CORRIGENDA FOR 'ARITHMETIC DUALITY THEOREMS FOR 1-MOTIVES'

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Abstract. The argument for proving Corollary 3.5 is insufficient; we fill in the gap here. Also, the first two statements of Proposition 4.1 may not be true in general in the case $i = 1$, but for the main results it suffices to use them over a sufficiently small U, where they hold. These results are used in the proof of Theorem 0.2, but its statement remains unchanged in the crucial case $i = 1$; in the (uninteresting) case $i = 0$ it needs to be modified slightly. We also correct a few other minor inaccuracies.

Our notation here follows that in [1]. All references to numbered theorems are to those in that paper.

List of corrections

p. 99, l. 21: read 'there is always a *surjection* $\mathbf{H}^{-1}(K, M)^\wedge \to \mathbf{H}^{-1}_\wedge(K, M)$...'

p.102: The first assertion of Proposition 2.8 is wrong: in general the map $\mathbf{H}^0(F, M^*) \to \mathbf{H}^0(F, M^*)^\wedge$ is not injective. Take e.g. $F = K = \mathbf{Q}_p$ and $M^* = [\mathbf{Z} \to \mathbf{G}_m]$ as on page 99, l. 16. The mistake in the proof is that in order to apply the results of the Appendix one would need the morphism $H^0(K, Y^*) \to G^*(K)$ to be strict, which is not necessarily the case. The only place where this statement was used is in the proof of the case $i = 0$ of Corollary 3.5 (see the modified statement below). The two other assertions of Proposition 2.8 and their proofs are unaffected.

p. 106: In exact sequence (8) there is a misprint at the beginning of the third term: it should be $\mathbf{H}^i(k_v, \mathcal{F}^\bullet)$, not $\mathbf{H}^i(\hat{k}_v, \mathcal{F}^\bullet)$.

pp. 109–110: In the proofs of Theorem 3.4 and Corollary 3.5 we are using implicitly that the maximal divisible subgroups involved equal the subgroups of divisible elements. This holds because we work with torsion groups *of finite cotype*.

p.110: There are several problems with the proof of Corollary 3.5.

Case $i = 0$. In the statement of this case the group $D^0(U, M)$ should be replaced by

$$D^0_\wedge(U, M) := H^0(U, M) \to \bigoplus_{v \in \Sigma} H^0(k_v, M)^\wedge.$$

The proof then runs as follows: consider the commutative exact diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & D^0_\wedge(U,M)\{\ell\} & \longrightarrow & H^0(U,M)\{\ell\} & \longrightarrow & \bigoplus_{v\in\Sigma} H^0(k_v,M)^{(l)} \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \overline{D^2(U,M^*)}\{\ell\}^D & \longrightarrow & \overline{H^2_c(U,M^*)}\{\ell\}^D & \longrightarrow & \bigoplus_{v\in\Sigma} H^1(k_v,M^*)\{\ell\}^D \end{array}$$

In the lower row the notation as in Theorem 3.4; its exactness comes from the fact that the groups being of finite cotype, we have

$$\overline{D^2(U,M^*)}\{\ell\} = D^2(U,M^*)\{\ell\}/l^N,\ \overline{H^2_c(U,M^*)}\{\ell\} = H^2_c(U,M^*)\{\ell\}/l^N$$

for N large enough. Now by Theorem 3.4 the middle vertical map is an isomorphism (recall that $H^0(U,M)\{\ell\}$ is finite by Lemma 3.2 (3)), and by Theorem 2.3 the right vertical map is injective.

Case $i = 1$. There is a more serious problem here. First we note that there is already a problem with [3], II.5.3, where the case $M = [0 \to A]$ is treated: the lower row of the diagram there does not make sense. In our case the diagram exists and gives rise to a map $D^1(U,M)\{\ell\} \to D^1(U,M^*)^D$ as claimed, but the argument does not show that the kernel C of the middle vertical map $\mathbf{H}^1(U,M)\{\ell\} \to \mathbf{H}^1_c(U,M^*)^D$ is divisible. The application of Theorem 3.4 only shows that the kernel of the map $\mathbf{H}^1(U,M)\{\ell\} \to (\mathbf{H}^1_c(U,M^*)\{\ell\})^D$ is divisible, but C may be a proper subgroup of this kernel. An additional statement is needed to justify the divisibility of C. This is the following:

Claim. *If n is a power of ℓ, and a is an element of $D^1(U,M)$ that is n-divisible in $\mathbf{H}^1(U,M)$ and orthogonal to $D^1(U,M^*)[n]$, then a is n-divisible in $D^1(U,M)$.*

To see this, we first state an analogue of [3], I.6.15.

