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LECTURE 1

What is small cancellation about ?

1. The basic setting

The basic problem tackled by small cancellation theory is the following one.

Problem. *Let G be a group, and $R_1, \ldots, R_n < G$ some subgroups. Give conditions under which you understand the normal subgroup $\langle\langle R_1, \ldots, R_n \rangle\rangle \lhd G$ and the quotient $G/\langle\langle R_1, \ldots, R_n \rangle\rangle$.*

In combinatorial group theory, there are various notions of small cancellation condition for a finite presentation $\langle S | r_1, \ldots, r_k \rangle$. In this case, G is the free group $\langle S \rangle$, and R_i is the cyclic groups $\langle r_i \rangle$. Essentially, these conditions ask that any common subword between two relators has to be short compared to the length of the relators.

More precisely, a *piece* is a word u such that there exist cyclic conjugates $\tilde{r}_1, \tilde{r}_2$ of relators r_{i_1}, r_{i_2} ($i_1 = i_2$ is allowed) such that $\tilde{r}_i = ub_i$ (as concatenation of words) with $b_1 \neq b_2$. Then the $C'(1/6)$ small cancellation condition asks that in this situation, $|u| < \frac{1}{6}|r_1|$ and $|u| < \frac{1}{6}|r_2|$. One can replace $\frac{1}{6}$ by any $\lambda < 1$ to define the $C'(\lambda)$ condition.

Then small cancellation theory says among other things that the group $\langle S | r_1, \ldots, r_k \rangle$ is a hyperbolic group, that it is torsion-free except if some relator is a proper power, and in this case that it is two dimensional (the 2-complex defined by the presentation is aspherical, meaning in some sense that there are no relations among relations).

There are many variants and generalizations of this condition. This started in the 50's with the work of Tartakovskii, Greendlinger, and continued with Lyndon, Schupp, Rips, Olshanskii, and many others [**Tar49, Gre60, LS01, Ol'91a, Rip82**]. Small cancellation theory was generalized to hyperbolic and relatively hyperbolic groups by Olshanskii, Delzant, Champetier, and Osin [**Ol'91b, Del96, Cha94, Osi10**]. An important variant is Gromov's *graphical* small cancellation condition, where the presentation is given by killing the loops of a labelled graph, and one asks for pieces in this graph to be small [**Gro03**]). This lecture will be about *geometric* small cancellation (or *very small* cancellation) as introduced by Delzant and Gromov, and further developped by Arzhantseva-Delzant, Coulon, and Dahmani-Guirardel-Osin [**DG08, AD, Cou11, DGO**].

There are other very interesting small cancellation theories, in particular, Wise's small cancellation theory for special cube complex.

2. Applications of small cancellation

Small cancellation is a large source of example of groups (the following list is very far from being exhaustive !).

Interesting hyperbolic groups.
The Rips construction allows to produce hyperbolic groups (in fact small cancellation groups) that map onto any given finitely presented group with finitely generated kernel. This allows to encode many pathologies of finitely presented groups into hyperbolic groups. For instance, there are hyperbolic groups whose membership problem is not solvable [**Rip82**]. There are many useful variants of this elegant construction.

Dehn fillings.
Given a relatively hyperbolic group with parabolic group P, and $N \lhd P$ a normal subgroup, then if N is *deep enough* (i. e. avoids a finite subset $F \subset P \setminus \{1\}$) given in advance, then P/N embeds in $G/\langle\langle N\rangle\rangle$, and $G/\langle\langle N\rangle\rangle$ is relatively hyperbolic with respect to P/N [**GM08, Osi07**].

Normal subgroups.
Small cancellation allows to understand the structure of the corresponding normal subgroup. For instance, Delzant shows that for any hyperbolic group G there exists n such that for any hyperbolic element $h \in G$, the normal subgroup generated by $\langle h^n\rangle$ is free [**Del96**]. This is because $\langle h^n\rangle$ is a small cancellation subgroup of G, so the small cancellation theorem applies. The same idea shows that if $h \in MCG$ is a pseudo-Anosov element of the mapping class group (or a fully irreducible automorphism of a free group), then for some $n \geq 1$, the normal subgroup generated by $\langle h^n\rangle$ is free and purely pseudo-Anosov [**DGO**]. This uses the fact that MCG act on the curve complex, which is a hyperbolic space [**MM99**], and that $\langle h^n\rangle$ is a small cancellation subgroup when acting on the curve complex. The argument in $\mathrm{Out}(F_n)$ uses the existence of a similar hyperbolic complex for $\mathrm{Out}(F_n)$ [**BF10**].

Many quotients.
Small cancellation theory allows to produce *many* quotients of any non-elementary hyperbolic group G: it is SQ-universal [**Del96, Ol′95**]. This means that if for any countable group A there exists a quotient $G \twoheadrightarrow Q$ in which A embeds (in particular, G has uncountably many non-isomorphic quotients). Small cancellation theory also allows to prove SQ universality of Mapping Class Groups, $\mathrm{Out}(F_n)$, and the Cremona group $Bir(\mathbb{P}^2)$ [**DGO**]. More generally, this applies to groups with *hyperbolically embedded subgroups* [**DGO**] (we will not discuss this notion in this lecture). Abundance of quotients makes it difficult for a group with few quotients to embed in such a group. This idea can be used to prove that lattices in higher rank Lie groups don't embed in mapping class groups, or $\mathrm{Out}(F_n)$ [**DGO, BW11**],

the original proof for mapping class group is due to Kaimanovich-Masur [**KM96**].

Monsters.
The following monsters are (or can be) produced as limits of infinite chains of small cancellation quotients.

(1) Infinite Burnside groups. For n large enough, $r \geq 2$, the free Burnside group $B(r,n) = \langle s_1, \dots, s_r | \forall w, w^n = 1\rangle$ is infinite [**NA68, Iva94, Lys96, DG08**].
(2) Tarski monster. For each prime p large enough, there is an infinite, finitely generated group all whose proper subgroups are cyclic of order p [**Ol′80**].
(3) Osin's monster. There is a finitely generated group not isomorphic to $\mathbb{Z}/2\mathbb{Z}$, with all whose non-trivial elements are conjugate [**Osi10**].
(4) Gromov's monster. This is a finitely generated group that contains a uniformly embedded expander, and which therefore does not uniformly embed in a Hilbert space [**Gro03, AD**]. This gives a counterexample to the strong form of the Baum-Connes conjecture [**HLS02**].

3. Geometric small cancellation

The goal of this lecture is to describe *geometric small cancellation*, introduced by Gromov in [**Gro01**]. The setting of geometric small cancellation is as follows. Let X be a Gromov δ-hyperbolic space, and G acts on X by isometries. Consider $\mathcal{Q} = (Q_i)_{i\in I}$ a family of *almost convex* [1] subspaces of X, and $\mathcal{R} = (R_i)_{i\in I}$ a corresponding family of subgroups such that R_i is a normal subgroup of the stabilizer of R_i. This data should be G-invariant: G acts on I so that $Q_{gi} = gQ_i$, and $R_{gi} = gR_ig^{-1}$. Let us call such a data a *moving family* $\mathcal{F}$.

Small cancellation hypothesis asks for a large injectivity radius and a small fellow travelling length. The injectivity radius measures the minimal displacement of all non-trivial elements of all R_i's:

$$\mathrm{inj}(\mathcal{F}) = \inf\{d(x,gx) | i \in I, x \in Q_i, g \in R_i \setminus \{1\}\}.$$

The fellow travelling constant between two subspaces Q_i, Q_j measures for how long they remain at a bounded distance from each other. Technically, $\Delta(Q_i,Q_j) = \mathrm{diam} Q_i^{+20\delta} \cap Q_j^{+20\delta}$, where Q^{+d} denotes the closed d-neighbourhood of Q (ie the set of points at distance at most d from Q). Because Q_i, Q_j are almost convex in a hyperbolic space, any point of Q_i that is far from $Q_i^{+20\delta} \cap Q_j^{+20\delta}$ is far from Q_j, so this really measures what we want. The fellow travelling constant of $\mathcal{F}$ is defined by

$$\Delta(\mathcal{F}) = \sup_{i\neq j} \Delta(Q_i, Q_j).$$

[1] i. e. $\forall x, y \in Q_i \exists x', y' \in Q_i \; s.t. \; d(x,x') \leq 8\delta, d(y,y') \leq 8\delta, [x,x'] \cup [x',y'] \cup [y',y] \subset Q_i$

Definition 1.1. *The moving family $\mathcal{F}$ satifies the (A,ε)-small cancellation condition if*

(1) *large injectivity radius:* $\mathrm{inj}(\mathcal{F}) \geq A\delta$
(2) *small fellow travelling compared to injectivity radius:* $\Delta(\mathcal{F}) \leq \varepsilon \mathrm{inj}(\mathcal{F})$.

Remark 1.2.
- It is convenient to say that some subgroup $H < G$ satisfies the (A,ε)-small cancellation condition if the family $\mathcal{G}$ of all conjugates of H together with a suitable family of subspaces of X, makes a small cancellation moving family.
- The (A,ε)-small cancellation hypothesis (for A large enough) implies that each R_i is torsion-free, because every element of $R_i \setminus \{1\}$ is hyperbolic.
- It is often convenient to take $I = \mathcal{Q}$, and to view $\mathcal{G}$ as a group attached to each subspace in $\mathcal{Q}$: $\mathcal{G} = (R_Q)_{Q\in\mathcal{Q}}$, or conversely, to take $I = \mathcal{G}$ and to view $\mathcal{Q}$ as a space attaced to each group in $\mathcal{G}$: $\mathcal{Q} = (Q_H)_{H\in\mathcal{G}}$.

Remark 1.3. Side remark on almost convexity: $Q \subset X$ is *almost convex* if for all $x, y \in Q$, there exist $x', y' \in Q$ such that $d(x,x') \leq 8\delta$, $d(y,y') \leq 8\delta$, and $[x',y'] \subset Q$. It follows that the path metric d_Q on Q induced by the metric d_X of X is close to d_X: for all $x, y \in Q$, $d_X(x,y) \leq d_Q(x,y) \leq d_X(x,y) + 32\delta$.

