On the regularity of the extinction probability of a branching process in varying and random environments

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Abstract

Abstract. We consider a supercritical branching process in time-dependent environment ξ. We assume that the offspring distributions depend regularly (C^k or real-analytically) on real parameters λ. We show that the extinction probability $q_\lambda(\xi)$, given the environment ξ "inherits" this regularity whenever the offspring distributions satisfy a condition of contraction-type. Our proof makes use of the Poincaré metric on the complex unit disk and a real-analytic implicit function theorem.

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1 Introduction

Consider a Galton-Watson process, $(G_n)_{n\geq 0}$, with $G_0 = 1$ and where the offspring distribution, p_λ, of the particles depends on external parameters, $\lambda \in U$, with $U \subset \mathbb{R}^n$ an open subset [in a growth-experiment λ could describe e.g. temperature, concentration of chemicals etc.]. For fixed λ it is well known that the extinction probability of the process $q_\lambda \in [0,1]$ is the smallest fixed point $q = \phi_{p_\lambda}(q) \in [0,1]$ of the probability generating function $\phi_{p_\lambda}(s) \equiv \sum_{k=0}^{\infty} p_\lambda(k)s^k$ for the offspring distribution (for details see e.g. [2, Chapter 1, Theorem 5.1]).

It is natural to ask how the fixed point depends upon λ. We are here interested in the regularity of this dependency. The answer is simple : If for $\lambda_0 \in U$ we have $q_{\lambda_0} < 1$ (and $\phi_{p_{\lambda_0}}$ is not the identity) then in a neighborhood of λ_0 the map $\lambda \mapsto q_\lambda$ is as smooth as the map $(\lambda, s) \mapsto \phi_{p_\lambda}(s)$. To see this note that $s = q_{\lambda_0}$ is a zero of the map $F(\lambda_0, s) = s - \phi_{p_{\lambda_0}}(s)$. Using $\phi_{p_{\lambda_0}}(1) = 1$, $\phi_{p_{\lambda_0}}(q_{\lambda_0}) = q_{\lambda_0} < 1$ and strict convexity we see that $\frac{\partial}{\partial s}_{|s=q_{\lambda_0}} \phi_{p_{\lambda_0}}(s) < 1$. The graph of $\phi_{p_{\lambda_0}}$ thus cuts the diagonal transversally[1], so that $\partial_s F(\lambda_0, q_{\lambda_0}) = 1 - \partial_s \phi_{p_{\lambda_0}}(q_{\lambda_0})$ is non-zero, whence invertible. Then (see e.g. [10, p 364, Theorem 2.1]) there is a locally defined implicit function q_λ which is as smooth as F and for which $F(\lambda, q_\lambda) = 0$ in a neighborhood of $\lambda = \lambda_0$. And this function determines precisely our extinction probability.

On the other hand, if $q_{\lambda_0} = 1$, a discontinuity may appear in the derivative of q_λ, as the following simple example shows (a geometric law): $\phi_{p_\lambda}(s) = \frac{\lambda}{1-(1-\lambda)s}$, $\lambda \in (0,1)$ for which $q_\lambda = 1$ for $\lambda \in [\frac{1}{2}, 1)$ but $q_\lambda = \lambda/(1-\lambda)$ for $\lambda \in (0, \frac{1}{2}]$. The transition occurs precisely at the border between what is known in the theory of branching processes as the subcritical and the supercritical regimes (where $\phi'_{p_\lambda}(1) - 1$ changes sign and the above-mentioned transversality

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[1] for more on the notion of transversality see e.g. [8, p. 289]

at the fixed point $s = 1$ is lost).

If in addition the offspring distribution may change from generation to generation in an environment, the difficulty of the above problem increases considerably. The extinction probability (depending on the environment, see e.g. [1]) then satisfies a functional equation (1). Nevertheless, we show (Theorem 2.2) that smoothness of the extinction probability persists when imposing Assumption 2.1 (a contraction property of the generating functions). A similar result is stated in section 6 for multitype branching processes. The main novelty in the proof is that transversality, which is needed for an implicit function theorem, is achieved by using the Poincaré metric from complex analysis. This idea may be useful when studying the regularity of solutions of other functional equations involving probability generating functions. We are grateful to anonymous referees for valuable suggestions and remarks.

