Spectral theory and semi-classical analysis for the complex Schrödinger operator
Talk at Luminy June 2017

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(after Almog, Henry, Grebenkov-Helffer-Henry, Grebenkov-Helffer, Almog-Grebenkov-Helffer,...)

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The transmission boundary condition which is considered in addition to the other conditions Dirichlet, Neumann and Robin appears in various exchange problems such as molecular diffusion across semi-permeable membranes [37, 34], heat transfer between two materials [12, 18, 9], or transverse magnetization evolution in nuclear magnetic resonance (NMR) experiments [20]. In the simplest setting of the latter case, one considers the local transverse magnetization $G(x, y\,; t)$ produced by the nuclei that started from a fixed initial point y and diffused in a constant magnetic field gradient g up to time t.

This magnetization is also called the propagator or the Green function of the Bloch-Torrey equation [39] (1956):

$$\frac{\partial}{\partial t} G(x, y\,; t) = (D\Delta - i\gamma g x_1)\, G(x, y\,; t)\,, \tag{1}$$

with the initial condition

$$G(x, y\,; t = 0) = \delta(x - y), \tag{2}$$

where D is the intrinsic diffusion coefficient, $\Delta = \partial^2/\partial x_1^2 + \ldots + \partial^2/\partial x_d^2$ the Laplace operator in $\mathbb{R}^d$, γ the gyromagnetic ratio, and x_1 the coordinate in a prescribed direction.

The 1D-situation

In the first part of the talk, we present the one-dimensional situation ($d = 1$), in which the operator

$$D_x^2 + ix = -\frac{d^2}{dx^2} + ix$$

is called the complex Airy operator and appears in many contexts: mathematical physics, fluid dynamics, time dependent Ginzburg-Landau problems and also as an interesting toy model in spectral theory (see [3]). We consider a suitable extension $\mathcal{A}_1^+$ of this differential operator and its associated evolution operator $e^{-t\mathcal{A}_1^+}$. The Green function $G(x, y\,; t)$ is the distribution kernel of $e^{-t\mathcal{A}_1^+}$.

For the problem on the line $\mathbb{R}$, an intriguing property is that this non self-adjoint operator, which has compact resolvent, has empty spectrum. However, the situation is completely different on the half-line $\mathbb{R}_+$. The eigenvalue problem

$$(D_x^2 + ix)u = \lambda u,$$

for a spectral pair (u, λ) with u in $H^2(\mathbb{R}_+)$, $xu \in L^2(\mathbb{R}_+)$ has been thoroughly analyzed for both Dirichlet ($u(0) = 0$) and Neumann ($u'(0) = 0$) boundary conditions.

The spectrum consists of an infinite sequence of eigenvalues of multiplicity one explicitly related to the zeroes of the Airy function (see [36, 27]).

The space generated by the eigenfunctions is dense in $L^2(\mathbb{R}_+)$ (completeness property) but there is no Riesz basis of eigenfunctions.

Finally, the decay of the associated semi-group has been analyzed in detail through Gearhard-Prüss like theorems.

The physical consequences of these spectral properties for NMR experiments have been first revealed by Stoller, Happer and Dyson [36], then by De Sviet et al. and D. Grebenkov [16, 19, 22].

In this talk, we not only consider the Dirichlet and Neumann problem but will consider another problem for the complex Airy operator on the line but with a transmission property at 0 which reads (cf Grebenkov [22]),

$$\left\{ \begin{array}{lcl} u'(0_+) & = & u'(0_-)\,, \\ u'(0) & = & \kappa \left(u(0_+) - u(0_-) \right)\,, \end{array} \right. \tag{3}$$

where $\kappa \geq 0$ is a real parameter.
The case $\kappa = 0$ corresponds to two independent Neumann problems on $\mathbb{R}_-$ and $\mathbb{R}_+$ for the complex Airy operator.
When κ tends to $+\infty$, the second relation in (3) becomes the continuity condition, $u(0_+) = u(0_-)$, and the barrier disappears.
Hence, the problem tends (at least formally) to the standard problem for the complex Airy operator on the line.

We summarize the main (1D)-result for the transmission case in the following:

Theorem

The semigroup $\exp(-t\mathcal{A}_1^+)$ is contracting. The operator $\mathcal{A}_1^+$ has a discrete spectrum $\{\lambda_n(\kappa)\}$. The eigenvalues $\lambda_n(\kappa)$ are simple and determined as (complex-valued) solutions of the equation

$$2\pi \mathrm{Ai}'(e^{2\pi i/3}\lambda)\mathrm{Ai}'(e^{-2\pi i/3}\lambda) + \kappa = 0, \tag{4}$$

where $\mathrm{Ai}'(z)$ is the derivative of the Airy function.
For all $\kappa \geq 0$, there exists N such that, for all $n \geq N$, there exists a unique eigenvalue of $\mathcal{A}_1^+$ in the ball $B(\lambda_n^\pm, 2\kappa|\lambda_n^\pm|^{-1})$, where $\lambda_n^\pm = e^{\pm 2\pi i/3}a_n'$, and a_n' are the zeros of $\mathrm{Ai}'(z)$.
Finally, for any $\kappa \geq 0$ the space generated by the eigenfunctions of the complex Airy operator with transmission is dense in $L^2(\mathbb{R}_-) \times L^2(\mathbb{R}_+)$.