Recall that $Q \subset X$ is K-*quasiconvex* if for all $x, y \in Q$, any geodesic $[x,y]$ is contained in the K-neighborhood of Q. This notion is weaker as it does say anything on d_Q (Q might be even be disconnected). If Q is K-quasiconvex, then for all $r \geq K$, Q^{+r} is almost convex (recall that Q^{+r} is the r-neighborhood of Q). Also note that an almost convex subset is 8δ-quasiconvex.

Relation with classical small cancellation

These small cancellation hypotheses (almost) include the classical small cancellation condition $C'(\lambda)$ (but with λ not explicit and far from optimal): G is the free group acting on its Cayley graph, $(R_i)_{i\in I}$ is the family of cyclic groups generated by the conjugates of the relators, and Q_i is the family of their axes. The first assumption is empty, and the second one is (a strenghtening of) the $C'(\varepsilon)$ small cancellation assumption.

Graphical small cancellation also fits in this context, in which case the groups R_i's need not be cyclic any more, they are conjugates of the subgroups of G defined by some labelled subgraphs.

The small cancellation Theorem

Theorem 1.4 (Small cancellation theorem)**.** *There exists A_0, ε_0 such that if $\mathcal{F}$ satisfies the (A,ε)-small cancellation hypothesis with $A \geq A_0, \varepsilon \leq \varepsilon_0$, then*

(1) $\langle\langle R_i | i \in I\rangle\rangle$ *is a free product of a subfamily of the R_i's,*
(2) $\mathrm{Stab}(Q_i)/R_i$ *embeds in* $G/\langle\langle R_i | i \in I\rangle\rangle$

(3) *small elements survive: there are constants* C_1, C_2 *such that any non-trivial element whose translation length is at most* $\mathrm{inj}(R)(1 - \max\{C_1\varepsilon, \frac{C_2}{A}\})$ *is not killed in* $G/ngrpR_i$.
(4) $G/\langle\langle R_i | i \in I\rangle\rangle$ *acts on a suitable hyperbolic space.*

In the setting of $C'(1/6)$ small cancellation, the groups R_i are conjugates of the cyclic groups generated by relators. Thus, if Q_i is the axis of some conjugate r of a relator, then $\mathrm{Stab}(Q_i)$ is $\langle\sqrt{r}\rangle$: $\mathrm{Stab}(Q_i)/R_i$ is trivial if r is not a proper power, and $\mathrm{Stab}(Q_i)/R_i \simeq \mathbb{Z}/k\mathbb{Z}$ otherwise if r is the k-th power of its primitive root.

It is difficult to state properties in this generality of the *suitable hyperbolic space* X' right now. But one of its main properties is that X' has a controlled geometry, including a controlled hyperbolicity constant. Also, if we start with X a proper cocompact hyperbolic space, then X' will also be proper cocompact if each $\mathrm{Stab}(Q_i)/R_i$ is finite, and I/G is finite. One of the main goal of this lecture will be to describe it.

LECTURE 2

Applying the small cancellation theorem

Assume that we have a group G acting on a space X. We are going to see how to produce small cancellation moving families, and how to use them.

1. When the theorem does not apply

Given a group G acting on a hyperbolic space, small cancellation families may very well not exist, except for trivial ones.

A first type of silly example is the solvable Baumslag-Solitar group $BS(1,n) = \langle a, t | tat^{-1} = a^n \rangle$, $n > 1$. This group acts on the Bass-Serre tree of the underlying HNN extension, but there is no small cancellation family.

Exercise 2.1. *Prove this assertion. Note that any two hyperbolic elements of $BS(1,n)$ share a half axis.*

Since we think as small cancellation families as a way to produce interesting quotients, one major obstruction to the existence of interesting such families occurs if G has very few quotients, for instance if it is simple. This the case for the simple group $G = \mathrm{Isom}^+(\mathbb{H}^n)$ for example. If we restrict to finitely generated groups, an irreducible lattice in $\mathrm{Isom}^+(\mathbb{H}^2) \times \mathrm{Isom}^+(\mathbb{H}^2)$ acts on $\mathbb{H}^2$ (in two ways), but any non-trivial quotient is finite by Margulis normal subgroup theorem. Similar, but more sophisticated examples include Burger-Moses simple group, a lattice in the product of two trees, viewed as a group acting on a tree, or some Kac-Moody groups when the twin buildings are hyperbolic.

Exercise 2.2. *What are* trivial *small cancellation families ? Here are examples:*

1. *the empty family.*
2. *Take $\mathcal{Q} = \{X\}$ consisting of the single subspace X, and $\mathcal{G} = \{N\}$ consists of a single normal subgroup of G, (including the case $N = \{1\}$ and $N = G$).*
3. *Another way is to take $\mathcal{Q}$ a G-invariant family of subspaces that satisfy the fellow travelling condition (for instance bounded subspaces), and take $(R_i)_{i\in\mathcal{Q}}$ a copy of the trivial group for each subspace.*

More generally, a trivial small cancellation family is a family such that $R_i = \{1\}$ except for at most one i.

Prove that if G is simple, then there exists A, ε such that any (A,ε)-small cancellation moving family is trivial in the above sense.

Hint: Consider a small cancellation moving family $(Q_i)_{i\in I}$, $(R_i)_{i\in I}$. Since G is simple, $R_i = \mathrm{Stab}(Q_i)$ by the small cancellation Theorem. If $h_1 \in R_{i_1} \setminus \{1\}$, $h_2 \in R_{i_2}$ for $i_1 \neq i_2$, prove that $h_1^N h_2^N$ satisfies the WPD property below, contradicting that G is simple.

2. Weak proper discontinuity

In hyperbolic groups, the easiest small cancellation family consists of the conjugates of a suitable power of a hyperbolic element. The proof is based on the properness of the action. In fact, a weaker notion, due to Bestvina-Fujiwara is sufficient.

Preliminaries about quasi-axes.
To make many statements simpler, we will always assume that X is a metric graph, all whose edges have the same length.

Define $[g] = \inf\{d(x,gx) | x \in G\}$ the translation length of g. Recall that g is *hyperbolic* if the orbit map $\mathbb{Z} \to X$ defined by $i \mapsto g^i x$ is a quasi-isometric embedding (for some x, equivalently for any x). This occurs if and only the stable norm of g, defined as $\|g\| = \lim_{i\to\infty} \frac{1}{i} d(x, g^i x)$ is strictly positive (the limit exists by subadditivity, and does not depend on x).

These are closely related as $[g] - 16\delta \leq \|g\| \leq [g]$ [**CDP90**, 10.6.4]. In particular, if $[g] > 16\delta$ then g is hyperbolic.

Consider a hyperbolic element g. Define $C_g = \{x | d(x,gx) = [g]\}$ the characteristic set of g (a non-empty set since X is a graph). We want to say that if $[g]$ is large enough, C_g is a good quasi-axis: it is close to be a bi-infinite line (with constants independant of g). Given $x \in C_g$, consider $l = l_{x,g} = \cup_{i\in\mathbb{Z}} [g^i x, g^{i+1} x]$. One easily checks that if $y \in [g^i x, g^{i+1} x]$, then $d(y, gy) = [g]$, so this bi-infinite path is contained in C_g, and it is $[g]$-local geodesic. By stability of 100δ-local geodesics, there exists a constant C such that if $[g] \geq 100\delta$, $l_{x,g}$ and $l_{y,g}$ are at Hausdorff distance at most C. Similar arguments show that if $[g] \geq 100\delta$, for any k, C_g and C_{g^k} are at Hausdorff distance at most C for some constant C independant of g and k.

In this sense, if $[g] \geq 100\delta$, C_g (or $l_{x,g}$), is a good quasi-axis for g. If g is hyperbolic with $[g] \leq 100\delta$, then there is k such $[g^k] \geq 100\delta$, and a better quasi-axis for g would be C_{g^k}. Finally, we want the quasi-axis to be almost convex. One easily checks that C_{g^k} is $2C + 4\delta$-quasiconvex. Thus, we define the quasi-axis of g as $A_g = C_{g^k}^{+2C+4\delta}$ where k is the smallest power of g such that $[g^k] \geq 100\delta$.

Lemma 2.3. *There exists a constant C such that if $[g] \geq 100\delta$, then for all $x \in A_g$ and all $i \in \mathbb{Z}$, $i\|g\| \leq d(x, g^i x) \leq i\|g\| + C$.*

This follows from the fact that quasi-axes A_g and A_{g^i} are at bounded Hausdorff distance, and from the inequality $[g^i] - 16\delta \leq \|g^i\| = i\|g\| \leq [g^i]$.

Weak proper discontinuity

Definition 2.4. *Say that $g \in G$, acting hyperbolically on X, satisfies WPD (*weak proper discontinuity*) if there exists r_0 such that for all pair of elements $x, y \in A_g$ at distance at least r_0, the set of all elements $a \in G$ that move both x and y by at most 100δ is finite:*

$$\#\{a \in G | d(x, ax) \leq 100\delta, d(y, ay) \leq 100\delta\} < \infty.$$

Obviously, if the action of G on X is proper, then any hyperbolic element g is WPD. In particular, any element of infinite order in a hyperbolic group satisfies WPD.

Here is an equivalent definition:

Definition 2.5. *g satisfies WPD if for all l, there exists r_l such that for all pair of elements $x, y \in A_g$ at distance at least r_l, the set of all elements $a \in G$ that move both x and y by at most l is finite:*

$$\#\{a \in G | d(x, ax) \leq l, d(y, ay) \leq l\} < \infty.$$

Exercise 2.6. *Prove that the definitions are equivalent.*

A lot of interesting groups have such elements.

Example 2.7.
(1) If G is hyperbolic or relatively hyperbolic, then any hyperbolic element satisfies WPD.
(2) If G is a right-angled Artin group that is not a direct product, then G acts on a tree in which there is a WPD element.
(3) If G is the mapping class group acting on the curve complex, any pseudo-anosov element is a hyperbolic element satisfying WPD [**BF07**].
(4) If $G = \mathrm{Out}(F_n)$, or G is the Cremona group $Bir(\mathbb{P}^2)$, then G acts on a hyperbolic space with a WPD element [**BF10, CL**].
(5) If G acts propertly on $CAT(0)$ space Y, and if g is a rank one hyperbolic element (the axis does not bound a half plane), there is a WPD element in some action of G on some hyperbolic space [**Sis11**]

Proposition 2.8. *Assume that g satisfies the WPD property.*

Then for all A, ε, there exists N such that the moving family consisting of the conjugates of $\langle g^N \rangle$, together with their quasi-axes, satisfies the (A, ε)-small cancellation condition.

