimality, and $h(r) = Cr^\alpha$ with $0 < \alpha \leq 1$ is a standard choice. This is a way to accommodate slightly more complicated functionals than just $\mathcal{H}^d$.

We say that the closed subset E of U is a d-dimensional almost minimal set in U, with gauge function h, if the following holds.

For each closed ball $B = \overline{B}(x,r) \subset U$, and each Lipschitz mapping $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\varphi(x) = x \quad \text{for } x \in \mathbb{R}^n \setminus B \quad \text{and } \varphi(B) \subset B, \tag{5.2}$$

then

$$\mathcal{H}^d(E \cap B) \leq \mathcal{H}^d(\varphi(E) \cap B) + r^d h(r); \tag{5.3}$$

we also demand that

$$\mathcal{H}^d(E \cap B) < +\infty \tag{5.4}$$

for all $B \subset U$ as above, to avoid stupidly large sets.

Notice again that φ is allowed not to be injective, and that since B is convex, we could easily connect φ with the identity by a one parameter family of mappings that satisfy (5.2) (take $\varphi_t(x) = t\varphi(x) + (1-t)x$).

It is clear that any solution of the sliding Plateau problem mentioned in Section 4.d is a minimal set in the complement of the $\partial\Gamma_j$. (We prefer to say $\partial\Gamma_j$, because some Γ_j may be n-dimensional and designed to contain E_0 and all its competitors.) We shall see in Section 7 that this also applies to minimizers of Reifenberg's problem, or supports of size minimizers.

The local regularity in U of the almost minimal sets was started by Almgren [Al4], and continued in [Ta2], [DS], [Da1,4,5], and others. For general dimensions and codimensions, we get that, modulo a set of vanishing $\mathcal{H}^d$-measure, the almost minimal set E is locally Ahlfors-regular, rectifiable, and even uniformly rectifiable with big pieces of Lipschitz graphs. When E is 2-dimensional in $\mathbb{R}^3$, Taylor [Ta2] proved that it is locally C^1-equivalent to a minimal cone (a plane, or a set Y or T as in Section 3.d), and this was partially extended to higher ambient dimensions in [Da4,5]. Also see [Da6] for a slightly more detailed survey of these results.

The reader may regret that this is not very smooth, but notice that the C^1 description of J. Taylor is nearly optimal, since after all the minimal cones above are almost minimal sets; the situation for $d > 2$ is widely open.

A last property of the notion of almost minimal sets that may turn out to be useful is its stability under limits [Da1]: if $\{E_k\}$ is a sequence of almost minimal sets, with the same gauge function h, and that converges (relative to the Hausdorff distance, and after cutting out unneeded sets of vanishing measure) to a limit E, then E is also almost minimal, with the same gauge function h. In addition,

$$\mathcal{H}^d(E \cap V) \leq \liminf_{k \to +\infty} \mathcal{H}^d(E_k \cap V) \tag{5.5}$$

for every relatively compact open subset V of U; that is, the restriction of $\mathcal{H}^d_{|V}$ to our sequence is lowersemicontinuous.

5.b. Regularity near the boundary

Not so many regularity results exist that go all the way to the boundary. The author knows about [All] (for varifold, near a flat point), and [HS] and [Wh] (for mass minimizers). Also see [LM2], Figure 5.3, or [Mo5], Figure 13.9.3 on page 137 in my third edition, for a conjecture about the types of singularities of a soap film near the boundary, and [Br3] for a description of soap films near a point where they leave a boundary curve.

It is nonetheless interesting to see what can be done near the boundary, in the context of Almgren almost minimal sets. Let the closed boundaries pieces Γ_j, $0 \leq j \leq j_{max}$, be chosen, as in Section 4.e, and let $E \subset \mathbb{R}^n$ be a closed set, with $\mathcal{H}^d(E) < +\infty$.

Definition 5.6. *We say that E is* <u>*almost minimal, with the sliding conditions*</u> *defined by the boundaries Γ_j, if*

$$\mathcal{H}^d(E \cap B) \leq \mathcal{H}^d(\varphi_1(E) \cap B) + r^d h(r) \tag{5.7}$$

