

Eigenfunctions for smooth expanding circle maps

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December 3, 2005

Abstract

We construct a real-analytic circle map for which the corresponding Perron-Frobenius operator has a real-analytic eigenfunction with an eigenvalue outside the essential spectral radius when acting upon C^1 -functions.

1 Introduction

Let T be a uniformly expanding C^2 map of the circle $S^1 = \mathbb{R}/\mathbb{Z}$. Consider an associated Perron Frobenius operator acting upon C^1 functions,

$$Pf(x) = \sum_{y:Ty=x} \frac{1}{|T'(y)|} f(y), \quad x \in S^1. \quad (1)$$

By the Ruelle-Perron-Frobenius Theorem, one is a simple eigenvalue and the operator has a ‘spectral gap’. It is also known (see e.g. [Ba]) that P has an essential spectrum which is contained in a disk centered at zero and of radius $r_{\text{ess}} = \vartheta$ with $\vartheta^{-1} = \liminf_k \inf_x \sqrt[k]{|(T^k)'(x)|}$. Outside this essential spectrum there can only be isolated eigenvalues of finite multiplicity. A natural question is then if there *really are* any eigenvalues outside the radius r_{ess} ?

This ‘non-essential’ spectrum is void in typical standard examples. For example, $Tx = 2x \pmod{1}$, with P acting upon real-analytic functions has 1 as a simple eigenvalue and the rest of the spectrum concentrated at zero. On C^1 functions the essential spectral radius is one-half and a small perturbation can thus not give rise to an example with a non-trivial non-essential spectrum. Even if we consider $Tx = 2x \pmod{1}$ with P acting on C^1 function on the interval $[0, 1]$ the largest eigenvalue below 1 is one-half, precisely on the boundary of the essential spectrum. More generally, if T is a smooth expanding circle map fixing say zero we may for $k \geq 1$ consider the Perron Frobenius operator as an operator P_{S^1} acting upon $C^k(S^1)$, or as $P_{[0,1]}$ acting upon $C^k([0, 1])$. The spectrum of $P_{[0,1]}$ is the same as the spectrum of P_{S^1} union a point

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[‡]G.K. acknowledges the support and hospitality of the Mathematics Department of the Université de Paris-Sud, France, where part of this work was done.

spectrum $\{|T'(0)|^{-m}, m \geq 1\}$. In particular, in the case $k = 1$ the spectrum outside the radius ϑ doesn't change. One may show this by adapting the proof of Corollary 2.11 in [Rue].

Recently, Liverani [Liv] suggested to us that when acting upon C^1 functions there may not be any non-trivial 'non-essential' spectrum. The present paper was conceived in an attempt to resolve this question and we have arrived at the following

Theorem 1 *We may construct a real-analytic circle map $T : S^1 \rightarrow S^1$ of derivative greater than $3/2$ for which P has a real-analytic eigenvector with eigenvalue greater than 0.75 in absolute value. In particular, this eigenvalue is greater than the essential spectral radius for $C^1(S^1)$ functions which does not exceed $2/3$.*

Our proof consists of a fairly explicit example homotopic to the $2x \bmod 1$ map. We will construct it in two steps, (1) we will find a piecewise linear map acting upon functions of bounded variation (BV) and (2) we apply a real-analytic smoothening of this map. The advantage when acting upon BV is that this space allows for discontinuities in the derivatives of the dynamical system and in the eigenfunctions of the associated operator. This makes it very easy to construct examples with very explicit spectral properties. The hard part is to recover a corresponding result in the smooth category. We do this through what may be called 'quasi-compact' perturbation theory for BV operators.

Acknowledgment: We thank Viviane Baladi and Carlangelo Liverani for bringing this problem to our attention and for stimulating discussions about it.