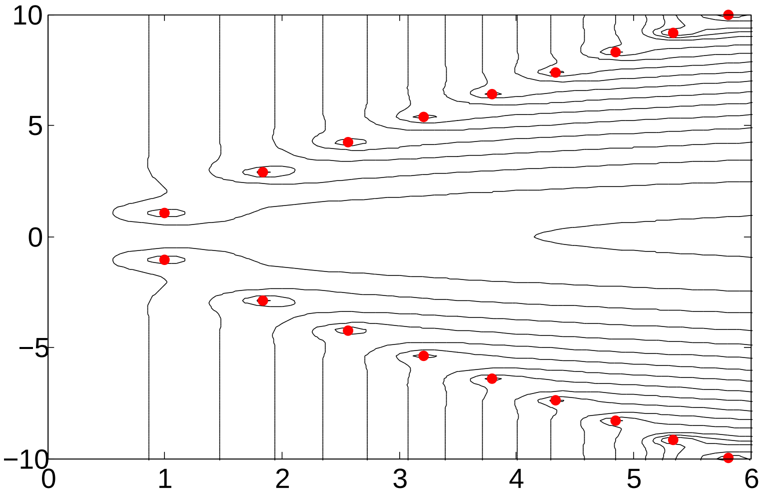

Figure: Numerically computed pseudospectrum in the complex plane of the complex Airy operator with the transmission boundary condition at the origin with $\kappa = 1$. The red points show the poles $\lambda_n^{\pm}(\kappa)$.

Basic properties of the Airy function

We recall that the Airy function is the unique solution of

$$(D_x^2 + x)u = 0\,,$$

on the line such that $u(x)$ tends to 0 as $x \to +\infty$ and $\mathrm{Ai}(0) = 1/\left(3^{\frac{2}{3}}\,\Gamma(\frac{2}{3})\right)$. This Airy function extends into an holomorphic function in $\mathbb{C}$.

Ai is positive decreasing on $\mathbb{R}_+$ but has an infinite number of zeros in $\mathbb{R}_-$. We denote by a_n ($n \in \mathbb{N}$) the decreasing sequence of zeros of Ai. Similarly we denote by a_n' the sequence of zeros of Ai'. Moreover

$$a_n \underset{n\to+\infty}{\sim} -\left(\frac{3\pi}{2}(n-1/4)\right)^{2/3}\,, \tag{5}$$

and

$$a_n' \underset{n\to+\infty}{\sim} -\left(\frac{3\pi}{2}(n-3/4)\right)^{2/3}\,. \tag{6}$$

$\mathrm{Ai}(e^{i\alpha}z)$ and $\mathrm{Ai}(e^{-i\alpha}z)$ (with $\alpha = 2\pi/3$) are two independent solutions of the differential equation

$$\left(-\frac{d^2}{dz^2} - iz\right) w(z) = 0\,.$$

Considering their Wronskian, one gets

$$e^{-i\alpha}\mathrm{Ai}'(e^{-i\alpha}z)\mathrm{Ai}(e^{i\alpha}z) - e^{i\alpha}\mathrm{Ai}'(e^{i\alpha}z)\mathrm{Ai}(e^{-i\alpha}z) = \frac{i}{2\pi}\,,\ \forall\, z \in \mathbb{C}\,. \quad (7)$$

Note the identity

$$\mathrm{Ai}(z) + e^{-i\alpha}\mathrm{Ai}(e^{-i\alpha}z) + e^{i\alpha}\mathrm{Ai}(e^{i\alpha}z) = 0\,,\ \forall\, z \in \mathbb{C}\,. \quad (8)$$

The Airy function and its derivative satisfy different asymptotic:
(i) For $|\arg z| < \pi$,

$$\mathrm{Ai}(z) = \frac{1}{2}\pi^{-\frac{1}{2}} z^{-1/4} \exp\left(-\frac{2}{3}z^{3/2}\right)\left(1+\mathcal{O}(|z|^{-\frac{3}{2}})\right), \tag{9}$$

$$\mathrm{Ai}'(z) = -\frac{1}{2}\pi^{-\frac{1}{2}} z^{1/4} \exp\left(-\frac{2}{3}z^{3/2}\right)\left(1+\mathcal{O}(|z|^{-\frac{3}{2}})\right). \tag{10}$$

(ii) For $|\arg z| < \frac{2}{3}\pi$,

$$\begin{aligned}\mathrm{Ai}(-z) = {} & \pi^{-\frac{1}{2}} z^{-1/4}\left(\sin\left(\frac{2}{3}z^{3/2}+\frac{\pi}{4}\right)\left(1+\mathcal{O}(|z|^{-\frac{3}{2}}\right)\right. \\ & \left. -\frac{5}{72}\left(\frac{2}{3}z^{\frac{3}{2}}\right)^{-1}\cos\left(\frac{2}{3}z^{3/2}+\frac{\pi}{4}\right)\left(1+\mathcal{O}(|z|^{-\frac{3}{2}}\right)\right)\end{aligned} \tag{11}$$

$$\begin{aligned}\mathrm{Ai}'(-z) = {} & -\pi^{-\frac{1}{2}} z^{1/4}\left(\cos(\frac{2}{3}z^{3/2}+\frac{\pi}{4})(1+\mathcal{O}(|z|^{-\frac{3}{2}}))\right., \\ & \left. +\frac{7}{72}\left(\frac{2}{3}z^{3/2}\right)^{-1}\sin(\frac{2}{3}z^{3/2}+\frac{\pi}{4})(1+\mathcal{O}(|z|^{-\frac{3}{2}}))\right).\end{aligned} \tag{12}$$

Analysis of the resolvent of $\mathcal{A}^+$ on the line for $\lambda > 0$

On the line $\mathbb{R}$, $\mathcal{A}^+$ is the closure of the operator $\mathcal{A}_0^+$ defined on $C_0^\infty(\mathbb{R})$ by $\mathcal{A}_0^+ = D_x^2 + ix$.
This is now standard. A detailed description of the properties of $\mathcal{A}^+$ can be found in my book in Cambridge University Press (2013).