(as in (5.3), but) whenever $B = \overline{B}(x,r) \subset \mathbb{R}^n$ and the $\{\varphi_t\} : E \to \mathbb{R}^n$, $0 \leq t \leq 1$, are such that

$$(t,x) \to \varphi_t(x) : [0,1] \times E \to \mathbb{R}^n \text{ is continuous,} \tag{5.8}$$

$$\varphi_0(x) = x \text{ for } x \in E, \tag{5.9}$$

$$\varphi_t(x) \in \Gamma_j \text{ when } 0 \leq j \leq j_{max} \text{ and } x \in E \cap \Gamma_j, \tag{5.10}$$

$$\varphi_1 \text{ is Lipschitz.} \tag{5.11}$$

as in (4.2)-(4.5), and, for $0 \leq t \leq 1$,

$$\varphi_t(x) = x \quad \text{for } x \in E \setminus B \text{ and } \varphi_t(B) \subset B. \tag{5.12}$$

Thus, when we take $h = 0$, we get that E is a minimizer if it solves the problem of Section 4.e with $E_0 = E$. The local almost minimality property above amounts to a little less than taking infinitely many boundary pieces, equal to the points of $\mathbb{R}^n \setminus U$, and applying the definition with (5.7)-(5.12).

Notice that the fact that we are now allowed to move points of the boundaries is good for us, because it allows more competitors and we can hope to get some direct information at the boundary.

Anyway, the author decided to run all the local regularity proofs he knows, and try to extend them to the boundary. So far (subject to additional proofreading), local Ahlfors-regularity, rectifiability, and limiting theorems seem to be under control, assuming for instance that the boundary pieces are all faces (possibly of all dimensions) of a dyadic grid, or of the image of a dyadic grid by a C^1 diffeomorphism. The next stages for an extension of J. Taylor's result would be to get some control on the blow-up limits of E

at a boundary point (a new, larger list of cones that we also need to determine), and some version of the monotonicity of density, including for balls that are centered near the boundary, but not exactly on the boundary. This last could be problematic (but worth trying).

Quite a few interesting things can happen at the boundary, even when $d = 2$ and the boundary is a smooth curve. See Figure 7 and other pictures of soap films over the Borromean Rings in K. Brakke's website, and a description [Br3] of what may happen when the minimal surface leaves the curve (as in the examples of Figures 2.d and 7).

6. AMNESIC SOLUTIONS OF THE PLATEAU PROBLEM

In this short section we try to clarify some issue about existence results for Plateau problems. To make things simpler, let us consider the case of sliding minimizers, with only one piece of boundary, a smooth closed curve $\Gamma \subset \mathbb{R}^3$. We parameterize Γ by $\gamma : \partial D \to \Gamma$, extend γ to the unit disk, and get an initial set E_0. Then we consider the set $\mathcal{F}(E_0)$ of deformations $E = \varphi_1(E_0)$ of E_0, where the φ_t preserve Γ (see near (4.2)). We would like to find $E \in \mathcal{F}(E_0)$ such that $\mathcal{H}^d(E) = m$, where

$$m = \inf \big\{\mathcal{H}^2(F)\,; F \in \mathcal{F}(E_0)\big\}. \tag{6.1}$$

Notice that we look for an absolute minimizer here, which in principle should make things simpler, but at the same time would forbid us from using solutions of other problems in other classes, such as mass minimizers.

An approach like the one we shall describe here would also make sense in more general situations, or for the Reifenberg problems of Section 2.b, but we shall not elaborate. Also, the reader may want to skip part of the construction of an amnesic solution just below, and go directly to one of the main points of the section, which is what the author means by amnesic solutions and why he thinks they are not entirely satisfactory.

Select a minimizing sequence $\{E_k\}$. That is, $E_k \in \mathcal{F}(E_0)$ and $\mathcal{H}^2(E_k) \leq m + \varepsilon_k$, where ε_k tends to 0 (and we may assume that $\varepsilon_k \leq 2^{-k}$). Let B be a large closed ball that contains Γ and E_0; we can assume that every E_k is contained in B (otherwise, project radially on B and you will get a competitor which is at least as good). Then replace $\{E_k\}$ with some subsequence that tends to a limit E (for the Hausdorff distance on compact sets).

We would like E to be a minimizer for our problem, but if we don't pay attention, this will surely not happen. Indeed, we can easily choose E_k with lots of long and thin hair, with almost no area, but so that E_k is 2^{-k}-dense in B. If we do this, we get $E = B$, which is not even be 2-dimensional.

But there is a way to pick the E_k carefully, so that the limit E represents a more respectable attempt. In [Re1], Reifenberg does this by carefully cutting the hair of an initial E_k. Let us say two words about a slightly different way; we refer to [Da6] for a more detailed account of the strategy and some applications.