Lemma. *Let n be an integer invertible on U, and $S_n(U,M)$ the kernel of the map $H^1(U,T_{\mathbf{Z}/n\mathbf{Z}}(M)) \to \oplus_{v\in\Sigma}\mathbf{H}^1(k_v,M)$. If a is an element of $\oplus_{v\in\Sigma}H^1(k_v,T_{\mathbf{Z}/n\mathbf{Z}}(M))$ orthogonal to the image of $S_n(U,M^*)$ in $\oplus_{v\in\Sigma}H^1(k_v,T_{\mathbf{Z}/n\mathbf{Z}}(M^*))$, then a is the sum of the coboundary of an element in $\oplus_{v\in S}H^0(k_v,M)$ and of the restriction of an element in $H^1(U,T_{\mathbf{Z}/n\mathbf{Z}}(M))$.*

The proof is an application of Poitou–Tate duality for finite modules and runs as in *loc. cit.*, except that the dual of $\mathbf{H}^1(k_v,M)$ is the profinite completion of $\mathbf{H}^0(k_v,M)$, but the image of both the completed and uncompleted groups in the finite group $\oplus_{v\in\Sigma}\mathbf{H}^1(k_v,T_{\mathbf{Z}/n\mathbf{Z}}(M))$ is the same. Also, in place of the map γ_1 there it is more convenient to use the composite of the coboundary map $\oplus_{v\in\Sigma}H^1(k_v,T_{\mathbf{Z}/n\mathbf{Z}}(M)) \to$

$H^2_c(U, T_{\mathbf{Z}/n\mathbf{Z}}(M))$ in the localization exact sequence for compact support cohomology with the Artin–Verdier isomorphism $H^2_c(U, T_{\mathbf{Z}/n\mathbf{Z}}(M)) \cong H^1(U, T_{\mathbf{Z}/n\mathbf{Z}}(M^*))^D$.

Now to prove the claim consider the commutative exact diagram

$$\begin{array}{ccccc}
 & & H^1_c(U,M) & \longrightarrow & H^1(U,M) \\
 & & \downarrow n & & \downarrow n \\
\bigoplus_{v\in\Sigma} H^0(k_v,M) & \longrightarrow & H^1_c(U,M) & \longrightarrow & H^1(U,M) \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{v\in\Sigma} H^1(k_v, T_{\mathbf{Z}/n\mathbf{Z}}(M)) & \longrightarrow & H^2_c(U, T_{\mathbf{Z}/n\mathbf{Z}}(M)) & \longrightarrow & H^2(U, T_{\mathbf{Z}/n\mathbf{Z}}(M))
\end{array}$$

Let a be an element of $D^1(U,M) = \mathrm{Im}\,(\mathbf{H}^1_c(U,M) \to \mathbf{H}^1(U,M))$ arising as $a = na_1$ with $a_1 \in H^1(U,M)$. By definition a comes from some $\tilde{a}$ in $H^1_c(U,M)$ whose image in $H^2_c(U, T_{\mathbf{Z}/n\mathbf{Z}}(M))$ will be denoted by a_2. By functoriality, given $a' \in D^1(U,M^*)[n]$, the value $\langle a, a'\rangle$ of the Cassels–Tate pairing equals that of the Artin–Verdier pairing $[a_2, b']$, where $b' \in H^1(U, T_{\mathbf{Z}/n\mathbf{Z}}(M^*))$ is a preimage of a'. A diagram chasing now shows that a_2 comes from $(c_v) \in \oplus_{v\in\Sigma} H^1(k_v, T_{\mathbf{Z}/n\mathbf{Z}}(M))$. It follows that $[a_2, b']$ equals the sum of the local pairings $\langle c_v, b'_v\rangle_v$ for $v \in S$, where b'_v is the image of b' in $H^1(k_v, T_{\mathbf{Z}/n\mathbf{Z}}(M^*))$.

