M2 Dynamical Systems. Lecture notes.

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1 Introduction

In this course we will mostly deal with dynamical systems in discrete time. Let (M, g) be a Riemmanian manifold and $f : M \to M$ either a continuous map or a smooth (at least C^1) self-map of M. We will also sometimes consider smooth invertible maps, i.e. $f \in \mathrm{Diff}^k(M)$ with $k \geq 1$.

For $x \in M$, the forward orbit of x is defined as: $\mathcal{O}^+(x) = (x, f(x), f^2(x), ...)$. When f is invertible, the backward orbit is: $\mathcal{O}^-(x) = (\cdots, f^{-2}(x), f^{-1}(x), x)$ and the (two-sided) orbit: $\mathcal{O}(f) = \left(f^k(x)\right)_{k\in\mathbb{Z}}$.

Definition 1.1 A point $x \in M$ is said to be a periodic point for f of period $k \geq 1$ if $f^k(x) = x$. The smallest such k is called the primitive or prime period of x.
The point $x \in M$ is a fixed point of f if $k = 1$, i.e. if $f(x) = x$.

Continuity and topology alone has a rather surprising consequence on existence of periodic orbits of a continuous interval map. First in a simple version known as: "Period 3 implies chaos":

Theorem 1.2 [Sharkovskii] Let I be an interval and $f : I \to I$ continuous. If f has a primitive period 3 orbit then f has a primitive period p orbit for any $p \geq 1$.

Proof: Let $x_1 \to x_2 \to x_3 \to x_1$ be a period 3 orbit (see figure). Our proof is for the shown order ($x_1 < x_2 < x_3$). It is easy to adjust the proof for the reverse orbit. Denote $I_0 = [x_1, x_2]$ and $I_1 = [x_2, x_3]$. It is clear that:

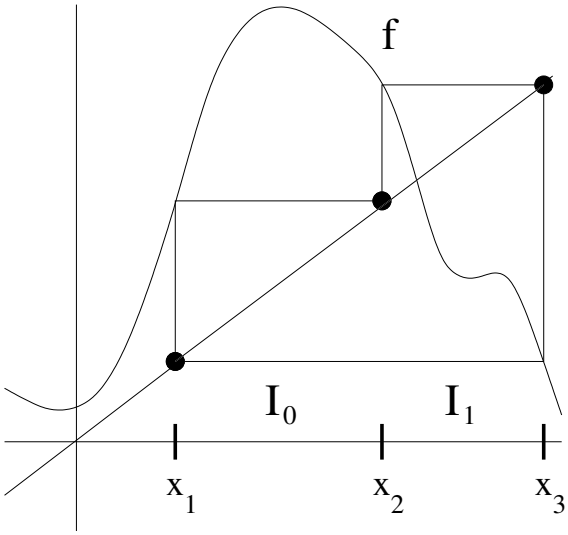

Figure 1: Map with period 3 orbit.

$$f(I_0) \supset I_1, \quad f(I_1) \supset I_0 \cup I_1 \tag{1.1}$$

One has the following two properties:

(P1) If J is a compact interval and $f(J) \supset J$ then f has a fixed point in J (The Intermediate Value Theorem applied to $f(x) - x$).

(P2) If $f(J) \supset K$ where J et K are compact, then there is $J^* \subset J$ so that $f(J^*) = K$.

With $K = [a, b]$, let $A = f^{-1}(a) \cap J \ni a_0$ and $B = f^{-1}(b) \cap J \neq b_0$ with e.g. $a_0 < b_0$. Set $a^* = \sup\{a \in A | a < b_0\}$ and then $b^* = \inf\{b \in A | a^* < b\}$. Then $J^* = [a^*, b^*]$ works.

Let $p = 1$. Since $f(I_1) \supset I_1$, (P1) implies the existence of a fixed point in I_1 (also seen in the fig-

ure), i.e. a period 1 orbit.

For a period 2 orbit, we note that since $f(I_1) \supset I_0$ implies (P2) the existence of a subinterval $I_{10} \subset I_1$ such that $f(I_{10}) = I_0$. Then $f^2(I_{10}) = f(I_0) \supset I_1 \supset I_{10}$ and (P1) imply the existence of $x \in I_{10}$ with $f^2(x) = x$ and $x \in I_1$ and $f(x) \in I_0$. It can not be a fixed point. Why?