We start from our initial $E_k \in \mathcal{F}(E_0)$ and first use a construction of Feuvrier [Fv2] to build a sort of dyadic grid $\mathcal{G}_k$ adapted to E_k. That is, instead of decomposing $\mathbb{R}^3$ into the almost disjoint union of dyadic cubes of size 2^{-l} for some very small l, we use polyhedra instead of cubes. The polyhedra will all have a diameter comparable to some 2^{-l} (the mesh of $\mathcal{G}_k$), and we make sure that $2^{-l} << 2^{-k}$ in the construction. We do not have a lower bound for the mesh, but we shall not need one.

We want to replace E_k with a Federer-Fleming projection on the grid $\mathcal{G}_k$, but since we are afraid of Γ, we shall only do this reasonably far from Γ. Denote by V_k the union of all the polyhedra of $\mathcal{G}_k$ that lie at distance at least 2^{-k} from Γ. On V_k, we replace E_k with its Federer-Fleming projection on the grid. That is, for each polyhedron $Q \subset V_k$, we take an interior point $x_Q \in Q \setminus E_k$, and replace $E_k \cap Q$ with its projection on the boundary ∂Q, where the projection is the radial projection centered at x_Q. Outside of V_k, we change nothing. This gives a new set E'_k.

Then we do the same construction in the 2-dimensional faces of the grid. Just consider the faces F that are contained in the interior of V_k, and such E'_k does not fill the interior of F. That is, such that we can find an interior point $x_F \in F \setminus E'_k$; we then use this point to project $E'_k \cap F$ to the boundary of F. These manipulations are independent, and we don't need to define the projections elsewhere, because now $E'_k \cap V_k$ is contained in the union of the 2-faces. We do this projection for all the faces F where this is possible, and get a new set $E_k^{(2)}$. Notice that $E_k^{(2)} \in \mathcal{F}(E_0)$, because $E_k \in \mathcal{F}(E_0)$, the Federer-Fleming projections are deformations, and we made sure not to move anything near Γ.

By the construction of the adapted grid (and maybe by choosing a little more carefully, by a Fubini argument, the place ∂V_k where we do the interface between the identity and the Federer-Fleming projections), we get that $E_k^{(2)} \leq \mathcal{H}^d(E_k) + 2^{-k} \leq m + 2^{-k+1}$; the general point is that we construct $\mathcal{G}_k$ so that E_k is often very close to 2-faces of the grid, so that the projections will not make $\mathcal{H}^2(E_k)$ much larger.

At last $E_k^{(2)}$ looks nicer inside V_k. That is, let $V'_k \subset V_k$ denote the union of the polyhedra Q of $\mathcal{G}_k$ such that every polyhedron R of the grid that touches Q is contained in V_k (we just remove something like one exterior layer of polyhedra). Then inside V'_k, $E_k^{(2)}$ is composed of entire 2-faces of the grid $\mathcal{G}_k$, plus possibly parts of 1-faces of $\mathcal{G}_k$. We could continue one more step to get rid of the 1-faces that are not entirely covered, but this will not be needed.

Now $E_k^{(2)}$ is not yet our cleaner competitor. We look for deformations F of $E_k^{(2)}$ inside V'_k, that are also composed, inside V'_k, of entire 2-faces of the grid, plus possibly parts of 1-faces of the grid, and for which $\mathcal{H}^2(F)$ is minimal. Here deformation of $E_k^{(2)}$ inside V'_k means that $F = \varphi_1(E_k^{(2)})$, where the φ_t, $0 \leq t \leq 1$, are such that $\varphi_t(x) = x$ for $x \in \mathbb{R}^3 \setminus V'_k$ and $\varphi_t(V'_k) \subset V'_k$, in addition to the usual (5.8), (5.9), and (5.11). Minimizers for this problem exist trivially, because modulo sets of dimension 1, there is only a finite number of sets F. We select a minimizer F_k, which is our cleaner competitor.

A second property of the adapted grids is that we have uniform lower bounds on the angles of the faces of the polyhedra that compose them. Because of this, the set F_k has some regularity inside V'_k: its minimality property implies that it is "Almgren quasiminimal" far from $\mathbb{R}^3 \setminus V'_k$, and this is enough to imply some lowersemicontinuity of $\mathcal{H}^d$, restricted to the sequence $\{F_k\}$. That is, (5.5) holds for the F_k, and for any open set V which is compactly contained in $\mathbb{R}^3 \setminus \Gamma$.

Here we are going a little fast; all these things need proofs, but we just want to give an idea of why it helps to make this complicated construction. We should also say that our restriction to $n = 3$ and $d = 2$ is not needed for this part of the argument; it just makes things more explicit. In other contexts (currents, varifolds), this stage could be less painful, because there exist powerful lowersemicontinuity results that can take care of the analogue of (5.5).

