Spectral problems related to a time dependent model in superconductivity with electric current. (after Almog, Almog-Helffer-Pan, ...)

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with numerical simulations of W. Bordeaux-Montrieux.

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This talk is mainly inspired by the reading of a paper of Yaniv Almog at Siam [2]. The main goal is to present proofs which will have some general character and will for example apply in a more physical model, for which we have obtained recently results together with Y. Almog and X. Pan.
After a presentation of the general problems and of our main results, we will come back to Almog's analysis. We will start from the complex Airy operator on $D_x^2 + ix$ on the line or on $\mathbb{R}^+$ and make a survey of what is known.

We also include some improvments coming of discussions with in particular F. Hérau, J. Martinet, F. Nier, other notes of Y.Almog, X. Pan and C. Han, W. Bordeaux-Montrieux who provides us also with numerical computations.

We also include some improvments coming of discussions with in particular F. Hérau, J. Martinet, F. Nier, other notes of Y.Almog, X. Pan and C. Han, W. Bordeaux-Montrieux who provides us also with numerical computations.
Similar results are presented in recent lectures of J. Sjöstrand in Evian (see the lectures of B. Davies and M. Zworski in this conference) and recover some questions we met already on the Fokker-Planck equation (Helffer-Herau-Nier). They are also related to the recent paper of Gallagher-Gallay-Nier.

Consider a superconductor placed in an applied magnetic field and submitted to an electric current through the sample. It is usually said that if the applied magnetic field is sufficiently high, or if the electric current is strong, then the sample is in a normal state. We are interested in analyzing the joint effect of the applied field and the current on the stability of the normal state.

Consider a superconductor placed in an applied magnetic field and submitted to an electric current through the sample. It is usually said that if the applied magnetic field is sufficiently high, or if the electric current is strong, then the sample is in a normal state. We are interested in analyzing the joint effect of the applied field and the current on the stability of the normal state.

To be more precise, let us consider a two-dimensional superconducting sample capturing the entire xy plane. We can assume also that a magnetic field of magnitude $\mathcal{H}^e$ is applied perpendicularly to the sample. Denote the Ginzburg-Landau parameter of the superconductor by κ and the normal conductivity of the sample by σ.

The physical problem is posed in a domain Ω with specific boundary conditions.
We will only analyze limiting situations where the domains possibly after a blowing argument become the whole space (or the half-space).
We will mainly work in $2D$ for simplification. $3D$ is also very important.

Then the time-dependent Ginzburg-Landau system (also known as the Gorkov-Eliashberg equations) is in $(0, T) \times \mathbb{R}^2$:

$$\begin{cases} \partial_t \psi + i\kappa \Phi \psi = \nabla^2_{\kappa \mathbf{A}} \psi + \kappa^2 (1 - |\psi|^2) \psi \,, \\ \kappa^2 \text{ curl }^2 \mathbf{A} + \sigma(\partial_t \mathbf{A} + \nabla \Phi) = \kappa \mathrm{Im}\,(\bar{\psi} \nabla_{\kappa \mathbf{A}} \psi) + \kappa^2 \text{ curl } \mathcal{H}^e \,, \end{cases} \tag{1}$$

where ψ is the order parameter, $\mathbf{A}$ is the magnetic potential, Φ is the electric potential, and $(\psi, \mathbf{A}, \Phi)$ also satisfies an initial condition at $t = 0$.

In order to solve this equation, one should also define a Gauge (Coulomb, Lorentz,...). The orbit of $(\psi, \mathbf{A}, \Phi)$ is $(\exp i\kappa q\, \psi, \mathbf{A} + \nabla q, \Phi - \partial_t q)$ where q is a function of (x, t). We refer to [5] for a discussion of this point. We will choose the Coulomb gauge which reads that we can add the condition $\operatorname{div} \mathbf{A} = 0$ for any t. Another possibility could be to take $\operatorname{div} \mathbf{A} + \sigma\Phi = 0$.

A solution $(\psi, \mathbf{A}, \Phi)$ is called a normal state solution if $\psi = 0$.

Stationary normal solutions

From (1) we see that if $(0, \mathbf{A}, \Phi)$ is a time-independent normal state solution then $(\mathbf{A}, \Phi)$ satisfies the equality

$$\kappa^2 \text{ curl }^2\mathbf{A} + \sigma\nabla\Phi = \kappa^2 \text{ curl } \mathcal{H}^e\,, \;\; \text{div } \mathbf{A} = 0 \quad \text{in } \mathbb{R}^2\,. \qquad (2)$$

(Note that if one identifies $\mathcal{H}^e$ to a function h, then $\text{curl } \mathcal{H}^e = (-\partial_y h\,,\, \partial_x h)$.)