Our assumption that $\langle a, a'\rangle = 0$ for all $a' \in D^1(U,M^*)[n]$ thus implies that (c_v) satisfies the assumptions of the lemma, and hence up to modifying it by an element of $\oplus_{v\in\Sigma} H^0(k_v, M)$ (which does not change a), we may assume that (c_v) comes from $H^1(U, T_{\mathbf{Z}/n\mathbf{Z}}(M))$, and hence $\tilde{a}$ maps to 0 in $H^2_c(U, T_{\mathbf{Z}/n\mathbf{Z}}(M))$. By the diagram this means that $\tilde{a}$ is divisible by n in $H^1_c(U,M)$, and hence so is a in $D^1(U,M)$.

Added 18/9/2013. There is another, simpler approach to prove the case $i = 1$ of Corollary 3.5. It is based on the following slight refinement of Theorem 3.4: for each prime number ℓ invertible on U, the pairing

$$H^1(U,M)\{\ell\} \times H^1_c(U,M^*)^{(\ell)} \to \mathbf{Q}/\mathbf{Z}$$

induced by (9) has divisible left kernel. The proof of this statement goes as in Theorem 3.4 once one has observed that the dual of $T_\ell H^2_c(U,M^*)$ is divisible (the group $H^2_c(U,M^*)\{\ell\}$ being torsion and of cofinite type). There is a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & D^1(U,M)\{\ell\} & \longrightarrow & H^1(U,M)\{\ell\} & \longrightarrow & \bigoplus_{v\notin U} H^1(k_v, M_k)\{\ell\} \\
 & & & & \downarrow g & & \downarrow h \\
0 & \longrightarrow & \overline{D^1(U,M^*)\{\ell\}}^D & \longrightarrow & (H^1_c(U,M^*)^{(\ell)})^D & \longrightarrow & \bigoplus_{v\notin U} (H^0(k_v, M_k^*)^{(\ell)})^D
\end{array}$$

To obtain the lower row, observe that $D^1(U,M^*)^{(\ell)} = \overline{D^1(U,M^*)\{\ell\}}$ (this holds because $D^1(U,M^*)$ is torsion and of cofinite type) and that the functor $^{(\ell)}$ preserves a right exact sequence $A \to B \to C \to 0$ of abelian groups if A/ℓ^n and B/ℓ^n are finite for every $n > 0$.

The diagram induces a map $f : D^1(U,M)\{\ell\} \to \overline{D^1(U,M^*)\{\ell\}}^D$. The map h is an isomorphism by local duality, hence $\ker f = \ker g$. But we have seen that $\ker g$ is divisible (note that it is not sufficient to know that the kernel of the map $H^1(U,M)\{\ell\} \to H^1_c(U,M^*)\{\ell\}^D$ is divisible, which was the problem in the original paper [1] as well as in [3], II.5.3). Thus $\ker f$ is divisible, whence the result by exchanging M and M^*.

p. 111: The first two statements of Proposition 4.1 are in general false for $i = 1$. The mistake comes from an incorrect interpretation of [3], II.4.14. There it is shown that for a U-torus T the group $H^1(U,T)\{\ell\}$ is isomorphic to $H^1(\Gamma_\Sigma, T(R_\Sigma))\{\ell\}$, where R_Σ is the normalization of $\mathcal{O}_{k,\Sigma}$ in k_Σ, and not to $H^1(\Gamma_\Sigma, T_k(k_\Sigma))\{\ell\}$. Nevertheless, for the remaining of the paper (except in Corollary 4.3, which is not used elsewhere; see below), it is sufficient to know that (1) and (2) hold (for $i = 1$) over *U sufficiently small*. To prove this, one first reduces to the case where $M = T$ is a torus, as explained on p. 112. Then one observes that the statement holds for a norm torus $R_{K|k}\mathbf{G}_m$ for some finite extension $K|k$, because $H^1(U, R_{K|k}\mathbf{G}_m) = \mathrm{Pic}(U \times_k K)$ is zero for U sufficiently small. The statement then follows for quasi-trivial tori, i.e. finite products of norm tori. Now let T be arbitrary. By Ono's lemma ([4], Theorem 1.5.1), there exist $m > 0$ and a quasi-trivial k-torus R_k such that $T_k^m \times R_k$ is isogenous to a quasi-trivial torus. As the statements to be proven are compatible with products and we have just shown them for R_k, we may replace T_k by $T_k^m \times R_k$ and therefore assume that there is an exact sequence

$$0 \to F \to R \to T \to 0$$

with F finite étale over U and R_k quasi-trivial. Now the result follows from the associated long exact sequence using the case $i = 2$, the case of a quasi-trivial torus, and [16], II.2.9.