Corollary 2.9. *If G contains a hyperbolic element with the WPD property, then G is not simple.*

Exercise 2.10. *Prove the proposition.*

Hints: First prove that there exists a constant Δ such that if A_g fellow travels with $A_{hgh^{-1}} = hA_g$ on a distance at least Δ, then hA_g is at finite hausdorff distance from A_g. For this, show that if the fellow-travelling distance $\Delta(A_g, A_{hgh^{-1}})$ is large, there there is a large portion of A_g that is moved a bounded amount by $g^i.hg^{\pm i}h^{-1}$ for many i's. Then apply WPD to deduce that h commutes with some power of g, hence maps A_g at finite Hausdorff distance. To conclude that this gives a small cancellation family, prove that the subgroup $E(g) = \{h \in G | d_H(h.A_g, A_g) < \infty\}$ is virtually cyclic. Now take N such that $[g^N]$ is large compared to $\Delta(A_g, A_{hgh^{-1}})$, and such that $\langle g^N \rangle \lhd E(g)$.

Remark 2.11 (Remark about torsion)**.** If G is not torsion-free, choosing N such that $[g^N]$ is large compared to $\Delta(A_g, A_{hgh^{-1}})$ is not sufficient, as shows the exercise below.

Exercise 2.12. *Let F be a finite group, $\varphi : F \to F$ a non-trivial automorphism, of order d. Let $G = (\mathbb{Z} \ltimes_\varphi F) * \mathbb{Z} =< a, b, F | \forall f \in F,\ afa^{-1} = \varphi(f) >$.*

Show that the family of conjugates of $\langle t^k \rangle$ does not satisfy any small cancellation condition if k is a multiple of d.

3. SQ-universality

We will greatly improve the result saying that G is not simple.

Definition 2.13. *A group G is* SQ-universal[1] *if for any countable group A, there exists a quotient Q of G in which A embeds.*

Since there are uncountably many 2-generated groups, and since a given finitely generated group has only countably many 2-generated subgroups, a SQ-universal group has uncountably many non-isomorphic quotients.

Theorem 2.14. *If G is not virtually cyclic, acts on a hyperbolic space X, and contains a WPD hyperbolic element, then G is SQ-universal.*

First step in the proof is to produce a free subgroup satisfying the small cancellation condition.

Proposition 2.15. *If G acts on a hyperbolic space X, and contains a WPD hyperbolic element h, and if G is not virtually cyclic, then for all (A, ε), there exists a two-generated free group $H < G$ satisfying the (A, ε) small cancellation condition, such that the stabilizer of the subspace Q_H corresponding to H in the moving family is* $\mathrm{Stab}(Q_H) = H \times F$ *for a finite subgroup F.*

Exercise 2.16. *Prove the proposition if G is torsion-free. Hint: prove that there is some conjugate k of h such that $\Delta(A_h, A_k)$ is finite. Replace h, k by large powers so that their translation length is large compared to $\Delta(A_h, A_k)$. The consider something like $a = h^{1000}k^{1000}h^{1001}k^{1001} \dots h^{1999}k^{1999}$, and $b = h^{2000}k^{2000}h^{2001}k^{2001} \dots h^{2999}k^{2999}$, and $H = \langle a, b \rangle$.*

Note that such H might fail witout further hypothesis in presence of torsion (there may be some element of finite order that almost fixes only half of the axis of a, so that $\Delta(A_a, A_{cac^{-1}})$ might be large.

[1] SQ stands for subquotient

Note that the above proof works if $E(h) = \mathbb{Z} \times F$ for some finite subgroup F, and if $F < E(k)$ for all conjugate k of h. To prove the proposition in full generality, construct h such that this holds.

Proof of the Theorem. It is a classical result that every countable group embeds in a two generated group [**LS01**]. Thus it is enough to prove that any two-generated group A embeds in some quotient of G. Let $F_2 \to A$ be an epimorphism, and N be its kernel. Let $H < G$ be a 2-generated free group satisfying the small cancellation hypothesis as in the proposition, and let $Q_H \subset X$ be the corresponding subspace in the moving family. View N as a normal subgroup of H.

We claim that N also satisfies the (A, ε)-small cancellation. Indeed, we assign the group N^g to the subspace $g.Q_H$. For this to be consistent, we need that N be normal in $\mathrm{Stab}(Q_H)$. This is true because $\mathrm{Stab}(Q_H) = H \times F$.

Applying the small cancellation theorem, we see that $\mathrm{Stab}(Q_H)/N$ embeds in $G/\langle\langle N \rangle\rangle$. It follows that $A \simeq H/N$ embeds in $G/\langle\langle N \rangle\rangle$. □

4. Dehn Fillings

Let G be a relatively hyperbolic group, relative to the parabolic group P (we assume that there is one parabolic only for notational simplicity). By definition, this means that G acts properly on a proper hyperbolic space X, such that there is a G-orbit of disjoint almost-convex horoballs $\mathcal{Q}$, such that the stabilizer of each horoball is a conjugate of P, and such that G acts cocompactly on the complement of the horoballs.

In fact, we can additionnally assume that the distance between any two distinct horoballs is as large as we want, in particular, $> 40\delta$. This means that the fellow travelling constant for $\mathcal{Q}$ is zero ! Given $R_0 \lhd P$, the family $\mathcal{G}$ of conjugates of R_0 defines a moving family $\mathcal{F} = (\mathcal{G}, \mathcal{Q})$.

Now for the small cancellation theorem to apply, we need the injectivity radius to be large. This clearly fails since elements of R_0 are parabolic, so their translation length is small. However, the following variant of the small cancellation theorem holds.

In the small cancellation hypothesis, replace the *large injectivity radius* (asking that all points of Q_i are moved a lot by each $g \in R_i \setminus \{1\}$), by the following one asking this only on the boundary of Q_i:

Theorem 2.17. *Consider a moving family on a hyperbolic space with the notations above. Assume that $\Delta(\mathcal{Q}) = 0$ (the Q_i's don't come close to each other), and that*

$$\forall i \in I, \forall g \in R_i \setminus \{1\}, \forall x \in \partial Q_i, d(x, gx) > A.$$

Then the conclusion of the small cancellation theorem still holds.

Let $*$ be a base point on the horosphere ∂Q preserved by P, and since P acts cocompactly on ∂Q, consider R such that $P.B(*, R) \supset \partial Q$. Now if R_0 avoids the finite set $S \subset P$ of all elements $g \in P \setminus \{1\}$ such that $d(*, g*) \leq 2R + A$ then R_0 satisfies this new assumption.

We thus get the Dehn filling theorem:

Theorem 2.18 ([**Osi07, GM08**]). *Let G be hyperbolic relative to P. Then there exists a finite set $S \subset P \setminus \{1\}$ such that for all $R_0 \lhd P$ avoiding S,*

- *P/R_0 embeds in $G/\langle\langle R_0\rangle\rangle$*
- *$G/\langle R_0\rangle$ is hyperbolic relative to P/R_0. In particular, if R_0 has finite index in P, then $G/\langle\langle R_0\rangle\rangle$ is hyperbolic.*

In fact, the proof allows to control the hyperbolicity constant of the hyperbolic space on which the quotient group acts. This is can be a very useful property.

LECTURE 3

Rotating families

1. Road-map of the proof of the small cancellation theorem

The goal of these lectures is to prove the geometric small cancellation theorem, and some of its applications.

There are essentially two main steps in the proof, each step involving only one of the two main hypotheses.

(1) Construct from the space X and the subspaces Q_i a cone-off $\dot{X}$ by coning all the subspaces Q_i, and prove its hyperbolicity. This step does not involve the groups R_i, so this is independent of the large injectivity radius hypothesis.

(2) Because the spaces Q_i have been coned, each subgroup R_i fixes a point in $\dot{X}$, and thus looks like a rotation. Our moving family becomes a *rotating family.* One studies the normal group $N = \langle\langle (R_i)_{i\in I} \rangle\rangle$ via its action on the cone-off. This is where the *large injectivity radius* assumption is used: it translates into a so-called *very-rotating* assumption. The group G/N naturally acts on the quotient space $\dot{X}/N$, and the hyperbolicity of the quotient space $\dot{X}/N$ is then easy to deduce.

2. Definitions

Assume that we are given a group G and a set of relators $r_1, \dots r_k \in G$. In what follows we assume that G acts on a hyperbolic space X and that each r_i fixes a point $x_i \in X$, in a rather special way, and we want to deduce information about the quotient group $G/\langle\langle r_1, \dots, r_k \rangle\rangle$, and about the quotient space $X/\langle\langle r_1, \dots r_k \rangle\rangle$. This will be done by studying the normal subgroup itself $\langle\langle r_1, \dots r_k \rangle\rangle$.

The situation is formalized below in the definition of a rotating family. Essentially, this means that the G-orbits of the point x_i remain far from each other, and every power r_i^k ($k \neq 0$) rotates by a very large angle.

Definition 3.1. *A rotating family $\mathcal{C} = (C, \{G_c, c \in C\})$ consists of a subset $C \subset X$, and a collection $\{G_c, c \in C\}$ of subgroups of G such that each G_c fixes c, and is G-invariant in the following sense: C is G-invariant, and $\forall g \in G \, \forall c \in C, G_{gc} = gG_cg^{-1}$.*

The set C is called the set of apices of the family, and the groups G_c are called the rotation subgroups of the family. Note that G_c is a normal subgroup of the stabilizer $\mathrm{Stab}(c)$ of $c \in C$.

One says that C (or $\mathcal{C}$) is *ρ-separated* if any two distinct apices are at distance at least ρ.

Definition 3.2 (Very rotating condition: local version)**.** *We say that the rotating family is* very rotating *if for all $c \in C, g \in G_c \setminus \{1\}$, and all $x, y \in X$ with $20\delta \leq d(x,c), d(y,c) \leq 40\delta$, and $(gx, y) < d(x,c) + d(c,y) - 10\delta$, then any geodesic between x and y contains c.*

The very rotating condition can be thought as a coarse version of a condition for the action on the link in a CAT(-1) space. Let $x \in X \setminus \{c\}$, and v in the link at c, the initial tangent vector of $[c,x]$. If $g \in G_c \setminus 1$ moves v by an angle at least π (in the natural metric of the link), then $[x,c] \cup [c,gx]$ is a geodesic. By uniqueness of geodesics, any geodesic between x and gx has to pass through c. If g moves v by an even larger angle, then the very rotating condition holds.