One can give the asymptotic control of the resolvent $(\mathcal{A}^+ - \lambda)^{-1}$ as $\lambda \to +\infty$.

We successively discuss the control in $\mathcal{L}(L^2(\mathbb{R}))$ and in the Hilbert-Schmidt space $\mathcal{C}^2(L^2(\mathbb{R}))$.

Note that the norm of the resolvent $(\mathcal{A}^+ - \lambda)^{-1}$ depends only on the real part of λ.

Control in $\mathcal{L}(L^2(\mathbb{R}))$.

Here we follow an idea present in an old paper of I. Herbst, the book of Davies and used in Martinet's PHD (see also my book at Cambridge University Press).

Proposition

For all $\lambda > \lambda_0$,

$$\|(\mathcal{A}^+ - \lambda)^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}))} \leq \sqrt{2\pi}\, \lambda^{-\frac{1}{4}} \exp\left(\frac{4}{3}\lambda^{\frac{3}{2}}\right)(1 + o(1))\,. \quad (13)$$

We can also have a control in the Hilbert-Schmidt norm (Martinet, Bordeaux-Montrieux). For complex λ, replace λ by $\mathrm{Re}\,\lambda$.

Analysis of the resolvent for the Dirichlet realization in the half-line.

It is not difficult to define the Dirichlet realization $\mathcal{A}^{\pm,D}$ of $D_x^2 \pm ix$ on $\mathbb{R}_+$ (the analysis on the negative semi-axis is similar). One can use for example the Lax Milgram theorem and take as form domain

$$V^D := \{u \in H_0^1(\mathbb{R}_+)\,,\, x^{\frac{1}{2}} u \in L^2_+\}\,.$$

It can also be shown that the domain is

$$\mathcal{D}^D := \{u \in V^D\,,\, u \in H^2_+\}\,.$$

This implies

Proposition

The resolvent $\mathcal{G}^{\pm,D}(\lambda) := (\mathcal{A}^{\pm,D} - \lambda)^{-1}$ is in the Schatten class C^p for any $p > \frac{3}{2}$ (see [17] for definition), where $\mathcal{A}^{\pm,D} = D_x^2 \pm ix$ and the superscript D refers to the Dirichlet case.

More precisely we provide the distribution kernel $\mathcal{G}^{-,D}(x,y\,;\lambda)$ of the resolvent for the complex Airy operator $D_x^2 - ix$ on the positive semi-axis with Dirichlet boundary condition at the origin. Matching the boundary conditions, one gets

$$\mathcal{G}^{-,D}(x,y\,;\lambda) = \mathcal{G}_0^{-}(x,y\,;\lambda) + \mathcal{G}_1^{-,D}(x,y\,;\lambda), \tag{14}$$

where $\mathcal{G}_0^{-}(x,y\,;\lambda)$ is the resolvent for the Airy operator $D_x^2 - ix$ on the whole line,

$$\mathcal{G}_0^-(x,y\,;\lambda) = \begin{cases} 2\pi \mathrm{Ai}(e^{i\alpha} w_x)\mathrm{Ai}(e^{-i\alpha} w_y) & (x<y), \\ 2\pi \mathrm{Ai}(e^{-i\alpha} w_x)\mathrm{Ai}(e^{i\alpha} w_y) & (x>y), \end{cases} \tag{15}$$

and

$$\mathcal{G}_1^{-,D}(x,y\,;\lambda) = -2\pi \frac{\mathrm{Ai}(e^{i\alpha}\lambda)}{\mathrm{Ai}(e^{-i\alpha}\lambda)} \mathrm{Ai}\big(e^{-i\alpha}(ix+\lambda)\big)\, \mathrm{Ai}\big(e^{-i\alpha}(iy+\lambda)\big). \tag{16}$$

The resolvent is compact. The poles of the resolvent are determined by the zeros of $\mathrm{Ai}(e^{-i\alpha}\lambda)$, i.e., $\lambda_n = e^{i\alpha} a_n$, where the a_n are zeros of the Airy function: $\mathrm{Ai}(a_n) = 0$. The eigenvalues have multiplicity 1 (no Jordan block).

As a consequence of the analysis of the numerical range of the operator, we have

Proposition

$$||\mathcal{G}^{\pm,D}(\lambda)|| \leq \frac{1}{|\mathrm{Re}\,\lambda|}\,, \quad \textit{if}\ \mathrm{Re}\,\lambda < 0\,; \tag{17}$$

and

$$||\mathcal{G}^{\pm,D}(\lambda)|| \leq \frac{1}{|\mathrm{Im}\,\lambda|}\,, \quad \textit{if}\ \mp\mathrm{Im}\,\lambda > 0\,. \tag{18}$$

This proposition together with the Phragmen-Lindelöf principle (see Agmon [2] or Dunford-Schwartz [17])

Proposition

The space generated by the eigenfunctions of the Dirichlet realization $\mathcal{A}^{\pm,D}$ of $D_x^2 \pm ix$ is dense in L^2_+.

It is proven by R. Henry in [29] that there is no Riesz basis of eigenfunctions.

