On spectral problems connected with the time dependent Ginzburg-Landau system. Course in Berder (July 2013)–NOSEVOL– Provisory notes

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Résumé et Abstract

Résumé:

Dans ce cours nous voudrions aborder quelques questions spectrales apparaissant dans l'étude de l'équation de Ginzburg-Landau dépendant du temps (due à Gorkov-Eliashberg) et plus spécialement la question de la stabilité globale des solutions stationnaires normales. Dans ce sujet encore en friche nous chercherons montrer le rôle du courant électrique en comparaison avec le rôle du champ magnétique extérieur pour le problème indépendant du temps (théorème de Giorgi-Phillips).
les théorèmes récents ont été obtenus en collaboration avec Y. Almog, X. Pan, R. Henry, K. Beauchard et L. Robbiano. Pour la fin de ce cours, voir l'exposé sur mon site web.

Abstract

In this course we would like to discuss spectral properties of non self-adjoint operators appearing in the analysis of the long time behavior of the solutions of the time dependent Ginzburg Landau system (due to Eliashberg-Gorkov) and to consider in particular the global stability of the stationary normal solutions.

In this subject where only preliminary results have been obtained, we will focus on the role of the electric current in comparison with the role of the exterior magnetic field for the time independent problem (Giorgii-Phillips theorem).

The recent theorems have been obtained in collaboration with Y. Almog, X. Pan, R. Henry, K. Beauchard and L. Robbiano. For the end of this course, see the talk on my website.

Chapter 1

The Ginzburg-Landau Functional

For details the reader is sent to the books of Fournais-Helffer and Sandier-Serfaty appearing in the series Progress in Non-Linear Analysis (Birkhäuser).

1.1 The problem in superconductivity

Let us describe the mathematical problem. It is naturally posed for domains in $\mathbb{R}^3$, but for cylindrical domains in $\mathbb{R}^3$, it is natural (though not completely justified mathematically) to consider a functional defined in a domain $\Omega \subset \mathbb{R}^2$, where Ω is the cross-section of the cylinder. This explains why we also consider models in $\mathbb{R}^2$. The behavior of a sample of material can be read off from the properties of the minimizers $(\psi, \mathbf{A})$ of the Ginzburg-Landau functional (free energy) $\mathcal{G}$ to be defined below.

1.1.1 The functional

Let us consider a domain $\Omega \subset \mathbb{R}^2$. In this course, we will always consider the cases where Ω is connected and simply connected. The Ginzburg-Landau functional is defined by

$$\begin{aligned}\mathcal{G}_{\Omega,\kappa,\sigma}(\psi, \mathbf{A}) = \int_\Omega |\nabla_{\kappa\sigma\mathbf{A}}\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 \, dx \\ + (\kappa\sigma)^2 \int_\Omega |\mathrm{rot}\, \mathbf{A} - \beta|^2 \, dx \,. \quad (1.1.1)\end{aligned}$$

Here the function ψ is called the order parameter (or sometimes the wave function) and $\mathbf{A}$ is a magnetic potential. The symbol β denotes a magnetic vector field and is called the external magnetic field or the applied magnetic field. In the case $d = 2$ which is the only one considered here, β is just a function in, say,

$L^2_{\rm loc}$. The parameter $\kappa > 0$ (the Ginzburg-Landau parameter) depends on the material, and $\sigma > 0$ (or rather the product $\kappa\sigma$) is a measure of the strength of the external magnetic field. In this course , we are concerned with the analysis of the asymptotic regime $\kappa \to +\infty$.

1.1.2 The two-dimensional functional

It will be convenient to subtract the constant $\frac{\kappa^2}{2}|\Omega|$ from the initial functional, and when Ω is simply connected, to consider

$$\mathcal{G}(\psi, \mathbf{A}) = \int_\Omega |\nabla_{\kappa\sigma\mathbf{A}}\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4\, dx \\ + (\kappa\sigma)^2 \int_\Omega |\mathrm{curl}\, \mathbf{A} - \beta|^2\, dx\,. \quad (1.1.2)$$

We will sometime write $\mathcal{G} = \mathcal{G}_{\kappa,\sigma}$ or even $\mathcal{G} = \mathcal{G}_{\kappa,\sigma,\beta}$, if we want to emphasize the choice of parameters involved in the definition of the functional. Note that if $\psi \equiv 0$ and $\mathbf{A}$ is such that $\mathrm{curl}\, \mathbf{A} = \beta$, then $\mathcal{G}(\psi, \mathbf{A}) = 0$. The above change of zero for the energy is motivated by the fact that we will, in particular, study the behavior of minimizers of $\mathcal{G}$ near such a state (called the normal state in physics).

The natural domain of the functional is $H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)$. However, due to the gauge invariance of $\mathcal{G}$, it is better to restrict the functional to the smaller set $H^1(\Omega, \mathbb{C}) \times H^1_{\rm div}(\Omega)$, where

$$H^1_{\rm div}(\Omega) = \Big\{\mathcal{V} = (V_1, V_2) \in H^1(\Omega, \mathbb{R})^2 \;\Big|\; \mathrm{div}\,\mathcal{V} = 0 \text{ in } \Omega\,,\ \mathcal{V}\cdot\nu = 0 \text{ on } \partial\Omega\Big\}\,. \quad (1.1.3)$$

The space $H^1_{\rm div}(\Omega)$ inherits the topology (norm) from $H^1(\Omega, \mathbb{R}^2)$. We will generally consider the functional on this space if not specified otherwise.

We define the Ginzburg-Landau ground state energy to be the infimum of the functional, i.e.

$$E(\kappa, \sigma) := \inf_{(\psi,\mathbf{A})\in H^1(\Omega)\times H^1_{\rm div}(\Omega)} \mathcal{G}_{\kappa,\sigma}(\psi, \mathbf{A})\,, \quad (1.1.4)$$

and we observe, using the previously mentioned gauge invariance, that

$$E(\kappa, \sigma) := \inf_{(\psi,\mathbf{A})\in H^1(\Omega)\times H^1(\Omega)} \mathcal{G}_{\kappa,\sigma}(\psi, \mathbf{A})\,. \quad (1.1.5)$$

As Ω is bounded, the existence of a minimizer is rather standard, so the infimum is actually a minimum. We will prove this existence in the next section. However, in general, one does not expect uniqueness of minimizers. A minimizer should satisfy the Euler-Lagrange equation, which is called in this context the Ginzburg-Landau system.

Using (1.1.5), this equation reads

$$\left.\begin{aligned} \nabla^2_{\kappa\sigma\mathbf{A}}\psi &= \kappa^2(1-|\psi|^2)\psi\,, \\ \text{curl}\,\left(\text{curl}\,\mathbf{A}-\beta\right) &= -\tfrac{1}{\kappa\sigma}\,\text{Re}\left(\overline{\psi}\,\nabla_{\kappa\sigma\mathbf{A}}\psi\right) \end{aligned}\right\}\quad \text{in}\quad \Omega\,, \tag{1.1.6a}$$

$$\left.\begin{aligned} \nu\cdot\nabla_{\kappa\sigma\mathbf{A}}\psi &= 0\,, \\ \text{curl}\,\mathbf{A}\ -\ \beta &= 0 \end{aligned}\right\}\quad \text{on}\quad \partial\Omega\,. \tag{1.1.6b}$$

Here, for $\mathbf{A}=(A_1,A_2)$, curl $\mathbf{A}=\partial_{x_1}A_2-\partial_{x_2}A_1$, and

$$\text{curl}\,{}^2\mathbf{A} = (\partial_{x_2}(\text{curl}\,\mathbf{A}), -\partial_{x_1}(\text{curl}\,\mathbf{A}))\,.$$

Notice that the weak formulation of (1.1.6) is

$$\text{Re}\int_\Omega \left(\overline{\nabla_{\kappa\sigma\mathbf{A}}\phi}\cdot\nabla_{\kappa\sigma\mathbf{A}}\psi-\kappa^2(1-|\psi|^2)\overline{\phi}\psi\right)\,dx = 0\,, \tag{1.1.7a}$$

$$\int_\Omega (\text{curl}\,\alpha)(\text{curl}\,\mathbf{A}-\beta)\,dx = -\frac{1}{\kappa\sigma}\int_\Omega \text{Re}\left(\overline{\psi}\,\nabla_{\kappa\sigma\mathbf{A}}\psi\right)\alpha\,dx\,, \tag{1.1.7b}$$

for all $(\phi,\alpha)\in H^1(\Omega)\times H^1(\Omega,\mathbb{R}^2)$.

The analysis of the system (1.1.6) can be performed by PDE techniques. We note that this system is nonlinear, that $H^1(\Omega)$ is, when Ω is bounded and regular in $\mathbb{R}^2$, compactly embedded in $L^p(\Omega)$ for all $p\in[1,+\infty[$, and that, if $\text{div}\,\mathbf{A}=0$, $\text{curl}\,{}^2\mathbf{A}=(-\Delta A_1,-\Delta A_2)$.

Actually, the nonlinearity is weak in the sense that the principal part is a linear elliptic system. One can show in particular that the solution in $H^1(\Omega,\mathbb{C})\times H^1_{\text{div}}(\Omega)$ of the elliptic system (1.1.6) is actually, when Ω is regular, in $C^\infty\left(\overline{\Omega}\right)$.

1.2 The existence of a minimizer

Using the discussion in the previous section, we can impose without loss of generality the condition that $\mathbf{A}\in H^1_{\text{div}}(\Omega)$.

Theorem 1.1.
Suppose that $\Omega\subset\mathbb{R}^2$ is bounded, smooth, and simply connected. For all $\kappa,\sigma\in\mathbb{R}^+$ and $\beta\in L^2(\Omega)$, the functional $\mathcal{G}$ on $H^1(\Omega)\times H^1_{\text{div}}(\Omega)$ has a minimizer. Furthermore, minimizers satisfy the Ginzburg-Landau systems in (1.1.6).

Proof.
Let $(\psi_n,\mathbf{A}_n)\in H^1(\Omega)\times H^1_{\text{div}}(\Omega)$ be a minimizing sequence, i.e.,

$$\lim_{n\to\infty}\mathcal{G}(\psi_n,\mathbf{A}_n) = \inf_{(\psi,\mathbf{A})\in H^1(\Omega)\times H^1_{\text{div}}(\Omega)}\mathcal{G}(\psi,\mathbf{A})\,. \tag{1.2.1}$$

Step 1. $\{(\psi_n,\mathbf{A}_n)\}$ is bounded in $H^1(\Omega)\times H^1(\Omega)$.
By using that $\widetilde{\mathcal{G}}$ is the sum of three positive terms, we get the existence of a constant $E_0>0$ such that

$$T_n\le E_0\,, \tag{1.2.2}$$

where T_n is any of the three terms

$$\int_\Omega |(\nabla + i\kappa\sigma\mathbf{A}_n)\psi_n|^2\,dx, \quad \int_\Omega (|\psi_n|^2-1)^2\,dx\,, \quad \int_\Omega |\text{curl }\mathbf{A}_n - \beta|^2\,dx\,.$$

Since β is a fixed function in $L^2(\Omega)$ and $\operatorname{div}\mathbf{A}_n = 0$, we get that $\{\mathbf{A}_n\}$ is uniformly bounded in $H^1(\Omega)$.