We now take a subsequence so that $\{F_k\}$ converges to a set F, and we would like to say that F is a solution of our problem. First, because of (5.5) (and because $\mathcal{H}^2(\Gamma) = 0$), we really get that

$$\mathcal{H}^2(F) \leq \liminf_{k\to+\infty} \mathcal{H}^2(F_k) \leq \liminf_{k\to+\infty} \mathcal{H}^2(E_k^{(2)}) \leq m, \tag{6.2}$$

so F looks like a good candidate. We can also say more about F in $U = \mathbb{R}^3 \setminus \Gamma$. From the fact that F_k nearly minimizes $\mathcal{H}^2$ in the class $\mathcal{F}(E_0)$, the fact that $\mathcal{F}(E_0)$ is stable under local deformations in compact balls $B \subset U$ (as in (5.2)), and the lowersemicontinuity property (5.5), one can deduce that

$$F \text{ is an Almgren minimal set in } U = \mathbb{R}^3 \setminus \Gamma. \tag{6.3}$$

That is, (5.3) holds, with $h(r) = 0$.

This is not so bad, but let us now say why we may call F an amnesic solution. In order to get a real solution to the initial problem, we would still need to check that $F \in \mathcal{F}(E_0)$, and this would involve finding a deformation $\{\varphi_t\}$ from E_0 to F. This will be hard if we keep the argument as it is suggested, because we have no control on the various deformations $\{\varphi_t^k\}$ associated to the F_k. The situation is not completely hopeless, and one can prove some existence results in similar contexts (but not yet for Plateau's problem with a curve); see [Da6].

Return to our set F. It looks like a solution of Plateau's problem, especially far from Γ; in addition to (6.3), something probably remains from the initial problem that was used to construct F. We know that F lies in the closure of $\mathcal{F}(E_0)$, and in some problems this may be enough information, but otherwise it is hard to say exactly what properties are preserved. In the specific sliding problem here, an intermediate information would be to prove that F is a sliding minimal set, i.e., a solution of the problem above, where we replace E_0 with F itself, but even then we will not be entirely happy, because some information may have been lost in the limit.

Part of the reason for this section was to insist on the difference, in the author's view, between a complete solution of a Plateau problem (like the one above, or some of the more classical variants of Section 2) and an amnesic solution, which is often easier to produce. The difference may be subtle though, because it may turn out that in fact all the amnesic solutions are true solutions. That is, it seems hard to find an example of an amnesic solution with some clearly missing feature.

7. REIFENBERG MINIMIZERS AND SUPPORTS OF SIZE MINIMIZERS ARE SLIDING MINIMAL

In this last section we record the fact that in many cases, the solutions of generalizations of the Reifenberg problem of Section 2.b, as well as the size minimizing current problem of Section 2.c, give sliding minimal sets. Thus boundary regularity results for sliding minimal sets may be used in these contexts as well, even though one may object that they may be easier to obtain directly.

Also notice that just by restricting to the complement U of the boundary set, we see that locally in U, these problems yield locally Almgren minimal sets, for which the interior regularity results of Section 5.a apply. This fact is apparently part of the folklore.

We start with variants of the Reifenberg problem. Choose integers $0 < d < n$, and a notion of homology (including the choice of an abelian group G). Any usual choice will do; we shall just use the homotopy invariance and the fact that ∂ is a natural map.

Also choose a closed boundary set $\Gamma \subset \mathbb{R}^n$, and a collection $\{\gamma_j\}$, $j \in J$, of elements of the homology group $H_{d-1}(\Gamma)$. Let us assume that

$$\mathcal{H}^d(\Gamma) = 0; \tag{7.1}$$

this will allow us to add Γ to our competitors F at no cost, and the definitions will be much easier to apply.

Then let $\mathcal{F}$ denote the class of closed sets $F \subset \mathbb{R}^3$ that contain Γ and for which $i_*(\gamma_j) = 0$ in $H_{d-1}(F)$ for all $j \in J$, where $i_* : H_{d-1}(\Gamma) \to H_{d-1}(F)$ is the natural map coming from the injection $i : \Gamma \to F$.