2 Rychlik's result

Although there are numerous treatments of piecewise expanding interval maps in various contexts, Rychlik's early paper [Ry] is still among the most elegant and general ones and seems most suitable as a base for our arguments. We recall his setting. Let X be the closed unit interval, m Lebesgue measure on X , $U \subseteq X$ open and dense with $m(U) = 1$ and denote $S = X \setminus U$. We assume that S contains the endpoints of X so that U is an at most countable union of open intervals. The closures of these intervals are then a countable family, call it β , of closed intervals with disjoint interiors such that $\bigcup_{B \in \beta} B \supset U$ and such that for any $B \in \beta$ the set $B \cap S$ contains exactly the endpoints of B .

The space of functions of bounded variation $f : X \rightarrow \mathbb{R}$ is denoted by BV , the variation of f over all of X by $\text{var}(f)$ and the variation over a subinterval $I \subseteq X$ by $\text{var}_I(f)$.

Let $T : U \rightarrow X$ be a continuous map such that for any $B \in \beta$ the restriction $T|_{B \cap U}$ admits an extension to a homeomorphism of B with some interval in X .

Suppose that $T|_U$ is differentiable in the following sense: There is a function $g : X \rightarrow \mathbb{R}_+$ with $\|g\|_\infty < \infty$, $\text{var}(g) < \infty$ and $g|_S = 0$ such that the transfer operator P defined by

$$Pf(x) = \sum_{y \in T^{-1}x} g(y)f(y)$$

preserves m which means $m(Pf) = m(f)$ for all bounded measurable $f : X \rightarrow \mathbb{R}$.

For iterates T^N of T we adopt the following notation: $S_N = \bigcup_{k=0}^{N-1} T^{-k}S$, $U_N = X \setminus S_N$ and $\beta_N = \bigcap_{k=0}^{N-1} T^{-k}\beta$. Next define $g_N : X \rightarrow \mathbb{R}_+$ by $g_N|_{S_N} = 0$ and $g_N|_{U_N} = g \circ T^{N-1} \dots \circ g \circ T \cdot g$. In order to see that $T^N, U_N, S_N, g_N, \beta_N$ satisfy our conditions for $N \geq 1$ the only nontrivial thing to verify is $\text{var}(g_N) < \infty$:

Lemma 1 [Ry, Lemma 2]

$$\text{var}(g_N) \leq 2^{N-1}(\text{var } g)^N.$$

(Indeed, since we assumed that the endpoints of X belong to S , inspection of the proof reveals that $\text{var}(g_N) \leq (\text{var } g)^N$.)

Lemma 2 [Ry, Lemma 6] For every $\epsilon > 0$ and every $N \geq 1$ there exists a finite partition α_N of X into intervals such that

$$\max_{A \in \alpha_N} \text{var}_A(g_N) < \|g_N\|_\infty + \epsilon.$$

Corollary 1 In the situation of the previous lemma there exists $\rho = \rho(\alpha_N) > 0$ such that actually $\text{var}_{U_\rho(A)}(g_N) < \|g_N\|_\infty + \epsilon$ where $U_\rho(A)$ denotes the ρ -neighbourhood of A .

Lemma 3 [Ry, Corollary 3 and its proof] For every $N \geq 1$ and $\lambda_N > 2\|g_N\|_\infty$ there exists $D_N \geq 0$ such that for every $f : X \rightarrow \mathbb{R}$

$$\text{var}(P^N f) \leq \lambda_N \text{var}(f) + D_N \|f\|_1$$

where D_N is determined as follows: By Lemma 2 there is a finite partition α_N such that $\max_{A \in \alpha_N} \text{var}_A(g_N) \leq \lambda_N - \|g_N\|_\infty$. Then $D_N = \max_{A \in \alpha_N} \text{var}_A(g_N)/m(A)$.