2 Branching process in varying and random environments

Consider in the following a branching process $(Z_n)_{n\geq 0}$ where the offspring distribution of the particles changes from one generation to the other (depending on an environment variable) and also depends on a parameter $\lambda \in U$ (where $U \subset \mathbb{R}^n$ is open and non-empty). The environment is described by a given sequence $\xi = (\xi_n)_{n\geq 0}$ with values in a dynamical system $(E, \mathcal{E}, T)$ where E is a set, $\mathcal{E}$ a σ-algebra and $T : E \to E$ an $\mathcal{E}$-measurable map. Here, $\xi_{n+1} = T\xi_n$.

We construct (Z_n) as follows : To each environment-value $\xi \in E$ and each parameter-value $\lambda \in U$ we associate a probability law $p_\lambda(\xi)$ on the set of non-negative integers $\mathbb{N}_0$. We write $\phi_{p_\lambda(\xi)}(s) = \sum_{k\geq 0} (p_\lambda(\xi))_k s^k$ for the probability generating function (pgf) associated to this law. We then consider a collection of particles that reproduce independently of each other. Call Z_0 the initial population (0th generation). Each particle reproduces and gives birth to particles following the law $p_\lambda(\xi)$. This gives the first generation denoted Z_1. Similarly, we denote Z_n the n-th generation. To go from the n-th generation to the $(n+1)$-th generation, each particle reproduces independently following the law $p_\lambda(T^n\xi)$, i.e. in the environment shifted n times, so that if we denote by $\mathbb{P}_{\lambda,\xi}$ the distribution of the branching process given λ and ξ and by $\mathbb{E}_{\lambda,\xi}$ the corresponding expectation, we have

$$\mathbb{E}_{\lambda,\,\xi}(s^{Z_{n+1}}|Z_n = 1) = \phi_{p_\lambda(T^n\xi)}(s).$$

Let $q_\lambda(\xi) = \mathbb{P}_{\lambda,\xi}(\lim_{n\to+\infty} Z_n = 0|Z_0 = 1)$ denote the extinction probability of the branching process given parameter λ and environment ξ. We will study the regularity of $q_\lambda(\xi)$ as a function of λ under the following :

Assumption 2.1 *$\forall\lambda \in U$ there is $\alpha = \alpha(\lambda) < 1$ so that $\sup_{\xi\in E} \phi_{p_\lambda(\xi)}(\alpha) < \alpha$.*

Under this assumption it is clear that non-certain extinction of the process occurs. Indeed, by setting $\alpha_1 = \sup_{\xi\in E} \phi_{p_\lambda(\xi)}(\alpha)$, we have $\phi_{p_\lambda(\xi)}([0, \alpha]) \subset [0, \alpha_1]$, $\xi \in E$, which implies that $q_\lambda(\xi) \leq \alpha_1 < \alpha(\lambda) < 1$.

A class of probability distributions satisfying this assumption may be described in the following way : Fix $0 < \delta < \frac{1}{4}$ and suppose that every probability distribution (uniformly in λ and ξ) satisfies $\sum_{n\geq 2} p_n \geq p_0 + \delta$ (where $p_n \equiv p_\lambda(\xi)(n)$, $n \in \mathbb{N}$). A straightforward calculation shows that $\phi_{p_\lambda(\xi)}(1 - \delta) \leq 1 - \delta - \delta^2/4$. Thus the above condition is verified with $\alpha = 1 - \delta$ and $\sup_{\xi\in E} \phi_{p_\lambda(\xi)}(\alpha) \leq \alpha - \delta^2/4 < \alpha$.

Theorem 2.2 *Suppose that Assumption 2.1 is satisfied. If $\lambda \in U \to \mathbf{p}_\lambda = (p_\lambda(\xi))_{\xi\in E} \in \mathcal{L}^\infty(E; \ell^1(\mathbb{N}_0))$ is of class $\mathcal{C}^k$ or is real-analytic (see section 4) then so is $\lambda \in U \to \mathbf{q}_\lambda = (q_\lambda(\xi))_{\xi\in E} \in \mathcal{L}^\infty(E)$.*

Remarks. Our setup covers e.g. the following two cases :

1. *Varying Environment:* $E = \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$, the set of non-negative integers and $T\xi = \xi + 1$ ($\mathcal{E}$ contains all subsets of $\mathbb{N}_0$). This corresponds to the model of branching process in a varying environment (BPVE), which is a generalization of the Galton-Watson process with no time-homogeneity reproduction assumption (see [6, 5]).