Using the Cauchy-Schwarz inequality and the inequality

$$2ab \le \epsilon a^2 + \epsilon^{-1} b^2 \text{ for any } \epsilon > 0\,,$$

notice that

$$\begin{aligned}\int_\Omega (|\psi_n|^2-1)^2\,dx &= \int_\Omega \left(|\psi_n|^4 - 2|\psi_n|^2 + 1\right)\,dx \\ &\ge \|\psi_n\|_4^4 - 2\|\psi_n\|_4^2\sqrt{|\Omega|} \ge \frac{1}{2}\|\psi_n\|_4^4 - 2|\Omega|\,.\end{aligned}$$

Therefore, $\{\psi_n\}$ is uniformly bounded in $L^4(\Omega)$, and therefore—again using the Cauchy-Schwarz inequality—in $L^2(\Omega)$.

The boundedness of $\{\mathbf{A}_n\}$ in $H^1(\Omega)$ implies, by the Sobolev embedding theorem, that $\{\mathbf{A}_n\}$ is uniformly bounded in $L^4(\Omega)$. Combined with the L^4-bound on ψ_n, this gives the uniform boundedness of $\{\mathbf{A}_n\psi_n\}$ in $L^2(\Omega)$. So, considering the uniform bound,

$$\int_\Omega |\nabla\psi_n + i\kappa\sigma\mathbf{A}_n\psi_n|^2\,dx \le E_0\,,$$

this implies that $\{\psi_n\}_n$ is uniformly bounded in $H^1(\Omega)$.

Step 2. A weak limit is a minimizer.

We now extract a subsequence, again denoted by $\{(\psi_n, \mathbf{A}_n)\}$, converging weakly in $H^1(\Omega)\times H^1(\Omega)$ to some $(\psi, \mathbf{A}) \in H^1(\Omega)\times H^1(\Omega)$.

Of course, by taking the limit, we obtain

$$\operatorname{div}\mathbf{A} = 0 \quad \text{in } \Omega\,, \tag{1.2.3}$$

in the sense of distributions.

Furthermore, since the inclusion of $H^1(\Omega)$ in $H^s(\Omega)$ is compact for all $s<1$ and the restriction $H^s(\Omega)\hookrightarrow L^2(\partial\Omega)$ is continuous for all $s > 1/2$, we also get

$$\mathbf{A}\cdot\nu = 0 \quad \text{on } \partial\Omega\,.$$

Thus, $\mathbf{A}\in H^1_{\text{div}}(\Omega)$. We can estimate:

$$\begin{aligned}\int_\Omega |\text{curl }\mathbf{A} - \beta|^2\,dx &= \lim_{n\to+\infty} \langle \text{curl }\mathbf{A} - \beta \mid \text{curl }\mathbf{A}_n - \beta\rangle_{L^2\times L^2} \\ &\le \|\text{curl }\mathbf{A} - \beta\|_2 \liminf_{n\to+\infty} \|\text{curl }\mathbf{A}_n - \beta\|_2\,.\end{aligned}$$

Therefore,

$$\int_\Omega |\text{curl } \mathbf{A} - \beta|^2 \, dx \leq \liminf_{n\to+\infty} \int_\Omega |\text{curl } \mathbf{A}_n - \beta|^2 \, dx \,. \tag{1.2.4}$$

The same type of calculation gives

$$\int_\Omega |(\nabla + i\kappa\sigma\mathbf{A})\psi|^2 \, dx \leq \liminf_{n\to+\infty} \int_\Omega |(\nabla + i\kappa\sigma\mathbf{A}_n)\psi_n|^2 \, dx \,. \tag{1.2.5}$$

The compactness of the Sobolev embedding

$$H^1(\Omega) \hookrightarrow L^p(\Omega) \;\text{ for } \frac{1}{p} > \frac{1}{2} - \frac{1}{d}$$

(if d is the dimension, here $d=2$), hence for $p = 2, 4$, implies that

$$\int_\Omega (|\psi|^2 - 1)^2 \, dx \;=\; \lim_{n\to+\infty} \int_\Omega (|\psi_n|^2 - 1)^2 \, dx \,. \tag{1.2.6}$$

Combining (1.2.1) with (1.2.3)-(1.2.6) shows that $(\psi, \mathbf{A})$ is a minimizer. This finishes the proof in the two-dimensional case. ∎

1.3 Basic properties for solutions of the Ginzburg-Landau equations

As we have seen, minimizers are solutions of the Ginzburg-Landau equations, but many properties are true for general solutions of these equations. The first important property which is based on the maximum principle is

Proposition 1.2.
If $(\psi, \mathbf{A}) \in H^1(\Omega) \times H^1(\Omega, \mathbb{R}^2)$ is a (weak) solution to (1.1.6), then

$$\|\psi\|_{L^\infty(\Omega)} \leq 1 \,. \tag{1.3.1}$$

Using Proposition 1.2, we can get various a priori estimates on solutions to the Ginzburg-Landau equations (1.1.6).

Lemma 1.3.
Let $\Omega \subset \mathbb{R}^2$ be bounded and smooth, and let $\beta \in L^2(\Omega)$ be given. Then for all $p \geq 2$, there exists a constant $C = C(p) > 0$ such that for all solutions $(\psi, \mathbf{A}) \in H^1(\Omega) \times H^1_{\text{div}}(\Omega)$ to (1.1.6), we have

$$\|\nabla^2_{\kappa\sigma\mathbf{A}}\psi\|_p \leq \kappa^2 \|\psi\|_p \,, \tag{1.3.2}$$

$$\|\nabla_{\kappa\sigma\mathbf{A}}\psi\|_2 \leq \kappa \|\psi\|_2 \,, \tag{1.3.3}$$

$$\|\mathit{curl}\ \mathbf{A} - \beta\|_{W^{1,p}(\Omega)} \leq \frac{C}{\kappa\sigma} \|\psi\|_\infty \, \|\nabla_{\kappa\sigma\mathbf{A}}\psi\|_p \,. \tag{1.3.4}$$

Furthermore, there exists a constant $C_2 > 0$ such that

$$\|\mathit{curl}\ \mathbf{A} - \beta\|_2 \leq \frac{C_2}{\sigma} \|\psi\|_2 \| \psi\|_4 \,. \tag{1.3.5}$$

Proof.
Since, by Proposition 1.2,

$$0 \leq 1 - |\psi|^2 \leq 1 \,, \tag{1.3.6}$$

the inequality (1.3.2) is immediate from (1.1.6a). Multiplying the equation for ψ in (1.1.6a) by $\overline{\psi}$ and integrating over Ω, one obtains (1.3.3), again using (1.3.6).

Since, by definition,

$$\text{curl }(\text{curl } \mathbf{A} - \beta) = \left(\partial_{x_2}(\text{curl } \mathbf{A} - \beta), -\partial_{x_1}(\text{curl } \mathbf{A} - \beta)\right) \,,$$

it follows immediately from the equation for $\mathbf{A}$ in (1.1.6a) that

$$\|\nabla(\text{curl } \mathbf{A} - \beta)\|_p \leq \frac{1}{\kappa\sigma}\, \|\psi\|_\infty \|\nabla_{\kappa\sigma\mathbf{A}}\psi\|_p \,. \tag{1.3.7}$$

Since curl $\mathbf{A} - \beta$ vanishes on $\partial\Omega$, (1.3.4) follows from (1.3.7) by the Poincaré inequality.

Finally, we prove (1.3.5). For this we use (1.1.7b) with $\alpha := \mathbf{A} - \mathbf{F}$. Here $\mathbf{F}$ is the unique vector field in $H^1_{\text{div}}(\Omega)$ such that

$$\text{rot}\, \mathbf{F} = \beta \quad \text{and} \quad \text{div}\, \mathbf{F} = 0 \quad \text{in } \Omega \,, \tag{1.3.8}$$

$$\mathbf{F} \cdot \nu = 0 \quad \text{on } \partial\Omega \,. \tag{1.3.9}$$

Applying Hölder's inequality yields

$$\|\text{curl } \mathbf{A} - \beta\|_2^2 \leq \frac{1}{\kappa\sigma}\, \|\psi\|_4 \, \|\nabla_{\kappa\sigma\mathbf{A}}\psi\|_2 \, \|\mathbf{A} - \mathbf{F}\|_4 \,.$$

Thus, using a Sobolev inequality and

$$||(A - F)||_{H^1(\Omega)} \leq C||\text{curl } A - \beta||_2 \,, \tag{1.3.10}$$

we get for another constant C

$$\|\text{curl } \mathbf{A} - \beta\|_2 \leq \frac{C}{\kappa\sigma}\, \|\psi\|_4 \, \|\nabla_{\kappa\sigma\mathbf{A}}\psi\|_2 \,. \tag{1.3.11}$$

The estimate (1.3.5) follows upon inserting (1.3.3) in (1.3.11). ∎

1.4 The result of Giorgi-Phillips

We observe that $(0, \mathbf{F})$ is a trivial critical point of the functional $\mathcal{G}$, i.e., a trivial solution of the Ginzburg-Landau system (1.1.6). The pair $(0, \mathbf{F})$ is often called the *normal state* or *normal solution.* It is natural to discuss—as a function of σ—whether this pair is a local or global minimizer. When σ is large, one will show that this solution is effectively the unique global minimizer. One says that in this case the superconductivity is destroyed. In other words, the order parameter is identically zero in Ω.

Let us give a rather simple proof of this result that roughly says (see Theorem 1.4 for the precise statement) that $(0, \mathbf{F})$ is the unique minimizer of the functional when the strength of the exterior magnetic field is sufficiently large. We will actually show this result for the solutions of the associated Ginzburg-Landau system.

So we assume that we have a **nonnormal** stationary point $(\psi, \mathbf{A})$ for $\mathcal{G}$. This means that $(\psi, \mathbf{A}) \in H^1(\Omega) \times H^1_{\mathrm{div}}(\Omega)$ is a solution of (1.1.6) and

$$\int_\Omega |\psi(x)|^2 \, dx > 0 \,. \tag{1.4.1}$$

By (1.3.3), (1.3.4), and (1.3.1), and using (1.3.10) for controlling $\|\mathbf{A} - \mathbf{F}\|^2$ in Ω by $\|\mathrm{curl}\ \mathbf{A} - \beta\|^2$, we get

$$\|\nabla_{\kappa\sigma\mathbf{A}}\psi\|_2^2 + (\kappa\sigma)^2 \|\mathbf{A} - \mathbf{F}\|_2^2 \leq C_\Omega \kappa^2 \|\psi\|_2^2 \,. \tag{1.4.2}$$

We now compare $\int_\Omega |(\nabla + i\kappa\sigma\mathbf{F})\psi|^2 \, dx$ and $\int_\Omega |(\nabla + i\kappa\sigma\mathbf{A})\psi|^2 \, dx$. A trivial estimate is

$$\int_\Omega |(\nabla + i\kappa\sigma\mathbf{F})\psi|^2 \, dx \leq 2 \, \|(\nabla + i\kappa\sigma\mathbf{A})\psi\|^2 + 2(\kappa\sigma)^2 \|(\mathbf{A} - \mathbf{F})|\psi| \, \|^2 \,. \tag{1.4.3}$$

Implementing (1.3.1) and (1.4.2) gives

$$\int_\Omega |(\nabla + i\kappa\sigma\mathbf{F})\psi|^2 \, dx \leq 2C_\Omega \kappa^2 \int_\Omega |\psi(x)|^2 \, dx \,. \tag{1.4.4}$$

Since ψ satisfies (5.4.1), we obtain

$$\lambda_1^N(\sigma\kappa\mathbf{F}) \leq 2C_\Omega \kappa^2 \,. \tag{1.4.5}$$

We observe that $\lambda_1^N(\sigma\kappa\mathbf{F}) > 0$. So by combining an analysis in the small B regime (perturbation theory) and for large B (see below) [and the continuity of $B \mapsto \lambda_1^N(B\mathbf{F})$], we get the existence of a constant $C_0 > 0$ such that

$$\lambda_1^N(\sigma\kappa\mathbf{F}) \geq \frac{1}{C_0} \min(\sigma\kappa, (\sigma\kappa)^2) \,. \tag{1.4.6}$$

Thus, we find that if a nontrivial stationary point $(\psi, \mathbf{A})$ exists, then

$$\sigma \leq C(1 + \kappa) \,.$$

This can be reformulated as the following theorem.