From now on we assume that $\|g_N\|_\infty < 1$ for some $N \geq 1$. Then

$$\vartheta := \lim_{n \rightarrow \infty} \|g_N\|_\infty^{1/n} < 1.$$

Lemma 4 [Ry, Proposition 1 and its proof] Given $\kappa \in (\vartheta, 1)$ we can find $F \geq 0$ such that for every $f : X \rightarrow \mathbb{R}$ and every $n \in \mathbb{N}$

$$\sum_{B \in \beta^n} \text{var } P^n(f \cdot \chi_B) \leq F \cdot (\kappa^n \text{var}(f) + \|f\|_1).$$

F is determined as follows: Fix M such that $\vartheta < \|2g_M\|_\infty^{1/M} < \kappa < 1$. Then $F := \max\{D/(1 - \kappa^M), \lambda/\kappa^{M-1}\}$ where $\lambda := \max\{\lambda_1, \dots, \lambda_M\}$ and $D := \max\{D_1, \dots, D_M\}$ with λ_i, D_i from Lemma 3.

Corollary 2 With F and κ as before we have for every $f \in BV$

$$\text{var}(P^n f) \leq F \cdot (\kappa^n \text{var}(f) + \|f\|_1).$$

It is this explicit knowledge of the dependence of F and κ on T and g which will allow us later to apply spectral perturbation theory to smooth approximations of piecewise linear maps.

3 Smoothing

We want to apply the Lasota-Yorke type estimate of Corollary 2 to piecewise linear circle maps and their smooth approximations. In terms of interval maps this will be full branched maps. To this end fix $p \in \mathbb{Z}$, $p \geq 2$, and let $\psi : \mathbb{R}/p\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$ be an increasing homeomorphism. It lifts to a continuous strictly increasing map $\tau : \mathbb{R} \rightarrow \mathbb{R}$ for which $\tau(x + p) = \tau(x) + 1$ for all x . Let $S = \tau(\mathbb{Z} \cap [0, p])$, $U = X \setminus S$, and denote by $T : U \rightarrow X$ the p -branched map defined by

$$T(x) = \tau^{-1}(x) \pmod{1}$$

(Of course T can be interpreted as a continuous p -fold covering map of \mathbb{T}^1 .)

In what follows we assume that τ is twice differentiable in the following sense: There is a function $\dot{\tau} : \mathbb{R} \rightarrow \mathbb{R}$ of locally bounded variation and with period p such that $\tau(x) = \int_0^x \dot{\tau}(u) du$ for all x . We assume that $0 < \inf \dot{\tau} \leq \sup \dot{\tau} \leq 1$, and we set $g|_U := \dot{\tau} \circ \tau^{-1}|_U$ and $g|_S = 0$. Then T , g and $\beta := \{[\tau(k-1), \tau(k)] : k = 1, \dots, p\}$ fit the setting of section 2.

Let $\varphi : \mathbb{R} \rightarrow [0, \infty)$ be a smooth convolution kernel, i.e. $\lim_{|x| \rightarrow \infty} \varphi(x) = 0$, $\int_{-\infty}^{\infty} \varphi(x) dx = 1$. To be definit we choose ϕ to be the density of the standard normal distribution, in which case ϕ is indeed real-analytic. For $\delta > 0$ define the rescaled kernels φ_δ by $\varphi_\delta(x) = \delta^{-1} \varphi(\delta^{-1}x)$. We use these kernels to define smooth approximations $\tau_\delta := \tau * \varphi_\delta - \tau * \varphi_\delta(0)$ to τ and $\dot{\tau}_\delta := \dot{\tau} * \varphi_\delta$ to $\dot{\tau}$. Obviously $\tau_\delta(0) = 0$, $\tau_\delta(x + p) = \tau_\delta(x) + 1$, and $\dot{\tau}_\delta(x + 1) = \dot{\tau}_\delta$. A simple calculation yields ¹

$$\tau_\delta(x) = \int_0^x \dot{\tau}_\delta(u) du \tag{2}$$

Denote by T_δ the p -branched transformation of X determined by τ_δ and define g_δ in terms of τ_δ as g was defined in terms of τ . Denote by P_δ the transfer operator for T_δ and g_δ .

