On the domain of a Schrödinger operator with complex potential – Old and New – (After H-Nourrigat (1985), H-Mohamed, Nourrigat, Guibourg, Mba Yébé, Shen,...,H-Nier, Almog-H, H-Nourrigat (2017)).

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Introduction

The aim of this talk is to review and compare the spectral properties of (the closed extension of) $-\Delta + U$ ($U \geq 0$) and $-\Delta + iV$ in $L^2(\mathbb{R}^d)$ for C^∞ potentials U or V with polynomial behavior.

The case with magnetic field is also considered. More precisely, the aim is to compare the criteria for:

- essential selfadjointness (**esa**) or maximal accretivity (**maxacc**)
- Compactness of the resolvent.
- Maximal inequalities,

for these operators.

The most recent results devoted to the Schrödinger operator with complex potential have been obtained in collaboration with Y. Almog (2016) and J. Nourrigat (2017).

Maximal inequalities

By L^2-maximal inequalities, we mean the existence of $C > 0$ s. t.

$$||u||^2_{H^2} + ||Uu||^2_{L^2} \leq C \left(||(-\Delta + U)u||^2_{L^2} + ||u||^2_{L^2} \right) \,, \forall u \in C_0^\infty(\mathbb{R}^d) \,, \tag{1}$$

or

$$||u||^2_{H^2} + ||Vu||^2 \leq C \left(||(-\Delta + iV)u||^2 + ||u||^2 \right) \,, \forall u \in C_0^\infty(\mathbb{R}^d) \,. \tag{2}$$

We will also discuss the magnetic case:

$$P_{\mathbf{A},\mathbf{V}} = -\Delta_A + W := \sum_{j=1}^{d} (D_{x_j} - A_j(x))^2 + W(x)\,,$$

(with $W = U + iV$) and the notion of maximal regularity is expressed in terms of the magnetic Sobolev spaces:

$$\begin{aligned} &||(D-\mathbf{A})\mathbf{u}||^2_{\mathbf{L}^2(\mathbb{R}^d,\mathbb{C}^d)} \\ &\quad + \sum_{j,\ell} ||(D_j - A_j)(D_\ell - A_\ell)u||^2_{L^2(\mathbb{R}^d)} \\ &\qquad + ||\,|W|u||^2_{L^2(\mathbb{R}^d)} \\ &\qquad\qquad \leq C\left(||P_{\mathbf{A},\mathbf{W}}u||^2_{L^2(\mathbb{R}^d)} + ||u||^2_{L^2(\mathbb{R}^d)}\right), \end{aligned} \tag{3}$$

The question of analyzing $-\Delta + iV$ or more generally $P_{\mathbf{A},\mathbf{iV}} := -\Delta_A + iV$ appears in many situations [2, 3, 1]:

- Time dependent Ginzburg-Landau theory leads for example to the spectral analysis of

$$D_x^2 + (D_y - \frac{x^2}{2})^2 + iy$$

Here curl $\mathbf{A} = \mathbf{x}$ vanishes along a line.

- Control theory (see Beauchard-Helffer-Henry-Robbiano (2015))
- Bloch-Torrey (complex Airy) equation

$$-\Delta + ix$$

- Fluid dynamics

Moreover, V does not satisfy necessarily a sign condition $V \leq 0$ as for dissipative systems.

One origin of our problem

The physical problem is posed in a domain Ω with specific boundary conditions. We will only analyze here limiting situations where the domain possibly after a blowing argument becomes the whole space (or the half-space). We work in dimension 2 for simplification. We assume that a magnetic field of magnitude $\mathcal{H}^e$ is applied perpendicularly to the sample and identified (via its intensity) with a function. We denote the Ginzburg-Landau parameter of the superconductor by κ ($\kappa > 0$) and the normal conductivity of the sample by σ.

