Sturm's theorem on the zeros of sums of eigenfunctions: Gelfand's strategy implemented (after P. Bérard and B. Helffer).

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In the second section "Courant-Gelfand theorem" of his last published paper: Topological properties of eigenoscillations in mathematical physics (2011), Arnold recounts Gelfand's strategy to prove that the zeros of any linear combination of the n first eigenfunctions of the Sturm-Liouville problem

$$-y''(s) + q(x)\,y(x) = \lambda\,y(x) \text{ in }]0,1[\,,$$

with

$$y(0) = y(1) = 0\,,$$

divide the interval into at most n connected components, and concludes that:

"The lack of a published formal text with a rigorous proof . . . is still distressing."

Inspired by Quantum mechanics, Gelfand's strategy consists in replacing the analysis of linear combinations of the n first eigenfunctions by that of their Slater determinant which is the first eigenfunction of the associated n-particle operator acting on Fermions.
We implement Gelfand's strategy, and give a complete proof of the above assertion. As a matter of fact, refining Gelfand's strategy, we prove a stronger property taking the multiplicity of zeros into account, a result which actually goes back to Sturm (1836).
This work has been done in collaboration with P. Bérard.

Introduction

On September 30, 1833, C. Sturm[1] presented a memoir on second order linear differential equations to the Paris Academy of Sciences. The main results are summarized in [24, 25], and were later published in the first volume of Liouville's journal (1836). In this talk, we shall consider the following particular case.

[1] Jacques Charles François STURM (1803–1855)

Sturm's Theorem (1836)

Let q be a smooth real valued function defined is a neighborhood of the interval $[0,1]$. The Dirichlet eigenvalue problem

$$\begin{cases} -y''(x) + q(x)\,y(x) = \lambda\,y(x) \text{ in }]0,1[\,, \\ y(0) = y(1) = 0\,, \end{cases} \tag{1}$$

has the following properties.

1. There exists an infinite sequence of (simple) eigenvalues

$$\lambda_1 < \lambda_2 < \cdots \nearrow \infty,$$

   with an associated orthonormal family of eigenfunctions $\{h_j, j \geq 1\}$.

2. For any $j \geq 1$, the eigenfunction h_j has exactly $(j-1)$ zeros in the interval $]0,1[$.

Extended Sturm's Property

3. Moreover, for any $1 \leq m \leq n$, any non trivial linear combination $U = \sum_{k=m}^{n} a_k \, h_k$ has the properties:

- 3a U has at most $(n-1)$ zeros in $]0,1[$, counted with multiplicities,
- 3b U changes sign at least $(m-1)$ times in $]0,1[$.

Sturm's motivations came from mathematical physics.
The novel point of view was to look for qualitative behavior of solutions rather than for explicit solutions.

To prove Assertions 1 and 2, he introduced the comparison and oscillation theorems which today bear his name.

Assertion 3 is less known but first appeared as a corollary of Sturm's investigation of the evolution of zeros of a solution $u(t, x)$ of the associated heat equation, with initial condition U, as times goes to infinity.

Remarks

- In the framework of Fourier series, Assertion 3b is often referred to as the Sturm-Hurwitz theorem. See Steinerberger for a quite recent quantitative version of this assertion.
- Sturm's theorem applies to more general operators, with more general boundary conditions (Fourier-Robin).

Courant's nodal theorem

R. Courant[2] partly generalized Assertion 2, in Sturm's theorem, to higher dimensions.

Courant's Theorem (1923)

Let $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \cdots \nearrow \infty$ be the Dirichlet eigenvalues of $-\Delta$ in a bounded domain of $\mathbb{R}^d$, listed in nondecreasing order, with multiplicities. Let u be any nontrivial eigenfunction associated with the eigenvalue λ_n, and let $\beta_0(u)$ denote the number of connected components of $\Omega \setminus u^{-1}(0)$ (*nodal domains*). Then,

$$\beta_0(u) \leq n\,.$$

[2] Richard COURANT (1888–1972).

A puzzling footnote

In a footnote of Courant-Hilbert 1953 (p. 454), Courant and Hilbert make the following statement.

