SCHEMES

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Contents

In this course, all rings are assumed to be commutative and with a unit element. A homomorphism $\varphi : A \to B$ of rings must satisfy $\varphi(1_A) = 1_B$; for a homomorphism of local rings $\psi : (A, \mathcal{M}_A) \to (B, \mathcal{M}_B)$, it is moreover required that $\psi^{-1}(\mathcal{M}_B) = \mathcal{M}_A$. A *domain* is a ring A with $A \neq \{0\}$, and such that $ab = 0$ (with $a, b \in A$) implies $a = 0$ or $b = 0$. An ideal $\wp$ of a ring A is *prime* if $A/\wp$ is a domain (in particular A is not a prime ideal of A).

The main reference will be the book "Algebraic Geometry" by R. Hartshorne (Graduate Texts in Mathematics, Springer-Verlag 1977), denoted by [H] in the following. For commutative algebra, one may use Matsumura's book [M] "Commutative Algebra".

1. Basic notions on schemes

1.1. First definitions and examples

Recall that the *spectrum* $\operatorname{Spec} A$ of a ring A is the set of its prime ideals, equipped with the *Zariski topology* (except in very specific cases, it is not Hausdorff) : the closed subsets are the $V(I)$ (for any ideal I of A), where by definition $V(I)$ is the set of all $\wp \in \operatorname{Spec} A$ such that $\wp \supset I$. A base of open subsets consists of the $D(f)$ for $f \in A$, where $D(f)$ is the set of all $\wp \in \operatorname{Spec} A$ such that $f \notin \wp$. Notice that if A is a domain, then any non empty open subset of $\operatorname{Spec} A$ contains the element $(0) \in \operatorname{Spec} A$, so in this case any non empty open subset is dense.

The topological space $\operatorname{Spec} A$ also comes with a sheaf of rings $\mathcal{O}$ such that $\mathcal{O}_\wp$ (the stalk of the sheaf at $\wp$) is isomorphic to $A_\wp$ (the localisation of the ring A with respect to the multiplicative set $A - \wp$), and $\mathcal{O}(D(f))$ (the ring of sections of $\mathcal{O}$ over the open subset $D(f)$) is isomorphic to A_f (the localisation of the ring A with respect to the multiplicative set $\{1, f, ..., f^n, ...\}$). For example the ring $\Gamma(\operatorname{Spec} A, \mathcal{O})$ of *global sections* of the sheaf $\mathcal{O}$ is just A. Intuitively, $\mathcal{O}_\wp$ corresponds to germs of functions defined "around $\wp$", and $\mathcal{O}(D(f))$ to functions defined on the open subet "$f \neq 0$".

One way to construct the sheaf $\mathcal{O}$ (see [H], II.2 for more details) is to define (for each open subset U) the ring $\mathcal{O}(U)$ as the ring of functions $U \to \coprod_{\wp \in U} A_\wp$ which are locally (for the Zariski topology) of the type $\frac{a}{f}$ with $a, f \in A$.

A *locally ringed space* is a topological space X, equipped with a sheaf of rings $\mathcal{O}_X$, such that for each $P \in X$ the stalk $\mathcal{O}_{X,P}$ is a local ring. A morphism $X \to Y$ of locally ringed spaces is a pair $(f, f^\#)$ such that $f : X \to Y$ is a continuous map, and $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is a morphism of sheaves of rings, with the additional condition that for each $P \in X$, the induced map $f_P^\# : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ is a homomorphism of local rings.

An *affine scheme* is a locally ringed space which is isomorphic to $\mathrm{Spec}\, A$ for some ring A. An important property (see [H], II.2.3) is that the contravariant functor $A \to \mathrm{Spec}\, A$ gives an anti-equivalence of categories between rings and affine schemes, the inverse functor being $X \to \Gamma(X, \mathcal{O}_X)$.

Examples of affine schemes.

1. $\mathrm{Spec}\,(\{0\}) = \emptyset$.

2. For a field k, $\mathrm{Spec}\, k$ consists of one single point, with structural sheaf k.

3. $\mathrm{Spec}\,(k[X_1, ..., X_n])$ is the *affine space* $\mathbf{A}_k^n$ over k. More generally, an *affine variety* over a field k is an affine scheme $\mathrm{Spec}\, A$, where the ring A is a finitely generated k-algebra. Since such an A is isomorphic to a quotient $k[X_1, ..., X_n]/(P_1, ..., P_r)$, where the P_i are polynomials, this means that such a variety is defined in the affine space by the equations "$P_i(X_1, ..., X_n) = 0$" for $1 \leq i \leq r$.

   For example on $\mathbf{A}_k^1$, you have the *generic point* η, corresponding to the ideal (0); the point η belongs to any non empty open subset, hence the closure of $\{\eta\}$ is the whole $\mathbf{A}_k^1$. The other points are closed (this corresponds to the fact that the associated ideals are maximal); some of them correspond to the elements of k (via the maximal ideal $(X - a) \subset k[X]$), but there might be other points if k is not algebraically closed : e.g. on $\mathbf{A}_{\mathbf{R}}^1$, there is the point $(X^2 + 1)$, although $x^2 + 1 = 0$ has no solution in $\mathbf{R}$. On $\mathbf{A}_k^2$, it is still much more complicated, because you also have the prime ideals $(\pi) \subset k[X_1, X_2]$, where π is an irreducible polynomial (the "generic point of the curve $\pi(X_1, X_2) = 0$"). Thus the language of schemes takes into account two new things : generic points, and points in field extensions of the ground field.

4. Let k be a field and set $k[\varepsilon] = k[X]/X^2$. The affine scheme $\mathrm{Spec}\,(k[\varepsilon])$ correspond to the equation "$x^2 = 0$" on the affine line. As a topological space, it is just one point, but as a scheme it is not the same as $\mathrm{Spec}\, k$. Actually $\mathrm{Spec}\,(k[\varepsilon])$ corresponds to a "double point". That's another

advantage of schemes : it is possibile to give a precise meaning to the intuitive notion of multiplicity.

5. The affine scheme $\operatorname{Spec} \mathbf{Z}$ consists of one generic point, and infinitely many closed points (one for each prime number).

Now we come to the general definition of a scheme :

Definition 1.1 A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ such that for any $P \in X$, there exists an open neighborhood U of P with the property that the locally ringed space $(U, (\mathcal{O}_X)_{|U})$ is an affine scheme.

For each point x of a scheme X, one defines its *residue field* $k(x)$ as the quotient of the local ring $\mathcal{O}_{X,x}$ by its maximal ideal $\mathcal{M}_x$.

Usually we shall simply write X for $(X, \mathcal{O}_X)$.

For example, on $\mathbf{A}^n_k$, the residue field of each closed point is a finite field extension of k. On any scheme X, you can evaluate an element $f \in \mathcal{O}_{X,x}$ (hence also an element of $\Gamma(X, \mathcal{O}_X)$) at x, taking the reduction of f modulo $\mathcal{M}_x$: you get an element of the residue field $k(x)$. E. g. on $\mathbf{A}^1_{\mathbf{R}} = \operatorname{Spec}(\mathbf{R}[T])$, the evaluation of T at the point $x = (T^2+1)$ is the element $\sqrt{-1}$ [1] of the residue field $\mathbf{C}$ of x.

So far we don't have any example of a non-affine scheme. An important remark is that an open subset U of a scheme X obviously is a scheme (with the restriction of $\mathcal{O}_X$ to U as structural sheaf), but an open subset of an affine scheme is not necessarily affine : take k a field, $X = A^2_k$, $U = X - \{(0,0)\}$ (U is obtained by removing the closed point corresponding to the maximal ideal $(T_1, T_2) \subset k[T_1, T_2]$). Then U is not affine. Indeed the inclusion map $i : U \subset X$ is not an isomorphism, but it is easy to check that the induced map $i^* : \Gamma(X, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_U)$ is (set $A = k[T_1, T_2]$, and $K = \operatorname{Frac} A = k(T_1, T_2)$; any element of $\Gamma(U, \mathcal{O}_U)$ must belong to the localisations A_{T_1} and A_{T_2}, and the intersection of these two subrings of K is A).

The most important class of examples of non affine schemes is obtained by means of *graded rings*. Let $S = \bigoplus_{d \geq 0} S_d$ be a graded ring, and set $S_+ = \bigoplus_{d>0} S_d$. One defines $\operatorname{Proj} S$ as the set of *homogeneous* prime ideals $\wp$ that do not contain S_+. For each homogeneous ideal I of S, one defines $V(I)$ as the set of $\wp \in \operatorname{Proj} S$ such that $\wp \supset I$. This gives $\operatorname{Proj} S$ a topology, a base of open subsets consisting of the $D_+(f)$ for f homogeneous element of S, where $D_+(f)$ is the set of $\wp \in \operatorname{Proj} S$ with $f \notin \wp$. One defines a sheaf

[1] that is, the image of T in $\mathbf{R}[T]/(T^2+1) \simeq \mathbf{C}$; note that here you cannot distinguish between $\sqrt{-1}$ and $-\sqrt{-1}$; more about this later.

$\mathcal{O}$ on $\operatorname{Proj} S$, with the properties : $\mathcal{O}_\wp$ is isomorphic to $S_{(\wp)}$ (the elements *of degree zero* in the localisation of S with respect to *homogeneous* elements of $S - \wp$) and $\mathcal{O}(D_+(f))$ is isomorphic to $S_{(f)}$ (the ring of elements of degree zero in the localisation S_f). With these properties, it is clear that $\operatorname{Proj} S$ is a scheme.

For example $\operatorname{Proj}(k[X_0, ..., X_n])$ is the *projective space* $\mathbf{P}^n_k$ over the field k. It is covered by the affine subsets $D_+(T_i)$ for $0 \leq i \leq n$, each of those isomorphic to $\mathbf{A}^n_k$. In particular this shows that $\Gamma(\mathbf{P}^n_k, \mathcal{O}_{\mathbf{P}^n_k}) = k$ because the intersection of the rings $S_{(T_i)}$ (where $S := k[X_0, ..., X_n]$) is k, hence $\mathbf{P}^n_k$ is not an affine scheme.

More generally, a *projective variety* over k is a scheme $\operatorname{Proj} S$, where S is the quotient of $k[X_0, ..., X_n]$ by a homogeneous ideal (this corresponds to equations $P_i(X_0, ..., X_n) = 0$, where the P_i are homogeneous polynomials).

1.2. Morphisms of schemes : first properties

Recall that a *morphism of schemes* is just a morphism of the underlying locally ringed spaces. Observe that if $f : Y \to X$ is a morphism of schemes, then for each $y \in Y$ with image $x = f(y)$, there is an induced homomorphism $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$, hence also a homomorphism between the residue fields $k(x) \to k(y)$. In particular if X is an affine or a projective k-variety, each residue field is an extension of k (and this extension is finite for, and only for, closed points).

Definition 1.2 Let S be a (fixed) scheme. An *S-scheme* (or scheme over S is a scheme X, equipped with a morphism $X \to S$. A *morphism of S-schemes* (or S-morphism) is a morphism $X \to Y$ which is compatible with the given morphisms $X \to S$ and $Y \to S$.

When X, Y are S-schemes, the piece of notation $\operatorname{Mor}_S(X, Y)$ stands for the set of S-morphisms from X to Y.

Definition 1.3 Let A be a ring and X an A-scheme (=$\operatorname{Spec} A$-scheme). Let B be an A-algebra. A *B-point* of X is an element of $\operatorname{Mor}_{\operatorname{Spec} A}(\operatorname{Spec} B, X)$.

Examples.

1. This notion is especially useful when $A = k$ is a field, and $B = L$ is a field extension of k. In this case, giving an L-point of a k-scheme X is the same as giving its image $x \in X$ (recall that the underlying topological space of $\operatorname{Spec} L$ is a singleton), plus a k-morphism $k(x) \to L$.

2. Let k be a field, $k[\varepsilon] = k[T]/T^2$. For an X-scheme X, giving a $k[\varepsilon]$-point of X is the same as giving a point $x \in X$ and a k-morphism $\mathcal{O}_{X,x} \to k[\varepsilon]$, that is a point x with residue field k plus a k-morphism $(\mathcal{M}_x/\mathcal{M}_x^2) \to k$. The dual $\mathrm{Hom}((\mathcal{M}_x/\mathcal{M}_x^2), k)$ of the k-vector space $(\mathcal{M}_x/\mathcal{M}_x^2)$ is the *tangent space* of X at x; hence a $k[\varepsilon]$-point of X consists of a point x with residue field k and a tangent vector at x.

Let S be a scheme, X and Y two S-schemes. There is a *fibred product* $X \times_S Y$; it is an S-scheme with S-morphisms $p_X : X \times_S Y \to X$ and $p_Y : X \times_S Y \to Y$ (first and second projection) satisfying the following universal property : for any S-scheme Z and any pair of S-morphisms $f_X : Z \to X$, $f_Y : Z \to Y$, there is a unique S-morphism $g : Z \to X \times_S Y$ such that $f_X = p_X \circ g$, $f_Y = p_Y \circ g$. For $X = \mathrm{Spec}\, A$, $Y = \mathrm{Spec}\, B$, $S = \mathrm{Spec}\, R$, one takes $X \times_S Y := \mathrm{Spec}\,(A \times_R B)$. The general construction of $X \times_S Y$ consists of a "globalisation" of this, see [H], II.3.3. for the details.