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(Note that if one identifies $\mathcal{H}^e$ to a function h, then $\text{curl } \mathcal{H}^e = (-\partial_y h\,, \partial_x h)$.)
This could be rewritten as the property that

$$\kappa^2(\text{ curl } \mathbf{A} - \mathcal{H}^e) + i\sigma\Phi\,,$$

is an holomorphic function.
In particular

$$\Delta\Phi = 0 \text{ and } \Delta(\text{ curl } \mathbf{A} - \mathcal{H}^e)) = 0\,.$$

Special situation: Φ affine

(1) has the following stationary normal state solution

$$\mathbf{A} = \frac{1}{2J}(Jx+h)^2 \hat{\imath}_y, \quad \Phi = \frac{\kappa^2 J}{\sigma} y\,. \qquad (3)$$

Note that

$$\text{curl } \mathbf{A} = (Jx+h)\, \hat{\imath}_z,$$

that is, the induced magnetic field equals the sum of the applied magnetic field $h\hat{\imath}_z$ and the magnetic field produced by the electric current $Jx\,\hat{\imath}_z$.

For this normal state solution, the linearization of (1) with respect to the order parameter is

$$\partial_t \psi + \frac{i\kappa^3 J y}{\sigma}\psi = \Delta\psi - \frac{i\kappa}{J}(Jx+h)^2\partial_y\psi - (\frac{\kappa}{2J})^2(Jx+h)^4\psi + \kappa^2\psi\,. \tag{4}$$

Applying the transformation $x \to x - J/h$, the time-dependent linearized Ginzburg-Landau equation takes the form

$$\frac{\partial\psi}{\partial t} + i\frac{J}{\sigma}y\psi = \Delta\psi - iJx^2\frac{\partial\psi}{\partial y} - \left(\frac{1}{4}J^2x^4 - \kappa^2\right)\psi\,. \tag{5}$$

Rescaling x and t by applying

$$t \to J^{2/3} t \; ; \; (x,y) \to J^{1/3}(x,y)\,, \tag{6}$$

yields

$$\partial_t u = -(\mathcal{A}_{0,c} - \lambda) u\,, \tag{7}$$

where

$$\mathcal{A}_{0,c} := D_x^2 + (D_y - \frac{1}{2}x^2)^2 + icy\,, \tag{8}$$

and

$$c = 1/\sigma \; ; \; \lambda = \frac{\kappa^2}{J^{2/3}} \; ; \; u(x,y,t) = \psi(J^{-1/3}x, J^{-1/3}y, J^{-2/3}t)\,.$$

Our main problem will be to analyze the long time property of the attached semi-group.

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We now apply the transformation

$$u \to u\, e^{icyt}$$

to obtain

$$\partial_t u = -\left(D_x^2 u + (D_y - \frac{1}{2}x^2 - ct)^2 u - \lambda u \right) . \qquad (9)$$

Note that considering the partial Fourier transform, we obtain

$$\partial_t \hat{u} = -D_x^2 \hat{u} - \left[\left(\frac{1}{2}x^2 + (ct - \omega) \right)^2 - \lambda \right] \hat{u} . \qquad (10)$$

This can be rewritten as the analysis of a family (depending on $\omega \in \mathbb{R}$) of time-dependent problems on the line

$$\partial_t \hat{u} = -\mathcal{L}_{\beta(t,\omega)} \hat{u} + \lambda \hat{u} \,, \tag{11}$$

with $\mathcal{L}_\beta$ being the well-known anharmonic oscillator (or Montgomery operator) :

$$\mathcal{L}_\beta = D_x^2 + (\frac{1}{2}x^2 + \beta)^2 \,, \tag{12}$$

and

$$\beta(t,\omega) = ct - \omega \,.$$

Note that in this change of point of view, we can after a change of time look at the family of problems

$$\partial_\tau v(x,\tau) = -(\mathcal{L}_{c\tau} v)(x,\tau) + \lambda v(x,\tau)\,, \qquad (13)$$

the initial condition at $t=0$ becoming at $\tau = -\frac{\omega}{c}$.