The Hilbert-Schmidt norm of the resolvent for $\lambda > 0$

At the boundary of the numerical range of the operator, it is interesting (and this will be important later for the transmission problem) to analyze the behavior of the resolvent. Numerical computations lead to the observation that, for λ real,

$$\lim_{\lambda \to +\infty} ||\mathcal{G}^{\pm,D}(\lambda)||_{\mathcal{L}(L^2_+)} = 0\,. \tag{19}$$

As a new result, we can prove [25]

Proposition

When λ tends to $+\infty$, we have

$$||\mathcal{G}^{\pm,D}(\lambda)||_{HS} \approx \lambda^{-\frac{1}{4}} (\log \lambda)^{\frac{1}{2}}\,. \tag{20}$$

Note that it gives a control of the resolvent at the boundary of the numerical range (See Dencker-Sjöstrand-Zworski for problems in $\mathbb{R}^n$).

About the proof

The Hilbert-Schmidt norm of the resolvent can be written as

$$||\mathcal{G}^{-,D}||_{HS}^2 = \int_{\mathbb{R}_+^2} |\mathcal{G}^{-,D}(x,y\,;\lambda)|^2 dxdy = 8\pi^2 \int_0^{\infty} Q(x;\lambda)dx, \tag{21}$$

where

$$Q(x;\lambda) = \frac{|\mathrm{Ai}(e^{-i\alpha}(ix+\lambda))|^2}{|\mathrm{Ai}(e^{-i\alpha}\lambda)|^2} \times \\ \times \int_0^x \left|\mathrm{Ai}(e^{i\alpha}(iy+\lambda))\mathrm{Ai}(e^{-i\alpha}\lambda) - \mathrm{Ai}(e^{-i\alpha}(iy+\lambda))\mathrm{Ai}(e^{i\alpha}\lambda)\right|^2 dy. \tag{22}$$

Using the identity (8), we observe that

$$\begin{aligned} &\mathrm{Ai}(e^{i\alpha}(iy+\lambda))\mathrm{Ai}(e^{-i\alpha}\lambda) - \mathrm{Ai}(e^{-i\alpha}(iy+\lambda))\mathrm{Ai}(e^{i\alpha}\lambda) \\ &\quad = e^{-i\alpha}\left(\mathrm{Ai}(e^{-i\alpha}(iy+\lambda))\mathrm{Ai}(\lambda) - \mathrm{Ai}(iy+\lambda)\mathrm{Ai}(e^{-i\alpha}\lambda)\right) . \end{aligned} \tag{23}$$

Hence we get

$$Q(x;\lambda) = |\mathrm{Ai}(e^{-i\alpha}(ix+\lambda))|^2 \int_0^x \left| \mathrm{Ai}(e^{-i\alpha}(iy+\lambda)) \frac{\mathrm{Ai}(\lambda)}{\mathrm{Ai}(e^{-i\alpha}\lambda)} - \mathrm{Ai}(iy+\lambda) \right|^2 dy . \tag{24}$$

Then you have to do a fine asymptotic analysis for this expression.

The complex Airy operator with a semi-permeable barrier: definition and properties

We consider the sesquilinear form a_ν defined for $u = (u_-, u_+)$ and $v = (v_-, v_+)$ by

$$\begin{aligned} a_\nu(u, v) \ &= \ \int_{-\infty}^{0} \Big(u_-'(x)\bar{v}_-'(x) + i\, x u_-(x)\bar{v}_-(x) + \nu\, u_-(x)\bar{v}_-(x) \Big)\, dx \\ &\quad + \int_{0}^{+\infty} \Big(u_+'(x)\bar{v}_+'(x) + i\, x u_+(x)\bar{v}_+(x) + \nu\, u_+(x)\bar{v}_+(x) \Big)\, dx \\ &\quad + \kappa\big(u_+(0) - u_-(0)\big)\big(\overline{v_+(0) - v_-(0)}\big)\,, \end{aligned} \tag{25}$$

where the form domain $\mathcal{V}$ is

$$\mathcal{V} := \Big\{ u = (u_-, u_+) \in H_-^1 \times H_+^1 \, : \, |x|^{\frac{1}{2}} u \in L_-^2 \times L_+^2 \Big\}\,,$$

and $\nu \in \mathbb{R}$.
The space $\mathcal{V}$ is endowed with the Hilbertian norm

$$\|u\|_{\mathcal{V}} := \sqrt{\|u_-\|_{H_-^1}^2 + \|u_+\|_{H_+^1}^2 + \||x|^{1/2} u\|_{L^2}^2}\,.$$

We first observe that for any $\nu \geq 0$, the sesquilinear form a_ν is continuous on $\mathcal{V}$.

As the imaginary part of the potential $V(x) = ix$ changes sign, it is not straightforward to determine whether the sesquilinear form a_ν is coercive.

Due to the lack of coercivity, the standard version of the Lax-Milgram theorem does not apply. We shall instead use the following generalization introduced in Almog-Helffer [6].