For example $\mathbf{A}_k^m \times \mathbf{A}_k^n = \mathbf{A}_k^{m+n}$. We can see that even for $m = n = 1$, the underlying topological space of $\mathbf{A}_k^2$ is *not* the product of the underlying topological space of $\mathbf{A}_k^1$ by itself, because it has more points (the generic points of the curves). Also, $\mathbf{P}_k^1 \times \mathbf{P}_k^1$ is not isomorphic to $\mathbf{P}_k^2$.

The first application of fibred product is the notion of *fibre* :

Definition 1.4 Let $f : X \to Y$ be a morphism of schemes. Let $y \in Y$ and $\mathrm{Spec}\,(k(y)) \to Y$ the corresponding morphism. The *fibre* of f at y is the scheme $X_y := X \times_Y \mathrm{Spec}\,(k(y))$.

It is easy to check (by reduction to the affine case) that the underlying topological space of $\mathrm{Spec}\,(k(y))$ is the inverse image $f^{-1}(\{y\})$; now we have a canonical $k(y)$-scheme structure on this inverse image.

Examples.

1. For a scheme X over $\mathrm{Spec}\,\mathbf{Z}$, there is a *generic fibre* $X_{\mathbf{Q}}$ over $\mathbf{Q}$, and *special fibres* X_p over each finite field $\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$ ("reduction modulo p"). For example if $X = \mathrm{Spec}\,\mathbf{Z}[T_1, T_2]/(T_1^2 + 2T_2^2 - 2)$, the generic fibre is a conic over $\mathbf{Q}$, but the reduction X_2 modulo 2 is $\mathbf{A}^1_{\mathbf{F}_2[\varepsilon]} = \mathrm{Spec}\,(\mathbf{F}_2[\varepsilon, T_2])$.

2. Let k be a field of characteristic $\neq 2$, $X = \mathrm{Spec}\,(k[T_1, T_2]/(T_2^2 - T_1)$, $Y = \mathbf{A}_k^1 = \mathrm{Spec}\,(k[T_1])$. Consider the morphism $X \to Y$, $(t_1, t_2) \mapsto t_1$ (that means that it is induced by the ring homomorphism $k[T_1] \to k[T_1, T_2]/(T_2^2 - T_1)$ that sends T_1 to the coset of T_1). Then the fibre at the closed point $t_1 = 0$ is $\mathrm{Spec}\,(k[\varepsilon])$ (one double point). The fibre

at the closed point $t_1 = a$ for $a \neq 0$ is Spec $(k[T_2]/(T_2^2 - a))$, that is either the spectrum of a quadratic field extension $k(\sqrt{a})$, or the union Spec $(k \oplus k)$ of two k-points.

Another application of the fibred product is *base extension*, which is the scheme-theoretic version of the extension of scalars associated to the tensor product. For example, if X is an affine or a projective variety over a field k, one can consider the L-variety $X_L := X \times_k L$ $(:= X \times_{\operatorname{Spec} k} \operatorname{Spec} L)$ for any field extension L/k. This corresponds to look at the same polynomial equations, but over the field L. For instance some points of $\mathbf{A}^1_{\mathbf{R}}$ have residue field $\mathbf{R}$, but any closed point of $\mathbf{A}^1_{\mathbf{C}}$ has residue field $\mathbf{C}$. Nevertheless there is always a bijection between L-points of X and X_L thanks to the adjunction property of the tensor product : $\operatorname{Hom}_k(A, L) = \operatorname{Hom}_L(A \otimes_k L, L)$ for any k-algebra A. The same is true if L is replaced by any L-algebra.

Now we come to special classes of morphisms.

Definition 1.5 An *open subscheme* U of a scheme X is an open subset, equipped with the restriction of the sheaf $\mathcal{O}_X$ to U. An *open immersion* is a morphism of schemes $X \to Y$ which induces an isomorphism from X to an open subscheme of Y.

The notion of closed subscheme is more complicated, because you have to define the locally ringed space structure on the closed subset, and there is no canonical one. First we have to define closed immersions.

Definition 1.6 A *closed immersion* is a morphism $f : X \to Y$ of schemes such that :

i) f induces a homeomorphism (i.e. a bicontinuous map) from X to a closed subset of X.

ii) The map of sheaves $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective. [2]

Typically, a closed immersion is a morphism that "looks locally" like Spec $(A/I) \to \operatorname{Spec} A$ for some ring A and some ideal I of A.

Definition 1.7 A *closed subscheme* of a scheme Y is a scheme X, equipped with a closed immersion $i : X \to Y$, where one identifies the pairs (Y, i) and (Y', i') if there exists an isomorphism of schemes $g : Y \to Y'$ such that $g \circ i = i'$.

[2] This means that $f^\#$ is surjective on stalks, not that the map $\mathcal{O}_Y(U) \to f_*\mathcal{O}_X(U) = \mathcal{O}_X(f^{-1}(U))$ is surjective for any open set $U \subset Y$.

Thus $\operatorname{Spec}(A/I)$ is a closed subscheme of $\operatorname{Spec} A$ with underlying topological space $V(I)$, but there might be several structures of closed subscheme on $V(I)$, e.g. $\operatorname{Spec}(k[\varepsilon])$ and $\operatorname{Spec} k$ are two different closed subschemes of $\mathbf{A}^1_k$ with underlying space $\{0\}$ (the closed point corresponding to the ideal $(T) \subset \operatorname{Spec}(k[T])$). Similarly, consider the closed subset $F = V(T_1)$ of the affine plane $\mathbf{A}^2_k = \operatorname{Spec}(k[T_1, T_2])$. You have the closed subscheme structure on F given by the ideal (T_1) (a line), but also one given by (T_1^2) (a doubled line), or by (T_1^2, T_1T_2) (a line with the origin doubled).

Now we deal with finiteness properties related to morphisms.

Definition 1.8 A morphism of schemes $f : X \to Y$ is *locally of finite type* if Y can be covered by open affine subsets $V_i = \operatorname{Spec} B_i$, with each $f^{-1}(V_i)$ covered by open affine subsets $U_{ij} = \operatorname{Spec} A_{ij}$, such that A_{ij} is a finitely generated B_i-algebra for any i, j. The morphism f is *of finite type* if moreover finitely many U_{ij} are sufficient for each i.

Examples.

1. A composition of two morphisms of finite type is of finite type.
2. In general a localisation $\operatorname{Spec} \mathcal{O}_{X,P} \to X$ is not of finite type because the localisation $A_\wp$ ($\wp \in \operatorname{Spec} A$) of a ring A is not a finitely generated A-algebra.
3. For A a ring and $f \in A$, the open immersion $\operatorname{Spec} A_f \to \operatorname{Spec} A$ is of finite type because A_f is generated as an A-algebra by $1/f$.
4. More generally $\operatorname{Spec} B \to \operatorname{Spec} A$ is of finite type if B is a finitely generated B-algebra. [3]
5. The notion of finite type morphism (and also of finite morphism, see below) is stable under base extension (this fact is not obvious).

There is a stronger notion :

Definition 1.9 A morphism $f : X \to Y$ is *finite* if Y can be covered by affine subsets $V_i = \operatorname{Spec} B_i$, such that $f^{-1}(V_i) = \operatorname{Spec} A_i$ is affine and satisfies : A_i is a finite type B_i-*module.*

[3]The converse is true : use the fact that if $f_1, ..., f_r$ are elements of B such that each B_{f_i} is f.g. over A, and $\operatorname{Spec} B$ is the union of the $D(f_i)$ (which means that the ideal generated by $(f_1, ..., f_r)$ is B), then B is f.g. over A.

Examples.

1. A closed immersion is finite : indeed in this case Y can be covered by affine subsets $\operatorname{Spec} B_i$, such that $f^{-1}(\operatorname{Spec} B_i)$ is isomorphic to $\operatorname{Spec} A_i$, where A_i is the quotient of B_i by some ideal.

2. An open immersion is *quasi finite* (that means that each fibre is finite), but not finite in general because A_f is not a finite type A-module for A a ring and $f \in A$. Notice that finite implies quasi finite (when B is a finite type A-module, there are only finitely many prime ideals of B lying over a given ideal of A).

3. Let X be the closed subscheme of $\mathbf{A}_k^2$ defined by $(T_2^2 - T_1)$. We have defined above a morphism $f : X \to \mathbf{A}_k^1$, $(t_1, t_2) \mapsto t_1$. This morphism is finite because $k[T_1, T_2]/(T_2^2 - T_1)$ is a finite $k[T_1]$-module. If one removes the closed point $(1, 1)$ from X, the restriction of f to $X - \{(1, 1)\}$ still is of finite type, surjective and quasi-finite but one can check that it is not finite anymore.

1.3. Some miscellaneous properties of schemes

Definition 1.10 A scheme X is *reduced* if for any open subset U of X, the ring $\mathcal{O}_X(U)$ has no nilpotent elements.

This is equivalent to saying that for each $P \in X$, the local ring $\mathcal{O}_{X,P}$ has no nilpotent elements, hence "reduced" is a local property. For example $\operatorname{Spec}(k[\varepsilon])$ is not reduced.

Definition 1.11 A scheme X is *irreducible* if it is not empty, and if for each decomposition $X = X_1 \cup X_2$ with X_1, X_2 closed subsets, one has $X_1 = X$ or $X_2 = X$.

Notice that this is a global property. It does not depend on the scheme structure on X, only on the topological structure. Intuitively, irreducible means that X doesn't break into smaller pieces. Equivalently, X irreducible means that the intersection of two non empty open subsets is not empty, or that any non empty open subset is dense. Also, any non empty open subscheme of an irreducible scheme is irreducible.

For instance $\operatorname{Spec}(k[\varepsilon])$ is irreducible, $\operatorname{Spec}(k[T_1, T_2]/T_1T_2)$ is not.

Definition 1.12 A scheme X is *integral* if it is both irreducible and reduced.

In particular $\operatorname{Spec} A$ is integral if and only if A is a domain (indeed irreducible corresponds to the fact that the nilradical $\bigcap_{\wp \in \operatorname{Spec} A} \wp$ is prime).

For example the affine space and the projective space over a field are integral. If X is an integral scheme, there is a unique *generic point* η which is dense in X (take $U = \operatorname{Spec} A$ an affine open subset of X, then A is a domain and one takes for η the point corresponding to the prime ideal $(0) \in \operatorname{Spec} A$). The local ring of X at η is a field (the quotient field of A), the *function field* of X. For example the function field of $\operatorname{Spec} \mathbf{Z}$ is $\mathbf{Q}$, the function field of $\mathbf{A}_k^n$ or $\mathbf{P}_k^n$ is $k(T_1, ..., T_n)$.

There is another characterisation of integral schemes :

Proposition 1.13 *A scheme X is integral if and only if for any open subset U of X, the ring $\mathcal{O}_X(U)$ is a domain.*

Proof : Suppose that $\mathcal{O}_X(U)$ is a domain. Then it has no nilpotent elements, so X is reduced. If U_1 and U_2 are two disjoint open subsets, then $\mathcal{O}_X(U) = \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2)$ (by definition of a sheaf), hence $\mathcal{O}_X(U_1)$ or $\mathcal{O}_X(U_2)$ must be zero (else $\mathcal{O}_X(U)$ would not be a domain), which implies that U_1 or U_2 is empty.

Conversely, assume that X is irreducible and reduced. Let U be an open subset, and take $f, g \in \mathcal{O}_X(U)$ such that $fg = 0$; we have to prove that either f or g is zero. Denote by Y the set of points $x \in U$ such that the restriction f_x to $\mathcal{O}_{X,x}$ belongs to the maximal ideal $\mathcal{M}_x$ (or equivalently : such that the evaluation $f(x) \in k(x)$ is zero). Let us show that Y is a closed subset of U. If $U = \operatorname{Spec} A$ is affine, this is clear because in this case Y is just the open subset $D(f) \simeq \operatorname{Spec} A_f$ of U. In the general case, cover U with affine open subsets U_i and use $U - Y = \bigcup(U_i - Y)$.

Similarly $Z := \{x \in U, g_x \in \mathcal{M}_x\}$ is a closed subset of U. We have $Y \cup Z = U$: indeed $fg = 0$, and the condition $f_x g_x = 0$ implies that either f_x or g_x belongs to $\mathcal{M}_x$. Since U is irreducible as an open subset of an irreducible space, we have for example $Y = U$. In particular, for any affine open subset $V = \operatorname{Spec} A$ of U, the restriction of f to $\mathcal{O}_X(V)$ is nilpotent (because f belongs to $\bigcap_{\wp \in \operatorname{Spec} A} \wp$, which is the nilradical of A), hence is zero (recall that X is reduced). Finally $f = 0$.

□

Recall that a ring A is *noetherian* if any ideal of A is finitely generated (or equivalently : if any non empty family of ideals has a maximal element, or any ascending chain of ideals is stationary). There is an analogue of this property for schemes :

Definition 1.14 A scheme X is

i) *quasi-compact* if every open cover of X has a finite subcover. [4]

ii) *locally noetherian* if X can be covered by open affine subsets $\operatorname{Spec} A_i$, where each A_i is a notherian ring.

iii) *noetherian* if it is both noetherian and quasi-compact.