Theorem 1.4 (Giorgi-Phillips)**.**
Let $\Omega \subset \mathbb{R}^2$ be smooth, bounded, and simply connected, and let the function β in (1.1.6) *be continuous and satisfy*

$$\beta(x) \geq c > 0 \,, \qquad \forall x \in \Omega \,.$$

Then there exists a constant $C = C(\Omega, c)$ such that if

$$\sigma \geq C \max\{\kappa, 1\} \,,$$

then the pair $(0, \mathbf{F})$ is the unique solution to (1.1.6) *in $H^1(\Omega) \times H^1_{\mathrm{div}}(\Omega)$.*

We emphasize that the result is true for any $\kappa > 0$.

We have indeed when the magnetic field is positive:

Proposition 1.5.

$$\lambda_1(BF) = B\min(b, \Theta_0 b') + o(B)\,,$$

where $\Theta_0 \in (0,1)$, $b = \inf_{x\in\Omega}\beta(x)$ *and* $b' = \inf_{x\in\partial\Omega}\beta(x)$.

Two models are indeed involved in the proof by localization: the model with constant magnetic fields

$$(D_x - \frac{B}{2}\beta(x_j,y_j)y)^2 + (D_y + \frac{B}{2}\beta(x_j,y_j)x)^2\,,$$

in $\mathbb{R}^2$ and the Neumann realization of the same operator in $\mathbb{R}^2_+$.
The bottom of the spectrum of the fist one is $B|\beta(x_j,y_j)|$ and the bottom of the spectrum of the second one is $\Theta_0 B|\beta(x_j,y_j)|$.

Remark 1.6. *More accurate estimates (Helffer, Morame, Kordyukov, N. Raymond, S. Vu-Ngoc,...).*
Possible generalizations when the magnetic field vanishes (Pan-Kwek), Corners (Bonnaillie, Dauge, Fournais,...), dimension 3 (Helffer, Morame, Pan,...)

Chapter 2

TGDL 1 : first models

2.1 The model in superconductivity

2.1.1 General context

The physical problem is posed in a domain Ω with specific boundary conditions. We will only analyze here limiting situations where the domain possibly after a blowing argument becomes the whole space (or the half-space). We will work in dimension 2 for simplification (corresponding to a cylindrical 3D problem). We assume that a magnetic field of magnitude $\mathcal{H}^e$ is applied perpendicularly to the sample and identified (via its intensity) with a function. We denote the Ginzburg-Landau parameter of the superconductor by κ ($\kappa > 0$) and the normal conductivity of the sample by σ. Then the time-dependent Ginzburg-Landau system (also known as the Gorkov-Eliashberg equations) is in $]0,T[\times\Omega$:

$$\begin{cases} \partial_t\psi + i\kappa\Phi\psi = \Delta_{\kappa\mathbf{A}}\psi + \kappa^2(1-|\psi|^2)\psi\,, \\ \kappa^2\mathrm{curl}\,^2\mathbf{A} + \sigma(\partial_t\mathbf{A} + \nabla\Phi) = \kappa\,\mathrm{Im}\,(\bar{\psi}\nabla_{\kappa\mathbf{A}}\psi) + \kappa^2\mathrm{curl}\;\mathcal{H}^e\,, \end{cases} \tag{2.1.1}$$

where ψ is the order parameter, $\mathbf{A}$ the magnetic potential, Φ the electric potential, $\nabla_{\kappa\mathbf{A}} = \nabla + i\kappa\mathbf{A}$ and $\Delta_{\kappa\mathbf{A}} = (\nabla + i\kappa\mathbf{A})^2$ is the magnetic Laplacian associated with magnetic potential $\kappa\mathbf{A}$. In addition $(\psi, \mathbf{A}, \Phi)$ satisfies an initial condition at $t = 0$. Note that many physicists are assuming that curl $\mathcal{H}_e = 0$.

In order to solve this equation, one should also define a gauge (Coulomb, Lorentz,...). The orbit of $(\psi, \mathbf{A}, \Phi)$ by the gauge group is

$$\{(\exp(i\kappa q)\,\psi, \mathbf{A} + \nabla q, \Phi - \partial_t q)\,|\,q \in \mathcal{Q}\}\,,$$

where $\mathcal{Q}$ is a suitable space of regular functions of (t, x, y). We refer to Bauman-Jadallah-Phillips [7] (Paragraph B in the introduction) for a discussion of this point. We will choose the Coulomb gauge which reads $\mathrm{div}\,\mathbf{A} = 0$ for any t. Another possibility could be to take $\mathrm{div}\,\mathbf{A} + \sigma\Phi = 0$ but this will not be further discussed. As in the analysis of the surface superconductivity, the "normal" solutions will play an important role. We recall that a solution $(\psi, \mathbf{A}, \Phi)$ is called a normal state solution if $\psi = 0$ in the whole sample.

2.2 From Ginzburg-Landau to TDGL

Let us try to make the parallel between the standard GL case and TDGL at the level of the models.

Schrödinger with constant magnetic field in $\mathbb{R}^2$ and in $\mathbb{R}^2_+$ are the basic models for analyzing $(hD_x - A)^2$ in Ω.

For TDGL, the models are

$$D_x^2 + D_y^2 + icy\,,$$

in $\mathbb{R}^2$

$$D_x^2 + D_y^2 + ic(x\cos\theta + y\sin\theta)$$

in $\mathbb{R}^2_+$ (affine case),

$$D_x^2 + (D_y - \alpha x^2)^2 + icy$$

in $\mathbb{R}^2$ analyzed in [3] and in $\mathbb{R}^2_+$ in [4, 5]

$$D_x^2 + (D_y - \alpha(x\sin\theta - y\cos\theta)^2)^2 + ic(x\cos\theta + y\sin\theta)$$

(for $\theta = \frac{\pi}{2}$).

The results obtained in these three papers corresponds in some sense to the results which can be obtained for the Schrödinger operator with constant magnetic field.

In the TDGL case, we are facing many new difficulties:

- Treat the spectral analysis of non self-adjoint problems. Already in the linear case, the decay of the associated semi-group does not depend uniquely of the knowledge of the spectrum, but also of resolvent estimates in the complex planes.
- The notion of stationary solutions has to be defined.
- The global existence of solutions has to be verified.
- The notion of stability has to be defined. Roughly speaking we hope to find conditions on the initial data and on the current implying the convergence of the solution to the stationary one and to measure the decay.
- The technical problems relative to the existence of corners has to be controlled...

2.2.1 Stationary normal solutions: first analysis

We now determine the stationary (i. e. time independent) normal solutions of the system. From (4.1.1), we see that if $(0, \mathbf{A}, \Phi)$ is such a solution, then $(\mathbf{A}, \Phi)$ satisfies the the system

$$\kappa^2 \text{curl (curl } \mathbf{A}) + \sigma\nabla\Phi = \kappa^2 \text{curl } \mathcal{H}^e\,,\ \text{div}\,\mathbf{A} = 0 \quad \text{in } \Omega\,. \tag{2.2.1}$$

Note that, identifying $\mathcal{H}^e$ with a function h, curl $\mathcal{H}^e = (-\partial_y h\,,\, \partial_x h)$. Interpreting these two equations as the Cauchy-Riemann equations, this can be rewritten (in addition to the divergence free condition) as the property that

$$\kappa^2(\mathrm{curl}\ \mathbf{A} - \mathcal{H}^e) + \, i\,\sigma\Phi\,,$$

is an holomorphic function in Ω. In particular, if $\sigma \neq 0$, Φ and curl $\mathbf{A} - \mathcal{H}^e$ are harmonic.

Chapter 3

Special situation: Φ affine

Here we follow the material of a paper published at Colloquia Mathematicae [30]. The reader can also look at the last chapters of the book [31], which I published in Cambridge University Press in 2013

3.1 Introduction

As simple natural example, we observe that, if $\Omega = \mathbb{R}^2$, (2.1.1) has the following stationary normal state solution

$$\mathbf{A} = \frac{1}{2J}(Jx+h)^2\hat{\imath}_y, \quad \Phi = \frac{\kappa^2 J}{\sigma}y\,. \tag{3.1.1}$$

Note that

$$\text{curl } \mathbf{A} = (Jx+h)\,\hat{\imath}_z,$$

that is, the induced magnetic field equals the sum of the applied magnetic field $h\hat{\imath}_z$ and the magnetic field produced by the electric current $Jx\,\hat{\imath}_z$.

For this normal state solution, the linearization of (4.1.1) with respect to the order parameter is

$$\partial_t\psi + \frac{i\kappa^3 Jy}{\sigma}\psi = \Delta\psi + \frac{i\kappa}{J}(Jx+h)^2\partial_y\psi - (\frac{\kappa}{2J})^2(Jx+h)^4\psi + \kappa^2\psi\,. \tag{3.1.2}$$

Applying the transformation $x \to x - h/J$ and taking for simplification $\kappa = 1$, the time-dependent linearized Ginzburg-Landau equation takes the form

$$\frac{\partial\psi}{\partial t} + i\frac{J}{\sigma}y\psi = \Delta\psi + iJx^2\frac{\partial\psi}{\partial y} - \Big(\frac{1}{4}J^2x^4 - 1\Big)\psi\,. \tag{3.1.3}$$

Rescaling x and t by applying

$$t \to J^{2/3}t\ ;\ (x,y) \to J^{1/3}(x,y)\,, \tag{3.1.4}$$

yields

$$\partial_t u = -(\mathcal{A}_{0,c} - \lambda) u \,, \tag{3.1.5}$$

where, with $D_x = -i\partial_x$, $D_y = -i\partial_y$,

$$\mathcal{A}_{0,c} := D_x^2 + (D_y + \frac{1}{2}x^2)^2 + i\,c\,y \,, \tag{3.1.6}$$

and

$$c = 1/\sigma \; ; \; \lambda = \frac{1}{J^{2/3}} \; ; \; u(x,y,t) = \psi(J^{-1/3}x, J^{-1/3}y, J^{-2/3}t) \,.$$

Our main problem will be to analyze the long time property of the attached semi-group.
We now apply the transformation

$$u \to u\, e^{icyt}$$

to obtain

$$\partial_t u = -\left(D_x^2 u + (D_y + \frac{1}{2}x^2 - ct)^2 u - \lambda u \right) \,. \tag{3.1.7}$$

Note that considering the partial Fourier transform with respect to the y variable, we obtain for the Fourier transform $\hat{u}$ of u:

$$\partial_t \hat{u} = -D_x^2 \hat{u} - \left[\left(\frac{1}{2}x^2 + (-ct + \omega) \right)^2 - \lambda \right] \hat{u} \,. \tag{3.1.8}$$

This can be rewritten as the analysis of a family (depending on $\omega \in \mathbb{R}$) of time-dependent problems on the line

$$\partial_t \hat{u} = -\mathcal{M}_{\beta(t,\omega)} \hat{u} + \lambda \hat{u} \,, \tag{3.1.9}$$

with $\mathcal{M}_\beta$ being the well-known anharmonic oscillator (also called the Montgomery operator in other contexts):

$$\mathcal{M}_\beta = D_x^2 + (\frac{1}{2}x^2 + \beta)^2 \,, \tag{3.1.10}$$

and

$$\beta(t,\omega) = -ct + \omega \,.$$

3.2 The results by Almog-Helffer-Pan [3]

The main point concerning the previously defined operator $\overline{\mathcal{A}_{0,c}}$ is to obtain an optimal control of the decay of the associated semi-group as $t \to +\infty$.

