Exam, M2

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For every abelian group A and every $n \in \mathbf{N}^*$, we denote by $A/n := A/nA$ the cokernel of multiplication by n, and by $A[n] = \{x \in A, nx = 0\}$ the n-torsion subgroup of A. Recall that A is said to be *divisible* if $A/nA = 0$ for all $n \in \mathbf{N}^*$. Recall also that if K is a p-adic field and M a finite Galois module over K, then the groups $H^i(K, M)$ are finite for all $i \in \mathbf{N}$.

Exercise 1 : Divisible Galois modules (7 points)

Let k be a field of characteristic zero with algebraic closure $\bar{k}$ and absolute Galois group $G = \mathrm{Gal}\,(\bar{k}/k)$. Let A be a divisible discrete G-module.

a) Show that for all $i, n \in \mathbf{N}^*$, there is an exact sequence

$$0 \to H^{i-1}(G, A)/n \to H^i(G, A[n]) \to H^i(G, A)[n] \to 0.$$

b) Assume further that the cohomological dimension $\mathrm{cd}(k)$ of k is at most 1. Show that $H^1(G, A)$ is divisible and $H^i(G, A) = 0$ for all $i \geq 2$.

c) Assume that $A = T(\bar{k})$, where T is an *algebraic k-torus*, that is a commutative algebraic group such that $T \times_k L$ is isomorphic to $\mathbf{G}_m^r$ for some finite field extension L of k and some $r \in \mathbf{N}$. Show that there exists a positive integer m such that the group $H^1(k, T) = H^1(G, A)$ is m-torsion.

d) Deduce that if $\mathrm{cd}(k) \leq 1$ (and $A = T(\bar{k})$ as in c), then $H^1(G, A) = 0$.

e) Does the result of d) still hold for an arbitrary divisible G-module A ?

Exercise 2 : Brauer group of a curve (7 points)

Let k be a field of characteristic zero with algebraic closure $\bar{k}$. Let X be a smooth and geometrically integral k-variety. Set $\overline{X} = X \times_k \bar{k}$ and

$$\mathrm{Br}_1 X = \ker[\mathrm{Br}\, X \to \mathrm{Br}\, \overline{X}].$$

a) Let $\alpha \in \mathrm{Br}_1 X$. Show that there exists a finite Galois field extension L of k such that the image of α in $\mathrm{Br}\, X_L$ is zero, where $X_L := X \times_k L$ (hint: use

example 4.13 d) of the lecture notes, about étale cohomology of a projective limit of schemes).

b) Let $m = [L : k]$. Show that $m\alpha = 0$ (hint: consider the field extension $L(X)/k(X)$, where $k(X)$ and $L(X)$ are the respective function fields of X and X_L).

From now on assume that X is a smooth and projective curve. Recall that we have an exact sequence of Galois modules:

$$0 \to D \to \operatorname{Pic} \overline{X} \stackrel{\deg}{\to} \mathbf{Z} \to 0,$$

where D is divisible (it is the group of $\bar{k}$-points of an abelian variety, the *Jacobian variety* of X) and satisfies $D[n]$ finite for every $n \in \mathbf{N}^*$ (here deg is the degree map). Assume further that the set $X(k)$ of k-points of X is non-empty.

c) Show that $\deg : H^0(k, \operatorname{Pic} \overline{X}) \to \mathbf{Z}$ is onto.

d) Show that $\operatorname{Br} X / \operatorname{Br} k$ is isomorphic to $H^1(k, D)$.

e) Assume that k is a p-adic field. Show that for all $n > 0$, the group $(\operatorname{Br} X)[n]$ is finite.

Exercise 3 : Brauer groups and number fields (7 points)

Let k be a number field. Denote by $\mathcal{O}_k$ its ring of integers and set $X = \operatorname{Spec}(\mathcal{O}_k)$ (it is a noetherian and regular scheme of dimension 1). Let $S = \{v_1, ..., v_m\}$ be a finite set of closed points of X (corresponding to non archimedean places of k). Set $U = X - S = \operatorname{Spec}(\mathcal{O}_{k,S})$.

a) Show that there is an exact sequence

$$0 \to \operatorname{Br} U \to \operatorname{Br} k \to \bigoplus_{v \in U^{(1)}} \mathbf{Q}/\mathbf{Z},$$

where $U^{(1)}$ stands for the set of closed points of U.

b) Show that $\operatorname{Br} X$ is isomorphic to $(\mathbf{Z}/2)^{r-1}$, where r is the number of real places of k (with the convention $(\mathbf{Z}/2)^{r-1} = 0$ for $r \leq 1$). Hint: use the Brauer-Hasse-Noether sequence (5) of paragraph 5.3. in the lecture notes.

c) Assume that k has no real place and $m \geq 1$. Show that $\operatorname{Br} U$ is isomorphic to $(\mathbf{Q}/\mathbf{Z})^{m-1}$.

d) Let T be an arbitrary set of places of k. Show that the diagonal map $\operatorname{Br} k \to \bigoplus_{v \in T} \operatorname{Br} k_v$ is onto if and only if there exists a non archimedean place v_0 of k with $v_0 \notin T$.