The very rotation condition is local around an apex. It implies the following global condition that implies that G_c acts freely on $X \setminus B(c, 24\delta)$.

Lemma 3.3 (Very rotating condition: global version)**.** *Consider $x_1, x_2 \in X$ such that there exists $q_i \in [c, x_i]$ with $d(q_i, c) \geq 20\delta$ and $h \in G_c \setminus \{1\}$, such that $d(q_1, hq_2) \leq d(q_1, c) + d(c, q_2) - 40\delta$. Then any geodesic between x_1 and x_2 contains c. In particular, for any choice of geodesics $[x_1, c]$, $[c, x_2]$, their concatenation $[x_1, c] \cup [c, x_2]$ is geodesic.*

Note that the very rotating conditions implies that G_c acts freely, and discretely on $X \setminus B(c, 25\delta)$.

Proof. Let $q'_i \in [c, q_i]$ at distance 20δ from c. By thinness of the triangle c, q_1, q_2, $d(q'_1, q'_2) \leq 4\delta$. The very rotating hypothesis says that $d(q'_1, c) + d(c, q'_2) = d(q'_1, q'_2)$. By thinness of the triangle x_1, c, x_2, any geodesic $[x_1, x_2]$ has to contain points q''_1, q''_2 at distance at most 4δ from q'_1, q'_2. In particular, $d(q'_1, hq'_2) \leq 12\delta$. Applying the local very rotating condition to q'_1, q'_2 shows that $[q'_1, q'_2]$ contains c, and so does $[x_1, x_2]$. □

3. Statements

Now we state some results describing the structure of the normal subgroup generated by the rotating family.

Theorem 3.4. *Let $(G_c)_{c \in C}$ be a ρ-separated very rotating family, with ρ large enough. Let $N = \langle\langle G_c | c \in C \rangle\rangle$. Then*

(1) *$\mathrm{Stab}(c)/G_c$ embeds in G/N. More generally, if $[g] < \rho$ and $g \in N$ then $g \in G_c$ for some G_c.*
(2) *There exists a subset $S \subset C$ such that N is the free product of the collection of $(G_c)_{c \in S}$.*
(3) *X/N is hyperbolic.*

Remark 3.5. $\rho \geq 100\delta$ is enough for the first two assertion. For the last one, ρ needs to be larger (see below).

The first assertion follows from the following form of the Greendlinger lemma which we are going to prove together with the theorem:

Theorem 3.6. *(Greendlinger lemma) Every element g in N that does not lie in any G_c is loxodromic in X, it has an invariant geodesic line l, this line contains a point $c \in C$ (in fact at least two) such that there is a shortening element for g in G_c (as defined below).*

A *shortening element* at $c \in l$ is an element $r \in G_c \setminus \{1\}$ such that there exists $q_1, q_2 \in l$, such that $24\delta \leq d(q_1, q_2) \leq 50\delta$, and $d(q_1, rq_2) \leq 20\delta$ Up to changing r to r^{-1} we can assume that q_1, q_2, gq_1 are aligned in this order in l. This implies that $d(q_1, rgq_1) \leq d(q_1, rq_2) + d(rq_2, rgq_1) \leq 20\delta + d(q_2, gq_1) = 20\delta + d(q_1, gq_1) - d(q_1, q_2) \leq [g] - 28\delta$, so $[rg] \leq [g] - 28\delta$. Thus, Greendlingers's lemma gives a form of (relative) linear isoperimetric inequality: every element g of N is the product of at most $[g]/36\delta$ elements of the rotating subgroups.

4. Proof

The proof is by an iterative process, described by Gromov in [**Gro01**]. To perform this iterative process, we construct a subset called windmill with a set of properties that remain true inductively.

Initially, we choose $c \in \mathcal{C}$, and we take as initial windmill W_0 a metric ball centered at c, and containing no other point of $\mathcal{C}$ in its 40δ-neighbourhood (for instance $W_0 = B(c, \rho/2)$). Corresponding to W there is a group G_W and the initial group is G_c. We will grow W and G_W, and at the limit of this infinite process, W will exhaust all X, and G_W will exhaust N.

Definition 3.7 (Windmill)**.** *A* windmill *is a subset $W \subset X$ satisfying the following axioms.*

1. *W is almost convex,*
2. *$W^{+40\delta} \cap C = W \cap C \neq \emptyset$,*
3. *the group G_W generated by $\bigcup_{c \in W \cap C} G_c$ preserves W,*
4. *there exists a subset $S_W \subset W \cap C$ such that G_W is the free product $*_{c \in S_W} G_c$.*
5. *(Greendlinger) every elliptic element of G_W lies in some $G_c, c \in W \cap C$, other elements of G_c have an invariant geodesic line l such that $l \cap C$ contains at least two g-orbits of points at which there is a shortening element.*

Proposition 3.8 (Inductive procedure)**.** *For any windmill W, there exists a windmill W' containing $W^{+10\delta}$ and $W^{+60\delta} \cap C$, and such that $G_{W'} = G_W * (*_{x \in S} G_c)$ for some $S \subset \mathcal{C} \cap (W' \setminus W)$.*

The proposition immediately implies the Theorem: define W_i from W_{i-1} applying the proposition. Since $\cup_i W_i = X$, $\cup_i G_{W_i} = N$. Greendlinger lemma follows, and so does the fact that N is a free product.

Proof. If $W^{+60\delta}$ does not intersect C, we just inflate W by taking $W' = W^{+10\delta}$. Otherwise, we construct W' in several steps.

Step 1: let $C_1 = C \cap W^{+60\delta}$. For each $c \in C_1$ choose a projection p_c of c in W, and a geodesic $[c, p_c]$. This choice can be done G_W-equivariantly because G_W acts freely on C_1 (by Greendlinger hypothesis, and the very rotating assumption). Define $W_1 = W \cup \bigcup_{c\in C_1}[c, p_c]$. Almost convexity of W_1 easily shows that W_1 is 12δ-quasiconvex.

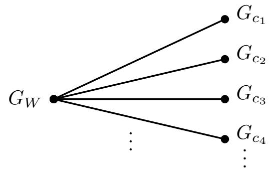
Step 2: The group $G' = G_{W_1}$ is the group generated by G_W and by $\{G_c | c \in C_1\}$. We define $W_2 = G_{W_1} W_1$. Abstract non-sense shows that $G' = G_{W_1} = G_{W_2}$.

Step 3: we take $W' = W_2^{+12\delta} = G'(W_1^{+12\delta})$. We have $W' \cap C = W_2 \cap C$, and even $W'^{+40\delta} \cap C = W_2 \cap C$ because there is no element of C at distance $\leq 52\delta$ from a segment $[c, p_c]$ because C is ρ-separated with $\rho > 92\delta$. In particular $G_{W'} = G_{W_2} = G'$, and Axiom 2 of windmill holds.

To prove that W' works, we first look at how W_1 is rotated around some $c \in C_1$. So take $h \in G_c \setminus \{1\}$, and look at $W_1 \cup hW_1$. Consider $x \in W_1 \setminus [c, p_c]$ and $y \in h(W_1 \setminus [c, p_c])$. Consider $q_x \in [c, x]$ and $q'_x \in [c, p_c]$, both at distance 24δ from c. Define $q_y \in [c, y]$ and $q'_y \in [c, hp_c]$ similarly. By thinness of the triangle c, x, p_c, $d(q_x, q'_x) \leq 4\delta$ (otherwise, there would be some $q''_x \in [p, x]$ such that $d(q_x, q''_x) \leq 4\delta$, and by 12δ-quasiconvexity of W_1, $d(q_x, W_1) \leq 16\delta$, so $d(c, W_1) \leq 40\delta$ a contradiction). Similarly, $d(q_y, q'_y) \leq 4\delta$, and since $h^{-1}q'_y = q'_x$, $d(q_x, h^{-1}q_y) \leq 8\delta$, the global very rotating condition implies that any geodesic from x to y contains c. By 12δ-quasiconvexity of W_1 and hW_1, we get that $[x, y]$ is in the 12δ-neighbourhood of $W_1 \cup hW_1$ so $W_1 \cup hW_1$ is 12δ-quasiconvex. We note that h is a shortening element of $[x, y]$ at c. We have proved:

Lemma 3.9 (Key lemma). *Fix $c \in C_1$, $h \in G_c \setminus 1$, $x \in W_1 \setminus [c, p_c]$ and $y \in h(W_1 \setminus [c, p_c])$. Then any geodesic from x to y contains c and h is a shortening element of $[x, y]$ at c.*

Now we prove that $G'.W_1$ has a treelike structure. Let Γ be the graph with vertex set $V = V_C \cup V_W$ where the elements of V_C are the apices gc for $c \in C_1$, $g \in G'$, and the elements of V_W are the sets gW for $g \in G'$. For any $g \in G'$ and $c \in C_1$, we put an edge between gW and gc. We consider $\tilde{C}_1 \subset C_1$ a set of representatives of the G_{W_1}-orbits. We define the free product $\hat{G} = G_W * (*_{c\in\tilde{C}_1} G_c)$, viewed as a tree of groups with trivial edge groups as in the figure below.



Let T the corresponding Bass-Serre tree. We have a natural morphism $\varphi : \hat{G} \to G'$ and a natural map $f : T \to \Gamma$ that is φ-equivariant.