Given any $p \geq 4$ we find successively I_{10}, I_{110}, I_{1110}, ... (all in I_1) with $f(I_{1^k0}) = I_{1^{k-1}0}$, $k \geq 1$. Then $f^p(I_{1^p0} = \cdots = f(I_0) \supset I_{1^{p-1}0}$ which by (P1) implies existence of a fixed point $x \in I_{1^p0}$ of f^p, whence a periodic orbit of length p. By the choice of symbols it must be a prime period (prove this). □

Exercise 1.2.1: Show that the f considered above admits at least two distinct period 5 orbits.

The more general version of Sharkovskii's theorem introduces the following total (Sharkovskii) ordering of the positive natural numbers:

$$\begin{gathered} 3 \prec 5 \prec 7 \prec 9 \prec 11 \prec \cdots \\ 2 \cdot 3 \prec 2 \cdot 5 \prec 2 \cdot 7 \prec 2 \cdot 9 \prec \cdots \\ 2^2 \cdot 3 \prec 2^2 \cdot 5 \prec 2^2 \cdot 7 \prec 2^2 \cdot 9 \prec \cdots \\ \cdots \\ \cdots \prec 2^4 \prec 2^3 \prec 2^2 \prec 2 \prec 1. \end{gathered}$$

The statement is then that when $\ell \prec m$ then the existence of a primitive period ℓ orbit implies the exostence of a primitive period m orbit.

The following converse equally holds true: When $\ell \prec m$ then you may find a map with a primitive period m orbit but not having a primitive period ℓ orbit.

Exercise 1.2.2: Let f be the continuous piecewise linear map of $[0, 4]$ into itself with the following period five orbit: $0 \mapsto 2 \mapsto 1 \mapsto 3 \mapsto 4 \mapsto 0$ and being bijective between each pair of succesive integers. Show that f does not have a primitive period 3 orbit.

2 Hyperbolic fixed points.

Definition 2.1 Let $f \in C^k(M, M)$ with $k \geq 1$ and let $x \in M$ be a fixed point of f. In a local chart at x, the derivative map $f_{*x} : T_xM \to T_xM$ takes the form of a matrix which depends upon the chart, but its eigenvalues do not. Let sp $(f_*) = \{\lambda_1, ...\lambda_d\}$ be the (possibly complex) eigenvalues. We say that the fixed point is:

- **hyperbolic** if all $|\lambda_i| \neq 1$.
- **attractive or stable** if all $|\lambda_i| < 1$
- **repelling or unstable** if all $\lambda_i| > 1$.

Note also that if $f^p(x) = x$, $p \geq 2$ is a fixed point of the iterated map, then the spectrum of $(f^p)_{*x}$ is independent of charts but also of which point in the orbit we use as a base point (because for products of square matrices, ABC and CAB have the same spectrum). We use the same terminology as above for the associated periodic orbit.

Example 2.1.1: $f(x, y) = (2x, y/2)$ has a hyperbolic fixed point at the origin which is neither attractive, nor repelling.

2.1 Attractive fixed points.

When $x_0 = f(x_0) \in M$ is an attractive fixed point, we have the following result (whence the use of the word"attractive"):

Proposition 2.2 There is $\theta < 1$, an open neighborhood U of x_0 and a metric d_h on U so that $f(U) \subset U$, $f(x_0) = x_0$ and

$$\forall x, y \in U: \ d_h(f(x), f(y)) \ \leq \ \theta \ d_h(x, y).$$

Proof: There are many ways to show this. We will here use the dynamics to adapt the metric tensor g at x_0. The same technique will prove useful when dealing with hyperbolic structures below.

A linear result: Recall that when A is a $d \times d$ matrix with all eigenvalues being strictly smaller

than η in absolute values, then for any norm $\|\cdot\|$ on $\mathbb{R}^d$ there is $C < +\infty$ so that for all $k \geq 0$ and $v \in \mathbb{R}^d$:

$$\|A^k v\| \leq C\eta^k \|v\|.$$

Consider a local chart where x_0 maps to the origin $0 \in V \subset \mathbb{R}^d$. In the chart, the metric tensor g_{x_0} maps to a scalar product g_0 which corresponds to a symmetric positif definite matrix or bilinear form, with an associated norm $\|\cdot\|_{g_0}$. The derivative map $A = Df(x_0)$ has eigenvalues strictly smaller than some $\eta < 1$ in absolute values. Thus, using the above linear result, there is $C < +\infty$ so that for all vectors $v \in \mathbb{R}^d$:

$$g_0(A^k v, A^k v) \ \leq \ C^2 \ \eta^{2k} \ g_0(v, v).$$

Now, picking $\rho \in (\eta, 1)$, we may define an adapted metric in the chart. For $v, w \in \mathbb{R}^d$ set