Notice that any affine scheme $\operatorname{Spec} A$ is quasi-compact because $\operatorname{Spec} A$ is covered by affine subsets $D(f_i)$ if and only if 1 belongs to the ideal generated by the f_i, and in this case finitely many f_i will do. Thus a scheme is noetherian if and only if it can be covered by finitely many $\operatorname{Spec} A_i$, with each A_i noetherian.

Proposition 1.15 *The scheme* $\operatorname{Spec} A$ *is noetherian if and only if the ring* A *is noetherian.*

In particular if X is locally noetherian, then any affine open subset $\operatorname{Spec} A$ of X satisfies: the ring A is noetherian.

Proof (sketch of) : The scheme $\operatorname{Spec} A$ can be covered by finitely many $\operatorname{Spec} A_i$, $i = 1, ..., r$, with each A_i noetherian. Refining the cover if necessary, we may assume that $A_i = A_{f_i}$, with $f_i \in A$. Now the result is a consequence of the following algebraic fact : if each A_{f_i} is noetherian and $1 \in (f_1, ..., f_r)$, then A is noetherian. See [H], II.3.2. for a proof of this.

□

Clearly a closed subscheme Y of a noetherian scheme X is noetherian (a closed subset of a quasi-compact topological space is quasi-compact, and X can be covered by affine subsets $\operatorname{Spec} A_i$, such that each $\operatorname{Spec} A_i \cap Y$ is isomorphic as a scheme to the spectrum of some quotient ring of A_i). The same statement for an open subscheme is not obvious, because an open subset of a quasi-compact space is not in general quasi-compact. It is nevertheless true :

Proposition 1.16 *Let* X *be a noetherian scheme. Then any open subscheme of* X *is noetherian.*

[4] Sometimes this is taken as the definition of a compact topological space, but usually the Hausdorff condition is required for this; moreover the good analogue of "compact" for a scheme of finite type over a field is "proper", not quasi-compact. See below.

Proof : Using the previous proposition, it is clear that U is locally noetherian. To prove that U is quasi-compact, we use

Lemma 1.17 *Let X be a noetherian scheme. Then the topological space X is noetherian.*

(A topological space is *noetherian* if any descending chain of closed subsets is stationary, or equivalently if any non empty family of closed subsets has a minimal element; notice that the converse of the statement of the lemma is false, e.g. for $\operatorname{Spec} A$ with $A=$ ring of integers of $\overline{\mathbf{Q}_p}$; then $\operatorname{Spec} A$ has two points but A is not noetherian).

Proof of the lemma : Covering X by finitely many affine open subsets, it is sufficient to show the result when X is the spectrum of a noetherian ring A. In this case a descending chain of closed subsets is of the form $V(I_1) \supset ... \supset V(I_n) \supset ....$ Replacing each I_n by its radical $\sqrt{I_n}$, we may assume that each I_n satisfies $\sqrt{I_n} = I_n$ (recall that the *radical* of an ideal I is the ideal consisting of those x such that some power of x belongs to I). Then the condition that the sequence $(V(I_n))$ is decreasing means exactly that the sequence (I_n) is increasing, hence is stationary because A is a noetherian ring.

□

Now we can prove the proposition. Obviously, an open subset U of the noetherian space X is a noetherian topological space as well. In particular it is quasi-compact : indeed any non empty family of open subsets of U has a maximal element; now if $(U_j)_{j \in J}$ is an open cover of U, then the family of the $\bigcup_{j \in F} U_j$ for F finite has a maximal element, which gives a finite subcover of U.

□

Corollary 1.18 *Let X be a notherian scheme. Then any open immersion $U \to X$ is of finite type.*

Proof : For any open affine subset $\operatorname{Spec} A$ of X, the open set $\operatorname{Spec} A \cap U$ is quasi-compact, hence can be covered by finitely many $D(f_i) = \operatorname{Spec} A_{f_i}$, and each A_{f_i} is a finitely generated A-algebra.

□

Corollary 1.19 *Let S be a scheme and let X, Y be two S-schemes. Assume that X is noetherian and of finite type over S. Then any S-morphism $f : X \to Y$ is of finite type.*

In particular any k-morphism between two affine or projective k-varieties is of finite type.

Proof : Since any open subset of X is noetherian (hence quasi-compact), it is sufficient to prove that f is locally of finite type. But this follows from the fact that if B is an A-algebra and C is a B-algebra, then the property that C is finitely generated over A implies that it is finitely generated over B.

□

Here is one last useful property of noetherian schemes (actually of noetherian topological spaces) :

Proposition 1.20 *Let X be a noetherian topological space. Then any closed subset Y can be written as a finite union $Y = \bigcup_{i=1}^{r} Y_i$, where each Y_i is irreducible and $Y_i \not\supset Y_j$ for $i \neq j$. The decomposition is unique up to permutation.*

One says that the Y_i are the *irreducible components* of Y. Caution : for a scheme X, there is in general no canonical closed subscheme structure on its irreducible components.

Proof : Existence:consider the set of closed subsets that do not admit such a decomposition. If this set is non empty, it has a minimal element Y. Then Y is not irreducible, and can be written $Y = Y_1 \cup Y_2$, where Y_1 and Y_2 are proper closed subsets. By minimality of Y, Y_1 and Y_2 have a decomposition into irreducible components, which contradicts the fact that Y has not. The condition $Y_i \not\supset Y_j$ for $i \neq j$ is obtained by removing some of the Y_i if necessary.

Unicity : if $Y = \bigcup_{i=1}^{r} Y_i = \bigcup_{i=1}^{s} Y_i'$, then $Y_1 = \bigcup_{i=1}^{s} (Y_i' \cap Y_1)$. Since Y_1 is irreducible, this means that Y_1 is one of the $(Y_i' \cap Y_1)$, namely $Y_1 \subset Y_i'$ for some i, for example $Y_1 \subset Y_1'$. Then by symmetry Y_1' is a subset of Y_l for some l, and the condition $Y_i \not\supset Y_j$ for $i \neq j$ implies that $l = 1$, that is $Y_1 = Y_1'$. One concludes by induction on r.

2. Dimension of a scheme

2.1. Definition, first properties

Definition 2.1 Let X be a scheme. The *dimension* of X (denoted $\dim X$) if the supremum (possibly $+\infty$) of all integers n such that there exists a chain

$$Y_0 \subset Y_1 \subset ... \subset Y_n$$

of distinct irreducible closed subsets of X.

Notice that $\dim X$ depends only on the topological space structure of X.

Proposition 2.2 *Let* $X = \operatorname{Spec} A$. *Then* $\dim X$ *is the Krull dimension* $\dim A$ *of* A.

Recall that the *Krull dimension* of a ring A is the supremum of $\operatorname{ht} \wp$ for $\wp \in \operatorname{Spec} A$, where the *height* $\operatorname{ht} \wp$ of a prime ideal $\wp$ is the supremum of all n such that there exists a chain

$$\wp_0 \subset \ldots \subset \wp_n = \wp$$

of distinct prime ideals of A. Also $\operatorname{ht} \wp = \dim A_\wp$.

Proof : An irreducible closed subset of $\operatorname{Spec} A$ is of the form $V(\wp)$ with $\wp$ prime. Now for ideals I, J equal to their radicals (e.g. prime ideals), the equality $V(I) \subset V(J)$ is equivalent to $I \supset J$, whence the result.

$\square$

Examples.

1. The dimension of $\mathbf{A}_k^n$ is n (this follows from the fact that for any noetherian ring A, $\dim A[X_1, \ldots, X_n] = n + \dim A$, see [M], chapter 5). The same is true for $\mathbf{P}_k^n$ (see theorem below).

2. The dimension of $\operatorname{Spec} \mathbf{Z}$ is 1 (likewise for any principal ideal domain, or even Dedekind domain).

3. The dimension of $\operatorname{Spec} k$ or $\operatorname{Spec} (k[\varepsilon])$ is zero for any field k.

4. Some rings of dimension 1 are not noetherian (take the ring of integers of $\overline{\mathbf{Q}_p}$), some noetherian rings are not of finite dimension (see Nagata's book "Local rings").

Definition 2.3 Let X be a scheme and Y an irreducible closed subset of X. The *codimension* $\operatorname{codim}(Y, X)$ of Y in X is the supremum of all integers n such that there exists a chain of distinct closed irreducible subsets

$$Y = Y_0 \subset \ldots \subset Y_n$$

For example the codimension of the irreducible subset $V(\wp)$ in $\operatorname{Spec} A$ is the dimension $\dim(A/\wp)$ of the domain $(A/\wp)$ ($\wp \in \operatorname{Spec} A$).

Caution. In general the equality $\dim Y + \operatorname{codim}(Y, X) = \dim X$ does not hold, even if X is an integral affine scheme. Take $X = \operatorname{Spec} A$ where $A = R[u]$ and $R = k[[t]]$. Then the prime ideal $\wp = (tu - 1)$ of A satisfies $\operatorname{ht} \wp = 1$, but $A/\wp \simeq R[1/t]$ is a field, hence is of dimension zero. Nevertheless $\dim A = \dim R + 1 = 2$.

Similarly the dimension of a dense open subset of $X = \operatorname{Spec} A$ might be strictly smaller than $\dim X$: take $A = k[[t]]$, then $\dim(A[1/t]) = 0$, thus the dimension of $D(t) \subset \operatorname{Spec} A$ is zero.

In the next subsection, we shall see that the situation is somewhat better for integral schemes of finite type over a field.

2.2. Dimension and schemes of finite type over a field

The main result is the following. It is a consequence of important (and difficult) results in commutative algebra.

Theorem 2.4 *Let X be an integral scheme of finite type over a field k, with function field K. Then*

1. $\dim X$ *is finite, equal to the transcendence degree* $\operatorname{trdeg}(K/k)$ *of K over k.*
2. *For any non empty open subset U of X,* $\dim X = \dim U$.
3. *For any closed point $P \in X$,* $\dim X = \dim \mathcal{O}_{X,P}$.

Proof : First of all, 2. follows from 1. because X and U have the same function field. To prove 1., we remark that if (U_i) is an open cover of X, then X and U_i have same function field and $\dim X = \sup_i(\dim U_i)$; thus it is sufficient to prove 1. when $X = \operatorname{Spec} A$ is affine, where A is a finitely generated k-algebra with quotient field K.

Now the formula $\dim A = \operatorname{trdeg}(K/k)$ is a classical result in commutative algebra (see [M], chapter 5). It is an easy consequence of *Noether's normalisation lemma* : there exists $y_1, ..., y_r \in A$, algebraically independent over k, such that A is a finite module over $k[y_1, ..., y_r]$.

To prove 3., one may assume (using 2.) that X is affine. then the result follows from the formula

$$\dim(A/\wp) + \operatorname{ht} \wp = \dim A$$

which holds for any finitely generated k-algebra A and any prime ideal $\wp$ of A (this is another consequence of Noether's normalisation lemma).

$\square$

Now we would like to be able to compute the dimension of a scheme of finite type over a field k without the assumption that it is integral. First af all, note that if X is any scheme, there is a reduced scheme X_{red}, equipped with a closed immersion $X_{\text{red}} \to X$, and with same topological space as X : for $X = \operatorname{Spec} A$, one just takes $X_{\text{red}} = \operatorname{Spec}(A_{\text{red}})$, where A_{red} is the quotient of A by its nilradical. The general case follows easily (see [H], II.3) fore more details.

If X is any scheme of finite type over a field k, write $X = \bigcup_{i=1}^{r} Y_i$ the decomposition of X into irreducible closed subsets, and give Y_i its structure of reduced scheme (starting from an arbitrary closed subscheme structure on Y_i). Then the dimension of each Y_i (which does not depend on its scheme structure) can be computed using the formula with the transcendance degree, because each Y_i can now be considered as an integral scheme. Then we have

$$\dim X = \sup_{1 \le i \le r} \dim Y_i$$

Indeed any closed irreducible subset Y of X satisfies $Y = \bigcup_{1 \le i \le r} (Y \cap Y_i)$, hence $Y \cap Y_i = Y$ for some i, that is $Y \subset Y_i$; therefore any descending chain of irreducible closed subsets of X is contained in some Y_i.

Here is another consequence of this principle, which extends Theorem 2.4 to non necessarily integral schemes. Let us say that a noetherian scheme is *pure* if each irreducible component of Y has the same dimension.