Observe that

$$0 < \text{ess inf } \dot{\tau} \leq \tau'_\delta \leq \text{ess sup } \dot{\tau} \leq 1$$

so that also

$$0 < \text{ess inf } g \leq g_\delta \leq \text{ess sup } g \leq 1 .$$

In particular

$$\|2g_{\delta, M}\|_\infty^{1/M} \leq 2^{1/M} \cdot \|g_M\|_\infty^{1/M} \tag{3}$$

so that the integer M in Lemma 4 can be chosen uniformly for $\delta \in (0, 1)$ provided $\kappa \in (\vartheta, 1)$. Then all λ_N in Lemma 3 can be chosen to be equal to 3 so that also $\lambda = 3$ in Lemma 4. In view of Corollary 1 the same partitions α_N can be used in Lemma 3 for all T_δ , g_δ provided $\delta > 0$ is sufficiently small ($N = 1, \dots, M$). Hence also the numbers D_N can be chosen uniformly for sufficiently small δ ($N = 1, \dots, M$). This proves:

¹ $\int_0^x \dot{\tau}_\delta(u) du = \int_0^x \int_{-\infty}^{\infty} \dot{\tau}(y) \varphi_\delta(u - y) dy du = \int_0^x \int_{-\infty}^{\infty} \tau(y) \varphi'_\delta(u - y) dy du = \int_{-\infty}^{\infty} \int_0^x \tau(y) \varphi'_\delta(u - y) du dy = \int_{-\infty}^{\infty} \tau(y) (\varphi_\delta(x - y) - \varphi_\delta(-y)) dy = \tau * \varphi_\delta(x) - \tau * \varphi_\delta(0) = \tau_\delta(x)$.

Proposition 1 *The constants F and κ in the Lasota-Yorke type inequality*

$$\text{var}(P_\delta^n f) \leq F \cdot (\kappa^n \text{var}(f) + \|f\|_1) .$$

for P_δ (compare to Corollary 2) can be chosen uniformly in δ provided $\delta > 0$ is small enough.

This Proposition opens the way to apply the spectral perturbation theorem of [KL] to P and P_δ . We just need to show that, for sufficiently small $\delta > 0$, the operator P_δ is close to P in a suitable sense. The following proposition provides the relevant estimate.

Proposition 2 *For all $\delta \in (0, 1)$ and all $f \in BV$:*

$$\int |P_\delta f(x) - Pf(x)| dx \leq 2C_0 \cdot \delta \cdot \|f\|_{BV}$$

where $\|f\|_{BV} := \text{var}(f) + \|f\|_1$.

For its proof we need some simple estimates.

Lemma 5 *There is a constant $C_0 > 0$ (depending on τ and on the kernel φ) such that for all $\delta \in (0, 1)$:*

$$a) \sup_{x \in X} |\tau_\delta^{-1}(x) - \tau^{-1}(x)| \leq C_0 \cdot \delta$$

$$b) \int_0^p |\tau'_\delta(y) - \dot{\tau}(y)| dy \leq C_0 \cdot \delta$$

Proof:

a) As $(\text{ess inf } \dot{\tau})^{-1}$ and $\text{ess sup } \dot{\tau}$ are Lipschitz constants for τ^{-1} and τ respectively, we have

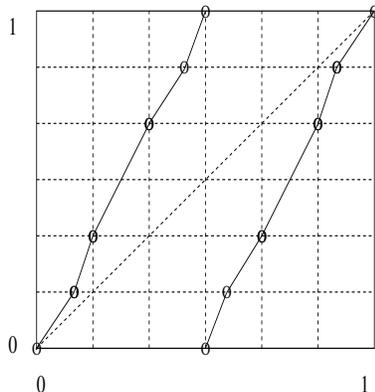
$$\begin{aligned} \sup_{x \in X} |\tau_\delta^{-1}(x) - \tau^{-1}(x)| &= \sup_{y \in [0, p]} |y - \tau^{-1}(\tau_\delta y)| \leq \frac{1}{\text{ess inf } \dot{\tau}} \cdot \sup_{y \in [0, p]} |\tau(y) - \tau_\delta(y)| \\ &\leq \text{const} \cdot \delta \cdot \frac{\text{ess sup } \dot{\tau}}{\text{ess inf } \dot{\tau}} =: C_0 \cdot \delta \end{aligned}$$

b) This follows from the assumption that $\dot{\tau}$ is of bounded variation.

q.e.d.