Theorem 3.1.
If $c \neq 0$, $\mathcal{A} = \overline{\mathcal{A}_{0,c}}$ has compact resolvent, empty spectrum, and there exists $C > 0$ such that

$$\| \exp(-t\mathcal{A}) \| \leq \exp\Big(-\frac{2\sqrt{2c}}{3} t^{3/2} + C t^{3/4} \Big) \,, \tag{3.2.1}$$

for any $t \geq 1$ and

$$\|(\mathcal{A}-\lambda)^{-1}\| \leq \exp\Big(\frac{1}{6c}(\,Re\,\lambda)^3 + C\,(\,Re\,\lambda)^{3/2}\Big)\,, \tag{3.2.2}$$

for all λ such that $Re\,\lambda \geq 1$.

Here a semi-classical analysis of the operator $\mathcal{M}_\beta$ as $|\beta| \to \pm\infty$ plays an important role. We refer to [3] for details and to [29] for the involved semi-classical analysis.

If we consider instead the Dirichlet realization $\mathcal{A}_c^D$ of $\mathcal{A}_{0,c}$ in $\{y > 0\}$, it is easily proven that $\mathcal{A}_c^D$ has compact resolvent if $c \neq 0$. We prove in [4] that if the spectrum of $\mathcal{A}_c^D$ is not empty then the decay of the semi-group $\exp -t\mathcal{A}_c^D$ is exponential with a rate corresponding to $\inf_{z\in\sigma(\mathcal{A}_c^D)} \mathrm{Re}\, z$. We will explain the argument in the case of a simpler model : the complex Airy operator. We also conjecture in [4] that $\sigma(\mathcal{A}_c^D)$ is not empty and give a proof of the statement for $|c|$ large enough and in [5] for $|c|$ small enough.

3.3 A simplified model : no magnetic field

We assume, following Almog [1], that a current of constant magnitude J is being flown through the sample in the x axis direction, and that there is no applied magnetic field: $h = 0$. Then (2.1.1) has (in some asymptotic regime) the following stationary normal state solution

$$\mathbf{A} = 0\,, \quad \Phi = Jx\,. \tag{3.3.1}$$

For this normal state solution, the linearization of (2.1.1) gives

$$\partial_t \psi + iJx\psi = \Delta_{x,y}\psi + \psi\,, \tag{3.3.2}$$

whose analysis is (see ahead) strongly related to the Airy equation.

3.3.1 The complex Airy operator in $\mathbb{R}$

This operator can be defined as the closed extension $\mathcal{A}$ of the differential operator on $C_0^\infty(\mathbb{R})$ $\mathcal{A}_0^+ := D_x^2 + i\,x$. We observe that $\mathcal{A} = (\mathcal{A}_0^-)^*$ with $\mathcal{A}_0^- := D_x^2 - i\,x$ and that its domain is

$$D(\mathcal{A}) = \{u \in H^2(\mathbb{R})\,,\, x\,u \in L^2(\mathbb{R})\}\,.$$

In particular $\mathcal{A}$ has compact resolvent.

It is also easy to see that

$$\mathrm{Re}\,\langle \mathcal{A}u \,|\, u\rangle \geq 0\,. \tag{3.3.3}$$

Hence $-\mathcal{A}$ is the generator of a semi-group S_t of contraction,

$$S_t = \exp -t\mathcal{A}\,. \tag{3.3.4}$$

Hence all the results of this theory can be applied.
In particular, we have, for $\operatorname{Re}\lambda < 0$

$$||(\mathcal{A}-\lambda)^{-1}|| \leq \frac{1}{|\operatorname{Re}\lambda|}\,. \tag{3.3.5}$$

A very special property of this operator is that, for any $a \in \mathbb{R}$,

$$T_a \mathcal{A} = (\mathcal{A} - ia) T_a\,, \tag{3.3.6}$$

where T_a is the translation operator $(T_a u)(x) = u(x-a)$.
As immediate consequence, we obtain that the spectrum is empty and that the resolvent of $\mathcal{A}$, which is defined for any $\lambda \in \mathbb{C}$ satisfies

$$||(\mathcal{A}-\lambda)^{-1}|| = ||(\mathcal{A}-\operatorname{Re}\lambda)^{-1}||\,. \tag{3.3.7}$$

One can also look at the semi-classical question, i.e. consider the operator

$$\mathcal{A}_h = h^2 D_x^2 + i\,x\,, \tag{3.3.8}$$

and observe that it is the toy model for some results of Dencker-Sjöstrand-Zworski [16]. The symbol is $(x,\xi) \mapsto p(x,\xi) = \xi^2 + ix$ and microlocally at $(0,0)$, we have $\{\operatorname{Re} p, \operatorname{Im} p\}(0,0) = 0$ and $\{\operatorname{Im} p, \{\operatorname{Re} p, \operatorname{Im} p\}\}(0,0) \neq 0$.
Of course in such an homogeneous situation one can go from one point of view to the other but it is sometimes good to look at what each theory gives on this very particular model. We refer for example to the lectures by J. Sjöstrand [**?**].

The most interesting property is the control of the resolvent for $\operatorname{Re}\lambda \geq 0$.

Proposition 3.2 (W. Bordeaux-Montrieux[9]).
As $Re\,\lambda \to +\infty$, we have

$$||(A-\lambda)^{-1}|| \sim \sqrt{\frac{\pi}{2}}(\,Re\,\lambda)^{-\frac{1}{4}} \exp\frac{4}{3}(\,Re\,\lambda)^{\frac{3}{2}}\,, \tag{3.3.9}$$

This improves a previous result by J. Martinet (see in [31]). The proof of the (rather standard) upper bound is based on the direct analysis of the semi-group in the Fourier representation. We note indeed that

$$\mathcal{F}(D_x^2 + i\,x)\mathcal{F}^{-1} = \xi^2 - \frac{d}{d\xi}\,. \tag{3.3.10}$$

Then we have

$$\mathcal{F} S_t \mathcal{F}^{-1} v = \exp(-\xi^2 t - \xi t^2 - \frac{t^3}{3}) v(\xi + t)\,, \tag{3.3.11}$$

and this implies immediately

$$||S_t|| = \exp\max_{\xi}(-\xi^2 t - \xi t^2 - \frac{t^3}{3}) = \exp(-\frac{t^3}{12})\,. \tag{3.3.12}$$

Then one can get an estimate of the resolvent by using, for $\lambda \in \mathbb{C}$, the formula

$$(\mathcal{A} - \lambda)^{-1} = \int_0^{+\infty} \exp -t(\mathcal{A} - \lambda)\, dt\,. \tag{3.3.13}$$

The right hand side can be estimated using (3.3.12) and the Laplace method. For a closed accretive operator, (3.3.13) is standard when $\operatorname{Re}\lambda < 0$, but estimate (3.3.12) on S_t gives immediately an holomorphic extension of the right hand side to the whole space, showing independently that the spectrum is empty (see Davies [15]) and giving for $\lambda > 0$ the estimate

$$||(\mathcal{A} - \lambda)^{-1}|| \leq \int_0^{+\infty} \exp(\lambda t - \frac{t^3}{12})\, dt\,. \tag{3.3.14}$$

The asymptotic behavior as $\lambda \to +\infty$ of this integral is immediately obtained by using the Laplace method and the dilation $t = \lambda^{\frac{1}{2}} s$ in the integral.

The proof by J. Martinet (see in [31]) of the lower bound is obtained by constructing quasimodes for the operator $(\mathcal{A} - \lambda)$ in its Fourier representation. We observe (assuming $\lambda > 0$), that

$$\xi \mapsto u(\xi; \lambda) := \exp\left(-\frac{\xi^3}{3} + \lambda\xi - \frac{2}{3}\lambda^{\frac{3}{2}}\right) \tag{3.3.15}$$

is a solution of

$$(-\frac{d}{d\xi} + \xi^2 - \lambda) u(\xi; \lambda) = 0\,. \tag{3.3.16}$$

Multiplying $u(\cdot;\lambda)$ by a cut-off function χ_λ with support in $]-\sqrt{\lambda}, +\infty[$ and $\chi_\lambda = 1$ on $]-\sqrt{\lambda}+1, +\infty[$, we obtain a very good quasimode, concentrated as $\lambda \to +\infty$, around $\sqrt{\lambda}$, with an error term giving almost the announced lower bound for the resolvent. The proof by W. Bordeaux-Montrieux is by introducing a Grushin's problem.

Of course this is a very special case of a result on the pseudo-spectra but this leads to an almost optimal result.

3.4 Pseudo-spectra and semi-groups.

We arrive now to the analysis of the properties of a contraction semi-group $\exp -t\mathcal{A}$, with $\mathcal{A}$ maximally accretive. As before, we have, for $\operatorname{Re}\lambda < 0$,

$$||(\mathcal{A} - \lambda)^{-1}|| \leq \frac{1}{|\operatorname{Re}\lambda|}\,, \tag{3.4.1}$$

If we add the assumption that $\operatorname{Im} < \mathcal{A}u, u > \geq 0$ for all u in the domain of $\mathcal{A}$ and if $\operatorname{Im}\lambda < 0$ one gets also a similar inequality, so the main remaining question is the analysis of the resolvent in the set $\operatorname{Re}\lambda \geq 0$, $\operatorname{Im}\lambda \geq 0$, which corresponds to the numerical range of the operator.

We recall that for any $\epsilon > 0$, we define the ϵ-pseudospectra by

$$\Sigma_\epsilon(\mathcal{A}) = \{\lambda \in \mathbb{C} \,|\, ||(\mathcal{A}-\lambda)^{-1}|| > \frac{1}{\epsilon}\}\,, \tag{3.4.2}$$

with the convention that $||(\mathcal{A}-\lambda)^{-1}|| = +\infty$ if $\lambda \in \sigma(\mathcal{A})$.
We have

$$\cap_{\epsilon>0}\Sigma_\epsilon(\mathcal{A}) = \sigma(\mathcal{A})\,. \tag{3.4.3}$$

We define, for any $\epsilon > 0$, the ϵ-pseudospectral abcissa by

$$\widehat{\alpha}_\epsilon(\mathcal{A}) = \inf_{z\in\Sigma_\epsilon(\mathcal{A})} \operatorname{Re} z\,, \tag{3.4.4}$$

and the growth bound of $\mathcal{A}$ by

$$\widehat{\omega}_0(\mathcal{A}) = \lim_{t\to+\infty} \frac{1}{t}\log||\exp -t\mathcal{A}||\,. \tag{3.4.5}$$

Of course, we have

$$\lim_{\epsilon\to+\infty} \widehat{\alpha}_\epsilon(\mathcal{A}) \leq \inf_{z\in\sigma(\mathcal{A})} \operatorname{Re} z\,, \tag{3.4.6}$$

but the equality is wrong in general. The right behavior is given by:

Theorem 3.3 (Gearhart-Prüss)**.**
Let $\mathcal{A}$ be a densely defined closed operator in an Hilbert space X such that $-\mathcal{A}$ generates a contraction semi-group, then

$$\lim_{\epsilon\to 0} \widehat{\alpha}_\epsilon(\mathcal{A}) = -\widehat{\omega}_0(\mathcal{A})\,. \tag{3.4.7}$$

We refer to [19] for a proof and to [32] for a more quantitative version of this theorem particularly useful when parameters are involved.