Then the time-dependent Ginzburg-Landau system (also known as the Gorkov-Eliashberg equations) is in $]0,T[\times\Omega$:

$$\begin{cases} \partial_t\psi + i\kappa\Phi\psi = \Delta_{\kappa\mathbf{A}}\psi + \kappa^2(1-|\psi|^2)\psi\,, \\ \kappa^2 \text{ curl }^2\mathbf{A} + \sigma(\partial_t\mathbf{A}+\nabla\Phi) = \kappa\mathrm{Im}\,(\bar\psi\nabla_{\kappa\mathbf{A}}\psi) + \kappa^2 \text{ curl } \mathcal{H}^e\,, \end{cases} \tag{4}$$

where ψ is the order parameter, $\mathbf{A}$ the magnetic potential, Φ the electric potential, $\nabla_{\kappa\mathbf{A}} = \nabla + i\kappa\mathbf{A}$ and $\Delta_{\kappa\mathbf{A}} = (\nabla + i\kappa\mathbf{A})^2$ is the magnetic Laplacian associated with magnetic potential $\kappa\mathbf{A}$. In addition $(\psi,\mathbf{A},\Phi)$ satisfies an initial condition at $t=0$.

The linearization of the first line near a time-independent solution leads to a magnetic Schrödinger operator with complex potential $i\Phi$ and magnetic potential $\kappa\mathbf{A}$.

Our goal

It seems therefore useful to present in a unified way, what is known on the subject. If we assume that the potential W is C^∞, we know that

- the operator is essentially selfadjoint (**esa**) starting from $C_0^\infty(\mathbb{R}^d)$ in the first situation ($W = U$)

  and
- the operator is maximally accretive (**maxacc**) in the second case ($W = U + iV$ with $U \geq 0$).

Hence in the two cases the closed operator in consideration is uniquely defined by its restriction to C_0^∞.
For the oldest contributions on the subject one can mention the papers by Ikebe-Kato (1962) [21], T. Kato (1972) [23] and the work of Avron-Herbst-Simon (1978) [5] which in particular popularizes the question of magnetic bottles. A very complete survey is in preparation by Barry Simon [40] (see the lectures of B. Simon in this conference).

Compactness of the resolvent

For the compactness of the resolvent, outside the easy case when $U \to +\infty$, the story starts around the eighties with the treatment of instructive examples (Simon [39] (1983), Robert [34] (1982)) and in the case with magnetic field [5] (the simplest example being for $d = 2$ and $U = 0$, when $B(x) \to +\infty$).
In the polynomial case, many results are deduced as a byproduct of the analysis of left-invariant operators on nilpotent groups (proof of the Rockland conjecture (1979)) see the book of Helffer-Nourrigat [17] (1985), at least in the case when U is a sum of square of polynomials.
Using Kohn's type inequalities (initially developed for the proof of hypoellipticity), B. Helffer and A. Morame (Mohamed) [15] (1988) obtain more general results which can be combined with the analysis of A. Iwatsuka [22] (1986).

Another family of results using the notion of capacity can be found in Kondratiev-Mazya-Shubin [27, 26]...

Maximal regularity

T. Kato proves, as a consequence of a contractive inequality, the inequality

$$||\Delta u||_{L^1} + ||Uu||_{L^1} \leq 3\,||(-\Delta + U)u||_{L^1}\,,\ \forall u \in C_0^\infty(\mathbb{R}^d)\,, \qquad (5)$$

under the condition that $U \geq 0$ and $U \in L^1_{loc}$.
The generalization to the L^p ($p > 1$) is only possible under stronger conditions on U.
We will mention some of the results with emphasis on L^2 estimates.

In the case, when $U = \sum_j U_j(x)^2$, the maximal L^2 estimate is obtained as a byproduct of the analysis of the hypoellipticity (see Hörmander [19] (1967), Rothschild-Stein (1977) [35] and the book by Helffer-Nourrigat [17] (including polynomial magnetic potentials) (1985) together with some idea of Folland (1977)).