We shall say that E is Reifenberg-minimal (relative to the choices above) if $E \in \mathcal{F}$ and

$$\mathcal{H}^d(E) = \inf_{F \in \mathcal{F}} \mathcal{H}^d(F) < +\infty. \tag{7.2}$$

Proposition 7.3. *If (7.1) holds and E is Reifenberg-minimal, then it is also a minimal set of dimension d, as in Definition 5.6, with the sliding conditions defined by the unique boundary piece Γ.*

Indeed, let B and the φ_t be as in Definition 5.6; we want to show, as in (5.7), that $\mathcal{H}^d(E \cap B) \leq \mathcal{H}^d(\varphi_1(E) \cap B)$. Since here $\mathcal{H}^d(E) < +\infty$, we can add $\mathcal{H}^d(E \setminus B)$ to both terms, and this simplifies to $\mathcal{H}^d(E) \leq \mathcal{H}^d(\varphi_1(E))$. Then set $F = \Gamma \cup \varphi_1(E)$; by (7.1), it is still enough to prove that $\mathcal{H}^d(E) \leq \mathcal{H}^d(F)$, and for this (and by (7.2)) we just need to prove that F lies in $\mathcal{F}$.

So let $j \in J$ be given, and let us check that $i_*(\gamma_j) = 0$ in $H_{d-1}(F)$, where $i_* : H_{d-1}(\Gamma) \to H_{d-1}(F)$ comes from the injection $i : \Gamma \to F$. Denote by $i_0 : \Gamma \to E$ the initial injection. Since $E \in \mathcal{F}$, $i_{0,*}(\gamma_j) = 0$ in $H_{d-1}(E)$, which means that there exists a chain σ in E such that $\partial\sigma = i_{0,*}(\gamma_j)$.

Apply the mapping φ_1 to this. Set $\sigma_1 = \varphi_{1,*}(\sigma)$; this is a chain in $\varphi_1(E)$, and

$$\partial\sigma_1 = \partial\varphi_{1,*}(\sigma) = \varphi_{1,*}(\partial\sigma) = \varphi_{1,*}(i_{0,*}(\gamma_j)) = (\varphi_1 \circ i_0)_*(\gamma_j) \tag{7.4}$$

because ∂ is natural and by definition of σ.

Notice that for $0 \leq t \leq 1$, φ_t is defined on Γ (because E contains Γ since $E \in \mathcal{F}$), and $\varphi_t(\Gamma) = \varphi_t(\Gamma \cap E) \subset \Gamma$ by (5.10). Call $\varphi : \Gamma \to \Gamma$ the restriction of φ_1 to Γ (to distinguish), and set

$$\gamma_j' = \varphi_*(\gamma_j) \in H_{d-1}(\Gamma). \tag{7.5}$$

Recall that $\varphi_0(x) = x$ on Γ (by (5.9)), so the φ_t, $0 \leq t \leq 1$, provide a homotopy from the identity to φ (among continuous mappings from Γ to Γ). Now we use the invariance of homology under homotopies, and get that $\gamma'_j = \gamma_j$ in $H_{d-1}(\Gamma)$ because φ and the identity induce the same mapping on $H_{d-1}(\Gamma)$.

Next denote by $i_1 : \Gamma \to \varphi_1(E)$ the inclusion, and notice that $\varphi_1 \circ i_0 = i_1 \circ \varphi$ (as a map from Γ to $\varphi_1(E)$). Then (7.4) yields

$$\partial\sigma_1 = (\varphi_1 \circ i_0)_*(\gamma_j) = (i_1 \circ \varphi)_*(\gamma_j) = i_{1,*}(\gamma'_j) = i_{1,*}(\gamma_j) \tag{7.6}$$

in $H_{d-1}(\varphi_1(E))$; hence $i_{1,*}(\gamma_j) = 0$ in $H_{d-1}(\varphi_1(E))$, because it is a boundary.

We compose with a last injection $i' : \varphi_1(E) \to F$, and get that $i_*(\gamma_j) = (i' \circ i_1)_*(\gamma_j) = i'_*(i_{1,*}(\gamma_j)) = 0$ in $H_{d-1}(F)$, as needed.

So $F \in \mathcal{F}$, and E is a minimal set, as promised. The main goal of the detailed computations above was to convince homology nincompoops such as the author that nothing goes wrong with the arrows. □

Remark 7.7. Our assumption (7.1) is not infinitely shocking (for instance, curves in $\mathbb{R}^n$ satisfy this when $d = 2$, and (7.1) still allows lots of room for chains of dimension $d - 1$), but it is not clear that it should really be there.

In the definition of $F \in \mathcal{F}$, it makes sense to take F closed, not necessarily containing Γ, and to say that $i_*(\gamma_j) = 0$ in $H_{d-1}(E \cup \Gamma)$, where i now denotes the injection from Γ to $E \cup \Gamma$.