CH-Statement

Any linear combination of the first n eigenfunctions divides the domain, by means of its nodes, into no more than n subdomains. See the Göttingen dissertation of H. Herrmann, Beiträge zur Theorie der Eigenwerten und Eigenfunktionen, 1932.

In the literature, CH-Statement is referred to as the "Courant-Herrmann theorem", "Courant-Herrmann conjecture", "Herrmann's theorem", or "Courant generalized theorem". In our recent works with P. Bérard, we call it the *Extended Courant property.*

Some story about Sturm and Liouville according to Lützen (1984)

In 1833 both Sturm and Liouville and their common friend Duhamel applied for the seat vacated by the death of Legendre. A fourth applicant was G. Libri-Carucci [...] On March 18th, Libri was elected with 37 votes against Duhamel 16 and Liouville 1. Nobody voted for Sturm. The next opportunity was offered after the death of Ampère in the summer of 1836. [...] Three weeks before the election [...] Liouville presented a paper to the Academy in which he praised Sturm's two memoires on the Sturm-Liouville theory as ranking with the best works of Lagrange. Supporting a rival in this way was rather unusual in the competitive Parisian academic circles, and it must have been shocking when on the day of the election, December 5th, Liouville and Duhamel withdrew their candidacies to secure the seat for their friend. Sturm was elected with an overwhelming majority.

Remarks

1. It can be shown (Pleijel (1956)) that the number $\beta_0(u_n)$ is asymptotically smaller than $\gamma(d)n$, where $\gamma(d) < 1$ for $d \geq 2$.
2. In dimension greater than or equal to 2, there is *no general lower bound* for $\beta_0(u_n)$, except the trivial ones. Examples were first given by A. Stern in her 1924 Göttingen thesis (see also H. Lewy (1977) and our recent papers with P. Bérard 2016-2017).
3. Nevertheless, there is a recent result of Steinerberger (ArXiv September 2018) where a metric Sturm-Liouville theory in two dimensions is proposed. More precisely, he got a lower bound of the length of the nodal set for a combination of eigenfunctions of the Laplacian corresponding to eigenvalues λ_j with $j \geq n$.

V. Arnold and R. Courant

In the early 1970's, V. Arnold[3] noticed that CH-Statement, would provide a partial answer to one of the problems formulated by D. Hilbert[4].

Citation from Arnold (2011)

I immediately deduced from the generalized Courant theorem [CH-Statement] new results in Hilbert's famous (16th) problem. . . . And then it turned out that the results of the topology of algebraic curves that I had derived from the generalized Courant theorem contradict the results of quantum field theory. . . . Hence, the statement of the generalized Courant theorem is not true (explicit counterexamples were soon produced by Viro). Courant died in 1972 and could not have known about this counterexample.

[3] Vladimir Igorevich ARNOLD (1937-2010).

[4] David HILBERT (1862–1943).

In the recent years 2017-2018, we (with P. Bérard) got many other counterexamples for the CH-statement (for example for the equilateral triangle, the cube and domains with cracks) (to appear in Documenta Mathematica).

V. Arnold and I. Gelfand

Arnold was very much intrigued by CH-Statement, as is illustrated by his last published paper, where he relates a discussion with I. Gelfand[5], which we transcribe below, using Arnold's words, in the form of an imaginary dialog.

(Gelfand) *I thought that, except for me, nobody paid attention to Courant's remarkable assertion. But I was so surprised that I delved into it and found a proof.*

(Arnold is quite surprised, but does not have time to mention the counterexamples before Gelfand continues.)

However, I could prove this theorem of Courant only for oscillations of one-dimensional media, where $m = 1$.

(Arnold) *Where could I read it?*

(Gelfand) *I never write proofs. I just discover new interesting things. Finding proofs (and writing articles) is up to my students.*

[5] Israel Moiseevich GELFAND (1913-2009).

Arnold then recounts Gelfand's strategy to prove CH-Statement in the *one-dimensional case*.

Quotations from Arnold

Nevertheless, the one-dimensional version of Courant's theorem is apparently valid. ... Gelfand's idea was to replace the analysis of the system of n eigenfunctions of the one-particle quantum-mechanical problem by the analysis of the first eigenfunction of the n-particle problem (considering as particles, fermions rather than bosons). ...