This was then generalized to the case when U is a positive polynomial by J. Nourrigat in 1990 (unpublished) and used in the PHD of D. Guibourg [12, 13] defended in 1992, which considers the case when the electric potential is real $W = U \geq 0$ and the magnetic potential $\mathbf{A}$ are polynomials (or some class of polynomial like potentials).
In his thesis Zhong (1993) proves the same result by showing that $\nabla^2(-\Delta + U)^{-1}$ is a Calderon-Zygmund operator.
We also mention the unpublished thesis of Nourrigat's student Mba-Yébé [28] (1995), whose techniques are re-used in our recent work.

Z. Shen (1995) [36] generalizes the result to the case when U is in the reverse Hölder class RH_q ($q \geq \frac{d}{2}$ and $d \geq 3$), a class which contains the positive polynomials.

Definition

For $1 < q < +\infty$, a locally L^q, a.e. strongly positive, function ω belongs to RH_q if
$\exists\ C > 0$ s.t. for any cube $Q \subset \mathbb{R}^d$

$$\left(\frac{1}{|Q|}\int \omega^q\,dx\right)^{\frac{1}{q}} \leq C\left(\frac{1}{|Q|}\int \omega\,dx\right).$$

There is a local version of this class (Shen) which could be sufficient.
The proof also involves techniques of C. Fefferman.

Extension to Schrödinger with magnetic field

This can be extended to the case with magnetic field.
The main conditions for maximal L^2-estimates are $U \geq 0$ and $U + |\text{ curl } \mathbf{A}| \in \mathbf{RH}_{\frac{d}{2}}$. Additional conditions on the magnetic field or on the structure of U are added, depending on the authors (Helffer-Nourrigat, Guibourg, Nourrigat, Shen, Auscher-Ben Ali) and on the proved result.

Kohn's approach–ESA-case

This approach was mainly used for getting the compactness of the resolvent. Except in a few cases, these estimates do not lead to the maximal regularity but are enough for getting the compactness. Here we mainly refer to [15] (see also [29], [16]).
We first analyze the problem for

$$\mathcal{P}_{\mathbf{A},U} = \sum_{j=1}^{d} (D_{x_j} - A_j(x))^2 + \sum_{\ell=1}^{p} U_\ell(x)^2 \,. \tag{6}$$

Under these conditions, the operator is **esa** on $C_0^\infty(\mathbb{R}^d)$. We note also that:

$$\mathcal{P}_{\mathbf{A},U} = \sum_{j=1}^{d+p} X_j^2 = \sum_{j=1}^{d} X_j^2 + \sum_{\ell=1}^{p} Y_\ell^2 \,,$$

with

$$X_j = (D_{x_j} - A_j(x))\,,\ j = 1, \dots, d\ ,\ \ Y_\ell = U_\ell\ ,\ \ell = 1, \dots, p\ .$$

In particular, the magnetic field is recovered by

$$B_{jk} = \frac{1}{i}[X_j, X_k] = \partial_j A_k - \partial_k A_j \ , \quad \text{for } j, k = 1, \dots, d \ .$$

We start with two trivial easy cases.
First we consider the case when $U \to +\infty$. In this case, it is well known that the operator has a compact resolvent.
On the opposite, if we assume that $U = 0$, $d = 2$ and $B(x) = B_{12} \geq 0$, one immediately gets:

$$\int B(x)|u(x)|^2 dx \leq ||X_1 u||^2 + ||X_2 u||^2 = \mathrm{Re}\, \langle \mathcal{P}_{\mathbf{A},iV} u \mid u \rangle \,. \quad (7)$$

Under the condition that $\lim_{|x|\to+\infty} B(x) = +\infty$, this implies that the operator has a compact resolvent.
Example:

$$A_1(x_1, x_2) = -x_2 x_1^2 \,,\; A_2(x_1, x_2) = +x_1 x_2^2 \,.$$

Here

$$B(x_1, x_2) = x_1^2 + x_2^2 \,.$$

In order to treat more general situations, we introduce (keeping $V = 0$ for the moment) the quantities:

$$\check{m}_q(x) = \sum_\ell \sum_{|\alpha|=q} |\partial_x^\alpha U_\ell| + \sum_{j<k} \sum_{|\alpha|=q-1} |\partial_x^\alpha B_{jk}(x)| \,. \tag{8}$$