2. *Random Environment:* $(E, \mathcal{E}, T, \pi)$ is a probability-space equipped with a measurable transformation T and such that the probability π is T-invariant and ergodic. Then $(Z_n)_{n\geq 0}$ is called a branching process in a random environment (BPRE). It has two levels of stochasticity, the first one is given by the realizations ξ of the environment while the second one, once ξ is fixed, is determined by the trajectories of a Markov chain (see [1]). One may replace in assumption 2.1, "supremum" by "π-essential supremum". All conclusions are then modulo sets of measure zero.

Let us here briefly outline the proof of the theorem:

Given a parameter λ and initial environment ξ write $\Phi^{(n)}_{\lambda,\xi}(s) \equiv \mathbb{E}_{\lambda,\xi}(s^{Z_n}|Z_0 = 1)$. Since $\mathbb{E}_{\lambda,\xi}(s^{Z_n}|Z_1) = \left(\Phi^{(n-1)}_{\lambda,T\xi}(s)\right)^{Z_1}$, we have the recurrence relation $\Phi^{(n)}_{\lambda,\xi}(s) = \phi_{p_\lambda(\xi)}(\Phi^{(n-1)}_{\lambda,T\xi}(s))$. As is well-known (we refer to e.g. [1] for details) the extinction probability $q_\lambda(\xi) = \{\lim_{n\to+\infty} Z_n = 0\}$ is obtained as the limit of the increasing (and bounded) sequence $\mathbb{P}_{\lambda,\xi}(Z_n = 0|Z_0 = 1) = \Phi^{(n)}_{\lambda,\xi}(0)$. Continuity of $\phi_{p_\lambda(\xi)}$ then implies that $q_\lambda(\xi)$ is a solution of the functional equation :

$$q_\lambda(\xi) = \phi_{p_\lambda(\xi)}\left(q_\lambda(T\xi)\right). \tag{1}$$

Setting $\alpha_1 = \sup_{\xi\in E} \phi_{p_\lambda(\xi)}(\alpha)$, Assumption 2.1 implies $\phi_{p_\lambda(\xi)}([0,\alpha]) \subset [0,\alpha_1]$, $\xi \in E$. Using Lemma 4.3 below, it follows that $q_\lambda(\xi)$ is the unique solution of (1) satisfying $0 \leq q_\lambda(\xi) \leq \alpha < 1$, $\xi \in E$.

In order to study the regularity of $\lambda \in U \to (q_\lambda(\xi))_{\xi\in E}$ we consider it as a fixed point of the map :

$$(\Gamma(\mathbf{p}_\lambda, \mathbf{q}))(\xi) = \phi_{p_\lambda(\xi)}\left(q(T\xi)\right),$$

acting upon $\mathbf{q} \in \mathcal{L}^\infty(E)$, with $\|\mathbf{q}\|_\infty \leq \alpha$. We will proceed in two steps :

1. For fixed λ we will use Assumption 2.1 and the Poincaré metric from complex analysis to show that the linearization $\frac{\partial}{\partial \mathbf{q}}\Gamma(\mathbf{p}_\lambda, \mathbf{q})$ of Γ_λ at the fixed point $\mathbf{q} = \mathbf{q}_\lambda$ has spectral radius strictly smaller than one.

2. The $\mathbf{q}$-derivative of the smooth map :

$$(\lambda, \mathbf{q}) \mapsto \mathbf{q} - \Gamma(\mathbf{p}_\lambda, \mathbf{q}) \tag{2}$$

is therefore invertible at $\mathbf{q} = \mathbf{q}_\lambda$. An implicit function theorem then implies that the solution q_λ inherits the regularity of the map (2) and we arrive at the wanted conclusion.

3 Examples

Let E be a set equipped with a σ-algebra $\mathcal{E}$ and let $T : E \to E$ be an $\mathcal{E}$-measurable map. Let $\xi \in E \to a_\xi \in (0,1]$ be a measurable map and $0 < \lambda < 1$ a parameter. As in the above, for given $\xi \in E$ and parameter $0 < \lambda < 1$, we denote by $p_\lambda(\xi)$ and $q_\lambda(\xi)$ the offspring distribution (on $\mathbb{N}_0$) and the extinction probability, respectively, of a branching process (Z_n) starting at time zero with one particle.