3.5 The complex Airy operator in $\mathbb{R}^+$

3.5.1 Spectral analysis

Here we mainly describe some results presented in [1], who refers to [38]. We consider the Dirichlet realization $\mathcal{A}^D$ of the complex Airy operator $D_x^2 + ix$ on the half-line, whose domain is

$$D(\mathcal{A}^D) = \{u \in H_0^1(\mathbb{R}^+), x^{\frac{1}{2}}u \in L^2(\mathbb{R}^+)\,,\, (D_x^2 + i\,x)u \in L^2(\mathbb{R}^+)\}\,, \tag{3.5.1}$$

and which is defined (in the sense of distributions) by

$$\mathcal{A}^D u = (D_x^2 + i\,x)u\,. \tag{3.5.2}$$

Moreover, by construction, we have

$$\operatorname{Re}\langle \mathcal{A}^D u \,|\, u\rangle \geq 0\,,\, \forall u \in D(\mathcal{A}^D)\,. \tag{3.5.3}$$

Again we have an operator, which is the generator of a semi-group of contraction, whose adjoint is described by replacing in the previous description $(D_x^2 + i\,x)$ by $(D_x^2 - i\,x)$, the operator is injective and as its spectrum contained in $\operatorname{Re}\lambda > 0$. Moreover, the operator has compact inverse, hence the spectrum (if any) is discrete.

Using what is known on the usual Airy operator, Sibuya's theory and a complex rotation, we obtain ([1]) that the spectrum of $\mathcal{A}^D$ is given by

$$\sigma(\mathcal{A}^D) = \cup_{j=1}^{+\infty}\{\lambda_j\} \tag{3.5.4}$$

with

$$\lambda_j = -(\exp i\frac{\pi}{3})\,\mu_j\,, \tag{3.5.5}$$

the μ_j's being real zeroes of the Airy function satisfying

$$0 > \mu_1 > \cdots > \mu_j > \mu_{j+1} > \cdots\,. \tag{3.5.6}$$

As can be recovered by Weyl's formula, there exists a constant $c \neq 0$ such that $\mu_j \sim c j^{\frac{2}{3}}$. It is also in [1] that the vector space generated by the corresponding eigenfunctions is dense in $L^2(\mathbb{R}^+)$. But there is no way to normalize these eigenfunctions for getting a good basis of $L^2(\mathbb{R}^+)$. See Almog [1], Davies [13] and Henry [**?**] who shows that the norm of the spectral projector π_n associated with the n-th eigenvalue increases exponentially like $\exp \alpha n$ for some $\alpha > 0$. Following E.B. Davies [13], we say in this case that $\mathcal{A}^D$ is spectrally wild.

3.5.2 Decay of the semi-group

We now apply Gearhardt-Pruss theorem to our operator $\mathcal{A}^D$ and our main theorem is

Theorem 3.4.

$$\widehat{\omega}_0(\mathcal{A}^D) = -\,Re\,\lambda_1\,. \tag{3.5.7}$$

This statement was established by Almog [1] in a much weaker form. Using the first eigenfunction it is easy to see that

$$||\exp -t\mathcal{A}^D|| \geq \exp -\operatorname{Re}\lambda_1\, t\,. \tag{3.5.8}$$

Hence we have immediately

$$0 \geq \widehat{\omega}_0(\mathcal{A}^D) \geq -\operatorname{Re}\lambda_1\,. \tag{3.5.9}$$

To prove that $-\operatorname{Re}\lambda_1 \geq \widehat{\omega}_0(\mathcal{A}^D)$, it is enough to show the following lemma.

Lemma 3.5.
For any $\alpha <\ Re\,\lambda_1$, there exists a constant C such that, for all λ s.t. $Re\,\lambda \leq \alpha$

$$||(\mathcal{A}^D - \lambda)^{-1}|| \leq C\,. \tag{3.5.10}$$

Proof : We know that λ is not in the spectrum. Hence the problem is just a control of the resolvent as $|\operatorname{Im}\lambda| \to +\infty$. The case, when $\operatorname{Im}\lambda < 0$ has already be considered. Hence it remains to control the norm of the resolvent as $\operatorname{Im}\lambda \to +\infty$ and $\operatorname{Re}\lambda \in [-\alpha, +\alpha]$.

This is indeed a semi-classical result[1].The main idea is that when $\operatorname{Im}\lambda \to +\infty$, we have to inverse the operator

$$D_x^2 + i(x - \operatorname{Im}\lambda) - \operatorname{Re}\lambda\,.$$

If we consider the Dirichlet realization in the interval $]0, \frac{\operatorname{Im}\lambda}{2}[$ of $D_x^2 + i(x - \operatorname{Im}\lambda) - \operatorname{Re}\lambda$, it is easy to see that the operator is invertible by considering the imaginary part of this operator and that this inverse $R_1(\lambda)$ satisfies

$$||R_1(\lambda)|| \leq \frac{2}{\operatorname{Im}\lambda}.$$

Far from the boundary, we can use the resolvent of the problem on the line for which we have a uniform control of the norm for $\operatorname{Re}\lambda \in [-\alpha, +\alpha]$.

3.5.3 Physical interpretation

Coming back to the application in superconductivity (with $\kappa = 1$), one is looking at the semi-group associated with $\mathcal{A}_J := D_x^2 + iJx - 1$ (where $J \geq 0$ is a parameter). The stability analysis leads to a critical value

$$J_c = (\operatorname{Re}\lambda_1)^{-\frac{3}{2}}\,, \tag{3.5.11}$$

such that :

- For $J \in [0, J_c[$, $||\exp -t\mathcal{A}_J|| \to +\infty$ as $t \to +\infty$.
- For $J > J_c$, $||\exp -t\mathcal{A}_J|| \to 0$ as $t \to +\infty$.

This improves Lemma 2.4 in Almog [1], who gets only this decay for $||\exp -t\mathcal{A}_J\psi||$, with ψ in a specific dense space.

3.6 Higher dimension problems relative to Airy

Here we follow (and extend) [1] (see also [33]).

3.6.1 The model in $\mathbb{R}^2$

We consider the operator

$$\mathcal{A}_2 := -\Delta_{x,y} + i\,x\,. \tag{3.6.1}$$

[1] After a dilation the operator becomes $\operatorname{Im}\lambda\left(h^2 D_x^2 + i(x-1) - \frac{\operatorname{Re}\lambda}{\operatorname{Im}\lambda}\right)$ with $h = |\operatorname{Im}\lambda|^{-\frac{3}{2}}$.

Proposition 3.6.

$$\sigma(\mathcal{A}_2) = \emptyset\,. \tag{3.6.2}$$

Proof : After a Fourier transform in the y variable, it is enough to show that

$$(\widehat{\mathcal{A}_2} - \lambda)$$

is invertible with

$$\widehat{\mathcal{A}_2} = D_x^2 + i\,x + \eta^2\,. \tag{3.6.3}$$

We have just to control for a given $\lambda \in \mathbb{C}$, $(D_x^2 + i\,x + \eta^2 - \lambda)^{-1}$ (whose existence is given by the 1D result) uniformly in $\mathcal{L}(L^2(\mathbb{R}))$ uniformly with respect to η.

3.6.2 The model in $\mathbb{R}^2_+$: perpendicular current.

Here it is useful to reintroduce the parameter J, which is assumed to be positive. Hence we consider the Dirichlet realization

$$\mathcal{A}_2^{D,\perp} := -\Delta_{x,y} + i\,Jx\,, \tag{3.6.4}$$

in $\mathbb{R}^2_+ = \{x > 0\}$.

Proposition 3.7.

$$\sigma(\mathcal{A}_2^{D,\perp}) = \cup_{r\geq 0, j\in\mathbb{N}^*}(\lambda_j + r)\,. \tag{3.6.5}$$

Proof : For the inclusion

$$\cup_{r\geq 0, j\in\mathbb{N}^*}(\lambda_j + r) \subset \sigma(\mathcal{A}_2^{D,\perp})\,,$$

we can use L^∞ eigenfunctions in the form

$$(x, y) \mapsto u_j(x)\,\exp iy\eta\,,$$

where u_j is the eigenfunction associated to λ_j. We have then to use the fact that L^∞-eigenvalues belong to the spectrum. This can be formulated in the following proposition.

Proposition 3.8.
Let $\Psi \in L^\infty(\mathbb{R}^2_+) \cap H^1_{loc}(\overline{\mathbb{R}^2_+})$ satisfying, for some $\lambda \in \mathbb{C}$,

$$-\Delta_{x,y}\Psi + iJx\Psi = \lambda\Psi \tag{3.6.6}$$

in $\mathbb{R}^2_+$ and

$$\Psi_{x=0} = 0\,. \tag{3.6.7}$$

Then either $\Psi = 0$ or $\lambda \in \sigma(\mathcal{A}_3^{D,\perp})$.

For the opposite inclusion, we observe that we have to control uniformly

$$(\mathcal{A}^D - \lambda + \eta^2)^{-1}$$

with respect to η under the condition that

$$\lambda \notin \cup_{r\geq 0, j\in\mathbb{N}^*}(\lambda_j + r)\,.$$

It is enough to observe the uniform control as $\eta^2 \to +\infty$ which results of (3.4.1).

3.6.3 The model in $\mathbb{R}_2^+$: parallel current

Here the models are the Dirichlet realization in $\mathbb{R}^2_+$:

$$\mathcal{A}_2^{D,\|} = -\Delta_{x,y} + i\,J\,y\,, \tag{3.6.8}$$

or the Neumann realization

$$\mathcal{A}_2^{N,\|} = -\Delta_{x,y} + i\,J\,y\,. \tag{3.6.9}$$

Using the reflexion (or antireflexion) trick we can see the problem as a problem on $\mathbb{R}^2$ restricted to odd (resp. even) functions with respect to $(x,y) \mapsto (-x,y)$. It is clear from Proposition 3.6 that in this case the spectrum is empty.

Remark 3.9.
The case when the current is neither parallel nor perpendicular has been treated by R. Henry [33, **?***]. The spectrum is actually empty..*

3.7 Almog's result and generalization by R. Henry

The analysis of the previous models permits actually the semi-classical analysis of the spectrum and of the resolvent for the Dirichlet realization of

$$-h^2\Delta + iV(x)$$

in $L^2(\Omega)$.
Here V is a C^∞ potential such that $\nabla V \neq 0$ in $\bar{\Omega}$.

Then using the results for the models, we can get a lower bound for

$$\liminf_{h\to 0} h^{-\frac{2}{3}} \left(\inf\, \mathrm{Re}\, \sigma(\mathcal{A}_h)\right)\,.$$

Although not motivated by superconductivity but by control's theory, we can also attack the case when V is a Morse function.

One can also measure the decay of the associated semi-group.

Chapter 4

Time Dependent Ginzburg-Landau equation II

The starting point on the mathematical side is a paper of Yaniv Almog at Siam J. Math. Appl. [1] . This work was continued in collaboration with Y. Almog and X. Pan [3, 4, 5] by the analysis of specific toy models. In [2] (in collaboration with Y. Almog) a rather general situation is considered showing how the toy models are involved in the question.