It is easy to reinterpret this quantity in terms of commutators of the X_j's.
When $q = 0$, the convention is that

$$\check{m}_0(x) = \sum_\ell |U_\ell(x)| \,. \tag{9}$$

Let us also introduce

$$\check{m}^r(x) = 1 + \sum_{q=0}^{r} \check{m}_q(x) \,. \tag{10}$$

Then a criterion (due to Helffer-Mohamed (1988)) is

Theorem

If there exists r and C s.t.

$$\check{m}_{r+1}(x) \leq C\, \check{m}^r(x)\,,\ \forall x \in \mathbb{R}^d\,, \tag{11}$$

and

$$\check{m}^r(x) \to +\infty\,,\ \ \text{as } |x| \to +\infty\,, \tag{12}$$

then $\mathcal{P}_{\mathbf{A},U}(h)$ has a compact resolvent.

It is shown in [29], that one can get the same result under the weaker assumption

$$\check{m}_{r+1}(x) \leq C[\check{m}^r(x)]^{1+\delta}\,, \tag{13}$$

where $\delta = \frac{1}{2^{r+1}-3}$ $(r \geq 1)$.

This result is optimal for $r = 1$ according to a counterexample by A. Iwatsuka [22], who gives an example of a Schrödinger operator which has a non compact resolvent and s.t. $\sum_{j<k} |\nabla B_{jk}(x)|$ has the same order as $\sum_{j<k} |B_{jk}|^2$.

Other generalizations are given in [36] (Corollary 0.11) .
One can for example replace $\sum_j V_j^2$ by U and the conditions on the m_j's can be reformulated in terms of the variation of U and B in suitable balls (Reverse Hölder property).
In particular A. Iwatsuka [22] showed that a necessary condition is:

$$\int_{B(x,1)} \left(V(x) + \sum_{j<k} B_{jk}(x)^2 \right) dx \to +\infty \text{ as } |x| \to +\infty \,, \quad (14)$$

where $B(x,1)$ is the ball of radius 1 centered at x.

The accretive case : maximal accretivness and compactness

There is there a general statement (see [16], [3]) about the maximal accretiveness of $\mathcal{P}_{\mathbf{A},W}$, when $U \geq 0$. Moreover

$$\mathcal{P}_{\mathbf{A},W} = (P_{\mathbf{A},\bar{W}})^* \,. \tag{15}$$

We can now extend the previous theorem to the family of operators:

$$\mathcal{P}_{\mathbf{A},W} = \sum_{j=1}^{d} (D_{x_j} - A_j(x))^2 + \sum_{\ell=1}^{p} U_\ell(x)^2 + iV(x) \,, \tag{16}$$

with $W = U + iV$ and $V \in C^\infty$.
We introduce the new quantity:

$$\check{m}_q(x) = \sum_{\ell} \sum_{|\alpha|=q} |\partial_x^\alpha U_\ell| + \sum_{j<k} \sum_{|\alpha|=q-1} |\partial_x^\alpha B_{jk}(x)| + \sum_{|\alpha|=q-1} |\partial_x^\alpha V| \,. \tag{17}$$

Then the new criterion is

Theorem à la Kohn

If there exist r and C such that

$$\check{m}_{r+1}(x) \leq C_0\ \check{m}^r(x)\ ,\ \forall x \in \mathbb{R}^n\ , \tag{18}$$

then there exist $\delta > 0$ and $C_1 := C_1(C_0)$ s. t.

$$||(\check{m}^r(x))^\delta u||^2 \leq C_1\left(||P_{A,W}u||^2 + ||u||^2\right). \tag{19}$$

The proof shows that we can take $\delta = 2^{-r}$ which is in general not optimal.