Unfortunately, [Gelfand's hints] do not yet provide a *proof* for this generalized theorem: many facts are still to be proved. ...

Gelfand did not publish anything concerning this: he only told me that he hoped his students would correct this drawback of his theory. ...

From Arnold continued

Viktor Borisovich Lidskii told me that "he knows how to prove all this". . . .

Although [Lidskii's] arguments look convincing, the lack of a published formal text with a proof of the Courant-Gelfand theorem is still distressing.

In a nice survey in 2015, Kuznetsov refers to CH-Statement as *Herrmann's theorem*, and relates that Gelfand's approach *so attracted Arnold that he included Herrmann's theorem for eigenfunctions of problem [(1)] together with Gelfand's hint into the 3rd Russian edition of his Ordinary Differential Equations*, see his Problem 9 in the "Supplementary problems" at the end of the book, which is formulated in the following way:

Arnold's Problem 9

The zeros of any linear combination of the n first eigenfunctions of the Sturm-Liouville problem (1) divide the interval into at most n connected components.

This statement is equivalent to saying that any linear combination of the n first eigenfunctions of (1) has at most $(n-1)$ zeros in the open interval. This is a weak form of Sturm's upper bound (Assertion 3a).

Our goal

Our goal is to implement Gelfand's strategy to solve Problem 9 *à la Gelfand*. Then we want to extend this strategy to take the multiplicities of zeros into account, and to prove Assertion 3a in Sturm's Theorem.
Inspired by Quantum mechanics, Gelfand's strategy consists in replacing the analysis of linear combinations of the n first eigenfunctions by that of their Slater determinant which is the first eigenfunction of the associated n-particle operator acting on Fermions.
Finally note that Assertion 3b can actually be deduced directly from Assertion 3a.

The Dirichlet Sturm-Liouville operator

We show how Gelfand's strategy can be applied to the general Dirichlet Sturm-Liouville problem. We consider the 1-particle operator

$$\mathfrak{h}^{(1)} := -\frac{d^2}{dx^2} + q(x)\,, \tag{2}$$

and, more precisely, its Dirichlet realization in $]0,1[$, i.e. the Dirichlet boundary value problem

$$\begin{cases} -\frac{d^2y}{dx^2} + q\,y = \lambda\,y\,, \\ y(0) = y(1) = 0\,. \end{cases} \tag{3}$$

Let $\{(\lambda_j, h_j), j \geq 1\}$ be the eigenpairs of $\mathfrak{h}^{(1)}$, and $\{h_j, j \geq 1\}$ an associated orthonormal basis of eigenfunctions.

We also consider the Dirichlet realization $\mathfrak{h}^{(n)}$ of the *n*-particle operator in $]0,1[^n$,

$$\mathfrak{h}^{(n)} := -\sum_{j=1}^{n} \left(\frac{\partial^2}{\partial x_j^2} + q(x_j)\right) = -\Delta + Q\,, \tag{4}$$

where $Q(x_1, \ldots, x_n) = q(x_1) + \cdots + q(x_n)$.

Denote by $\vec{k} = (k_1, \cdots, k_n)$ a vector with positive integer entries, and by $\vec{x} = (x_1, \cdots, x_n)$ a vector in $]0,1[^n$. The eigenpairs of $\mathfrak{h}^{(n)}$ are the $(\Lambda_{\vec{k}}, H_{\vec{k}})$, with

$$\begin{cases} \Lambda_{\vec{k}} = \lambda_{k_1} + \cdots + \lambda_{k_n}\,, \text{ and} \\ H_{\vec{k}}(\vec{x}) = h_{k_1}(x_1) \cdots h_{k_n}(x_n)\,, \end{cases} \tag{5}$$

where $H_{\vec{k}}$ is seen as a function in $L^2(]-1,+1[^n, dx)$ identified with $\widehat{\bigotimes} L^2(]-1,+1[, dx_j)$.

