Cones and gauges in complex spaces : Spectral gaps and complex Perron-Frobenius theory.

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Abstract

We introduce complex cones and associated projective gauges, generalizing a real Birkhoff cone and its Hilbert metric to complex vector spaces. We deduce a variety of *spectral gap* theorems in complex Banach spaces. We prove a *dominated* complex cone-contraction Theorem and use it to extend the classical Perron-Frobenius Theorem to complex matrices, Jentzsch's Theorem to complex integral operators, a Kreĭn-Rutman Theorem to compact and quasi-compact complex operators and a Ruelle-Perron-Frobenius Theorem to complex transfer operators in dynamical systems. In the simplest case of a complex n by n matrix $A \in M_n(\mathbb{C})$ we have the following statement : Suppose that $0 < c < +\infty$ is such that $|\mathrm{Im}\, A_{ij}\overline{A}_{mn}| < c \leq \mathrm{Re}\, A_{ij}\overline{A}_{mn}$ for all indices. Then A has a 'spectral gap'.

1 Introduction

The Perron-Frobenius Theorem [Per07, Fro08] asserts that a real square matrix with strictly positive entries has a 'spectral gap', i.e. the matrix has a positive simple eigenvalue and all other eigenvalues are strictly smaller in modulus. More generally, let A be a bounded linear operator acting upon a real or complex Banach space and write $r_{\mathrm{sp}}(A)$ for its spectral radius. We say that A has a spectral gap if (1) it has a simple isolated eigenvalue λ the modulus of which equals $r_{\mathrm{sp}}(A)$ and (2) the remaining part of the spectrum is contained in a disk centered at zero and of radius strictly smaller than $r_{\mathrm{sp}}(A)$.

Jentzsch generalized in [Jen12] the Perron-Frobenius Theorem to integral operators with a strictly positive continuous kernel. The proof uses the Schauder-Tychonoff Theorem to produce a dual eigenvector and then a contraction on the kernel of this eigenvector to get a spectral gap. Kreĭn-Rutman [KR50, Theorem 6.3] (see also [Rut40] and [Rot44]) gave an abstract setting of this result by considering a punctured real closed cone mapped to its interior by a compact operator. Compactness of the operator essentially reduces the problem to finite dimensions.

Birkhoff, in a seminal paper [Bir57], developed a more elementary and intuitive (at least in our opinion) Perron-Frobenius 'theory' by considering the projective contraction of a cone equipped with its associated Hilbert metric. Birkhoff noted that this projective metric satisfies a *contraction principle*, i.e. any linear map preserving the cone is a contraction for the metric and the contraction is strict and uniform if the image of the cone has finite projective diameter.

All these results, or rather their proofs, make use of the 'lattice'-structure induced by a real cone on a real Banach space (see [Bir67] and also [Mey91]). On the other hand, from complex analysis we know that the Poincaré metric on the unit disk, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and the induced metric on a hyperbolic Riemann surface enjoy properties similar to the Hilbert metric, in particular a *contraction principle* with respect to conformal maps. More precisely, if $\phi : U \to V$ is a conformal map between hyperbolic Riemann surfaces then its hyperbolic derivative never exceeds one. The map is a strict contraction unless it is a bijection (see e.g. [CG93, Chapter I.4: Theorems 4.1 and 4.2]). By considering analytic images of complex discs Kobayashi [Kob67, Kob70] (see also [Ves76]) constructed a hyperbolic metric on complex (hyperbolic) manifolds, a tool with many applications also in infinite dimensions (see e.g. [Rug02, Appendix D]).

Given a real cone contraction, perturbation theory allows on abstract grounds to consider 'small' complex perturbations but uniform estimates are usually hard to obtain. Uniform complex estimates are needed e.g. when (see e.g. [NN87, BR98]) proving local limit theorems and refined large deviation theorems for Markov additive processes and also (see e.g. [Rue79, Rug02]) for studying the regularity of characteristic exponents for time-dependent and/or random dynamical systems (see section 10 below). It is desirable to obtain a description of a projective contraction and, in particular, a spectral gap condition for complex operators without the restrictions imposed by perturbation theory. We describe in the following how one may accomplish this goal.

In section 2 we introduce families of $\mathbb{C}$-invariant cones in complex Banach spaces and a theory for the projective contraction of such cones. The central idea is simple, namely to use the Poincaré metric as a 'gauge' on 2-dimensional affine sections of a complex cone. At first sight, this looks like the Kobayashi construction. A major difference, however, is that we only consider disk images in 2-dimensional subspaces. Also we do not take infimum over chains (so as to obtain a triangular inequality, see Appendix A). This adapts well to the study of linear operators and makes computations much easier than for the general Kobayashi metric. Lemma 2.3 shows that this gauge is indeed projective. The contraction principle for the Poincaré metric translates into a contraction principle for the gauge and, under additional regularity assumptions, developed in section 3, into a projective contraction, and finally a spectral gap, with respect to the Banach space norm.

In sections 4 and 5 we consider real cones and define their *canonical complexification*. For example, $\mathbb{C}^n_+ = \{u \in \mathbb{C}^n : |u_i + u_j| \geq |u_i - u_j|, \forall\, i, j\} = \{u \in \mathbb{C}^n : \mathrm{Re}\, u_i \overline{u}_j \geq 0, \forall\, i, j\}$ is the canonical complexification of the standard real cone, $\mathbb{R}^n_+$. We show that our complex cone contraction yields a genuine extension of the Birkhoff cone contraction : A real Birkhoff cone is *isometrically* embedded into its canonical complexification. It enjoys here the same contraction properties with respect to linear operators. We obtain then in section 6 one of our main results: When a complex operator is *dominated* by a sufficiently regular real cone-contraction (Assumption 6.1) then (Theorem 6.3) the complex operator has a spectral gap. It is of interest to note that the conditions on the complex operator are expressed in terms of a real cone and, at least in some cases, very easy to verify. Sections 7-9 thus present a selection of complex analogues of well-known real cone contraction theorems : A Perron-Frobenius Theorem for complex matrices (as stated at the end of the abstract), Jentzsch's Theorem for complex integral operators, a Kreĭn-Rutman Theorem for compact and quasi-compact complex operators and a Ruelle-Perron-Frobenius Theorem for complex transfer operators. In section 10 we prove results on the regularity of characteristic exponents of products of random complex cone-contractions. Finally, in section 11 we discuss how our results compare to those of perturbation theory.

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2 Complex cones and gauges

Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denote the Riemann sphere. When $U \subset \widehat{\mathbb{C}}$ is an open connected subset avoiding at least three points one says that the set is *hyperbolic*. We write d_U for the corresponding hyperbolic metric. We refer to [CG93, Chapter I.4] or [Mil99, Chapter 2] for the properties of the hyperbolic metric which we use in the present paper. As normalization we use $ds = 2|dz|/(1-|z|^2)$ on the unit disk $\mathbb{D}$ and the metric d_U on U induced by a Riemann mapping $\phi : \mathbb{D} \to U$. One then has :

$$d_{\mathbb{D}}(0,z) = \log \frac{1+|z|}{1-|z|}, \qquad |z| = \tanh \frac{d_{\mathbb{D}}(0,z)}{2}. \tag{2.1}$$

Let E be a complex topological vector space. We denote by $\mathrm{Span}\{x,y\} = \{\lambda x + \mu y : \lambda, \mu \in \mathbb{C}\}$ the complex subspace generated by two vectors x and y in E.

Definition 2.1

(1) We say that a subset $\mathcal{C} \subset E$ is a **closed complex cone** *if it is closed in E, $\mathbb{C}$-invariant (i.e. $\mathcal{C} = \mathbb{C}\,\mathcal{C}$) and $\mathcal{C} \neq \{0\}$.*

(2) We say that the closed complex cone $\mathcal{C}$ is **proper** *if it contains no complex planes, i.e. if x and y are independent vectors then $\mathrm{Span}\{x,y\} \not\subset \mathcal{C}$.*

Throughout this paper we will simply refer to a proper closed complex cone as a **$\mathbb{C}$-cone**.

Let $\mathcal{C}$ be a $\mathbb{C}$-cone. Given a pair of non-zero vectors, $x, y \in \mathcal{C}^* \equiv \mathcal{C} - \{0\}$, we consider the subcone : $\mathrm{Span}\{x,y\} \cap \mathcal{C}$. We wish to construct a 'projective distance' between the complex lines $\mathbb{C}x$ and $\mathbb{C}y$ within this subcone. We do this by considering the complex affine plane through $2x$ and $2y$, choosing coordinates as follows (Lemma 2.3 below implies that the choice of affine plane is of no importance) :

$$D(x,y) \equiv D(x,y;\mathcal{C}) = \{\lambda \in \widehat{\mathbb{C}} : (1+\lambda)x + (1-\lambda)y \in \mathcal{C}\} \subset \widehat{\mathbb{C}}, \tag{2.2}$$

with the convention that $\infty \in D(x,y)$ iff $x - y \in \mathcal{C}$. The interior of this "slice" is denoted $D^o(x,y)$ (for the spherical topology on $\widehat{\mathbb{C}}$). Continuity of the canonical mapping $\mathbb{C}^2 \to \mathrm{Span}\{x,y\}$ implies that $D = D(x,y)$ is a closed subset of $\widehat{\mathbb{C}}$. As the cone is proper, $D \subset \widehat{\mathbb{C}}$ is a strict subset so that $\widehat{\mathbb{C}} - D$ is open and non-empty, whence contains (more than) 3 points. If, in addition, D^o is connected it is a hyperbolic Riemann surface ([CG93, Theorem I.3.1]).

Definition 2.2 *Given a $\mathbb{C}$-cone, we define the* **gauge**, *$d_{\mathcal{C}} : \mathcal{C}^* \times \mathcal{C}^* \to [0, +\infty]$, between two points $x, y \in \mathcal{C}^*$ as follows : When two vectors are co-linear we set $d_{\mathcal{C}}(x,y) = 0$. If they are linearly independent and -1 and 1 belong to the same connected component U of $D^o(x,y)$ we set :*

$$d_{\mathcal{C}}(x,y) \equiv d_U(-1,1) > 0. \tag{2.3}$$

In all remaining cases, we set $d_{\mathcal{C}} = \infty$.

When $V \subset \mathcal{C}$ is a (sub-)cone of the $\mathbb{C}$-cone $\mathcal{C}$ we write $\mathrm{diam}_{\mathcal{C}}(V^*) \equiv \sup_{x,y \in V^*} d_{\mathcal{C}}(x,y) \in [0, +\infty]$ for the projective **'diameter'** *of V in $\mathcal{C}$. We call it a diameter even though the gauge need not verify the triangular inequality, whence need not be a metric (see Appendix A for more on this issue).*

Lemma 2.3 *Let $\mathcal{C}$ be a $\mathbb{C}$-cone. The gauge on the cone is symmetric and projective, i.e. for $x, y \in \mathcal{C}^*$ and $a \in \mathbb{C}^*$:*

$$d_{\mathcal{C}}(y,x) = d_{\mathcal{C}}(x,y) = d_{\mathcal{C}}(ax,y) = d_{\mathcal{C}}(x,ay).$$

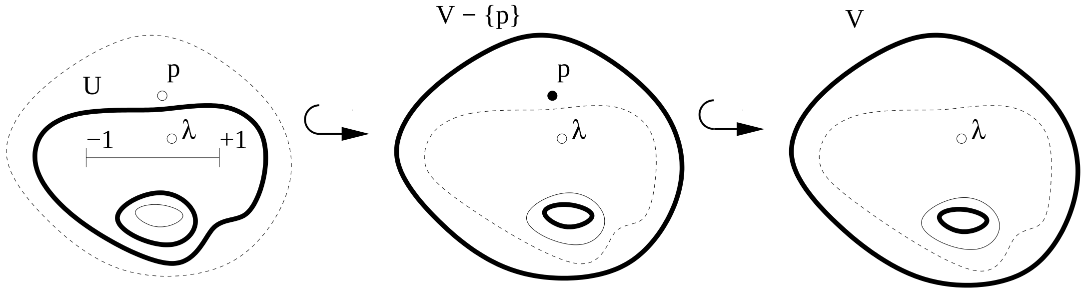

Figure 1: The sequence of inclusions $U \hookrightarrow V - \{p\} \hookrightarrow V$ in the proof of Lemma 2.4

Proof: For $(1+\mu)a + (1-\mu) \neq 0$ we write

$$(1+\mu)ax + (1-\mu)y = \frac{(1+\mu)a + (1-\mu)}{2}((1+R)x + (1-R)y)$$

with

$$R = R_a(\mu) = \frac{(1+\mu)a - (1-\mu)}{(1+\mu)a + (1-\mu)}.$$

Then R_a extends to a conformal bijection $R_a : \mu \in D(ax,y) \mapsto R_a(\mu) \in D(x,y)$ (a Möbius transformation of $\widehat{\mathbb{C}}$) preserving -1 and 1. The hyperbolic metric is invariant under such transformations so indeed $d_{\mathcal{C}}(x,y) = d_{\mathcal{C}}(ax,y)$ (but both could be infinite). Similarly, the map $\lambda \mapsto -\lambda$ yields a conformal bijection between the domains $D(x,y)$ and $D(y,x)$, interchanging -1 and 1 and the symmetry follows. $\square$

Lemma 2.4 *Let $T : E_1 \to E_2$ be a complex linear map between topological vector spaces and let $\mathcal{C}_1 \subset E_1$ and $\mathcal{C}_2 \subset E_2$ be $\mathbb{C}$-cones for which $T(\mathcal{C}_1^*) \subset \mathcal{C}_2^*$. Then the map,*

$$T : (\mathcal{C}_1^*, d_{\mathcal{C}_1}) \to (\mathcal{C}_2^*, d_{\mathcal{C}_2}),$$

is a contraction. If the image has finite diameter, i.e. $\Delta = \mathrm{diam}_{\mathcal{C}_2^*} T\mathcal{C}_1^* < \infty$, then the contraction is strict and uniform. More precisely, there is $\eta = \eta(\Delta) < 1$ (depending on Δ only) for which

$$d_{\mathcal{C}_2}(Tx, Ty) \leq \eta \; d_{\mathcal{C}_1}(x,y), \quad \forall x, y \in \mathcal{C}_1^*.$$

Proof : Let $x, y \in \mathcal{C}_1^*$ and set $D_1 = D(x,y;\mathcal{C}_1)$ and $D_2 = D(Tx,Ty;\mathcal{C}_2)$ for which we have

$$\{-1, 1\} \subset D_1 \subset D_2 \subset \widehat{\mathbb{C}}.$$

Suppose that $Tx, Ty \in \mathcal{C}_2^*$ are linearly independent and that D_2 and D_1 are hyperbolic (if not, $d_{\mathcal{C}_2}(Tx,Ty)$ vanishes and we are through). Since shrinking a domain increases hyperbolic distances, it follows that $d_{\mathcal{C}_2}(Tx,Ty) \leq d_{\mathcal{C}_1}(x,y)$ (although both could be infinite).

Suppose now that $\Delta < +\infty$. Then -1 and 1 belong to the same connected component, V, of $D^o(Tx,Ty)$. We may suppose that -1 and 1 also belong to the same connected component, U, of $D^o(x,y)$ (or else $d_{\mathcal{C}_1}(x,y) = \infty$) and we are through). Our assumptions imply that $U \subset V$ is a strict inclusion and that $\mathrm{diam}_V(U) \leq \Delta$. Choose $\lambda \in U$ and [Dou04] pick $p \in V \setminus U$ for which $d_V(\lambda, p) \leq \Delta$ (this is possible as the inclusion $U \subset V$ is strict and the diameter of U did not exceed Δ). The inclusion $U \hookrightarrow V - \{p\}$ is non-expanding and the inclusion $V - \{p\} \hookrightarrow V$ is a contraction which has hyperbolic derivative uniformly smaller than some $\eta = \eta(\Delta) < 1$ on the punctured Δ-neighborhood, $B_V(p,\Delta)^*$, of p (see Remark 2.5). In particular, the composed map (see Figure 1) $U \hookrightarrow V - \{p\} \hookrightarrow V$ has hyperbolic derivative smaller than $\eta(\Delta)$ at $\lambda \in B_V(p,\Delta)^*$. As $\lambda \in U$ was arbitrary this is true at any point along a geodesic joining -1 and 1 in U so that

$$d_{\mathcal{C}_2}(Tx,Ty) = d_V(-1,1) \leq \eta\, d_U(-1,1) = \eta\, d_{\mathcal{C}_1}(x,y).\ \Box$$

Remark 2.5 *An explicit bound may be given using the expression* $ds = |dz|/(|z| \log \frac{1}{|z|})$ *for the metric on the punctured disk at* $z \in \mathbb{D}^*$ *(see e.g. [Mil99, Example 2.8]). Denoting,* $t = \tanh \Delta/2$*, we obtain the bound,* $\eta(\Delta) = \frac{2t}{1-t^2} \log \frac{1}{t} = \sinh(\Delta)\ \log(\coth \frac{\Delta}{2}) < 1$*. Often, however, it is possible to improve this bound. For example, suppose that* U *is contractible in* V *(e.g. if* V *is simply connected) and that* U *is contained in a hyperbolic ball of radius* $0 < R < \infty$*. Lifting to the universal cover we may assume that* $V = \mathbb{D}$ *and that* $U = \{z \in \mathbb{D} : |z| < t\}$ *with* $0 < t = \tanh \frac{R}{2} < 1$*. The inclusion* $(U, d_U) \hookrightarrow (\mathbb{D}, d_{\mathbb{D}})$ *has hyperbolic derivative* $t\frac{1-|z|^2/t^2}{1-|z|^2} \leq t$ *for* $z \in U$*. We may thus use* $\eta = \tanh \frac{R}{2} < 1$ *for the contraction constant. Recall that for a real Birkhoff cone [Bir57] one may take* $\eta = \tanh \frac{\Delta}{4}$ *(an open interval in* $\mathbb{R}$ *of diameter* Δ *is a ball of radius* $\Delta/2$ *in* $\mathbb{R}$*).*

3 Complex Banach spaces and regularity of $\mathbb{C}$-cones

Let X be a complex Banach space and let $\mathcal{C} \subset X$ be a $\mathbb{C}$-cone (Definition 2.1). We denote by X' the dual of X and we write $\langle \cdot, \cdot \rangle$ for the canonical duality $X' \times X \to \mathbb{C}$. We will consider a bounded linear operator $T \in L(X)$ which preserves $\mathcal{C}^*$ and is a strict and uniform contraction with respect to our gauge on $\mathcal{C}$. We seek conditions that assure : (1) The presence of an invariant complex line (existence of an eigenvector of non-zero eigenvalue) and (2) A spectral gap. In short, an invariant line appears when the cone is not too 'wide' and the spectral gap when, in addition, the cone is not too 'thin'.

Definition 3.1 *Let* $\mathcal{C} \subset X$ *be a closed complex cone in a complex Banach space (in section 4 we will use the very same definition for a real cone in a real Banach space). When* $m \in X'$ *is a non-zero functional, bounded on the vector space generated by* $\mathcal{C}$*, we define the aperture of* $\mathcal{C}$ *relative to* m *:*

$$K(\mathcal{C}; m) = \sup_{u \in \mathcal{C}^*} \frac{\|m\|\ \|u\|}{|\langle m, u\rangle|} \in [1, +\infty].$$

We define the aperture of $\mathcal{C}$ *to be :* $K(\mathcal{C}) = \inf\limits_{m \in X', m \neq 0} K(\mathcal{C}; m) \in [1, +\infty]$*.*

When $K(\mathcal{C}) < +\infty$ *we say that* $\mathcal{C}$ *is of bounded aperture (or of* K*-bounded aperture with* $K(\mathcal{C}) \leq K < +\infty$ *if we want to emphasize a value of the bounding constant).*

Definition 3.2

(1) We call $\mathcal{C}$ inner regular *if it has non-empty interior in* X*.*

We say that $\mathcal{C}$ *is* reproducing *(or* generating*) if there is a constant* $g < +\infty$ *such that for every* $x \in X$ *we may find* $x_1, x_2 \in \mathcal{C}$ *for which* $x = x_1 + x_2$ *and*

$$\|x_1\| + \|x_2\| \leq g\ \|x\|. \tag{3.4}$$

We that $\mathcal{C}$ *is* T-reproducing *(or* T-generating*) if there are constants* $g < +\infty$*,* $q \in \mathbb{N} \cup \{0\}$ *and* $2 \leq p < +\infty$ *such that for every* $x \in X$ *and* $\epsilon > 0$ *there are* $y_1, \ldots, y_p \in \mathcal{C}$ *with* $\|y_1\| + \|y_2\| + \cdots + \|y_p\| \leq g\|x\|$ *and* $\|y_1 + y_2 + \cdots + y_p - T^q x\| < \epsilon$*.*

(2) *We say that $\mathcal{C}$ is* outer regular *if $K(\mathcal{C}) < +\infty$.*

We say that $\mathcal{C}$ is of K-bounded sectional aperture or of bounded sectional aperture *(with a bounding constant $1 \le K < +\infty$) iff for every pair $x, y \in X$, the subcone* $\mathrm{Span}\{x,y\} \cap \mathcal{C}$ *is of K-bounded aperture, i.e. there is a non-zero linear functional, $m = m_{\{x,y\}} \in \mathrm{Span}\{x,y\}'$, such that*

$$|\langle m, u\rangle| \ge \frac{1}{K}\|u\|\,\|m\|, \quad \forall u \in \mathrm{Span}\{x,y\} \cap \mathcal{C}. \tag{3.5}$$

(3) *We say that $\mathcal{C}$ is* regular *iff the cone is inner and outer regular.*

Remarks 3.3 *When a cone is of bounded aperture then the cone has a bounded global transverse section not containing the origin. This is often a too strong requirement. For example, in L^1-spaces this is usually OK but not in L^p with $1 < p \le +\infty$ unless we are in finite dimensions. Being inner regular means containing an open ball and this typically fails in L^p for $1 \le p < +\infty$, again with the exemption of the finite dimensional case. The notions of being (T-)reproducing and of bounded sectional aperture, respectively, are more flexible and may circumvent the two above-mentioned restrictions. We illustrate this in Example 4.9 and Theorem 7.2. Obviously, 'inner regular' $\Rightarrow$ reproducing $\Rightarrow$ T-reproducing. Also, 'outer regular' $\Rightarrow$ bounded sectional aperture.*

It is necessary to create a passage between the cone-gauge and the Banach space norm. The regularity properties defined above will enable us to do so through the following two Lemmas :