We prove that φ and f are isomorphisms. Assume for instance that f identifies two points $c \neq c' \in T$ corresponding to apices in $G'C_1$. Consider the segment $[c, c'] \subset T$, and let $c = c_1, c_2, \dots, c_n = c'$ be the points in $[c, c'] \cap V_C$. Consider the path γ in X defined as a concatenation of geodesics $[c_1, c_2], [c_2, c_3], ...[c_{n-1}, c_n]$. The key lemma (applied around c_2) shows that any geodesic from c_1 to c_3 contains c_2 so in particular, $\gamma_3 = [c_1, c_2] \cup [c_2, c_3]$ is a geodesic and there is a shortening element at c_2 for $[c_1, c_2] \cup [c_2, c_3]$. Similarly, $[c_2, c_3] \cup [c_3, c_4]$ is geodesic, and there is a shortening element at c_3 for $[c_2, c_3] \cup [c_3, c_4]$. Then the global very rotating condition applies to $\gamma_3 \cup [c_3, c_4]$ and shows that $\gamma_3 \cup [c_3, c_4]$ is geodesic. We clearly get inductively that γ is geodesic so $c \neq c'$. A similar argument applies to prove that if $W_a \neq W_b \in V_W(T)$, then $f(W_a) \neq f(W_b)$, by considering a path of the form $[x, c_1].[c_1, c_2] \dots [c_n, y]$ where $x \in W_a$, $y \in W_b$ and $c_1, .., c_n = [W_a, W_b] \cap V_C$. This proves that f is injective, and injectivity of φ follows since an element of $\ker \varphi$ has to fix T pointwise, and is therefore trivial.

The segments $[x, c_1].[c_1, c_2] \dots [c_n, y]$ considered above also have shortening pairs at c_i. The very rotating condition implies that any geodesic segment between x and y has to contain c_i and is therefore of this form. It follows that $W_2 = G'.W_1$ is 12δ-quasiconvex. It follows that W' is almost convex, and Axiom 1 holds.

The Greendlinger Axiom is similar: if $g \in G_{W'}$ is elliptic in T, there is nothing to prove. If it is hyperbolic, its axis contains a vertex in $c \in V_C$. Let $c = c_1, c_2, \dots, c_n = gc$ be the points in $[c, gc] \cap V_C$. Then the g-translates of $[c_1, c_2].[c_2, c_3] \dots [c_{n-1}, gc]$ form a bi-infinite geodesic and there is a shortening element in at each c_i. In the special case where $[c, gc] \cap V_C = \{c, gc\}$, this gives only one shortening element, one checks that there is a shortening element at some point in (c, gc). □

5. Hyperbolicity of the quotient

The goal of this section is to prove the hyperbolicity of the quotient space X/N. We will prove local hyperbolicity, and use Cartan-Hadamard Theorem.

5.1. Cartan Hadamard Theorem

Cartan-Hadamard theorem allows to deduce global hyperbolicity from local hyperbolicity. A version of this result can be derived from a theorem by Papasoglu saying a local subquadratic isoperimetry inequality implies a global linear isoperimetry [**Pap96**], see [**DG08**], or [**OOS09**, Th 8.3].

Theorem 3.10 (Cartan-Hadamard Theorem)**.** *There exist universal constants C_1, C_2 such that for each δ the constants $R_{CH} = C_1\delta$ and $\delta_{CH} = C_2\delta$ are such that for every geodesic space X such that*

- *X is R_{CH}-locally δ-hyperbolic*
- *X is 32δ-simply connected*

then X is (globally) δ_{CH}-hyperbolic.

The assumption that X is locally δ-hyperbolic asks that for any subset $\{a, b, c, d\} \subset X$ whose diameter is at most R_{CH}, the 4-point inequality holds: $d(a, b) + d(c, d) \leq \max\{d(a, c) + d(b, d), d(a, d) + d(b, c)\} + 2\delta$.

The assumption that X is 32δ-simply connected means that the fundamental group of X is normally generated by free homotopy classes of loops of diameter at most 32δ. This is equivalent to ask that the Rips complex $P_{32\delta}(X)$ is simply connected. Any δ-hyperbolic space is 4δ-simply connected [**CDP90**, Section 5].

Note that since N is generated by isometries fixing a point, and X is 32δ-simply connected, so is X/N. Indeed, let $\overline{\gamma}$ be a loop in X/N. Lift it to γ in X, joining x to gx with $g \in N$. Write $g = g_n...g_1$ with g_i fixing a point. One can homotope γ rel endpoints to ensure that γ contains a fixed point c of g_n (just insert a path and its inverse). Then $\gamma = \gamma_1.\gamma_2$ where the endpoint of γ_1 and the initial point of γ_2 are c. Downstairs, this is gives a homotopy. Now change γ_2 to $g_n^{-1}\gamma_2$. Downstairs, this does not change the path. The new path $\gamma_1.g_n^{-1}\gamma_2$ joins x to $g_{n-1} \dots g_1 x$. Repeating, we can assume $g = 1$ so that γ is a loop in X. By hypothesis, $\gamma = \prod_i p_i l_i p_i^{-1}$ where p_i is a path with origin at x, and l_i is a loop of diameter $\leq 32\delta$. Projecting downstairs, we get the same property for the projection.

Thus we will prove only local hyperbolicity of the quotient.

5.2. Proof of local hyperbolicity

We will only prove the proposition in the particular case where X is a cone-off of radius ρ (see Corollary 4.2 in the next lecture). The main simplification is that in this case, the neighbourhood of an apex is a hyperbolic cone over a graph, and so is its quotient. Thus we can apply Proposition 4.6 saying that such a hyperbolic cone is locally $2\delta_{\mathbb{H}^2}$-hyperbolic, where $\delta_{\mathbb{H}^2}$ is the hyperbolicity constant of $\mathbb{H}^2$.

Proposition 3.11. *Assume that X and the rotating family are obtained by coning-off a small cancellation moving family, as described in next section, where ρ is the radius of the cone-off. We denote $\overline{X} = X/N$, and $\overline{C}$ the image of C in X/N.*

Assume that $\rho \geq 10R_{CH}(\delta)$ (where we assume $\delta \geq 2\delta_{\mathbb{H}_2}$). Let N be the group of isometries generated by the rotating family. Then

1. *For each apex $\overline{c} \in \overline{C}$, the $9\rho/10$-neighbourhood $\overline{c}$ in $\overline{X}$ is $\rho/10$-locally $\delta_{\mathbb{H}_2}$-hyperbolic.*
2. *The complement of the $8\rho/10$-neighbourhood $\overline{C}$ in $\overline{X}$ is $\rho/10$-locally δ-hyperbolic. In fact, any subset of diameter $\leq \rho/10$ in $X \setminus C^{+8\rho/10}$. isometrically embeds in X/N.*
3. *X/N is $\delta_{CH}(\delta)$-hyperbolic.*

In the general case, one can also prove that X/N is locally hyperbolic with worse constants.

Proof. The first assertion is a direct consequence of the fact that the hyperbolic cone over a graph is $2\delta_{\mathbb{H}_2}$-hyperbolic (Proposition 4.6.

For the second assertion, let $E \subset X \setminus C^{+8\rho/10}$ be a subset of diameter $\leq \rho/10$, and let E' be its $\rho/10$-neighbourhood. We claim that E' injects into $\overline{X}$, so that E isometrically embeds in $\overline{X}$. Now assume on the contrary that there are $x, y \in E'$ and $g \in N \setminus \{\rho\}$ such that $y = gx$. In particular $[g] \leq \rho$, so by assertion 2 of Theorem 3.4, $g \in G_c$ for some $c \in C$. Then the very rotating condition implies that any geodesic $[x, y]$ contains c, a contradiction.

To conclude, we have shown that X/N is $\rho/10$-locally δ-hyperbolic. Since X/N is 4δ-simply connected, and since $\rho > 10R_{CH}(\delta)$, the Cartan-Hadamard Theorem says that X/N is globally $\delta_{CH}(\delta)$-hyperbolic. □

6. Exercises

Exercise 3.12. *Assume that $\rho >> \delta$. Let $E \subset X$ be an almost convex subset, and assume that E does not intersect the $\rho/10$-neighbourhood of C.*

Prove that E isometrically embeds in X/N.

Hint: prove that any subset of E of diameter $\rho/100$ isometrically injects in X/N. Then say that a $\rho/100$-local geodesic in X/N is close to a global geodesic.

Exercise 3.13. *Assume that G is torsion-free, and that for all c, $\mathrm{Stab}(c)/G_c$ is torsion-free. Prove that G/N is torsion-free.*

Hint: given $g \in G/N$, look for a lift in G with smallest translation length.

LECTURE 4

The cone-off

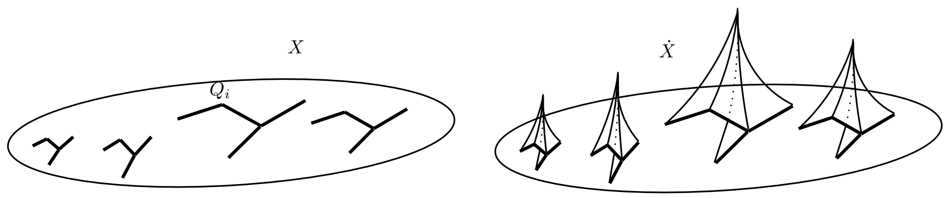
1. Presentation

The goal of this part is, given a hyperbolic space X and a family $\mathcal{Q}$ of almost convex subspaces, to perform a coning construction of these subspaces, thus obtaining a new space $\dot{X}$ called the *cone-off* space. The goal is to transform a small cancellation moving family on a hyperbolic space into a very rotating family on this new hyperbolic space $\dot{X}$.

This construction has been introduced by Delzant and Gromov, and further developed by Azhantseva, Delzant and Coulon. A construction of this type was introduced before by Bowditch in the context of relatively hyperbolic groups (with cone points at infinity). See also Farb's and Groves-Manning's constructions [**Far98, GM08**]. We follow [**Cou11**], with minor modifications and simplifications.

For simplicity we assume that X is a metric graph, all whose edges have the same length. This is no loss of generality: if X is a length space, the graph Y with vertex set X where one connects x to y by an edge of length l if $d(x,y) \leq l$ satisfies $\forall x, y \in X$, $d_X(x,y) \leq d_Y(x,y) \leq d_X(x,y) + l$.

Topologically, this coning construction consists in coning a family of subspaces $\mathcal{Q}$, and one geometrizes the added triangles as sectors of $\mathbb{H}^2$ of some fixed radius ρ and such that arclength of the boudary arc is l. Thus ρ is a parameter of this construction, to be chosen.



The assumptions will be that X is δ-hyperbolic with δ very small, and that we have a family $\mathcal{Q}$ of almost convex subspaces having a small fellow travelling length.