4.1 Introduction to the boundary conditions.

Consider a superconductor placed at a temperature lower than the critical one. It is well-understood from numerous experimental observations, that a sufficiently strong current, applied through the sample, will force the superconductor to arrive at the normal state. To explain this phenomenon mathematically, we use the time-dependent Ginzburg-Landau model which is defined by the following system of equations, and will be referred to as (TDGL1) (Time Dependent

Ginzburg-Landau equation). (TDGL1)

$$\frac{\partial\psi}{\partial t} + i\phi\psi = (\nabla - iA)^2\,\psi + \psi\left(1 - |\psi|^2\right)\,, \qquad \text{in } \mathbb{R}_+ \times \Omega\,, \tag{4.1.1a}$$

$$\kappa^2 \text{curl}\,^2 A + \sigma\left(\frac{\partial A}{\partial t} + \nabla\phi\right) = \text{Im}\,(\bar{\psi}\,\cdot\,(\nabla - iA)\psi)\,, \qquad \text{in } \mathbb{R}_+ \times \Omega\,, \tag{4.1.1b}$$

$$\psi = 0\,, \qquad \text{on } \mathbb{R}_+ \times \partial\Omega_c\,, \tag{4.1.1c}$$

$$(\nabla - iA)\psi\cdot\nu = 0\,, \qquad \text{on } \mathbb{R}_+ \times \partial\Omega_i\,, \tag{4.1.1d}$$

$$\sigma\left(\frac{\partial A}{\partial t} + \nabla\phi\right)\cdot\nu = J\,, \qquad \text{on } \mathbb{R}_+ \times \partial\Omega_c\,, \tag{4.1.1e}$$

$$\sigma\left(\frac{\partial A}{\partial t} + \nabla\phi\right)\cdot\nu = 0\,, \qquad \text{on } \mathbb{R}_+ \times \partial\Omega_i \tag{4.1.1f}$$

$$\frac{1}{|\partial\Omega|}\int_{\partial\Omega} \text{curl}\; A(t,x)\,ds = h_{ex}\,, \qquad \text{on } \mathbb{R}_+\,,\ (1g) \tag{4.1.1g}$$

$$\psi(0,x) = \psi_0(x)\,, \qquad \text{in } \Omega\,,\ (1h) \tag{4.1.1h}$$

$$A(0,x) = A_0(x)\,, \qquad \text{in } \Omega\,,\ (1i)\,. \tag{4.1.1i}$$

In the above ψ denotes the order parameter, A is the magnetic potential, ϕ is the electric potential, κ denotes the Ginzburg-Landau parameter, which is a material property, and the normal conductivity of the sample is denoted by σ. ds denotes the induced measure on $\partial\Omega$. The domain $\Omega \subset\subset \mathbb{R}^2$, occupied by the superconducting sample, has a smooth interface, denoted by $\partial\Omega_c$, with a conducting metal which is at the normal state.

We require that ψ would vanish on $\partial\Omega_c$ in (4.1.1c), and allow for a smooth current $J = hJ_r$ satisfying

$$(J1) \quad J_r \in C^2(\overline{\partial\Omega_c}), \tag{4.1.2}$$

to enter the sample in (4.1.1e).
We further require that

$$(J2) \quad \int_{\partial\Omega_c} J_r\,ds = 0\,, \tag{4.1.3}$$

and

$$(J3) \quad \text{the sign of } J_r \text{ is constant on each connected component of } \partial\Omega_c\,. \tag{4.1.4}$$

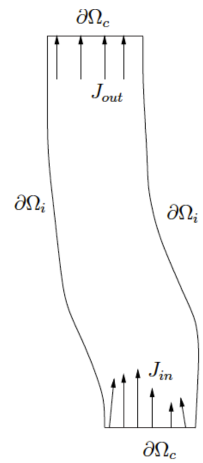

FIGURE 1. Typical superconducting sample. The arrows denote the direction of the current flow (J_{in} for the inlet, and J_{out} for the outlet).

We allow for $J \neq 0$ at the corners. (By convention, $J = 0$ on $\partial\Omega \setminus \partial\Omega_c$).

The rest of the boundary, denoted by $\partial\Omega_i$ is adjacent to an insulator. To simplify some of our arguments (or simply have a proof) we introduce the following geometrical assumption on $\partial\Omega$:

$$(R1) \left\{ \begin{array}{l} (a)\ \partial\Omega_i \text{ and } \partial\Omega_c \text{ are of class } C^3\,; \\ (b)\quad \text{Near each edge, } \partial\Omega_i \text{ and } \partial\Omega_c \text{ are flat} \\ \quad\ \text{and meet with an angle of } \frac{\pi}{2}\,. \end{array} \right. \tag{4.1.5}$$

We also require:

$$(R2) \qquad \text{Both } \partial\Omega_c \text{ and } \partial\Omega_i \text{ have two components.} \tag{4.1.6}$$

Figure 1 presents a typical sample with properties (R1) and (R2). Most wires would fall into the above class of domains.

We assume, for the initial conditions (4.1.1h,i), that

$$\psi_0 \in H^1(\Omega, \mathbb{C}) \text{ and } A_0 \in H^1(\Omega, \mathbb{R}^2)\,, \tag{4.1.7}$$

and:

$$\|\psi_0\|_\infty \leq 1\,. \tag{4.1.8}$$

We mainly consider Coulomb gauge solutions of (4.1.1):

$$\operatorname{div} A = 0 \ \text{ in } \Omega,\ A \cdot \nu = 0 \text{ on } \partial\Omega\,. \tag{4.1.9}$$

Note that for the proof of existence of solutions it is better to consider first solutions in the Lorentz gauge:

$$\phi = \omega \operatorname{div} A\,,$$

keeping the condition $A \cdot \nu = 0$ on $\partial\Omega$.

Equivalent boundary conditions.
Instead of considering the boundary conditions (4.1.1e,f,g), it is possible to use an equivalent boundary condition where we prescribe instead the magnetic field. By (4.1.1b,e,f), on each point on $\partial\Omega$, except for the corners, we have

$$\frac{\partial}{\partial\tau}\text{curl } A(t,\cdot) = \frac{1}{\kappa^2} J(\cdot)\,, \tag{4.1.10}$$

where $\partial/\partial\tau$ denotes the tangential derivative along $\partial\Omega$ in the positive direction. For convenience we set

$$J_r(x) \equiv 0 \text{ on } \partial\Omega_i\,. \tag{4.1.11}$$

Thus, if we introduce on the boundary the function B by

$$\text{curl } A(t,x) = h\, B_r(t,x) \quad \text{on } \partial\Omega\,, \tag{4.1.12}$$

where h denotes a parameter measuring the intensity of the magnetic field.

One can recover the magnetic field $B(t,\cdot)$

$$B_r(t,x) = h_r - \frac{1}{\kappa^2\,|\partial\Omega|}\int_{\partial\Omega} |\Gamma(\tilde{x},x)|\, J_r(\tilde{x}) ds(\tilde{x}) \text{ for } x \in \partial\Omega\,. \tag{4.1.13}$$

where $h_r = h_{ex}/h$ and $|\Gamma(\tilde{x},x)|$ is the length inside the boundary between x and $\tilde{x}$.
In [2], it appears useful in order to get a κ-independent model to take $J_r = \kappa^2 \widetilde{J}_r$. This shows that

$$B_r(t,x) = B_r(x)$$

on the boundary, hence is time independent.
Note also that the condition (4.1.10) gives:

$$\textit{The magnetic field } B \textit{ is constant along each component of } \partial\Omega_i\,. \tag{4.1.14}$$

Hence the system (TGDL1) is equivalent to the system (TGDL2) (same equations except (1e-1g) replaced by)

$$\text{curl } A(t,x) = hB_r(x)\,, \text{ on } \mathbb{R}_+ \times \partial\Omega\,, \tag{4.1.15}$$

where B is given by (4.1.13).

Of course functional spaces should be introduced to give a precise mathematical sense to this statement of equivalence.

Conversely, a solution of (TGDL2) must satisfy (TGDL1) with

$$J_r = \kappa^2 \frac{\partial B_r}{\partial\tau} \text{ on } \partial\Omega\,,$$

and

$$h_r = \frac{1}{|\partial\Omega|}\int_{\partial\Omega} B_r(x) ds\,.$$

4.2 Stationary normal solutions.

If we assume time independence and a solution of (TDGL1) $(0, A_n, \phi_n)$, we get for the magnetic and electric normal potentials A_n and ϕ_n. These equations are obtained by setting $\psi \equiv 0$ in (4.1.1b), yielding

$$\begin{cases} -c\,\mathrm{curl}\,^2 A_n + \nabla\phi_n = 0 & \text{in } \Omega\,, \\ -\sigma \frac{\partial \phi_n}{\partial \nu} = J_r & \text{on } \partial\Omega\,, \\ \frac{1}{|\partial\Omega|}\int_{\partial\Omega} \mathrm{curl}\; A_n\, ds = h_r\,, \end{cases}$$

in which $c = \kappa^2/\sigma$ and $J_r = J/h$ and $h_r = h_{ex}/h$ respectively denote some reference electric current and magnetic field, where h is a positive parameter representing the applied fields intensity. For convenience we set $J_r \equiv 0$ on $\partial\Omega_i$.

If we fix the Coulomb gauge for A_n, we can prove the existence, uniqueness, and regularity of solutions to the above problem.

Note that ϕ_n is a solution of

$$\Delta\phi_n = 0$$

$$\int_\Omega \phi_n dx = 0\,,$$

and

$$-\sigma \frac{\partial \phi_n}{\partial \nu} = J_r\,.$$

This is Neumann but for a problem with corners ! H^2-regularity is OK when the angles are $\frac{\pi}{2}$.
See Kondratev, Grisvard, Dauge for these questions of regularity.
The next assumption (which can be expressed in term of J and h_{ex}), is

$$(B) \quad B_n \neq 0 \;\text{ at the corners}\,, \tag{4.2.1}$$

where $B_n = \mathrm{curl}\; A_n$.
For some of the results, we assume for technical reasons

$$(C) \quad \nabla\phi_n \perp \partial\Omega \;\text{ on } B_n^{-1}(0) \cap \partial\Omega\,. \tag{4.2.2}$$

To recover A_n we first determine $B_n = \mathrm{curl}\; A_n$ modulo a constant. The constant is fixed by the mean value. We recover A_n uniquely by chosing the Coulomb gauge.

4.3 The strong solution in the Coulomb gauge

We fix the Coulomb gauge, i.e., we look for global solutions in $L^2_{loc}([0, +\infty), H^1(\Omega, \mathbb{R}^2))$ of (4.1.1) satisfying

$$\mathrm{div}\, A(t,\cdot) = 0 \text{ in } L^2_{loc}([0,+\infty), L^2(\Omega))\,, A(t,\cdot)\cdot\nu|_{\partial\Omega} = 0 \;\text{ in } L^2_{loc}([0,+\infty), H^{\frac{1}{2}}(\partial\Omega))\,, \tag{4.3.1}$$

and we also assume:

$$\int_\Omega \phi(t,x)\,dx = 0 \text{ in } L^2_{loc}([0,+\infty))\,. \tag{4.3.2}$$

Suppose first that the initial condition A_0 satisfies

$$\operatorname{div} A_0 = 0 \text{ in } \Omega\,, A_0\cdot\nu = 0 \text{ on } \partial\Omega\,, \tag{4.3.3}$$

where

$$A_0 \in H^2(\Omega,\mathbb{R}^2). \tag{4.3.4}$$

We further assume that

$$\psi_0 \in H^2(\Omega,\mathbb{C})\,, \tag{4.3.5}$$

and (4.1.8).