$$h(v, w) = \sum_{k \geq 0} \rho^{-2k} g_0(A^k v, A^k w).$$

Then, $g_0(v, v) \leq h(v, v) \leq \frac{C^2}{1-(\eta/\rho)^2} g_0(v, v)$ shows that the associated norms $\|\cdot\|_{g_0}$ and $\|\cdot\|_h$ are equivalent. Furthermore, we see that for any $v \in \mathbb{R}^d$:

$$h(Av, Av) = \rho^2 h(v, v) - g_0(v, v) \leq \rho^2 h(v, v)$$

Thus, we have a uniform contraction in the associated norm: $\|Av\|_h \leq \rho \|v\|_h$.

The above works a priori only at 0. We extend the metric h identically to all points of $V \subset \mathbb{R}^d$. Let $\theta \in (\rho, 1)$ The map f is C^1 so there is $\delta > 0$ so that f maps the ball (for the h norm) $U = B_h(0, \delta) \subset V$ into V and such that the derivative matrix (by continuity of Df) verifies $\|Df(x)v\|_h \leq \theta \|v\|_h$ for all vectors v. U is a convex set and by the Mean Value Theorem, $\|f(x) - f(y)\|_h \leq \theta \|x - y\|_h$ for all $x, y \in U$. As $f(0) = 0$, we conclude that f is a strict θ-Lipschitz contraction of U for the metric $d_h(x, y) = \|x - y\|_h$ and with fixed point 0. □

2.2 Repelling fixed points

Let $x_0 = f(x_0)$ be a repelling fixed point of f. As there are no zero eigenvalues, in a local chart $Df(0)$ is invertible so by the inverse function theorem, f is a local diffeormorphism (not necessarily globally). The local inverse map, f_{loc}^{-1} also has x_0 as a fixed point but attractive since eigenvalues of $Df_{\text{loc}}^{-1}(x_0) = (Df(x_0))^{-1}$ are the reciprocals of those of $Df(x_0)$. We may apply the result from the previous subsection and construct an adapted metric h in a local chart which is then being uniformly θ-Lipschitz contracted by f_{loc}^{-1} in some neighborhood of x_0 and for some $0 < \theta < 1$.

2.3 Hyperbolic fixed points

Let $x_0 = f(x_0)$ be a (genuine) hyperbolic fixed point, genuine meaning that there are both stable and unstable eigenvalues but no eigenvalue of unit absolute value. Let $d^s \geq 1$ be the number of stable eigenvalues, $d^u \geq 1$ the number of unstable eigenvalues and let $T_{x_0}M = E^u_{x_0} \oplus E^s_{x_0}$ be a splitting of the tangent space at x_0 into the subspaces correcponding to the unstable and the stable eigenvalues, respectively.

It is practical to make the two subspaces orthogonal. In order to achieve this let g_0 be the Riemannian metric at x_0 and $\|\cdot\|_0$ the associated norm. By Cauchy-Schwarz we have for two tangent vectors: $|g_0(v, w)| \leq \|v\|_0 \ \|w\|_0$ with equality iff the vectors are linearly dependent. Let $\overline{B}^u$ and $\overline{B}^s$ be the closed unit balls in $E^u_{x_0}$ and $E^s_{x_0}$, respectively and $D = \overline{B}^u \times \overline{B}^s$. Then define:

$$\kappa = \sup_{(v^u, v^s) \in D} \{ \ | \ g_0(v^u, v^s) \ | \ \}. \tag{2.2}$$

By compactness this number is a maximum and since $E^s_{x_0}$ and $E^u_{x_0}$ are transversal, $\kappa < 1$. We also have $|g_0(v^u, v^s)| \leq \frac{1}{2}\kappa \left(\|v^u\|_0^2 + \|v^s\|_0^2\right)$ which implies : $\|v^u + v^s\|_0^2 \geq (1 - \kappa)\left(\|v^u\|_0^2 + \|v^s\|_0^2\right)$. Define now a new scalar product at x_0 by setting: $\widetilde{g}_0(v^u, v^s) \equiv 0$ for $v^u \in E^u_0$, $v^s \in E^s_0$ and letting $g_1 \equiv g_0$ when both are in E^u_0 or both in E^s_0. Then one has the following bounds for the associated norms and any $v \in T_{x_0}M$:

$$(1 - \kappa)\|v\|_1^2 \leq \|v\|_0^2 \leq (1 + \kappa)\|v\|_1^2.$$

In the subspace $E^s_{x_0}$ we may use the procedure from the above section 2.1 on attractive fixed points

to adapt the metric on $E^s_{x_0}$ and on the subspace $E^u_{x_0}$ that of section 2.2.