Recent results by Almog-Helffer-Pan

If $c \neq 0$, $\mathcal{A} = \overline{\mathcal{A}_{0,c}}$ has compact resolvent, empty spectrum, and

$$\|\exp(-t\mathcal{A})\| \leq \exp\left(-\frac{2\sqrt{2c}}{3}t^{3/2} + Ct^{3/4}\right). \tag{14}$$

There exists C such that, for all λ such that $\mathrm{Re}\,\lambda \geq 0$,

$$\|(\mathcal{A}-\lambda)^{-1}\| \leq \exp\left(\frac{1}{6c}\mathrm{Re}\,\lambda^3 + C\,\mathrm{Re}\,\lambda^{3/2}\right). \tag{15}$$

Simplified model: no magnetic field

We assume, following Almog, that a current of constant magnitude J is being flown through the sample in the x axis direction, and $h = 0$.
Then (1) has (in some asymptotic regime) the following stationary normal state solution

$$\mathbf{A} = 0\,, \quad \Phi = Jx\,. \tag{16}$$

For this normal state solution, the linearization of (1) gives

$$\partial_t \psi + iJx\psi = \Delta_{x,y}\psi + \psi\,, \tag{17}$$

whose analysis is strongly related to the Airy equation.

The Airy operator in $\mathbb{R}$

The operator $D_x^2 + ix$ can be defined as the closed extension $\mathcal{A}$ of the differential operator on $C_0^\infty(\mathbb{R})$:

$$\mathcal{A}_0^+ := D_x^2 + ix\,. \tag{18}$$

We observe that

$$\mathcal{A} = (\mathcal{A}_0^-)^* \quad \text{with } \mathcal{A}_0^- := D_x^2 - ix\,. \tag{19}$$

and,

$$D(\mathcal{A}) = \{u \in H^2(\mathbb{R})\,,\, xu \in L^2(\mathbb{R})\}\,. \tag{20}$$

In particular $\mathcal{A}$ has compact resolvent.

It is also easy to see that

$$\mathrm{Re}\,\langle \mathcal{A}u \,|\, u\rangle \geq 0\,. \tag{21}$$

Hence $-\mathcal{A}$ is the generator of a semi-group S_t of contraction,

$$S_t = \exp -t\mathcal{A}\,. \tag{22}$$

Hence all the results of this theory can be applied.
In particular, we have, for $\mathrm{Re}\,\lambda < 0$

$$||(\mathcal{A}-\lambda)^{-1}|| \leq \frac{1}{|\mathrm{Re}\,\lambda|}\,. \tag{23}$$

One can also show that the operator is maximally accretive.

A very special property of this operator is that, for any $a \in \mathbb{R}$,

$$T_a \mathcal{A} = (\mathcal{A} - ia) T_a \,, \tag{24}$$

where T_a is the translation operator :

$$(T_a u)(x) = u(x-a) \,. \tag{25}$$

As immediate consequence, we obtain that the spectrum is empty

$$\sigma(\mathcal{A}) = \emptyset \tag{26}$$

and that the resolvent of $\mathcal{A}$, which is defined for any $\lambda \in \mathbb{C}$ satisfies

$$||(\mathcal{A} - \lambda)^{-1}|| = ||(\mathcal{A} - \mathrm{Re}\,\lambda)^{-1}|| \,. \tag{27}$$

One can also look at the semi-classical question, i.e. look

$$\mathcal{A}_h = h^2 D_x^2 + ix \,. \tag{28}$$

Of course in an homogeneous situation one can go from one point of view to the other but it is sometimes good to look at what each theory gives on this very particular model.
This could for example interact with the first part of the lectures by J. Sjöstrand.

The most interesting property is the control of the resolvent for $\mathrm{Re}\,\lambda \geq 0$.

Proposition

There exist two positive constants C_1 and C_2, such that

$$C_1 |\mathrm{Re}\,\lambda|^{-\frac{1}{4}} \exp \frac{4}{3} \mathrm{Re}\,\lambda^{\frac{3}{2}} \leq ||(A-\lambda)^{-1}|| \leq C_2 |\mathrm{Re}\,\lambda|^{-\frac{1}{4}} \exp \frac{4}{3} \mathrm{Re}\,\lambda^{\frac{3}{2}}, \tag{29}$$

(see Martinet [17] for this fine version).

The proof of the (rather standard) upper bound is based on the direct analysis of the semi-group in the Fourier representation. We note indeed that

$$\mathcal{F}(D_x^2 + ix)\mathcal{F}^{-1} = \xi^2 + \frac{d}{d\xi}\,. \tag{30}$$

Then we have

$$\mathcal{F} S_t \mathcal{F}^{-1} v = \exp(-\xi^2 t + \xi t - \frac{t^3}{3}) v(\xi - t)\,, \tag{31}$$

and this implies immediately

$$||S_t|| = \exp \max_{\xi}(-\xi^2 t + \xi t - \frac{t^3}{3}) = \exp(-\frac{t^3}{12})\,. \tag{32}$$

Note that this implies that the spectrum of $\mathcal{A}$ is empty.

