M2 Dyn Sys

3. The Hilbert metric & Birkhoff's inequality

Let E be a topological vector space of dim ≥ 2. (real vector space)

Let $u, v \in E$ be independent vectors
Let $x, y \in V = \text{Vect}\{u, v\}$, $x, y \neq 0$
We may write

$$x = au + bv \qquad a, b, c, d \in \mathbb{R}$$
$$y = cu + dv$$

Def 3.1 Given $u, v \in E$ as above and x, y non-zero vectors in $\text{Vect}\{u, v\}$ with no three vectors being parallel. Then we define the CROSS-RATIO (fr: bi-rapport) of x, y with respect to u, v:

$$[x, y\ ;\ u, v] = \frac{a \cdot d}{b \cdot c}$$

Prop 3.2 The cross-ratio is a projective invariant $[\lambda x, \mu y\ ;\ \alpha u, \beta v] = [x, y, u, v]$
$\mathbb{R} \ni \lambda, \mu, \alpha, \beta \neq 0$

$$[\lambda x, y\ ;\ u, v] = \frac{\lambda a \cdot d}{\lambda b \cdot c} = \frac{a \cdot d}{b \cdot c}$$

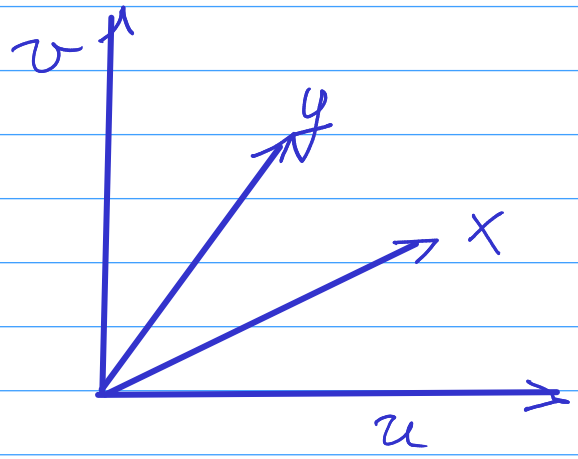
Prop 3.3 The cross-ratio is a linear invariant. For $A \in L(E)$ such that Au and Av are linearly invariant

then $[Ax, Ay; Au, Av] = [x, y; u, v]$

Proof $\left.\begin{aligned} Ax &= a\,Au + b\,Av \\ Ay &= c\,Au + d\,Av \end{aligned}\right\} \Rightarrow$ same C-R.

Prop 3.4 (symmetries):

$$[y, x; v, u] = [x, y; u, v] = \frac{1}{[x, y; v, u]}$$



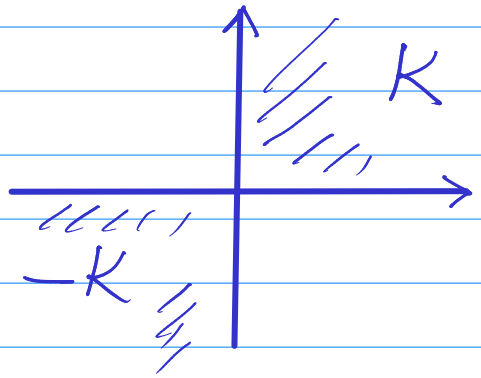
Def 3.5 A subset $K \subset E$ (containing ≥ 2 pts) is called a proper, convex, closed cone (in the following called an IR-cone) iff

1) $\mathbb{R}_+ K = K$
2) $K + K = K$
3) $K \cap (-K) = \{0\}$

(and K is closed)

What you might have in mind:

$K = \mathbb{R}^n_+ \subset E = \mathbb{R}^n$



The Hilbert metric (projective metric) for an $\mathbb{R}$- cone

Def 3.6 Let $x, y \in K^* = K \setminus \{0\}$ and set

$$\beta(x,y) = \inf\{t > 0 : tx - y \in K\} \in [0, +\infty]$$

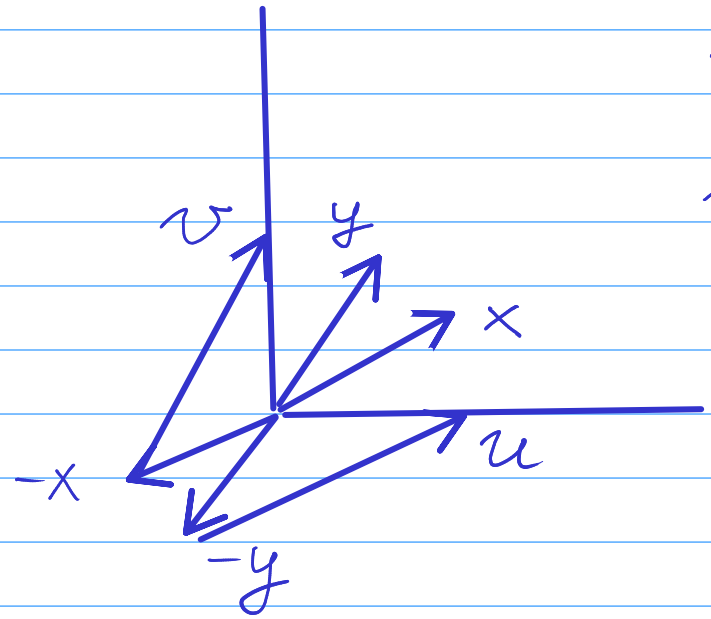
and then

$$d_K(x,y) = \ln\left(\beta(x,y)\,\beta(y,x)\right)$$

Prop 3.7 d_K defines a projective metric on K^*: $x, y, z \in K^*$, $\lambda \in \mathbb{R}_+^*$

1) $d_K(x,y) \geq 0$ and $= 0$ iff $x \parallel y$
2) $d_K(\lambda x, y) = d(x,y)$
3) $d_K(x,y) = d_K(y,x)$
4) $d_K(x,z) \leq d_K(x,y) + d_K(y,z)$

Proof



$t_1 = \beta(x,y)$

$u = t_1 x - y$

$t_2 = \beta(y,x)$

$v = t_2 y - x$

$t_1, t_2 > 0$ $\quad u = t_1 x - y \in K \quad v = t_2 y - x \in K$

$$t_2 u + v = t_2 t_1 x - t_2 y + t_2 y - x$$
$$= \underbrace{(t_2 t_1 - 1)}_{\in \mathbb{R}_+} x \in K \Rightarrow t_1 t_2 \geq 1$$

$d(x,y) \geq 0$

if $d(x,y)=0$ $\quad t_1 t_2 = 1$ $\quad t_1 = 1/t_2$

$u = \frac{1}{t_2}x - y \in K$ $\quad v = t_2 y - x \in K$

$= \underbrace{-t_2\left(\frac{1}{t_2}x - y\right)}_{\in -K} \in K \Rightarrow \boxed{v = 0}$

exo : proove the triangular inequality

For 2) : $\beta(\lambda x, y) = \frac{1}{\lambda}\beta(x,y) \quad (\lambda > 0)$

$\beta(y, \lambda x) = \lambda\, \beta(x,y)$

$\Rightarrow \underline{d_K(\lambda x; y) = d_K(x,y)}$

Given independent vectors $u, v \in E^*$

we write

$\underline{K = \mathbb{R}_+(u,v) = \{au + bv \mid a, b \geq 0\}}$

Prop 3.8 : Given indep vectors $u, v \in E$ and $x, y \in (\mathbb{R}_+(u,v))^*$ with at least one of x, y not parallel to u, nor v. Then