The symmetric group $\mathfrak{S}_n$ acts on $]-1,+1[^n$ by

$$\sigma(\vec{x}) = \left(x_{\sigma(1)}, \cdots, x_{\sigma(n)}\right).$$

It consequently acts on $L^2(]-1,+1[^n)$. A fundamental domain of the action of $\mathfrak{S}_n$ on $]-1,+1[^n$ is the n-simplex

$$\Omega_n := \{0 < x_1 < x_2 < \cdots < x_n < 1\}. \tag{6}$$

We introduce the Slater determinant $\mathfrak{s}_n$ defined by,

$$\mathfrak{s}_n(x_1, \ldots, x_n) = \begin{vmatrix} h_1(x_1) & h_1(x_2) & \ldots & h_1(x_n) \\ h_2(x_1) & h_2(x_2) & \ldots & h_2(x_n) \\ \vdots & \vdots & & \vdots \\ h_n(x_1) & h_n(x_2) & \ldots & h_n(x_n) \end{vmatrix}. \tag{7}$$

For $\overrightarrow{c} = (c_1, \dots, c_{n-1}) \in]-1, +1[^{n-1}$, we consider the function

$$x \mapsto \mathfrak{s}_n(c_1, \dots, c_{n-1}, x)\,.$$

Developing the determinant with respect to the last column, we see that this function is a linear combination of the functions $h_1, \dots, h_n$ we get...

$$S_{s(\overrightarrow{c})}(x) = \sum_{j=1}^{n} s_j(\overrightarrow{c})\, h_j(x) \tag{8}$$

where $s(\overrightarrow{c}) = \left(s_1(\overrightarrow{c}), \dots, s_n(\overrightarrow{c})\right)$, and

$$s_j(\overrightarrow{c}) = s_j(c_1, \dots, c_{n-1}) = (-1)^{n+j} \begin{vmatrix} h_1(c_1) & \dots & h_1(c_{n-1}) \\ \vdots & & \vdots \\ h_{j-1}(c_1) & \dots & h_{j-1}(c_{n-1}) \\ h_{j+1}(c_1) & \dots & h_{j+1}(c_{n-1}) \\ \vdots & & \vdots \\ h_n(c_1) & \dots & h_n(c_{n-1}) \end{vmatrix} \tag{9}$$

so that $s(\overrightarrow{c})$ is computed in terms of Slater determinants of size $(n-1) \times (n-1)$.

We now prove the weak Sturm's Statement using Gelfand's strategy. We first observe.

Lemma A

The function $\mathfrak{s}_n$ is not identically zero.

The proof relies on the fact that the functions h_j, $1 \leq j \leq n$ are linearly independent.

Lemma B

$\mathfrak{s}_n$ is the first Dirichlet eigenfunction of $\mathfrak{h}^{(n)}$ in Ω_n, with corresponding eigenvalue $\Lambda^{(n)} := \lambda_1 + \cdots + \lambda_n$.
In particular, $\mathfrak{s}_n$ does not vanish in Ω_n. One can choose the signs of the functions h_j, $1 \leq j \leq n$, such that $\mathfrak{s}_k$ is positive in Ω_k for $1 \leq k \leq n$.
As a consequence, for any $c_1 < \cdots < c_n$ in $]-1,+1[$, the vectors $\vec{h}(c_1), \ldots, \vec{h}(c_n)$, are linearly independent.

This Lemma was the first step in Gelfand's strategy.

Proof

It is standard that the (Fermionic) ground state energy of $\mathfrak{h}_F^{(n)}$ is $\Lambda^{(n)}$. The restriction of $\mathfrak{S}_{\Omega_n}$ to Ω_n satisfies the Dirichlet condition on $\partial\Omega_n$, and is (using Lemma A) an eigenfunction of $\mathfrak{h}_F^{(n)}$ corresponding to $\Lambda^{(n)}$. Suppose that $\mathfrak{S}_{\Omega_n}$ is not the ground state. Then, it has a nodal domain ω strictly included in Ω_n. Define the function U which is equal to $\mathfrak{S}_{\Omega_n}$ in ω, and to 0 elsewhere in Ω_n. It is clearly in $H_0^1(\Omega_n)$. Using $\mathfrak{S}_n$, extend the function U to a Fermi state U_F on $]-1,+1[^n$. Its energy is $\Lambda^{(n)}$ which is the bottom of the spectrum of $\mathfrak{h}_F^{(n)}$. It follows that U_F is an eigenfunction of $\mathfrak{h}_F^{(n)}$, and a fortiori of $\mathfrak{h}^{(n)}$. But U_F cannot vanish in an open set. The fact that one can choose the $\mathfrak{s}_n$ to be positive in Ω_n follows immediately.