The features of the obtained space are as follow:

(1) $\dot{X}$ is a hyperbolic space, whose hyperbolicity constant is *good* (meaning: a universal constant; in particular, ρ can be chosen to be very large compared to this hyperbolicity constant).

(2) If a group G acts on X, preserving $\mathcal{Q}$, and if to each $Q \in \mathcal{Q}$ corresponds a group G_Q (in an equivariant way) preserving Q and with injectivity radius large enough, then $(G_Q)_{Q\in\mathcal{Q}}$ is a very rotating family on $\dot{X}$.

With quantifiers:

Theorem 4.1. *There exist constants $\delta_c, \Delta_c, \rho_0, \delta_U$ such that if X is δ_c-hyperbolic, and $\Delta(\mathcal{Q}) \leq \Delta_c$, then for all $\rho \geq \rho_0$, the corresponding cone-off $\dot{X}$ satisfies:*

(1) *$\dot{X}$ is locally $2\delta_{\mathbb{H}^2}$-hyperbolic (where $\delta_{\mathbb{H}^2}$ is the hyperbolicity constant of $\mathbb{H}^2$)*
(2) *it is globally δ_U-hyperbolic (where $\delta_U = \delta_{CH}(2\delta_{\mathbb{H}^2}$ does not depend on ρ, nor on X)*
(3) *if $((G_Q)_{Q\in\mathcal{Q}}, \mathcal{Q})$ is a moving family whose injectivity radius is at least $\mathrm{inj}_\rho := 2\pi\sinh(\rho)$, then $((G_Q)_{Q\in\mathcal{Q}}, \mathcal{Q})$ is a very rotating family on $\dot{X}$.*

The first two hypotheses of the theorem can be achieved by rescaling the metric if the fellow travelling constant $\Delta(\mathcal{Q})$ of $\mathcal{Q}$ is finite. However, if $((G_Q)_{Q\in\mathcal{Q}}, \mathcal{Q})$ is a moving family, this rescaling scales down the injectivity radius accordingly. In order to get a large injectivity radius after rescaling, the initial injectivity radius has to be large compared to the initial hyperbolicity constant and the initial fellow travelling constant. This is exactly what the small cancellation hypothesis asks. Thus, one immediately deduces:

Corollary 4.2. *There exists $A_0, \varepsilon_0 > 0$ such that if $((G_Q)_{Q\in\mathcal{Q}}, \mathcal{Q})$ is an (A, ε)-small cancellation moving family on X, then $((G_Q)_{Q\in\mathcal{Q}})$ is a very rotating family on the cone-off $\dot{X}$ of a rescaled version of X.*

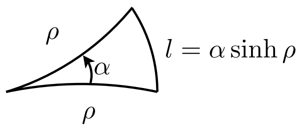
The small cancellation Theorem then follows from the theorem about very rotating families. Assertion (3) saying that small elements survive, require a bit more work, based on Greendlinger's lemma. We won't prove it in these notes.

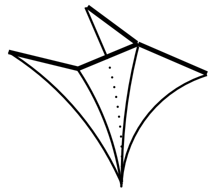
In fact, the geometry of the cone-off is even nicer than this δ_U-hyperbolicity. Indeed, this space is locally $CAT(-1;\varepsilon)$, meaning in a precise sense "almost $CAT(-1)$". This property introduced in [**Gro01**] implies hyperbolicity with a hyperbolicity constant close to $\delta_{\mathbb{H}^2}$, but gives in particular a much better control of bigons that in a standard $\delta_{\mathbb{H}^2}$-hyperbolic space. We will not discuss this property here.

It is important that ρ is large compared to the hyperbolicity constant δ_U of $\dot{X}$ to apply the theorem about rotating families. We have the freedom to do so since δ_U is independant of ρ.

2. The hyperbolic cone of a graph

Given $\rho > 0$, and $\alpha \in (0, \pi)$, consider a hyperbolic sector in $\mathbb{H}^2$, of radius ρ and angle α in $\mathbb{H}^2$. The arclength of its boundary arc of circle is $l = \alpha \sinh \rho$.





If Q is a metric graph, all whose edges have length l, the hyperbolic cone over Q is the triangular 2-complex $C(Q) = [0, \rho] \times Q/\sim$ where $\sim$ identifies to a point $Q \times \{0\}$. The cone point $c = Q \times \{0\}$ is also called the *apex* of $C(Q)$. We define a metric on each triangle of $C(Q)$ by identifying it with the hyperbolic sector of radius ρ and arclength l. We identify Q with $Q \times \{1\}$, but we distinguish the original metric d_Q from the new metric $d_{C(Q)}$. If we want to emphasize the dependance in ρ, we will denote the cone by $C_\rho(Q)$.

Remark 4.3. We don't want to assume local compactness of Q. The fact that Q is a graph whose edges have the same length is used to ensure that Q and $C(Q)$, (and later the cone-off) are geodesic spaces. Indeed, a theorem by Bridson shows that any connected simplicial complex whose cells are isometric to finitely many convex simplices in $\mathbb{H}^n$, and glued along their faces using isometries, is a geodesic space [**BH99**, Th 7.19]. This can be easily adapted to our situation where 2-cells are all isometric to the same 2-dimensional sector.

For $t \in [0, \rho], x \in Q$ we denote by tx the image of (t, x) in $C(Q)$. There are explicit formulas for the distance in $C(Q)$ [**BH99**, Def 5.6 p.59]:

$$\cosh d(tx, t'x') = \cosh t \cosh t' - \sinh t \sinh t' \cos(\min\{\pi, \frac{d_Q(x, x')}{\sinh(\rho)}\}).$$

These formulas allow to define the hyperbolic cone over any metric space. We shall not use these formulas directly.

Proposition 4.4. (1) *For each $x \in Q$, the radial segment $\{tx | x \in [0, \rho]\}$ is the only geodesic joining c to x;*
(2) *For each $x, y \in Q$ such that $d_Q(x, y) \geq \pi \sinh \rho$, then for any $t, s \in [0, \rho]$ the only geodesic joining tx to sy is the concatenation of the two radial segments $[tx, c] \cup [c, sy]$.*
(3) *For each $x, y \in Q$ such that $d_Q(x, y) < \pi \sinh \rho$, and all $s, t \in (0, \rho]$, there is a bijection between the set of geodesics between x and y in Q and the set of geodesics between tx and sy in $C(Q)$. None of these geodesics go through c.*

The map $C(Q) \setminus \{c\} \to Q$ defined by $tx \mapsto x$ is called the *radial projection.*

Exercise 4.5. *Prove that the restriction of the radial projection to $C(Q) \setminus B(c, \varepsilon)$ is locally Lipschitz. Prove that it is not globally Lipschitz in general. Prove that the Lipschitz constant goes to 1 as one gets closer to Q.*

The hyperbolic cone on a tripod is $CAT(-1)$ because it is obtained by gluing $CAT(-1)$ spaces over a convex subset. It follows that the cone over an $\mathbb{R}$-tree is $CAT(-1)$ since any geodesic triangle is contained in the cone over a tripod. In particular, such a cone is $\delta_{\mathbb{H}^2}$-hyperbolic. One can also view this fact as a particular case of Beretosvkii's theorem saying that the hyperbolic cone over any $CAT(1)$-space is $CAT(-1)$ [**BH99**, Th 3.14 p188].

The definition generalizes naturally to $\rho = \infty$, where one glues on each edge a sector of horoball with arclength l (explicitly, each triangle is isometric to $[0, l] \times [1, \infty)$ in the half-plane model of $\mathbb{H}^2$). The same argument shows that such a horospheric cone over an $\mathbb{R}$-tree is also $\delta_{\mathbb{H}^2}$-hyperbolic.

Proposition 4.6. *The hyperbolic cone of any radius, over any graph, is $2\delta_{\mathbb{H}^2}$-hyperbolic.*

This very simple proof is due to Coulon.

Proof. Let C be such a cone, and c its apex. One checks the 4-point inequality. Since any three point set is isometric to a subset of a tree, and since the cone over a tree is $CAT(-1)$, for any 3 points $u, v, w \in C$, we know that u, v, w, c satisfy the $\delta_{\mathbb{H}^2}$-hyperbolic 4-point inequality. Consider $x, y, z, t \in C$, and we want to prove that one of the inequations

$$L : xy + zt \leq xz + yt + 2\delta_{\mathbb{H}_2} \quad R : xy + zt \leq xt + yz + 2\delta_{\mathbb{H}_2}$$

holds. Consider the inequalities

$$\begin{aligned}
L_1 : xy + zc \leq xz + yc + \delta_{\mathbb{H}_2}, &\quad R_1 : xy + zc \leq xc + yz + \delta_{\mathbb{H}_2}\\
L_2 : xy + ct \leq xc + yt + \delta_{\mathbb{H}_2}, &\quad R_2 : xy + ct \leq xt + yc + \delta_{\mathbb{H}_2}\\
L_3 : xc + zt \leq xz + ct + \delta_{\mathbb{H}_2}, &\quad R_3 : xc + zt \leq xt + cz + \delta_{\mathbb{H}_2}\\
L_3 : cy + zt \leq cz + yt + \delta_{\mathbb{H}_2}, &\quad R_3 : cy + zt \leq ct + yz + \delta_{\mathbb{H}_2}.
\end{aligned}$$

We know that for each i, either L_i or R_i holds. If L_1 and R_3 hold, then L holds. Similarly, if L_i and $R_{i+2 \mod 4}$ hold, we are done. Thus, if L_i holds, we can assume that R_{i+2} does not hold and therefore that L_{i+2} holds. Similarly, we can assume $R_i \implies R_{i+2}$. Up to exchanging the role of x and y, we may also assume that L_1 holds. Then either L_1, L_2, L_3, L_4 hold, in which case summing them gives L, or L_1, R_2, L_3, R_4 which implies that $R + L$ holds, so either R or L holds. □

Lemma 4.7 (Very rotating condition)**.** *Recall that c is the apex of $C(Q)$. Let $\delta \geq 2\delta_{\mathbb{H}_2}$, and given $\rho > 0$, define $\mathrm{inj}_\rho = 2\pi \sinh \rho$.*

Assume that some group G acts on Q, and that $d_Q(y, gy) \geq \mathrm{inj}_\rho$ for all $y \in Q, g \in G \setminus \{1\}$

Then if $d(x_1, gx_2) < d(x_1, c) + d(x_2, c)$, then any geodesic from x_1 to x_2 contains c. In particular, G satisfies the very rotating condition on $C(Q)$.