Proof of Proposition 2: The proof is similar to the one of Lemma 13 in [Ke]. Let $\psi := \text{sign}(P_\delta f - Pf)$. Then

$$\begin{aligned} \int_X |P_\delta f(x) - Pf(x)| dx &= \int_X (P_\delta f(x) - Pf(x)) \psi(x) dx \\ &= \int_X f \cdot (\tilde{\psi}(\tau_\delta^{-1}x) - \tilde{\psi}(\tau^{-1}x)) dx \end{aligned} \tag{4}$$

Figure 1: The piecewise linear map T

where $\tilde{\psi}$ is the periodic extension of ψ from X to all of \mathbb{R} .

Observe now that for each $\varphi : X \rightarrow \mathbb{R}$ which is differentiable in the sense that $\varphi(x) = \int_0^x \dot{\varphi}(u) du$ for some bounded measurable $\dot{\varphi} : X \rightarrow \mathbb{R}$ we have $\int_X f(x)\dot{\varphi}(x) dx \leq \text{var}(f) \sup_X |\varphi|$. Therefore we can continue (4) by

$$\begin{aligned} \int_X |P_\delta f(x) - Pf(x)| dx &\leq \|f\|_{BV} \cdot \sup_{x \in X} \left| \int_0^x \tilde{\psi}(\tau_\delta^{-1}u) - \tilde{\psi}(\tau^{-1}u) du \right| \\ &= \|f\|_{BV} \cdot \sup_{x \in X} \left| \int_0^{\tau_\delta^{-1}x} \tilde{\psi}(y)\tau'_\delta(y) dy - \int_0^{\tau^{-1}x} \tilde{\psi}(y)\dot{\tau}(y) dy \right| \\ &\leq \|f\|_{BV} \cdot \left(\sup_{x \in X} |\tau_\delta^{-1}x - \tau^{-1}x| + \int_0^P |\tau'_\delta(y) - \dot{\tau}(y)| dy \right) \\ &\leq 2C_0 \cdot \delta \cdot \|f\|_{BV} \end{aligned}$$

q.e.d.

4 A piecewise linear example

Let now $p = 2$ and denote by $\psi : \mathbb{R}/2\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$ the homeomorphism fixing zero and having successive slopes $\frac{2}{3}, \frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}$ on the intervals $(\frac{k-1}{6}, \frac{k}{6}) \pmod{2}$, $k = 1, \dots, 12$. The lift $\tau : \mathbb{R} \rightarrow \mathbb{R}$ of ψ satisfies $\tau(x+2) = \tau(x) + 1$. Also let $\pi : \mathbb{R}/2\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$ be the canonical projection. The map $T = \pi \circ \psi^{-1} : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$ is a piecewise linear uniformly expanding map of the circle (cf. Figure 1).

Denote $I_k = (\frac{k-1}{6}, \frac{k}{6}) \pmod{\mathbb{Z}}$, $k = 1, \dots, 6$ and $C = \frac{1}{6}\mathbb{Z}/\mathbb{Z}$. We write $S = \psi(\frac{1}{6}\mathbb{Z}/2\mathbb{Z})$ for the 12 points on the circle where $(T'(x))^{-1} = \psi' \circ \psi^{-1}(x)$ is not defined. We set

$$g(x) = \begin{cases} \psi' \circ \psi^{-1}(x) & x \in \mathbb{R} \setminus S, \\ 0 & x \in S. \end{cases} \quad (5)$$