Lemma 3.4 *Let $\mathcal{C}$ be a closed complex cone of K-bounded sectional aperture. Then $\mathcal{C}$ is proper, whence a $\mathbb{C}$-cone (Definition 2.1). If $x, y \in \mathcal{C}^*$ and $m = m_{\{x,y\}}$ is a functional associated to the subcone* $\mathrm{Span}\{x,y\} \cap \mathcal{C}$ *as in equation (3.5) then :*

$$\left\|\frac{x}{\langle m, x\rangle} - \frac{y}{\langle m, y\rangle}\right\| \le \frac{4K}{\|m\|}\tanh\frac{d_{\mathcal{C}}(x,y)}{4} \le K\,\frac{d_{\mathcal{C}}(x,y)}{\|m\|}.$$

Proof: We normalize the functional so that $\|m\| = K$. Then $\|u\| \le |\langle m, u\rangle| \le K\|u\|$ for all $u \in \mathrm{Span}\{x,y\} \cap \mathcal{C}$. Denote $\widehat{x} = \frac{x}{\langle m,x\rangle}$ and $\widehat{y} = \frac{y}{\langle m,y\rangle}$ and consider, as a function of $\lambda \in \mathbb{C}$, the point $u_\lambda = (1+\lambda)\widehat{x} + (1-\lambda)\widehat{y}$. When $u_\lambda \in \mathcal{C}$ the properties of m show that $\|u_\lambda\| \le |\langle m, u_\lambda\rangle| = |(1+\lambda) + (1-\lambda)| \equiv 2$ and therefore,

$$|\lambda|\,\|\widehat{x} - \widehat{y}\| \le \|u_\lambda\| + (\|\widehat{x}\| + \|\widehat{y}\|) \le 4.$$

Setting $R = \frac{4}{\|\widehat{x}-\widehat{y}\|} \in [2, +\infty]$ we see that $D(\widehat{x},\widehat{y}) \subset \overline{B(0,R)}$. The radius R is bounded iff x and y are independent so the cone is proper. Enlarging a domain decreases hyperbolic distances so

$$d_{\mathcal{C}}(x,y) = d_{D^o(\widehat{x},\widehat{y})}(-1,1) \ge d_{B(0,R)}(-1,1) = d_{\mathbb{D}}\left(\frac{1}{R}, -\frac{1}{R}\right) = 2\log\frac{1+\frac{1}{R}}{1-\frac{1}{R}}.$$

Therefore, $\dfrac{\|\widehat{x}-\widehat{y}\|}{4} = \dfrac{1}{R} \le \tanh\dfrac{d_{\mathcal{C}}(x,y)}{4} \le \dfrac{d_{\mathcal{C}}(x,y)}{4}$, and the stated bound follows. □

Lemma 3.5 *Let $\mathcal{C}$ be a $\mathbb{C}$-cone of K-bounded sectional aperture and let $x \in \mathcal{C}^*$, $y \in X$. Suppose that there is $r > 0$ such that $x + t\,y \in \mathcal{C}^*$ for all $t \in \mathbb{C}$ with $|t| < r$. Then*

$$d_{\mathcal{C}}(x, x + s\,y) \le \frac{2}{r}|s| + o(|s|) \quad \textit{as} \quad s \to 0 \tag{3.6}$$

and

$$\|y\| \le \frac{K}{r}\|x\|. \tag{3.7}$$

Proof: Let $|s| < r$. Using the scale-invariance of the cone we see that $D(x, x + s\,y) = \{\lambda \in \widehat{\mathbb{C}} : (1+\lambda)x + (1-\lambda)(x+sy) \in \mathcal{C}\} = \{\lambda \in \widehat{\mathbb{C}} : x + (1-\lambda)\,\frac{s}{2}\,y \in \mathcal{C}\}$. Our hypothesis implies that $D(x, x+s\;y)$ contains a disc of radius $\dfrac{2r}{|s|}$, centered at 1. Shrinking a domain increases hyperbolic distances, whence

$$d_{\mathcal{C}}(x, x+s\,y) \leq d_{B(1,\frac{2r}{|s|})}(-1,1) = d_{\mathbb{D}}(0, \frac{|s|}{r}) = \log \frac{1+|s|/r}{1-|s|/r} = \frac{2}{r}|s| + o(|s|).$$

Choose m associated to the subcone $\mathrm{Span}\{x,y\} \cap \mathcal{C}$ as in equation (3.5), normalized so that $\|m\| = K$. As m is non-zero on the punctured subcone, $0 < |\langle m, x + t\,y\rangle| = |\langle m, x\rangle + t\langle m, y\rangle|$ for all $|t| < r$, and this implies that $r|\langle m, y\rangle| \leq |\langle m, x\rangle|$. Possibly after multiplying x and u with complex phases we may assume that $\langle m, x\rangle \geq \langle m, ry\rangle > 0$. But then $2r\|y\| \leq \|x+ry\| + \|x - ry\| \leq \langle m, x+ry+x-ry\rangle \leq 2K\|x\|$. □

Theorem 3.6 *Let $\mathcal{C}$ be a $\mathbb{C}$-cone of K-bounded sectional aperture. Let $T \in L(X)$ be a strict cone-contraction, i.e. $T : \mathcal{C}^* \to \mathcal{C}^*$ with $\Delta = \mathrm{diam}_{\mathcal{C}} T(\mathcal{C}^*) < \infty$. Let $\eta < 1$ be as in Lemma 2.4. Then :*

(1) $\mathcal{C}$ contains a unique T-invariant complex line, $\mathbb{C}h$.

We define $\lambda \in \mathbb{C}^*$ by setting $Th = \lambda h$.

(2) There are constants $R, \tilde{C} < +\infty$ and a map $c : \mathcal{C} \to \mathbb{C}$ so that for any $x \in \mathcal{C}$ and $n \geq 1$:

$$\|\lambda^{-n}T^n x - c(x)h\| \;\leq\; \tilde{C}\,\eta^{n-1}\|x\| \tag{3.8}$$

$$\|c(x)h\| \;\leq\; R\,\|x\| \tag{3.9}$$

Proof: Let $x_0 \in \mathcal{C}^*$ and set $e_1 = Tx_0/\|Tx_0\| \in T(\mathcal{C}^*) \subset \mathcal{C}^*$. We will construct a Cauchy-sequence $(e_n)_{n\in\mathbb{N}}$ recursively. Given e_n, $n \geq 1$ choose, as in Definition 3.2 (2), a functional $m_n \in X'$ normalized so that $\|m_n\| = K$, associated to the subcone $\mathrm{Span}\{e_n, Te_n\} \cap \mathcal{C}$. Set $\lambda_n = \langle m_n, Te_n\rangle / \langle m_n, e_n\rangle$ (for which we have the bound $0 < |\lambda_n| \leq \|T\|\,K$) and define the next element in our recursion :

$$e_{n+1} = \frac{\lambda_n^{-1}Te_n}{\|\lambda_n^{-1}Te_n\|} \;\in\; T^{n+1}\mathcal{C}^*.$$

Using Lemma 3.4 and then Lemma 2.4 (with a contraction constant $\eta < 1$) we obtain for $n \geq 1$:

$$\|\frac{e_n}{\langle m_n, e_n\rangle} - \frac{Te_n}{\langle m_n, Te_n\rangle}\| \;\leq\; d_{\mathcal{C}}(e_n, Te_n) \;\leq\; \mathrm{diam} T^n\mathcal{C}^* \leq \Delta\eta^{n-1}. \tag{3.10}$$

As $1 \leq |\langle m_n, e_n\rangle| \leq K$ and $|\langle m_n, Te_n\rangle| \leq \|T\|\,K$ we get :

$$\|e_n - \lambda_n^{-1}Te_n\| \;\leq\; K\,\Delta\,\eta^{n-1} \quad \text{and} \quad \|\lambda_n e_n - Te_n\| \;\leq\; \|T\|\,K\,\Delta\,\eta^{n-1}. \tag{3.11}$$

Noting that $\|e_n\| = 1$, the first inequality implies :

$$\|e_n - e_{n+1}\| \leq 2K\,\Delta\,\eta^{n-1}. \tag{3.12}$$

The sequence, $(e_n)_{n\in\mathbb{N}}$, is therefore Cauchy, whence has a limit,

$$h = \lim_n e_n \in \mathcal{C}^*, \;\; \|h\| = 1. \tag{3.13}$$

The limit belongs to $\mathcal{C}$ because the cone was assumed closed. Writing $(\lambda_{n+1}-\lambda_n)e_{n+1} = (T-\lambda_n)e_n + (\lambda_{n+1}-T)e_{n+1} + (T-\lambda_n)(e_{n+1}-e_n)$ and using the second inequality in (3.11) as well as (3.12) and $|\lambda_n| \leq \|T\|\, K$ we obtain

$$|\lambda_n - \lambda_{n+1}| \;\leq\; (1+\eta+(2+2K))\,\|T\|\;K\;\Delta\;\eta^{n-1}, \tag{3.14}$$

so also the limit $\lambda = \lim_n \lambda_n$ exists. But $\|Th-\lambda h\| = \lim_n \|Te_n - \lambda_n e_n\| = 0$ shows that $Th = \lambda h \in \mathcal{C}^*$ which implies that $\lambda \neq 0$, whence that $\mathbb{C}h \subset \mathcal{C}$ is a T-invariant complex line. Suppose that also $\mathbb{C}k \subset \mathcal{C}$ (with $k \neq 0$) is T-invariant. Then $d_{\mathcal{C}}(h,k) = d_{\mathcal{C}}(Th,Tk) \leq \eta\, d_{\mathcal{C}}(h,k) \leq \eta\Delta < +\infty$ and this implies $d_{\mathcal{C}}(h,k) = 0$ so the two vectors must be linearly dependent. Thus, $\mathbb{C}h$ is unique.

To see the second part, let $x \in \mathcal{C}^*$ and define for $n \geq 1$: $x_n = T^n x$. This time we pick $m_n \in X'$ associated to the subcone $\mathrm{Span}\{x_n, h\} \cap \mathcal{C}$ and set $c_n = \langle m_n, \lambda^{-n} x_n\rangle / \langle m_n, h\rangle$ for which $|c_n| \leq K\|\lambda^{-n}x_n\|$. As in (3.10) we get

$$\|\frac{x_n}{\langle m_n, x_n\rangle} - \frac{h}{\langle m_n, h\rangle}\| \;\leq\; \frac{K}{\|m_n\|}\, d_{\mathcal{C}}(T^n x, T^n h) \;\leq\; \frac{K}{\|m_n\|}\mathrm{diam} T^n \mathcal{C}^* \leq \frac{K}{\|m_n\|}\Delta\eta^{n-1}. \tag{3.15}$$

Thus, $\|\lambda^{-n}x_n - c_n h\| \leq K\|\lambda^{-n}x_n\|\,\Delta\,\eta^{n-1}$ and

$$\|\lambda^{-n-1}x_{n+1} - \lambda^{-n}x_n\| = \|\lambda^{-1}T(\lambda^{-n}x_n - c_n h) + (\lambda^{-n}x_n - c_n h)\| \leq (1+\|\lambda^{-1}T\|)\,K\,\|\lambda^{-n}x_n\|\,\Delta\,\eta^{n-1}.$$

Then $\|\lambda^{-n-1}x_{n+1}\| \leq \left(1+(1+\|\lambda^{-1}T\|)\,K\,\Delta\,\eta^{n-1}\right)\|\lambda^{-n}x_n\|$ so we get the following uniform bound in $n \geq 1$:

$$\|\lambda^{-n}x_n\| \;\leq\; \prod_{k\geq 0}\left(1+(1+\|\lambda^{-1}T\|)\,K\,\Delta\,\eta^k\right)\|\lambda^{-1}x_1\| \tag{3.16}$$

$$\leq\; \exp\left((1+\|\lambda^{-1}T\|)\frac{K\Delta}{1-\eta}\right)\|\lambda^{-1}\;T\|\;\|x\| \equiv R\,\|x\|. \tag{3.17}$$

Writing $c_{n+1}h - c_n h = (c_{n+1}h - \lambda^{-n-1}x_{n+1}) + (\lambda^{-n-1}x_{n+1} - \lambda^{-n}x_n) - (c_n h - \lambda^{-n}x_n)$ we obtain

$$|c_{n+1} - c_n| = \|c_{n+1}h - c_n h\| \leq \left(\eta + (1+\|\lambda^{-1}T\|) + 1\right)\;K\,\Delta\,\eta^{n-1}R\,\|x\|.$$

Therefore, $c^* = \lim c_n \in \mathbb{C}$ exists and by re-summing to ∞,

$$|c^* - c_n| \leq \frac{2+\eta+\|\lambda^{-1}T\|}{1-\eta}\,K\,\Delta\,\eta^{n-1}\,R\,\|x\|.$$

Then also

$$\|\lambda^{-n}x_n - c^* h\| \leq \frac{3+\|\lambda^{-1}T\|}{1-\eta}\,K\,\Delta\,\eta^{n-1}\,R\,\|x\| \leq \tilde{C}\,\eta^{n-1}\|x\|$$

which implies that $c(x) \equiv c^*$ depends only on x (and not of the choice of m_n's). We also have : $|c(x)| = \|c(x)h\| = \lim_n \|\lambda^{-n}x_n\| \leq \sup_n \|\lambda^{-n}x_n\| \leq R\,\|x\|$. □

Theorem 3.7 *Let $T \in L(X)$ and let $\mathcal{C}$ be a $\mathbb{C}$-cone of K-bounded sectional aperture and which, in addition, is reproducing. Suppose that T is a strict cone-contraction, i.e. $T : \mathcal{C}^* \to \mathcal{C}^*$ with $\Delta = \mathrm{diam}_{\mathcal{C}} T(\mathcal{C}^*) < \infty$. Then T has a spectral gap.*

Proof: Let $x \in X$ and let g be the 'reproducing'-constant from (3.4). Pick $x_1, x_2 \in \mathcal{C}$ with $x = x_1 + x_2$ and $\|x_1\| + \|x_2\| \leq g\|x\|$. We apply the previous Theorem to x_1 and x_2 and set $c^* = c(x_1) + c(x_2)$ for which $|c^*| \leq \tilde{A}\,\|x_1\| + \tilde{A}\,\|x_2\| \leq g\tilde{A}\|x\|$. In a similar way we obtain $\|\lambda^{-n}T^n x - c^* h\| \leq g\,\tilde{C}\,\eta^{n-1}\|x\|$, $n \geq 1$. Letting $n \to \infty$, we see that $c(x) \equiv c^*$ depends on x but not on the choice of the decomposition. Linearity of T furthermore implies that the mapping $x \to \langle \nu, x\rangle \equiv c(x) \in \mathbb{C}$ must be linear, and, as a linear functional, $\nu \in X'$ is bounded in norm by $A \equiv g\tilde{A}$. We have shown that

$$\left\|\lambda^{-n}T^n x - h\langle \nu, x\rangle\right\| \leq\; C\eta^{n-1}\|x\|, \quad \forall\, x \in X,\; n \geq 1, \tag{3.18}$$

with $C < +\infty$. Therefore, λ is a simple eigenvalue of T corresponding to the eigenprojection, $x \to h\langle \nu, x\rangle$ and the remainder has spectral radius not exceeding $\eta\,|\lambda|$.

In the more general case, when $\mathcal{C}$ is T-reproducing, we let g, p, q be constants from Definition 3.2. For $x \in X$ and fixed $k \geq 1$, set $\epsilon = 1/k$ and choose : $y_1, \ldots, y_p \in \mathcal{C}$ with $\|y_1\| + \|y_2\| + \cdots + \|y_p\| \leq g\|x\|$ and $\|y_1 + y_2 + \cdots + y_p - T^q x\| < \frac{1}{k}$. Setting $c_k^* = c(y_1) + \cdots + c(y_p)$ we obtain in this way a sequence $(c_k^*)_{k\geq 1}$ for which $\|\lambda^{-n}T^{n+q}x - c_k^* h\| \leq g\tilde{C}\eta^{n-1}\|x\| + \frac{1}{k}\|\lambda^{-n}T^n x\|$, $n \geq 1$. The sequence is bounded so we may extract a convergent subsequence $c_{k_m}^* \to c^*$ and conclude that $\|\lambda^{-n}T^{n+q}x - c^* h\| \leq g\tilde{C}\eta^{n-1}\|x\|$, $\forall\, n \geq 1$. Again, $\langle \nu, x\rangle \equiv c^*$ depends linearly on x and T has a spectral gap. $\square$

Remark 3.8 *Some explicit estimates for the constants in Theorem 3.7 when $\mathcal{C}$ is reproducing :*

$$\|c^*\| \leq A \equiv g\,K\,\|\lambda^{-1}\,T\|\,\exp\left(\left(1 + \|\lambda^{-1}\,T\|\right)\frac{K\,\Delta}{1-\eta}\right) \;\; \textit{and} \;\; C = \left(3 + \|\lambda^{-1}\,T\|\right)\,\frac{\Delta}{1-\eta}A. \tag{3.19}$$

Note, however, that in this setting there is no a priori lower bound on $|\lambda|$. In particular, to get an explicit bound on $\|\lambda^{-1}\,T\|$ one needs further information on the map T and the cone $\mathcal{C}$.

Example 3.9 *Let X be a complex Banach space and consider $e \in X$, $\ell \in X'$ with $\langle \ell, e\rangle = 1$. We write $P = e \otimes \ell$ for the associated one dimensional projection. For $0 < \sigma < +\infty$ we set*

$$\mathcal{C}_\sigma = \Big\{x \in X : \|(1-P)x\| \leq \sigma\|Px\|\Big\}. \tag{3.20}$$

Then $B\left(e, \dfrac{\sigma\|e\|}{1 + (1+\sigma)\|P\|}\right) \subset \mathcal{C}_\sigma$ and $K(\mathcal{C}_\sigma) \leq (1+\sigma)\|P\|$ so that $\mathcal{C}_\sigma$ is a regular $\mathbb{C}$-cone. Furthermore, if $0 < \sigma_1 < \sigma < +\infty$ a calculation shows that $\mathrm{diam}_{\mathcal{C}_\sigma}\mathcal{C}^*_{\sigma_1} < +\infty$.

Remark 3.10 *We have the following characterization of the spectral gap property : A bounded linear operator, $T \in L(X)$, has a spectral gap iff it is a strict contraction of a regular $\mathbb{C}$-cone. Sketch of proof: One direction is the content of Theorem 3.7 (since a regular cone in particular is of uniformly bounded sectional aperture). For the other direction one uses the spectral gap projection P to construct an adapted norm (equivalent to $\|\cdot\|$) : $\|x\|_\theta = \|Px\| + \sum_{k\geq 0}\theta^{-k}\|T^k(1-P)x\|$ for some fixed choice of $\theta \in (\eta, 1)$. Using this norm to define the cone family in (3.20), it is not difficult to see that T is a strict and uniform contraction of $\mathcal{C}_\sigma$, $\sigma > 0$.*

4 Real cones

Let $X_{\mathbb{R}}$ denote a real Banach space. Recall that a subset $\mathcal{C}_{\mathbb{R}} \subset X_{\mathbb{R}}$ is called a (real) proper closed convex[1] cone if it is closed and convex and if

[1]Convexity of a real cone often is a useful property that *a fortiori* is lost when dealing with complex cones.

$$\mathbb{R}_+\mathcal{C}_\mathbb{R} = \mathcal{C}_\mathbb{R}, \tag{4.21}$$
$$\mathcal{C}_\mathbb{R} \cap -\mathcal{C}_\mathbb{R} = \{0\}. \tag{4.22}$$

The real cone is said to be reproducing (or generating) provided $\mathcal{C}_\mathbb{R} + (-\mathcal{C}_\mathbb{R}) = X_\mathbb{R}$. Using Baire's Theorem and convexity of $\mathcal{C}_\mathbb{R}$ it is not difficult to see that this is equivalent to the existence of $g < +\infty$ such that every $x \in X_\mathbb{R}$ decomposes into $x = x_+ - x_-$ with $x_+, x_- \in \mathcal{C}_\mathbb{R}$ and such that

$$\|x_+\| + \|x_-\| \leq g\,\|x\|. \tag{4.23}$$

Remark 4.1 *A possible generalization: As in Definition 3.2 (1) we may say that $\mathcal{C}_\mathbb{R}$ is T-reproducing (or T-generating) if there is $g < +\infty$ and $q \in \mathbb{N}$ so that for every $x \in X_\mathbb{R}$ and $\epsilon > 0$ there are $y_+, y_- \in \mathcal{C}_\mathbb{R}$ ($\mathcal{C}_\mathbb{R}$ is convex) with*

$$\|y_+\| + \|y_-\| \leq g\,\|x\| \quad \text{and} \quad \|y_+ - y_- - T^q x\| < \epsilon. \tag{4.24}$$

In the following, we will refer to a real proper closed convex cone as an $\mathbb{R}$-*cone*. We assume throughout that such a cone is non-trivial, i.e. not reduced to a point. Given an $\mathbb{R}$-cone one associates a projective (Hilbert) metric for which we here give two equivalent definitions (for details we refer to [Bir57, Bir67]). The first, originally given by Hilbert, uses cross-ratios and is very similar to our complex cone gauge : Let $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ denote the extended real line (topologically a circle). For $x, y \in {\mathcal{C}_\mathbb{R}}^* \equiv \mathcal{C}_\mathbb{R} - \{0\}$, we write

$$\ell(x,y) = \left\{ t \in \widehat{\mathbb{R}} : (1+t)x + (1-t)y \in \mathcal{C}_\mathbb{R} \cup -\mathcal{C}_\mathbb{R} \right\} \tag{4.25}$$

with the convention that $\infty \in \ell(x,y)$ iff $x - y \in \mathcal{C}_\mathbb{R} \cup -\mathcal{C}_\mathbb{R}$. Properness of the cone implies that $\ell(x,y) = \widehat{\mathbb{R}}$ iff x and y are co-linear. In that case we set their distance to zero. Otherwise, $\ell(x,y)$ is a closed (generalized) segment $[a,b] \subset \widehat{\mathbb{R}}$ containing the segment $[-1;1]$, see Figure 3 in section 5.

The logarithm of the cross-ratio of $a, -1, 1, b \in \widehat{\mathbb{R}}$,

$$d_{\mathcal{C}_\mathbb{R}}(x,y) = R(a,-1,1,b) = \log \frac{a-1}{a+1}\,\frac{b+1}{b-1}, \tag{4.26}$$

then yields the Hilbert projective distance between x and y. Birkhoff [Bir57] found an equivalent definition of this distance : For $x, y \in {\mathcal{C}_\mathbb{R}}^* \equiv \mathcal{C}_\mathbb{R} - \{0\}$, one defines

$$\beta(x,y) = \inf\{\lambda > 0 : \lambda x - y \in \mathcal{C}_\mathbb{R}\} \in (0, +\infty] \tag{4.27}$$

in terms of which :

$$d_{\mathcal{C}_\mathbb{R}}(x,y) = \log\left(\beta(x,y)\beta(y,x)\right) \in [0, +\infty]. \tag{4.28}$$

A simple geometric argument shows that indeed the two definitions are equivalent.