But if $\mathcal{H}^d(\Gamma) > 0$, we have some problems with the proof above. We used the fact that E contains Γ to say that our φ_t are defined on Γ (and not just $E \cap \Gamma$), and this was used to compute $i_*(\gamma_j)$. We don't want to include Γ if $H^d(\Gamma) > 0$, because $E \cap \Gamma$ should not be minimal (maybe a big part of Γ is just useless).

If for some reason we know that our mapping $(t, x) \to \varphi_t(x)$, from $[0, 1] \times (E \cap \Gamma)$ to Γ, has a continuous extension from $[0, 1] \times \Gamma$ to Γ, we can compute as above.

Otherwise, and if we can choose chains in Γ that represent the γ_j, and whose supports are all contained in a close set $\Gamma' \subset \Gamma$, with $\mathcal{H}^d(\Gamma') = 0$, we can also try to finesse the issue by showing that $E \cup \Gamma'$ is minimal, with the sliding condition associated to Γ, but this is hard if the support of σ is not contained in $E \cup \Sigma'$ (but has a significant piece in the rest of Γ). Let us not try to get a statement.

Next we turn to size-minimizing currents, as in Section 2.c, but again we shall consider a slightly more general problem.

Let $0 < d < n$ be integers, and let Γ be a compact subset of $\mathbb{R}^n$. Also let S be an integral current of dimension $d - 1$ in $\mathbb{R}^n$, with $\partial S = 0$ and with support in Γ. This last just means that $\langle S, \omega \rangle = 0$ when the $(d - 1)$-form ω is supported in $\mathbb{R}^n \setminus \Gamma$.

Next let $\mathcal{S}$ denote the collection of all the integral current S' of dimension $d - 1$ that are supported on Γ and homologous to S on Γ. This last means that there exists a d-dimensional current V, supported on Γ, and such that $\partial V = S - S'$.

Finally denote by $\mathcal{T}$ the set of integral currents T such that $\partial T \in \mathcal{S}$. This is our set of competitors, and we want to minimize $Size(T)$ over $\mathcal{T}$.

Notice that this setting includes the simple example of Section 2. Indeed, let Γ be a smooth orientable surface of dimension $d - 1$, and let S be the current of integration on Σ.

Notice that $\mathcal{S} = \{S\}$, because if V is a d-dimensional current supported on Γ, then $V = 0$. In this case we just want to minimize $Size(T)$ over the solutions of $\partial T = S$, as above.

But we could also take for Γ some 2-dimensional torus along a closed curve, pick a closed loop on Γ, let S be the current of integration over that loop, and $\mathcal{S}$ will allow the current of integration over any Lipschitz deformation of that loop in Γ (see the proof below). This is a way to encode a possibly sliding boundary.

Proposition 7.8. *Let Γ, $\mathcal{S}$, and $\mathcal{T}$ be as above, and let $T \in \mathcal{T}$ be such that*

$$Size(T) = \inf_{R\in\mathcal{T}} Size(R) < +\infty. \tag{7.9}$$

Denote by Z the closed support of ∂T, and assume that

$$\mathcal{H}^d(Z) = 0. \tag{7.10}$$

Also assume that Γ is a Lipschitz neighborhood retract. Let E denote the support of T; then $E \cup Z$ is a minimal set of dimension d, as in Definition 5.6, with the sliding conditions defined by the single boundary piece Γ.

Probably we can prove (7.10) (instead of assuming it) in some cases, but let us not bother. Our conclusion that $E \cup Z$ is minimal, rather than just E, looks unpleasant (the Hausdorff measures are the same, but there is a difference because the boundary constraint (5.10) for the competitors may be different). Under fairly weak assumptions, one can show that if F is minimal (with some sliding boundary conditions), then the closed support of the restriction of $\mathcal{H}^d$ to F is minimal too (with the same sliding boundary conditions), which could be simpler.

Our neighborhood retract assumption is just that there is a small neighborhood W of Γ, and a mapping $h : W \to \Gamma$, which is Lipschitz and such that $h(x) = x$ for $x \in \Gamma$. The reader should not pay too much attention to all these details, which are mostly technical.