Proof. For $i = 1, 2$, denote $x_i = t_i y_i$ with $t_i \geq 20\delta$, and $y_i \in Q$. To prove that any geodesic $[x_1, x_2]$ contains the apex c, we have to check that $d_Q(y_1, y_2) \geq \pi \sinh \rho$. By triangular inequality, no geodesic $[x_1, gx_2]$ contains

c so $d_Q(y_1, gy_2) \le \pi \sinh \rho$. By hypothesis on g, $d_Q(y_1, y_2) \ge d_Q(y_2, gy_2) - d_Q(gy_2, y_1) \ge \mathrm{inj}_\rho - \pi \sinh \rho \ge \pi \sinh \rho$. $\square$

3. Cone-off of a space over a family of subspaces

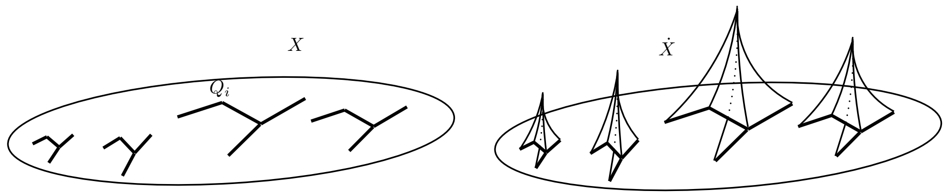
Let X be a graph, and $\mathcal{Q} = (Q_i)_{i \in I}$ a family of subgraphs. We fix some radius $\rho > 0$.

Definition 4.8. *The hyperbolic cone-off of X over $\mathcal{Q}$, of radius ρ, is the 2-complex*

$$\dot{X} = \left(X \sqcup \coprod_{i \in I} (C(Q_i)) \right) / \sim$$

where $\sim$ is the equivalence relation that identifies for each i the natural images of Q_i in X and in $C(Q_i)$.

The metric on $\dot{X}$ is the corresponding path metric.



Recall that we assume that every $Q_i \in \mathcal{Q}$ is *almost convex* in the following sense: for all $x, y \in Q$, there exist $x', y' \in Q$ such that $d(x, x') \le 8\delta$, $d(y, y') \le 8\delta$ and all geodesics $[x, x'], [x', y'], [y', y]$ are contained in Q. In particular, for all $x, y \in Q$, $d_X(x, y) \le d_Q(x, y) \le d_X(x, y) + 32\delta$.

Assertion 3 of Theorem 4.1 is then immediate from Lemma 4.7. Indeed if $x \in Q$, and $\mathrm{inj}_X(G) \ge 2\pi \sinh \rho$, then $d_Q(x, gx) \ge d_X(x, gx) \ge 2\pi \sinh \rho$ and Lemma 4.7 concludes that the very rotating property holds.

We note that if X is δ-hyperbolic, then it is 4δ-simply connected, hence so is $\dot{X}$ by Van Kampen theorem. Thus, by the Cartan-Hadamard Theorem, to prove Theorem 4.1, it is enough to check R-local $2\delta_{\mathbb{H}^2}$-hyperbolicity, with $R = R_{CH}(2\delta_{\mathbb{H}^2})$. Thus it is enough to check that the following theorem holds.

Theorem 4.9. *Fix $R = R_{CH}(2\delta_{\mathbb{H}^2})$ as above.*

There exists $\delta_c, \Delta_c > 0$ such that for all δ_c-hyperbolic space X and all Δ_c fellow-traveling family $\mathcal{Q}$ of almost convex subspaces of X, and all $\rho > 3R$, the hyperbolic cone-off of radius ρ of X over $\mathcal{Q}$ is R-locally $2\delta_{\mathbb{H}^2}$-hyperbolic.

The limit case of the theorem is as follows.

Lemma 4.10. *Let T be an $\mathbb{R}$-tree, $\mathcal{Q}$ be a family of closed subtrees of T, two of which intersect in at most one point. Then the cone-off $\dot{T}$ of T over $\mathcal{Q}$ is $\delta_{\mathbb{H}^2}$-hyperbolic (in fact $CAT(-1)$).*

Proof of the lemma. If $\mathcal{Q}$ is finite and T is a finite metric tree, then $\dot{T}$ is $\delta_{\mathbb{H}^2}$-hyperbolic. For instance, this follows by induction on $\#\mathcal{Q}$ using the fact that the space obtained by gluing two $\delta_{\mathbb{H}^2}$-hyperbolic spaces over a point is $\delta_{\mathbb{H}^2}$-hyperbolic (see also [**BH99**, Th II.11.1]). For the general case, consider $a_1, a_2, a_3, a_4 \in \dot{T}$, and write $\dot{T}$ as an increasing union of cone-offs $\dot{S}_n$ of finite trees, with $\{a_1, a_2, a_3, a_4\} \subset \dot{S}_n$ for all n, such that $d_{\dot{S}_n}(a_i, a_j) \to d_{\dot{T}}(a_i, a_j)$. The 4-point inequality of $\dot{S}_n$ thus implies the 4-point inequality for $\dot{T}$. □

3.1. Ultralimits to prove local hyperbolicity

Let $\omega : 2^{\mathbb{N}} \to \{0,1\}$ be a non-principal ultrafilter. By definition, this is a finitely additive "measure" defined on all subsets of $\mathbb{N}$, such that $\omega(\mathbb{N}) = 1$, $\omega(F) = 0$ for every finite subset $F \subset \mathbb{N}$. Zorn Lemma shows that for any infinite subset E, there is a non-principal ultrafilter such that $\omega(E) = 1$. Given some property P_i depending on $i \in \mathbb{N}$, we say that P_i holds ω-*almost everywhere* (or equivalently for almost every i), if $\omega(\{i | P_i \ true\}) = 1$. Since ω takes values in $\{0,1\}$, if a property does not hold ω-almost everywhere, its negation holds ω-almost everywhere. If $(t_i)_{i\in\mathbb{N}}$ is a sequence of real numbers, then $\lim_\omega t_i$ is always well defined in $[-\infty, \infty]$: this is the only $l \in [-\infty, \infty]$ such that for any neighborhood U of l, $t_i \in U$ for ω-almost every i.

Let $(X_i, *_i)_{i\in\mathbb{N}}$ be a sequence of pointed metric spaces. Let $B \subset \prod_i X_i$ be the set of all sequences of points $(x_i)_{i\in\mathbb{N}}$ such that $d(x_i, *_i)$ is bounded ω-almost everywhere. The ultralimit of $(X_i, *_i)$ for ω is defined as the metric space $X_\infty = B/\sim$ where $(x_i)_{i\in\mathbb{N}} \sim (y_i)_{i\in\mathbb{N}}$ if $\lim_\omega d(x_i, y_i) = 0$.

If $x_i \in X_i$ is a sequence of points such that $d(x_i, *_i)$ is bounded ω-almost everywhere, we define the ultralimit of x_i as the image of $(x_i)_{i\in\mathbb{N}}$ in X_∞.

We will use ultralimits in the following fashion. Note that we do not rescale our metric spaces, contrary to what one does in the construction of asymptotic cones. Assume that $(X_i)_{i\in\mathbb{N}}$ is a sequence of metric space such that any ultralimit of X_i is δ-hyperbolic (for any ultrafilter, and any base point $*_i$). Then for all $R, \varepsilon > 0$, X_i is R-locally $\delta + \varepsilon$-hyperbolic for i large enough. Indeed, if this does not hold, then there is a subsequence X_{i_k} and a subset $\{x_{i_k}, y_{i_k}, z_{i_k}, t_{i_k}\} \subset X_{i_k}$ of diameter at most R that contradicts the 4-point $\delta + \varepsilon$-hyperbolicity condition. Taking ω a non-principal ultrafilter such that $\omega(\{i_k\}_{k\in\mathbb{N}}) = 1$, we get a ultralimit X_∞ in which the ultralimit of the points $\{x_{i_k}, y_{i_k}, z_{i_k}, t_{i_k}\}$ contradicts δ-hyperbolicity.

3.2. Proof of the local hyperbolicity of the cone-off

Proof of Theorem 4.9. Let R be given. We need to prove that any 4-point set $\{x, y, z, t\}$ of diameter $\leq R$ satisfies the $2\delta_{\mathbb{H}_2}$-hyperbolic inequality. Although $C(Q)$ may fail to be isometrically embedded in $\dot{X}$, any subset of $B(c_Q, \rho - R)$ of diameter $\leq R$ is isometrically embedded in $\dot{X}$. Since $C(Q)$ is $2\delta_{\mathbb{H}^2}$-hyperbolic, we are done if $\{x, y, z, t\} \subset B(c_Q, \rho - R)$

There remains to check that there exists Δ_c, δ_c such that the $2R$-neighborhood of X in $\dot{X}$ is R-locally $2\delta_{\mathbb{H}^2}$-hyperbolic. If not, there is $\delta_i \to 0$, $\Delta_i \to 0$, and some δ_i-hyperbolic spaces X_i $\mathcal{Q}_i$ almost convex subsets in X_i with $\Delta(\mathcal{Q}_i) \leq \Delta_i$, and a subset $\{a_i, b_i, c_i, d_i\} \subset \dot{X}_i$ of diameter $\leq R$ for which the 4-point $2\delta_{\mathbb{H}^2}$-hyperbolicity inequality fails. Let $*_i \in X_i$ be a point at distance $\leq 2R$ from a_i. We note that $\{a_i, b_i, c_i, d_i\} \subset B(*_i, 3R)$.

Let ω be a non-principal ultrafilter, and $\dot{X}_\infty$ the ultralimit of $\dot{X}_i$ pointed at $*_i$. Let $a, b, c, d \in \dot{X}_\infty$ be the ultralimit of the points a_i, b_i, c_i and d_i. Since $2\delta_{\mathbb{H}^2} > \delta_{\mathbb{H}^2}$, to get a contradiction, it is enough to prove that a, b, c, d satisfy the 4-point $\delta_{\mathbb{H}^2}$-hyperbolicity inequality.