Finally, if the vectors $\vec{h}(c_1),\ldots,\vec{h}(c_n)$ were linearly dependent, $\mathfrak{s}_n$ would vanish at $(c_1,\ldots,c_n)\in\Omega_n$, a contradiction. □

The following proposition provides a *weak form* of Sturm's upper bound on the number of zeros of a linear combination of eigenfunctions of (3) ("weak" in the sense that the multiplicities of zeros are not accounted for).

Proposition C

Let $\vec{b} \in \mathbb{R}^n$, with $\vec{b} \neq \vec{0}$. Then, the linear combination

$$S_{\vec{b}:} = \sum_j b_j\, h_j\,,$$

has a most $(n-1)$ distinct zeros in $]-1,+1[$.
If $S_{\vec{b}}$ has exactly $(n-1)$ zeros in $]-1,+1[$, $c_1 < \cdots < c_{n-1}$, then there exists $C \neq 0$ such that

$$S_{\vec{b}}(x) = C\,\mathfrak{s}_n(c_1,\ldots,c_{n-1},x)\,.$$

Furthermore, each zero c_j has order 1.

Comment

The second part of the statement corresponds to the second step of Gelfand's strategy according to Arnold but it was wrongly formulated in Arnold who says that one has to prove:
For any $\vec{b} \neq 0$*, there exists* $C \neq 0$ *such that*

$$S_{\vec{b}}(x) = C\, \mathfrak{s}_n(c_1, \dots, c_{n-1}, x)\,.$$

This statement is incorrect !

Proof

Given $\vec{b}$, assume that $S_{\vec{b}}$ has at least n distinct zeros $c_1 < \cdots < c_n$ in $]-1,+1[$. This means that the n components $b_j, 1 \leq j \leq n$, satisfy the system of n equations,

$$\begin{cases} b_1 h_1(c_1) + \cdots + b_n h_n(c_1) = 0, \\ \ldots \\ b_1 h_1(c_n) + \cdots + b_n h_n(c_n) = 0. \end{cases}$$

By Lemma B, the determinant of this system is positive, and hence the unique possible solution is $\vec{0}$. This proves the first assertion.

Assume that $S_{\vec{b}}$ has precisely $(n-1)$ distinct zeros, $c_1 < \cdots < c_{n-1}$, in $]-1,+1[$. By Lemma B, the vectors $\vec{h}(c_1), \ldots, \vec{h}(c_{n-1})$, are linearly independent. Then, $x \mapsto \mathfrak{s}_n(c_1, \ldots, c_{n-1}, x)$ can be written as the linear combination $S_{\vec{s}(\overrightarrow{c})}$, where the vector $\vec{s}(\overrightarrow{c})$ is given by (9). It follows that the vectors $\vec{b}$ and $\vec{s}(\overrightarrow{c})$ are both orthogonal to the family $\vec{h}(c_1), \ldots, \vec{h}(c_{n-1})$, and must therefore be proportional. This proves the second assertion.

Assume that $x \mapsto \mathfrak{s}_n(c_1, \dots, c_{n-1}, x)$ vanishes at order at least 2 at c_1. Then

$$\frac{d}{dx}\Big|_{x=c_1} \mathfrak{s}_n(x, c_1, c_2, \dots, c_{n-1}) = 0.$$

This implies that $\frac{\partial \mathfrak{s}_n}{\partial x_1}(c_1, c_1, c_2, \dots, c_{n-1}) = 0$, and hence that $\frac{\partial \mathfrak{s}_n}{\partial \nu}(c_1, c_1, c_2, \dots, c_{n-1})$, where ν is the unit normal to the boundary $\partial\Omega_n$, which contradicts Hopf's lemma. This proves the last assertion. □

For completeness, we state the following immediate corollary.