Then one can get an estimate of the resolvent by using, for λ real, the formula

$$(\mathcal{A}-\lambda)^{-1} = \int_0^{+\infty} \exp -t(\mathcal{A}-\lambda)\, dt\,. \tag{33}$$

We immediately obtain

$$||(\mathcal{A}-\lambda)^{-1}|| \leq \int_0^{+\infty} \exp(\lambda t - \frac{t^3}{12})\, dt\,. \tag{34}$$

The asymptotic behavior as $\lambda \to +\infty$ of this integral is immediately obtained by using the Laplace method and the dilation $t = \lambda^{\frac{1}{2}} s$ in the integral.

The proof of the lower bound is obtained by constructing quasimodes for the operator $(\mathcal{A}-\lambda)$ in its Fourier representation. We observe that

$$u(\xi;\lambda) := \exp -\frac{\xi^3}{3} + \lambda\xi \tag{35}$$

is a solution of

$$(\partial_\xi + \xi^2 - \lambda)u(\xi;\lambda) = 0\,. \tag{36}$$

Multiplying u by a cut-off function in such a way that at the cut-off u is much smaller than at its maximum, we obtain a very good quasimode with an error term giving the stated lower bound for the resolvent.
Of course this is a very special case of a result on the pseudo-spectrum but this leads to an almost optimal result.

The Airy complex operator in $\mathbb{R}^+$

Here we mainly describe some results presented in [2], who refers to [16]. We can then associate the Dirichlet realization $\mathcal{A}^D$ of $D_x^2 + ix$ on the half space, whose domain is

$$D(\mathcal{A}^D) = \{u \in L^2, x^{\frac{1}{2}} u \in L^2\,,\,(D_x^2 + ix)u \in L^2(\mathbb{R}^+)\}\,, \qquad (37)$$

and which is defined (in the sense of distributions) by

$$\mathcal{A}^D u = (D_x^2 + ix)u\,. \qquad (38)$$

Moreover, by construction, we have

$$\mathrm{Re}\,\langle \mathcal{A}^D u \,|\, u\rangle \geq 0\,,\, \forall u \in D(\mathcal{A}^D)\,. \tag{39}$$

Again we have an operator, which is the generator of a semi-group of contraction, whose adjoint is described by replacing in the previous description $(D_x^2 + ix)$ by $(D_x^2 - ix)$, the operator is injective and as its spectrum contained in $\mathrm{Re}\,\lambda > 0$. Moreover, the operator has compact inverse, hence the spectrum (if any) is discrete.

Using what is known on the usual Airy operator, Sibuya's theory and a complex rotation, we obtain ([2]) that

$$\sigma(\mathcal{A}^D) = \cup_{j=1}^{+\infty}\{\lambda_j\} \tag{40}$$

with

$$\lambda_j = \exp i\frac{\pi}{3}\,\mu_j\,, \tag{41}$$

the μ_j's being real zeroes of the Airy function satisfying

$$0 < \mu_1 < \cdots < \mu_j < \mu_{j+1} < \cdots\,. \tag{42}$$

It is also shown in [2] that the vector space generated by the corresponding eigenfunctions is dense in $L^2(\mathbb{R}^+)$.

We arrive now to the analysis of the properties of the semi-group and the estimate of the resolvent.
As before, we have, for $\mathrm{Re}\,\lambda < 0$,

$$||(\mathcal{A}^D - \lambda)^{-1}|| \leq \frac{1}{|\mathrm{Re}\,\lambda|}\,, \tag{43}$$

If $\mathrm{Im}\,\lambda < 0$ one gets also a similar inequality, so the main remaining question is the analysis of the resolvent in the set $\mathrm{Re}\,\lambda \geq 0\,,\ \mathrm{Im}\,\lambda \geq 0$, which corresponds to the numerical range of the symbol.