We want to compare $\dot{X}_\infty$ with the cone-off on an $\mathbb{R}$-tree. Let T be the ultralimit of X_i pointed at $*_i$ (this an $\mathbb{R}$-tree). To define a cone-off of T, we need to define a family of subtrees $\mathcal{Q}$ of T. Denote by J_i the index set of $\mathcal{Q}_i$ so that $\mathcal{Q}_i = (Q_j)_{j \in J_i}$. Given a sequence of indices $j = (j_i) \in \prod_{i \in \mathbb{N}} J_i$, say that the sequence of subspaces $(Q_{j_i})_{i \in \mathbb{N}} \subset X_i$ is *non escaping* if there exists $x_i \in Q_{j_i}$ such that $d(x_i, *_i)$ is bounded ω-almost everywhere. Let $J_\infty \subset (\prod_{i \in \mathbb{N}} J_i)/ \sim_\omega$ be the set of non-escaping sequences up to equality ω-almost everywhere. Given $j \in J_\infty$ a non-escaping sequence, let Q_j be the ultralimit of $(Q_{j_i})_{i \in \mathbb{N}}$ based at x_i. There is a natural map $Q_j \to T$ induced by the inclusions $Q_{j_i} \to X_i$. This map is an isometry because the inclusion $Q_{j_i} \to X_i$ is an isometry up to an additive constant bounded by $32\delta_i$. Thus we identify Q_j with its image in T, we define $\mathcal{Q} = (Q_j)_{j \in J_\infty}$, and we consider $\dot{T}$ the corresponding cone-off with radius $\rho = \lim_\omega \rho_i$.

Lemma 4.11. (1) *For $j \neq j'$, $Q_j \cap Q_{j'}$ contains at most one point. In particular $\dot{T}$ is $\delta_{\mathbb{H}^2}$-hyperbolic.*
(2) *There is a natural map 1-Lipschitz $\dot{\psi} : \dot{T} \to \dot{X}_\infty$ that maps isometrically $B_{\dot{T}}(*, 3R)$ to $B_{\dot{X}_\infty}(*, 3R)$.*

The lemma allows to conclude the proof: $\{a, b, c, d\} \subset B_{\dot{X}_\infty}(*, 3R)$, which is isometric to a subset of the $\delta_{\mathbb{H}^2}$-hyperbolic space $\dot{T}$, so a, b, c, d satisfy the 4-point $\delta_{\mathbb{H}^2}$-hyperbolicity inequality. $\square$

Proof of Lemma 4.11. For Assertion 1, consider $x, y \in Q_j \cap Q_{j'}$. Since $x \in Q_j \cap Q_{j'}$, there are sequences $(x_i)_{i \in \mathbb{N}}$, $(x'_i)_{i \in \mathbb{N}}$ representing x such that $x_i \in Q_{j_i}, x'_i \in Q_{j'_i}$. In particular $\lim_\omega d(x_i, x'_i) = 0$. Similarly, consider $y_i \in Q_{j_i}, y'_i \in Q_{j_i}$ representing y, so that in particular, $\lim_\omega d(y_i, y'_i) = 0$. If $x \neq y$, then $d(x, y) > 0$, so $d(x_i, y_i)$ and $d(x'_i, y'_i)$ are bounded below by $d(x, y)/2$ for ω-almost every i. By almost convexity, we see that Q_{j_i} fellow travels Q_{j_i} by at least $d(x, y)/4$ for ω-almost every i. Since $\Delta_i \to 0$, we get $j_i = j'_i$ for almost every i, so $j = j'$ a contradiction.

Now we define the map $\dot{\psi} : \dot{T} \to \dot{X}_\infty$. Inclusions $\varphi_{X_i} : X_i \to \dot{X}_i$ are 1-Lipschitz and define naturally a 1-Lipschitz map $\psi : T \to \dot{X}_\infty$. Similarly, the inclusions $\varphi_{C(Q_{j_i})} : C_{\rho_i}(Q_{j_i}) \to \dot{X}_i$ induce 1-Lipschitz maps $\varphi_{C(Q_j)} : C_\rho(Q_j) \to \dot{X}_\infty$ for all $j \in J_\infty$. Since φ_{X_j} coincides with $\varphi_{C(Q_j)}$ in restriction to Q_j, these maps induce a 1-Lipschitz map $\dot{\psi} : \dot{T} \to X_\infty$. Note that in general, $\dot{\psi}$ may be not onto.

To prove Assertion 2, we define a partial inverse ψ' of $\dot\psi$. Given $x \in B_{\dot X_\infty}(*, 3R)$, represent x by a sequence $x_i \in \dot X_i$ with $d_{\dot X_i}(x_i, *_i) \leq 3R$.

If x_i lies in X_i (i. e. not in the interior of a cone) for ω-almost every i, we want to define $\psi'(x)$ as the ultralimit of x_i. For this ultralimit to exist, we have to prove that $d_{X_i}(x_i, *_i)$ is bounded. But since $\rho_i > 3R$, any geodesic $[*_i, x_i]$ avoids the $\rho_i - 3R$ neighbourhood of any apex. Now there exists M such that the radial projection is M-Lipschitz (independently of ρ_i, see Exercise 4.13). It follows that the radial projection of this geodesic has length bounded by $3RM$, so the ultralimit of x_i in T exists.

Similarly, if x_i lies in a cone for ω-almost every i, write $x_i = t_i y_i$ for some $t_i < \rho_i$, and $y_i \in X_i$. The argument above shows that $d_{X_i}(*_i, y_i)$ is bounded, so the ultralimit $y \in T$ of y_i exists. Moreover, the sequence of cones Q_{j_i} containing x_i is non-escaping, so we can define $\psi'(x)$ as ty in the cone Q_j, where $j = (j_i)_{i\in\mathbb{N}} \in J_\infty$ and $t = \lim_\omega t_i$.

It is clear from the definition that $\psi'(x)$ is a preimage of x under $\dot\psi$. There remains to show that ψ' is 1-Lipschitz. It is based on the following technical fact, proved below.

Fact 4.12. *For any $\rho_0, \varepsilon, R_0 > 0$, there exists $n \in \mathbb{N}$ such that for any graph X and any cone-off $\dot X$ of radius $\rho \geq \rho_0$, and any pair of points $x, y \in \dot X$ with $d(x, y) \leq R_0$, there is a path in $\dot X$ joining x to y, of length at most $d_{\dot X}(x, y) + \varepsilon$, and that is a concatenation of at most n paths, each of which is either contained in X or in a cone $C(Q)$.*

To conclude, fix $\varepsilon > 0$, $\rho_0 = 3R$ and $R_0 = 6R + 3\varepsilon$, and consider n given by the fact. Consider $x, y \in B_{\dot X_\infty}(*, 3R)$, write x and y as an ultralimit of sequences $x_i, y_i \in B_{\dot X_i}(*_i, 3R+\varepsilon)$. Consider p_i a path joining x_i, y_i of length at most $d_{\dot X_i}(x_i, y_i) + \varepsilon$ and which is a concatenation of n sub-paths as in the fact. Then the sub-paths of p_i give a well defined concatenation of paths in $\dot T$ joining $\psi'(x)$ to $\psi'(y)$, and whose length is at most $\lim_\omega d_{\dot X_i}(x_i, y_i) + \varepsilon = d_{\dot X}(x, y)$. □

Proof of Fact 4.12. We claim that the radial projection of short paths is almost 1-Lipschitz in the following sense: for all $\lambda > 1$, there exists η such that if $p \subset C(Q)$ is a geodesic path whose endpoints are in Q, and whose length $l(p)$ is at most η, then its radial projection has length L bounded by $\lambda l(p)$, where η does not depend on ρ as long as $\rho \geq R_0$. This follows from the relation $\cosh l = \cosh^2 \rho = \sinh^2 \rho \cos(\frac{L}{\sinh \rho})$: if ρ is fixed, this simply follows from the estimate $1 + l^2/2 + o(l^2) = 1 + L^2/2 + o(L^2)$. On the other hand, at L fixed, it is an exercise to check that the ratio L/l decreases with ρ as long as $L/\sinh(\rho) < \pi$, so the estimate for $\rho = R_0$ is valid for all $\rho \geq R_0$.

To prove the fact, consider a path p in $\dot X$ joining x to y, of length $d(x, y) + \varepsilon/2$. We can assume that p is a concatenation of paths $p_1, \ldots, p_k$ where each p_i is either contained in a cone, or contained in X. If two consecutive path are contained in the same cone or are both contained in

X, we can replace them by their concatenation to decrease k. Thus for each $i \in \{2, \dots, k-1\}$, if p_i is contained in a cone $C(Q)$, then the endpoints of p_i are in Q. Let $\lambda = \frac{R_0+eps}{R_0+\varepsilon/2} \leq \frac{d(x,y)+eps}{d(x,y)+\varepsilon/2}$, and consider η as above. For each $i \in \{2, \dots, k-1\}$ such that p_i is contained in a cone and has length at most η, we replace it by its radial projection p_i'. The length of the obtained new path p' is at most $\lambda(d(x,y) + \varepsilon/2) \leq d(x,y) + \varepsilon$. Since each p_i' that is contained in a cone has length at least η, there are at most $n_0 = (R_0 + \varepsilon)/\eta$ such sub-paths. By concatenation of consecutive paths contained in X, we get that p' is a concatenation of at most $2n_0 + 3$ paths, each of which is either contained in a cone, or contained in X. $\square$

Exercise 4.13. *Given $\rho > 0$ denote by $p_\rho : B_{\mathbb{H}^2}(0,\rho)\backslash 0 \to S(0,\rho)$ the radial projection on $S(0,\rho)$, the circle of radius ρ.*

Prove that given $r, \rho_0 > 0$, there is a constant M such that for any $\rho \in [\rho_0+r, \infty)$, the restriction of the radial projection to of $B(0,\rho)\backslash\mathring{B}(0,\rho-r)$ is locally M-Lipschitz.

Hint: Since the closest point projection $\mathbb{H}^2 \to B(0,\rho-r)$ is distance decreasing, it is enough to bound the Lipschitz constant of the restriction of p_ρ to the circle of radius $\rho - r$. Using polar coordinates, prove that this follows from the fact that $\frac{\sinh\rho}{\sinh(\rho-r)}$ decreases with ρ.

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