We show that the solution (ψ_d, A_d, ϕ_d) with $\widehat{A}_0 = A_0$ and $\widehat{\psi}_0 = \psi_0$ is gauge-equivalent to the solution of (4.1.1) and (4.3.1).
To this end we define the gauge function ω as the solution of

$$\begin{cases} -\Delta\omega = \operatorname{div} A_d & \text{in } (0,+\infty)\times\Omega\,, \\ \frac{\partial\omega}{\partial\nu} = 0 & \text{on } (0,+\infty)\times\partial\Omega\,, \\ \int_\Omega \omega(t,x)\,dx = 0 & \text{in } (0,+\infty)\,. \end{cases} \tag{4.3.6}$$

As $A_d \in C([0,+\infty); W^{1+\alpha,2}(\Omega,\mathbb{R}^2))$ for any $0<\alpha<1$, it follows by Sobolev embeddings and using the regularity results for problem with corners that $\omega \in C([0,+\infty); W^{2,p}(\Omega))$ for all $p\geq 2$. Furthermore, since $\operatorname{div} A_d \in L^2_{loc}([0,+\infty), H^1(\Omega))$ we get also by regularity

$$\omega \in L^2_{loc}([0,+\infty), H^3(\Omega))\,. \tag{4.3.7}$$

Next, we observe that the projector π_1 (projecting a vector field on is component in H^1_{dvi}) extends (by tensor product) to a projector Π_1 in $H^1_{loc}([0,+\infty); L^2(\Omega,\mathbb{R}^2))$ and that by the uniqueness of the decomposition established in the proposition and (4.3.6):

$$-\nabla\omega = \Pi_1 A_d\,, \tag{4.3.8}$$

in $\mathcal{D}'(0,+\infty; L^2(\Omega,\mathbb{R}^2))$, where $\mathcal{D}'(0,+\infty; L^2(\Omega;\mathbb{R}^2))$ denotes the space of distributions on $(0,+\infty)$ with value in $L^2(\Omega,\mathbb{R}^2)$.
Note that (4.3.8) simply reads

$$-\nabla(\int \omega(t,\cdot)\phi(t)dt) = \pi_1(\int A_d(t,\cdot)\phi(t)dt)\,, \tag{4.3.9}$$

for all $\phi \in C_0^\infty(0,+\infty)$.
The right hand side of (4.3.8) being in $H^1_{loc}([0,+\infty); L^2(\Omega,\mathbb{R}^2)$, this implies that $\nabla\omega \in H^1_{loc}([0,+\infty); L^2(\Omega,\mathbb{R}^2))$, and hence

$$\partial_t\omega \in L^2_{loc}([0,+\infty); H^1(\Omega,\mathbb{R}^2))\,. \tag{4.3.10}$$

It can now be readily verified from (4.3.7) and (4.3.10) that the Coulomb gauge solution $(\psi_c, A_c, \phi_c) = G_\omega(\psi_d, A_d, \phi_d)$ satisfies:

$$\psi_c \in C([0,+\infty); W^{1+\alpha,2}(\Omega,\mathbb{C})) \cap H^1_{loc}([0,+\infty); L^2(\Omega,\mathbb{C}))\,,\ \forall \alpha < 1\,, \quad (4.3.11)$$

$$A_c \in C([0,+\infty); W^{1,p}(\Omega,\mathbb{R}^2)) \cap H^1_{loc}([0,+\infty); L^2(\Omega,\mathbb{R}^2))\,, \forall p \geq 1\,, \quad (4.3.12)$$

which follows from the fact that by (4.3.8) $\nabla\omega \in C([0,+\infty); W^{1,p}(\Omega,\mathbb{R}^2))$, and

$$\phi_c \in L^2_{loc}([0,+\infty); H^1(\Omega))\,. \quad (4.3.13)$$

We can now state:

Theorem 4.1. *Suppose that Ω satisfies condition (R1) and that B is in $H^{\frac{1}{2}}(\partial\Omega)$ (on each regular component of $\partial\Omega$). Suppose further that (ψ_0, A_0) satisfies* (4.3.4)*,* (4.3.3)*,* (4.3.5) *and* (4.1.8)*.*
Then, there exists a unique weak solution (ψ_c, A_c, ϕ_c) of (TGDL2) in the Coulomb gauge. Moreover, this solution is strong in the sense that it satisfies (4.3.11)*-*(4.3.13) *and*

$$\|\psi_c(t,\cdot)\|_\infty \leq 1\,,\ \forall t > 0\,. \quad (4.3.14)$$

Finally, let $A_1 = A_c - hA_n$ where A_n is the previously constructed normal solution. Then

$$A_1 \in L^2_{loc}([0,+\infty); H^2(\Omega,\mathbb{R}^2))\,. \quad (4.3.15)$$

We can now return to the solution of (TGDL1).

Theorem 4.2. *Under the assumptions of the previous theorem, assuming that* $\mathbf{j}$ *is given by* (4.1.2)*-*(4.1.3)*, and B by* (4.1.13)*, the solution of (TDGL2) has the additional property that $\phi_c \in C([0,+\infty); W^{1,p}(\Omega))$ for all finite p, and is a solution of (TDGL1).*

Proof. Let (ψ_c, A_c, ϕ_c) denote a solution of (TDGL2) and (4.3.1). One has to clarify first the sense in which the trace condition (4.1.1e)-(4.1.1f) is satisfied. By Theorem 4.1 we have that $\partial_t A_c + \nabla\phi_c$ belongs to $L^2_{loc}([0,+\infty), L^2(\Omega,\mathbb{R}^2))$. Hence, we can use the fact (see for example Theorem 2.2 in [26]) that for a vector field V in $L^2_{loc}(0,+\infty; L^2(\Omega;\mathbb{R}^2))$
with $\operatorname{div} V \in L^2_{loc}([0,\infty); L^2(\Omega))$, the normal component of its trace, $V\cdot\nu|_{\partial\Omega}$, belongs to $L^2_{loc}([0,+\infty); H^{-\frac{1}{2}}(\partial\Omega))$.
Consider then $V = \partial_t A_c + \nabla\phi_c$. By (4.1.15b) and (4.3.1) we obtain:

$$\sigma \operatorname{div} V = \sigma \operatorname{div} \nabla\phi_c = \operatorname{Im} \operatorname{div}(\bar{\psi}_c \cdot \nabla_{A_c}\psi_c)\,. \quad (4.3.16)$$

It is easy to show that the left hand side is in $L^2_{loc}([0,+\infty); L^2(\Omega))$. As $\Delta_{A_c}\psi_c \in L^2_{loc}([0,+\infty); L^2(\Omega))$ we can use (4.3.14) to conclude that $\psi_c\Delta_{A_c}\psi_c \in L^2_{loc}([0,+\infty); L^2(\Omega))$. Furthermore, $\nabla\psi_c \in C([0,+\infty); L^4(\Omega,\mathbb{R}^2))$ and $A_c \in C([0,+\infty)\times\overline{\Omega})$ in view of (4.3.11) and (4.3.12) , hence $\nabla\psi_c\cdot\nabla_{A_c}\psi_c \in L^2_{loc}([0,+\infty); L^2(\Omega))$. Consequently, $V\cdot\nu$ is well defined in $L^2_{loc}([0,\infty); H^{-1/2}(\partial\Omega))$, and we can discern that

$$V\cdot\nu|_{\partial\Omega} = \partial_\nu\phi\,,$$

due to the fact that $\partial_t A_c \cdot \nu = 0$ in $\mathcal{D}'(0,+\infty; H^{\frac{1}{2}}(\partial\Omega))$ by (4.3.1).
Consider again (4.1.15b). Each term of the equality have a meaningful normal component for its trace and hence, as the right hand side has a zero "normal" trace,

$$V \cdot \nu = \kappa^2 \partial_\tau \text{curl } A_c = \kappa^2 h \, \partial_\tau B = J \,, \tag{4.3.17}$$

in $L^2_{loc}([0,+\infty); H^{-\frac{1}{2}}(\partial\Omega))$, as expected. ∎

4.4 The question of stability

One possible mechanism which contributes to the breakdown of superconductivity by a strong current is the magnetic field induced by the current. In the absence of electric current, it was proved by Giorgi-Phillips in [24] that, when a sufficiently strong magnetic field is applied on the sample's boundary (or when h_{ex} is sufficiently large), the normal state, for which $\psi \equiv 0$, becomes the unique solution for the steady-state version of (4.1.1) (cf. also Fournais-Helffer [23] and the references therein).

For the time-dependent Ginzburg-Landau equations it was proved in Feireisl-Takac [21] that every solution must reach an equilibrium in the long-time limit. When combined with the results in [24] it follows that when the applied magnetic field is sufficiently large the normal state becomes globally stable.

No such result was available in the presence of electric currents. The results in [21] are based on the fact that, in the absence of currents, the Ginzburg-Landau energy functional serves as a Lyapunov functional. In the presence of a current one has to take account of the work it produces, which does not necessarily decrease the energy (cf. [45] for instance).

Moreover, the magnetic field is not the only mechanism which forces the sample into the normal state when the electric current is sufficiently large.

Consider the reduced model where one neglects the induced magnetic field and set $A \equiv 0$ in (2.1.1). It has been proved in [38, 46, 1] that the normal state is at least locally stable when the current is sufficiently strong. In a recent contribution [4], together with Almog and Pan, we show that the critical current where the normal state looses its local stability tends to the critical value for the reduced model [38] in the small conductivity limit, or when $c \to \infty$. This result suggests that stability is being forced not only by the magnetic field that the current induces, but also by the potential term in (4.1.1a).

In [2] we prove global stability of the normal state, as a solution of (4.1.1), for sufficiently large currents. We begin by proving global existence and uniqueness of solutions for (4.1.1) and obtain their regularity. While these questions have previously addressed (cf. [10], [22], and [17] to name just a few references) the fact that the boundary is not smooth at the corners requires some additional attention.

4.5 A non self-adjoint operator.

Let

$$\mathcal{L}_h = -\nabla^2_{hA_n} + i\, h\phi_n \,,$$

be defined over the domain

$$D(\mathcal{L}_h) = \{u \in H^2(\Omega) \mid u|_{\partial\Omega_c} = 0 \,;\ \nabla u \cdot \nu|_{\partial\Omega_i} = 0\,\}\,.$$

We prove that a proper bound on the resolvent of $\mathcal{L}_h$, which is the elliptic operator in (4.1.1a) linearized near $(0, hA_n, h\phi_n)$ gives the stability.

Theorem 4.3. *Let $\nu \geq 0$. There exists $\kappa_0 > 0$ and $C_1 > 0$ such that, if for some $\kappa > \kappa_0$ we have*

$$\sup_{\gamma\in\mathbb{R}} \|(\mathcal{L}_h - i\gamma - \nu)^{-1}\| < 1 - \frac{C_1}{\kappa^2}\,, \tag{4.5.1}$$

then, any solution of (4.1.1) *must satisfy*

$$\int_0^\infty e^{2\nu t}\, \|\psi(t,\cdot)\|_2^2\, dt < \infty\,. \tag{4.5.2}$$

Assumption (4.5.1) does not guarantee that the semigroup necessarily becomes a contraction in the long-time limit. The above stability is proved in the large κ limit both for (4.1.1) and we treat the same system, scaled with respect to the penetration depth, which is obtained by applying the transformation $x \to x/\kappa$ in (4.1.1).

As the resolvent of $\mathcal{L}_h$ in an arbitrary domain is difficult to control, we provide an estimate of its norm for large values of h, which can be applied for either large domains, or large κ values.

4.6 Large domains Ω_R

Our aim is to show that the norm of the resolvent can be controlled from two approximated problems, with constant current defined either in $\mathbb{R}^2$ or in $\mathbb{R}^2_+$ with Dirichlet boundary conditions.

From resolvent estimates, together with the results of Almog-Helffer-Pan in [3, 4, 5] we deduce that the critical current, for which the normal state looses its local stability, can be approximated by the same critical current obtained for the above $\mathbb{R}^2_+$ problem. Before to state the result let us describe the toy models.

Two toy models We now give the definitions of these model operators in $\mathbb{R}^2$ and $\mathbb{R}^2_+ = \{y > 0\}$.