Proposition 2.5 *Let X be a scheme of finite type over a field k. Then*

1. *For a non empty open subset U, we have* $\dim U = \dim X$ *if U is dense or if X is pure.*

2. *If X is pure, any closed irreducible subset Y of X satisfies*

$$\dim Y + \operatorname{codim}(Y, X) = \dim X$$

Proof : 1. If X is irreducible, we may assume it is reduced (replacing it by X_{red} if necessary), hence integral and we apply Theorem 2.4. In general, let $X = \bigcup_{1 \le i \le r} Y_i$ be the decomposition of X into irreducible subsets. Then any non empty open subset U of X meets Y_i for some i. If X is pure, then $\dim X = \dim Y_i$ and $\dim U = \dim(U \cap Y_i)$ because $U \cap Y_i$ is a non empty open subset of the irreducible scheme Y_i (which is an integral scheme of finite type over k when equipped with its reduced structure). Now assume that U is dense (but X not necessarily pure). Then $U \cap Y_i \neq \emptyset$ for any $i = 1, ..., r$ because each Y_i contains a non empty open subset of X (the complement in X of the union of Y_j for $j \neq i$). Thus $\dim(U \cap Y_i) = \dim Y_i$ by the previous argument. Since $\dim U = \sup_{1 \leq i \leq r} \dim(U \cap Y_i)$ and $\dim X = \sup_{1 \leq i \leq r} \dim Y_i$, we are done.

2. Since Y is contained in some irreducible component of X and X is pure, we may assume X irreducible. Let U be an affine open subset of X containing some point of Y, then $\dim X = \dim U$ and $\dim Y = \dim(Y \cap U)$ by 1. Moreover $\operatorname{codim}(Y, X) = \operatorname{codim}(Y \cap U, X \cap U)$: indeed $Z \mapsto Z \cap U$ is a strictly increasing bijection between irreducible closed subsets of X containing Y and irreducible closed subsets of U containing $Y \cap U$ (if Y_1 contains strictly Y_2, with Y_1 and $Y_2 \supset Y$ irreducible, then $Y_1 \cap U$ is dense in Y_1, hence meets $Y_1 - Y_2$). Therefore we may assume $X = \operatorname{Spec} A$ affine and $Y = V(\wp)$ with $\wp \in \operatorname{Spec} A$. Now the formula follows from

$$\dim(A/\wp) + \operatorname{ht} \wp = \dim A$$

which holds for any k-algebra of finite type.

□

Remark : As we have seen, the codimension formula is false, even for an integral and affine scheme X, if we do not assume that X is of finite type over a field. On the other hand, the formula is clearly false if X is not pure (take the disjoint union of a line and a point in the affine plane).

One last result about schemes of finite type over a field :

Proposition 2.6 *Let X be a scheme of finite type over a field k. Then the closed points of X are dense.*

Again, this is false in general, e.g. $X = \operatorname{Spec}(k[[t]])$.

Proof : We can assume that X is integral (thanks to the decomposition of X into irreducible components). Let $U = \operatorname{Spec} A$ be an affine open subset of X. Then A has a maximal ideal, that is there exists a point $x \in U$ which is closed *in* U. Now by Theorem 2.4, we have $\dim \mathcal{O}_{X,P} = \dim \mathcal{O}_{U,P} = \dim U = \dim X$. This shows that x is closed in *any* open affine subset $\operatorname{Spec} B$ of X (thanks to the formula $\dim(B/\wp) + \dim B_\wp = \dim B$, applied to the prime ideal $\wp \in \operatorname{Spec} B$ corresponding to x). Therefore x is closed in X, and any non empty open subset of X has a closed point. Another approach consists of using the fact that a point is closed in X if and only if its residue field is a finite extension of k.

□

2.3. Morphisms and dimension

Here again, "intuitive" results are false in general. For example a morphism $f : Y \to X$ might be surjective with $\dim Y < \dim X$, e.g. $Y = \mathrm{Spec}\,(k((t)) \oplus k)$, $X = \mathrm{Spec}\,(k[[t]])$; then X is of dimension 1, Y is of dimension zero, but the morphism $Y \to X$ induced by the homomorphism $k[[t]] \to k((t)) \oplus k$, $f(t) \mapsto (f(t), f(0))$ is surjective. The situation is somewhat better in two cases: finite morphisms and morphisms between schemes of finite type over a field.

Theorem 2.7 *Let $f : Y \to X$ be a finite and surjective morphism of noetherian schemes. Then* $\dim X = \dim Y$.

The assumption "surjective" is of course necessary (for example a closed immersion is a finite morphism).

Proof : One reduces immediately to the case when X, Y are affine. Let $f : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ be a finite and surjective morphism, we have to show that $\dim A = \dim B$. We can suppose that $\mathrm{Spec}\, B$ and $\mathrm{Spec}\, A$ are reduced, replacing the homomorphism $i : A \to B$ by $A_{\mathrm{red}} \to B_{\mathrm{red}}$ (which is finite as well). Then the assumption that f is surjective implies that i is injective[5]: indeed any prime ideal $\wp$ of A which does not contain the kernel I of i is not in the image of f; and the assumption that A has no nilpotents implies that such a prime ideal exists if $I \neq 0$ because the intersection of all prime ideals of A is zero.

Now the result follows from the so-called "Cohen-Seidenberg Theorem" in commutative algebra ([M], chapters 2 and 5). Basically, it is related to the "going-up" theorem : if A is a subring of B with B/A finite, then for any pair of ideals $p_1 \subset p_2$ of A, and any ideal $\mathcal{P}_1$ of B lying over p_1, there is an ideal $\mathcal{P}_2 \supset \mathcal{P}_1$ of B lying over p_2.

$\square$

Here is another result coming from commutative algebra :

Theorem 2.8 *Let $f : Y \to X$ be a morphism of noetherian schemes. Let $y \in Y$ and $x = f(y)$. Let Y_x be the fibre of Y at x. Then*

$$\dim \mathcal{O}_{Y,y} \leq \dim \mathcal{O}_{X,x} + \dim_x Y_x$$

[5] One can check that for A reduced, i injective is equivalent to $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ *dominant*, that is the image of $\mathrm{Spec}\, B$ is dense in $\mathrm{Spec}\, A$)

($\dim_x Y_x$ means the dimension of the local ring of the fibre Y_x at x). A special case is when x, y are closed points of integral schemes of finite type over a field. Then the inequality means $\dim Y - \dim X \leq \dim Y_x$ ("the dimension of the fibre is at least the relative dimension"). We shall see later that the equality holds in Theorem 2.8 in one important case : when the morphism f is *flat*.

Proof : One reduces immediately to the affine case $Y = \operatorname{Spec} B$, $X = \operatorname{Spec} A$; then it is the formula ([M], chapter 5) :

$$\dim B_\wp \leq \dim A_p + \dim(B_\wp \otimes_A k(p))$$

which holds for each prime ideal $\wp$ of B and its inverse image $p \in \operatorname{Spec} A$ (with residue field $k(p) := \operatorname{Frac}(A/p)$).

□

For schemes of finite type over a field, there is a more precise result :

Theorem 2.9 *Let $f : Y \to X$ be a dominant morphism of integral schemes of finite type over a field k. Set $e = \dim Y - \dim X$. Then there is a non empty open subset U of Y such that for any $x \in f(U)$, the dimension of the fibre U_x is e.*

Recall that *dominant* means that the image $f(Y)$ is dense in X, or equivalently that the generic point of Y is mapped to the generic point of X.

Proof : Shrinking Y and X if necessary, we may assume that $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$ are affine. Notice that the generic fibre Y_η is of finite type[6] over the function field K of X; it is an integral scheme with same function field L as Y. Since

$$\operatorname{trdeg}(L/k) = \operatorname{trdeg}(L/K) + \operatorname{trdeg}(K/k)$$

we obtain that $\dim Y_\eta = e$ by Theorem 2.4, that is L/K is of transcendance degree e. Let $t_1, ..., t_e$ be a transcendance base of L/K. Localising A and B if necessary, we can assume that $t_1, ..., t_e \in B$ and that B is a finite module over $A[t_1, ..., t_e]$ because L is a finite field extension of $K(t_1, ..., t_e)$. Set $X_1 = \operatorname{Spec}(A[t_1, ..., t_e])$, then the morphism f factorises through a finite morphism $Y \to X_1$. Now for $x \in X$, the fibre Y_x has a finite and surjective morphism to the fibre of $X_1 \to X$ at x; the latter is isomorphic to $\mathbf{A}^e_{k(x)}$, hence is of dimension e. We conclude with Theorem 2.7.

□

[6] One must be careful here: Y_η is not of finite type over k, because $\operatorname{Spec} K \to \operatorname{Spec} k$ is not a morphism of finite type.

3. Separated, proper, projective morphisms

The usual notions of separated (that is Hausdorff) topological space, and of compact topological space are not convenient for schemes: for example $\operatorname{Spec} A$ is almost never Hausdorff, but is always quasi-compact. Nevertheless, one would like to say that for example $\mathbf{A}^n_{\mathbf{C}}$ is separated but not compact because its set of complex points $\mathbf{C}^n$ is Hausdorf and not compact; similarly $\mathbf{P}^n_{\mathbf{C}}$ should be compact. That's the motivation for the following definitions. Since we might have to deal with quite general schemes, it is useful to define the notions of "separated" and "proper" in a relative context, that is to define them for morphisms, not for schemes.

3.1. Separated morphisms

Let $f : X \to Y$ be a morphism of schemes. The *diagonal morphism* is the morphism $\Delta : X \to X \times_Y X$ which induces the identity map on both components (the fibred product is relative to the morphism f).

Definition 3.1 *The morphism f is said to be separated if Δ is a closed immersion.*

Notice that it is a relative notion (any scheme is separated over itself !).

Proposition 3.2 *Any morphism $f : X \to Y$ of affine schemes is separated.*

Proof : For $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$, the diagonal morphism comes from the homomorphism $B \otimes_A B \to B$, $b \otimes b' \mapsto bb'$, which is surjective.

$\square$

A non separated morphism is somewhat pathological. To have an example, we need the notion of *glueing* of two schemes. Let X_1, X_2 be two schemes, U_1, U_2 two open subsets (resp. of X_1, X_2), and assume that there is an isomorphism of schemes $i : U_1 \to U_2$. Then we define the scheme X obtained by *glueing X_1 and X_2 along U_1 and U_2* as follows. As a topolgical space, X is the quotient of the disjoint union $X_1 \coprod X_2$ by the equivalence relation "$x_1 \sim i(x_1)$" (x_1 arbitrary in U_1), whence maps $j_1 : X_1 \to X$ and $j_2 : X_2 \to X$, such that a set V is open in X iff $j_1^{-1}(V)$, $j_2^{-1}(V)$ are resp. open subsets of X_1, X_2. Now the sheaf $\mathcal{O}_X$ is defined as : $\mathcal{O}_X(V)$ consists of the pairs (s_1, s_2) with $s_1 \in \mathcal{O}_{X_1}(j_1^{-1}(V))$, $s_2 \in \mathcal{O}_{X_2}(j_2^{-1}(V))$, such that s_1 and s_2 "coincide on $U_1 = U_2$", that is the image of the restriction of s_1 to $j_1^{-1}(V) \cap U_1$ by the isomorphism i is the restriction of s_2 to $j_2^{-1}(V) \cap U_2$.

Now take $X_1 = X_2 = \mathbf{A}^1_k$, $U_1 = U_2 = \mathbf{A}^1_k - \{P\}$, where P is the closed point corresponding to the origin. Then X is "the affine line with two origins".[7] It is not separated over k because $X \times_k X$ is the plane with both axes doubled and four origins; the image of Δ is the usual diagonal, which is not closed because it contains only two of the four origins.

Remark : If the image of the diagonal morphism is closed, then the morphism is separated; indeed in this case Δ induces a bicontinuous map from X to a closed subset of $X \times_Y X$, and the condition about the surjectivity of the associated map of sheaves holds thanks to Proposition 3.2.

Roughly speaking, separated means that there should be some "unicity of the limit", that is if Z is a scheme and $z \in Z$, a morphism f from $Z - \{z\}$ to a separated scheme X should have *at most* one extension to Z (at least if the local ring $\mathcal{O}_{Z,z}$ is "reasonable", for example if Z is a nonsingular curve). The formalisation of this idea is the following theorem:

Theorem 3.3 (Valuative criterion) *Let $f : X \to Y$ be a morphism of schemes with X noetherian. Then f is separated if and only if the following condition holds. Let R be any valuation ring with quotient field K, then for any commutative diagram*

$$\begin{array}{ccc} \operatorname{Spec} K & \xrightarrow[g]{} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} R & \longrightarrow & Y \end{array}$$

(where i is the morphism induced by the inclusion $R \to K$) there is at most one extension $\operatorname{Spec} R \to X$ *of the morphism g making the diagram commutative.*

For example if $Y = \operatorname{Spec} k$ and C is a nonsingular integral curve over k, the ring $\mathcal{O}_{C,P}$ of C at any point P is a discrete valuation ring with quotient field K (the function field of C); any k-morphism from an open subset of C to a k-scheme X is defined at the generic point, hence yields a k-morphism $\operatorname{Spec} K \to X$. The theorem says that if X is separated over k, then this morphism has at most one extension to P.

The proof is highly technical, see [H], II.4.