So let T be a size minimizer, as in the statement. Recall that T has an expression like (2.6), namely,

$$\langle T, \omega\rangle = \int_A m(x)\, \omega(x) \cdot \tau(x)\, d\mathcal{H}^d(x), \tag{7.11}$$

when ω is a (smooth) d-form, where A is a rectifiable set, $\tau(x)$ is a d-vector that spans the approximate tangent d-plane to A at x, and m is an integer-valued multiplicity function on A, integrable against $\mathbf{1}_E d\mathcal{H}^d$. Also recall that

$$Size(T) = \mathcal{H}^d(A'), \text{ with } A' = \big\{x \in A\,;\, m(x) \neq 0\big\}. \tag{7.12}$$

The Borel support A' may be strictly smaller than E, which is in fact equal to the closure of A', so we need to be careful about $\mathcal{H}^d(E \setminus A')$. We claim that

$$\mathcal{H}^d(E \setminus (Z \cup A')) = 0. \tag{7.13}$$

That is, away from the support Z of ∂T, $E \backslash A'$ has vanishing Hausdorff measure. Let merely sketch the proof here. There is a monotonicity formula, which says that for $x \in E \setminus Z$, the density function $r \to r^{-d}\mathcal{H}^d(A' \cap B(x,r))$ is nondecreasing on $(0, \mathrm{dist}(x,Z))$. This is, classically, obtained by replacing T on any small ball $B(x,r)$ with the cone over its slice on $\partial B(x,r)$, comparing, and then integrating the result. Next let ω_d denote the $\mathcal{H}^d$-measure of the unit ball in $\mathbb{R}^d$; since A' is rectifiable,

$$\lim_{r\to 0} r^{-d}\mathcal{H}^d(A' \cap B(x,r)) = \omega_d \text{ for } \mathcal{H}^d\text{-almost every } x \in A'; \tag{7.14}$$

see [Ma], for instance, for this and the next density results. Hence (by monotonicity)

$$\mathcal{H}^d(A' \cap B(x,r)) \geq \omega_d r^d \text{ for } 0 < r < \mathrm{dist}(x,Z) \tag{7.15}$$

for almost-every $x \in A'$, and hence (take a limit) for every $x \in E$ too. But by a standard density theorem,

$$\lim_{r\to 0} \mathcal{H}^d(A' \cap B(x,r)) = 0 \text{ for } \mathcal{H}^d\text{-almost every } x \in \mathbb{R}^n \setminus A', \tag{7.16}$$

and (7.13) follows by comparing (7.16) and (7.15).

We are now ready to check that $E' = E \cup Z$ is minimal, with sliding conditions associated to Γ. Let B and the φ_t be as in Definition 5.6 (for E'); we want to show that $\mathcal{H}^d(E' \cap B) \leq \mathcal{H}^d(\varphi_1(E') \cap B)$, and since

$$\begin{aligned} \mathcal{H}^d(E') &= \mathcal{H}^d(E \cup Z) = \mathcal{H}^d(E \setminus Z) + \mathcal{H}^d(Z) = \mathcal{H}^d(E \setminus Z) \\ &\leq \mathcal{H}^d(E \setminus (Z \cup A')) + \mathcal{H}^d(A') = \mathcal{H}^d(A') < +\infty \end{aligned} \tag{7.17}$$

by (7.10), (7.13), and (7.12), we can add or subtract $\mathcal{H}^d(E' \setminus B) = \mathcal{H}^d(\varphi_1(E') \setminus B)$, and so we just need to check that

$$\mathcal{H}^d(E') \leq \mathcal{H}^d(\varphi_1(E')). \tag{7.18}$$

We want to build a competitor for T, and logically we use $\varphi = \varphi_1$ to push T forward and set $T_1 = \varphi_\sharp T$. Recall (for instance from the end of 4.1.7 in [Fe1]) that when φ is smooth, $\varphi_\sharp T$ is defined by $\langle \varphi_\sharp T, \omega \rangle = \langle T, \varphi^\sharp \omega \rangle$, for every d-form ω, where at each point x, $(\varphi^\sharp\omega)(x)$ is obtained by applying to $\omega(x)$ the (d-linear version of the) differential of φ at x. Since T is given by (7.11), we get that

$$\langle T_1, \omega \rangle = \langle T, \varphi^\sharp \omega \rangle = \int_A m(x)\, (\varphi^\sharp\omega)(x) \cdot \tau(x)\, d\mathcal{H}^d(x), \tag{7.19}$$

which we may transform into an integral on $\varphi(A)$, with a multiplicity $m(y)$ that is a sum of multiplicities $m(x)$, $x \in \varphi^{-1}(y)$. For general Lipschitz functions φ, $\varphi_\sharp T$ is defined by a limiting argument (see 4.1.14 in [Fe1]), but it is not hard to see that $T_1 = \varphi_\sharp T$ is an integrable current, associated to the rectifiable set $A_1 = \varphi_1(A)$, and whose Borel support