Theorem

Let $\mathcal{V} \subset \mathcal{H}$ *be two Hilbert spaces s.t. that* $\mathcal{V}$ *is continuously embedded in* $\mathcal{H}$ *and dense in* $\mathcal{H}$. *Let* a *be a continuous sesquilinear form on* $\mathcal{V} \times \mathcal{V}$, *and* $\exists \alpha > 0$ *and two bounded linear operators* Φ_1 *and* Φ_2 *on* $\mathcal{V}$ *s.t.* $\forall u \in \mathcal{V}$,

$$\left\{ \begin{array}{lcl} |a(u,u)| + |a(u,\Phi_1 u)| & \geq & \alpha \, \|u\|_{\mathcal{V}}^2 \,, \\ |a(u,u)| + |a(\Phi_2 u,u)| & \geq & \alpha \, \|u\|_{\mathcal{V}}^2 \,. \end{array} \right. \tag{26}$$

Assume further that Φ_1 *extends to a bounded linear operator on* $\mathcal{H}$. *Then* $\exists$ *a closed, densely-defined operator* S *on* $\mathcal{H}$ *with domain*

$$\mathcal{D}(S) = \left\{ u \in \mathcal{V} : v \mapsto a(u,v) \;\; \textit{can be extended continuously on} \;\; \mathcal{H} \right\},$$

s.t. $\forall u \in \mathcal{D}(S)$, $\forall v \in \mathcal{V}$,

$$a(u,v) = \langle Su, v \rangle_{\mathcal{H}} \,.$$

Moreover, from the characterization of the domain, we deduce the stronger

Proposition

There exists λ_0 *(*$\lambda_0 = 0$ *for* $\kappa > 0$*) such that* $(\mathcal{A}_1^+ - \lambda_0)^{-1}$ *belongs to the Schatten class* $\mathcal{C}^p$ *for any* $p > \frac{3}{2}$*.*

Note that if it is true for some λ_0 it is true for any λ in the resolvent set.

The following statement summarizes the previous discussion.

Proposition

The operator $\mathcal{A}_1^+$ *acting as*

$$u \mapsto \mathcal{A}_1^+ u = \left(-\frac{d^2}{dx^2}u_- + ixu_-, -\frac{d^2}{dx^2}u_+ + ixu_+\right)$$

on the domain

$$\mathcal{D}(\mathcal{A}_1^+) = \{u \in H_-^2 \times H_+^2 : xu \in L_-^2 \times L_+^2 \\ \textit{and } u \textit{ satisfies transmission conditions } (3)\}$$

is a closed operator with compact resolvent.

$\exists \lambda > 0$ *s. t.* $\mathcal{A}_1^+ + \lambda$ *is maximal accretive.*

Remark

We have

$$\Gamma \mathcal{A}_1^+ = \mathcal{A}_1^- \, \Gamma \,, \tag{27}$$

where Γ *denotes the complex conjugation:*

$$\Gamma(u_- \, , \, u_+) = (\bar{u}_- \, , \, \bar{u}_+) \, .$$

Remark (PT-Symmetry)

If (λ, u) *is an eigenpair, then* $(\bar{\lambda}, \bar{u}(-x))$ *is also an eigenpair.*

Integral kernel of the resolvent

Lengthy but elementary computations give:

$$\mathcal{G}^-(x,y\,;\lambda,\kappa) = \mathcal{G}_0^-(x,y\,;\lambda) + \mathcal{G}_1(x,y\,;\lambda,\kappa)\,, \tag{28}$$

where $\mathcal{G}_0^-(x,y\,;\lambda)$ is the distribution kernel of the resolvent of the operator $\mathcal{A}_0^* := -\frac{d^2}{dx^2} - ix$ on $\mathbb{R}$ and

$$\mathcal{G}_1(x,y\,;\lambda,\kappa) = \begin{cases} -4\pi^2 \frac{e^{2i\alpha}[\mathrm{Ai}'(e^{i\alpha}\lambda)]^2}{f(\lambda)+\kappa} \mathrm{Ai}(e^{-i\alpha}w_x)\mathrm{Ai}(e^{-i\alpha}w_y)\,, & \text{for } x>0\,, \\ -2\pi \frac{f(\lambda)}{f(\lambda)+\kappa} \mathrm{Ai}(e^{i\alpha}w_x)\mathrm{Ai}(e^{-i\alpha}w_y)\,, & \text{for } x<0\,, \end{cases} \tag{29}$$

for $y>0$, and

$$\mathcal{G}_1(x,y\,;\lambda,\kappa) = \begin{cases} -2\pi \frac{f(\lambda)}{f(\lambda)+\kappa} \mathrm{Ai}(e^{-i\alpha}w_x)\mathrm{Ai}(e^{i\alpha}w_y)\,, & x>0\,, \\ -4\pi^2 \frac{e^{-2i\alpha}[\mathrm{Ai}'(e^{-i\alpha}\lambda)]^2}{f(\lambda)+\kappa} \mathrm{Ai}(e^{i\alpha}w_x)\mathrm{Ai}(e^{i\alpha}w_y)\,, & x<0\,, \end{cases} \tag{30}$$

for $y<0$.

Hence the poles are determined by the equation

$$f(\lambda) = -\kappa\,, \tag{31}$$

with f defined by

$$f(\lambda) := 2\pi \mathrm{Ai}'(e^{-i\alpha}\lambda)\mathrm{Ai}'(e^{i\alpha}\lambda)\,. \tag{32}$$

Remark

For $\kappa = 0$, one recovers the conjugated pairs associated with the zeros a_n' of Ai'.

We have indeed

$$\lambda_n^+ = e^{i\alpha} a_n'\,, \quad \lambda_n^- = e^{-i\alpha} a_n'\,, \tag{33}$$

where a_n' is the n-th zero of Ai'.