Example 1. (Binary offspring distribution) Let

$$(p_\lambda(\xi))_0 = \lambda a_\xi,\ (p_\lambda(\xi))_2 = 1 - \lambda\, a_\xi,\ (p_\lambda(\xi))_k = 0 \text{ for } k \in \mathbb{N}_0 \setminus \{0,2\}.$$

As a function of λ we write $\mathbf{p}_\lambda = (p_\lambda(\xi))_{\xi\in E} \in X \equiv \mathcal{L}^\infty(E;\ell^1(\mathbb{N}_0))$ in the polynomial form : $\mathbf{p}_\lambda = \mathbf{f}_0 + \lambda \mathbf{f}_1$ with $\mathbf{f}_0 = (0,0,1,0,\ldots)_{\xi\in E} \in X$ and $\mathbf{f}_1 = (a_\xi, 0, -a_\xi, 0, \ldots)_{\xi\in E} \in X$ being measurably bounded and real-valued. The map $\lambda \in (0,1) \mapsto \mathbf{p}_\lambda = (p_\lambda(\xi))_{\xi\in E} \in X$ is real analytic in λ.

The associated pgf is $\phi_{p_\lambda(\xi)}(s) = \lambda\, a_\xi(1-s^2)+s^2$. When $\lambda \in (0,\frac{1}{2})$ we may find α such that $\frac{\lambda}{1-\lambda} < \alpha < 1$. Our Assumption 2.1 is satisfied since uniformly in $\xi \in E$:

$$\phi_{p_\lambda(\xi)}(\alpha) \leq \lambda(1-\alpha^2)+\alpha^2 < \frac{\alpha}{1+\alpha}(1-\alpha^2)+\alpha^2 < \alpha.$$

Consequently, by Theorem 2.2 the map $\lambda \in (0,\frac{1}{2}) \to \mathbf{q}_\lambda \in \mathcal{L}^\infty(E)$ is real-analytic. In general, $\mathbf{q}_\lambda$ may not admit an analytic extension at $\lambda = \frac{1}{2}$. To see this, simply consider $a_\xi \equiv 1$, $\forall \xi \in E$ for which $q_\lambda(\xi) = \min\{1, \frac{\lambda}{1-\lambda}\}$ and this function has a discontinuous derivative at $\lambda = \frac{1}{2}$.

Example 2 (Geometric offspring distribution) Let

$$(p_\lambda(\xi))_k = (\lambda\, a_\xi)(1-\lambda\, a_\xi)^k,\;\; k=\{0,1,2,\ldots\}. \tag{3}$$

The associated pgf is

$$\phi_{p_\lambda(\xi)}(s) = \frac{\lambda\, a_\xi}{1-s\,(1-\lambda\, a_\xi)}.$$

Suppose that $\lambda \in U \equiv (0,\frac{1}{2})$. Setting $\alpha = \frac{1}{2(1-\lambda)} \in (\frac{1}{2},1)$ we see that $\phi_{p_\lambda(\xi)}(\alpha) \leq \frac{\lambda}{1-\alpha\,(1-\lambda)} = 2-\frac{1}{\alpha} < \alpha$ so that Assumption 2.1 is verified. To apply our Theorem we need to show that the map $\lambda \in U \mapsto \mathbf{p}_\lambda = (p_\lambda(\xi))_{\xi\in E} \in X = \mathcal{L}^\infty(E;\ell^1(\mathbb{N}_0))$ is real-analytic. One shows (we omit the somewhat lengthy calculations) that $\mathbf{p}_\lambda$ is a uniform limit in X of λ-polynomials, whence admits an analytic extension, on the domain

$$\widetilde{U} \equiv \{x+iy \in \mathbb{C} : |y| < x,\; x^2+y^2 < x\}$$

which indeed forms an open neighborhood of U in $\mathbb{C}$.

We may apply our Theorem to conclude that the map $\lambda \in (0,\frac{1}{2}) \to \mathbf{q}_\lambda \in \mathcal{L}^\infty(E)$ is real-analytic. As in the previous example, $\mathbf{q}_\lambda$ may not admit an analytic extension at $\lambda = \frac{1}{2}$ (consider as in the previous example $a_\xi \equiv 1$, $\forall \xi \in E$ so that $q_\lambda(\xi) = \min\{1, \frac{\lambda}{1-\lambda}\}$).