We recall that for any $\epsilon > 0$, we define the ϵ-pseudospectrum by

$$\Sigma_\epsilon(\mathcal{A}^D) = \{\lambda \in \mathbb{C} \,|\, ||(\mathcal{A}^D - \lambda)^{-1}|| > \frac{1}{\epsilon}\}\,, \tag{44}$$

with the convention that $||(\mathcal{A}^D - \lambda)^{-1}|| = +\infty$ if $\lambda \in \sigma(\mathcal{A}^D)$.
We have

$$\cap_{\epsilon>0}\Sigma_\epsilon(\mathcal{A}^D) = \sigma(\mathcal{A}^D)\,. \tag{45}$$

We define, for any accretive operator, for $\epsilon > 0$,

$$\widehat{\alpha}_\epsilon(\mathcal{A}) = \inf_{z \in \Sigma_\epsilon(\mathcal{A})} \mathrm{Re}\, z\,. \tag{46}$$

We also define

$$\widehat{\omega}_0(\mathcal{A}) = \lim_{t \to +\infty} \frac{1}{t} \log || \exp -t\mathcal{A}|| \tag{47}$$

$$\widehat{\alpha}_\epsilon(\mathcal{A}) \leq \inf_{z \in \sigma(\mathcal{A})} \mathrm{Re}\, z\,. \tag{48}$$

Theorem (Gearhart-Prüss)

Let $\mathcal{A}$ be a densely defined closed operator in an Hilbert space X such that $-\mathcal{A}$ generates a contraction semi-group and let $\widehat{\alpha}_\epsilon(\mathcal{A})$ and $\widehat{\omega}_0(\mathcal{A})$ denote the ϵ-pseudospectral abcissa and the growth bound of $\mathcal{A}$ respectively. Then

$$\lim_{\epsilon \longrightarrow 0} \widehat{\alpha}_\epsilon(\mathcal{A}) = -\widehat{\omega}_0(\mathcal{A}) \,. \tag{49}$$

We refer to [8] for a proof.

This theorem is interesting because it reduces the question of the decay, which is basic in the question of the stability to an analysis of the ϵ-spectrum of the operator.

We apply this theorem to our operator $\mathcal{A}_D$ and our main theorem is

Theorem

$$\widehat{\omega}_0(\mathcal{A}_D) = -\mathrm{Re}\,\lambda_1\,. \tag{50}$$

This statement was established by Almog in a much weaker form. Using the first eigenfunction it is easy to see that

$$||\exp -t\mathcal{A}^D|| \geq \exp -\mathrm{Re}\,\lambda_1\,t\,. \tag{51}$$

Hence we have immediately

$$0 \geq \widehat{\omega}_0(\mathcal{A}^D) \geq -\mathrm{Re}\,\lambda_1\,. \tag{52}$$

To prove that $-\mathrm{Re}\,\lambda_1 \geq \widehat{\omega}_0(\mathcal{A}^D)$, it is enough to show the following lemma.

Lemma

For any $\alpha < \mathrm{Re}\,\lambda_1$, there exists a constant C such that, for all λ s.t. $\mathrm{Re}\,\lambda \leq \alpha$

$$||(\mathcal{A}^D - \lambda)^{-1}|| \leq C\,. \tag{53}$$

Proof

We know that λ is not in the spectrum. Hence the problem is just a control of the resolvent as $|\mathrm{Im}\,\lambda| \rightarrow +\infty$. The case, when $\mathrm{Im}\,\lambda < 0$ has already be considered. Hence it remains to control the norm of the resolvent as $\mathrm{Im}\,\lambda \rightarrow +\infty$ and $\mathrm{Re}\,\lambda \in [-\alpha, +\alpha]$.

This is indeed a semi-classical result ! The main idea is that when $\operatorname{Im}\lambda \to +\infty$, we have to inverse the operator

$$D_x^2 + i(x - \operatorname{Im}\lambda) - \operatorname{Re}\lambda\,.$$

If we consider the Dirichlet realization in the interval $]0, \frac{\mathrm{Im}\,\lambda}{2}[$ of $D_x^2 + i(x - \mathrm{Im}\,\lambda) - \mathrm{Re}\,\lambda$, it is easy to see that the operator is invertible by considering the imaginary part of this operator and that this inverse $R_1(\lambda)$ satisfies

$$||R_1(\lambda)|| \leq \frac{2}{\mathrm{Im}\,\lambda}.$$

Far from the boundary, we can use the resolvent of the problem on the line for which we have a unifom control of the norm for $\mathrm{Re}\,\lambda \in [-\alpha, +\alpha]$.

Application

Coming back to the application in superconductivity, one is looking at the semigroup associated with $\mathcal{A}_J := D_x^2 + iJx - 1$ (where $J \geq 0$ is a parameter). The stability analysis leads to a critical value J_c with

$$J_c = (\mathrm{Re}\,\lambda_1)^{-\frac{3}{2}}\,. \tag{54}$$

For $J \in [0, J_c[$ the associated operator $\exp -t\mathcal{A}_J$ satisfies $||\exp -t\mathcal{A}_J|| \to +\infty$ as $t \to +\infty$.
For $J > J_c$, $||\exp -t\mathcal{A}_J|| \to 0$.
This improves Lemma 2.4 in Almog [2], who gets only this decay for $||\exp -t\mathcal{A}_J\psi||$ for ψ in a specific dense space.