As before we have a corresponding Perron Frobenius operator

$$Pf(x) = \sum_{y:Ty=x} g(y)f(y). \quad (6)$$

P maps BV into the subspace

$$E_C = \{\phi \in BV : \phi|_C \equiv 0\}. \quad (7)$$

P also preserves the subspace of step functions

$$L_C = \left\{ \phi = \sum_{i=1}^6 c_i \mathbf{1}_{I_i} : c = (c_i)_{i=1..6} \in \mathbb{R}^6 \right\} : \quad (8)$$

acting with the operator P upon L_C we obtain $P \sum_i c_i \mathbf{1}_{I_i} = \sum_{ij} c_i M_{ij} \mathbf{1}_j$ where M is the doubly stochastic matrix

$$M = \begin{pmatrix} 2/3 & 1/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2/3 & 1/3 \\ 1/3 & 2/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 2/3 \end{pmatrix}$$

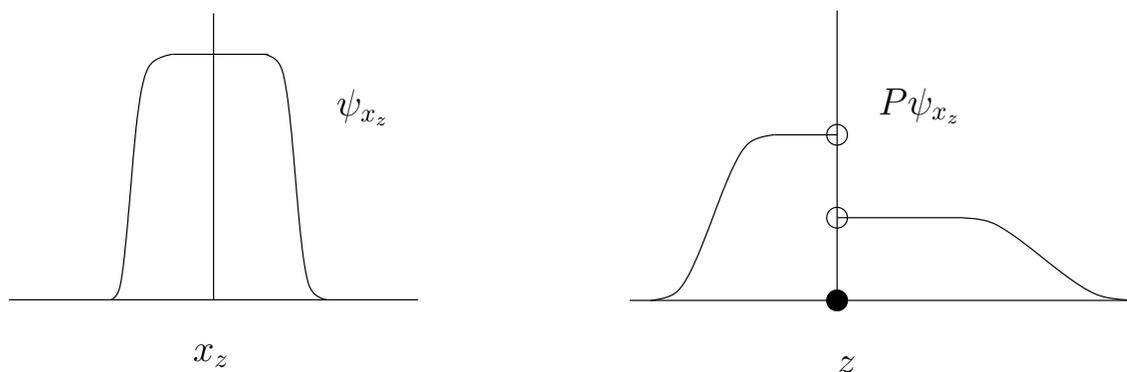
Therefore, if v is a left eigenvector of this matrix, $\lambda v = vM$, then it induces an eigenfunction ϕ_v of $P : BV \rightarrow BV$ with the same eigenvalue λ . More precisely, $\phi_v = \sum_i \mu_i \mathbf{1}_{I_i}$. In particular, $P1 = 1$.

The matrix M has simple eigenvalues $\lambda_1 = 1$, $\lambda_2 = -\frac{1}{6} - \frac{\sqrt{13}}{6} \approx -0.7676$, $\lambda_3 = \frac{2}{3}$, $\lambda_4 = -\frac{1}{6} + \frac{\sqrt{13}}{6} \approx 0.4343$ and a double eigenvalue at 0. As 0 is a fixed point with slope $\frac{3}{2}$, the essential spectral radius of P is $\vartheta = \frac{2}{3}$. It follows that the spectrum of $P : BV \rightarrow BV$ is contained in the set $\{1, \lambda_2\} \cup \{|z| \leq \vartheta\}$, see e.g. [BK, Lemma 3.1]. Indeed, a straightforward generalization of the the proof of that lemma shows that λ_2 is a simple eigenvalue of P because it is a simple eigenvalue of M of modulus larger than ϑ . Let $\Phi_2 : BV \rightarrow BV$ be the corresponding spectral projector.

The unique (up to normalization) left eigenvector of M with eigenvalue λ_2 is $v_2 = (1, \frac{3+\sqrt{13}}{2}, -\frac{5+\sqrt{13}}{2}, -\frac{5+\sqrt{13}}{2}, \frac{3+\sqrt{13}}{2}, 1)$. The associated eigenfunction ϕ_{v_2} is piecewise constant and its set of essential discontinuities is $D = \{\frac{1}{6}, \frac{2}{6}, \frac{4}{6}, \frac{5}{6}\}$. Observe that these are all essential discontinuities also if ϕ_{v_2} is considered as a function from S^1 to \mathbb{R} . (The discontinuity at $\frac{1}{2}$, where $\phi_{v_2}(\frac{1}{2}) = 0$ by definition, is inessential, because the left and right limits of ϕ_{v_2} at this point are both $-\frac{5+\sqrt{13}}{2}$. Similarly the discontinuity at $0 \in S^1$ is inessential.)