In the end we obtain an adapted metric $\widetilde{g}_{x_0}$ at x_0 and $\theta < 1$ such that $\forall\ (v^u, v^s) \in E^u_{x_0} \times E^s_{x_0}$:

$$\begin{aligned} \|Df(x_0)v^u\| &\geq 1/\theta\ \|v^u\|, & (2.3)\\ \|Df(x_0)v^s\| &\leq \theta\ \|v^s\|, & (2.4)\\ \widetilde{g}(v^s, v_u) &= 0. & (2.5) \end{aligned}$$

Consider now a local chart around x_0 mapping a neighborhood of x_0 to a neighborhood $V \in \mathbb{R}^{d^u} \times \mathbb{R}^{d_s}$ of the origin and with basis vectors being an orthonormal basis with respect to our adapted metric. We use coordinates (x^u, x^s) so that $E^u_{x_0}$ and $E^s_{x_0}$ maps to respectively $E^u_0 = \mathbb{R}^{d^u} \times \{0\}$ and $E^s_0 = \{0\} \times \mathbb{R}^{d_s}$. Let $\delta > 0$ be small and let $\overline{B}^{\,u}_\delta = \overline{B}^{\,u}(0, \delta)$ and $\overline{B}^{\,s}_\delta = \overline{B}^{\,s}(0, \delta)$ be the closed δ-balls at the origin of E^u_0 and E^s_0, respectively and with respect to our adapted metric. We set $R = R_\delta = \overline{B}^{\,u}_\delta \times \overline{B}^{\,s}_\delta$ For $\delta_0 > 0$ small enough $R \subset V$ and $f(R) \subset V$.

In our coordinate-system $(0, 0)$ is a fixed point and we have for $(x^u, x^s) \in R$:

$$f(x^u, x^s) = (\lambda^u x_u, \lambda^s x^s) + \epsilon_\delta(x^u, x^s),$$

where $\lambda^u \in M_{d^u}(\mathbb{R})$ is invertible with $\|(\lambda^u)^{-1}\| \leq \theta$, $\lambda^s \in M_{d^s}(\mathbb{R})$ has norm $\|\lambda^s\| \leq \theta$ and ϵ_δ is a C^1 function with C^1-norm that goes to zero as $\delta \to 0$.

2.4 Stable and unstable manifolds.

Given a (genuine) hyperbolic fixed point as above we will now construct associated local stable and unstable manifolds for the fixed point. In the adapted coordinates and the closed neighborhood R_δ above, the unstable manifold $W^u_{\rm loc}(0)$ will be the graph of a map $\gamma^u \in C^k(\overline{B}^{\,u}, \overline{B}^{\,s})$ and $W^s_{\rm loc}(0)$ the graph of a map $\gamma^s \in C^k(\overline{B}^{\,s}, \overline{B}^{\,u})$.

For the unstable manifold, consider the space of Lipschitz maps $X = {\rm Lip}(\overline{B}^{\,u}, \mathbb{R}^{d^s})$ with norm $\|\phi\| = \|\phi\|_{C^0} + {\rm Lip}(\phi)$. The subset $A = \{\phi \in X \mid \phi(\overline{B}^{\,u}) \subset \overline{B}^{\,s}, {\rm Lip}(\phi) \leq 1\}$ is a closed subset of X.

Given a function $\gamma \in A$ we will construct $\widetilde{\gamma} \in A$ in the following way: For $y^u \in \overline{B}^{\,u}$ we seek $x^u \in \overline{B}^{\,u}$ so that $y^u = f^u(x^u, \gamma(x^u)) = \lambda^u x^u + \epsilon^u_\delta(x^u, \gamma(x^u))$ or, equivalently,

$$x^u = (\lambda^u)^{-1} y^u - (\lambda^u)^{-1} \epsilon_\delta(x^u, \gamma^u).$$

The RHS depends upon x^u but is (very) strongly contracting in x^u, so admits a unique fixed point which itself must be $(\theta + \epsilon_\delta)$-Lipschitz in y^u. Then $\widetilde{\gamma}(y^u) = f^s(x^u, \gamma(x^u)) = \lambda^s x^u$ with the x^u just found. One has

$$\widetilde{\gamma}(y^u) = \lambda^s \gamma((\lambda^u)^{-1} y^u + \epsilon_\delta) + \epsilon_\delta,$$

where the ϵ_δ-terms depends upon y^u and γ but can be made arbitrarily small by choosing $\delta > 0$ small enough.

