TD 02 : Dynamical Systems and invariant measures : The doubling map

Let $I = [0, 1]$, let $\mathcal{B}$ be the Borel σ-algebra of I and let λ be the Lebesque measure on I. We consider the (discontinuous) *doubling map* T on I defined by $T : x \mapsto 2x \bmod 1$. Recall that $\lambda(T^{-1}A) = \lambda(A)$ for every $A \in \mathcal{B}$ (i.e. that λ is T-invariant). Below we consider two ways of proving the strong mixing (whence ergodicity) property of T.

Composing f with T^n

1. Let $a, b \in I$ with $a < b$, let $J = [a, b]$ and let $f = \mathbf{1}_J$ be the characteristic function of J. Write $f \circ T^n$ as a sum of characteristic functions.

2. Prove that for every $g \in C^0([0, 1])$, we have
$$\lim_{n\to+\infty} \int_I f \circ T^n \; g \; d\lambda = \int_I f \; d\lambda \int_I g \; d\lambda \,.$$
(Hint: Reformulate the first integral as a suitable Riemann sum):

3. Prove that the above limit also holds, first for all $f \in L^\infty(I, \lambda)$ and $g \in C^0(I)$, then for all $f \in L^\infty(I, \lambda)$ and $g \in L^1(I, \lambda)$.

4. We denote by δ_z the unit Dirac mass at any point $z \in I$. Fix $a \in [0, 1]$ and for every $n \in \mathbb{N}$, let
$$f_n = \frac{1}{2^n} \sum_{\xi \in I \,:\, T^n(\xi) = a} \delta_\xi \,.$$
Prove that for every $g \in C^0(I)$ we have $\lim_{n\to+\infty} \int_I f_n \; g = \int_I g \; d\lambda$, but that this may fail if $g \in L^1(I)$.
Why is this situation different from the previous one where we integrated against $f \circ T^n$?

Composing by pre-images of T

5. Let $I, \mathcal{B}, \lambda$ be as above, and consider the transfer operator $L : L^1(I, \lambda) \to L^1(I, \lambda)$ defined, for all $\phi \in L^1(I, \lambda)$ and $y \in I$ by
$$L\phi(y) = \frac{1}{2}\left(\phi\left(\frac{y}{2}\right) + \phi\left(\frac{1+y}{2}\right)\right) .$$

6. Prove that for $\phi \in C^1(I)$, we have $\left\| L^n\phi(x) - \displaystyle\int_0^1 \phi(t) \; dt \right\|_\infty \leq \frac{1}{2^n} \|\phi'\|_\infty$. (Hint: Use the mean value theorem).

7. Use the previous result to prove that for all $f \in L^\infty(I, \lambda)$ and $g \in C^1(I)$, we have
$$\lim_{n\to+\infty} \int_I f \circ T^n \; g \; d\lambda = \lim_{n\to+\infty} \int_I f \; (L^n g) \; d\lambda = \int_I f \; d\lambda \int_I g \; d\lambda \,.$$
Prove that for all $f \in L^\infty(I, \lambda)$ and $g \in L^1(I, \lambda)$, we have
$$\lim_{n\to+\infty} \int_I f \circ T^n \; g \; d\lambda = \int_I f \; d\lambda \int_I g \; d\lambda \,.$$