We also know that the eigenvalues for the Neumann problem are simple. Hence by the local inversion theorem we get the existence of a solution close to each $\lambda_n^\pm$ for κ small enough (possibly depending on n) if we show that $f'(\lambda_n^\pm) \neq 0$. For λ_n^+, we have, using the Wronskian relation (7) and $\mathrm{Ai}'(e^{-i\alpha}\lambda_n^+) = 0$,

$$\begin{aligned} f'(\lambda_n^+) & = 2\pi\, e^{-i\alpha}\ \mathrm{Ai}''(e^{-i\alpha}\lambda_n^+)\mathrm{Ai}'(e^{i\alpha}\lambda_n^+) \\ & = 2\pi e^{-2i\alpha}\lambda_n^+\mathrm{Ai}(e^{-i\alpha}\lambda_n^+)\mathrm{Ai}'(e^{i\alpha}\lambda_n^+) \\ & = -i\lambda_n^+ \,. \end{aligned} \tag{34}$$

Similar computations hold for λ_n^-. We recall that

$$\lambda_n^+ = \overline{\lambda_n^-}\,.$$

Remark Very recently, we prove [AGH] that the eigenvalues are always simple.

Completion of the proof of the properties of the operator for the semi-permeable case

For the analysis of the resolvent, it is enough to compare the resolvent for some κ, with the resolvent for $\kappa = 0$, and to show that the asymptotic behavior as $\lambda \to +\infty$ is the same. For $\kappa = 0$, it is easy to see that we are reduced to the Neumann case on $\mathbb{R}^+$. We can also observe that the behavior of the resolvent are the same for Dirichlet and Neumann as $\lambda \to +\infty$.

Finally, we observe that we have the control of the resolvent on enough rays (the other rays being chosen outside of the numerical range) and the Phragmen-Lindelöf argument can now be used.

Then a general theorem (see the book of Agmon) gives us the completeness.

Applications to (2D)-problems

In higher dimension, an extension of the complex Airy operator is the differential operator that we call the Bloch-Torrey operator or simply the BT-operator:

$$-D\Delta + igx_1\,,$$

where $\Delta = \partial^2/\partial x_1^2 + \ldots + \partial^2/\partial x_d^2$ is the Laplace operator in $\mathbb{R}^d$, and D and g are real parameters. More generally, we will study the spectral properties of some realizations of the differential operator

$$\mathcal{A}_h^\# = -h^2\Delta + i\,V(x)\,, \qquad (35)$$

in an open set Ω, where h is a real parameter and $V(x)$ a real-valued potential with controlled behavior at ∞, and the superscript $\#$ distinguishes Dirichlet (D), Neumann (N), Robin (R), or transmission (T) conditions.

More precisely we discuss

1. the case of a bounded open set Ω with Dirichlet, Neumann or Robin boundary condition;
2. the case of a complement $\Omega := \complement\overline{\Omega}_-$ of a bounded set Ω_- with Dirichlet, Neumann or Robin boundary condition;
3. the case of two components $\Omega_- \cup \Omega_+$, with $\Omega_- \subset \overline{\Omega}_- \subset \Omega$ and $\Omega_+ = \Omega \backslash \overline{\Omega}_-$, with Ω bounded and transmission conditions at the interface between Ω_- and Ω_+;
4. the case of two components $\Omega_- \cup \complement\,\overline{\Omega}_-$, with Ω_- bounded and transmission conditions at the boundary.

The state u (in the first two items) or the pair (u_-, u_+) in the last items should satisfy some boundary or transmission condition at the interface. We consider the following situations:

- the Dirichlet condition: $u_{|\partial\Omega} = 0$;
- the Neumann condition: $\partial_\nu u_{|\partial\Omega} = 0$, where $\partial_\nu = \nu \cdot \nabla$, with ν being the outwards pointing normal;
- the Robin condition: $h^2 \partial_\nu u_{|\partial\Omega} = -\mathcal{K} u_{|\partial\Omega}$, where $\mathcal{K} \geq 0$ denotes the Robin parameter;
- the transmission condition:

$$h^2 \partial_\nu u_{+\,|\partial\Omega_-} = h^2 \partial_\nu u_{-\,|\partial\Omega_-} = \mathcal{K}(u_{+\,|\partial\Omega_-} - u_{-\,|\partial\Omega_-}) \,,$$

  where $\mathcal{K} \geq 0$ denotes the transmission parameter. In the last case, we should add a boundary condition at $\partial\Omega_+$ which can be Dirichlet or Neumann.

$\Omega^\#$ denotes Ω if $\# \in \{D, N, R\}$ and Ω_- if $\# = T$.
$L^2_\#$ denotes $L^2(\Omega)$ if $\# \in \{D, N, R\}$ and $L^2(\Omega_-) \times L^2(\Omega_+)$ if $\# = T$.

In the first part of this talk, we have described various realizations of the complex Airy operator $A_0^\# := -\frac{d^2}{d\tau^2} + i\tau$ in the four cases.
The boundary conditions read respectively:

- $u(0) = 0$ (Dirichlet)
- $u'(0) = 0$ (Neumann)
- $u'(0) = \kappa\, u(0)$ (Robin)
- $u'_-(0) = u'_+(0) = \kappa\,(u_+(0) - u_-(0))$ (Transmission)

(with $\kappa \geq 0$ in the last items).

For all these cases, we have proven the existence of a discrete spectrum and the completeness of the corresponding eigenfunctions.

We have started the analysis of the spectral properties of the BT operator in dimension 2 or higher that are relevant for applications in superconductivity theory (Almog, Almog-Helffer-Pan, Almog-Helffer), in fluid dynamics (Martinet), in control theory (Beauchard-Helffer-Henry-Robbiano) and in diffusion magnetic resonance imaging (Grebenkov) . The main questions are

- definition of the operator,
- construction of approximate eigenvalues in some asymptotic regimes,
- localization of quasimode states near certain boundary points,
- numerical simulations.