We can then show (I omit the rather lengthy calculations) that the above so-called graph-transformation, $\Gamma^u : \gamma \mapsto \widetilde{\gamma}$ is a $(\theta + \epsilon_\delta)$-contraction of A which can be made smaller than one when choosing δ small enough. The graph transform then has a unique fixed point, $\gamma^u \in A$.
Up to an ϵ_δ contribution one has:

$$\gamma^u(y^u) = \lambda^s \gamma^u((\lambda^u)^{-1}\ y^u\ ) + \epsilon_\delta. \qquad (2.6)$$

We admit here that if f is a C^k function then so is this fixed point of Γ^u. The graph of γ^u is the local unstable manifold, denoted $W^u_R(0)$ or $W^u_{\rm loc}(0)$

One has $f(W^u_R(0)) \supset W^u_R(0)$ as well as $\bigcap_{k\geq 0} f^k R = W_R(0)$. When f is (locally) invertible then also $\bigcap_{k\geq 0} f^{-k} W^u_R(0) = \{0\}$.

For the local stable manifold we proceed in a similar way, looking at graphs of Lipschitz maps, $\gamma : \overline{B}^{\,s} \to \overline{B}^{\,u}$, with Lipschitz constant not exceeding 1. Given such a graph, consider for $x^s \in \overline{B}^{\,s}$: $f(x^u, x^s) = (y^u, y^s) = (\gamma(y^s), y^s)$. We want to solve this for y^s and x^u, after which we pose $\widetilde{\gamma}(x^s) = x^u$. To do this, note that $y^s = \lambda^s x^s + \epsilon^s_\delta(x^s, x^u)$ and $\gamma(y^s) = \lambda^u x^u + \epsilon^u_\delta(x^s, x^u)$ which yields

$$x^u = (\lambda^u)^{-1}(\gamma(\lambda^s x^s - \epsilon^s_\delta) - \epsilon^u_\delta.$$

x^u appears on the RHS but again trough a strongly contraction. The unique fixed point exists $\gamma^s \in A^s$ and depends Lipschitz upon x^s. Thus, up to an ϵ_δ contribution one has (compare with (2.6):

$$\gamma^s(x^u)\ =\ (\lambda^u)^{-1}\ \gamma^s(\lambda^s x^s) + \epsilon_\delta \qquad (2.7)$$

3 Anosov Diffeomorphisms.

Let (M, g) be a compact Riemmannian manifold. A diffeomorphism of M is said to be Anosov if it admits a socalled hyperbolic structure:

Definition 3.1 A hyperbolic structure on (M, g) for $f \in \mathrm{Diff}^k(M)$ is a continuous splitting of the tangentbundle of M into f-invariant subbundles, $T_xM = E^u_x \oplus E^s_x$, $x \in M$, for which there are constants $C < +\infty$, $\theta < 1$ so that for $n \geq 0$:

1. $\|f^{-n}_{*x} v^u\|_{f^{-n}(x)} \leq C\theta^n \|v^u\|_x$, for $v^u \in E^u_x$.
2. $\|f^{n}_{*x} v^s\|_{f^{n}(x)} \leq C\theta^n \|v^s\|_x$, for $v^s \in E^s_x$.

Proposition 3.2 we may adapt the metric and get an equivalent metric for which $C = 1$ in the above and the sub-bundles are orthogonal. More precisely we have for such an adapted metric:

1. $\|f_{*x} v^u\|_{f(x)} \geq \frac{1}{\theta} \|v^u\|_x$, for $v^u \in E^u_x$.
2. $\|f_{*x} v^s\|_{f(x)} \leq \theta \|v^s\|_x$, for $v^s \in E^s_x$.
3. $g_x(v^u, v^s) = 0$, for $v^u \in E^u_x$, $v^s \in E^s_x$.

Proof: For orthogonality note that the $\kappa = \kappa_x$ defined in equation (2.2) depends continuously upon x so has a maximum on the compact manifold. For e.g. the stable direction define

$$\widetilde{g}_x(v^s, v^s) = \sum_{k \geq 0} g_{f^k x}(f^k_{*x} v^s, f^k_{*x} v^s),$$

and proceed as in the previous section. □

An example on the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ is multiplication by the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ which has one unstable and one stable direction. The unstable and stable bundles are direct products of $\mathbb{T}^2$ with the subspaces generated. The natural Riemannian metric is then already adopted (see exercise sheet).