4 The functional setup

Throughout, a C^k-map refers to the standard notion from differential calculus of a C^k-map between Banach spaces, see e.g. [10, Chap XIII]. Analytic means that the function admits locally norm-convergent power-series expansions. Real-analytic, that in addition it takes real values on real vectors and parameters.

Let $\ell^1(\mathbb{N}_0) = \{x : \mathbb{N}_0 \to \mathbb{C} : \|x\|_{\ell^1} = \sum_{n\geq 0} |x_n| < +\infty\}$ denote the space of summable sequences. To each element $x \in \ell^1(\mathbb{N}_0)$ we associate a generating function $\phi_x(s) = \sum x_n s^n$, $|s| \leq 1$. This function is holomorphic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and has a continuous extension to the closed disc. We will consider measurable maps from E into either $\ell^1(\mathbb{N}_0)$ or $\mathbb{C}$. More precisely, we consider the Banach spaces (the maps considered being measurable) :

$$X = \mathcal{L}^\infty(E;\ell^1(\mathbb{N}_0)) = \{\mathbf{x} = (x(\xi))_{\xi\in E} : E \to \ell^1(\mathbb{N}_0) \,:\, \|\mathbf{x}\|_X < +\infty\}$$

with $\|\mathbf{x}\|_X = \sup_{\xi\in E} \|x(\xi)\|_{\ell^1}$ and $Y \equiv \mathcal{L}^\infty(E)$. We identify $\Delta = \{p : \mathbb{N}_0 \to [0,1] : \sum_n p_n = 1\}$ with the space of probability measures on $\mathbb{N}_0$ and write

$$\Delta(E) = \{\mathbf{p} : E \to \Delta \;\;\text{(meas.)}\} \subset X$$

for the space of measurable maps of E into probability measures on $\mathbb{N}_0$. We define a transformation, Γ of $(\mathbf{x},\mathbf{q}) \in X \times B_Y(0,1)$ as follows :

$$\Gamma(\mathbf{x},\mathbf{q})(\xi) \equiv \phi_{x(\xi)}\left(q(T\xi)\right), \quad \xi \in E. \tag{4}$$

Lemma 4.1 *The map $\Gamma : X \times B_Y(0,1) \to Y$ is real-analytic.*

Proof: Measurability of $\xi \to \Gamma(\mathbf{x},\mathbf{q})(\xi)$ follows because the map is a composition of continuous or measurable maps. The image is bounded in norm by $\|\mathbf{x}\|_X$ so is an element of Y. Γ is linear in $\mathbf{x}$ (since ϕ_x is linear in $x \in \ell^1(\mathbb{N}_0)$), whence analytic in $\mathbf{x}$. For $|s|<1$, $\xi \in E$, the function $\phi_{x(\xi)}(s+h)$ admits a convergent and uniformly bounded power-series expansion for $|h| \leq 1-s$. Then also $\Gamma(\mathbf{x},\mathbf{q}+\mathbf{h})$ admits a norm-convergent power-series expansion for $\|\mathbf{h}\|_\infty \leq 1-\|\mathbf{q}\|_\infty$. So Γ is indeed analytic. When $\mathbf{x}$ and $\mathbf{q}$ are real-valued, then so is $\Gamma(\mathbf{x},\mathbf{q})$. □

Write $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$, $r>0$. We will need the following technical Lemma.

Lemma 4.2 *Let $\psi : \mathbb{D}_\alpha \to \mathbb{D}_\alpha$ (with $0<\alpha<1$) be a holomorphic map and let*

$$D\psi(z) = \left|\frac{d\psi(z)}{dz}\right| \frac{\alpha^2 - |z|^2}{\alpha^2 - |\psi(z)|^2}, \tag{5}$$

be the conformal derivative with respect to the Poincaré metric $d_{\mathbb{D}_\alpha}$ on $\mathbb{D}_\alpha$. Then

(a) $D\psi(z) \leq 1$ for all $z \in \mathbb{D}_\alpha$.