Given a linear functional, $m \in X'_\mathbb{R}$, the image of the cone, $\langle m, \mathcal{C}_\mathbb{R} \rangle$, equals either $\{0\}$, $\mathbb{R}_+$, $\mathbb{R}_-$ or $\mathbb{R}$. One defines the dual cone as

$$\mathcal{C}'_\mathbb{R} = \{m \in X'_\mathbb{R} : m_{|\mathcal{C}_\mathbb{R}} \geq 0\}. \tag{4.29}$$

Using Mazur's Theorem, cf. e.g. [Lang93, p. 88], one sees that the $\mathbb{R}$-cone itself may be recovered from :

$$\mathcal{C}_\mathbb{R} = \{x \in X_\mathbb{R} : \langle m, x \rangle \geq 0, \ \forall m \in \mathcal{C}'_\mathbb{R}\}. \tag{4.30}$$

Given an $\mathbb{R}$-cone $\mathcal{C}_\mathbb{R}$ we use Definition 3.1 (replacing $\mathbb{C}$ by $\mathbb{R}$, complex by real) to define the aperture of $\mathcal{C}_\mathbb{R}$. It is given as the infimum of K-values for which there exists a linear functional $m \in X'_\mathbb{R}$ satisfying (see Figure 2)

$$\|u\| \leq \langle m, u \rangle \leq K\|u\|, \quad u \in \mathcal{C}_\mathbb{R}. \tag{4.31}$$

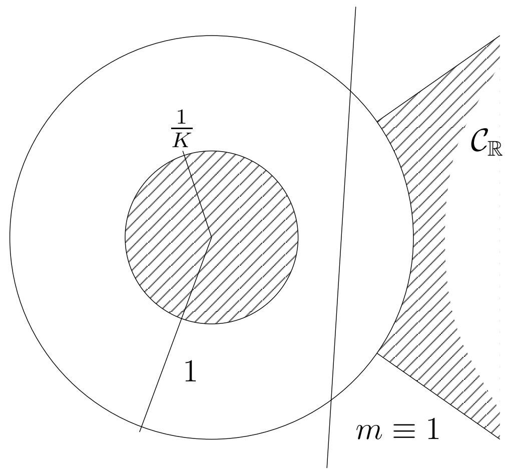

Figure 2: A real cone $\mathcal{C}_\mathbb{R}$ of K-bounded sectional aperture

Lemma 4.2 *The aperture, $K(\mathcal{C}_\mathbb{R}) \in [1, +\infty]$, of an $\mathbb{R}$-cone, $\mathcal{C}_\mathbb{R} \subset X_\mathbb{R}$, is determined by*

$$\frac{1}{K(\mathcal{C}_\mathbb{R})} = \inf \left\{ \frac{\|x_1 + \cdots + x_n\|}{\|x_1\| + \cdots + \|x_n\|} \; : \; x_i \in \mathcal{C}^*_\mathbb{R}, n \geq 1 \right\}. \tag{4.32}$$

Proof: Let $x_1, \ldots, x_n \in \mathcal{C}^*_\mathbb{R}$ and note that $a = \sum_1^n x_i / \sum_1^n \|x_i\|$ belongs to $A \equiv \mathrm{Conv}(\mathcal{C}_\mathbb{R} \cap \partial B(0,1))$, the convex hull of cone-elements of norm one. The reciprocal of the right hand side in (4.32) therefore equals $r = \inf\{\|a\| : a \in A\} \in [0,1]$. Suppose that $r > 0$. Then $B(0,r)$ and $\mathrm{Cl}\, A$ are disjoint convex subsets. The vector difference, $Z = \{a - b : a \in \mathrm{Cl}\, A, b \in B(0,r)\}$, is open, convex and does not contain the origin, whence [Lang93, Lemma 2.2, p.89] there is $\ell \in X'_\mathbb{R}$ whose kernel does not intersect Z. We may normalize ℓ so that

$$B(0,r) \subset \{\ell < 1\} \quad \text{and} \quad \mathrm{Cl}\, A \subset \{\ell \geq 1\}.$$

Then

$$\|x\| \leq \langle \ell, x \rangle \leq \frac{\|x\|}{r}, \quad \forall x \in {\mathcal{C}_\mathbb{R}}^*, \tag{4.33}$$

and therefore $K(\mathcal{C}_\mathbb{R}) \leq \frac{1}{r}$. To get the converse inequality let m be positive and verify (4.31). Then $\sum \|x_i\| \leq \sum \langle m, x_i \rangle = \langle m, \sum x_i \rangle \leq K \| \sum x_i \|$. □

Lemma 4.3 *Let $\mathcal{C}_\mathbb{R} \subset X_\mathbb{R}$ be a d-dimensional $\mathbb{R}$-cone of K-bounded sectional aperture. Then $\mathcal{C}_\mathbb{R}$ itself is of dK-bounded aperture.*

Proof: Let $F \subset \mathbb{R}^d$. By a theorem of Caratheodory, a point in the convex hull, $\mathrm{Conv}F$, is *a fortiori* in the convex hull of $d+1$ points in F (see e.g. [Rud91, p.73]). If $x \in \partial\, \mathrm{Conv}F$, we may even write it as a limit of convex combinations of d points in F. Now, apply this to the set A in the proof of the previous Lemma. In the formula, (4.32) it thus suffices to consider d cone-elements which we may order decreasingly according to their norm, $\|x_1\| \geq \|x_2\| \geq \cdots \geq \|x_d\|$. Using Lemma 4.2 with $n = 2$, the K-bounded sectional aperture implies that

$$\|x_1 + \cdots + x_d\| \geq \frac{1}{K}\|x_1\| + \frac{1}{K}\|x_2 + \cdots + x_d\| \geq \frac{1}{K}\|x_1\| \geq \frac{1}{K}\frac{\|x_1\| + \cdots + \|x_d\|}{d}.$$

Thus

$$\frac{\|x_1 + \cdots + x_d\|}{\|x_1\| + \cdots + \|x_d\|} \geq \frac{1}{dK},$$

and in view of Lemma 4.2, we see that $\mathcal{C}_\mathbb{R}$ is of dK bounded aperture. □

Lemma 4.4 *Let $\mathcal{C}_\mathbb{R} \subset \mathbb{R}^d$ be an $\mathbb{R}$-cone. Then $\mathcal{C}_\mathbb{R}$ is outer regular.*

Proof : As in the previous Lemma it suffices to look at the supremum in (4.32) over d-tuples. The set $A \equiv \{x_1, \ldots, x_d \in \mathcal{C}_\mathbb{R} : \|x_1\| + \cdots + \|x_d\| = 1\}$ is compact and $\|x_1 + \cdots + x_d\|$ is continuous and non-vanishing on A, whence has a minimum, $r > 0$. It follows that $K(\mathcal{C}_\mathbb{R}) \leq \frac{1}{r} < +\infty$. □

Remark 4.5 *In the literature an $\mathbb{R}$-cone $\mathcal{C}_\mathbb{R}$ is said to be norm-directed (with a constant $1 \leq K < \infty$) if $\|x - y\| \leq K\|x + y\|$, $\forall x, y \in \mathcal{C}_\mathbb{R}$. For an $\mathbb{R}$-cone our notion of uniformly bounded sectional aperture is equivalent (up to a small unavoidable loss in constants) to that of being norm-directed. To see this note that if $\mathcal{C}_\mathbb{R}$ is of K-bounded sectional aperture and ℓ verifies (4.33) then $\forall x, y \in \mathcal{C}_\mathbb{R}$:*

$$\|x - y\| \leq \|x\| + \|y\| \leq \langle \ell, x\rangle + \langle \ell, y\rangle = \langle \ell, x + y\rangle \leq K\|x + y\|,$$

which shows that $\mathcal{C}_\mathbb{R}$ is K-norm-directed. Conversely, if $\mathcal{C}_\mathbb{R}$ is K-norm-directed then

$$\|x\| + \|y\| \leq \|x + y\| + \|x - y\| \leq (1 + K)\|x + y\|$$

and Lemma 4.2 shows that $\mathcal{C}_\mathbb{R}$ is of $(K+1)$-bounded sectional aperture. For example, $(\mathbb{R}^d_+, \|\cdot\|_1)$ is 1-norm directed and of 1-bounded aperture, whereas $(\mathbb{R}^d_+, \|\cdot\|_\infty)$ is 1-norm directed, of 2-bounded sectional aperture but only of d bounded aperture. Lemma 3.5 is a complex cone-analogue of being norm-directed.

Theorem 4.6 *Let $A \in L(X_\mathbb{R})$ and let $\mathcal{C}_\mathbb{R} \subset X_\mathbb{R}$ be a reproducing $\mathbb{R}$-cone which is K-norm-directed. Suppose that A is a strict cone-contraction, i.e. $A : \mathcal{C}^*_\mathbb{R} \to \mathcal{C}^*_\mathbb{R}$ with $\Delta_A = \mathrm{diam}_{\mathcal{C}_\mathbb{R}} A(\mathcal{C}_\mathbb{R}{}^*) < +\infty$. Then A has a spectral gap. More precisely, there is $\lambda > 0$ and a one dimensional projection P for which $\lambda^{-1}A - P$ has spectral radius not greater than $\tanh \frac{\Delta_A}{4} < 1$.*

Proof: The statement of this Theorem is very close to Birkhoff's Theorem 4 in [Bir57]. The proof of that Theorem may be adapted to the present case. Alternatively, we may here simply use Remark 5.10 below. Our Theorem 4.6 also generalizes easily to the case when 'reproducing' is replaced by 'T-reproducing', equation (4.24). We omit the proof.

Corollary 4.7 *Let $\mathcal{C}_\mathbb{R}$ be an $\mathbb{R}$-cone in $\mathbb{R}^d$, $d < +\infty$ and suppose that $A \in L(X_\mathbb{R})$ verifies $A(\mathcal{C}^*_\mathbb{R}) \subset \mathrm{Int}\, \mathcal{C}_\mathbb{R}$. Then A has a spectral gap.*

Proof: Implicitly it is assumed that $\mathcal{C}_{\mathbb{R}}$ has non-empty interior. Lemma 4.4 shows that $\mathcal{C}_{\mathbb{R}}$ is outer regular, in particular, norm-directed. As is easily shown, the map $x, y \in \mathcal{C}_{\mathbb{R}}{}^* \mapsto d_{\mathcal{C}_{\mathbb{R}}}(x, y)$ is continuous. Compactness of $\mathcal{C}_{\mathbb{R}} \cap \{|x| = 1\}$ then implies that $\operatorname{diam}_{\mathcal{C}_{\mathbb{R}}} A(\mathcal{C}_{\mathbb{R}}{}^*) < +\infty$ so the Corollary follows from Theorem 4.6. □

Let $\mathcal{C}_{\mathbb{R}} \subset X_{\mathbb{R}}$ be an $\mathbb{R}$-cone. It is standard to write $x \preceq y \Leftrightarrow y - x \in \mathcal{C}_{\mathbb{R}}$ for the induced partial ordering of $x, y \in X_{\mathbb{R}}$. For $A, B \in L(X_{\mathbb{R}})$, we also write $A \preceq B \Leftrightarrow \forall x \in \mathcal{C}_{\mathbb{R}} \ : \ A(x) \preceq B(x)$.

The following dominated cone contraction theorem is trivial in the context of an $\mathbb{R}$-cone contraction. In section 6 we show that a similar (non-trivial) result holds in the complex case.

Theorem 4.8 *Let $A, P \in L(X_{\mathbb{R}})$ be contractions of the $\mathbb{R}$-cone $\mathcal{C}_{\mathbb{R}}$. Suppose that there are constants $0 < \alpha \leq \beta < +\infty$ for which $\alpha P \preceq A \preceq \beta P$. Then*

$$\operatorname{diam}_{\mathcal{C}_{\mathbb{R}}} A(\mathcal{C}_{\mathbb{R}}{}^*) \leq 2 \log \frac{\beta}{\alpha} + \operatorname{diam}_{\mathcal{C}_{\mathbb{R}}} P(\mathcal{C}_{\mathbb{R}}{}^*).$$

Proof: Given $x, y \in \mathcal{C}_{\mathbb{R}}{}^*$, suppose that $\lambda, \lambda' > 0$ are such that $\lambda P x - P y \in \mathcal{C}_{\mathbb{R}}$, $\lambda' P y - P x \in \mathcal{C}_{\mathbb{R}}$. Then also $\lambda \beta A x - \alpha A y \in \mathcal{C}_{\mathbb{R}}$ and $\lambda' \beta A y - \alpha A x \in \mathcal{C}_{\mathbb{R}}$ so that

$$d_{\mathcal{C}_{\mathbb{R}}}(Ax, Ay) \leq 2 \log \frac{\beta}{\alpha} + \log(\lambda \lambda')$$

and the claim follows by Birkhoff's characterization (4.28). □

Example 4.9

(1) Let $A \in M_n(\mathbb{R})$ and suppose that $0 < \alpha \leq A_{ij} \leq \beta < +\infty$ for all indices. Setting $P_{ij} \equiv 1$ we see that

$$P\left((\mathbb{R}^n_+)^*\right) = \{(t, \ldots, t) : t > 0\}$$

which is of zero projective diameter in $\mathbb{R}^n_+$. By Theorem 4.8 we recover the standard result :

$$\operatorname{diam}_{\mathbb{R}^n_+} A\left((\mathbb{R}^n_+)^*\right) \leq \Delta_A = 2 \log \frac{\beta}{\alpha}.$$

The cone $\mathbb{R}^n_+$ is regular so Theorem 4.6 applies. If $\lambda_1 > 0$ and $|\lambda_2|$ denote the leading eigenvalue and the second largest eigenvalue (in absolute value), respectively, then $\frac{|\lambda_2|}{\lambda_1} \leq \tanh \frac{\Delta_A}{4} = \frac{\beta - \alpha}{\beta + \alpha}$.

(2) The standard Perron-Frobenius Theorem generalizes to integral operators, cf. Jentzsch's Theorem [Jen12] and the generalization given by Birkhoff in [Bir57]. We present a somewhat different generalization : Let (Ω, μ) be a measure space and let $X_{\mathbb{R}} = L^p \equiv L^p(\Omega, \mu)$, $1 \leq p \leq +\infty$. Let $h \in L^p_+$ ($h > 0$, a.e.) and $m \in L^q_+$ ($m > 0$, a.e.) with $q = p/(p-1) \in [1, +\infty]$ being the conjugated exponent so that $0 < \int_\Omega h \, m \, d\mu < +\infty$. Let $k_A : \Omega \times \Omega \to \mathbb{R}_+$ be a $\mu \otimes \mu$-measurable map. We suppose there are constants $0 < \alpha \leq \beta < +\infty$ so that for μ-almost all $x, y \in \Omega$:

$$\alpha \, h(x) m(y) \leq k_A(x, y) \leq \beta \, h(x) m(y).$$

Let $A \in L(X_{\mathbb{R}})$ be the integral operator defined by $A\phi(x) = \int_\Omega k_A(x, y) \phi(y) \, d\mu(y)$. Then A has a spectral gap (again with a contraction rate given by $\frac{\beta - \alpha}{\beta + \alpha}$).

Proof: We write $\mathcal{C}_\mathbb{R} = L^p_+(\Omega,\mu)$ for the cone of positive L^p-functions ($\phi \geq 0$, a.e.) and compare the operator A with the one-dimensional projection $P\phi = h\,\int_\Omega m\,\phi\,d\mu$. Our assumption, $\int h\,m\,d\mu > 0$ shows that $P : \mathcal{C}^*_\mathbb{R} \to \mathcal{C}^*_\mathbb{R}$ and that $\Delta_P = 0$. By Theorem 4.8, $\mathrm{diam}_{\mathcal{C}_\mathbb{R}} A(\mathcal{C}_\mathbb{R}{}^*) \leq 2\,\log\frac{\beta}{\alpha}$.

$\mathcal{C}_\mathbb{R}$ is (trivially) reproducing with a constant $g = 2^{1-1/p} \leq 2$. To see that $\mathcal{C}_\mathbb{R}$ is of uniformly bounded sectional aperture let $f, g \in \mathcal{C}_\mathbb{R}$ be of unit norm in L^p_+, $1 \leq p < +\infty$ and pick $\tilde f, \tilde g \in L^q_+(\Omega)$ with $\|\tilde f\|_q = \|\tilde g\|_q = 1$ (the case $p = \infty$, $q = 1$ should be treated separately; we leave this to the reader) and $\int \tilde f\, f\, d\mu = \int \tilde g\, g\, d\mu = 1$. The functional $m(u) = \int (\tilde f + \tilde g) u\, d\mu$ then verifies $\|u\|_p \leq m(u) \leq 2\|u\|_p$ for all $u \in \mathcal{C}_\mathbb{R} \cap \mathrm{Span}\{f, g\}$. Thus $\mathcal{C}_\mathbb{R}$ is of 2-bounded sectional aperture.

5 The canonical complexification of a real Birkhoff cone

A complex cone yields a genuine extension/generalization of the cone contraction described by Birkhoff [Bir57, Bir67]. More precisely, we will show that any Birkhoff cone may be isometrically embedded in a complex cone, enjoying qualitatively the same contraction properties.

Let $X_\mathbb{R}$ be a Banach space over the reals. A complexification $X_\mathbb{C}$ of $X_\mathbb{R}$ is a complex Banach space, equipped with a bounded anti-linear complex involution, $J : X_\mathbb{C} \to X_\mathbb{C}$, $J^2 = Id$ (the identity map), $J(\lambda x) = \overline{\lambda} J(x)$, $J(x+y) = J(x) + J(y)$, $\lambda \in \mathbb{C}$, $x, y \in X_\mathbb{C}$, for which $X_\mathbb{R} = \frac{1}{2}(Id + J)X_\mathbb{C}$ is the real part. Then $X_\mathbb{C} = X_\mathbb{R} \oplus iX_\mathbb{R}$ is a direct sum. [Note that this is not the same as regarding $X_\mathbb{C}$ as a real Banach space. For example, $\mathbb{C}^n$ is a complexification of $\mathbb{R}^n$ for any ℓ^p-norm, $1 \leq p \leq \infty$, while the real dimension of $\mathbb{C}^n$ is $2n$].

For simplicity we will assume that J is an isometry on $X_\mathbb{C}$ in which case the canonical projections, $\mathrm{Re} = \frac{1}{2}(Id + J)$ and $\mathrm{Im} = \frac{1}{2i}(Id - J)$, have norm one. We note that any real Banach space, $(X_\mathbb{R}, \|\cdot\|_\mathbb{R})$, admits a complexification, $X_\mathbb{C} = X_\mathbb{R} \oplus iX_\mathbb{R}$ as follows : We adopt the obvious rules for multiplying by complex numbers, set $J(x+iy) = x - iy$ and introduce a norm e.g. using real functionals,

$$\|x+iy\|_\mathbb{C} = \sup\{|\langle \ell, x\rangle + i\langle \ell, y\rangle| : \ell \in X'_\mathbb{R}, \|\ell\|_\mathbb{R} \leq 1\}.$$

The latter norm is equivalent (within a factor of 2) to any other conjugation invariant norm on $X_\mathbb{C}$ having as real part the given space $(X_\mathbb{R}, \|\cdot\|_\mathbb{R})$. For the rest of this section $X_\mathbb{R}$ will denote the real part of a complex Banach space $X_\mathbb{C}$. A real linear functional, $m \in X'_\mathbb{R}$, extends to a complex linear functional by setting $\langle m, x+iy\rangle = \langle m, x\rangle + i\langle m, y\rangle$ for $x + iy \in X_\mathbb{R} \oplus iX_\mathbb{R}$.

Definition 5.1 *Given an $\mathbb{R}$-cone $\mathcal{C}_\mathbb{R} \subset X_\mathbb{R}$ we define its canonical complexification :*

$$\mathcal{C}_\mathbb{C} = \{u \in X_\mathbb{C} : \mathrm{Re}\ \langle m, u\rangle\,\overline{\langle \ell, u\rangle}\ \geq\ 0,\quad \forall m, \ell \in \mathcal{C}'_\mathbb{R}\}. \tag{5.34}$$

Proposition 5.2 *We have the following* **polarization** *identity :*

$$\mathcal{C}^*_\mathbb{C} = \{\lambda(x+iy) : \lambda \in \mathbb{C}, x \pm y \in \mathcal{C}^*_\mathbb{R}\} \tag{5.35}$$

Proof: Let $u \in \mathcal{C}^*_\mathbb{C}$. Our defining condition (5.34) means that $\langle m, u\rangle$ (assume here it is non-zero) must have an argument that vary within a $\pi/2$ angle as $m \in \mathcal{C}'_\mathbb{R}$ varies. Normalizing appropriately, we may write $u = \lambda v$ with $\lambda \in \mathbb{C}^*$ and $\mathrm{Arg}\langle m, v\rangle \leq \pi/4$. If we set $v = x + iy$ with $x, y \in X_\mathbb{R}$

then $|\langle m, y\rangle| \leq \langle m, x\rangle$ for all $m \in \mathcal{C}_{\mathbb{R}}'$. Hence, $\langle m, x \pm y\rangle \geq 0$ for all such functionals and by (4.30) this is equivalent to $x \pm y \in \mathcal{C}_{\mathbb{R}}$. If $x = y$ (or $x = -y$) then we may write $u = \lambda(1+i)\, x$ (or $u = \lambda(1-i)\, x$) so we may always assume $x \pm y \in \mathcal{C}_{\mathbb{R}}^*$. ∎

Lemma 5.3 *Let $\mathcal{C}_{\mathbb{R}}$ be an $\mathbb{R}$-cone of K-bounded aperture. Then its canonical complexification, $\mathcal{C}_{\mathbb{C}}$, is of $2\sqrt{2}\,K$-bounded aperture.*

Proof: Let $\ell \in X_{\mathbb{R}}'$ satisfy $\|x\| \leq \langle \ell, x\rangle \leq K\|x\|$, $x \in \mathcal{C}_{\mathbb{R}}$ and extend ℓ to a complex linear functional. When $u \in \mathcal{C}_{\mathbb{C}}$ we use polarization, Lemma 5.2, to write $u = \lambda(x+iy)$ with $\langle \ell, x \pm y\rangle \geq 0$. Then $\|x \pm y\| \leq \langle \ell, x\rangle \pm \langle \ell, y\rangle \leq K\|x \pm y\|$, from which $\|x\| \leq \langle \ell, x\rangle$ and $\|y\| \leq \langle \ell, x\rangle$ so that $\frac{1}{2}\|x+iy\| \leq \langle \ell, x\rangle \leq |\langle \ell, x\rangle + i\langle \ell, y\rangle|$. As $|\langle \ell, y\rangle| \leq \langle \ell, x\rangle$ we also have $|\langle \ell, x+iy\rangle| \leq \sqrt{2}\,\langle \ell, x\rangle \leq \sqrt{2}\,K\|x\| \leq \sqrt{2}\,K\|x+iy\|$. Therefore, $\frac{1}{2}\|u\| \leq |\langle \ell, u\rangle| \leq \sqrt{2}\,K\ \|u\|$ and the result follows. ∎

Proposition 5.4 *Let $\mathcal{C}_{\mathbb{R}}$ be an $\mathbb{R}$-cone. If $\mathcal{C}_{\mathbb{R}}$ is (1) inner regular / (2) reproducing / (3) outer regular/ (4) of bounded sectional aperture then so is its canonical complexification.*

Proof: (1) One checks that if $\mathcal{C}_{\mathbb{R}}$ contains an open ball $B_{X_{\mathbb{R}}}(h, r)$ then $\mathcal{C}_{\mathbb{C}}$ contains $B_{X_{\mathbb{C}}}(h, r/2)$.