3.1 Structural stability

Let f_0 be a hyperbolic diffeo on M with the s/u-splitting $TM = E^u \oplus E^s$ and an adapted metric for some uniform contraction constant $\theta < 1$. .

For $(v^u, v^s) \in E^u_x \times E^s_x$ we have $Df_0(x).(v^u, v^s) = (\lambda^u_0 v^u, \lambda^s_0 v^s)$ with $\|(\lambda^u)^{-1}\|, \|\lambda^s\| \leq \theta < 1$ uniformly in $x \in M$.

Theorem 3.3 [Anosov] For sufficiently small (for the C^1-topology) deformations of f_0, the perturbed map f is still Anosove and the (u/s)-splitting of the tangent-bundle behaves continuously on the deformation

So let f be C^1 close to f (how close to be discussed below). Then for the the pertubed map:

$$Df(x) \begin{pmatrix} v^u \\ v^s \end{pmatrix} = \begin{pmatrix} \lambda^u & \epsilon^u \\ \epsilon^s & \lambda^s \end{pmatrix} \begin{pmatrix} v^u \\ v^s \end{pmatrix}.$$

with λ^u, λ^s close the unperturbed values, and ϵ^u, ϵ^s uniformly small, all with respect to matrix norms and our adopted metric.

For $x \in M$ and $A_x \in K = L(E^u_x; E^s_x)$ the graph of A_x is given by $G_x(A_x) = \{ \ (v^u, A_x v^u) \mid v^u \in E^u_x \ \}$. When A is zero, it simply describes E^u_x and for small A a (linear) deformation of that subspace. Define the u-cone field which to each $x \in M$ associates:

$$C^u_x = \{ \ G_x(A_x) \mid \|A_x\| \leq 1 \ A \in X_k \}.$$

The derivative map f_{*x} at x maps a graph $G_x(A_x)$ into a graph at $f(x)$ of some $\tilde{A}_{fx} \in L(E^u_{fx}, E^s_{fx})$. Our goal is to make sure that $\tilde{A}_{fx}$ lies in the u-cone C^u_{fx} and that $\Gamma_x : A_x \mapsto \tilde{A}_{fx}$ is a strict and uniform contraction.

A tangent vector $(v^u, A_x v^u) \in E^u_x \times E^s_x$ maps to: $(D(A_x)v^u, N(A_x)v^u)$ where $D(A_x) = \lambda^u + \epsilon^u A_x$)

and $N(A_x) = \epsilon^s + \lambda^s A_x$. We want to write this as $\Gamma_{f(x)}(A_x) = (\widetilde{v}^u, \widetilde{A}_{fx}\widetilde{v}^u)$ for some $\widetilde{v}^u \in E^u_{fx}$. This

is possible provided $D(A_x)$ is invertible and we get the formula:

$$\Gamma_{f(x)}(A_x) = \widetilde{A}_{fx} = N(A_x)(D(A_x))^{-1}.$$

As $D(A_x)$ is close to the invertible map λ_x^u indeed this is possible for small pertubations. The norm of both N and D^{-1} are even smaller than $\theta + \epsilon$ with ϵ uniformly small for uniformly small perturbations. So taking sufficiently uniformly small perturbations, we may assure that $\Gamma_{fx}(A_x) \in C^u_{fx}$.

To obtain contraction we will apply the mean value theorem. The derivative of $\Gamma_{f(x)}(A_x)$ with respect to A_x in a direction $H \in L(E^u_x; E^s_x)$ is given by

$$\Gamma'_{f(x))}(A).H = (\lambda^s - N(A_x)D^{-1}\epsilon^u)HD^{-1}.$$

Again for a small enough perturbation we have $\|\Gamma'(A)\| \leq (\theta + \epsilon)^2$ which can be made strictly smaller than $\theta' < 1$ for small enough perturbations. By the mean-value theorem and convexity with respect to A_x in the domain for C^u_x we get for A_1, A_2 $\|\Gamma(A_1) - \Gamma(A_2)\| \leq \theta'\|A_1 - A_2\|$ as claimed.

Now define $\Gamma : C^u \to C^u$ to be the map:

$$\Gamma(A)_x = \Gamma_{f^{-1}(x)}(A_{f^{-1}x}).$$

By the above this is a uniform contraction of the A-field (for the uniform norm over M) so starting e.g. with the zero field and iterating we get a fixed point field A^*. Since a continuous field A is mapped to a continuous field, so is the uniform limit A^*. Finally at each $x \in M$ the new f-invariant u-bundle is given by $E^u_x(f) = G_x(A^*_x)$.

