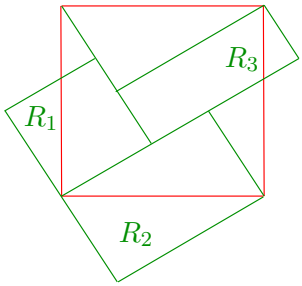
TD 3 : Hyperbolic maps, symbolic dynamics and zeta-functions.

(1) Let $A = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, let $f = f_A : \mathbb{T}^2 \to \mathbb{T}^2$ be the induced linear automorphism of the 2-torus $\mathbb{T}^2$ and let $\gamma = \frac{1+\sqrt{5}}{2}$. Setting $e^u = \left(\begin{smallmatrix} 1 \\ 1/\gamma \end{smallmatrix}\right)$ and $e^s = \left(\begin{smallmatrix} -1/\gamma \\ 1 \end{smallmatrix}\right)$, we have $A(e^u) = \gamma\, e^u$ and $A(e^s) = -\frac{1}{\gamma}\, e^s$.

(a) Recall how to construct a Markov partition for f consisting of the three closed rectangles in the figure, say R_1, R_2, R_3, bounded by pieces of the unstable/stable manifolds of the origin, such that they have disjoint interior and their union covers $\mathbb{T}^2$.



(b) Let $t = (t_{ij})_{1\le i,j\le 3}$ be the 3x3 transition matrix defined for every $i, j \in [\![1, 3]\!]$ by $t_{ij} = 1$ if $f(\overset{\circ}{R_i}) \cap \overset{\circ}{R_j} \neq \emptyset$, and zero otherwise. Compute the eigenvalues of t. Let $\Sigma_t = \{\xi = (\xi_k)_{k\in\mathbb{Z}} \in \Sigma_3 = [\![1, 3]\!]^{\mathbb{Z}} : \forall\ k \in \mathbb{Z},\ t_{\xi_k,\xi_{k+1}} = 1\}$ and let σ_t be the restriction of the full shift to Σ_t. Prove that (Σ_t, σ_t) is semi-conjugated to $(\mathbb{T}^2, f_A)$ by a surjective continuous map $\pi : \Sigma_t \to \mathbb{T}^2$.

(c) Prove that the periodic points in the two spaces are in 1-1 correspondence except for the fixed point 0 of f. Compute $\lim\limits_{n\to\infty} \dfrac{1}{n} \log\big(\operatorname{card}(\operatorname{Fix}(f^n))\big)$.

(d) The Artin-Mazur zeta-function for a hyperbolic map h is defined, for z small enough, as:

$$\zeta_h(z) = \exp\Big(\sum_{n=1}^{\infty} \frac{z^n}{n} \operatorname{card}(\operatorname{Fix}(h^n))\Big).$$

Compute ζ_f and prove that it defines a rational function in z.

Hints : First compute ζ_{σ_t}, using that for every $n \in \mathbb{N}$, the trace of t^n is the sum of its eigenvalues raised to the power n. Then correct the 'wrong' counting at the origin.

(2) Define the rectangles $A = [0, 1/3] \times [0, 1]$ and $B = [2/3, 1] \times [0, 1]$. Define the map $g : A \cup B \to [0, 1] \times [0, 1]$ (make a drawing) by

$$g : (x_1, x_2) \mapsto \begin{cases} (\ 3x_1\ ,\ x_2/3\ ) & \text{if } x \in A \\ (\ 3 - 3x_1\ ,\ 1 - x_2/3\ ) & \text{if } x \in B \end{cases} \tag{0.1}$$

Let $\Omega = \bigcap_{k\in\mathbb{Z}} g^k(A \cup B)$ denote the maximal invariant set of g. Then $\Omega = g\Omega = g^{-1}\Omega$.

(a) Prove that $R_A = A \cap \Omega$ and $R_B = B \cap \Omega$ are products of (subsets of) pieces of unstable and stable manifolds of g. The pair $\{R_A, R_B\}$ is called a *Markov partition* for (Ω, g).

(b) Prove that there exists a bijective map $j : \Omega \to \Sigma_2 = \{A, B\}^{\mathbb{Z}}$ which conjugates (Ω, g) to the full shift (Σ_2, σ).

(c) Compute the Artin-Mazur zeta-function ζ_g.