(b) If $\psi(\mathbb{D}_\alpha) \subset \mathbb{D}_{\alpha_1}$ with $0<\alpha_1<\alpha$ then $D\psi(z) \leq \alpha_1/\alpha$ for all $z \in \mathbb{D}_\alpha$.

Proof: The first part is a standard result in complex analysis. We refer to e.g. [4, Chapter I.4] for standard properties of the Poincaré metric on Riemann surfaces. The Poincaré metric $d_{\mathbb{D}_\alpha}$ on $\mathbb{D}_\alpha$ is given by the line element $ds = 2|dz|/(\alpha^2-|z|^2)$ (unique up to a constant factor). The conformal derivative with respect to this metric is then given by formula (5). By [4, Theorem I.4.1] its value can not exceed one, as claimed. To see part (b) note that the conformal derivative of $R(z) = \alpha_1 z/\alpha$ verifies :

$$DR(z) = \frac{\alpha_1}{\alpha} \frac{1-|z|^2}{1-\alpha_1^2|z|^2/\alpha^2} \leq \frac{\alpha_1}{\alpha}.$$

Under the hypothesis of (b), $w = R^{-1} \circ \psi(z)$ defines a holomorphic map of $\mathbb{D}_\alpha$ into itself so that $(DR^{-1} \circ \psi)\ D\psi \leq 1$ and therefore

$$D\psi(z) \leq DR(w) \leq \alpha_1/\alpha. \ \square$$

Lemma 4.3 *Let $\Delta(E,\alpha)$ denote the set of $\mathbf{p} \in \Delta(E)$ that verifies*

$$\sup_{\xi \in E} \sum_n (p(\xi))_n \alpha^n < \alpha. \tag{6}$$

Let $\mathbf{p} \in \Delta(E,\alpha)$. Then, the map $\mathbf{q} \in B_Y(0,\alpha) \to \Gamma(\mathbf{p},\mathbf{q}) \in B_Y(0,\alpha)$ has a unique fixed point, $\mathbf{q}_\mathbf{p} \in \mathcal{L}^\infty(E;[0,\alpha]) \subset B_Y(0,\alpha)$. At the fixed point the $\mathbf{q}$-partial derivative $\partial_\mathbf{q}\Gamma(\mathbf{p},\mathbf{q}_\mathbf{p}) \in L(Y)$ has spectral radius strictly smaller than one.

The map $\mathbf{p} \in \Delta(E,\alpha) \to \mathbf{q}_\mathbf{p} \in \mathcal{L}^\infty(E;[0,\alpha])$ is real-analytic.

Proof: Given $\xi \in E$ we write $g_n = \phi_{p(T^n\xi)}$, $n \geq 0$ for the sequence of generating functions along the orbit of ξ. Let α_1 denote the supremum on the left hand side in (6). Then $g_n(\mathbb{D}_\alpha) \subset \mathbb{D}_{\alpha_1}$. As each $g_n : \mathbb{D}_\alpha \to \mathbb{D}_\alpha$ is holomorphic Lemma 4.2 shows that the conformal derivative given by equation (5) verifies : $Dg_n \leq \alpha_1/\alpha$. The composed map, $g^{(n)} = g_0 \circ \cdots \circ g_{n-1}$, then has conformal derivative not exceeding $(\alpha_1/\alpha)^n$. Taking the restriction to $z \in \overline{\mathbb{D}_{\alpha_1}}$ and noting that $w = g^{(n)}(z) \in \mathbb{D}_{\alpha_1}$ we may use (5) to convert back to the standard Euclidean metric and obtain :

$$\left|\frac{dg^{(n)}(z)}{dz}\right| \leq \left(\frac{\alpha_1}{\alpha}\right)^n \frac{\alpha^2-|w|^2}{\alpha^2-|z|^2} \leq \left(\frac{\alpha_1}{\alpha}\right)^n \frac{\alpha^2}{\alpha^2-\alpha_1^2}. \tag{7}$$