As an easy Corollary, we obtain

[7] It should not be taken for "the affine line with one doubled point", which is the affine scheme Spec $(k[x,y]/(x^2, xy)$

Corollary 3.4 *Assume all schemes noetherian. Then*

1. *Open and closed immersions are separated.*

2. *A composition of two separated morphisms is separated.*

3. *A separated morphism remains separated after base extension.*

Definition 3.5 Let k be a field. A *k-variety* is a scheme, separated and of finite type over k.[8]

Here is another property looking like "unicity of the limit":

Proposition 3.6 *Let S be a scheme, X a reduced and noetherian S-scheme, Y a separated S-scheme. Let U be a dense open subset of X, and f, g two S-morphisms $X \to Y$ which coincide on U. Then $f = g$.*

Proof : Consider the morphism $h : X \to Y \times_S Y$ given by (f, g). Since f and g coincide on U, the image $f(U)$ of U is contained in the image $\Delta(Y)$ of the diagonal map $\Delta : Y \to Y \times_S Y$. The map f is continuous, which implies that the image $f(\overline{U})$ of the closure of U is a subset of $\overline{f(U)}$. But $\overline{f(U)}$ is contained in $\Delta(Y)$ ($\Delta(Y)$ being closed in $Y \times_S Y$ because Y is separated over S) and $f(\overline{U}) = f(X)$ (U is dense in X by assumption). This shows that the *set-theoretic* maps f and g agree on X. It remains to show that the corresponding $f^\#$ and $g^\#$ coincide as well.

Since this is a local question, we can assume that all relevant schemes are affine. Set $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$. Let φ, ψ the homomorphisms $A \to B$ resp. associated to f, g. Let $a \in A$, set $b = \varphi(a) - \psi(a)$. By assumption the restriction of b to U is zero; thus $U \subset V(bB)$, hence $V(bB) = \operatorname{Spec} B$ because $V(bB)$ is a closed and dense subset of $\operatorname{Spec} B$. Therefore b is a nilpotent element of B, that is $b = 0$ (X is reduced). Finally $\varphi = \psi$.

□

Notice this last statement about separated morphisms : if X is separated over an affine scheme S, and U, V are two affine open subsets of X, then $U \cap V$ is affine (it is easy to see that $U \cap V \to U \times_S V$ is a closed immersion).

[8]Some authors require further that a variety is integral.

3.2. Proper morphisms

Recall that a map between two topological spaces is *closed* if the image of any closed subset is a closed subset. Such is the case for example for a continuous map between (Hausdorff) compact topological spaces. This suggests the following definition.

Definition 3.7 A morphism $f : X \to Y$ is *proper* if it is separated, of finite type, and universally closed (the latter means that for any morphism $Y' \to Y$, the corresponding map $X \times_Y Y' \to Y'$ is closed).

For example, the affine line $\mathbf{A}^1_k$ is not proper over $\operatorname{Spec} k$. Indeed the projection $\mathbf{A}^1_k \times_k \mathbf{A}^1_k = \mathbf{A}^2_k \to \mathbf{A}^1_k$ is not closed: the image of the closed subset $xy = 1$ (the hyperbola) is $\mathbf{A}^1_k - \{0\}$, which is not closed (it is a dense open subset of the affine line). We shall see that *projective* k-varieties are proper over $\operatorname{Spec} k$.

Proposition 3.8 *Any finite morphism $f : X \to Y$ (with X noetherian) is proper.*

Proof : Cover X with affine subsets $U_i = \operatorname{Spec} A_i$, such that the inverse image $f^{-1}(U_i)$ is isomorphic to $\operatorname{Spec} B_i$, with B_i of finite type as an A_i-module. Using the valuative criterion and the fact that a morphism of affine schemes is separated, it is immediate that f is separated. Now for any homomorphism of rings $\varphi : A \to B$ with B finite over A, the corresponding morphism $g : \operatorname{Spec} B \to \operatorname{Spec} A$ is closed (if B is noetherian) thanks to the going-up theorem[9], which shows in particular that $g(V(\wp)) = V(\varphi^{-1}(\wp))$ for each prime ideal $\wp$ of B; thus the image of an irreducible closed subset is closed, and any closed subset is a finite union of irreducible closed subsets because B is noetherian. Applying this to each $\operatorname{Spec} B_i \to \operatorname{Spec} A_i$, we see that any finite morphism is closed. Since finite morphisms are stable under base extension, we are done.

□

As for separated morphisms, there is a valuative criterion.

Theorem 3.9 (Valuative criterion of properness) *Let $f : X \to Y$ be a morphism of finite type with X noetherian. Then f is proper if and only if the following condition holds. Let R be any valuation ring with quotient field K, then for any commutative diagram*

[9] φ is not necessarily injective, but one can just apply the Cohen-Seidenberg Theorem to the induced homomorphism $A/\ker\varphi \to B$.

$$\begin{array}{ccc} \operatorname{Spec} K & \xrightarrow[g]{} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} R & \longrightarrow & Y \end{array}$$

(where i is the morphism induced by the inclusion $R \to K$) there is a unique extension $\operatorname{Spec} R \to X$ *of the morphism g making the diagram commutative.*

For a projective variety X over a field k, the valuative criterion corresponds to the following intuitive fact: take R a valuation ring, K its quotient field, $\operatorname{Spec} K \to X$ a K-point. Then you can make it an R-point by "removing the denominators", because on $\mathbf{P}_k^n$, the K-points with homogeneous coordinates $(x_0, ..., x_n)$ and $(dx_0, ..., dx_n)$ are the same for any $d \neq 0$.

Here are some consequences of the valuative criterion of properness:

Corollary 3.10

1. *Closed immersions are proper.*
2. *The composition of two proper morphisms is proper.*
3. *Properness is stable by base extension.*
4. *If X is proper over S and Y is separated over S, then any S-morphism $X \to Y$ is proper (this will work for example to projective k-varieties).*

The most important class of proper morphisms is the class of projective morphisms; this is a relative version of projective varieties over a field.

3.3. Projective morphisms

For any scheme Y, set $\mathbf{P}_Y^n = \mathbf{P}_{\mathbf{Z}}^n \times_{\operatorname{Spec} \mathbf{Z}} Y$, where $\mathbf{P}_{\mathbf{Z}}^n = \operatorname{Proj}(\mathbf{Z}[x_0, ... x_n])$.

Definition 3.11 A morphism of schemes $f : X \to Y$ is *projective* if it has a factorisation $f = p \circ i$, where $i : X \to \mathbf{P}_Y^n$ is a closed immersion, and $p : \mathbf{P}_Y^n \to Y$ is the projection.

For example let A be a ring, and $S = A[x_0, ..., x_n]/I$ a graded ring, where I is some homogeneous ideal of $S' := A[x_0, ..., x_n]$. Then the natural morphism $\operatorname{Proj} S \to \operatorname{Spec} A$ is projective. Indeed the morphism $\operatorname{Proj} S \to \operatorname{Proj} S' = \mathbf{P}_A^n$ is a closed immersion.

Here is the main theorem about projective morphisms:

Theorem 3.12 *A projective morphism of noetherian schemes is proper.*

Proof : Properness is stable by base extension, and a closed immersion is proper, hence it is sufficient to prove the result for the projection $\pi : X = \mathbf{P}^n_{\mathbf{Z}} \to \operatorname{Spec} \mathbf{Z}$. Write X as the union of the open affine subsets $D_+(x_i) = \operatorname{Spec} \mathbf{Z}[x_0/x_i, ..., x_n/x_i]$. Each one is isomorphic to the affine space over $\mathbf{Z}$, so π is of finite type. We apply the valuative criterion: let R be a valuation ring with quotient field K, $v : K^* \to (G, +)$ the corresponding valuation. Consider a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} R & \longrightarrow & \operatorname{Spec} Z \end{array}$$

Let $\xi \in X$ the image of the point of $\operatorname{Spec} K$; we may assume that ξ belongs to every V_i (else we reduce to the case of $\mathbf{P}^{n-1}_{\mathbf{Z}}$ because $\mathbf{P}^n_{\mathbf{Z}} - V_i$ is isomorphic to $\mathbf{P}^{n-1}_{\mathbf{Z}}$). This means that each x_i/x_j is invertible in $\mathcal{O}_{X,\xi}$, hence has an image (still denoted x_i/x_j) in $k(\xi)^*$, where $k(\xi)$ is the residue field of ξ.

We have an inclusion of fields $i : k(\xi) \to K$ corresponding to the K-point ξ on X. Let $f_{ij} = i(x_i/x_j) \in K^*$, and set $g_i = v(f_{i0})$ for $i = 0, ..., n$. Denote by g_k the smallest g_i for the ordering in G. Then $v(f_{ik}) = g_i - g_k$ is non negative, hence $f_{ik} \in R$ for each i (we have removed the denominators, using homogeneity). Therefore the K-point ξ extends to an R-point via the homomorphism

$$\mathbf{Z}[x_0/x_k, ..., x_n/x_k] \to R$$

obtained by sending each x_i/x_k to f_{ik}, which gives a morphism $\operatorname{Spec} R \to V_k \subset X$. Unicity is obvious.

$\square$

The question of deciding whether a proper variety is projective is in general quite difficult. One should be aware that:

- Any proper curve is projective.
- Every non singular proper surface is projective, but there are counterexamples for singular surfaces.
- In dimension ≥ 3, there exist proper and non projective regular varieties (e.g. among the so-called toric varieties).

4. Quasi-coherent and coherent sheaves on schemes

The goal of this section is to extend the properties of the structural sheaf $\mathcal{O}_X$ of a scheme X to more general sheaves of modules. That's the same idea as generalising properties of rings to modules, especially finite type modules.

4.1. First definitions and properties

Definition 4.1 Let X be a scheme. An *$\mathcal{O}_X$-module* on X is a sheaf (of abelian groups) $\mathcal{F}$, such that for any subset U, the abelian group $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, with the obvious compatibility with the restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$ and $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$ (where $V \subset U$ is an inclusion of open subsets).

A *morphism of $\mathcal{O}_X$-modules* is a morphism $\mathcal{F} \to \mathcal{G}$ of sheaves, such that for each open set U the map $\mathcal{F}(U) \to \mathcal{G}(U)$ is a homomorphism of $\mathcal{O}_X(U)$-modules. The kernel, cokernel, and image of such a morphism still are $\mathcal{O}_X$-modules, likewise for the quotient of an $\mathcal{O}_X$-module by a sub-$\mathcal{O}_X$-module. A *sheaf of ideals* is a sub-$\mathcal{O}_X$-module of $\mathcal{O}_X$.

The *tensor product* $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ (or simply $\mathcal{F} \otimes \mathcal{G}$ if $\mathcal{O}_X$ is understood) is the sheaf associated to the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$. In particular its stalk at $P \in X$ is $\mathcal{F}_P \otimes_{\mathcal{O}_{X,P}} \mathcal{G}_{\mathcal{P}}$. An $\mathcal{O}_X$-module $\mathcal{F}$ is *free of rank r* if it is isomorphic to the direct sum of r copies of $\mathcal{O}_X$. It is *locally free of rank r* if X can be covered by open subset such that the restriction of $\mathcal{F}$ to each of these is free of rank r. An *invertible sheaf*[10] is a locally free sheaf of rank 1.

Definition 4.2 Let $f : X \to Y$ be a morphism of schemes. The *direct image* of an $\mathcal{O}_X$-module $\mathcal{F}$ is the sheaf $f_*\mathcal{F}$, the $\mathcal{O}_Y$-module structure being given thanks to the morphism of sheaves of rings $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$.
The *inverse image* of an $\mathcal{O}_Y$-module $\mathcal{G}$ is defined as $f^*\mathcal{G} = f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$. Here $f^{-1}\mathcal{G}$ is the sheaf associated to the presheaf $U \mapsto \lim_{V \supset f(U)} \mathcal{G}(V)$, and the canonical map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ comes from $f^\#$.

Now for a ring A and an A-module M, we define an $\mathcal{O}_X$-module $\widetilde{M}$ on $X = \operatorname{Spec} A$ just the same the structural sheaf $\mathcal{O}_X$ is defined : the ring A is simply replaced everywhere by the module M. In particular $\widetilde{M}_\wp = M_\wp =$

[10] One can check that $\mathcal{F}$ is invertible if and only if $\mathcal{F} \otimes \mathcal{G} = \mathcal{O}_X$ for some $\mathcal{O}_X$-module $\mathcal{G}$, whence the name.

$M \otimes_A A_\wp$ for each $\wp \in \operatorname{Spec} A$, and $\widetilde{M}(D(f)) = M_f = M \otimes_A A_f$ for each $f \in A$.

There are obvious formulas : $\widetilde{M \otimes_A N} = \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$, $\widetilde{M \oplus N} = \widetilde{M} \oplus \widetilde{N}$. If $f : \operatorname{Spec} B \to \operatorname{Spec} A$ is a morphism, then $f_*(\widetilde{N}) = (\widetilde{{}_A N})$ (where ${}_A N$ means N viewed as an A-module), $f^*(\widetilde{M}) = (\widetilde{M \otimes_A B})$.

As for going from affine schemes to arbitrary schemes, we extend this definition as follows:

Definition 4.3 Let X be a scheme. An $\mathcal{O}_X$-module $\mathcal{F}$ is *quasi-coherent* if X can be covered by open affine subsets $U_i = \operatorname{Spec} A_i$, such that for each i the restriction of $\mathcal{F}$ to U_i is isomorphic to $\widetilde{M_i}$ for some A_i-module M_i. When X is noetherian,[11] $\mathcal{F}$ is said to be *coherent* if each M_i can be taken of finite type over A_i.