$$d_K(x,y) = \left|\ln\, [x, y; u, v]\right|$$

proof : $t_1 = \beta(x,y)$, $t_2 = \beta(y,x)$

$u' = t_1 x - y$ $\quad v' = t_2 y - x$

with u' and v' parallel to either u or v

say $u' \parallel u$, $v' \parallel v$

$t_2 u' + 1\, v' = t_2 t_1 x - t_2 y + t_2 y - x = (t_1 t_2 - 1)x$

$1\, u' + t_2 v' = (t_1 t_2 - 1)\, y$

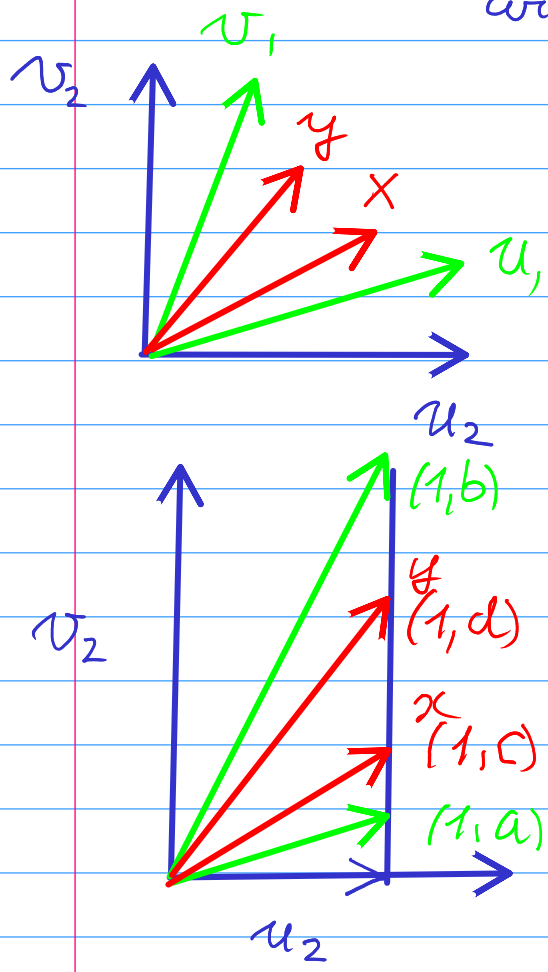
Cross ratios : $\frac{t_2 \cdot t_1}{1 \cdot 1} = [x, y; u, v]$

$\ln [x, y; u, v] = \ln (t_2 \cdot t_1) = d_K(x, y)$

Garrett '56, '57

Thm 3.9 [Birkhoff's inequality]

Suppose that $x, y \in \mathbb{R}_+(u_1, v_1) \subset \mathbb{R}_+(u_2, v_2)$
with x, y independent
($\Rightarrow u_1, v_1$ indep & u_2, v_2 indep)



Let $\Delta = d_{\mathbb{R}_+(u_2, v_2)}(u_1, v_1) \in [0, +\infty]$

Then we have :

$$d_{\mathbb{R}_+(u_2,v_2)}(x,y) \leq \tanh \frac{\Delta}{4}\, d_{\mathbb{R}_+(u_1,v_1)}(x,y)$$

≤ 1 and < 1 if $\Delta < +\infty$

suppose ordered s.t
$0 < a < c < d < b < +\infty$

$u_1 = u_2 + a v_2$
$v_1 = u_2 + b v_2$

$$\Delta = d_{\mathbb{R}_+(u_2,v_2)}(u_1, v_1) = \left| \log \frac{1 \cdot b}{1 \cdot a} \right| = \log \frac{b}{a} > 0$$

$$d_{\mathbb{R}_+(u_2,v_2)}(x,y) = \log \frac{1 \cdot d}{1 \cdot c} = \log \frac{d}{c}$$

$$\hookrightarrow = \int_c^d \frac{1}{t}\, dt \qquad (**)$$

The difficult part : dist of x, y wrt u_1, v_1 :

Claim $x = \frac{b-c}{b-a} u_1 + \frac{c-a}{b-a} v_1$
$y = \frac{b-d}{b-a} u_1 + \frac{d-a}{b-a} v_1$ } (exo)

The Hilbert dist:

$$d_{R_+(u_1,v_1)}(x,y) = \left| \log \frac{(b-c)(d-a)}{(c-a)(b-d)} \right|$$

$$= \left| \log \frac{d-a}{b-d} - \log \frac{c-a}{b-c} \right|$$

$\log\left(\frac{t-a}{b-t}\right)'$

$$= \int_c^d \frac{b-a}{(b-t)(t-a)}\, dt \quad (*)$$

Ratio of integrands in (*) and (**)

$$\eta = \sup_{a \le t \le b} \frac{1}{t} \frac{(b-t)(t-a)}{(b-a)}$$ max for $t^* = \sqrt{ab}$ (exo)

$$= \frac{1}{\sqrt{ab}} \frac{(b-\sqrt{ab})(\sqrt{ab}-a)}{(\sqrt{b}-\sqrt{a})(\sqrt{b}+\sqrt{a})}$$