Lemma 6 *There is a trigonometric polynomial $f : X \rightarrow \mathbb{R}$ such that $\Phi_2(f) \neq 0$.*

Proof: The difficulty arises from the fact that smooth functions are not dense in BV . Thus a non-zero linear functional on BV may vanish when acting upon any smooth function. We need to show that this is not the case for the eigenprojection Φ_2 .

Figure 2: A test-function and its image under P

Consider the space $E_C^\infty = \{\phi \cdot \mathbf{1}_{S^1 \setminus C} : \phi \in C^\infty(S^1)\}$. We claim that $\phi_{v_2} \in E_C^\infty + PE_C^\infty$. To see this let $\psi \in C^\infty(S^1)$ be a test function with support in a ‘sufficiently small’ neighborhood of 0. For example, we may suppose that $\psi(x) \equiv 1$ when $d(x, 0) \leq 1/100$ and that $\psi(x) = 0$ for $d(x, 0) \geq 1/40$. For $x \in S^1$ we write $\psi_x(y) = \psi(x + y)$ for the translated test-function.

Let D be the above set of essential discontinuities of ϕ_{v_2} . For each $z \in D$ we denote by x_z the unique \mathbb{T} -preimage of z in $(0, \frac{1}{2})$. Consider now the action of P upon our test-function ψ_{x_z} . The support of ψ_{x_z} is so small (this is the meaning of ‘sufficiently small’ above) that $P\psi_{x_z}$ has an essential discontinuity at z but is otherwise C^∞ , see Figure 2. Since ϕ_{v_2} is locally constant on each side of z we may find $\alpha_z \neq 0$ such that $\phi_{v_2} - \alpha_z P\psi_{x_z}$ is locally constant in a punctured neighborhood of z (simply let α_z be the ratio of essential discontinuities at z of the two functions). Doing this for all $z \in D$ we see that

$$\tilde{\phi} = \phi_{v_2} - \sum_{z \in D} \alpha_z P\psi_{x_z} \quad (9)$$

is locally constant at each punctured neighborhood of $z \in C$, vanishes at $z \in C$, and is C^∞ elsewhere. In other words $\tilde{\phi} \in E_C^\infty$ and we have obtained the desired decomposition

$$\phi_{v_2} = \tilde{\phi} + P\phi, \quad \tilde{\phi}, \phi \in E_C^\infty. \quad (10)$$

Assume now that $C^\infty(S^1)$ was in the kernel of Φ_2 . Functions with support in C are in the kernel of P , and therefore also of Φ_2 . But then every $\phi \in E_C^\infty$ is in the kernel of Φ_2 . The calculation,

$$\phi_{v_2} = \Phi_2 \phi_{v_2} = \Phi_2 \tilde{\phi} + \Phi_2 P\phi = 0 + P\Phi_2 \phi = 0 \quad (11)$$

is a contradiction, showing that Φ_2 can not vanish upon the space of C^∞ functions. Since C^∞ functions can be approximated in BV -norm by trigonometric polynomials, there is also a trigonometric polynomial f with $\Phi_2(f) \neq 0$. q.e.d.