(2) Note that when $u, v \in \mathcal{C}_{\mathbb{R}}$ then by Proposition 5.2,

$$u + iv = (1+i)\left(\frac{1-i}{2}u + \frac{1+i}{2}v\right) = (1+i)\left(\frac{u+v}{2} + i\frac{u-v}{2}\right) \in \mathcal{C}_{\mathbb{C}},$$

because $\frac{u+v}{2} \pm \frac{u-v}{2} \in \mathcal{C}_{\mathbb{R}}$. Now, let $w = u + iv \in X_{\mathbb{R}} \oplus iX_{\mathbb{R}}$ and use that $\mathcal{C}_{\mathbb{R}}$ is reproducing in $X_{\mathbb{R}}$ to write $u = u_+ - u_-$ and $v = v_+ - v_-$ with $u_+, u_-, v_+, v_- \in \mathcal{C}_{\mathbb{R}}$ and $\|u_+\| + \|u_-\| \leq g\|u\|$ and $\|v_+\| + \|v_-\| \leq g\|v\|$. Then $w = (u_+ + iv_+) - (u_- + iv_-) \in \mathcal{C}_{\mathbb{C}} + \mathcal{C}_{\mathbb{C}}$ and $\|u_+ + iv_+\| + \|u_- + iv_-\| \leq \|u_+\| + \|u_-\| + \|v_+\| + \|v_-\| \leq g(\|u\| + \|v\|) \leq 2g\|w\|$.

(3) As shown in Lemma 5.3, if $\mathcal{C}_{\mathbb{R}} \subset X_{\mathbb{R}}$ is of K-bounded aperture then $\mathcal{C}_{\mathbb{C}}$ is of $2\sqrt{2}\,K$-bounded aperture.

(4) Let $u_1, u_2 \in \mathcal{C}_{\mathbb{C}}^*$ and write $W = \mathrm{Span}_{\mathbb{C}}\{u_1, u_2\} \cap \mathcal{C}_{\mathbb{C}}$ for the subcone generated by these two elements. We also write $F = \mathrm{Span}_{\mathbb{R}}\{\mathrm{Re}\, u_1, \mathrm{Im}\, u_1, \mathrm{Re}\, u_2, \mathrm{Im}\, u_2\}$ and $V_{\mathbb{R}} = F \cap \mathcal{C}_{\mathbb{R}}$ which is an at most 4 and at least 1-dimensional $\mathbb{R}$-subcone of $\mathcal{C}_{\mathbb{R}}$. Now, if $w \in W$ then $w = \lambda'(x' + iy')$ with $x' \pm y' \in \mathcal{C}_{\mathbb{R}}$ and clearly also $x', y' \in F$. But then $x' \pm y' \in V_{\mathbb{R}}$ so that also $w \in V_{\mathbb{C}}$, with $V_{\mathbb{C}}$ being the complexification of $V_{\mathbb{R}}$. By Lemma 4.3, $V_{\mathbb{R}}$ is of $4\,K$ bounded aperture so by Lemma 5.3, $V_{\mathbb{C}}$ and therefore also W are of $8\sqrt{2}\,K$ bounded (complex) aperture. ∎

Theorem 5.5 *Let $\mathcal{C}_{\mathbb{R}}$ be an $\mathbb{R}$-cone and let $\mathcal{C}_{\mathbb{C}}$ denote its canonical complexification (5.34). Then $\mathcal{C}_{\mathbb{C}}$ is a $\mathbb{C}$-cone (Definition 2.1). Writing $d_{\mathcal{C}_{\mathbb{C}}}$ for our projective gauge on the complex cone, the natural inclusion,*

$$(\mathcal{C}_{\mathbb{R}}^*, d_{\mathcal{C}_{\mathbb{R}}}) \hookrightarrow (\mathcal{C}_{\mathbb{C}}^*, d_{\mathcal{C}_{\mathbb{C}}}),$$

is an isometric embedding.

Proof: The set $\mathcal{C}_{\mathbb{C}}$ is clearly $\mathbb{C}$-invariant. Consider independent vectors, $x, y \in {\mathcal{C}_{\mathbb{R}}}^*$. By Lemma 4.4 any finite dimensional subcone of $\mathcal{C}_{\mathbb{R}}$ is outer regular, so in particular of uniformly

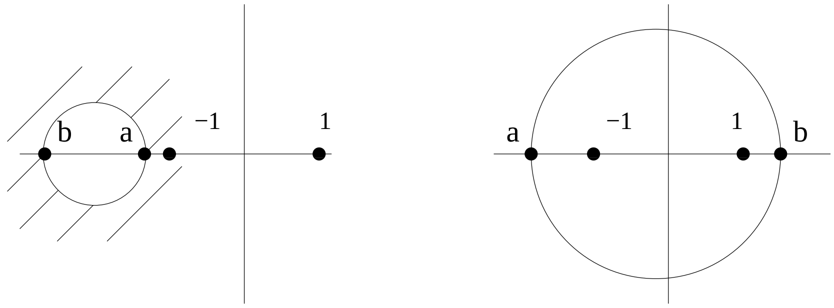

Figure 3: Illustration of two possible configurations of $D(x,y)$

bounded sectional aperture. Our previous Lemma shows that the corresponding complex cone is of bounded sectional (complex) aperture. But then $\mathcal{C}_\mathbb{C}$ must be proper by Lemma 3.4.

Regarding the embedding we may normalize the points so that $\ell(x,y)=[a,b]$ is a bounded segment in $\mathbb{R}$. Define the 'boundary' points, $x_0=(1+a)x+(1-a)y$ and $y_0=(1+b)x+(1-b)y$. For any $\epsilon>0$ the point $-\epsilon x_0+(1+\epsilon)y_0$ is outside the closed convex cone $\mathcal{C}_\mathbb{R}$. By Mazur's Theorem, [Lang93, p.88], we may separate this point from $\mathcal{C}_\mathbb{R}$ by a functional $\ell\in\mathcal{C}_\mathbb{R}'$. For any $\epsilon>0$ we may then find $m,\ell\in\mathcal{C}_\mathbb{R}'$ for which

$$\langle m,x_0\rangle=\langle \ell,y_0\rangle=\epsilon \quad\text{and}\quad \langle m,y_0\rangle=\langle \ell,x_0\rangle=1.$$

Then $u=\mu x_0+\lambda y_0\in\mathcal{C}_\mathbb{C}$ only if $\operatorname{Re}(\epsilon\mu+\lambda)(\epsilon\overline{\lambda}+\overline{\mu})\geq 0$ for any $0<\epsilon\leq 1$ whence only if

$$\operatorname{Re}\lambda\,\overline{\mu}\geq 0 \quad\Leftrightarrow\quad |\lambda+\mu|^2\geq|\lambda-\mu|^2.$$

Conversely, when $\operatorname{Re}\lambda\overline{\mu}\geq 0$ and $m,\ell\in\mathcal{C}_\mathbb{R}'$ then

$$\operatorname{Re}\langle m,u\rangle\,\overline{\langle \ell,u\rangle}\;\geq\;\operatorname{Re}(\lambda\overline{\mu})\,(\,\langle m,y_0\rangle\,\langle \ell,x_0\rangle\;+\;\langle m,x_0\rangle\,\langle \ell,y_0\rangle\,)\;\geq\;0,$$

so this condition is also sufficient. We thus have : $D(x_0,y_0)=\overline{\mathbb{D}}$. Therefore $D=D(x,y)\subset\widehat{\mathbb{C}}$ is a generalized disc, symmetric under complex conjugation for which $\ell(x,y)=D(x,y)\cap\widehat{\mathbb{R}}$ (see Figure 3). In this situation we know explicit formulas for both (4.26) the real and (2.1) the complex hyperbolic metrics in terms of cross-ratios so we get $d_{\mathcal{C}_\mathbb{C}}(x,y)=d_{D^o(x,y)}(-1,1)=\log R(a,-1,1,b)=d_{\mathcal{C}_\mathbb{R}}(x,y)$. □

Corollary 5.6 *For $n\geq 1$ the set*

$$\mathbb{C}^n_+=\{u\in\mathbb{C}^n:\operatorname{Re}u_i\overline{u_j}\geq 0,\ \forall i,j\}=\{u\in\mathbb{C}^n:|u_i+u_j|\geq|u_i-u_j|,\ \forall i,j\}$$

is a regular $\mathbb{C}$-cone. The inclusion, $((\mathbb{R}^n_+)^*,d_{\mathbb{R}^n_+})\hookrightarrow((\mathbb{C}^n_+)^*,d_{\mathbb{C}^n_+})$ is an isometric embedding.

Proof: Let $\ell_i\in(\mathbb{R}^n)'$, $i=1,\ldots,n$ denote the canonical coordinate projections then $\mathbb{R}^n_+=\{x\in\mathbb{R}^n:\langle\ell_i,x\rangle\geq 0,\ \forall\ i\}$ and $\mathbb{C}^n_+=\{u\in\mathbb{C}^n:\operatorname{Re}\langle\ell_i,u\rangle\overline{\langle\ell_j,u\rangle}\geq 0,\ \forall\ i,j\}$. Thus, $\mathbb{C}^n_+$ is the canonical complexification of the standard real cone $\mathbb{R}^n_+$. See Figure 4 for an illustration of $\mathbb{C}^2_+$. □

Below we shall need the following complex polarization

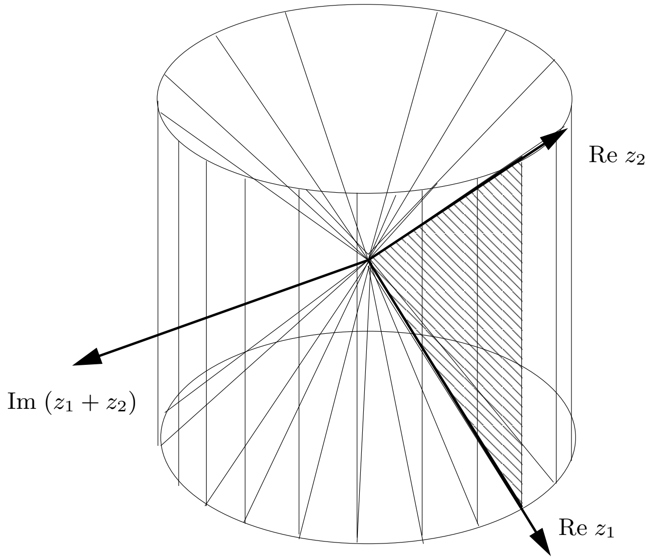

Figure 4: An attempt to illustrate the canonical complexification $\mathbb{C}_+^2$ of $\mathbb{R}_+^2$ in the coordinate system (Re z_1, Re z_2, Im (z_1+z_2)) and setting Im $(z_1 - z_2) = 0$. We show the part of the cone contained in the region $(\text{Re }(z_1+z_2))^2 + (\text{Im }(z_1+z_2))^2 \leq 1$ The shaded region shows the intersection with the real cone, $\mathbb{R}_+^2$.

Lemma 5.7 *Let $x \pm y \in \mathcal{C}_\mathbb{R}^*$ be at a distance $\Delta = d_{\mathcal{C}_\mathbb{R}}(x-y, x+y) < +\infty$. We may then find $\alpha \in \mathbb{R}$ so that the 'rotated' vector $x' + iy' = e^{i\alpha}(x+iy)$, or equivalently :*

$$\begin{aligned} x' &= x\ \cos\alpha - y\ \sin\alpha \\ y' &= x\ \sin\alpha + y\ \cos\alpha \end{aligned}$$

verifies:

$$x' - ty' \in \mathcal{C}_\mathbb{R}, \quad \forall\ |t| \leq \coth\frac{\Delta}{4}.$$

In particular, $x' \pm y' \in \mathcal{C}_\mathbb{R}^*$ and we have for all $\ell \in \mathcal{C}_\mathbb{R}'$:

$$|\langle \ell, y'\rangle| \leq \left(\tanh\frac{\Delta}{4}\right)\ \langle \ell, x'\rangle \quad \textit{and} \quad \frac{1}{\sqrt{2}}\langle \ell, x\rangle \leq \langle \ell, x'\rangle \leq \sqrt{2}\langle \ell, x\rangle. \tag{5.36}$$

Proof: We have that

$$\begin{aligned} \ell(x-y, x+y) &= \{t \in \widehat{\mathbb{R}} : (1+t)(x-y) + (1-t)(x+y) \in \mathcal{C}_\mathbb{R} \cup -\mathcal{C}_\mathbb{R}\} \\ &= \{t \in \mathbb{R} : x - ty \in \mathcal{C}_\mathbb{R} \cup -\mathcal{C}_\mathbb{R}\}. \end{aligned}$$

Now write $t = \tan(\theta)$, $\theta \in \mathbb{R}$ and set

$$\begin{aligned} \Theta(x,y) &= \{\theta \in \mathbb{R} : x - \tan\theta\ y \in \mathcal{C}_\mathbb{R} \cup -\mathcal{C}_\mathbb{R}\} \\ &= \{\theta \in \mathbb{R} : \cos\theta\ x - \sin\theta\ y \in \mathcal{C}_\mathbb{R} \cup -\mathcal{C}_\mathbb{R}\} \\ &= \{\theta \in \mathbb{R} : \text{Re } e^{i\theta}(x+iy) \in \mathcal{C}_\mathbb{R} \cup -\mathcal{C}_\mathbb{R}\} \end{aligned} \tag{5.37}$$

Let $[\theta_1, \theta_2]$ denote the connected component of $\Theta(x,y)$ containing $[-\pi/4; \pi/4]$. Then $-\theta_1, \theta_2 \in]\frac{\pi}{4}, \frac{3\pi}{4}[$ and $\theta_2 - \theta_1 < \pi$. Inserting $a = \tan\theta_1$ and $b = \tan\theta_2$ in equation (4.26) (see Figure 5)

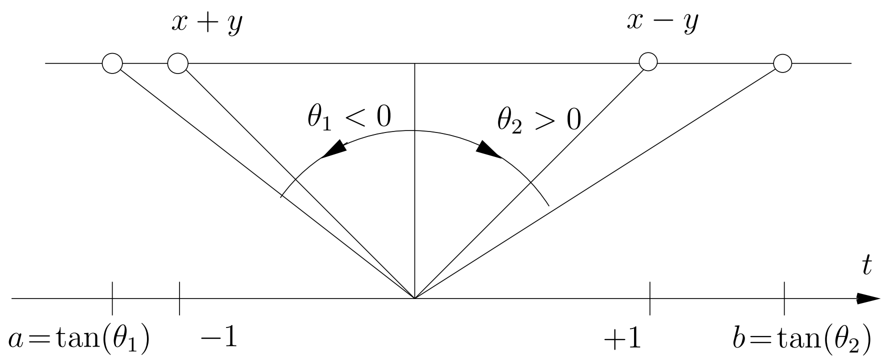

Figure 5: The subcone $\mathcal{C}_{\mathbb{R}} \cap \mathrm{Span}\{x, y\}$ viewed in the x,y-coordinate system.

standard trigonometric formulae show that the projective distance between $x-y$ and $x+y$ may be written as

$$d_(\theta_1, \theta_2) \equiv \log\left(\frac{\sin\theta_2 + \cos\theta_2}{\sin\theta_2 - \cos\theta_2} \times \frac{\sin(-\theta_1) + \cos(-\theta_1)}{\sin(-\theta_1) - \cos(-\theta_1)}\right).$$

If we do a complex rotation, $x' + iy' = e^{i\alpha}(x + iy)$ then the last expression in (5.37) shows that $\Theta(x', y') = [\theta_1 - \alpha, \theta_2 - \alpha]$. Here, the interval J of allowed α-values is such that $\Theta(x', y')$ contains $[-\pi/4, \pi/4]$. One has $J =]\theta_1 + \frac{\pi}{4}, \theta_1 + \frac{3\pi}{4}[\cap]\theta_2 - \frac{3\pi}{4}, \theta_2 - \frac{\pi}{4}[$. The derivative of $\alpha \in J \to d_{\mathcal{C}_{\mathbb{R}}}(x' - y', x' + y') = d(\theta_1 - \alpha, \theta_2 - \alpha)$ equals

$$\frac{2}{\cos(2(\theta_1 - \alpha))} - \frac{2}{\cos(2(\theta_2 - \alpha))},$$

which vanishes precisely at $\alpha = -(\theta_2 + \theta_1)/2$. So the minimal distance between $x' - y'$ and $x' + y'$ is obtained for this value of α and corresponds to a symmetric configuration in which $\ell(x' - y', x' + y') = [-L, L]$ with $L = \tan\frac{\theta_2 - \theta_1}{2} > 1$ and $\Delta \geq d(x' - y', x' + y') = 2\log\frac{L+1}{L-1}$ or equivalently,

$$L \geq \coth\frac{\Delta}{4}.$$

Thus $x' - ty' \in \mathcal{C}_{\mathbb{R}}$ whenever $|t| \leq L$ and we have obtained the first claim. Since $\langle \ell, x' - ty' \rangle \geq 0$ for all $-L \leq t \leq L$ we also get the first inequality in (5.36). To see the last inequality note that $\langle \ell, x \pm y \rangle \geq 0$ implies that $\langle \ell, x' \rangle = \cos\alpha\ \langle \ell, x \rangle - \sin\alpha\ \langle \ell, y \rangle \leq \sqrt{2}\ \langle \ell, x \rangle$. Similarly $\langle \ell, x \rangle = \cos\alpha\ \langle \ell, x' \rangle + \sin\alpha\ \langle \ell, y' \rangle \leq \sqrt{2}\ \langle \ell, x' \rangle$ (because $\langle \ell, x' \pm y' \rangle \geq 0$). ∎

Lemma 5.8 *Let $x_1, x_2 \in \mathcal{C}_{\mathbb{R}}$ be at a distance $\Delta = d_{\mathcal{C}_{\mathbb{R}}}(x_1, x_2) < +\infty$. Through a positive real rescaling, e.g. replacing x_1 by tx_1 for a suitable $t > 0$, we may assure that $\forall \ell \in \mathcal{C}'_{\mathbb{R}}$:*

$$\langle \ell, x_1 \rangle \leq e^{\Delta/2} \langle \ell, x_2 \rangle,\ \langle \ell, x_2 \rangle \leq e^{\Delta/2} \langle \ell, x_1 \rangle \quad and \quad |\langle \ell, x_1 - x_2 \rangle| \leq \left(\tanh\frac{\Delta}{4}\right)\ \langle \ell, x_1 + x_2 \rangle.$$

Proof: From the Birkhoff characterization (4.28) of the projective distance we may rescale, say x_1, to obtain $e^{\Delta/2} x_1 - x_2 \in \mathcal{C}_{\mathbb{R}}$ and $e^{\Delta/2} x_2 - x_1 \in \mathcal{C}_{\mathbb{R}}$. Then $\langle \ell, x_1 \rangle \leq e^{\Delta/2} \langle \ell, x_2 \rangle$ and $\langle \ell, x_2 \rangle \leq e^{\Delta/2} \langle \ell, x_1 \rangle$. From this we get : $(e^{\Delta/2} - 1)(\langle \ell, x_1 \rangle + \langle \ell, x_2 \rangle) - (e^{\Delta/2} + 1)(\langle \ell, x_1 \rangle - \langle \ell, x_2 \rangle) = 2(e^{\Delta/2} \langle \ell, x_2 \rangle - \langle \ell, x_1 \rangle) \geq 0$ and similarly with x_1 and x_2 interchanged. Rearranging terms the claim follows. ∎

Proposition 5.9 *Let $\mathcal{C}^1_{\mathbb{R}} \subset \mathcal{C}_{\mathbb{R}}$ be an inclusion of $\mathbb{R}$-cones and denote by $\mathcal{C}^1_{\mathbb{C}} \subset \mathcal{C}_{\mathbb{C}}$ the inclusion of the corresponding complexified cones. Let $\Delta_{\mathbb{R}} = \mathrm{diam}_{\mathcal{C}_{\mathbb{R}}}(\mathcal{C}^1_{\mathbb{R}})^* \in [0, +\infty]$ and $\Delta_{\mathbb{C}} = \mathrm{diam}_{\mathcal{C}_{\mathbb{C}}}(\mathcal{C}^1_{\mathbb{C}})^* \in [0, +\infty]$ be the projective diameters of the respective inclusions. Then $\Delta_{\mathbb{R}}$ is finite iff $\Delta_{\mathbb{C}}$ is finite.*

Proof: From the embedding in Theorem 5.5 we see that $\Delta_{\mathbb{R}} \leq \Delta_{\mathbb{C}}$ which implies one direction. To see the converse, suppose that $\eta = \tanh \Delta_{\mathbb{R}}/4 < 1$ and let $u_1, u_2 \in {\mathcal{C}^1_{\mathbb{C}}}^*$. We write $u_1 = \lambda_1(\tilde{x}_1 + i\tilde{y}_1)$ with $\tilde{x}_1 \pm \tilde{y}_1 \in \mathcal{C}^1_{\mathbb{R}}$ (and similarly for u_2). Possibly after applying a real rescaling of e.g. u_1 we may by Lemma 5.8 assume that :

$$\langle \ell, \tilde{x}_1 \rangle \leq e^{\Delta_{\mathbb{R}}/2} \langle \ell, \tilde{x}_2 \rangle \quad \text{and} \quad \langle \ell, \tilde{x}_2 \rangle \leq e^{\Delta_{\mathbb{R}}/2} \langle \ell, \tilde{x}_1 \rangle \tag{5.38}$$