For example, $\mathcal{O}_X$, and more generally every locally free sheaf of rank r, is coherent. If $i : Y \to X$ is a closed immersion, then $i_*\mathcal{O}_Y$ is a coherent $\mathcal{O}_X$-module.

For affine schemes, we have the following theorem :

Theorem 4.4 *Let A be a ring and $X = \operatorname{Spec} A$. Then a sheaf $\mathcal{F}$ is quasi-coherent on X if and only if it is isomorphic to $\widetilde{M}$ for some A-module M. If A is noetherian, then $\mathcal{F}$ is coherent if and only if M is of finite type over A. Namely $M \mapsto \widetilde{M}$ is an equivalence of categories between between A-modules (resp. finite type A-modules) and quasi-coherent sheaves (resp. coherent sheaves).*

For a detailed proof, see [H], II.5. The key-lemma is the following (it has several variants, which will be used in this section) :

Lemma 4.5 *Let $X = \operatorname{Spec} A$, $s \in \Gamma(X, \mathcal{F})$ a global section of a quasi-coherent sheaf $\mathcal{F}$. Then if the restriction of s to an open set $D(f)$ is zero, there exists $n > 0$ such that $f^n s = 0$. If $t \in \Gamma(D(f), \mathcal{F})$ is a section of $\mathcal{F}$ over $D(f)$, then $f^n t$ lifts to a global section of $\mathcal{F}$ over X for some $n > 0$.*

Notice that the inverse functor of $M \mapsto \widetilde{M}$ is $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$.

4.2. Direct and inverse images

Proposition 4.6 *Let $f : X \to Y$ be a morphism of schemes. If $\mathcal{G}$ is a quasi-coherent (resp. coherent) $\mathcal{O}_Y$-module, then $f^*\mathcal{G}$ is a quasi-coherent (resp. coherent) $\mathcal{O}_X$-module.*

[11] Our definition is not the good one for non noetherian schemes, because a submodule of a finite type A-module is not necessarily of finite type if the ring A is not noetherian.

Proof : The question is local on X and Y, so we we can assume $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$. Then by Theorem 4.4, we have $\mathcal{G} = \widetilde{M}$ for some A-module M (and M is of finite type if $\mathcal{G}$ is coherent). Since $f^*\mathcal{G} = \widetilde{M \otimes_A B}$, the result follows.

$\square$

For direct image, things are a bit more complicated. We start with a very easy statement :

Proposition 4.7 *Let X be a scheme. Then the kernel, cokernel, and image of a morphism of quasi-coherent sheaves is quasi-coherent. The same holds with coherent sheaves if X is noetherian. A direct sum (resp. finite direct sum) of quasi-coherent sheaves (resp. coherent sheaves) is coherent (resp. quasi-coherent).*

Proof : Again we can assume that $X = \operatorname{Spec} A$; then the result follows from the equivalence of categories described in Theorem 4.4, and (for the coherent case) from the property that any submodule of a finite type A-module is of finite type when the ring A is noetherian.

$\square$

Now we can show that the direct image of a quasi-coherent sheaf is (under very weak assumptions) quasi-coherent. Observe that for coherent, we cannot expect such a thing : the direct image of the structural sheaf by a morphism $f : \operatorname{Spec} B \to \operatorname{Spec} A$ is not coherent, unless B is finite over A (use the formula $f_*(\widetilde{B}) = \widetilde{({}_AB)}$. We shall see later that the situation is better for *projective* morphisms.

Theorem 4.8 *Let $f : X \to Y$ be a morphism of schemes, with X noetherian (or X quasi compact and f separated). Then the image $f_*\mathcal{F}$ of a quasi-coherent sheaf $\mathcal{F}$ is quasi-coherent. If f is finite and $\mathcal{F}$ coherent, then $f_*\mathcal{F}$ is coherent.*

Proof : The case when f is finite and $\mathcal{F}$ coherent is easy: indeed cover Y by affine subsets $U_i = \operatorname{Spec} A_i$, such that the inverse image $V_i = f^{-1}(U_i)$ is isomorphic to $\operatorname{Spec} B_i$ with B_i of finite type as an A_i-module. Then the restriction of $\mathcal{F}$ to each V_i is isomorphic to $\widetilde{M_i}$ for some finite type B_i-module M_i, hence the restriction of $f_*\mathcal{F}$ to each U_i is isomorphic to $\widetilde{N_i}$, where N_i is M_i viewed as an A_i-module. In particular N_i is of finite type over A_i because M_i/B_i and B_i/A_i are of finite type.

For the general case, we can assume that $Y = \operatorname{Spec} A$ is affine (but not that X is affine, because the inverse image of an affine subset of Y is not necessarily affine). Cover X with finitely many open affine sets U_i, then with the assumptions made each $U_i \cap U_j$ can again be covered by finitely many affine open subsets U_{ijk} (if f is separated, $U_i \cap U_j$ is already affine). Now by definition of a sheaf the following sequence is exact :

$$0 \to f_* \mathcal{F} \to \bigoplus_i f_*(\mathcal{F}_{|U_i}) \to \bigoplus_{i,j,k} f_*(\mathcal{F}_{|U_{ijk}})$$

(for V an open subset of X, the image of $s_{i_0} \in f_*(\mathcal{F}_{|U_{i_0}})(V)$ in $f_*(\mathcal{F}_{|U_{ijk}})(V)$ is defined as: the restriction of s_{i_0} if $i_0 = i$, the restriction of $-s_{i_0}$ if $i_0 = j$, zero if i_0 is neither i nor j; observe that this would not work if there were infinitely many U_{ijk}, because the map would not land to the direct sum).

Now the previous proposition shows that $f_* \mathcal{F}$ is quasi-coherent as the kernel of a morphism between two quasi-coherent sheaves (since the U_i and the U_{ijk} are affine, it is obvious that $f_*(\mathcal{F}_{|U_i})$ and $f_*(\mathcal{F}_{|U_{ijk}})$ are quasi-coherent).

□

4.3. Quasi-coherent sheaves and exact sequences

Usually an exact sequence of sheaves does not remain exact if one takes global sections. With quasi-coherent sheaves on an affine scheme, the situation is better.

Theorem 4.9 *Let $X = \operatorname{Spec} A$ and $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ an exact sequence of $\mathcal{O}_X$-modules, with $\mathcal{F}'$ quasi-coherent. Then the sequence*

$$0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'') \to 0$$

is exact

Proof (sketch of) : Only the surjectivity of the last map is not trivial. Let $s \in \Gamma(X, \mathcal{F}'')$. For each open subset $D(f)$, the restriction of s to $D(f)$ lifts to a section t of $\mathcal{F}$ over $D(f)$. Now a variant of the key-lemma 4.5 shows that for some $n > 0$, $f^n s$ lifts to a global section of $\mathcal{F}$. Doing this for $f_1, ..., f_r$ such that X is covered by the $D(f_i)$, we obtain global sections $t_1, ..., t_r$ and $n > 0$, such that the image of t_i in $\Gamma(X, \mathcal{F}'')$ is $f_i^n s$. Now $(f_1, ..., f_r) = A$, hence we can write $1 = \sum_{i=1}^r a_i f_i^n$ with $a_i \in A$, and $\sum_{i=1}^r a_i t_i$ lifts s.

□

4.4. Quasi-coherent sheaves on $\operatorname{Proj} S$

Like for $\operatorname{Spec} A$, there is a correspondance between quasi-coherent sheaves over $\operatorname{Proj} S$ and graded S-modules, but it is more complicated. In this section S ia a graded ring such that S *is finitely generated by* S_1 *as an* S_0*-algebra* (typically S is a quotient of the graded ring $A[X_0, ..., X_n]$, where $A = S_0$).

For any graded S-module M, we define an $\mathcal{O}_X$-module $\widetilde{M}$ the same way the structural sheaf of $\operatorname{Proj} S$ is defined (replacing S by M everywhere). In particular $\widetilde{M}_\wp = M_{(\wp)}$ and $(\widetilde{M})_{D_+(f)} = \widetilde{M_{(f)}}$; thus $\widetilde{M}$ is quasi-coherent, and coherent for S noetherian and M of finite type over S.

Definition 4.10 Let S be as above $X = \operatorname{Proj} S$. For any $n \in \mathbf{Z}$, define $\mathcal{O}_X(n) = \widetilde{S(n)}$, where $S(n)$ denotes the graded module S with dimension shifted: $S(n)_i = S_{n+i}$. The sheaf $\mathcal{O}_X(1)$ is the *twisting sheaf of Serre*. If $\mathcal{F}$ is any $\mathcal{O}_X$-module, we set $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

Proposition 4.11 *The sheaf* $\mathcal{O}_X(n)$ *is invertible. For any graded* S*-module* M*, we have* $\widetilde{M}(n) = \widetilde{M(n)}$*, e.g.* $\mathcal{O}_X(m) \otimes \mathcal{O}_X(n) = \mathcal{O}_X(m+n)$*.*

Proof : Since S is generated by S_1 as an S_0-algebra, it is sufficient to show that the restriction of $\mathcal{O}_X(n)$ to each $D_+(f)$ *with* f *of degree 1* is invertible. This follows from the fact that $S(n)_{(f)}$ is isomorphic to $S_{(f)}$ via $s \mapsto f^{-n}s$. Similarly the second statement is a consequence of $(M \otimes_S N)_{(f)} = M_{(f)} \otimes_{S_{(f)}} N_{(f)}$.

□

Now we would like to recover S from $X = \operatorname{Proj} S$. Unlike the affine case, taking $\Gamma(X, \mathcal{O}_X)$ usually does not work because there are to few global sections (ex. $X = \mathbf{P}^n_k$, all global sections are constant). Whence the following construction:

Definition 4.12 Let S be as above, $X = \operatorname{Proj} S$. For each $\mathcal{O}_X$-modules $\mathcal{F}$, define the *graded* S*-module*

$$\Gamma_*(\mathcal{F}) = \bigoplus_{n \in \mathbf{Z}} \Gamma(X, \mathcal{F}(n))$$

(notice that if $s \in S_d$ and $t \in \Gamma(X, \mathcal{F}(n))$, then s is a global section of $\mathcal{O}_X(d)$, hence $s.t$ makes sense because $\mathcal{O}_X(d) \otimes \mathcal{F}(n) = \mathcal{F}(n+d)$).

Theorem 4.13 *Let* $X = \operatorname{Proj} S$*. Then for any coherent sheaf on* X*, the sheaf* $\widetilde{\Gamma_*(\mathcal{F})}$ *and* $\mathcal{F}$ *are isomorphic.*

For the proof, see [H], II.5. It uses a lemma similar to the key-lemma 4.5. The isomorphism is defined as follows: over $D_+(f)$ (with $\deg f = 1$), a section of $\widetilde{\Gamma_*(\mathcal{F})}$ is of the form m/f^d with $m \in \Gamma(X, \mathcal{F}(d))$. Viewing f^{-d} as a section of $\mathcal{O}_X(-d)$, $m.f^{-d}$ is a section of $\mathcal{F}(d) \otimes \mathcal{O}_X(-d) = \mathcal{F}$.

Corollary 4.14 *Let A be a ring. Then if Y is a closed subscheme of $\mathbf{P}^r_A$, there is a homogeneous ideal I of $S = A[X_0, ..., X_r]$ such that Y is the closed subscheme* $\mathrm{Proj}\,(S/I) \hookrightarrow \mathrm{Proj}\,S$.

Proof : Let i be a closed immersion $Y \to \mathbf{P}^r_A$ corresponding to Y, and $\mathcal{F} := \ker i^\# : \mathcal{O}_X \to i_*\mathcal{O}_Y$. Then $\mathcal{F}$ is a quasi-coherent subsheaf of $\mathcal{O}_X$ (using Theorem 4.8). By Theorem 4.13, we have $\mathcal{F} = \widetilde{I}$, where $I = \Gamma_*(\mathcal{F})$, and $\Gamma_*(\mathcal{F}) \subset \Gamma_*(\mathcal{O}_X)$. But in this special case, it is easy to check that $\Gamma_*(\mathcal{O}_X) = S$. Therefore I is a homogeneous ideal of S and $Y \simeq \mathrm{Proj}\,(S/I)$.

□

4.5. Sheaves of modules over $\mathbf{P}^r_A$

Definition 4.15 Let X be a scheme and $\mathcal{F}$ an $\mathcal{O}_X$-module. $\mathcal{F}$ is sait to be *generated by its global sections* if there exists a family $(s_i)_{i \in i}$ of global sections such that for each $x \in X$, the stalk $\mathcal{F}_x$ is generated as an $\mathcal{O}_{X,x}$-module by the images of s_i.

For example, a quasi-coherent sheaf on an *affine* scheme is generated by its global sections. The sheaf $\mathcal{O}_X(1)$ on $X = \mathrm{Proj}\,S$ is generated by its global sections (but not $\mathcal{O}_X$ in general, e.g. $X = \mathbf{P}^n_k$) for S a graded ring generated by S_1 as an S_0-algebra.