By the Mean-value theorem $g^{(n)}_{|\mathbb{D}_{\alpha_1}}$ is $\frac{\alpha^2}{\alpha^2-\alpha_1^2}(\alpha_1/\alpha)^n$-Lipschitz on $\overline{\mathbb{D}_{\alpha_1}}$. The (decreasing) intersection $\cap_{n\geq 0} g^{(n)}(\overline{\mathbb{D}_{\alpha_1}})$ therefore contains a unique element $q_{\mathbf{p}}(\xi) \in \overline{\mathbb{D}_{\alpha_1}}$. Since $g^{(n)}$ maps $[0,\alpha]$ into $[0,\alpha_1] \subset [0,\alpha]$. The unique fixed point must be real-valued and thus verify $0 \leq q_{\mathbf{p}}(\xi) \leq \alpha_1 < \alpha$. Being a limit of measurable functions $\xi \in E \to q_{\mathbf{p}}(\xi)$ is measurable and thus a fixed point of $\mathbf{q} \mapsto \Gamma(\mathbf{p},\mathbf{q})$ with $\mathbf{q_p} \in \mathcal{L}^\infty(E,[0,\alpha[)$. Let $A = \partial_{\mathbf{q}}\Gamma(\mathbf{p},\mathbf{q_p})$ be the $\mathbf{q}$-derivative at the fixed point. The uniform bound (7) implies that

$$\| A^n \|_{L(Y)} \leq \left(\frac{\alpha_1}{\alpha}\right)^n \frac{\alpha^2}{\alpha^2-\alpha_1^2}, \quad n \geq 0. \tag{8}$$

This shows that the spectral radius $\rho_{\rm sp}(A)$ verifies

$$\rho_{\rm sp}(A) = \lim_{n\to\infty} \left(\| A^n \|_{L(Y)}\right)^{1/n} \leq \frac{\alpha_1}{\alpha} < 1. \tag{9}$$

Now, write $F(\mathbf{x},\mathbf{q}) = \mathbf{q} - \Gamma(\mathbf{x},\mathbf{q})$. The map $F : X \times B_Y(0,1) \to Y$, is real-analytic and $F(\mathbf{p},\mathbf{q_p}) = 0$. As the operator A has spectral radius strictly smaller than one, $\partial_{\mathbf{q}} F(\mathbf{p},\mathbf{q_p}) = \mathbf{1} - A$ is invertible. By the implicit function theorem, the equation $F(\mathbf{x},\mathbf{q}) = \mathbf{q} - \Gamma(\mathbf{x},\mathbf{q}) = \mathbf{0}$ gives rise to a real-analytic implicit function $\mathbf{x} \mapsto \mathbf{q}(\mathbf{x}) \in X$ defined in a neighborhood of $\mathbf{p}$ and such that $\mathbf{q}(\mathbf{p}) = \mathbf{q_p}$ (see e.g. [10, Theorem 2.1, p. 364] for the implicit function theorem, and [3, Section 5.6.7] for the real-analytic part). □

5 Proof of Theorem 2.2

Fix $\lambda_0 \in U$ and $\alpha_0 = \alpha(\lambda_0) < 1$ (from Assumption 2.1). The hypotheses of the theorem states that the map $\lambda \in U \subset \mathbb{R}^n \mapsto \mathbf{p}_\lambda \in X$ is of class C^k (respectively, real-analytic). Assumption 2.1 also means that $\mathbf{p}_{\lambda_0}$ belongs to $\Delta(E,\alpha) \subset X$. By Lemma 4.3 the map $\mathbf{p} \in \Delta(E,\alpha) \mapsto \mathbf{q_p} \in \mathcal{L}^\infty([0,\alpha_0])$ is real-analytic at $\mathbf{p}_{\lambda_0}$, so the composed map $\lambda \in U \mapsto \mathbf{q}_\lambda \equiv \mathbf{q}_{\mathbf{p}_\lambda} \in Y$ is then also C^k (respectively, real-analytic) in a neighborhood of λ_0. Being the unique fixed point of $\mathbf{q} \mapsto \Gamma(\mathbf{p}_\lambda,\mathbf{q})$ the function $q_\lambda(\xi) = q_{\mathbf{p}_\lambda}(\xi)$, $\xi \in E$ verifies equation (1) as we wished to show. □

6 Multitype Branching Process

The above analysis applies also to Multitype Branching Processes [7]. Such a branching process $(\mathbf{Z}_n)_{n\geq 0}$ takes values in $\mathbb{N}_0^d$, with $d < +\infty$ being the number of different types. In an environment ξ and for a parameter λ each particle of type $i = 1,\ldots,d$ reproduces independently according to a pgf

$$\phi_{p_\lambda(i,\xi)}(\mathbf{s}) = \sum_{k_1\geq 0} \cdots \sum_{k_d\geq 0} (p_\lambda(i,\xi))_{k_1,\ldots,k_d} s_1^{k_1} \cdots s_d^{k_d}, \quad \mathbf{s} = (s_1,\ldots,s_d),$$

where $p_\lambda(i,\xi)$ is probability on $\mathbb{N}_0^d$.