Rotating the complex polarization of $\tilde{x}_1 + i\tilde{y}_1$ and $\tilde{x}_2 + i\tilde{y}_2$, we may by Lemma 5.7 assume that $u_1 = x_1 + iy_1$ and $u_2 = x_2 + iy_2$ with

$$|\langle \ell, y_1 \rangle| \leq \eta \langle \ell, x_1 \rangle \quad \text{and} \quad |\langle \ell, y_2 \rangle| \leq \eta \langle \ell, x_2 \rangle \tag{5.39}$$

for all $\ell \in \mathcal{C}'_{\mathbb{R}}$. The complex rotation may, however, push x_1 and x_2 out of the small cone[2] but using the second inequality in (5.36) as well as (5.38) we still have the bound :

$$\langle \ell, x_1 \rangle \leq 2e^{\Delta_{\mathbb{R}}/2} \langle \ell, x_2 \rangle \quad \text{and} \quad \langle \ell, x_2 \rangle \leq 2e^{\Delta_{\mathbb{R}}/2} \langle \ell, x_1 \rangle. \tag{5.40}$$

Proceeding as in the proof of Lemma 5.8 we see that

$$|\langle \ell, x_1 - x_2 \rangle| \leq \kappa \langle \ell, x_1 + x_2 \rangle \quad \text{with} \quad \kappa \equiv \frac{2e^{\Delta_{\mathbb{R}}/2} - 1}{2e^{\Delta_{\mathbb{R}}/2} + 1} < 1. \tag{5.41}$$

Let us write $u_\lambda = (1+\lambda)u_1 + (1-\lambda)u_2$, $\lambda \in \mathbb{C}$ and similarly for x_λ and y_λ. In order to prove our claim it suffices to find a fixed open neighborhood $U = U(\Delta_{\mathbb{R}})$ of the segment $[-1;1] \subset \mathbb{C}$, depending on $\Delta_{\mathbb{R}}$ but not upon u_1 and u_2, such that $u_\lambda \in \mathcal{C}_{\mathbb{C}}$ for every $\lambda \in U$. Let $-1 \leq t \leq 1$. Then $|\langle \ell, y_t \rangle| \leq \eta \langle \ell, x_t \rangle$ and we get (with $\ell_1, \ell_2 \in \mathcal{C}'_{\mathbb{R}}$) (a) :

$$\mathrm{Re}\langle \ell_1, u_t \rangle \langle \ell_2, \overline{u}_t \rangle = \langle \ell_1, x_t \rangle \langle \ell_2, x_t \rangle + \langle \ell_1, y_t \rangle \langle \ell_2, y_t \rangle \geq (1 - \eta^2) \langle \ell_1, x_t \rangle \langle \ell_2, x_t \rangle$$

as well as (b) : $|\langle \ell, u_t \rangle| \leq \sqrt{1+\eta^2} \langle \ell, x_t \rangle$.
We also obtain the estimates (c) : $|\langle \ell, u_1 - u_2 \rangle| \leq \sqrt{\eta^2 + \kappa^2}\, \langle \ell, x_1 + x_2 \rangle$ and (d) : $|\langle \ell, x_t \rangle| \geq (1 - \kappa|t|) \langle \ell, x_1 + x_2 \rangle \geq (1-\kappa) \langle \ell, x_1 + x_2 \rangle$. Let us write $\lambda = t + z$ with $-1 \leq t \leq 1$ and $z \in \mathbb{C}$. Using the expansion $\langle \ell, u_\lambda \rangle = \langle \ell, u_t \rangle + z \langle \ell, u_1 - u_2 \rangle$ and inserting the estimates (a)-(d) we obtain

$$\begin{aligned}
&\mathrm{Re}\langle \ell_1, u_\lambda \rangle \langle \ell_2, \overline{u_\lambda} \rangle \\
&\geq \; (1-\eta^2) \langle \ell_1, x_t \rangle \langle \ell_2, x_t \rangle - \\
&\qquad |z| \sqrt{\eta^2 + \kappa^2} \sqrt{1 + \eta^2}\, (\langle \ell_1, x_t \rangle \langle \ell_2, x_1 + x_2 \rangle + \langle \ell_1, x_1 + x_2 \rangle \langle \ell_2, x_t \rangle) \\
&\qquad - |z|^2 (\eta^2 + \kappa^2) \langle \ell_1, x_1 + x_2 \rangle \langle \ell_2, x_1 + x_2 \rangle \\
&\geq \; \langle \ell_1, x_t \rangle \langle \ell_2, x_t \rangle \left(2 - \left(\sqrt{1+\eta^2} + |z| \frac{\sqrt{\eta^2 + \kappa^2}}{1 - \kappa} \right)^2 \right).
\end{aligned}$$

This remains positive when $|z| \leq r_0$ where $r_0 = \frac{\sqrt{2} - \sqrt{1+\eta^2}}{\sqrt{\eta^2 + \kappa^2}} (1 - \kappa) > 0$ depends upon $\Delta_{\mathbb{R}}$ only. The set, U, of such $\lambda = t + z$-values is the r_0-neighborhood, of the segment $[-1;1] \subset \mathbb{C}$. Since enlarging a domain decreases hyperbolic distances, we conclude that $\Delta_{\mathbb{C}} \leq d_U(1, -1) < \infty$. □

[2] I am grateful to Loïc Dubois for having pointed this out to me

Remark 5.10 *Suppose that T is a real bounded linear operator, that $\mathcal{C}_{\mathbb{R}}$ is a reproducing K-norm-directed cone and that $T\mathcal{C}^*_{\mathbb{R}}$ has finite projective diameter in $\mathcal{C}_{\mathbb{R}}$. By Theorem 4.6 the operator has a spectral gap. In view of Remark 4.5 and the properties of the canonical complexification shown above, the same conclusion follows when considering the complexified operator acting on the canonically complexified cone. Our complex cone contraction thus contains the real contraction as a special case (but, of course, with a more complicated proof).*

6 Dominated complex cone-contractions

A real operator P which contracts a real cone $\mathcal{C}_{\mathbb{R}}$ contracts *a fortiori* the corresponding complexified cone $\mathcal{C}_{\mathbb{C}}$ (easy). It is then natural to ask if this complex contraction may be preserved when adding an imaginary part to the operator. Several of our applications below are cast over this idea and has lead us to state an abstract assumption for the action upon $\mathcal{C}_{\mathbb{R}}$ and a corresponding complex contraction Theorem for complexified cones :

Assumption 6.1 *Let $P \in L(X_{\mathbb{R}})$ be a contraction of an $\mathbb{R}$-cone $\mathcal{C}_{\mathbb{R}}$. Let $M \in L(X_{\mathbb{C}})$ be an operator acting upon the corresponding complex Banach space. We say that M is dominated by P with constants $0 \leq \gamma < \alpha \leq \beta < +\infty$ provided that for all $\ell_1, \ell_2 \in \mathcal{C}'_{\mathbb{R}}$ and $x_1, x_2 \in \mathcal{C}_{\mathbb{R}}$:*

$$\mathrm{Re}\langle \ell_1, Mx_1\rangle\langle \ell_2, \overline{M}x_2\rangle \geq \alpha\langle \ell_1, Px_1\rangle\langle \ell_2, Px_2\rangle \qquad (6.42)$$

$$\mathrm{Re}\langle \ell_1, Mx_1\rangle\langle \ell_2, \overline{M}x_2\rangle \leq \beta\langle \ell_1, Px_1\rangle\langle \ell_2, Px_2\rangle \qquad (6.43)$$

$$|\mathrm{Im}\langle \ell_1, Mx_1\rangle\langle \ell_2, \overline{M}x_2\rangle| \leq \gamma\langle \ell_1, Px_1\rangle\langle \ell_2, Px_2\rangle \qquad (6.44)$$

Remark 6.2 *The above conditions are $\mathbb{R}_+$-invariant and also stable when taking convex combinations. It thus suffices to verify that these conditions hold for subsets, $V \subset \mathcal{C}_{\mathbb{R}}$ and $W \subset \mathcal{C}'_{\mathbb{R}}$ which are generating for the cone and the dual cone, respectively, i.e. for which :*

$$\mathcal{C}_{\mathbb{R}} = \mathrm{Cl\ Conv}(\mathbb{R}_+ \times V) = \{x \in X_{\mathbb{R}} : \langle \ell, x\rangle \geq 0, \quad \forall\, \ell \in W\}\,.$$

When $\gamma = 0$ an operator M verifying the above assumption is essentially real. Possibly after multiplication with a complex phase the operator maps $\mathcal{C}_{\mathbb{R}}$ into $\mathcal{C}_{\mathbb{R}}$ itself. The above condition then reduces to the real cone-dominated condition of Theorem 4.8. Our goal is here to show that the conclusion of that Theorem also applies when M is allowed to have a non-trivial imaginary part. It turns out that the allowed 'amount' of imaginary part depends on the rate of contraction of P.

Theorem 6.3 *Let $\mathcal{C}_{\mathbb{R}} \subset X_{\mathbb{R}}$ be a proper convex cone and let $P : \mathcal{C}^*_{\mathbb{R}} \to \mathcal{C}^*_{\mathbb{R}}$ be a strict cone-contraction, i.e. $\Delta_P = \mathrm{diam}_{\mathcal{C}_{\mathbb{R}}} P(\mathcal{C}^*_{\mathbb{R}}) < +\infty$. Write $\mathcal{C}_{\mathbb{C}}$ for the canonical complexification of $\mathcal{C}_{\mathbb{R}}$. Suppose that $M \in L(X_{\mathbb{C}})$ is P-dominated (Assumption 6.1) with constants that satisfy :*

$$\gamma\ \cosh\frac{\Delta_P}{2} < \alpha. \qquad (6.45)$$

Then $M : \mathcal{C}^*_{\mathbb{C}} \to \mathcal{C}^*_{\mathbb{C}}$ and $\mathrm{diam}_{\mathcal{C}_{\mathbb{C}}} M(\mathcal{C}^*_{\mathbb{C}}) < +\infty$. If, in addition, $\mathcal{C}_{\mathbb{R}}$ is reproducing and of uniformly bounded sectional aperture then M has a spectral gap.

Proof: Let $u \in \mathcal{C}^*_{\mathbb{C}}$ and $\ell_1, \ell_2 \in \mathcal{C}'_{\mathbb{R}}$. We write $\eta = \tanh \frac{\Delta_P}{4} < +\infty$. The first step is to establish the following inequality (which, in particular, implies that $M : \mathcal{C}^*_{\mathbb{C}} \to \mathcal{C}^*_{\mathbb{C}}$) :

$$\mathrm{Re}\langle \ell_1, Mu\rangle\langle \ell_2, \overline{Mu}\rangle \geq \left(\frac{\alpha}{\cosh(\Delta_P/2)} - \gamma\right) |\langle \ell_1, Pu\rangle||\langle \ell_2, Pu\rangle|. \tag{6.46}$$

We will use polarization twice to achieve this. First, write $u = e^{i\theta}(x+iy)$ with $\theta \in \mathbb{R}$ and $x \pm y \in \mathcal{C}^*_{\mathbb{R}}$. Then

$$\begin{aligned}
&\langle \ell_1, Mu\rangle\overline{\langle \ell_2, Mu\rangle} \\
&\quad= \langle \ell_1, M(x+iy)\rangle\langle \ell_2, \overline{M}(x-iy)\rangle = \\
&\quad= \left[\langle \ell_1, Mx\rangle\langle \ell_2, \overline{M}x\rangle + \langle \ell_1, My\rangle\langle \ell_2, \overline{M}y\rangle\right] + i\left[\langle \ell_1, My\rangle\langle \ell_2, \overline{M}x\rangle - \langle \ell_1, Mx\rangle\langle \ell_2, \overline{M}y\rangle\right] \\
&\quad= \frac{1}{2}\left[\langle \ell_1, M(x+y)\rangle\langle \ell_2, \overline{M}(x+y)\rangle + \langle \ell_1, M(x-y)\rangle\langle \ell_2, \overline{M}(x-y)\rangle\right] + \\
&\qquad \frac{i}{2}\left[\langle \ell_1, M(x+y)\rangle\langle \ell_2, \overline{M}(x-y)\rangle - \langle \ell_1, M(x-y)\rangle\langle \ell_2, \overline{M}(x+y)\rangle\right] \\
&\quad\equiv \frac{1}{2}[A] + \frac{i}{2}[B].
\end{aligned}$$

Since $x \pm y \in \mathcal{C}_{\mathbb{R}}$ we may use inequality (6.42) of our assumption to deduce :

$$\begin{aligned}
\mathrm{Re}\, A &\geq \alpha\, \langle \ell_1, P(x+y)\rangle\langle \ell_2, P(x+y)\rangle + \alpha\, \langle \ell_1, P(x-y)\rangle\langle \ell_2, P(x-y)\rangle \\
&= 2\alpha\, \langle \ell_1, Px\rangle\langle \ell_2, Px\rangle + \alpha\langle \ell_1, Py\rangle\langle \ell_2, Py\rangle \\
&= 2\alpha\, \mathrm{Re}\, \langle \ell_1, P(x+iy)\rangle\langle \ell_2, P(x-iy)\rangle \\
&= 2\alpha\, \mathrm{Re}\, \langle \ell_1, Pu\rangle\langle \ell_2, P\overline{u}\rangle.
\end{aligned}$$

For the second term we have by (6.44)

$$\begin{aligned}
|\, \mathrm{Im}\, B| &\leq \gamma\, \langle \ell_1, P(x+y)\rangle\langle \ell_2, P(x-y)\rangle + \gamma\, \langle \ell_1, P(x-y)\rangle\langle \ell_2, P(x+y)\rangle \\
&= 2\gamma\, (\langle \ell_1, Px\rangle\langle \ell_2, Px\rangle - \langle \ell_1, Py\rangle\langle \ell_2, Py\rangle) \\
&\leq 2\gamma\, |\langle \ell_1, P(x+iy)\rangle|\, |\langle \ell_2, P(x-iy)\rangle| \\
&= 2\gamma\, |\langle \ell_1, Pu\rangle|\, |\langle \ell_2, Pu\rangle|,
\end{aligned}$$

where for the last inequality we used Schwarz' inequality. From these two estimates we get :

$$\mathrm{Re}\langle \ell_1, Mu\rangle\overline{\langle \ell_2, Mu\rangle} \geq \alpha\, \mathrm{Re}\langle \ell_1, Pu\rangle\langle \ell_2, P\overline{u}\rangle - \gamma\, |\langle \ell_1, Pu\rangle|\, |\langle \ell_2, Pu\rangle|. \tag{6.47}$$

We note that (6.47) is here independent of the choice of polarization. Since $x \pm y \in \mathcal{C}^*_{\mathbb{R}}$ we see that the elements $P(x+y) \in \mathcal{C}^*_{\mathbb{R}}$ and $P(x-y) \in \mathcal{C}^*_{\mathbb{R}}$ are at a projective distance not exceeding Δ_P. We may then use Lemma 5.7 to rotate the polarization again and write $Pu = e^{i\alpha}(x' + iy')$ where $|\langle \ell, y'\rangle| \leq \eta\langle \ell, x'\rangle$ for all $\ell \in \mathcal{C}'_{\mathbb{R}}$. But then

$$\mathrm{Re}\langle \ell_1, Pu\rangle\langle \ell_2, P\overline{u}\rangle \geq \langle \ell_1, x'\rangle\langle \ell_2, x'\rangle + \langle \ell_1, y'\rangle\langle \ell_2, y'\rangle \geq (1-\eta^2)\langle \ell_1, x'\rangle\langle \ell_2, x'\rangle.$$

We also obtain $|\langle \ell, Pu\rangle| = \sqrt{\langle \ell, x'\rangle^2 + \langle \ell, y'\rangle^2} \leq \sqrt{1+\eta^2}\, \langle \ell, x'\rangle$ so that

$$\mathrm{Re}\langle \ell_1, Pu\rangle\langle \ell_2, P\overline{u}\rangle \geq \frac{1-\eta^2}{1+\eta^2}|\langle \ell_1, Pu\rangle|\, |\langle \ell_2, Pu\rangle| = \left(\cosh\frac{\Delta_P}{2}\right)^{-1} |\langle \ell_1, Pu\rangle|\, |\langle \ell_2, Pu\rangle|.$$

Together with (6.47) this establishes (6.46).

In order to obtain an estimate for $\mathrm{diam}_{\mathcal{C}_\mathbb{C}} M(\mathcal{C}_\mathbb{C}^*)$ we also need the following inequality :

$$|\langle \ell, Mu\rangle| \leq \sqrt{\beta+\gamma}\ \ |\langle \ell, Pu\rangle|, \quad \forall \ell \in \mathcal{C}_\mathbb{R}^{'}, u \in \mathcal{C}_\mathbb{C}. \tag{6.48}$$

This follows by setting $\ell_1 = \ell_2 = \ell$ in the expression for A and B above and using the upper bounds (6.43) and (6.44) of our Assumption :

$$A \leq \beta(\langle \ell, P(x+y)\rangle^2 + \langle \ell, P(x-y)\rangle^2) = 2\beta(\langle \ell, Px\rangle^2 + \langle \ell, Py\rangle^2) = 2\beta\ |\langle \ell, Pu\rangle|^2$$

and the bound $|B| = |\mathrm{Im} B| \leq 2\gamma\ |\langle \ell, Pu\rangle|^2$ as before.

Consider $u_1, u_2 \in \mathcal{C}_\mathbb{C}^*$. Using the polarization identity, Proposition 5.35, we may assume that $u_1 = x_1 + iy_1$ with $x_1 \pm y_1 \in \mathcal{C}_\mathbb{R}^*$ so that $|\langle \ell, Py_1| \leq \langle \ell, Px_1\rangle$. Then also $\langle \ell, Px_1\rangle \leq |\langle \ell, Pu_1\rangle| \leq \sqrt{2}\langle \ell, Px_1\rangle$ and with the same bounds for $u_2 = x_2+iy_2$. Through a real rescaling, Lemma 5.8, we may also assume that $|\langle \ell, P(x_1-x_2)\rangle| \leq \eta\langle \ell, P(x_1+x_2)\rangle$. We also write $u_\lambda = (1+\lambda)u_1+(1-\lambda)u_2$ with $\lambda = t+z$, $-1 \leq t \leq 1$ (and similarly for x_λ and y_λ). By the choice of polarization $x_t \pm y_t \in \mathcal{C}_\mathbb{R}$ so that $u_t \in \mathcal{C}_\mathbb{C}$, i.e. belongs to the complex cone for all $-1 \leq t \leq 1$. We want to show that when $|z|$ is small enough the same is true for Mu_{t+z}.

First note that $\langle \ell, Px_t\rangle \geq (1-\eta|t|)\ \langle \ell, P(x_1+x_2)\rangle$. Applying the inequality (6.48) we deduce that $|\langle \ell, Mu_t\rangle| \leq \sqrt{2\ (\beta+\gamma)}\ \langle \ell, Px_t\rangle$ and $|\langle \ell, M(u_1-u_2)\rangle| \leq \sqrt{2\ (\beta+\gamma)}\langle \ell, P(x_1+x_2)\rangle \leq \sqrt{2\ (\beta+\gamma)}\ \dfrac{\langle \ell, Px_t\rangle}{1-\eta|t|}$. Using (6.46) on u_t and the expansion $\langle \ell, Mu_{t+z}\rangle = \langle \ell, Mu_t\rangle + z\langle \ell, M(u_1-u_2)\rangle$, we obtain the inequality

$$\begin{aligned}
&\mathrm{Re}\ \langle \ell_1, Mu_{t+z}\rangle\ \langle \ell_2, \overline{Mu_{t+z}}\rangle \qquad (6.49)\\
\geq\ & \mathrm{Re}\ \langle \ell_1, Mu_t\rangle\ \langle \ell_2, \overline{Mu_t}\rangle \\
& -|z|\ \left(\ |\langle \ell_1, Mu_t\rangle|\ |\langle \ell_2, M(u_1-u_2)\rangle|\ +\ |\langle \ell_2, Mu_t\rangle|\ |\langle \ell_1, M(u_1-u_2)\rangle|\ \right) \\
& -|z|^2\ |\langle \ell_1, M(u_1-u_2)\rangle|\ |\langle \ell_1, M(u_1-u_2)\rangle| \\
\geq\ & \langle \ell_1, Px_t\rangle\ \langle \ell_2 Px_t\rangle\ \times \left(\frac{\alpha}{\cosh(\Delta_P/2)} - \gamma + 2(\beta+\gamma) - 2(\beta+\gamma)\left(1+\frac{|z|}{1-\eta|t|}\right)^2 \right).
\end{aligned}$$

When condition (6.45) is satisfied, the set of $t+z$ values, $-1 \leq t \leq 1$ for which the quantity (6.50) is non-negative contains an open neighborhood,

$$U = U(\frac{\beta}{\alpha}, \frac{\gamma}{\alpha}, \Delta_P) \tag{6.50}$$

of the segment $[-1;1] \subset \mathbb{C}$. It follows that $\mathrm{diam}_{\mathcal{C}_\mathbb{C}} M(\mathcal{C}_\mathbb{C}^*) \leq d_U(-1,1) < +\infty$.

When $\mathcal{C}_\mathbb{R}$ is reproducing and of uniformly bounded sectional aperture then so is $\mathcal{C}_\mathbb{C}$ by Proposition 5.4. Thus, the conclusion follows from our spectral gap theorem, Theorem 3.7. □

7 Applications

The most striking application is also the simplest. A complex Perron-Frobenius Theorem (see also Appendix 11) :

Theorem 7.1 *Let $A \in M_n(\mathbb{C})$ and suppose there is $0 < c < +\infty$ for which $|\mathrm{Im}\, A_{ij}\overline{A}_{mn}| < c \leq \mathrm{Re}\, A_{ij}\overline{A}_{mn}$ for all indices. Then A has a spectral gap.*

Proof : The cone $\mathcal{C}_{\mathbb{R}} = \mathbb{R}^n_+$ is regular in $\mathbb{R}^n$. By Corollary 5.6, $\mathcal{C}_{\mathbb{C}} = \mathbb{C}^n_+$ is regular in $\mathbb{R}^n$. We will compare M with the constant matrix $P_{ij} \equiv 1$ with respect to the real cone $\mathbb{R}^n_+$. As in Example 4.9(1), $\Delta_P = 0$. The canonical basis and its dual generates the cone and its dual, respectively, cf. remark 6.2. The constants from Assumption 6.1 then become (sups and infs over all indices) (a) $\alpha = \alpha(A) = \inf\ \mathrm{Re} A_{ij}\overline{A}_{kl}$, (b) $\beta = \beta(A) = \sup\ \mathrm{Re} A_{ij}\overline{A}_{kl}$ and (c) $\gamma = \gamma(A) = \sup\ |\mathrm{Im} A_{ij}\overline{A}_{kl}|$. Our spectral gap condition of Theorem 6.3 simply reads $\gamma < \alpha$ and by finiteness of n this is equivalent to the stated assumptions on A. We also note that the 'contraction constant' for the spectral gap, $\eta = \eta(\beta/\alpha, \gamma/\alpha) < 1$ from Lemma 2.4, cf. equations (6.50) and (6.50), only depends on the ratios $\beta/\alpha \geq 1$ and $\gamma/\alpha < 1$. □

In the following, denote by $\mathrm{osc}(h) = \operatorname{ess\,sup}(h) - \operatorname{ess\,inf}(h)$ the essential oscillation of a real valued function h on a measured space. Theorem 7.1 may (almost) be viewed as a special case of the following complex version of a result of Jentzsch [Jen12] :

Theorem 7.2 *Let (Ω, μ) be a measure space and let $X = L^p(\Omega, \mu)$, with $1 \leq p \leq +\infty$. Let $h \in L^p$, $h > 0$ a.e. and $m \in L^q$, $m > 0$ a.e. with $\frac{1}{p} + \frac{1}{q} = 1$ so that $0 < \int h\, m\, d\mu < +\infty$, cf. Example 4.9(2). Given $g \in L^\infty(\Omega \times \Omega)$ we define the integral operator, $M_g \in L(X)$:*

$$M_g\phi(x) = h(x) \int_\Omega e^{g(x,y)} \phi(y)\, m(y)\, \mu(dy). \tag{7.51}$$

Set $\theta = \mathrm{osc}(\mathrm{Im}\, g)$ and $\Lambda = \mathrm{osc}(\mathrm{Re}\, g)$. Suppose that $\theta < \pi/4$ and that $\tan\theta < \exp(-2\Lambda)$. Then M_g has a spectral gap.