$$= \frac{(\sqrt{b}-\sqrt{a})^2}{(\sqrt{b}-\sqrt{a})(\sqrt{b}+\sqrt{a})} = \frac{\sqrt{b}-\sqrt{a}}{\sqrt{b}+\sqrt{a}}$$

$$= \frac{\sqrt{\frac{b}{a}}-1}{\sqrt{\frac{b}{a}}+1} = \frac{e^{\Delta/2}-1}{e^{\Delta/2}+1} = \frac{e^{\Delta/4}-e^{-\Delta/4}}{e^{\Delta/4}+e^{-\Delta/4}}$$

$$= \tanh \frac{\Delta}{4}$$

$$d_{R_+(u_2,v_2)}(x,y) = \int_c^d \frac{1}{t}\, dt \le \int_c^d \eta \frac{b-a}{(b-t)(t-a)}\, dt$$

$$= \tanh \frac{\Delta}{4}\, d_{R_+(u_1,v_1)}(x,y)$$ //

Thm 3.10 [G. Birkhoff]

Let $T : V \to V$ be a linear map

Let $K_1 \subset K_2 \subset V$ be embedded $\mathbb{R}$-cones such that $T(K_2^*) \subset K_1^*$

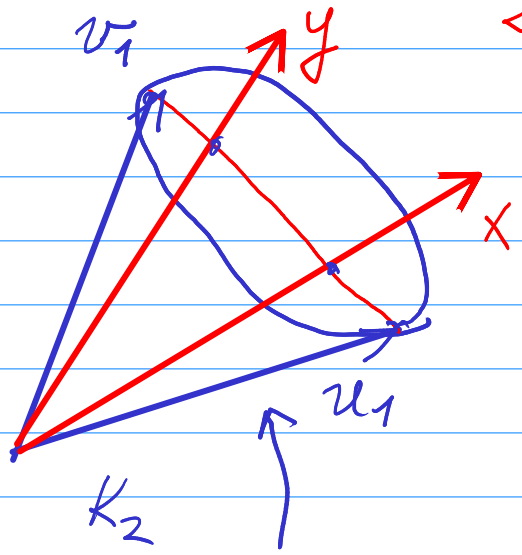
Set $\Delta = \operatorname{diam}_{K_2}(K_1) = \sup_{x,y \in K_1^*} d_{K_2}(x,y) \leq +\infty$

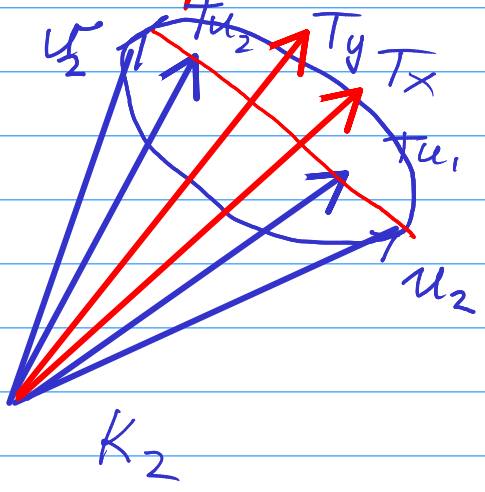
Then for every $x, y \in K_2^*$:

$$d_{K_2}(Tx, Ty) \leq \tanh\frac{\Delta}{4}\, d_{K_2}(x,y)$$

$< 1 \Rightarrow$ uniform contraction

Assume x, y lin. indep





$$\operatorname{span}(x,y) \cap K_2 = \mathbb{R}_+(u_1, v_1)$$

$$\operatorname{span}(Tx, Ty) \cap K = \mathbb{R}_+(u_2, v_2)$$

Birkhoff's ineq:

$$d_{K_2}(Tx, Ty) = d_{\mathbb{R}_+(u_2,v_2)}(Tx, Ty) \leq$$

$$\tanh\left(\frac{1}{4} d_{\mathbb{R}_+(u_2,v_2)}(Tu_1, Tu_2)\right) d_{\mathbb{R}_+(Tu_1,Tu_2)}(Tx, Ty) \leq$$

$$\tanh\frac{\Delta}{4}\; d_{\mathbb{R}_+(u_1,v_1)}(x,y) = \tanh\frac{\Delta}{4}\, d_{K_2}(x,y)$$

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