Corollary

If in addition

$$\check{m}^r(x) \to +\infty\,, \quad \text{as } |x| \to +\infty\,. \tag{20}$$

Then $\mathcal{P}_{\mathbf{A},W}(h)$ has a compact resolvent.

When $\mathbf{A} = U = 0$, the choice of δ can be improved (Almog-Helffer) leading to optimal regularity for $r \leq 2$.

Proof of the theorem (sketch)

We can replace $\check{m}^r(x)$ by an equivalent C^∞ function $\Psi(x)$ which has the property that, $\exists C > 0$ and $\forall \alpha$, $\exists C_\alpha$ s.t.

$$\begin{array}{l} \frac{1}{C}\Psi(x) \leq \check{m}^r(x) \leq C\Psi(x)\,, \\ |D_x^\alpha \Psi(x)| \leq C_\alpha \Psi(x)\,. \end{array} \tag{21}$$

In the same spirit as in Kohn's proof, let us introduce

Definition

For $s > 0$, M^s is the space of C^∞ functions T s.t. $\exists C_s$ s.t.

$$||\Psi^{-1+s} Tu||^2 \leq C_s \left(||P_A u||\, ||u|| + ||u||^2\right)\,,\ \forall u \in C_0^\infty(\mathbb{R}^d)\,. \tag{22}$$

We first observe that

$$U_\ell \in M^1\,, \tag{23}$$

$$[X_j, X_k] \in M^{\frac{1}{2}}\,,\ \forall j, k = 1, \dots, d\,, \tag{24}$$

and

$$V \in M^{\frac{1}{2}}\,. \tag{25}$$

Another claim (integration by part) is:
If T is in M^s and $|\partial_x^\alpha T| \leq C_\alpha \Psi$, then $[X_k, T] \in M^{\frac{s}{2}}$, when $|\alpha| = 1$ or $|\alpha| = 2$.
Assuming these two properties, it is clear that

$$\Psi(x) \in M^{2^{-r}} .$$

The claim and (23) lead to

$$\partial_x^\alpha U_\ell \in M^{2^{-|\alpha|}} ,$$

and we deduce:

$$\partial_x^\alpha B_{jk} \in M^{2^{-(|\alpha|+1)}} .$$

The proof of the theorem then becomes easy.

Maximal estimates: Main assumptions

For $V \in C^\infty$, we introduce:

- $(H1)$ $\exists C_2 \geq 1$ and $\exists r \in \mathbb{N}$ s.t. , $\forall x \in \mathbb{R}^d$, $\forall R > 0$,

$$\frac{1}{C_2} \sup_{|y-x|\leq R} |V(y)| \leq \sum_{|\alpha|\leq r} R^{|\alpha|} |\partial^\alpha V(x)| \leq C_2 \sup_{|y-x|\leq R} |V(y)| \,.$$

- $(H2(r))$ $\exists C_0 > 0$ and $\exists r \in \mathbb{N}$ s.t.

$$\max_{|\beta|=r+1} |D_x^\beta V(x)| \leq C_0 \, m(x) \,,$$

where

$$m(x) := m_V^{(r)}(x) = \sqrt{\sum_{|\alpha|\leq r} |D_x^\alpha V(x)|^2 + 1} \,.$$

We note that any polynomial of degree r satisfies these conditions. With an extra effort (see [18]) we can remove (H1).

Main theorem

Theorem (Helffer-Nourrigat 2017)

If V satisfies for some $r \in \mathbb{N}$ assumptions (H1) and (H2), there exists $C > 0$ s.t. $\forall u \in C_0^\infty$

$$\|Vu\|^2 + \||V|^{1/2}\nabla u\|^2 \leq C\left(\|P_{iV}\, u\|^2 + ||u||^2\right) . \qquad (26)$$

One gets the complete regularity statement using the regularity of the Laplacian.

Hörmander's metrics and partition of unity.