Theorem 4.16 *Let X be a projective scheme over a noetherian ring A. Denote by $\mathcal{O}_X(1)$ the inverse image of the sheaf $\mathcal{O}(1)$ on $\mathbf{P}^r_A$ (for some closed immersion $i : X \hookrightarrow \mathbf{P}^r_A$). Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then for n sufficiently large, the sheaf $\mathcal{F}(n) = \mathcal{F} \otimes \mathcal{O}_X(n)$ can be generated by a finite number of global sections.*

Proof : Since i is a finite morphism, the sheaf $i_*\mathcal{F}$ is coherent on $\mathbf{P}^r_A$. By Corollary 4.14, X is the closed subscheme of $\mathbf{P}^r_A$ given by some homogeneous ideal I of $S = A[X_0, ..., X_r]$. In particular $i_*(\mathcal{F}(n)) = (i_*\mathcal{F})(n)$ and we reduce to the case $X = \mathbf{P}^r_A$ because the global sections of $\mathcal{F}(n)$ and $i_*(\mathcal{F}(n))$ are by definition the same.

Cover X with $D_+(x_i)$, the restriction of $\mathcal{F}$ to each affine open set $D_+(x_i)$ can be written $\widetilde{M_i}$, with M_i of finite type over $B_i = A[x_0/x_i, ...x_r/x_i]$. Take generators s_{ij} for each M_i. Then a variant of the key-lemma 4.5 shows that for some $n > 0$, every $x_i^n s_{ij}$ extends to a global section t_{ij} of $\mathcal{F}(n)$. Now the restriction of $\mathcal{F}(n)$ to $D_+(x_i)$ is isomorphic to $\widetilde{M_i'}$, with M_i isomorphic to M_i' via multiplication by x_i^n. Therefore the global sections t_{ij} generate $\mathcal{F}(n)$.

□

Corollary 4.17 *Let X be a projective scheme over a noetherian ring A. Then any coherent sheaf $\mathcal{F}$ is a quotient of a sheaf $\mathcal{E} = \bigoplus_{i=1}^r \mathcal{O}_X(n_i)$, $n_i \in \mathbf{Z}$.*

Proof : By the theorem, $\mathcal{F}(n)$ is a quotient of such a sheaf for some n. Then take the tensor product with $\mathcal{O}_X(-n)$.

□

The previous Theorem has the following generalisation

Theorem 4.18 *Let X be a projective scheme over a Nagata ring A. Let $\mathcal{F}$ be a coherent sheaf on X. Then $\Gamma(X, \mathcal{F})$ is a finite type A-module.*

A noetherian ring A is *Nagata* (or *universally japanese*) if for any $\wp \in \operatorname{Spec} A$ and any finite extension L of the quotient field of $A/\wp$, the integral closure of $A/\wp$ in L is a finite $(A/\wp)$-module. For example any finitely generated k-algebra over a field k is Nagata.

There are essentially three steps in the proof (see [H], II.5). Using Theorem 4.16, one reduces to showing that if M is of finite type over A, then $\Gamma(X, \widetilde{M})$ is of finite type over A. Then one reduces to showing this when $S_0 = A$ is a (Nagata) domain, S is a domain, and $M = S(n)$ with $n \geq 0$; this step uses the filtration of M by modules with successive quotients of the type $(S/\wp)(n)$, $\wp \in \operatorname{Proj} S$. The last step consists of showing that $S' = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{O}_X(n))$ is integral over S, which implies that S'_n is integral over A, hence finite with the assumption that A is Nagata.

Corollary 4.19 *Let $f : X \to Y$ be a projective morphism of k-varieties. Let $\mathcal{F}$ be a coherent sheaf on X. Then $f_*\mathcal{F}$ is coherent.*

Proof : We can assume $Y = \operatorname{Spec} A$. Then $f_*\mathcal{F} = \widetilde{M}$ for some M (Theorem 4.8). By Theorem 4.18, $M = \Gamma(Y, f_*\mathcal{F}) = \Gamma(X, \mathcal{F})$ is of finite type over A.

□

As a Corollary, any projective morphism $f : X \to Y$ between two affine k-varieties is finite ($f_*\mathcal{O}_X$ being a coherent $\mathcal{O}_Y$-module). For example, only finite k-varieties are both affine and projective.

We end this section with two additional definitions.

Definition 4.20 Let $f : X \to Y$ be a morphism. An *immersion* $i : X \to \mathbf{P}^r_Y$ is a Y-morphism which induces an isomorphism from X to an open subscheme of a closed subscheme of $\mathbf{P}^r_Y$. An invertible sheaf $\mathcal{L}$ on X is *very ample* (relatively to Y) if $\mathcal{L} = i^*(\mathcal{O}(1))$ for some immersion i.

Thus f is projective if and only if: f is proper and there exists a very ample sheaf for X (relative to Y).

Definition 4.21 A k-variety X is *quasi-projective* if there exists an immersion $i : X \to \mathbf{P}^r_k$ for some r.

For example, affine and projective k-varieties, but also open subschemes of affine and projective k-varieties, are quasi-projective. It is not easy to construct non quasi-projective k-varieties (that's basically the same problem as finding proper but non-projective varieties).

5. Flat morphisms, smooth morphisms

The notion of flat morphism, albeit not very intuitive, gives usually the best picture for a "family of varieties".

5.1. Definition of a flat morphism

Let A be a ring. Recall that an A-module M is *flat* if the functor $N \mapsto M \otimes_A N$ is exact on A-modules. It is always right eaxct, so this simply means that for any injective homomorphism $N \to N'$, the associated homomorphism $M \otimes_A N \to M \otimes_A N'$ is injective.

Examples.

1. If $A = k$ is a field, any A-module is flat.
2. Any localisation $S^{-1}A$ is a flat A-module.
3. If A is a principal ideal domain, then M is flat if and only if it is torsion free.
4. If A is a local ring and M is of finite type, then M is flat if and only if it is free.

5. If B is an A-algebra, then B is flat if and only if $B_\wp$ is flat over A_p for every prime ideal $\wp \in \operatorname{Spec} B$ with inverse image p. In particular, if B is finite over A, B is flat if and only if it is locally free.[12]

6. If M is flat over B and B is a flat A-algebra, then M is flat over A.

7. Flatness is stable by base extension.

The "global" version of flatness is the following:

Definition 5.1 Let $f : X \to Y$ be a morphism of schemes. Let $x \in X$, $y = f(x)$. f is said to be *flat at* x if $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,y}$-module. The morphism f is *flat* if it is flat at any $x \in X$.

For example, an open immersion is flat, but not a closed immersion in general: indeed if A is a domain, a flat A-module M is torsion-free (multiplication by $a \neq 0$ is injective on A, hence on M if M is flat), so for example A/aA is not flat over A. The composition of two flat morphisms is flat, and flatness is stable by base change.

5.2. Flatness and dimension

The most important result about flatness is the "invariance of the dimension of the fibres of a flat morphism". Namely the inequality in Theorem 2.8 becomes an equality in the case of a flat morphism.

Theorem 5.2 *Let $f : X \to Y$ be a flat morphism of schemes of finite type over a field k. Let $x \in X$ and $y = f(x)$. Then*

$$\dim_x(X_y) = \dim_x X - \dim_y Y$$

(if s is a point of a scheme S, then $\dim_s S$ means the dimension of the local ring $\mathcal{O}_{S,s}$). This result is not specific to schemes of finite type over a field (see [M], Chapter 5), but in this special case we can give a direct proof, and the most convenient application of the Theorem is the next corollary (which does not hold for more general schemes).

[12] Considering the morphism $f : \operatorname{Spec} B \to \operatorname{Spec} A$, this simply means that the direct image of the structural sheaf is locally free. Using Nakayama's lemma, one can show that this is equivalent to: the A-module B is *projective*.

Proof : Set $Y' = \operatorname{Spec} \mathcal{O}_{Y,y}$. Making the base change with Y', we can assume that $Y = \operatorname{Spec} R$ with R local and y is the closed point of Y. Now the proof is by induction on $n = \dim Y$. For $n = 0$, the closed immersion $X_y \to X$ corresponds to nilpotent ideals, so $\dim_x X = \dim_x X_y$ and the formula holds.

If $\dim Y > 0$, making a base extension $Y_{\mathrm{red}} \to Y$ does not change anything, hence we can assume that Y is reduced. Let $t \in \mathcal{M}_y$ which is not a zero divisor.[13] Then its image in $\mathcal{O}_{X,x}$ is in $\mathcal{M}_x$, but is not a zero divisor by flatness. Set $Y' = \operatorname{Spec}(\mathcal{O}_{Y,y}/t)$, by the dimension formula

$$\dim A = \dim A_\wp + \operatorname{ht} \wp$$

(when A is a finitely generated k-algebra) and Krull's Hauptidealsatz, we have $\dim Y' = \dim Y - 1$. Similarly $\dim_x X' = \dim_x X - 1$, where $X' = X \times_Y Y'$. The result follows by induction, as flatness is preserved by base extension.

□

Corollary 5.3 *Let $f : X \to Y$ be a flat morphism of schemes of finite type over a field. Assume Y irreducible. Then the following are equivalent:*

1. *Every irreducible component of X has dimension* $\dim Y + n$.

2. *For any $y \in Y$, every irreducible component of X_y has dimension n.*

Proof : Assume 1. Let $y \in Y$, $Z \subset X_y$ an irreducible component, $x \in Z$ a closed point which is not in any other irreducible component of the fibre X_y. By the previous result, $\dim_x Z = \dim_x X - \dim_y Y$ (indeed the local ring of Z at x is $\mathcal{O}_{X,x}$). Let F be the closure of $\{x\}$ in X, G the closure of $\{y\}$ in Y. Since X is pure, $\dim_x X = \dim X - \dim F$ by Proposition 2.5 (since F has the generic point x, it is irreducible and of codimension $\dim \mathcal{O}_{X,x}$). Similarly $\dim_y Y = \dim Y - \dim G$. Now $\dim F = \dim G$ because the field extension $k(x)/k(y)$ is finite (x being closed in the fibre X_y, which is of finite type over $k(y)$), and $k(x)$, $k(y)$ are the respective function fields of F, G (with their reduced structure), so one can apply the formula with the transcendance degree (Theorem 2.4).

□

[13] The existence of t is not obvious; the point is that if $\wp_1, \ldots, \wp_r$ are the minimal prime ideals of the local ring R, the union of the $\wp_i$ is not the whole $\mathcal{M}_y$ by [M], pages 2–3. Now a t in $\mathcal{M}_y$ which is not in this union is not a zero divisor because the intersection of the $\wp_i$ is zero, R being reduced.

5.3. Flat and non flat morphisms

Except for the criteria coming directly for commutative algebra, [14] it is usually quite difficult to show that a morphism is flat. Here is one statement.

Theorem 5.4 *Let $f : X \to Y$ be a morphism of noetherian schemes, with Y integral, regular of dimension 1, and X reduced. If every irreducible component of X dominates Y, then f is flat.*

(a scheme Y is *regular* if all its local rings are regular. A noetherian local ring R with maximal ideal I is *regular* if the dimension of the (R/I)-vector space I/I^2 is $\dim R$. Intuitively, this means that the dimension of the tangent space is the right one.)

Proof : We give the proof only in the case when X, Y are both integral (it is usually sufficient for the applications). We may assume $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$, with f dominant. To show that $f : \operatorname{Spec} B \to \operatorname{Spec} A$ is flat, it is sufficient to show that $B_\wp$ is flat over A_p for each $\wp \in \operatorname{Spec} B$ with inverse image p. But f is dominant, hence $f^\# : A \to B$ is injective; therefore $B_\wp$ is a torsion-free A_p-module, so it is flat because A_p is a principal ideal domain (it is a discrete valuation ring: indeed it is regular of dimension 1).

□

Here are two counter-examples:

a) Take a noetherian domain A with quotient field K, such that A is not integrally closed in K. Let B the integral closure of A in K, e.g. $A = k[x,y]/(y^2 - x^3 - x^2)$. Then $\operatorname{Spec} B \to \operatorname{Spec} A$ is not flat. Indeed if it were, B would be locally free of rank one (at ordinary points $B_\wp = A_p$), but at the singular point $p = (x,y)$, the A_p-module $B_\wp$ is of rank 2. Here the problem comes from the fact that $Y = \operatorname{Spec} A$ is not regular.

b) Take $Y = \mathbf{A}^2_k$, and X the *blowing-up of Y at $P = (0,0)$*. Namely X is defined in $\mathbf{A}^2_k \times \mathbf{P}^1_k$ by the equations $x_i y_j = x_j y_i$, $i,j = 1,2$, where x_1, x_2 are the affine coordinates on $\mathbf{A}^2_k$ and y_1, y_2 the homogeneous coordinates on $\mathbf{P}^1_k$. Then $X \mapsto Y$ is not flat, because the dimension of the fiber at 0 is 1, and the other fibres have dimension 0.

5.4. Smooth morphisms

Smooth is the relative version of regular. In this subsection, we will just say a few words (without proofs) about this notion.

[14]For example a finite and surjective morphism between two regular schemes is flat by [M], Th. 46 in chapter 6.