Some of these questions have been already analyzed by Y. Almog (see [3] (2008) and references therein for earlier contributions), R. Henry in his PHD (2013) (+ ArXiv paper 2014) and Almog-Henry (2015) but they were mainly devoted to the case of a Dirichlet realization in bounded domains in $\mathbb{R}^2$ or particular unbounded domains like $\mathbb{R}^2$ and $\mathbb{R}^2_+$, these two last cases playing an important role in the local analysis of the global problem.

We consider $\mathcal{A}_h$ and the corresponding realizations in Ω are denoted by $\mathcal{A}_h^D$, $\mathcal{A}_h^N$, $\mathcal{A}_h^R$ and $\mathcal{A}_h^T$. These realizations will be properly under the condition that, when Ω is unbounded, there exists $C > 0$ such that

$$|\nabla V(x)| \leq C\sqrt{1 + V(x)^2}\,. \tag{36}$$

Our main construction in [Grebenkov-Helffer 2016] is local and summarized in the following

Main (2D)-theorem

Let $\Omega \subset \mathbb{R}^2$ as above, $V \in \mathcal{C}^\infty(\overline{\Omega};\mathbb{R})$ and $x^0 \in \partial\Omega^\#$ such that

$$\nabla V(x^0) \neq 0\,, \quad \nabla V(x^0) \wedge \nu(x^0) = 0\,, \tag{37}$$

where $\nu(x^0)$ denotes the outward normal on $\partial\Omega$ at x^0.
Assume that, in the local curvilinear coordinates, the second derivative $2\, v_{20}$ of the restriction of V to the boundary at x^0 satisfies

$$v_{20} \neq 0\,.$$

For the Robin and transmission cases, we assume that for some $\kappa > 0$

$$\mathcal{K} = h^{\frac{4}{3}}\kappa\,. \tag{38}$$

Main theorem continued

If $\mu_0^\#$ is a simple eigenvalue of the realization "$\#$" $-\frac{d^2}{dx^2} + ix$ in $L^2_\#$,
and μ_2 is an eigenvalue of Davies operator $-\frac{d^2}{dy^2} + iy^2$ on $L^2(\mathbb{R})$,
then there exists a pair $(\lambda_h^\#, u_h^\#)$ with $u_h^\#$ in the domain of $\mathcal{A}_h^\#$, such that

$$\lambda_h^\# = i\,V(x^0) + h^{\frac{2}{3}} \sum_{j\in\mathbb{N}} \lambda_{2j}^\#\, h^{\frac{j}{3}} + \mathcal{O}(h^\infty)\,, \tag{39}$$

$$(\mathcal{A}_h^\# - \lambda_h^\#)\, u_h^\# = \mathcal{O}(h^\infty) \text{ in } L^2_\#(\Omega)\,, \quad ||u_h^\#||_{L^2} \sim 1\,, \tag{40}$$

where

$$\lambda_0^\# = \mu_0^\#\, |\, v_{01}|^{\frac{2}{3}} \exp\left(i\frac{\pi}{3}\mathrm{sign}\, v_{01}\right)\,, \quad \lambda_2 = \mu_2 |v_{20}|^{\frac{1}{2}} \exp\left(i\frac{\pi}{4}\mathrm{sign}\, v_{20}\right)\,, \tag{41}$$

with $v_{01} := \nu \cdot \nabla V(x^0)$.

We can also discuss a physically interesting case when κ in (38) depends on h and tends to 0 .

The proof of this theorem provides a general scheme for quasimode construction in an arbitrary planar domain with smooth boundary $\partial\Omega$. In particular, this construction allowed us to retrieve and further generalize the asymptotic expansion of eigenvalues obtained by de Swiet and Sen for the Bloch-Torrey operator in the case of a disk. The generalization is applicable for any smooth boundary, with Neumann, Dirichlet, Robin, or transmission boundary condition. Moreover, since the analysis is local, the construction is applicable to both bounded and unbounded components.
We prove in Almog-Grebenkov-Helffer the existence of the eigenvalues and the rate of the associated semi-group.

Main results for the exterior problem

We consider the Dirichlet or Neumann exterior problem and use the notation $\#$ for D (Dirichlet) or N (Neumann). We introduce at ∞ the Assumption that

$$\sum_{1\leq|\alpha|\leq 2} |D_x^\alpha V(x)| \leq C\,. \tag{42}$$

This is for example satisfied if

$$V(x) = \hat{g} x_1\,, \tag{43}$$

outside a ball $B(0,R)$.
We add at the (bounded) boundary $\partial\Omega$ the assumptions of [4].
The potential V satisfies

$$|\nabla V(x)| > c > 0\,,\; \forall x \in \overline{\Omega}\,. \tag{44}$$

Let $\partial\Omega_\perp$ denote the subset of $\partial\Omega$ where ∇V is orthogonal to $\partial\Omega$:

$$\partial\Omega_\perp = \{x \in \partial\Omega^\# : \nabla V(x) = (\nabla V(x)\cdot\vec{\nu}(x))\,\vec{\nu}(x)\}\,, \tag{45}$$

where $\vec{\nu}(x)$ denotes the outward normal on $\partial\Omega$ at x.