5 Proof of Theorem 1

We will construct the example announced in Theorem 1 as a smooth perturbation of the above piecewise linear map T . Indeed, for $\vartheta = \frac{2}{3} < \kappa := 0.7 < \lambda_2$ we can apply the reasoning of section 3 to T . The spectral perturbation theorem in [KL] then shows that for sufficiently small $\delta > 0$ the transfer operator P_δ of the smooth map T_δ (as an operator on BV) has essential spectral radius at most κ and has exactly two simple eigenvalues 1 and $\lambda_\delta \equiv \lambda_{2,\delta}$ close to λ_2 outside the essential spectrum. The associated eigenprojector $\Phi_{2,\delta}$ is a small perturbation (in BV -norm) of the eigenprojector Φ_2 . In particular it does not vanish on the space of trigonometric polynomials if $\delta > 0$ is sufficiently small, see Lemma 6. We want to show that associated to λ_δ there is an eigenfunction of P_δ which is real-analytic. This will prove our Theorem.

First note that there is at least one left eigenvector, $\ell_\delta \in BV^*$ associated to λ_δ , i.e. $\ell_\delta P_\delta = \lambda_\delta \ell_\delta$. (One may take $\ell_\delta(f) = \int \Phi_{2,\delta}(f) h_{2,\delta} dm$ where $h_{2,\delta}$ is the eigenfunction of P_δ with eigenvalue $\lambda_{2,\delta}$.) Since $\Phi_{2,\delta}$ and $h_{2,\delta}$ are small perturbations of the corresponding objects Φ_2 and $h_2 = \phi_{v_2}$, we may choose $\delta > 0$ so small that ℓ_δ does not vanish on the space of trigonometric polynomials on S^1 . We fix from now on a value of $\delta > 0$ for which the above holds.

The map τ_δ , smoothed by convolution with the Gaussian kernel, is real-analytic and has derivative smaller than $2/3$ (because, a.e. the derivative of ψ varies between $1/3$ and $2/3$). We may therefore find $\rho > 0$ and $\vartheta < 1$ (close to $2/3$) for which $|\tau'_\delta(z)| \leq \vartheta$ for all z in the annulus $A_\rho^p = \{z \in \mathbb{C}/p\mathbb{Z} : |\operatorname{Im} z| < \rho\}$. Then τ_δ is a contraction from A_ρ^2 into $A_{\vartheta\rho}^1$. Let $E_\rho = C^\omega(A_\rho) \cap C^0(\overline{A_\rho})$ be the Banach space of analytic functions on the open annulus extending continuously to its closure.

Let $j : E_\rho \rightarrow BV$ denote the natural injection. It is continuous because by a Cauchy estimate, $\|f\|_\infty + \int_{S^1} |f'| dx \leq (1 + \frac{1}{\rho}) \|f\|_{E_\rho}$. Write P^E for the restriction of P_δ to E_ρ . Then

$$j \circ P^E = P_\delta \circ j. \quad (12)$$

The contraction property of τ shows that $P^E : E_\rho \rightarrow E_{\vartheta\rho}$ is norm-bounded (by 2ϑ), and since the natural injection $E_{\vartheta\rho} \rightarrow E_\rho$ is compact (in fact nuclear) the operator P^E is compact when acting upon E_ρ .

Returning now to the above left eigenfunction, ℓ_δ we see that

$$0 = \ell_\delta(\lambda_\delta - P_\delta) \circ j = \ell_\delta \circ j \circ (\lambda_\delta - P^E). \quad (13)$$

The functional $\ell_\delta \circ j$ is therefore in the kernel of $\lambda_\delta - (P^E)^*$. On the other hand, it does not vanish when acting upon the test functions \mathcal{S} (because ℓ_0 did not and δ is small enough) so that λ_δ must be in the spectrum of P^E as well. But P^E is compact so $\lambda_\delta \neq 0$ is necessarily an isolated eigenvalue of finite multiplicity. Hence, there must be a corresponding right eigenfunction, $\phi^E \in E_\rho$. But then

$$(\lambda_\delta - P_\delta) \circ j(\phi^E) = j \circ ((\lambda_\delta - P^E)\phi^E) = 0 \quad (14)$$

shows that $j(\phi^E)$ is indeed the corresponding eigenfunction in BV for the eigenvalue λ_δ . And $j(\phi^E)$ is manifestly real-analytic on the circle. This finishes the proof of Theorem 1.

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