A'_1 is contained in $\varphi_1(A')$; it may be strictly smaller because two different pieces of A' may be sent to the same piece of $\varphi_1(A')$, and the multiplicities may end up canceling. So

$$Size(T_1) \leq \mathcal{H}^d(\varphi_1(A')). \tag{7.20}$$

Our next task is to show that $T_1 \in \mathcal{T}$, because as soon as we do this, the minimality of T will yield

$$\mathcal{H}^d(E') \leq \mathcal{H}^d(A') = Size(T) \leq Size(T_1) \leq \mathcal{H}^d(\varphi_1(A')) \leq \mathcal{H}^d(\varphi_1(E')) \tag{7.21}$$

by (7.17) and because $A' \subset E'$; then (7.18) and the conclusion will follow.

We know from 4.1.14 in [Fe1] that $\partial T_1 = \partial(\varphi_\sharp T) = \varphi_\sharp(\partial T)$. Since T_1 is an integral current, we just need to show that $\partial T_1 \in \mathcal{S}$. The support of ∂T_1 is contained in $\varphi(Z)$. Let us check that

$$Z \subset (E \cup Z) \cap \Gamma = E' \cap \Gamma \ \text{ and } \ \varphi(Z) \subset \Gamma; \tag{7.22}$$

indeed Z, the closed support of ∂T, is contained in Γ because $\partial T \in \mathcal{S}$; the first part now follows from the definition of E', and then $\varphi(Z) = \varphi_1(Z) \subset \Gamma$ by (5.10) for E'.

So ∂T_1 is supported in Γ. We still need to show that ∂T_1 is homologous to S on Γ. Since $T \in \mathcal{T}$, ∂T is homologous to S, and it is enough to show that ∂T_1 is homologous to ∂T.

We would like to use the homotopy, from the origin to $\varphi = \varphi_1$, given by the φ_t, but there is a minor difficulty, because we did not assume $\varphi_t(x)$ to be a Lipschitz function of x and t. This is where we shall need a smoothing argument and our assumption that Γ is a Lipschitz neighborhood retract.

Recall that φ_t is defined on Z, because $Z \subset E' \cap \Gamma$ (by (7.22)), and in addition $\varphi_t(Z) \subset \Gamma$, again by (5.10) as in (7.22).

Let $\varepsilon > 0$ be so small that W contains a 2ε-neighborhood of Γ, and let $\psi : [0,1] \times Z \to \mathbb{R}^n$ be a smooth function such that $|\psi(t,x) - \varphi_t(x)| \leq \varepsilon$ for $(t,x) \in [0,1] \times Z$. Such a function is easy to obtain: first extend $(x,t) \to \varphi_t$, and then smooth it. Next define a homotopy $\{\psi_t\}$, $0 \leq t \leq 3$, from the identity (on Z) to φ by

$$\begin{aligned} &\psi_t(x) = (1-t)x + t\psi(0,x) && \text{for } 0 \leq t \leq 1, \\ &\psi_t(x) = \psi(t-1,x) && \text{for } 1 \leq t \leq 2, \\ &\psi_t(x) = (t-2)\varphi(x) + (3-t)\psi(1,x) && \text{for } 2 \leq t \leq 3. \end{aligned} \tag{7.23}$$

Recall that $\varphi_t(x) \in \Gamma$ for $x \in Z$, so $\mathrm{dist}(\psi_t(x), \Gamma) \leq \varepsilon$ for $(t,x) \in [0,3] \times Z$, and we can define $\xi(t,x) = \xi_t(x) = h(\psi_t(x))$ for $(t,x) \in [0,3] \times Z$, where $h : W \to \Gamma$ is the Lipschitz retraction of the statement, and this defines a Lipschitz homotopy from the identity to φ, with values in Γ. Now the homotopy formula for currents (4.1.9 and 4.1.14 in [Fe1]) says that

$$T_1 - T = \varphi_\sharp T - T = \partial \xi_\sharp([0,3] \times T) + \xi_\sharp([0,3] \times \partial T); \tag{7.24}$$

we take the boundary and get that

$$\partial T_1 - \partial T = \partial \xi_\sharp([0,3] \times \partial T), \tag{7.25}$$

which is fine because $\xi_\sharp([0,3] \times \partial T)$ is a current of degree d supported in Γ. Thus ∂T_1 is homologous to ∂T on Γ, $T \in \mathcal{T}$, and this completes our proof of Proposition 7.8. $\square$

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