Let $\# \in \{D, N\}$ and $\mathfrak{D}^\#$ be defined in the following manner

$$\begin{cases} \mathfrak{D}^\# = \{u \in H^2_{loc}(\overline{\mathbb{R}_+}) \,|\, u(0) = 0\} & \# = D \\ \mathfrak{D}^\# = \{u \in H^2_{loc}(\overline{\mathbb{R}_+}) \,|\, u'(0) = 0\} & \# = N\,. \end{cases} \tag{46}$$

Then, we define the operator

$$\mathcal{L}^\#(\mathbf{j}) = -\frac{d^2}{dx^2} + i\,\mathbf{j}\,x\,,$$

whose domain is given by

$$D(\mathcal{L}^\#(\mathbf{j})) = H^2(\mathbb{R}^+) \cap L^2(\mathbb{R}^+; |x|^2 dx) \cap \mathfrak{D}^\#\,, \tag{47}$$

and set

$$\lambda^\#(\mathbf{j}) = \inf \mathrm{Re}\,\sigma(\mathcal{L}^\#(\mathbf{j}))\,. \tag{48}$$

Next, let

$$\Lambda_m^\# = \inf_{x\in\partial\Omega_\perp} \lambda^\#(|\nabla V(x)|)\,, \tag{49}$$

In all cases we denote by $\mathcal{S}^\#$ the set

$$\mathcal{S}^\# := \{x \in \partial\Omega_\perp \,:\, \lambda^\#(|\nabla V(x)|) = \Lambda_m^\#\,\}\,. \tag{50}$$

When $\# \in \{D, N\}$ it can be verified by a dilation argument that, when $\mathbf{j} > 0$,

$$\lambda^{\#}(\mathbf{j}) = \lambda^{\#}(1)\,\mathbf{j}^{2/3}\,. \tag{51}$$

Hence

$$\Lambda_m^{\#} = \lambda^{\#}(\mathbf{j}_m)\,,\ \text{with } \mathbf{j}_m := \inf_{x\in\partial\Omega_\perp} (|\nabla V(x)|)\,, \tag{52}$$

and $\mathcal{S}^{\#}$ is actually independent of $\#$:

$$\mathcal{S}^{\#} = \mathcal{S} := \{x \in \partial\Omega_\perp\ :\ |\nabla V(x)| = \mathbf{j}_m\}\,. \tag{53}$$

We next make the following additional assumption:
At each point x of $\mathcal{S}^\#$,

$$\alpha(x) = \det D^2 V_\partial(x) \neq 0 \,, \tag{54}$$

where V_∂ denotes the restriction of V to $\partial\Omega$, and $D^2 V_\partial$ denotes its Hessian matrix.
It can be easily verified that (44) implies that $\mathcal{S}^\#$ is finite. Equivalently we may write

$$\alpha(x) = \Pi_{i=1}^{n-1} \alpha_i(x) \neq 0 \,, \tag{55a}$$

where

$$\{\alpha_i\}_{i=1}^{N-1} = \sigma(D^2 V_\partial) \,, \tag{55b}$$

where each eigenvalue is counted according to its multiplicity.

The main results are

Theorem

Under the previous Assumptions, we have

$$\lim_{h\to 0} \frac{1}{h^{2/3}} \inf\{\mathrm{Re}\ \sigma(\mathcal{A}_h^D)\} = \Lambda_m^D\,, \qquad \Lambda_m^D = \frac{|a_1|}{2}\mathbf{j}_m^{2/3}\,, \tag{56}$$

where $a_1 < 0$ is the rightmost zero of the Airy function Ai.
Moreover, for every $\varepsilon > 0$, there exist $h_\varepsilon > 0$ and $C_\varepsilon > 0$ such that

$$\forall h \in (0, h_\varepsilon), \qquad \sup_{\substack{\gamma \leq \Lambda_m^D \\ \nu \in \mathbb{R}}} \|(\mathcal{A}_h^D - (\gamma - \varepsilon)h^{2/3} - i\nu)^{-1}\| \leq \frac{C_\varepsilon}{h^{2/3}}\,. \tag{57}$$

In its first part, this result is essentially a reformulation of the result stated in [3]. Note that the second part provides, with the aid of the Gearhart-Prüss theorem, an effective bound (with respect to both t and h) of the decay of the associated semi-group as $t \to +\infty$. The theorem holds in particular in the case $V(x) = x_1$ where Ω is the complementary of a disk (and hence S^D consists of two points). Note that $\mathbf{j}_m = 1$ in this case.

A similar result can be proved for the Neumann case.

In the case of the Dirichlet problem, this theorem was obtained in [7, Theorem 1.1] for the interior problem and under the stronger assumption that, at each point x of $\mathcal{S}^D$, the Hessian of $V_\partial := V_{/\partial\Omega^\#}$ is positive definite if $\partial_\nu V(x) < 0$ or negative definite if $\partial_\nu V(x) > 0$, with $\partial_\nu V := \nu \cdot \nabla V$. This was extended in [4] to the interior problem without the signe condition of the Hessian.

Approximating models in $\mathbb{R}^d$ or $\mathbb{R}^d_+$

The models are ($\mathbb{R}^d$ or $\mathbb{R}^d_+$)

$$-h^2\Delta + i\mathbf{J}\cdot x$$

and (in $\mathbb{R}^d_+$)

$$-h^2\Delta + i\left(\mathbf{j}x_1 + \sum_{j\geq 2}\alpha_j x_j^2\right)$$

with $\alpha_j \neq 0$.
Because these operators are with separate variables, we can use the semi-group estimates for the $1D$-problems and get an estimate for the semi-group defined by tensor product for the dD-problem. The operators are then defined as infinitesimal generators of the semi-group. Resolvent estimates are then deduce from the estimates for the semi-group and direct estimates.
We refer to [3, 29, 25, 4] for the spectral analysis of these models.

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