Here we reproduce numerical computations for us by W. Bordeaux Montrieux.

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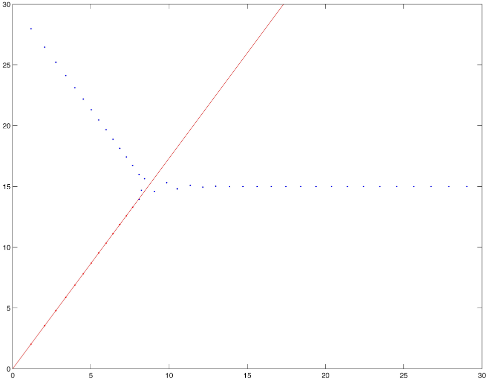
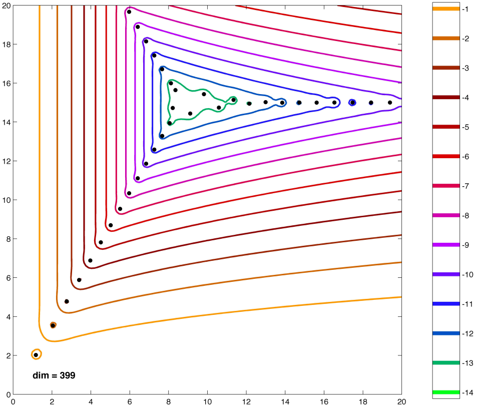
W. Bordeaux Montrieux is using a programme of Trefethen using mapple.

Airy with Dirichlet condition: Spectrum



As usual for this kind of computation for non self adjoint operators, we observe, in addition to the spectrum lying on the halfline of argument $\frac{\pi}{3}$, a parasite spectrum starting from the fifteen-th eigenvalue.
This was already observed for the complex harmonic oscillator $D_x^2 + ix$ considered by B. Davies.
This is immediately connected with the accuracy of Maple.

Airy with Dirichlet condition : pseudospectra



The computation is done on an interval $[0, L]$ with Dirichlet conditions at 0 and L using 400 “grid points”. We represent the levels $||(A - z)^{-1} = \frac{1}{\epsilon}$ corresponding to the boundary of the ϵ-pseudopsectra. The right column gives the correspondence between the color and $\log_{10}(\epsilon)$.

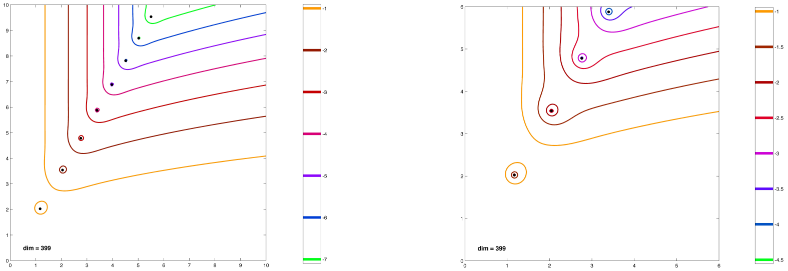


Figure: Zooms 1 and 2

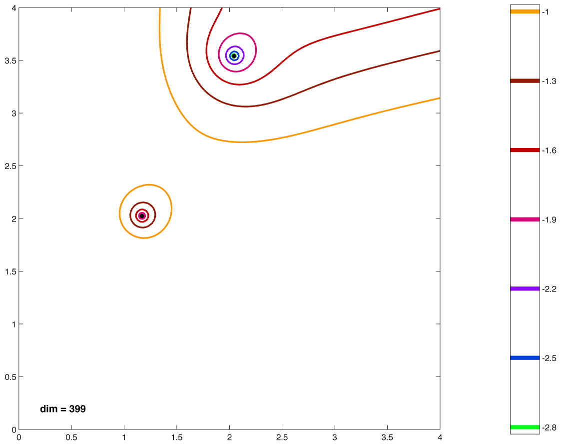
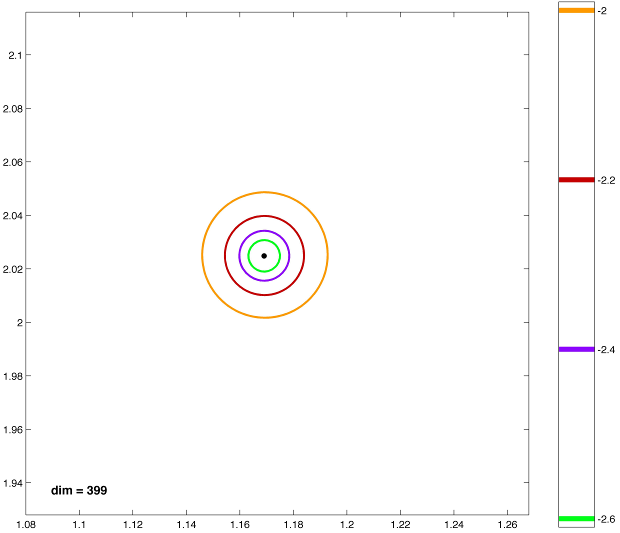