Denote $p_\lambda(\xi) = (p_\lambda(1,\xi),\ldots,p_\lambda(d,\xi))$ and let $e_1,\ldots,e_d$ denote the standard canonical basis in $\mathbb{R}^d$. Then there are d extinction probabilities given by $q_\lambda(i,\xi) = \mathbb{P}_{\lambda,\xi}(\lim_{n\to+\infty} \mathbf{Z}_n = 0|\mathbf{Z}_0 = e_i)$, i.e. depending on the type of the 0th generation particle. Define the vector valued function $Q_\lambda(\xi) = (q_\lambda(1,\xi),\ldots,q_\lambda(d,\xi))$ and the vector valued map

$$\Phi_{p_\lambda(\xi)}(\mathbf{s}) = (\phi_{p_\lambda(1,\xi)}(\mathbf{s}),\ldots,\phi_{p_\lambda(d,\xi)}(\mathbf{s})) : [0,1]^d \to [0,1]^d.$$

Then we have the following functional equation, which may be obtained in the same way as (1),

$$Q_\lambda(\xi) = \Phi_{p_\lambda(\xi)}\left(Q_\lambda(T\xi)\right), \quad \xi \in E. \tag{10}$$

Assumption 6.1 *$\forall\lambda \in U$ there are $\widetilde{\boldsymbol{\alpha}}, \boldsymbol{\alpha} \in [0,1)^d$ with $\widetilde{\boldsymbol{\alpha}} \prec \boldsymbol{\alpha}$ so that for all $\xi \in E$: $\Phi_{p_\lambda(\xi)}(\boldsymbol{\alpha}) \prec \widetilde{\boldsymbol{\alpha}}$.*

Here $\mathbf{s} = (s_1, \ldots, s_d) \prec \mathbf{t} = (t_1, \ldots, t_d)$ is the lattice ordering meaning that every $s_i < t_i$, $i = 1, \ldots, d$. Under this assumption we have the following

Theorem 6.2 *Suppose that Assumption 6.1 is satisfied. If each map $\lambda \in U \to (p_\lambda(i,\xi))_{\xi\in E} \in \mathcal{L}^\infty(E; \ell^1(\mathbb{N}_0^d))$, $i = 1, \ldots, d$ is of class $\mathcal{C}^k$ or is real-analytic then so is the map $\lambda \in U \to \mathbf{Q}_\lambda = (Q_\lambda(\xi))_{\xi\in E} \in \mathcal{L}^\infty(E; [0,1)^d)$.*

The proof is very similar to the one presented in section 4. Set $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$ and define the poly-disk $\mathbb{P}_{\boldsymbol{\alpha}} = \mathbb{D}_{\alpha_1} \times \cdots \times \mathbb{D}_{\alpha_d}$. Consider the Kobayashi (or Caratheodory) distance (see e.g. [9]) between points $\mathbf{s}, \mathbf{t} \in \mathbb{P}_{\boldsymbol{\alpha}}$. In this particular case it is given by $d_{\mathbb{P}_{\boldsymbol{\alpha}}}(\mathbf{s}, \mathbf{t}) = \max\{d_{\mathbb{D}_{\alpha_1}}(s_1, t_1), \ldots, d_{\mathbb{D}_{\alpha_d}}(s_d, t_d)\}$ where each $d_{\mathbb{D}_{\alpha_i}}$ is the Poincaré metric on $\mathbb{D}_{\alpha_i}$. Assumption 6.1 implies that $\Phi_{p_\lambda(\xi)}$ is an L-Lipschitz contraction on $(\mathbb{P}_{\boldsymbol{\alpha}}, d_{\mathbb{P}_{\boldsymbol{\alpha}}})$ with $L = \max_i \tilde{\alpha}_i/\alpha_i < 1$, precisely as we showed in Lemma 4.2 for the Poincaré metric. The proof continues from there and an implicit function theorem concludes. $\square$

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