Proof: As in Example 4.9(2) we consider the $\mathbb{R}$-cone $\mathcal{C}_{\mathbb{R}} = \{\phi \in X_{\mathbb{R}} : \phi \geq 0 \text{ (a.e.)}\}$ and we compare with $P\phi = h \int_\Omega \phi\, m\, d\mu$. We have that $P : \mathcal{C}^*_{\mathbb{R}} \to \mathcal{C}^*_{\mathbb{R}}$ and $\Delta_P = 0$. We obtain the following estimate for the constants

$$\mathrm{Re}\, e^{g(x,y)+\overline{g(x',y')}} \geq \alpha \equiv e^{2\operatorname{ess\,inf} \mathrm{Re}\, g} \cos\theta \quad \text{and} \quad \mathrm{Im}\, e^{g(x,y)+\overline{g(x',y')}} \leq \gamma \equiv e^{2\operatorname{ess\,sup} \mathrm{Re}\, g} \sin\theta.$$

The cone $\mathcal{C}_{\mathbb{R}}$ is reproducing. As shown in Example 4.9 (2) the real cone has uniformly bounded sectional aperture. The spectral gap condition of Theorem 6.3 then translates into the stated condition on θ and Λ. □

8 A complex Kreĭn-Rutman Theorem

Let X be a complex Banach space. We denote by $\mathrm{Gr}_2(X)$ denote the set of complex planes in X, i.e. subsets of the form $\mathbb{C}x + \mathbb{C}y$ with x and y independent vectors in X. If we write $S(X)$ for the unit sphere in X then

$$d_2(F, F') = \mathrm{dist}_H \left(F \cap S(X), F' \cap S(X)\right), \quad F, F' \in \mathrm{Gr}_2(X)$$

defines a metric on $\mathrm{Gr}_2(X)$. In the following let us fix a norm on $\mathbb{C}^n$. The choice may affect the constants below but is otherwise immaterial. The space $(\mathrm{Gr}_2(\mathbb{C}^n), d_2)$ is then a compact metric space.

Lemma 8.1 *Let $V \subset X$ be a $\mathbb{C}$-cone and let $F \in \mathrm{Gr}_2(X)$. Suppose there is $u \in F$, $r > 0$ such that $B(u, r) \cap V = \emptyset$. Then $V_F = F \cap V$ has at most $1 + \frac{\|u\|}{r}$ bounded aperture.*

Proof: Let $m \in (F)'$ be a linear functional with $u \in \ker m$ and $\|m\| = 1$. Choose $x \in F$ for which $|\langle m, x\rangle| = \|x\|$. If $ax + bu \in V_F$ then $u + \frac{a}{b}x \notin B(u, r)$ so that $|b| \leq \frac{|a|}{r}\|x\|$ and therefore,

$$\|ax + bu\| \leq |a|\ \|x\| \left(1 + \frac{\|u\|}{r}\right) = |\langle m, ax + bu\rangle| \left(1 + \frac{\|u\|}{r}\right).$$

The 2-dimensional space F is spanned by u and x so $K(V_F) \leq 1 + \dfrac{\|u\|}{r}$. □

Lemma 8.2 *Let $V \subset \mathbb{C}^n$ be a $\mathbb{C}$-cone. Then there is $K < \infty$ so that V is of K-bounded sectional aperture.*

Proof: Suppose that this is not the case. Then we may find a sequence F_n of planes for which the aperture $K(V \cap F_n)$ diverges. Taking a subsequence we may assume that F_n converges in $\mathrm{Gr}_2(X)$ to a plane F. As V is proper, $V \cap F$ is a strict subset of F. Thus there is $u \in F - V$. But V is closed in $\mathbb{C}^n$ so there is $r > 0$ so that $B(u, r)$ is disjoint from V as well. Given another complex plane, F', we may find $u' \in F'$ for which $\|u - u'\| \leq \|u\| d_2(F, F')$. When F and F' are close enough, $r' = r - \|u\| d_2(F, F') > 0$ and $B(u', r')$ is also disjoint from V. By our previous Lemma, $V \cap F'$ is of aperture not exceeding $1 + \|u\|/(r - \|u\| d_2(F, F'))$. But this contradicts the divergence of $K(V \cap F_n)$ as $F_n \to F$. □

Lemma 8.3 *Let $V \subset \mathbb{C}^n$ be a $\mathbb{C}$-cone and let $W \subset V$ be a closed complex subcone with $W^* \subset$* Int V*. Then there is $\Delta = \Delta(W, V) < +\infty$ such that for $x, y \in W^*$:*

$$d_W(x, y) < +\infty \Rightarrow d_V(x, y) \leq \Delta.$$

Proof: We denote by $\pi : \mathbb{C}^n - \{0\} \to \mathbb{C}P^{n-1}$ the canonical projection to complex projective space. We consider $\mathbb{C}P^{n-1}$ as a metric space with the metric $d_{\mathbb{C}P^{n-1}}$ as in equation (A.82). The projected image, $\pi(W^*)$, is compact in the open set $\pi(\mathrm{Int}\ V^*) \subset \mathbb{C}P^{n-1}$ so there is $\epsilon = \epsilon(W, V) > 0$ for which the ϵ-neighborhood of $\pi(W^*)$ is contained in $\pi(V^*)$.

Let $x, y \in W^*$ be linearly independent and suppose that $d_W(x, y) < +\infty$. Let $F \in \mathrm{Gr}_2(\mathbb{C}^n)$ be the complex plane containing x and y. Denote by C the connected set in $\pi(F^*)$ containing x and y. Let $\xi_i \in C$, $i \in J$ be an $\epsilon/3$-maximally separated set in C. Thus, the balls $B(\xi_i, \frac{\epsilon}{6})$, $i \in J$ are all disjoint and $\bigcup_{i \in J} B(\xi_i, \frac{\epsilon}{3}) = \mathbb{C}P^{n-1}$. The cardinality of J is bounded by a constant depending on ϵ only. Then $B(\xi_i, \frac{2\epsilon}{3}) \subset \pi(V^*)$, $i \in J$ so by Lemma A.1(2) each $B(x_i, \frac{\epsilon}{3})$, $i \in J$ is of radius not greater than $\log \frac{1+1/2}{1-1/2} = \log 3$ for the d_V-metric. Also $\cup_{i \in J} B(\xi, \frac{\epsilon}{3})$ contains C which is connected. It follows that $d_V(x, y)$ does not exceed $2 \log 3\ \mathrm{Card}(J)$. which is bounded by a constant depending on ϵ only. □

Theorem 8.4 *Let $V \subset \mathbb{C}^n$ be a closed subset which is $\mathbb{C}$-invariant and contains no complex planes (in terms of Definition 2.1, V is a $\mathbb{C}$-cone). Suppose that $A : \mathbb{C}^n \to \mathbb{C}^n$ is a linear map for which $A(V^*) \subset$* Int V*. Then A has a spectral gap.*

Proof : We write $W = A(V)$ for the image of V and use the notation and constants from the two previous Lemmas. First note that for $x, y \in V$,

$$d_V(x, y) < \infty \Rightarrow d_V(Ax, Ay) \leq \eta\ d_V(x, y) \quad \text{and} \quad d_V(Ax, Ay) \leq \Delta.$$

To see this note that when $d_V(x,y)<\infty$ then $d_V(Ax,Ay)<\infty$ so by Lemma 8.3, $d_V(Ax,Ay)\leq \Delta$. If $\mathcal{C}^*$ denotes the connected component of $F\cap V^*$ containing x and y then also $\mathrm{diam}_V A(\mathcal{C}^*)\leq \Delta$. By Lemma 2.4, $d_V(Ax,Ay)\leq \eta\, d_{\mathcal{C}}(x,y)\leq \eta d_V(x,y)$. Iterating this argument we see that $\mathrm{diam}_V A^n(\mathcal{C}^*)\leq \Delta\eta^{n-1}$, $n\geq 1$. By Lemma 8.2, V is of K-bounded sectional aperture, so Lemma A.1(1) assures that $\mathrm{diam}_{\mathbb{C}P^{n-1}} A^n(\mathcal{C}^*)\leq 2K\Delta\eta^{n-1}$. Fix $n_1<+\infty$ so that $2K\Delta\eta^{n_1-1}\leq \epsilon/3$.

Now let ξ_i, $i\in J$ be an $\epsilon/3$-maximally separated set in W. Setting $V_i=\pi^{-1}B(\xi_i,\epsilon)$ with $i\in J$ we see that $\mathrm{diam}_{\mathbb{C}P^{n-1}} A^n V_i^*\leq \epsilon/3$, $n\geq n_1$. It follows that there is a map, $\tau: J\to J$ so that $A^n V_i^*\subset W_{\tau(i)}\equiv \pi^{-1}B(\xi_{\tau(i)},2\epsilon/3)$, $n\geq n_1$. Since J is of finite cardinality, τ must have a cycle. Thus, there are $i_1\in J$ and $n_1<+\infty$ for which $A^{n_1}(V_{i_1})\subset W_{i_1}$. The cone W_{i_1} is regular (easy) and of bounded diameter in V_{i_1} so A^{n_1} has a spectral gap and therefore also A. □

When the operator is sufficiently regular one may weaken the assumptions on the contraction and the outer regularity of the cone. This is illustrated by the following complex version of a theorem of Kreĭn and Rutman [KR50, Theorem 6.3] :

Theorem 8.5 *Let $\mathcal{C}\subset X_{\mathbb{C}}$ be an inner regular $\mathbb{C}$-cone in the Banach space $X_{\mathbb{C}}$. Let $A\in L(X_{\mathbb{C}})$ be a quasi-compact operator or a compact operator of strictly positive spectral radius and suppose that $A\,\mathcal{C}^*\subset \mathrm{Int}\,\mathcal{C}$. Then A has a spectral gap.*

Proof : Let P be the spectral projection associated with eigenvalues on the spectral radius circle, $\{\lambda\in\mathbb{C}: |\lambda|=r_{\mathrm{sp}}(A)\}$. By hypothesis $\mathrm{im}P$ is finite dimensional and we may find $\theta\in\mathbb{R}$ such that

$$r_{\mathrm{sp}}(A(1-P))<\theta<r_{\mathrm{sp}}(A).$$

We claim that $\mathcal{C}^*\cap \mathrm{im}P$ is non-empty : Let $x\in\mathcal{C}^*$ and define $e_n=A^n x/\|A^n x\|\in\mathcal{C}^*$, $n\in\mathbb{N}$.

Suppose first that $Px\neq 0$. Then $\lim_{n\to\infty}\|A^n(1-P)x\|/\|A^nPx\|=0$ so that the distance between e_n and $\mathrm{im}P$ tends to zero. Since $\mathrm{im}\,P$ is locally compact and e_n is bounded we may extract a convergent subsequence $e^*=\lim e_{n_k}\in \mathrm{im}P\cap\mathcal{C}^*$. Suppose instead that $Px=0$ then $Ax\in\mathrm{Int}\,\mathcal{C}$ so there is $r>0$ for which $B(Ax,r)\in\mathcal{C}$. We may then replace x by $Ax+u$ where $u\in\mathrm{im}P$, $\|u\|<r$ and we are back in the first case. Thus $\mathcal{C}^*_P=\mathcal{C}^*\cap\mathrm{im}P\neq\emptyset$. Now,

$$A:\mathcal{C}^*_P\to (A\,\mathcal{C}^*)\cap \mathrm{im}P\subset \mathrm{Int}\,\mathcal{C}\cap\mathrm{im}P=\mathrm{Int}\,\mathcal{C}_P,$$

the latter for the topology in $\mathrm{im}P$. In particular, $\mathrm{Int}\,\mathcal{C}_P$ is non-empty so $\mathcal{C}_P$ is an inner regular $\mathbb{C}$-cone in a finite dimensional space and $A:\mathcal{C}^*_P\to\mathrm{Int}\,\mathcal{C}_P$. We may then apply the finite dimensional contraction theorem, Theorem 8.4, to $A_P=A_{|\mathrm{imP}}\in L(\mathrm{im}P)$. It follows that A_P, whence also our original operator A has a spectral gap. □

Remark 8.6 *In the real cone version (replacing $\mathbb{C}$ by $\mathbb{R}$) of theorem 8.5 it is not necessary to assume that the spectral radius of A is strictly positive. This forms part of the conclusion. To see this pick $x\in\mathcal{C}^*$ of norm one. Then $Ax\in\mathrm{Int}\,\mathcal{C}$ so there is $\lambda>0$ for which $B(Ax,\lambda)\subset\mathcal{C}$. Therefore, $Ax-\lambda x\in\mathcal{C}$ and then also $B(A^2x,\lambda^2)=A(Ax-\lambda x)+\lambda B(Ax,\lambda)\subset\mathcal{C}$ by the properties of an $\mathbb{R}$-cone. More generally, $B(A^nx,\lambda^n)\subset\mathcal{C}$. As $0\in\partial\mathcal{C}$ it follows that*

$$r_{\mathrm{sp}}(A)\geq \limsup \sqrt[n]{|A^nx|}\geq\lambda>0.$$

The fact that this conclusion is non-trivial is illustrated e.g. by the operator, $A\phi(t)=\int_0^s\phi(s)\,ds$, $0\leq t\leq 1$, which is compact when acting upon $\phi\in X=C^0([0,1])$. It contracts (but not strictly) the cone of positive elements but has spectral radius zero.

In the complex setup, if one assumes that $\mathcal{C}$ is of K-bounded sectional aperture then strict positivity of $r_{\mathrm{sp}}(A)$ also comes for free : Suppose that $x\in\mathcal{C}$, $|x|=1$ and $B(Ax,r)\subset\mathcal{C}$, $r>0$.

Then $Ax + \lambda x \in \mathcal{C}^*$, $\forall |\lambda| < r$ and also $A^{n+1}x + \lambda A^n x \in \mathcal{C}^*$ for such λ-values. By Lemma 3.5. we see that $|A^{n+1}x| \geq \frac{r}{K}|A^n x| > 0$ from which $r_{\rm sp}(A) \geq \frac{r}{K} > 0$.

9 A complex Ruelle-Perron-Frobenius Theorem

The Ruelle-Perron-Frobenius Theorem, [Rue68, Rue69, Rue78] (see also [Bow75]), ensures a spectral gap for certain classes of real, positive operators with applications in statistical mechanics and dynamical systems. Ferrero and Schmitt [FS79, FS88] used Birkhoff's Theorem on cone contraction to give a conceptually new proof of the Ruelle-Perron-Frobenius Theorem. See also [Liv95] and [Bal00] for further applications in dynamical systems. We present here a generalization to a complex setup.

Let (Ω, d) be a metric space of finite diameter, $D < +\infty$. We write $C^0(\Omega)$ for the Banach space of (real- or complex-valued) continuous functions on Ω under the supremum norm, $|\cdot|_0$. When $\phi : \Omega \to \mathbb{R}$ (or $\mathbb{C}$) we write $\mathrm{Lip}(\phi) = \sup_{x \neq y} |\phi_x - \phi_y|/d(x,y) \in [0, +\infty]$ for the associated Lipschitz constant. Then $X_{\mathbb{R}} = \{\phi : \Omega \to \mathbb{R} \mid \|\phi\| \equiv |\phi|_0 + \mathrm{Lip}(\phi) < +\infty\}$ (and similarly for $X_{\mathbb{C}}$) is a Banach algebra.

Let $U \subset \Omega$ and let $f : U \to \Omega$ be an unramified covering map of Ω which is uniformly expanding. For simplicity, we will take it to be of finite degree (it is an instructive exercise to extend Theorem 9.1 below to maps of countable degree). More precisely, we assume that there is $0 < \rho < 1$ and a finite index set J so that for every couple $y, y' \in \Omega$ we have a pairing $\mathcal{P}(y, y') = \{(x_j, x'_j) : j \in J\}$ of the pre-images, $f^{-1}(y) = \{x_j\}_{j \in J} \subset U$ and $f^{-1}(y') = \{x'_j\}_{j \in J} \subset U$, for which $d(x_j, x'_j) \leq \rho\, d(y, y')$, $j \in J$.

Fix an element $g \in X_{\mathbb{C}}$ and define for $\phi \in C^0(\Omega)$ (or $\phi \in X_{\mathbb{C}}$) :

$$M_g \phi(y) = \sum_{x : f(x) = y} e^{g(x)} \phi(x), \;\; y \in \Omega.$$

The norm of M_g when acting upon $C^0(\Omega)$ (in the uniform norm) is given by

$$\|M_g\|_0 = \sup_{y \in \Omega} \sum_{x : f(x) = y} e^{\mathrm{Re} g(x)},$$

and a straight-forward calculation shows that $M_g \in L(X_{\mathbb{C}})$ with $\|M\| \leq \|M\|_0 (1 + \rho \,\mathrm{Lip}\, g)$.

Theorem 9.1 *Denote $a = \mathrm{Lip}\ \mathrm{Re}\ g$, $b = \mathrm{Lip}\ \mathrm{Im}\ g$ and $\theta = \mathrm{osc}\ \mathrm{Im}\ g$. Suppose that*

$$\left(\theta + \frac{2\rho^2 D\, b}{1 - \rho + \rho^2 D\, a} \right) \exp\left(1 + \rho \frac{1+\rho}{1-\rho} D\, a \right) \frac{4}{1-\rho} < 1.$$

Then $M_g \in L(X_{\mathbb{C}})$ has a spectral gap.

Proof : We will compare M_g with the real operator $P = M_{\mathrm{Re}\, g}$. For $\sigma > 0$ the set,

$$\mathcal{C}_{\sigma,\mathbb{R}} = \{\phi : \Omega \to \mathbb{R}_+ \mid \langle \ell_{y,y'}, \phi \rangle \equiv \phi(y) - e^{-\sigma d(y,y')} \phi(y') \geq 0, \; \forall y, y' \in \Omega\}, \tag{9.52}$$

defines a proper convex cone in $X_{\mathbb{R}}$ which in addition is regular. Inner regularity : Let $\mathbf{1}(x) \equiv 1$, $x \in \Omega$ and $h \in X_{\mathbb{R}}$. Then $\mathbf{1} + h \in \mathcal{C}_{\sigma,\mathbb{R}}$ provided $\mathrm{Lip}\, h/(1 - |h|_0) \leq \sigma$. Whence $B(\mathbf{1}, \min(\sigma, 1)) \subset \mathcal{C}_{\sigma,\mathbb{R}}$. Outer regularity : Pick $x_0 \in \Omega$ and set $\ell_0(\phi) = \phi(x_0)$. For $\phi \in \mathcal{C}_{\sigma,\mathbb{R}}$ we have $\mathrm{Lip}\, \phi \leq \sigma |\phi|_0$ so that $\|\phi\| \leq (1 + \sigma)|\phi|_0 \leq (1 + \sigma) e^{\sigma D} \ell_0(\phi)$, and this shows outer regularity.

Let $0 < \sigma' < \sigma$ and $\phi_1, \phi_2 \in \mathcal{C}^*_{\sigma',\mathbb{R}}$. As in (4.27) let $\beta_\sigma(\phi_1,\phi_2) = \inf\{\lambda > 0 : \lambda\phi_1 - \phi_2 \in \mathcal{C}_{\sigma,\mathbb{R}}\}$. A calculation using the defining properties of the cone-family yields :

$$\beta_\sigma(\phi_1,\phi_2) \leq \sup_{d>0} \frac{1-\exp(-(\sigma+\sigma')d)}{1-\exp(-(\sigma-\sigma')d)} \sup_{y\in\Omega} \frac{\phi_2(y)}{\phi_1(y)} \leq \frac{\sigma+\sigma'}{\sigma-\sigma'} \sup_{y\in\Omega} \frac{\phi_2(y)}{\phi_1(y)},$$

and we get the following bound for the diameter $\Delta_\mathbb{R} = \mathrm{diam}_{\mathcal{C}_{\sigma,\mathbb{R}}} \mathcal{C}^*_{\sigma',\mathbb{R}}$, cf. (4.28) :

$$\Delta_\mathbb{R} \leq 2 \log \frac{\sigma+\sigma'}{\sigma-\sigma'} + \sup_{y,y'\in\Omega} \log \frac{\phi_2(y)}{\phi_1(y)} \frac{\phi_1(y')}{\phi_2(y')} \leq 2 \log \frac{\sigma+\sigma'}{\sigma-\sigma'} + 2\, D\, \sigma' < +\infty.$$

The injection $\mathcal{C}_{\sigma',\mathbb{R}} \hookrightarrow \mathcal{C}_{\sigma,\mathbb{R}}$ is thus a uniform contraction for the respective projective metrics. Given $\phi \in \mathcal{C}_{\sigma,\mathbb{R}}$ and using the pairing $\mathcal{P}(y,y')$ we get for the operator $P = M_{\mathrm{Re}\, g}$:

$$P\phi(y) = \sum_{x:f(x)=y} e^{\mathrm{Re}\, g(x)} \phi(x) \geq \sum_{x':f(x')=y'} e^{\mathrm{Re}\, g(x') - (a+\sigma)d(x,x')} \phi(x') \geq e^{-\rho(a+\sigma)d(y,y')} P\phi(y').$$

This implies that $P : \mathcal{C}_{\sigma,\mathbb{R}} \to \mathcal{C}_{\sigma',\mathbb{R}}$ with $\sigma' = \rho(a+\sigma)$. If we choose $\sigma > a\rho/(1-\rho)$ then P becomes a strict cone contraction of the regular cone $\mathcal{C}_{\sigma,\mathbb{R}}$. We also get the estimate (to obtain an a priori estimate for the contraction one may here try to optimize for the value of σ) :

$$\frac{\Delta_P}{2} \leq \log \frac{\sigma + \rho(\sigma+a)}{\sigma - \rho(\sigma+a)} + D\, \rho\, (\sigma + a). \tag{9.53}$$

By Theorem 4.6, $P \in L(X_\mathbb{R})$ has a spectral gap (see [Rue68, FS79] and also [Liv95]).