Figure: Zoom3


2.1
2.08
2.06
2.04
2.02
2
1.98
1.96
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dim = 399
1.08
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1.12
1.14
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1.18
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1.22
1.24
1.26
-2
-2.2
-2.4
-2.6

Higher dimension problems relative to Airy

Here we follow (and extend) [2] Almog. We consider the operator

$$\mathcal{A}_2 := -\Delta_{x,y} + ix\,. \tag{55}$$

Proposition

$$\sigma(\mathcal{A}_2) = \emptyset\,. \tag{56}$$

Proof

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After a Fourier transform in the y variable,
it is enough to show that

$$(\widehat{\mathcal{A}_2} - \lambda)$$

is invertible with

$$\widehat{\mathcal{A}_2} = D_x^2 + ix + \eta^2 \,. \tag{57}$$

We have just to control the resolvent uniformly in η.

The model in $\mathbb{R}^2_+$: perpendicular current.

Here it is useful to reintroduce the parameter J which is assumed to be positive. Hence we consider the Dirichlet realization

$$\mathcal{A}_2^{D,\perp} := -\Delta_{x,y} + iJx\,, \tag{58}$$

in $\mathbb{R}^2_+ = \{x > 0\}$.

Proposition

$$\sigma(\mathcal{A}_2^{D,\perp}) = \cup_{r\geq 0, j\in\mathbb{N}^*}(\lambda_j + r)\,. \tag{59}$$

Proof

For the inclusion

$$\cup_{r\geq 0, j\in\mathbb{N}^*}(\lambda_j + r) \subset \sigma(\mathcal{A}_2^{D,\perp}),$$

we can use L^∞ eigenfunctions in the form

$$(x,y,z) \mapsto \exp i(y\eta + z\zeta)u_j(x)$$

where u_j is the eigenfunction associated to λ_j.
We have then to use the fact that L^∞-eigenvalues belong to the spectrum.

For the opposite inclusion, we observe that we have to control uniformly

$$(\mathcal{A}^D - \lambda + \eta^2)^{-1}$$

with respect to η under the condition that

$$\lambda \notin \cup_{r\geq 0, j\in\mathbb{N}^*} (\lambda_j + r)\,.$$

It is enough to observe the uniform control as $\eta^2 \to +\infty$ which results of (43). □

Then we can get the following information on the L^∞-spectrum.

Proposition

Let $\Psi \in L^\infty(\mathbb{R}^2_+) \cap H^1_{loc}(\mathbb{R}^2_+)$ satisfying, for some $\lambda \in \mathbb{C}$,

$$-\Delta_{x,y}\Psi + iJx\Psi = \lambda\Psi \tag{60}$$

in $\mathbb{R}^2_+$ and

$$\Psi_{x=0} = 0\,. \tag{61}$$

Then either $\Psi = 0$ or $\lambda \in \sigma(\mathcal{A}_3^{D,\perp})$.

The model in $\mathbb{R}_2^+$: parallel current.

Here the models are the Dirichlet realization in $\mathbb{R}^2_+$:

$$\mathcal{A}_2^{D,\|} = -\Delta_{x,y} + iJy\,, \tag{62}$$

or the Neumann realization

$$\mathcal{A}_2^{N,\|} = -\Delta_{x,y} + iJy\,. \tag{63}$$

Using the reflexion (or antireflexion) trick we can see the problem as a problem on $\mathbb{R}^2$ restricted to odd (resp. even) functions with respect to $(x, y) \mapsto (-x, y)$. It is clear from Proposition 1 that in this case the spectrum is empty.