As in the PHD of Mba Yébé, we introduce a parameter $\mu \geq 1$ to be determined later and an associate metrics.
For any $x \in \mathbb{R}^d$, $\exists R > 0$ unique, denoted by $R(x, \mu)$, s.t.

$$\sup_{|y-x| \leq R} |V(y)| = \frac{\mu}{R^2}\,. \tag{27}$$

Proposition (slow variation)

With $K = C_2\, 2^{r/2}$,

$$|y - x| \leq \frac{R(x, \mu)}{K} \Longrightarrow \frac{1}{K} \leq \frac{R(y, \mu)}{R(x, \mu)} \leq K\,.$$

This proposition shows that the metric defined on $\mathbb{R}^d$ by $g_x(t) = |t|^2/R(x, \mu)^2$ ($x \in \mathbb{R}^d$, $t \in \mathbb{R}^d$), is slowly varying in the sense of Hörmander.
Moreover, the constant in the definition is independent of μ.

We deduce from Lemma 18.4.4 in the book of Hörmander

Proposition: Partition of unity

For any $\mu \geq 1$, there exist (φ_j) in C_0^∞, and (x_j) in $\mathbb{R}^d$, s.t. :

- $$\sum_j \varphi_j(x)^2 = 1\,,\ \forall x \in \mathbb{R}^d\,. \tag{28}$$

- With K the constant of previous proposition ,

$$\operatorname{supp} \varphi_j \subset B(x_j, R(x_j,\mu)/K)\,. \tag{29}$$

Partition of unity (continued)

- $\forall \alpha$, $\exists \hat{C}_\alpha > 0$, independent of μ, s.t.

$$\sum_j |\partial^\alpha \varphi_j(x)|^2 \leq \frac{\hat{C}_\alpha}{R(x,\mu)^{2|\alpha|}} \,. \tag{30}$$

- $\exists \hat{C} > 0$, independent of μ, s. t., $\forall u \in C_0^\infty$,

$$\int_{\mathbb{R}^d} \frac{|u(x)|^2}{R(x,\mu)^4} dx \leq \hat{C} \left(\|u\|^2 + \sum_{R(x_j,\mu)\leq 1} \int_{\mathbb{R}^d} \frac{\varphi_j(x)^2 |u(x)|^2}{R(x_j,\mu)^4} dx \right) . \tag{31}$$

Proof of Main Theorem.

Proposition

For $\mu \geq 1$ let (x_j) be a sequence in $\mathbb{R}^d$ as above. $\exists \mu_0 > 1$ and $\exists C_3$ (depending only on C_0 and C_2) s.t., $\forall \mu \geq \mu_0$, $\forall j$ s.t. $R(x_j, \mu) \leq 1$, and $\forall f \in C_0^\infty$ supported in $B_j = B(x_j, R(x_j, \mu)/K)$,

$$\frac{\mu^\delta}{R(x_j, \mu)^2} \|f\| + \frac{\mu^{\delta/2}}{R(x_j, \mu)} \|\nabla f\| \leq C_3 \|P_{iV} f\| , \qquad (32)$$

where $\delta > 0$ (coming from Kohn's estimate).

Idea of the proof:

Apply Kohn's like estimate with

$$V_j(y) = R_j^2 \, V(x_j + R_j \, y) \text{ with } R_j = R(x_j, \mu) .$$

Observe that the V_j satisfy the condition for this estimate with constants which are independent of j and that the resulting estimates are obtained with constants independent of j.

End of the proof of Main Theorem.

Let $u \in C_0^\infty(\mathbb{R}^d)$. For all $\mu \geq 1$, we apply (31) and obtain:

$$\int_{\mathbb{R}^d} \left[\frac{|u(x)|^2}{R(x,\mu)^4} + \frac{|\nabla u(x)|^2}{R(x,\mu)^2} \right] dx \leq \hat{C} \left(\|u\|^2 + \|\nabla u\|^2 \right) + R \,,$$

with

$$R = \hat{C} \sum_{R(x_j,\mu) \leq 1} \frac{\|\varphi_j u\|^2}{R(x_j,\mu)^4} + \frac{\|\nabla(\varphi_j u)\|^2}{R(x_j,\mu)^2} \,.$$