We may apply the same using the inverse map for the s-bundle thus prooving the theorem. □

3.2 Local s/u-manifolds

We admit in the following that the procedure for defing local unstable/stable manifolds for a fixed point may be adopted to the Anosov situation and for each $x \in M$ produce locally defined submanifolds $W^u_{\text{loc}}(x)$ and $W^s_{\text{loc}}(x)$ with the property that $f(W^s_{\text{loc}}(x)) \subset W^s_{\text{loc}}(fx)$ and $f^{-1}(W^u_{\text{loc}}(fx)) \subset W^u_{\text{loc}}(x)$

Note that at each point $y \in W^u_{\text{loc}}(x)$ the tangent space to the sub-manifold is precisely $E^u_x(f)$ found above (and similarly for the stable direction). Also $x \in W^u_{\text{loc}}(y)$ when x and y are close enough then

$$y \in W^u_{\text{loc}}(x) \;\Leftrightarrow\; x \in W^u_{\text{loc}}(y).$$

3.3 Markov partitions

Definition 3.4 A rectangle in M is a compact subset $R \subset M$ with non-empty interior, such that whenever $x, y \in R$ then the intersection $W^u_{\text{loc}}(x) \cap W^s_{\text{loc}}(y)$ contains exactly one point, denoted $[x, y] \in R$.

We assume tacitly that the rectangle is of small enough diameter for this definition to make sense. We write

$$W^u_R(x) = R \cap W^u_{\text{loc}}(x) \quad \text{and} \quad W^s_R(x) = R \cap W^s_{\text{loc}}(x)$$

for the restriction of the local s/u-manifolds to R.

Definition 3.5 A Markov partition of M is a finite partition of M into rectangles $R_1, \ldots, R_p$ with the following properties:

1. $\text{Int}(R_i) \cap \text{Int}(R_j) = \emptyset$ for $i \neq j$.
2. For $x \in \text{Int}(R_i)$ and $f(x) \in \text{Int}(R_j)$:
   (a) $f(W^u_{R_i}) \supset W^u_{R_j}$,
   (b) $f(W^s_{R_i}) \subset W^s_{R_j}$.

   We write $t_{ij} = 1$ when such an x exists and $t_{ij} = 0$, otherwise.

Example : Let $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$. The map $f : \mathbb{T}^2 \to \mathbb{T}^2$, given by $(x_1, x_2) \mapsto (x_1 + x_2, x_1) \mod \mathbb{Z}^2$ t is Anosov and admits e.g. a 3 element Markov partition obtained by cutting along the s/u-manifold of the fixed point at $(0, 0)$ (see exercise sheet).

Remark 3.5.1: We note that the boundary of a rectangle R_i in a Markov partition is itself constituted of stable/unstable manifold pieces, $\partial^u R_i$ and $\partial^s R_i$. When $t_{ij} = 1$ then we have $f(\partial^u R_i) \supset \partial^u R_j)$ and $f^{-1}(\partial^s R_j) \supset \partial^s R_i)$.

3.4 Symbolic dynamics

Given a Markov transition matrix t of size $p \times p$ we define the associated shift-space:

$$\Sigma_t = \{\xi : \mathbb{Z} \to \{1, \cdots, p\} \mid t_{\xi_k, \xi_{k+1}} = 1,\ \forall k \in \mathbb{Z}\}$$

The so-called shift map is defined as the bijective map $\sigma : \Sigma_t \to \Sigma_t$, such that $(\sigma(\xi))_k = \xi_{k+1}$, for all $k \in \mathbb{Z}$. For the map ξ we may represent it as an ordered list of its values:

$$\xi = (\cdots\ \xi_{-2}\ \xi_{-1}\ .\ \xi_0\ \xi_1\ \xi_2\ \cdots)$$

where $t_{\xi_k, \xi_{k+1}} = 1$ for all k and with the *dot* indicating the place of ξ_0. The shift map takes the whole sequence and shift it one step to the left or, equivalently, shift the dot one step to the right. We write:

$$\xi^+ = (\ \xi_0\ \xi_1\ \xi_2\ \cdots) \text{ and } \xi^- = (\cdots\ \xi_{-2}\ \xi_{-1}\ .\ \xi_0\ )$$

for the part from ξ_0, and up to ξ_0, respectively (both including ξ_0). One should think of ξ^+ as the future visits of rectangles and of ξ^- as the past visits.