Returning to the complex operator, M_g, let us fix $y, y' \in \Omega$ and the corresponding pairing of pre-images $\mathcal{P}(y,y')$ as described above. Let $\phi \in \mathcal{C}^*_{\sigma',\mathbb{R}}$ and write $\langle \ell_{y,y'}, M_g\phi\rangle = \sum_j \langle \mu_j(g), \phi\rangle$ with

$$\langle \mu_j(g), \phi\rangle \equiv e^{g(x_j)}\phi(x_j) - e^{-\sigma d(y,y') + g(x'_j)}\phi(x'_j), \quad j \in J.$$

In order to compare with the real operator, we define complex numbers w_j, $j \in J$, through the relation

$$\langle \mu_j(g), \phi\rangle = e^{i\, \mathrm{Im}\, g(x_j)}\, w_j\, \langle \mu_j(\mathrm{Re}\, g), \phi\rangle.$$

Equivalently (when the denominator is non-zero) :

$$e^{i\, \mathrm{Im}\, g(x_j)}\, w_j = \frac{e^{g(x_j)}\phi(x_j) - e^{-\sigma d(y,y') + g(x'_j)}\phi(x'_j)}{e^{\mathrm{Re}\, g(x_j)}\phi(x_j) - e^{-\sigma d(y,y') + \mathrm{Re}\, g(x'_j)}\phi(x'_j)}.$$

We may apply Lemma 9.3 below with the bounds $\mathrm{Re}(z_1 - z_2) \geq (\sigma - \rho(\sigma+a))\, d(y,y')$ and $|\mathrm{Im}(z_1 - z_2)| \leq \rho b\, d(y,y')$ to deduce that

$$|\mathrm{Arg}\, w_j| \leq s_0 \equiv \frac{\rho b}{\sigma - \rho(\sigma+a)}. \quad \text{and} \quad 1 \leq |w_j|^2 \leq 1 + s_0^2. \tag{9.54}$$

Given $i, j \in J$ and $\phi_1, \phi_2 \in \mathcal{C}^*_{\sigma',\mathbb{R}}$ we obtain :

$$\langle \mu_j(g), \phi_1\rangle \overline{\langle \mu_i(g), \phi_2\rangle} = \left(e^{i(\mathrm{Im}\, g(x_j) - \mathrm{Im}\, g(x_i))} w_j \overline{w_i}\right) \langle \mu_j(\mathrm{Re}\, g), \phi\rangle\, \langle \mu_i(\mathrm{Re}\, g), \phi\rangle.$$

The two last factors are real and non-negative (because $\sigma - \rho(\sigma+a) > 0$) and the complex pre-factor belongs to the set

$$A = \{re^{iu} : 1 \leq r \leq 1 + s_0^2, \quad |u| \leq \theta + 2s_0\}.$$

Summing over all indices we therefore obtain

$$\langle \ell_{y,y'}, M_g\phi_1\rangle\langle \ell_{w,w'}, \overline{M}_g\phi_2\rangle = Z\langle \ell_{y,y'}, P\phi_1\rangle\langle \ell_{w,w'}, P\phi_2\rangle,$$

in which Z is an average of numbers in A whence belongs to $\mathrm{Conv}(A)$, the convex hull of A.

When $\theta + 2s_0 < \pi/4$ we conclude that the bounds in Assumption 6.1 are verified for the constants $\alpha = \cos(\theta+2s_0)$, $\gamma = (1+s_0^2)\sin(\theta+2s_0)$ and $\beta = 1+s_0^2$. The spectral gap condition in Theorem 6.3 then reads as follows :

$$(1+s_0^2)\ \tan(\theta+2s_0)\ \cosh\frac{\Delta_P}{2} < 1. \tag{9.55}$$

Now, in order to get a more tractable and explicit formula we make the following (not optimal) choice for σ :

$$\sigma = \frac{2a\rho}{1-\rho} + \frac{1}{\rho D}.$$

Then $\sigma' = \rho(a+\sigma) \leq \frac{1+\rho}{2}\sigma$ so that $(\sigma+\rho(a+\sigma))/(\sigma-\rho(a+\sigma)) \leq (3+\rho)/(1-\rho)$. Using (9.53) we obtain

$$\cosh\frac{\Delta_P}{2} \leq e^{\Delta_P/2} = \frac{3+\rho}{1-\rho}\exp\left(1+2a\,D\rho\frac{1+\rho}{1-\rho}\right).$$

One also checks that $(\theta+2s_0)\frac{4}{1-\rho} < 1$ implies that $(1+s_0^2)\tan(\theta+2s_0)\frac{3+\rho}{1-\rho} < 1$ so we may replace (9.55) by the stronger condition

$$(\theta+2\,s_0)\ \exp\left(1+\rho\,\frac{1+\rho}{1-\rho}\,D\,a\right)\frac{4}{1-\rho} < 1.$$

Finally inserting $s_0 = \rho^2\,D\,b/(1-\rho+\rho^2\,D\,a)$ we obtain the claimed condition which thus suffices to ensure a spectral gap. ∎

Remark 9.2 *In the literature, one often includes a statement on Gibbs measures as well. If we let $\lambda h \otimes \mu$ denote the leading spectral projection of $P = M_{\mathrm{Re}\,g}$, then positivity of P implies that the 'state' $\phi \in X_{\mathbb{R}} \mapsto \nu(\phi) = \mu(\phi h)$ is uniformly bounded with respect to $|\phi|_0$. By continuity, ν extends to a linear functional on $C^0(\Omega)$. If, in addition, we assume Ω compact, then by Riesz, this functional defines a Borel probability measure $d\nu$ on Ω. The measure is invariant and strongly mixing for f. It is known as a Gibbs measure for f and the weight g. This part of the theorem, however, needs the partial ordering induced by the cone of positive continuous functions and does not extend to a complex setup (in general, it is even false there).*

In the proof we made use of the following complex estimate :

Lemma 9.3 *Let $z_1, z_2 \in \mathbb{C}$ be such that $\mathrm{Re}\,z_1 > \mathrm{Re}\,z_2$ and define $w \in \mathbb{C}$ through*

$$e^{i\,\mathrm{Im}\,z_1}w \equiv \frac{e^{z_1}-e^{z_2}}{e^{\mathrm{Re}\,z_1}-e^{\mathrm{Re}\,z_2}}.$$

Then

$$|\mathrm{Arg}\,w| \leq \frac{|\mathrm{Im}\,(z_1-z_2)|}{\mathrm{Re}\,(z_1-z_2)} \quad \textit{and} \quad 1 \leq |w^2| \leq 1 + \left(\frac{\mathrm{Im}\,(z_1-z_2)}{\mathrm{Re}\,(z_1-z_2)}\right)^2.$$

Proof: Writing $t = \mathrm{Re}\,(z_1-z_2) > 0$ and $s = \mathrm{Im}\,(z_1-z_2)$ we have :

$$w = \frac{1-e^{-t-is}}{1-e^{-t}}.$$

Taking real and imaginary parts, $\mathrm{Re}\ w = \frac{1-e^{-t}\cos s}{1-e^{-t}}$ and $\mathrm{Im}\ w = \frac{e^{-t}\sin s}{1-e^{-t}}$, we get $|w|^2 = 1+\frac{\sin^2(s/2)}{\sinh^2(t/2)} \leq 1+(\frac{s}{t})^2$. Also $|\frac{\partial}{\partial s}\,\log w| = |\frac{1}{w}\frac{\partial w}{\partial s}| = \frac{e^{-t}}{|1-e^{-t-is}|} \leq \frac{e^{-t}}{1-e^{-t}} \leq \frac{1}{t}$ so that $|\mathrm{Arg}\ w| \leq \frac{|s|}{t}$. ∎

10 Random products of cone contractions

A conceptual difference between standard perturbation theory and cone contractions is the behavior under compositions. Composing a sequence of operators that uniformly contracts the same cone, one obtains again a contraction, and even with a sub-multiplicative bound for the contraction rate. This is extremely useful when studying time-dependent and/or random products of such operators as it allows for the use of an implicit function theorem. In [Rue79], Ruelle showed the real-analytic behavior of the characteristic exponent of a product of random positive matrices. He did not use Birkhoff's cone contractions (which would have simplified some estimates and avoided some unnecessary assumptions) but the central part of his proof may still be viewed as a an argument based upon real 'cone-contractions'. We will here show how results similar to [Rue79] hold for complex cone-contractions. The resulting theorems and examples of this section may not deduced from neither real cone contractions, nor standard analytic perturbation theory.

In the following we will assume that the $\mathbb{C}$-cone $\mathcal{C}$ is regular (Definition 3.2). This is convenient (if not necessary). In particular, for $\rho > 0$ sufficiently small the (closed) subcone

$$\mathcal{C}(\rho) \equiv \{\phi \in \mathcal{C} : B(\phi, \rho \, \|\phi\|) \subset \mathcal{C}\} \tag{10.56}$$

is non-trivial (not reduced to $\{0\}$). We fix such a value of $\rho > 0$ in the following. Also let $\Delta < +\infty$ be arbitrary but fixed. We write $\eta = \eta(\Delta) < 1$ for the contraction constant from Lemma 2.4.

Definition 10.1 *Let $\mathcal{M} = \mathcal{M}(\Delta, \rho) \subset L(X)$ be the (non-empty) family of cone contractions : $M \in L(X)$, $M : \mathcal{C}^* \to \mathcal{C}^*$ subject to the following uniform bounds : $\mathrm{diam}_{\mathcal{C}} M(\mathcal{C}^*) \leq \Delta$ and $M(\mathcal{C}) \subset \mathcal{C}(\rho)$.*

Let (Ω, μ) be a probability-space and $\tau : \Omega \to \Omega$ a μ-ergodic transformation. We denote by $\mathbb{E}$ an average taken with respect to μ. When A is a subset of some Banach space Y we write $\mathcal{E}(\Omega, A)$ for the set of Bochner-measurable maps from Ω into A (the image of a set of full measure has a countable dense subset). We write $\mathcal{B}(\Omega, Y) \subset \mathcal{E}(\Omega, Y)$ for the Banach space of (μ-essentially) bounded measurable maps equipped with the (μ-essential) uniform norm of Y. For $\mathbf{M} \in \mathcal{E}(\Omega, \mathcal{M})$ we write $\mathbf{M}^{(n)}_\omega = \mathbf{M}_\omega \cdots \mathbf{M}_{\tau^{n-1}\omega}$ for the product of operators along the τ-orbit of $\omega \in \Omega$. It is again an element of $\mathcal{E}(\Omega, \mathcal{M})$. Our goal here is to show the following :

Theorem 10.2 *Let $t \in \mathbb{D} \mapsto \mathbf{M}(t) \in \mathcal{E}(\Omega, \mathcal{M})$ be a map for which*

1. *$(t, \omega) \in \mathbb{D} \times \Omega \mapsto \mathbf{M}_\omega(t) \in L(X)$ is measurable and $\forall\, \omega \in \Omega : t \mapsto \mathbf{M}_\omega(t)$ is analytic.*
2. *$\sup \left\{ \left\| \frac{d}{dt} \mathbf{M}_\omega(t) \right\| \;/\; \|\mathbf{M}_\omega(t)\| \;:\; \omega \in \Omega, t \in \mathbb{D} \right\} < +\infty$.*
3. *$\mathbb{E}\left(\Big| \log \|\mathbf{M}_\omega(0)\| \Big| \right) < +\infty$*

Then for each $t \in \mathbb{D}$ the following limit exists μ-a.s. and is μ-a.s. independent of $\omega \in \Omega$:

$$\chi(t) = \lim \frac{1}{n} \; \log \; \left\| \mathbf{M}^{(n)}_\omega(t) \right\| . \tag{10.57}$$

The function $t \in \mathbb{D} \mapsto \chi(t) \in \mathbb{R}$ is real-analytic and harmonic.

We first use our theory for complex cone contractions to get some necessary uniform bounds. Using outer regularity we find (and fix throughout) $m \in X'$ of norm one and $K < +\infty$, such that

$$\|u\| \;\geq\; |\langle m, u\rangle| \;\geq\; \frac{1}{K}\|u\|, \quad \forall u \in \mathcal{C}. \tag{10.58}$$

For $M : \mathcal{C}^* \to \mathcal{C}^*$ a (linear) cone-contraction, we write

$$\pi_M(u) = \frac{Mu}{\langle m, Mu\rangle}, \quad u \in \mathcal{C}^* \tag{10.59}$$

for the natural projection of Mu onto the (bounded) subset : $\mathcal{C}_{m=1} \equiv \mathcal{C} \cap \{\langle m, u\rangle = 1\}$.

Lemma 10.3 *Given a sequence of matrices,* $(M_n)_{n\in\mathbb{N}} \subset \mathcal{M}$*, let* $M^{(n)} = M_1 \cdots M_n$ *and write* $\pi^{(n)} \equiv \pi_{M^{(n)}} = \pi_{M_1} \circ \cdots \circ \pi_{M_n}$*. Suppose that* $h \in \mathcal{C}(\rho)$ *and* $|\langle m, h\rangle| = 1$*. Then for* $\phi_1, \phi_2 \in \mathcal{C}^*$ *and* $\phi \in X$ *:*

$$\|\pi^{(n)}(\phi_1) - \pi^{(n)}(\phi_2)\| \leq K\,\eta^{n-1}\,\Delta. \tag{10.60}$$

$$\|\pi^{(n)}(h+\phi) - \pi^{(n)}(h)\| \leq K\,\eta^{n}\left(\frac{2}{\rho}\,\|\phi\| + o(\|\phi\|)\right), \;\; \textit{as} \;\; \phi \to 0. \tag{10.61}$$

Proof: Using Lemma 3.4 and then Lemma 2.4 we see that the left hand side in (10.60) is bounded by $K\,d_{\mathcal{C}}(M^{(n)}\phi_1, M^{(n)}\phi_2) \leq K\,\eta^{n-1}\,\Delta$. For the second inequality, equation (10.58) and our assumption on h show that $\|h\| \geq 1$ and then that $B(h,\rho) \subset \mathcal{C}$. The first part of Lemma 3.5 implies that $d_{\mathcal{C}}(h+\phi, h) \leq \frac{2}{\rho}\|\phi\| + o(\|\phi\|)$. We then use Lemmas 3.4 and 2.4 as before. □

Lemma 10.4 *For any* $M \in \mathcal{M}$ *and* $u \in \mathcal{C}[\rho]^*$ *we have*

$$\frac{1}{K}\left|\frac{\langle m, Mu\rangle}{\langle m, u\rangle}\right| \;\leq\; \|M\| \;\leq\; \frac{K^2}{\rho}\left|\frac{\langle m, Mu\rangle}{\langle m, u\rangle}\right| \tag{10.62}$$

Proof: Let $\phi \in X^*$. The assumption on u implies that $u + t\,\phi \in \mathcal{C}^*$ when $|t|\;\|\phi\| < \rho\|u\|$, i.e. for $|t| < r = \rho\|u\|/\|\phi\|$. Then also $Mu + tM\phi \in \mathcal{C}^*$ whenever $|t| < r$. By Lemma 3.5, $\|M\phi\| \leq \frac{K}{r}\|Mu\| = \|\phi\|\;\frac{K}{\rho}\,\frac{\|Mu\|}{\|u\|} \leq \|\phi\|\;\frac{K^2}{\rho}\;\left|\frac{\langle m, Mu\rangle}{\langle m, u\rangle}\right|$ where the last inequality is a consequence of the properties in (10.58) of m. We also have : $|\langle m, Mu\rangle| \;\leq\; \|M\|\;\|u\| \;\leq\; \|M\|\;|\langle m, u\rangle|\;K$. □

Lemma 10.5 *Let* $M \in \mathcal{M}$ *and* $h \in \mathcal{C}$ *with* $\langle m, h\rangle = 1$*. Suppose that* $U \in L(X)$ *and* $\phi \in X$ *verify* $\|U\|/\|M\| \leq \dfrac{\rho}{4K^3}$ *and* $\phi \leq \dfrac{\rho}{4K^2}$*. Then*

$$|\langle m, (M+U)(h+\phi)\rangle| \geq \|M\|\,\frac{\rho}{4K^2} \tag{10.63}$$

and

$$\|\pi_{M+U}(h+\phi)\| \leq \frac{16K^3}{\rho}. \tag{10.64}$$

Proof: We need to show that the denominator in (10.59) stays uniformly bounded away from zero : $|\langle m, (M+U)(h+\phi)\rangle| \geq |\langle m, Mh\rangle| - \|M\| \, \|\phi\| - \|U\|(\|h\| + \|\phi\|) \geq \|M\| \, \frac{\rho}{K^2}(1 - \frac{1}{4} - \frac{1}{4}(1+1)) \geq \|M\| \, \frac{\rho}{4K^2}$ (we have used : $\|h\| \leq K|\langle m, h\rangle| = K$ and $\|\phi\| \leq \rho/4K^2 \leq K$). Then :

$$\|\pi_{M+U}(h+\phi)\| \leq \frac{\|M\| \cdot 2 \cdot 2K}{\|M\| \, (\rho/4K^2)} = \frac{16K^3}{\rho}. \square$$

We define for $t \in \mathbb{D}$ the (measurable) map $\boldsymbol{\pi}_t : \mathcal{E}(\Omega, \mathcal{C}^*) \to \mathcal{E}(\Omega, \mathcal{C}_{m=1})$ through :

$$(\boldsymbol{\pi}_t(\mathbf{h}))_\omega = \pi_{M_\omega(t)}(\mathbf{h}_{\tau\omega}), \quad \omega \in \Omega, \quad \mathbf{h} \in \mathcal{E}(\Omega, \mathcal{C}^*). \tag{10.65}$$

Lemma 10.6 *For each $t \in \mathbb{D}$ the map $\boldsymbol{\pi}_t$ has a unique fixed point $\mathbf{h}^*(t)$ in $\mathcal{E}(\Omega, \mathcal{C}_{m=1}) \subset \mathcal{B}(\Omega, X)$.*

Proof : The subsets $\boldsymbol{\pi}_t^n(\mathcal{E}(\Omega, \mathcal{C}_{m=1}))$, $n \geq 1$ form a decreasing sequence in $\mathcal{E}(\Omega, \mathcal{C}_{m=1}) \subset \mathcal{B}(\Omega, X)$. By (10.60) the diameters verify : diam $\boldsymbol{\pi}_t^n(\mathcal{E}(\Omega, \mathcal{C}_{m=1})) \leq K\eta^{n-1}\Delta$. Pick $h^0 \in \mathcal{C}_{m=1}$ and define $(\mathbf{h}^0(t))_\omega = h^0$, $\omega \in \Omega$. The sequence, $\mathbf{h}^{n+1}(t) = \boldsymbol{\pi}_t(\mathbf{h}^n(t)) \in \mathcal{B}(\Omega, X)$, $n \geq 0$, is thus Cauchy so the map has a (clearly unique) fixed point $\mathbf{h}^*(t) = \boldsymbol{\pi}_t(\mathbf{h}^*(t)) \in \mathcal{E}(\Omega, \mathcal{C}_{m=1})$. $\square$

Recall that $\langle m, \mathbf{h}_\omega(t)\rangle = 1$ for all $\omega \in \Omega$. We define the map $\mathbf{p} : t \in \mathbb{D} \to \mathcal{E}(\Omega, \mathbb{C})$ by

$$\mathbf{p}_\omega(t) = \frac{\langle m, \mathbf{M}_\omega(t)\mathbf{h}_{\tau\omega}(t)\rangle}{\langle m, \mathbf{h}_\omega(t)\rangle} = \langle m, \mathbf{M}_\omega(t)\mathbf{h}_{\tau\omega}(t)\rangle, \quad \omega \in \Omega. \tag{10.66}$$

Lemma 10.7 *We have for every $t \in \mathbb{D}$: $\chi(t) = \int \log |\mathbf{p}_\omega(t)| \, d\mu(\omega)$.*

Proof : By Lemma 10.4, $|\mathbf{p}_\omega(t)|$ is equivalent to $\|\mathbf{M}_\omega(t)\|$ (within uniformly bounded constants). Assumption (2) in Theorem 10.2 implies that it is also equivalent to $\|\mathbf{M}_\omega(0)\|$ which, by Assumption (3) of that Theorem, is log integrable. It follows that $(\omega \in \Omega \mapsto \log |\mathbf{p}_\omega(t)|) \in L^1(\Omega, \mu)$. Our uniform bounds in Lemma 10.4 show that for every $\omega \in \Omega$:

$$\frac{1}{n}\log \left\|\mathbf{M}_\omega^{(n)}(t)\right\| = \frac{1}{n}\log \left|\frac{\langle m, \mathbf{M}_\omega^{(n)}(t)\mathbf{h}_{\tau^n\omega}(t)\rangle}{\langle m, \mathbf{h}_\omega(t)\rangle}\right| + O(\frac{1}{n}) = \frac{1}{n}\sum_{k=0}^{n-1}\log|\mathbf{p}_{\tau^k\omega}(t)| + O(\frac{1}{n}).$$

The latter sum is a 'Birkhoff'-average of the L^1-function $\log |\mathbf{p}(t)|$ so the almost sure convergence follows from Birkhoff's Ergodic Theorem. This a.s. limit is (a.s.) independent of $\omega \in \Omega$ since τ was supposed ergodic. $\square$

Lemma 10.8 *The map $t \in \mathbb{D} \mapsto \mathbf{h}^*(t) \in \mathcal{B}(\Omega, X)$ is analytic.*

Proof : Pick $t_0 \in \mathbb{D}$. It suffices to show that $\mathbf{h}^*(t)$ is analytic in a neighborhood of t_0. Using Assumption (2) of Theorem 10.2 we may find $\delta > 0$ so that for $|t - t_0| < \delta$:

$$\|\mathbf{M}_\omega(t) - \mathbf{M}_\omega(t_0)\| \leq \frac{\rho}{4K^3}\|\mathbf{M}_\omega(t_0)\|, \quad \omega \in \Omega. \tag{10.67}$$

By Lemma 10.5 the map,

$$(t, \mathbf{h}) \in B(t_0, \delta) \times B(\mathbf{h}^*(t_0), \frac{\rho}{4K^2}) \mapsto \boldsymbol{\pi}_t(\mathbf{h}) \in \mathcal{B}(\Omega, X) \tag{10.68}$$

is analytic (because it is fiber-wise analytic and uniformly bounded). We denote by $T_0 = D_{\mathbf{h}}\boldsymbol{\pi}_{t_0}(\mathbf{h}^*(t_0)) \in L(\mathcal{B}(\Omega, X))$ the derivative of $\boldsymbol{\pi}_{t_0}$ at the fixed point $\mathbf{h}^*(t_0)$. By linearization of the uniform bound in (10.61) we see that for each $n \geq 1$, $D_{\mathbf{h}}\boldsymbol{\pi}_{t_0}^{(n)}(\mathbf{h}^*(t_0)) = T_0^n \in L(\mathcal{B}(\Omega, X))$ verifies : $\|T_0^n\| \leq (2K/\rho)\eta^n$. It follows that $r_{\mathrm{sp}}(T_0) \leq \eta < 1$. The derivative, $\mathbf{1} - T_0 \in L(\mathcal{B}(\Omega, X))$ of $\mathbf{h} \mapsto \mathbf{h} - \boldsymbol{\pi}_{t_0}(\mathbf{h})$ at the fixed point $\mathbf{h}^*(t_0)$ is therefore invertible.