We have also used (30). For any j s.t. $R(x_j, \mu) \leq 1$, we apply the local estimate (32) to $f = \varphi_j u$.
We get for $\mu \geq \mu_0$

$$
\begin{aligned}
R &\leq C\mu^{-2\delta} \textstyle\sum_{R(x_j,\mu)\leq 1} \|P_{iV}(\varphi_j u)\|^2 \\
&\leq C\mu^{-2\delta} \|P_{iV} u\|^2 + C\mu^{-2\delta} \textstyle\sum_j \left[\|\nabla\varphi_j \cdot \nabla u\|^2 + \|u(\Delta\varphi_j)\|^2\right].
\end{aligned}
$$

Using (30), we obtain for a new $C > 0$:

$$R \leq C\mu^{-2\delta}\|P_{iV}u\|^2 + C\mu^{-2\delta}\int_{\mathbb{R}^d}\left[\frac{|u(x)|^2}{R(x,\mu)^4} + \frac{|\nabla u(x)|^2}{R(x,\mu)^2}\right]dx\,.$$

For $\mu \geq \mu_1$ (with $\mu_1 \geq \mu_0$ large enough), we get for some new $C > 0$

$$\int_{\mathbb{R}^d}\left[\frac{|u(x)|^2}{R(x,\mu)^4} + \frac{|\nabla u(x)|^2}{R(x,\mu)^2}\right]dx \leq C(\|u\|^2+\|\nabla u\|^2)+C\mu^{-2\delta}\|P_{iV}u\|^2\,.$$

Using

$$\|\nabla f\|^2 \leq \|P_{iV} f\| \, \|f\| \,,$$

we then get

$$\int_{\mathbb{R}^d} \left[\frac{|u(x)|^2}{R(x,\mu)^4} + \frac{|\nabla u(x)|^2}{R(x,\mu)^2} \right] dx \leq C\|u\|^2 + C(1+\mu^{-2\delta})\|P_{iV} u\|^2 \,.$$

Main Theorem follows by observing (see (27)) that

$$|V(x)| \leq R(x,\mu)^{-2} \mu \,.$$

It is actually possible (Helffer-Nourrigat) to extend our main Theorem to the case with magnetic field:

$$W = \sum_{\ell} U_\ell^2 + iV \,,$$

and the associated complex Schrödinger operator $P_{\mathbf{A},W}$.

Main theorem with magnetic fields (Helffer-Nourrigat (2017)

Under the assumptions of the theorem "à la Kohn", there exists $C > 0$ such that, for all $u \in C_0^\infty(\mathbb{R}^d)$:

$$\| |W| u \|^2 \leq C \left(\| P_{\mathbf{A},W}\, u \|^2 + ||u||^2 \right) \,. \qquad (33)$$

We introduce, for $t \in [0,1]$ and $x \in \mathbb{R}^d$,

$$\begin{aligned} \Phi(x,t) \; &= \sum_\ell \sum_{|\alpha|\leq r} t^{|\alpha|+1} \, |\partial_x^\alpha U_\ell(x)| \\ &\quad + \sum_{j<k} \sum_{|\alpha|\leq r-1} t^{|\alpha|+2} \, |\partial_x^\alpha B_{jk}(x)| \\ &\qquad + \sum_{|\alpha|\leq r-1} t^{|\alpha|+2} |\partial_x^\alpha V(x)| \, . \end{aligned} \tag{34}$$

This time for $\mu \geq 1$ we define

$$R(x,\mu) = \sup\{t \in [0,1], \quad \Phi(x,t) \leq \mu\} \, .$$

Then the proof goes on along the same scheme as before, with additional technicalities.

To complete the regularity result, we should use a "self-adjoint" statement of optimal regularity for $P_{\mathbf{A},|W|}$ or for the magnetic Laplacian $P_{\mathbf{A},\mathbf{0}}$ and use either Helffer-Nourrigat (if $\mathbf{A}$ is a polynomial) or Shen (see above) for more general cases.

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