Proposition 3.6 There is a surjective map: $\pi : \Sigma_t \to M$ so that for every $\xi \in \Sigma_t$ there is a unique point $x = \pi(\xi) \in M$ so that $f^k(x) \in R_{\xi_k}$ for every $k \in \mathbb{Z}$.

Conversely, if $x \in M$ then there is a (not necessarily unique) $\xi(x) \in \Sigma_t$ for which $\pi(\xi(x)) = x$.

One has: $f(\pi(\xi)) = \pi(\sigma(\xi))$, $\forall \xi \in \Sigma_t$.

Furthermore, if $x = \pi(\xi) \in R = R_{\xi_0}$ then

1. $y \in W^s_R(x)$ iff $\xi^+(x) = \xi^+(y)$,
2. $y \in W^u_R(x)$ iff $\xi^-(x) = \xi^-(y)$.

Proof: Suppose first that all iterates $f^k(x) \in \mathrm{Int} R_{\xi_k}$, $k \in \mathbb{Z}$. By definition $t_{\xi_k, \xi_{k+1}} = 1$ so $\xi \in \Sigma_t$. Clearly, also $f \circ \pi(\xi) = \pi \circ \sigma(\xi)$.

By the Markov property (b) we have $f(W^s_{\mathbb{R}_{\xi_k}}) \subset W^s(R_{\xi_{k+1}})$ for all $k \in \geq 0$, so every point of $W^s_{R_{\xi_0}}(x)$ has the same future symbolic orbit as x. If $y \in R_{\xi_0} \setminus W^s_{R_{\xi_0}}$ then $z = [y, x] \in W^s_{R_{\xi_0}}$ but is distinct from y, separated by a piece of unstable manifold. Iterating forward in time this piece of unstable manifold will expand and at some point $f^k(z)$ and $f^k(y)$ will no longer be in the same reactangle. Thus $W^s_{R_{\xi_0}}(x)$ is precisely the set of points in R_{ξ_0} which has the same forward symbolic orbit as x.

Similarly for the past symbolic orbit and $W^u_{R_{\xi_0}}(x)$.

It follows that x is the unique point in M which has precisely the symbolic orbit given by $\xi = \xi(x)$.

If $f^k(x)$ at some point is on the s-boundary of a rectangle then in the future, by the Markov properties, it has also to be on the s-boundary of a rectangle, Similarly if on a u-boundary of a rectangle, then in the past is also has to be on the u-boundary of a rectangle. □.

3.5 Periodic points

When $x = f^n(x)$ then for every $\xi \in \Sigma_y$ such that $\pi(\xi) = x$ we have $\sigma^n(\xi) = \xi$. We write $(\bar{(}\xi_0\xi_1 \ldots \xi_{n-1})$ for the corresponding periodic orbit in Σ_t. On the other hand this is a periodic orbit in Σ_t iff $t_{\xi_0\xi_1} \cdots t_{\xi_{n-1}\xi_0} = 1$. Thus, in order to count the number of periodic orbits of length n we may compute it as follows:

$$\#\mathrm{Fix}(\sigma^n) = \sum\{t_{\xi_0\xi_1} \cdots t_{\xi_{n-1}\xi_0} \mid \xi \in \Sigma_t\} = \mathrm{tr}(t^n).$$

3.6 Artin-Mazur zeta function

The so-called Artin-Mazur zeta-function, orginally introduced in the contex of Algebraic Geometry is given by as the formal power series in the number of fixed points of a map f:

$$\zeta_f(z) = \exp\left(\sum_{n \geq 1} \frac{z^n}{n} \#\mathrm{Fix}(f^n)\right).$$

For the symbolic map we simply have

$$\zeta_\sigma(z) = \exp\left(\sum_{n\geq 1} \frac{z^n}{n} \#\mathrm{Fix}\binom{n}{\sigma}\right).$$

If ${}^{(}t) = \{\lambda_1, \ldots, \lambda_d\}$ we have $\mathrm{tr}(t^n) = \sum_i \lambda_i^n$ which yields

$$(\zeta_\sigma(z))^{-1} = \prod_{i=1}^{d} (1 - \lambda_i z).$$

For the map f you have take into account that the fixed point which is used to created the Markov partition will be badly counted so you have to correct for this. This is part of an exercise sheet. (Here we assume dimension 1+1. The same holds true in higher dimension but is considerably more difficult to prove.