Corollary C1

Given $c_1 < \cdots < c_{n-1}$ in $]-1,+1[$, the function

$$x \mapsto \mathfrak{s}_n(c_1,\ldots,c_{n-1},x)\,,$$

vanishes exactly at order 1, changes sign at each c_j, and does not vanish elsewhere in $]-1,+1[$.

Local behaviour of $\mathfrak{s}_n$ near a zero, i.e. at the boundary of Ω_n.

Let $\vec{c} \in \partial\Omega_n$ be a boundary point, i.e.
$\vec{c} = (\bar{c}_1, \dots, \bar{c}_1, \; \dots \;, \bar{c}_p, \dots, \bar{c}_p)$,
where p is a positive integer, $\bar{c}_1 < \bar{c}_2 < \cdots < \bar{c}_p$, are in $]-1,+1[$, and $\vec{c}$ is such that $\bar{c}_j$ is repeated k_j times, with $k_1 + \cdots + k_p = n$.

We write $\vec{x} = \vec{c} + \vec{\xi}$, with $\vec{\xi}$ close to 0. The function $\mathfrak{s}_n$ is an eigenfunction of the operator $-\Delta + Q$, and vanishes at the point $\vec{c} \in]-1,+1[^n$. By Bers's theorem (1955), there exists a *harmonic* homogeneous polynomial $\widehat{P}_k$, of degree k, such that

$$\mathfrak{s}_n(\vec{c} + \vec{\xi}) = \widehat{P}_k(\vec{\xi}) + \omega_{k+1}(\vec{\xi}) \,, \tag{10}$$

where the function $\omega_{k+1}(\vec{\xi})$ is a function of $\vec{\xi}$, depending on $\vec{c}$, such that $\omega_{k+1}(t\vec{\xi}) = O(t^{k+1})$. Note that, for the time being, we have no a priori information on the degree k.

We relabel the coordinates of $\vec{\xi}$, according to above, and write this vector as

$$\vec{\xi} = \left(\xi^{(1)}, \ldots, \xi^{(p)}\right), \tag{11}$$

where $\xi^{(j)} = (\xi_{j,1}, \ldots, \xi_{j,k_j})$.

The permutation group $\mathfrak{S}_{k_j}$ acts by permuting the entries of $\xi^{(j)}$. Given $\sigma_j \in \mathfrak{S}_{k_j}, 1 \leq j \leq p$, we denote by $\sigma = (\sigma_1, \ldots, \sigma_p) \in \mathfrak{S}_{k_1} \times \cdots \times \mathfrak{S}_{k_p}$ the permutation in $\mathfrak{S}_n$ which permutes the entries of $\xi^{(j)}$ by σ_j.

VanderMonde Polynomials

We recall that the VanderMonde polynomial is defined by

$$\begin{cases} P_1(x_1) & = 1\,, \text{ and, for } n \geq 2, \\ P_n(x_1, \dots, x_n) & = \prod_{1 \leq i < j \leq n} (x_i - x_j)\,. \end{cases} \tag{12}$$

We note that P_n has degree $n(n-1)/2$.
Note also that the Slater determinant associated to the n first eigenfunctions of the harmonic oscillator has the form

$$P_n(x_1, \dots, x_n) \exp -\frac{|x|^2}{2}\,.$$

For the same vector $\vec{c}$, we look at the local behavior of the Vandermonde polynomial P_n, and get

$$P_n(\vec{c} + \vec{\xi}) = \rho_1(\vec{c})\, P_{k_1}(\xi^{(1)}) \cdots P_{k_p}(\xi^{(p)}) + \omega_\ell(\vec{\xi})\,, \tag{13}$$

with $\ell := 1 + \sum_{j=1}^{p} k_j(k_j - 1)/2$.

Lemma D

The polynomial $\widehat{P}_k$ given by (10) has the following properties.