We may apply the implicit function theorem and conclude that there is $0 < \delta_1 < \delta$ and an analytic function

$$t \in B(t_0, \delta_1) \mapsto \mathbf{h}^*(t) \in B(\mathbf{h}^*(t_0), \frac{\rho}{4K^2}) \subset \mathcal{B}(\Omega, X) \tag{10.69}$$

for which $\mathbf{h}^*(t) - \boldsymbol{\pi}_t(\mathbf{h}^*(t)) = 0 \in \mathcal{B}(\Omega, X)$ for all $|t - t_0| < \delta_1$. $\square$

Proof (of Theorem 10.2): For fixed $\omega \in \Omega$ the map $t \in \mathbb{D} \mapsto \mathbf{p}_\omega(t) \in \mathbb{C}$ is holomorphic (being a continuous bilinear form composed with analytic functions). The difficulties here are that the images need not be uniformly bounded (with respect to ω) and that we want to define a complex logarithm in a consistent way. We proceed as follows : For $t \in B(t_0, \delta_1)$,

$$\begin{aligned} |\mathbf{p}_\omega(t) - \mathbf{p}_\omega(t_0)| &\leq \|\mathbf{M}_\omega(t) - \mathbf{M}_\omega(t_0)\| \cdot \|\mathbf{h}_{\tau\omega}(t)\| + \|\mathbf{M}_\omega(t_0)\| \cdot \|\mathbf{h}_{\tau\omega}(t) - \mathbf{h}_{\tau\omega}(t_0)\| \\ &\leq \left(\frac{\rho}{4K^3}K + \frac{\rho}{4K^2}\right) \|\mathbf{M}_\omega(t_0)\| \\ &= \frac{\rho}{2K^2}\|\mathbf{M}_\omega(t_0)\|. \end{aligned}$$

Lemma 10.4 shows that $|\mathbf{p}_\omega(t_0)| \geq \frac{\rho}{K^2} \|\mathbf{M}_\omega(t_0)\|$ so we conclude that

$$\left| \frac{\mathbf{p}_\omega(t)}{\mathbf{p}_\omega(t_0)} - 1 \right| \leq \frac{1}{2}, \quad ,\omega \in \Omega, |t - t_0| < \delta_1. \tag{10.70}$$

The difference,

$$\chi(t) - \chi(t_0) = \int \log \left| \frac{\mathbf{p}_\omega(t)}{\mathbf{p}_\omega(t_0)} \right| \, d\mu(\omega), \tag{10.71}$$

is thus the real part of the following holomorphic function (with the usual logarithm on $\mathbb{C} - \mathbb{R}_-$) :

$$H(t) = \int \log \left(\frac{\mathbf{p}_\omega(t)}{\mathbf{p}_\omega(t_0)} \right) d\mu(\omega), \quad |t - t_0| < \delta_1. \tag{10.72}$$

Therefore, $\chi(t)$ is harmonic, whence real-analytic. $\square$

Theorem 10.2 applies to certain classes of dominated complex cone-contractions as defined in section 6. We need to impose a further :

Assumption 10.9

1. *We assume that $\mathcal{C}_{\mathbb{R}} \subset X_{\mathbb{R}}$ is a regular cone in a real Banach space. [The real-subcones $\mathcal{C}_{\mathbb{R}}(\rho)$, $\rho > 0$ are then defined analogously to (10.56)].*
2. *Let $P \in L(X_{\mathbb{R}})$, $P : \mathcal{C}_{\mathbb{R}}^* \to \mathcal{C}_{\mathbb{R}}^*$. We assume that there is $\rho_0 > 0$ such that $P(\mathcal{C}_{\mathbb{R}}) \subset \mathcal{C}_{\mathbb{R}}(\rho_0)$. (And also that $\Delta_P = \mathrm{diam}_{\mathcal{C}_{\mathbb{R}}} P(\mathcal{C}_{\mathbb{R}}^*) < +\infty$).*

Given a complex operator $M \in L(X_{\mathbb{C}})$, which is dominated by P, we write $\alpha(M)$, $\beta(M)$ and $\gamma(M)$ for the optimal constants in Assumption 6.1. Now, let $0 < \kappa < 1$ and define the following subset of complex operators :

$$\mathcal{M}_\kappa = \left\{ M \in L(X_{\mathbb{C}}) \; : , 1 - \frac{\alpha(M)}{\beta(M)} < \kappa \; , \; 0 \leq \frac{\gamma(M)}{\alpha(M)} \cosh \frac{\Delta_P}{2} \; < \; \kappa \right\}. \tag{10.73}$$

Theorem 10.10 *Suppose that Assumption 10.9 holds. The class of operators $\mathcal{M}_\kappa \subset L(X_\mathbb{C})$ defined in (10.73) verifies the uniform bound in Definition 10.1 (for suitable values of Δ and ρ). So Theorem 10.2 applies when setting $\mathcal{M} = \mathcal{M}_\kappa$.*

Proof : The bound (6.50) in the proof of Theorem 6.3 shows that there is $\Delta = \Delta(\kappa, \Delta_P) < \infty$ (depending upon κ and Δ_P only) so that for any $M \in \mathcal{M}_\kappa$: $\mathrm{diam}_{\mathcal{C}_\mathbb{C}} M(\mathcal{C}^*_\mathbb{C}) \leq \Delta$. We still need to show that $M \in \mathcal{M}_\kappa$ maps $\mathcal{C}_\mathbb{C}$ uniformly into its interior : $\mathcal{C}_\mathbb{R}$ is assumed regular. Lemma 5.4 shows that $\mathcal{C}_\mathbb{C}$ is then also regular. We may assume that we have found $m \in \mathcal{C}'_\mathbb{R}$, extended to $m \in X'_\mathbb{C}$ which verifies (10.58). By the assumption on P we have for any $u \in \mathcal{C}_\mathbb{R}$: $B(Pu, \rho_0\|Pu\|) \subset \mathcal{C}_\mathbb{R}$. When $\ell \in \mathcal{C}_\mathbb{R}{}'$ is of norm one this implies : $\langle \ell, Pu\rangle \geq \rho_0 \|Pu\|$. Using the bound (6.48) we then get for $x \in \mathcal{C}_\mathbb{R}$:

$$\|Mx\| \leq K|\langle m, Mx\rangle| \leq K\sqrt{\beta+\gamma}\|Px\| \leq \frac{K\sqrt{\beta+\gamma}}{\rho_0}\langle \ell, Px\rangle.$$

Let $u \in \mathcal{C}^*_\mathbb{C}$ and (by Proposition 5.2) write $u = e^{i\theta}(x+iy)$ with $\theta \in \mathbb{R}$ and $x \pm y \in \mathcal{C}^*_\mathbb{R}$. Then

$$\|Mu\| \leq \|M(x+y)\| + \|M(x-y)\| \leq \frac{K\sqrt{\beta+\gamma}}{\rho_0}\langle \ell, P(2x)\rangle \leq \frac{2K\sqrt{\beta+\gamma}}{\rho_0}|\langle \ell, Pu\rangle|.$$

Denote $\sigma^2 = (\frac{\alpha}{\cosh\frac{\Delta_P}{2}} - \gamma)/(\beta+\gamma) \in \,]0,1[$ and let $\ell_1, \ell_2 \in \mathcal{C}'_\mathbb{R}$ be of norm 1. Given $\phi \in X_\mathbb{C}$ we use (6.46) to obtain :

$$\begin{aligned}\mathrm{Re}\langle \ell_1, Mu+\phi\rangle\langle \ell_2, \overline{Mu}+\overline{\phi}\rangle &\geq (\frac{\alpha}{\cosh\frac{\Delta_P}{2}} - \gamma)|\langle \ell_1, Pu\rangle||\langle \ell_2, Pu\rangle| - 2\|\phi\|\,|\langle \ell, Mu\rangle| - \|\phi\|^2 \\ &\geq \sigma^2\frac{\rho^2}{4K^2}\|Mu\|^2 - 2\|\phi\|\,\|Mu\| - \|\phi\|^2.\end{aligned}$$

This is non-negative when $\|\phi\|/\|Mu\| \leq \rho \equiv \sqrt{1+\sigma^2\frac{\rho_0^2}{4K^2}} - 1$. Thus, $B(Mu, \rho\|Mu\|) \in \mathcal{C}_\mathbb{C}$. □

Corollary 10.11 *In the case of finite dimensional matrices Assumption 10.9 is indeed verified. The class of matrices in (10.73) then reduces to (see Theorem 7.1) :*

$$\mathcal{M}_\kappa = \left\{ A \in M_n(\mathbb{C}) \ : \ 1 - \frac{\inf\ \mathrm{Re} A_{ij}\overline{A}_{kl}}{\sup\ \mathrm{Re} A_{ij}\overline{A}_{kl}} < \kappa \ , \ 0 \leq \frac{\sup\ |\mathrm{Im} A_{ij}\overline{A}_{kl}|}{\inf\ \mathrm{Re} A_{ij}\overline{A}_{kl}} < \kappa \right\}.$$

Theorem 10.2 thus applies when we set $\mathcal{M} = \mathcal{M}_\kappa$.

Example 10.12 *Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d random variables with values in $\mathbb{D}$. For $t \in \mathbb{D}$ define :*

$$M_n(t) = \begin{pmatrix} 8-t & 7+\xi_n \\ 7-i\xi_n & 6+it \end{pmatrix}$$

One checks that (see e.g. the calculation leading to (11.78) below) each $M_n(t) \in \mathcal{M}_{\kappa=0.75}$, $t \in \mathbb{D}$. By the previous Corollary, a.s. $\chi(t) = \lim \frac{1}{n} \log \|M_1(t)\cdots M_n(t)\|$ exists and defines a harmonic function of $t \in \mathbb{D}$.

For a closely related result we mention [Rue79] in which Ruelle was able to show that the characteristic exponent of a product of random real matrices (under additional assumptions) behaves real-analytically with respect to the matrices. Using Theorem 10.11 above, it is possible to recover the result of Ruelle together with an explicit estimate for the analytic extension.

It seems plausible that a result similar to Theorem 10.2 should hold for integral operators of the type given in our Theorem 7.2 (a complex generalization of Jentzsch's Theorem), but our proofs are insufficient as the cones needed in Theorem 10.2 are not regular.

11 Complex cone contraction versus perturbation theory

The applications in section 7 and 9 were based upon Theorem 6.3 on dominated complex cone contractions. In Theorem 6.3 it is conceivable to view the complex operator M as a 'perturbation' of a real operator P. It is natural to ask what are then the benefits from our above-mentioned applications relative to applying standard analytic perturbation theory when looking for a spectral gap for one fixed (real) operator. We address this question here.

We consider the case of an operator T contracts a regular real cone so that $T = \lambda P + R$ where P is a projection of rank 1, $\lambda > 0$ and the residual operator R commutes with P and verifies $\|\lambda^{-n} R^n\| \leq C\eta^{n-1}$, $n \geq 1$ for some $C < +\infty$ and $0 \leq \eta < 1$. Now, recall what can be obtained within the framework of perturbation theory, see e.g. Kato [Kato80, III.§6.4 and IV.§3.1] : If we perturb T then the spectral gap persists provided that the perturbation is 'small' compared to the resolvent on a suitable separating circle. To be more precise, for $z \notin \sigma_{sp}(T)$ we have :

$$R(z,T) = (z-T)^{-1} = (z-\lambda)^{-1}P + (z-R)^{-1}(1-P). \tag{11.74}$$

When $\lambda > |z| > \eta\,\lambda$, one has :

$$R(z,T) = (z-\lambda)^{-1}P + z^{-1}(1-P) + \sum_{n\geq 1} z^{-n-1}R^n. \tag{11.75}$$

Using the estimate for the residual operator and re-summing we get :

$$\|R(z,T)\| \leq \frac{\|P\|}{|\lambda - z|} + \frac{\|1-P\|}{|z|} + \frac{C}{|z|\,(|z| - \lambda\,\eta)}. \tag{11.76}$$

Consider now the closed curve (our not necessarily optimal choice for a separating circle), $\Gamma = \{z \in C : |z| = \lambda\frac{1+\eta}{2}\}$. For $z \in \Gamma$:

$$\|R(z,T)\| \leq \frac{2\|P\|}{\lambda(1-\eta)} + \frac{2\|1-P\|}{\lambda(1+\eta)} + \frac{4C}{\lambda\,(1-\eta^2)} \equiv \frac{1}{\rho^*}. \tag{11.77}$$

When $\|S\| < \rho^*$, the von Neumann series $(z-T-S)^{-1} = R(z,T) + R(z,T)SR(z,T) + \cdots$ converges normally on Γ. It follows that [Kato80, II,§1.3 and IV.§3.1] the spectral projections on the two components of $C - \Gamma$ depend analytically on S. In particular, that the algebraic dimension of the spectral projection on the unbounded component stays constant, i.e. equals one. In other words the spectral gap persists.

To be more concrete, consider then the case of a real matrix $T \in M_d(\mathbb{R})$ and constants $0 < m \leq M < +\infty$ such that $m \leq T_{ij} \leq M$ for all indices. Let us perturb by a purely imaginary matrix iS with $S \in M_d(\mathbb{R})$ and such that $|S_{ij}| < r$ for all indices. The matrix $A = T + iS$ then verifies $\mathrm{Re}\, A_{ij}\,\overline{A_{kl}} \geq m^2 - r^2$ and $\mathrm{Im}\, A_{ij}\,\overline{A_{kl}} \leq 2Mr$. Thus, the condition in Theorem 7.1 simply reads: $2rM \leq m^2 - r^2$. Consequently if

$$r \leq \frac{m^2}{M + \sqrt{M^2 + m^2}}, \tag{11.78}$$

then A has a spectral gap. For example if $m = 3 \leq T_{ij} \leq M = 4$ then $|S_{ij}| < r = \frac{9}{\sqrt{25}+4} = 1$ suffices to get a spectral gap. Also, one cannot take r to be bigger than 3 or else $\begin{pmatrix} 3+3i & 3-3i \\ 3-3i & 3+3i \end{pmatrix}$ provides a counter-example of a matrix without a spectral gap.

Perturbation theory works well in a special case, namely when T is itself of rank one. For example, suppose that $T_{ij} = 1$, so that (using Euclidean norm) $m = M = 1$, $\|P\| = \|1-P\| = 1$ and $\lambda = d$. In this case $C = \eta = 0$ so setting $|z| = d/2$ we obtain from (11.76) :

$$\|R(z,T)\| \leq \frac{\|P\|}{|\lambda - z|} + \frac{\|1-P\|}{|z|} \leq \frac{4}{d} \equiv \frac{1}{\rho^*}. \tag{11.79}$$

Thus, if we add $S \in M_d(\mathbb{R})$ with $\|S\| < \rho^* = d/4$ then $T + iS$ has a spectral gap. In particular, when $|S_{ij}| < r$ then $\|S\| \leq rd$ so one needs $r < 1/4$ in order to apply this result. By comparison, the bound obtained from the complex cone contraction (11.78) is $r < \frac{1}{1+\sqrt{2}}$. On the other hand, and in favor of the perturbation result, note that it applies to some perturbations which are not immediately seen by the complex cone contraction, e.g. when only one element of S_{ij} is non-zero, and this element is strictly smaller than $d/4$.

Consider now the case when the original matrix T is not of rank one. In order to make a computation within perturbation theory note that the matrix contracts the real standard cone $\mathbb{R}^n_+$ and given the constants $0 < m \leq M < +\infty$ from above one has (see Example 4.9) : $\mathrm{diam}_{\mathbb{R}^n_+}(T\mathbb{R}^n_+) \leq \Delta = 2\log\frac{M}{n}$. From this, $\eta = \tanh\dfrac{\Delta}{4} = \dfrac{M-m}{M+m}$ and $1 - \eta = \dfrac{2m}{M+m}$. Using $\|T\| \leq Md$ and $\lambda \geq md$ we also see that $\|\lambda^{-1}T\| \leq M/m$. One also has $\|P\| \leq \frac{M}{m}$. In order to get a bound on C we may use the constants in Remark 3.8 equation (3.19), which were obtained for the complex cone but apply equally well to the real case. With the Euclidean norm on $\mathbb{R}^n$ one has $K = g = \sqrt{2}$ and we obtain

$$A = \left(\sqrt{2}\right)^2 \frac{M}{m} \exp\left((1 + \frac{M}{m}) \frac{(\sqrt{2})\ 2\ \log\frac{M}{m}}{\frac{2m}{M+m}} \right), \quad C = \left(3 + \frac{M}{m}\right) \frac{2\log\frac{M}{m}}{\frac{2m}{M+m}} A. \tag{11.80}$$

When $|z| = \lambda\frac{1+\eta}{2}$ we have the bound:

$$\|R(z,T)\| \leq \frac{M(M+m)}{2dm^3} + \frac{(M+m)^2}{2dMm^2} + \frac{C(M+m)^2}{4dMm^2} \equiv \frac{1}{\rho^*}. \tag{11.81}$$

When $m = 3$ and $M = 4$ we obtain from these estimates that $r = \rho^*/d = 0.02825...$ ensures a spectral gap. This is within two orders of magnitude to $r = 1$ which we obtained above from equation (11.78).

Increasing, however, the ratio of M to m substantially deteriorates the perturbative bounds. When e.g. $m = 10$ and $M = 100$ we obtain $r \approx 2 \cdot 10^{-175}$ (!!) from (11.80) and (11.81). This should be compared to the bound $r = 0.4987...$ obtained from equation (11.78) when using the complex cone-contraction.

A Projective space

Let X be a complex Banach space. Given non-zero elements $x, y \in X^* \equiv X - \{0\}$ we write $x \sim y$ iff $\mathbb{C}x = \mathbb{C}y$. Let $\pi : X^* \to X^*/\sim$ denote the quotient map and write $[x] = \mathbb{C}^* x$ for the equivalence class of $x \in X^*$. We equip the quotient space $\pi(X^*)$ with the following metric

$$d_{\pi(X^*)}([x],[y]) = \mathrm{dist}_H(\mathbb{C}x \cap S, \mathbb{C}y \cap S) = \inf\left\{ \left\| \frac{\mu x}{\|\mu x\|} - \frac{\nu y}{\|\nu y\|} \right\| : \mu, \nu \in \mathbb{C}^* \right\}, \quad x, y \in X^* \tag{A.82}$$

in which dist_H is the Hausdorff distance between non-empty sets and $S = S(X)$ is the unit-sphere.

Lemma A.1

1. Let $\mathcal{C} \subset X$ be a $\mathbb{C}$-cone of K-bounded sectional aperture. Then for all $x, y \in \mathcal{C}^*$:

$$d_{\pi(X^*)}([x],[y]) \leq 2K d_{\mathcal{C}}(x,y).$$

2. Let $x \in X^*$, $r > 0$ and set $V = \pi^{-1} B_{\pi(X^*)}([x], r)$. Then for all $y \in V^*$

$$d_V(x,y) \leq \log \frac{r + d_{\pi(X^*)}([x],[y])}{r - d_{\pi(X^*)}([x],[y])}.$$

Proof: Using the inequality

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq 2\|x-y\| \min\left\{ \frac{1}{\|x\|}, \frac{1}{\|y\|} \right\}, \quad x, y \in X^* \tag{A.83}$$

we obtain from Lemma 3.4 :

$$\left\| \frac{x/\langle m,x\rangle}{\|x/\langle m,x\rangle\|} - \frac{y/\langle m,y\rangle}{\|y/\langle m,y\rangle\|} \right\| \leq 2\|m\| \left\| \frac{x}{\langle m,x\rangle} - \frac{y}{\langle m,y\rangle} \right\| \leq 2K d_{\mathcal{C}}(x,y)$$

and the first conclusion follows. For the second claim, normalize so that $d_{\pi(X^*)}([x],[y]) = \|x-y\| < r$ and $\|x\| = \|y\| = 1$. Let $u_\lambda = \frac{1+\lambda}{2} x + \frac{1-\lambda}{2} y = x + \frac{1-\lambda}{2}(y-x)$. By (A.83), $\|\frac{u_\lambda}{\|u_\lambda\|} - x\| \leq \frac{|1-\lambda|}{2}\|y-x\|$ which remains smaller than r when $|1-\lambda| < \frac{2r}{\|x-y\|} \equiv 2R \in (2, +\infty]$. Then $d_V(x,y) \leq d_{B_{\mathbb{C}}(1,2R)}(-1,1) = d_{\mathbb{D}}(0, \frac{1}{R}) = \log \frac{R+1}{R-1}$ □

Given any two points $x, y \in \mathcal{C}^*$ we may follow Kobayashi [Kob67, Kob70] and define a projective pseudo-distance between x and y through :

$$\tilde{d}_{\mathcal{C}}(x,y) = \inf\{\sum d_{\mathcal{C}}(x_i, x_{i+1}) : x_0 = x, x_1, \ldots, x_n = y \in \mathcal{C}^*\}.$$

Since $d_{\pi(X^*)}$ is a (projective) metric, the previous Lemma implies that

Theorem A.2 *Suppose that $\mathcal{C}$ is of K-bounded sectional aperture in X. Then the inclusion map, $(\mathcal{C}^*, \tilde{d}_{\mathcal{C}}) \to (\mathcal{C}^*, d_{\pi(X^*)})$ is $2K$-Lipschitz.*

In other words, this new distance does not degenerate when taking the inf over finite chains, so distinct complex lines in $\mathcal{C}$ have a non-zero $\tilde{d}_{\mathcal{C}}$-distance. This is conceptually very nice, but, in our context, not particularly useful. The reason is that even if $T \in L(X)$ maps $\mathcal{C}^*$ into a subset of finite diameter in $\mathcal{C}^*$ for the metric $\tilde{d}$ this does not seem to imply a uniform contraction of T, i.e. no spectral gap. We leave a further study of this metric to the interested reader.

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