p. 113: For the statement of Corollary 4.3 to hold, the definition of $Ш^1_\Sigma(M)$ has to be changed. With the notation above it should be defined as the kernel of the map

$$\mathbf{H}^1(\Gamma_\Sigma, M(R_\Sigma)) \to \prod_{v\in\Sigma} \mathbf{H}^1(\hat{k}_v, M_k).$$

This corollary is not used elsewhere.

p. 115: Proposition 4.12 is not justified and should be suppressed, together with the remarks following it (though the argument goes through in the case $M_k = Y_k[1]$, justifying Remark 4.13 (1)). In any case, this

statement is not very interesting; Proposition 5.1 is much more useful. The latter statement also implies the duality between the groups $Ш^1(Y_k)$ and $Ш^2(T_k^*)$, where T_k^* is the torus with character module Y_k.

Concerning Remark 4.13 (3), the finiteness of $Ш^2(M_k)$ was proven in important mixed cases in the 2009 PhD thesis of P. Jossen, among them 1-motives of the form $[Y_k \to A_k]$, with A_k a geometrically simple abelian variety. He also proved that in these cases there is a perfect pairing

$$Ш^0(M_k) \times Ш^2(M_k^*) \to \mathbf{Q}/\mathbf{Z}$$

of finite groups (in fact, the finiteness of $Ш^0(M_k)$ can be proven to hold in general). However, he recently produced an example where $Ш^2(M_k)$ is infinite!

p. 120: At the end of the proof of Proposition 5.4., an additional argument is required to check that the morphism θ_0 is strict because it is not clear that the groups $\mathbf{P}^0(M_k)_\wedge$ and $H^0(k, M_k)_\wedge$ are locally compact in general (the difficulty is related to the toric part of M_k), so [2], 5.29 does not apply directly. But for every $n > 0$, the groups $H^0(k, M_k)/n$ and $\mathbf{P}^0(M_k)/n$ are locally compact (the latter thanks to Lemma 5.3); moreover the morphism $f_n : H^0(k, M_k)/n \to \mathbf{P}^0(M_k)/n$ is strict for every $n > 0$: indeed the morphisms $H^0(k, M_k)/n \to H^1(k, T_{\mathbf{Z}/n\mathbf{Z}}(M_k))$ and $\mathbf{P}^0(M_k)/n \to \mathbf{P}^1(T_{\mathbf{Z}/n\mathbf{Z}}(M_k))$ are strict (thanks to [2], 5.29 and to the diagram on bottom of page 118), hence the image of $H^0(k, M_k)/n$ into $\mathbf{P}^0(M_k)/n$ identifies to a subspace of the image of the group $H^1(k, T_{\mathbf{Z}/n\mathbf{Z}}(M_k))$ into $\mathbf{P}^1(T_{\mathbf{Z}/n\mathbf{Z}}(M_k))$, which is discrete (using Poitou-Tate exact sequence for finite modules we get that this image is a closed subset of $\mathbf{P}^1(T_{\mathbf{Z}/n\mathbf{Z}}(M_k))$, hence the morphism $H^1(k, T_{\mathbf{Z}/n\mathbf{Z}}(M_k)) \to \mathbf{P}^1(T_{\mathbf{Z}/n\mathbf{Z}}(M_k))$ is strict by [2], 5.29). Now θ_0 is obtained as the projective limit of the strict morphisms f_n with $\ker f_n$ finite and $H^0(k, M_k)/n$ discrete; it is not difficult to check that this implies that θ_0 is strict (see for example http://www.math.u-psud.fr/~harari/recherche/ topolog.pdf).

p. 123: Since $\operatorname{coker} \theta_0$ (with the quotient topology) is already profinite, the morphism γ_0' is strict and we don't need $\mathbf{P}^0(M_k)_\wedge$ locally compact here (the point is that its quotient by the image of $H^0(k, M_k)_\wedge$ is compact).

p. 127: In the proof of (3), when replacing B by B/A' we implicitly used the fact that every quotient of B by a closed subgroup is again Hausdorff, locally compact, compactly generated, and completely disconnected (the latter by [2], Theorems 3.5. and 7.11).

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References

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