TD 01 : Introduction to Dynamical Systems

Toral endo- and automorphisms

Let $\pi : \mathbb{R}^n \to \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the canonical projection. We denote by $[x] = x + \mathbb{Z}^n = \pi(x)$ the equivalence class of $x \in \mathbb{R}^n$. Let $\mathrm{L}_n = \mathcal{M}_n(\mathbb{Z})$ be the ring of $n \times n$ matrices with integer entries. We denote by $\mathrm{GL}_n(\mathbb{Z})$ the multiplicative group of invertible elements of L_n, and by I_n its identity element.

(1) For every $A \in L_n$, prove that $A \in \mathrm{GL}_n(\mathbb{Z})$ if and only if $|\det(A)| = 1$.

(2) Prove that for every element $A \in \mathrm{L}_n$ there is a unique map $f_A : \mathbb{T}^n \to \mathbb{T}^n$ making the diagram to the right commutative. Prove that if $A \in \mathrm{GL}_n(\mathbb{Z})$ then f_A is invertible.

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{A} & \mathbb{R}^n \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathbb{T}^n & \xrightarrow[f_A]{} & \mathbb{T}^n \end{array}$$

(3) Prove that if $[x] \in \mathrm{Per}_p(f_A)$ then $(I_n - A^p)x \in \mathbb{Z}^n$.

(4) Prove that $\mathrm{Per}_p(f_A)$ has infinite cardinality if and only if 1 is an eigenvalue of A^p.

We assume from now on that every eigenvalue of A has modulus different from 1.

(5) Prove that if $[x] \in \mathrm{Per}_p(f_A)$ then $x = \frac{1}{q}\overrightarrow{m}$ where $q = \det(I_n - A^p)$ and $\overrightarrow{m} \in \mathbb{Z}^n$.

(6) Prove that if $x = \frac{1}{q}\overrightarrow{m}$ with $q \in \mathbb{Z} \smallsetminus \{0\}$ and $\overrightarrow{m} \in \mathbb{Z}^n$ then $[x]$ is a periodic point for f_A.

(7) Prove that the periodic points are dense in $\mathbb{T}^n$.

From now on, we consider the case $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

(8) Prove that there exist two unique one-dimensional A-invariant linear subspaces $E^u \subset \mathbb{R}^2$ and $E^s \subset \mathbb{R}^2$ such that

$$\forall x \in E^s : \lim_{k\to+\infty} A^k x = 0 \quad \text{and} \quad \forall x \in E^u : \lim_{k\to-\infty} A^k x = 0.$$

(9) Prove that $W^s(0) = \pi(E^s) \subset \mathbb{T}^2$ and $W^u(0) = \pi(E^u) \subset \mathbb{T}^2$ are f_A-invariant, that

$$\xi \in W^s(0) \text{ if and only if } \lim_{k\to+\infty} f_A^k(\xi) = 0$$

and that

$$\xi \in W^u(0) \text{ if and only if } \lim_{k\to-\infty} f_A^k(\xi) = 0\,.$$

(10) Prove that $W^s(0)$ and $W^u(0)$ are both dense in $\mathbb{T}^2$.

(11) Describe the (forward and backward) orbit of $\xi \in W^s(0) \cap W^u(0)$.

Topological dynamical systems

A (discrete time) *topological dynamical system* is a pair (X, ϕ) with X a topological space and $\phi : X \to X$ a continuous map. It is *invertible* if ϕ is an homeomorphism. A point $x \in X$ is *positively recurrent* if x belongs to its *ω-limit set* $\omega(x) = \bigcap_{n\in\mathbb{N}} \overline{\{\phi^k(x) \ : \ k \geq n\}}$. We say that (X, ϕ) is *topologically transitive* if (X, ϕ) is invertible and if for all nonempty open subsets U et V in X, there exists $n \in \mathbb{Z}$ such that the intersection $U \cap \phi^{-n}(V)$ is non empty. We say that (X, ϕ) is *topologically mixing* if for all nonempty open subsets U et V in X, there exists $n \in \mathbb{N}$ such that for every $k \geq n$, the intersection $U \cap \phi^{-k}(V)$ is non empty. We say that (X, ϕ) is *minimal* if (X, ϕ) is invertible and if for every $x \in X$, its orbit $\mathcal{O}(x) = \{\phi^n(x) : n \in \mathbb{Z}\}$ is dense. A *semi-conjugation* from (X, ϕ) to a topological dynamical system (X', ϕ') is a continuous surjective map $h : X \to X'$ such that $h \circ \phi = \phi' \circ h$.

(1) Let $h : X \to X'$ be a semi-conjugation between two topological dynamical systems (X, ϕ) and (X', ϕ'). Among the following properties, which ones are satisfied by (X', ϕ') if they are by (X, ϕ) ?

- having its periodic points dense,
- having a positively recurrent point,
- being topologically transitive,
- being topologically mixing
- being minimal.

(2) Prove that an invertible topological dynamical system (X, ϕ) is minimal if and only if the only nonempty closed ϕ-invariant subset of X is X.

(3) If X is a complete separable metric space, prove that an invertible topological dynamical system (X, ϕ) is topologically transitive if and only if there exists a point in X whose orbit is dense in X.