Definition 5.5 A morphism $f : X \to Y$ of schemes of finite type over a field is *smooth* if

- f is flat
- For each point $y \in Y$, the fibre X_y is pure of dimension n, and *geometrically regular.*

Here geometrically regular means that $X_y \times_{k(y)} \overline{k(y)}$ is regular, where $\overline{k(y)}$ is the algebraic closure of $k(y)$.

For example, over a perfect field, a variety is smooth if and only if it is regular, but over $\mathbf{Z}/p\mathbf{Z}(t)$, the affine curve $y^2 = x^p - t$ is regular and not smooth (it becomes singular over $\mathbf{Z}/p\mathbf{Z}(t^{1/p})$). The affine and the projective space are smooth over $\operatorname{Spec} k$. An open immersion is smooth, and smoothness is preserved by base change and by composition.

Smooth is the analogue of "submersive" in differential geometry, as shows:

Theorem 5.6 *Let $f : X \to Y$ be a morphism of regular integral varieties over an algebraically closed field k, $n = \dim Y - \dim X$. Then f is smooth of relative dimension n if and only if for any closed point $x \in X$, the map between tangent spaces $T_x \to T_y$ is surjective.*

(Recall that T_x is the dual of the k-vector space $\mathcal{M}_x/\mathcal{M}_x^2$).

Finally, one result which is specific to the characteristic zero:

Theorem 5.7 *Let $f : X \to Y$ be a dominant morphism of integral schemes of finite type over a field of characteristic zero. Then $f : U \to Y$ is smooth for some non empty open subset U of X.*

(This fails in positive characteristic, consider the morphism $\mathbf{A}^1_k \to \mathbf{A}^1_k$ given by $f \mapsto f^p$ on $k[t]$ if the characteristic of k is p). For the proof of the two previous results, see [H], III.10.

Remark : Smoothness is an interesting property for arithmetic purposes: for example if X is an integral k-variety with $k = \mathbf{R}$ or $k = \mathbf{Q}_p$ (a p-adic field), the existence of a k-point on X implies the existence of a k-point on U for any non empty Zariski open subset U, *provided X is smooth.* This fails in general, ex. $x^2 + y^2 = 0$ over $\mathbf{R}$.

6. Divisors

So far if we want to show that two varieties are not isomorphic, we have essentially only one invariant, the dimension, which is rather coarse. In this section, we introduce a more subtle invariant, the *divisor class group*, and another related invariant, the *Picard group*. In many cases, these two invariants coincide.

6.1. Weil divisors

The groupe of Weil divisors is the easiest to define, but this construction works well only for schemes X satisfying the following property:

(*) X is noetherian, integral, separated, and regular in codimension 1.

regular in codimension 1 means that any local ring $\mathcal{O}_{X,x}$ of dimension 1 is regular (hence is a discrete valuation ring). For example this condition is satisfied if the integral scheme X is *normal*, that is all local rings of X are integrally closed in the function field of X.

In this subsection, we assume that X satisfies (*).

Definition 6.1 A *prime divisor* on X is a closed integral subscheme of codimension 1.[15] A *Weil divisor* is an element of the abelian free group $\operatorname{Div} X$ generated by the prime divisors.

Let K be the function field of X. For each prime divisor Y, there is an associated valuation $v_Y : K^* \to \mathbf{Z}$ (indeed the local ring of X at the generic point of Y is a discrete valuation ring by assumption). The *divisor* of a function $f \in K^*$ is the element $\operatorname{Div} f := \sum_Y v_Y(f)Y$. To see that this is well defined, one has to show the following

Lemma 6.2 *Let $f \in K^*$. Then $v_Y(f) = 0$ for all but finitely many prime divisors Y.*

Proof : Let $U = \operatorname{Spec} A$ be an affine open subscheme of X such that $f \in A$. Only finitely many prime divisors are disjoint from U (because X is noetherian). Thus one reduces to prove the statement for $X = \operatorname{Spec} A$ and $f \in A$. Now $v_Y(f) \geq 0$ in any case, and $v_Y(f) > 0$ implies that $Y \subset V(fA)$. But $f \neq 0$, hence $V(fA)$ is a strict closed subset of $\operatorname{Spec} A$; therefore it contains only finitely many irreducible closed subsets of codimension 1.

□

[15] Or equivalently, a closed irreducible subset of codimension 1, because there is only one reduced subscheme structure on such a subset.

Definition 6.3 A divisor D is *principal* if $D = \operatorname{Div} f$ for some $f \in K^*$. Two divisors D_1, D_2 are *linearly equivalent* if $D_1 - D_2$ is principal. The divisor class group $\operatorname{Cl} X$ of X is the quotient of $\operatorname{Div} X$ by the group $\operatorname{Div}_0 X$ of principal divisors.

This is a subtle invariant, in general quite difficult to compute. Here are some special cases:

Proposition 6.4 *Let A be an integrally closed domain, $X = \operatorname{Spec} A$. Then $\operatorname{Cl} X = 0$ if and only if A is a unit factorisation domain.*

For a proof, see [H], II.6. The basic fact is that A is the intersection of $A_\wp$ for $\wp$ prime of height 1 ("a function defined in codimension 1 is defined everywhere"). Notice that it is clear from the definition that for a Dedekind ring A, the group $\operatorname{Cl}(\operatorname{Spec} A)$ is the ideal class group, as it is defined in number theory.

Examples.

1. $\operatorname{Cl}(\mathbf{A}_k^n) = 0$.

2. $\operatorname{Cl}(\mathbf{P}_k^n) = \mathbf{Z}$, the group being generated by the class of the hyperplane $H : x_0 = 0$. To see that, the first remark is that any prime divisor is defined as $P(x_0, ..., x_n) = 0$, where P is one single homogeneous polynomial (this follows from Krull's Hauptidealsatz). Then one defines the degree of a prime divisor as the degree of the corresponding P, and for a divisor $D = \sum_Y n_Y Y$, the degree of D as $\deg D = \sum_Y n_Y \deg Y$. Finally one has to prove the two following classical facts: the degree of a principal divisor is zero, and any divisor of degree d is linearly equivalent to dH.

3. Similarly, $\operatorname{Cl}(\mathbf{P}_k^1 \times \mathbf{P}_k^1) = \mathbf{Z} \oplus \mathbf{Z}$. In particular, $\mathbf{P}_k^1 \times \mathbf{P}_k^1$ is not isomorphic to $\mathbf{P}_k^2$.

4. If E is an elliptic curve over an algebraically closed field k, then there is an exact sequence

$$0 \to E(k) \to \operatorname{Cl} E \to \mathbf{Z} \to 0$$

(the last map is given by the degree of a divisor $D = \sum_i n_i P_i$, that is $\sum_i n_i$; here the P_i are just closed points of E). Thus $\operatorname{Cl} E$ is quite big, and E cannot be isomorphic to $\mathbf{P}_k^1$.

6.2. Cartier divisors

This is a generalisation. Roughly speaking, a Cartier divisor is something which looks locally like the divisor of a function.

Let A be a ring. The *total quotient ring* $K(A)$ of A is the localisation of A with respect to the set of elements that are not divisors of zero. For any scheme X, one defines $\mathcal{K}$ as the sheaf associated to the presheaf $U \mapsto K(\mathcal{O}_X(U))$. Similarly, we have sheaves $\mathcal{K}^*$ and $\mathcal{O}^*$.

Definition 6.5 A *Cartier divisor* on X is a global section of the sheaf $\mathcal{K}^*/\mathcal{O}^*$. It is *principal* if it is in the image of the canonical map $\Gamma(X, \mathcal{K}^*) \to \Gamma(X, \mathcal{K}^*/\mathcal{O}^*)$. The quotient of the group of Cartier divisors by the principal one is denoted $\operatorname{CaCl} X$.

Thus giving a Cartier divisor is equivalent to give an open cover (U_i) of X, and elements $f_i \in \Gamma(U_i, \mathcal{K}^*)$, such that the restriction of f_i/f_j to $\Gamma(U_i \cap U_j, \mathcal{K}^*)$ belongs to $\Gamma(U_i \cap U_j, \mathcal{O}^*)$ for each i, j.

Theorem 6.6 *Assume that X is integral, separated, noetherian, and locally factorial. Then there is a 1-1 correspondance between the group* $\operatorname{Div} X$ *of Weil divisors and the group* $\Gamma(X, \mathcal{K}^*/\mathcal{O}^*)$ *of Cartier divisors. Moreover principal Weil divisors correspond to principal Cartier divisors. Thus the group* $\operatorname{Cl} X$ *is isomorphic to the group* $\operatorname{CaCl} X$.

A scheme is *locally factorial* if all its local rings are unit factorisation domains.

Proof : Since X is integral, $\mathcal{K}$ is just the constant sheaf K, where K is the field of functions of X. Let $D = (U_i, f_i)$ be a Cartier divisor; then $f_i \in K^*$. For each prime divisor Y, set $v_Y(D) = v_Y(f_i)$, where i is chosen such that U_i meets Y. This is well defined because f_i/f_j is invertible on $U_i \cap U_j$. Then $\sum_Y v_Y(D)$ is a Weil divisor (the sum is finite, X being noetherian).

Conversely, let D be a Weil divisor and $x \in X$. Then its restriction D_x to $\operatorname{Spec} \mathcal{O}_{X.x}$ is a Weil divisor, which has to be principal thanks to Proposition 6.4 ($\mathcal{O}_{X.x}$ is a UFD). Write $D_x = \operatorname{Div} f_x$, with $f_x \in K^*$. Then D and $\operatorname{Div} f_x$ coincide over an open neighborhood U_x of x, and it remains to cover X with the U_x to obtain a Cartier divisor : indeed on $U_x \cap U_{x'}$, $f_x/f_{x'}$ is invertible (it is invertible in codimension 1).

Clearly the two constructions are inverse from each other, and principal Cartier divisors correspond to principal Weil divisors.

6.3. Invertible sheaves

Let X be a scheme. Recall that an $\mathcal{O}_X$-module $\mathcal{L}$ is an *invertible sheaf* if it is locally free of rank 1. The tensor product $\mathcal{L} \otimes \mathcal{M}$ of two invertible sheaves is again an invertible sheaf, and for each invertible sheaf $\mathcal{L}$, the sheaf $\mathcal{M} = \mathcal{H}om(\mathcal{L}, \mathcal{O}_{\mathcal{X}})$ satisfies $\mathcal{L} \otimes \mathcal{M} \simeq \mathcal{O}_X$. Whence the following definition

Definition 6.7 The *Picard group* $\operatorname{Pic} X$ of X is the group of isomorphism classes of invertible sheaves with the composition law $\otimes$.

For example for any local ring R, we have $\operatorname{Pic} R = 0$. Let $D = (U_i, f_i)$ be a Cartier divisor on X. Then one defines an invertible sheaf $\mathcal{L}(D)$ as follows: $\mathcal{L}(D)$ is the sub-$\mathcal{O}_X$-module of $\mathcal{K}$ generated by f_i^{-1} on each U_i (this makes sense because f_i/f_j is invertible on $U_i \cap U_j$, hence f_i and f_i generate the same submodule).

Proposition 6.8 *1. $D \mapsto \mathcal{L}(D)$ is a bijection between Cartier divisors and the invertible subsheaves of $\mathcal{K}$.*

2. $\mathcal{L}(D_1 - D_2) \simeq \mathcal{L}(D_1) - \mathcal{L}(D_2)$.

3. $D_1 - D_2$ is principal if and only if the $\mathcal{O}_X$-modules $\mathcal{L}(D_1)$ and $\mathcal{L}(D_2)$ are isomorphic (as abstract sheaves, not as subsheaves of $\mathcal{K}$).

Proof : 1. Let $\mathcal{L}$ be an invertible subsheaf of $\mathcal{K}$ and (U_i) an open cover of X such that the restriction of $\mathcal{L}$ to each U_i is free of rank one. Then one just takes $D = (U_i, f_i)$, where f_i is the inverse of a generator of $\mathcal{L}$ on U_i.

2. This follows from $\mathcal{L}(D_1 - D_2) = \mathcal{L}(D_1).\mathcal{L}(D_2)^{-1}$. The latter is isomorphic (as an abstract sheaf) to $\mathcal{L}(D_1) \otimes \mathcal{L}(D_2)^{-1}$.

3. follows from 2.

□

Corollary 6.9 *The map $D \mapsto \mathcal{L}(D)$ induces an injective homomorphism $i : \operatorname{CaCl} X \to \operatorname{Pic} X$.*

Proposition 6.10 *If X is integral, then i is an isomorphism.*

Proof : In this case $\mathcal{K} = K$ is constant. If $\mathcal{L}$ is an invertible sheaf, then $\mathcal{L} \otimes \mathcal{K}$ is locally isomorphic to $\mathcal{K}$, hence isomorphic because X is irreducible. Thus $\mathcal{L}$ is a subsheaf of $\mathcal{L} \otimes \mathcal{K} \simeq \mathcal{K}$.

□

In particular, if X is noetherian, separated, integral and locally factorial, the three groups: $\operatorname{Cl} X$, $\operatorname{CaCl} X$, and $\operatorname{Pic} X$ coincide. For example on $\mathbf{P}^n_k$, any invertible sheaf is isomorphic to $\mathcal{O}(l)$ for some $l \in \mathbf{Z}$.