1. For any permutation
   $\sigma = (\sigma_1, \dots, \sigma_p) \in \mathfrak{S}_{k_1} \times \cdots \times \mathfrak{S}_{k_p} \subset \mathfrak{S}_n$,

$$\widehat{P}_k(\sigma \cdot \vec{\xi}) = \varepsilon(\sigma)\, \widehat{P}_k(\vec{\xi})\,. \tag{14}$$

2. The zero set of $\widehat{P}_k$ is characterized by

$$\widehat{P}_k(\vec{\xi}) = 0 \Leftrightarrow \prod_{j=1}^{p} P_{k_j}(\xi^{(j)}) = 0\,. \tag{15}$$

3. There exists a nonzero constant $\rho(\vec{c}\,)$ such that

$$\widehat{P}_k(\vec{\xi}) = \rho(\vec{c})\, P_{k_1}(\xi^{(1)}) \dots P_{k_p}(\xi^{(p)})\,. \tag{16}$$

Continued

This means that $\widehat{P}_k$ has degree $k = \sum_j \frac{k_j(k_j-1)}{2}$, and that

$$\mathfrak{s}_n(\vec{c} + \vec{\xi}\,) = \rho(\vec{c}\,)\, P_{k_1}(\xi^{(1)}) \dots P_{k_p}(\xi^{(p)}) + \omega_{k+1}(\vec{\xi}\,)\,, \qquad (17)$$

where the function $\omega_{k+1}(\vec{\xi}\,)$ tends to zero like $\mathcal{O}(|\xi|^{k+1})$ when $\vec{\xi}$ tends to zero.

Assertion 3. Notice that the polynomials $\widehat{P}_k(\xi)$ and $\prod_{j=1}^{p} P_{k_j}(\xi^{(j)})$ are both harmonic and homogeneous, with the same zero set in a neighborhood of 0. We can then apply a division Lemma for harmonic functions, which implies that they divide each other, so that these polynomials must be proportional. The lemma is proved. $\square$

Division lemma (Murdoch (1964), Logunov-Malinnikova (2015))

Let P, Q be polynomials in $\mathbb{R}[X_1, \dots, X_n]$. Assume that Q is harmonic and homogenous. If the set of real zeros of Q is contained in the set of real zeros of P,

$$\{x \in \mathbb{R}^n \mid Q(x) = 0\} \subset \{x \in \mathbb{R}^n \mid P(x) = 0\},$$

then Q divides P, i.e. there exists R in $\mathbb{R}[X_1, \dots, X_n]$ such that $P = QR$.

Strong upper bound

We can now prove Assertion 3a in Sturm's Theorem, using Gelfand's strategy.

Proposition

Let $\vec{b} \in \mathbb{R}^n \setminus \{0\}$.
Call $\bar{c}_1 < \cdots < \bar{c}_p$ the zeros of the linear combination $S_{\vec{b}}$ of the first n eigenfunctions of problem (3). Call k_j the order of vanishing of $S_{\vec{b}}$ at $\bar{c}_j$.
Call $\vec{c}$ the vector $(\bar{c}_1, \ldots, \bar{c}_1, \ \ldots \ , \bar{c}_p, \ldots, \bar{c}_p)$, where $c_j, 1 \leq j \leq p$ is repeated k_j times.
Then, ...

Continued

1. $k_1 + \cdots + k_p \leq (n-1)$,
2. If $k_1 + \cdots + k_p = (n-1)$, then there exists a nonzero constant C such that

$$S_{\vec{b}} = C\, S_{\vec{s}(\vec{c})}\,,$$

where the linear combination $S_{\vec{s}(\vec{c})}$ is given by developing the determinant

$$\left| \vec{h}(c_1) \ldots \vec{h}^{(k_1-1)}(c_1)\ \ldots\ \vec{h}(c_p) \ldots \vec{h}^{(k_p-1)}(c_p) \vec{h}(x) \right|\,, \qquad (18)$$

and where $\vec{h}^{(m)}(a)$ is the vector $\left(h_1^{(m)}(a), \ldots, h_n^{(m)}(a)\right)$ of the m-th derivatives of the h_j's evaluated at the point a.

Conclusion and complementary discussion

The Gelfand's strategy (as attributed by V. Arnold) appears earlier in various contexts.
E. Lieb mentions to us his paper with D.C.Mattis (1962) which involves similar considerations.

This has also strong links with the theory of oscillation matrices and kernels for which we refer to the book of F. Gantmacher and M. Krein (2002). This is related to the notion of Chebyshev-Haar systems in the theory of discontinuous Kellog kernels. Kellog's work was done during the first world war !

Happy birthday Rafael.

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