Other models

The goal is to treat more general situations were we no more know explicitely the spectrum like for complex Airy or complex harmonic oscillator.
At least for the case without boundary this is close to the problematic of the lectures of J. Sjöstrand.
The operators we have in mind are

$$D_x^2 + (D_y - \frac{1}{2}x^2)^2 + icy\,, \tag{64}$$

and the next one could be

$$(D_x + \frac{x^3}{3})^2 + (D_y - x^2 y)^2 + ic(x^2 - y^2)\,. \tag{65}$$

More generally :

$$B(x,y) = \mathrm{Re}\,\psi(z)\,,\;\Phi(x,y) = c\mathrm{Im}\,\psi(z)\,, \tag{66}$$

with ψ holomorphic will work. If ψ is a non constant polynomial and $c \neq 0$ then the operator will have compact resolvent.

Maximal accretivity

All the operators can be placed in the following more general context. We consider in $\mathbb{R}^n$ (or in an open set)

$$-\Delta_A + V\,, \tag{67}$$

with

$$\mathrm{Re}\, V \geq 0 \ \text{ and } \ V \in C^\infty(\mathbb{R}^n)\,.$$

Then the operator is maximally accretive.

A criterion for compactness of the resolvent

Here we follow the proof of Helffer-Mohamed , actually inspired by Kohn's proof in Subellipticity. We will analyze the problem for the family of operators :

$$P_{A,V} = \sum_{j=1}^{n} (D_{x_j} - A_j(x))^2 + V(x)\,. \tag{68}$$

Here the magnetic potential $A(x) = (A_1(x), A_2(x), \cdots, A_n(x))$ is supposed to be C^∞ and the electric potential

$$V(x) = (\sum_j V_j(x)^2) + iQ(x)$$

is C^∞.

We note also that it has the form :

$$P_{A,V} = \sum_{j=1}^{n+p} X_j^2 = \sum_{j=1}^{n} X_j^2 + \sum_{\ell=1}^{p} Y_\ell^2 + iX_0 \,,$$

with

$$X_j = (D_{x_j} - A_j(x))\,,\, j = 1, \dots, n\,,\ Y_\ell = V_\ell\,,\ \ell = 1, \dots, p\,,\ X_0 = Q.$$

In particular, the magnetic field is recovered by observing that

$$B_{jk} = \frac{1}{i}[X_j, X_k] = \partial_j A_k - \partial_k A_j\,,\quad \text{for } j, k = 1, \dots, n\,.$$

In order to treat more general situations, we introduce the quantities :

$$m_q(x) = \sum_\ell \sum_{|\alpha|=q} |\partial_x^\alpha V_\ell| + \sum_{j<k} \sum_{|\alpha|=q-1} |\partial_x^\alpha B_{jk}(x)| + \sum_{|\alpha|=q-1} |\partial_x^\alpha Q| \,. \tag{69}$$

It is easy to reinterpret this quantity in terms of commutators of the X_j's.
When $q=0$, the convention is that

$$m_0(x) = \sum_\ell |V_\ell(x)| \,. \tag{70}$$

Let us also introduce

$$m^r(x) = 1 + \sum_{q=0}^{r} m_q(x) \,. \tag{71}$$

Then the criterion is

Theorem

Let us assume that there exists r and a constant C such that

$$m_{r+1}(x) \leq C\, m^r(x)\,,\ \forall x \in \mathbb{R}^n\,, \tag{72}$$

and

$$m^r(x) \to +\infty\,,\quad \text{as } |x| \to +\infty\,. \tag{73}$$

Then $P_{A,V}(h)$ has a compact resolvent.

About the L^∞-spectrum

The question we are analyzing can be reformulated as a comparison between the L^∞-spectrum and the L^2-spectrum. Assuming that we have determined the L^2-spectrum in the previous proposition, we would like to analyze the consequence of an admissible pair (with bounded ψ).

Proposition

We assume that $V \in C^0$ and $\mathrm{Re}\, V \geq 0$ If (ψ, λ) is an admissible L^∞ pair for $P_{\mathbf{A},V}$, i.e. if

$$(P_{\mathbf{A},V} - \lambda)\psi = 0 \text{ in } \mathcal{D}'(\mathbb{R}^n)\,,$$

with $\psi \in L^\infty$ (or L^2), then λ is in the spectrum of $\mathcal{P} = \overline{P_{A,V}}$.

The proof is reminiscent of the so-called Schnol's theorem.

"In progress" with Y. Almog and X. Pan

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Analyze

$$\frac{1}{t} \ln || \exp -t\overline{P_{\mathbf{A},V}} || \,,$$

as $t \to +\infty$ for our specific examples

1. in the case of the whole space,
2. in the case of the half space
3. and apply these results to the stability question of problem with boundary in various asymptotic limits.

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