These models depend on two real parameters $c \neq 0$ and $\mathfrak{j}$.
The first one is

$$\mathcal{A}(\mathfrak{j}, c) = D_x^2 + (D_y - \mathfrak{j}x^2)^2 + ic\mathfrak{j}y\,, \tag{4.6.1}$$

defined on

$$D(\mathcal{A}) = \{u \in L^2(\mathbb{R}^2)\,|\,\mathcal{A}u \in L^2(\mathbb{R}^2)\}\,. \tag{4.6.2}$$

It has empty spectrum and we have a good control of the resolvent depending only of the real part of the spectral parameter.

The second one is $\mathcal{A}_+(\mathfrak{j}, c)$, which is defined (via the Lax-Milgram theorem) by the same differential formula of $\mathcal{A}$ but on the domain

$$D(\mathcal{A}_+) = \{u \in \tilde{V}\ :\ \mathcal{A}_+ u \in L^2(\mathbb{R}^2_+, \mathbb{C})\}, \tag{4.6.3}$$

where

$$\tilde{V} = H_0^{1,\mathrm{mag}}(\mathbb{R}^2_+, \mathbb{C}) \cap L^2(\mathbb{R}^2_+, \mathbb{C}; y\, dxdy)\,. \tag{4.6.4}$$

Here the analysis of the spectrum is more difficult. The guess is that it is non-empty. This is only proven for $|c|$ large enough or small enough.

Towards the last theorem Set, for $z \in \bar{\Omega}$,

$$\mathfrak{j}(z) := h|\nabla B_n(z)| = \frac{h}{c}|\nabla \phi_n(z)|\,, \tag{4.6.5}$$

and then define,

$$\mathcal{A}(z) = \mathcal{A}(\mathfrak{j}(z), c) \quad ; \quad \mathcal{A}_+(z) = \mathcal{A}_+(\mathfrak{j}(z), c) \tag{4.6.6}$$

Under all of the above assumptions $B_n^{-1}(0)$ is either empty, or else consists of a single curve Γ connecting between the two connected components of $\partial\Omega_c$. We treat the second case. We denote the two points of intersection by z_1 and z_2 and then set

$$\nu_m(z_1, z_2, c) = \min_{i=1,2}\ \inf_{\lambda \in \sigma(\mathcal{A}_+(z_i))} \mathbb{R}e\lambda\,. \tag{4.6.7}$$

Large domain limit Let then $R > 0$. We denote by Ω_R the image of Ω under the dilation $x \to R\,x$ We assume that the domain Ω has the property (R1)-(R2) and that assumptions (J1)-(J3), (B) and (C) are met.
Denote the transformed electric field by ϕ_R. It satisfies the problem

$$\begin{cases} \Delta\phi_R = 0 & \text{in } \Omega_R\,, \\ \frac{\partial\phi_R}{\partial\nu} = -\frac{J_R(x)}{\sigma} & \text{on } \partial\Omega_R\,, \end{cases}$$

where

$$J_R(x) = J_r(x/R)\,.$$

Note that

$$\phi_R(x) = R\,\phi_n(x/R)\,.$$

The transformed magnetic potential, which we denote by A_R then satisfies

$$A_R(x) = R^2\, A_n(x/R)\,.$$

Let then

$$\mathcal{L}_h^R = -\nabla_{hA_R}^2 + ih\phi_R\,, \tag{4.6.8}$$

and let

$$\mu(R) = \inf_{\lambda\in\sigma(\mathcal{L}_h^R)} \mathbb{R}e\lambda \quad\text{and}\quad \mu_\infty = \liminf_{R\to\infty} \mu(R)\,. \tag{4.6.9}$$

We can now state

Theorem 4.4. *Under the previous assumptions, $\mu(R)$ has a limit as $R \to +\infty$, which is given by*

$$\mu_\infty = \nu_m\,.$$

Furthermore, let $\nu < \mu_\infty$. Then, $\exists\, R_0,\, C,$ such that, for $R \geq R_0$,

$$\begin{aligned}\sup_{\gamma\in\mathbb{R}} \|(\mathcal{L}_h^R - \nu - i\gamma)^{-1}\| \leq \\ \max\Big(\sup_{z_0\in\Gamma} \|(\mathcal{A}(z_0)-\nu)^{-1}\|, \sup_{\substack{\gamma\in\mathbb{R}\\ i=1,2}} \|(\mathcal{A}_+(z_i)-\nu-i\gamma)^{-1}\|\Big)\Big(1+\frac{C}{R^{1/4}}\Big) \\ + \frac{C}{R^{1/4}}\,.\end{aligned} \tag{4.6.10}$$

One can deduce from (4.6.10) an upper bound for the critical current where the normal state $(0, hA_n, h\phi_n)$ becomes globally stable. Let

$$\mathrm{j}_m = \inf_{z\in\Gamma} \mathrm{j}(z)\,, \tag{4.6.11a}$$

and

$$\mathrm{j}_+ = \inf_{i=1,2} \mathrm{j}(z_i)\,. \tag{4.6.11b}$$

When the domain size is multiplied by R, the resolvent norm of $\mathcal{L}_h$ is given by the left-hand-side of (4.6.10). By (4.5.1) it then follows that if R and κ are sufficiently large, and if

$$\mathrm{j}_m > \|\mathcal{A}^{-1}(1,c)\|^{3/2} \tag{4.6.12a}$$

and

$$\mathrm{j}_+ > \sup_{\gamma\in\mathbb{R}} \|(\mathcal{A}_+(1,c)-i\gamma)^{-1}\|^{3/2}\,, \tag{4.6.12b}$$

then the normal state must be globally stable. The above conditions serve as an upper bound for the critical current where the normal state becomes globally stable.

On the semiclassical side This corresponds to the spectral analysis of

$$\sum_j (\hbar D_{x_j} - A_j)^2 + i\hbar\phi(x)\,,$$

in the limit $\hbar \to 0$. With $\phi = 0$, this analysis plays an important role in the analysis of the superconductivity. In the above questions, we have $\nabla\phi \cdot \nabla \mathrm{curl}\, A = 0$ and the zeros of curl A consists in a curve Γ joining two points of the boundary where the Dirichlet condition is assumed.

When $\mathbf{A} = \mathbf{0}$, a connected problem is to determine the bottom of the (real part of the) spectrum under the following assumptions ϕ is a Morse function and has no critical point at the boundary. The answer depends on the presence or not of critical sets inside Ω. When there are no critical points, the case is treated in the paper of Y. Almog at Siam [1] (see also Henry). One should look at all the points where $\nabla\phi$ is orthogonal to the boundary. Assuming that these points are isolated, we will get the result by looking at the transversal Airy operators computed at these points. That is looking at

$$\hbar^2 D_t^2 + i\hbar|\nabla\phi(x_\ell)|t + i\hbar\phi(x_\ell)$$

in $\mathbb{R}^+$, with Dirichlet condition at 0.
With $j(x_\ell) = |\nabla\phi(x_\ell)|$, the smallest real part is $j(x_\ell)^{\frac{2}{3}}\hbar^{\frac{4}{3}} \cos\frac{\pi}{3}\,\alpha$, where α is the lowest eigenvalue of the standard Airy operator on $\mathbb{R}^+$. Actually, depending of the angle of $\nabla\phi$ with the normal, we get a model in $\mathbb{R}^2_+$:

$$\hbar^2(D_t^2 + D_s^2) + iJ(\cos\theta t + \sin\theta s)\,,$$

with boundary condition at $t = 0$.
As we have seen in the study of models, the only case when spectrum is present is the case when $\theta = 0$.

In the case where there are critical points in Ω, we consider the quadratic approximation of ϕ at the various critical points:

$$\hbar^2(D_x^2 + D_y^2) + i\hbar\langle \mathrm{Hess}\,\phi(x_\ell)(x,y)\,,\,(x,y)\rangle + i\hbar\phi(x_\ell)$$

in $\mathbb{R}^2$.
In this case the bottom is of order $\mathcal{O}(h^{\frac{3}{2}})$ and this explains why these points will have the dominant role. (connected work of K. Pravda Starov)

In the $(1D)$-case, this question appears also in control theory (Beauchard, Helffer, Henry, and Robbiano) for two models $-\frac{d^2}{dx^2} + ix$ in $]-R,+R[$ and $-\frac{d^2}{dx^2} + ix^2$.

Around Dencker-Sjöstrand-Zworski criterion
Dencker-Sjöstrand-Zworski[16] will probably give some information. We divide by $\hbar$ and get $\hbar D_x^2 + i\phi(x)$ this is again an h-semiclassical problem with $\hbar = h^2$. The h-symbol is $p(x,\xi) = \xi^2 + i\phi(x)$. One can look at the brackets

$\{\operatorname{Re} p, \operatorname{Im} p\} = 2\xi \cdot \nabla\phi(x)$ and $\{\operatorname{Im} p, \{\operatorname{Re} p, \operatorname{Im} p\}\} = 2|\nabla\phi(x)|^2$. This can be applied at $\xi = 0$ for some point x. This gives an information about the existence of the resolvent at points with $\operatorname{Re}\lambda = 0$ and as a consequence the absence of spectrum in the region $\operatorname{Re}\lambda \leq Ch^{\frac{2}{3}}$. This is far from optimal.

Chapter 5

Elements of proofs.

5.1 $1D$-models (extracted of Beauchard-Helffer-Henry-Robbiano)

In this section, we are interested in the spectrum of the operators

$$\mathcal{A}_{[-R,R]} = -\frac{d^2}{dy^2} + iy \quad and \quad \mathcal{H}_{[-R,R]} = -\frac{d^2}{dy^2} + iy^2$$

defined on the segment $[-R, R]$, $R > 0$, with Dirichlet boundary conditions at $y = \pm R$, with domains

$$\mathcal{D}(\mathcal{A}_{[-R,R]}) = \mathcal{D}(\mathcal{H}_{[-R,R]}) = H^1_0([-R,R] \cap H^2(]-R,R[;\mathbb{C}).$$

More precisely, we study the asymptotic behaviour, as $R \to +\infty$, of the bottom of the spectrum of $\mathcal{A}_{[-R,R]}$ and $\mathcal{H}_{[-R,R]}$ and we prove the following two theorems, in subsections 5.1.1 and 5.1.2 respectively.

Theorem 5.1. *Let $\mu_1 < 0$ be the first zero of the Airy function. Then,*

$$\lim_{R\to\infty} \left(\inf \; Re\,\sigma(\mathcal{A}_{[-R,R]})\right) = \frac{|\mu_1|}{2}\,, \tag{5.1.1}$$

where $\sigma(\mathcal{A}_{[-R,R]})$ denotes the spectrum of $\mathcal{A}_{[-R,R]}$.

Now, let us consider the case of Davies operator (or 'complex hamonic oscillator')

Theorem 5.2.

$$\lim_{R\to\infty} \left(\inf \; Re\,\sigma(\mathcal{H}_{[-R,R]})\right) = \frac{\sqrt{2}}{2}, \tag{5.1.2}$$

where $\sigma(\mathcal{H}_{[-R,R]})$ denotes the spectrum of $\mathcal{H}_{[-R,R]}$.

Analogous questions have been considered in [1, 3, 4, 5] and [2]. We study these two operators thanks to technics developed in these references. The study of more general cases (dimension 2) complementary to those studied in [1] and [2] will be done by R. Henry in [37].

5.1.1 Semi classical analysis of the complex Airy operator ($\gamma = 1$)

We introduce three model-operators, that have well known spectral and pseudospectral behaviour. Let $\mathcal{A}_{[-R,+\infty[}$, $\mathcal{A}_{\mathbb{R}}$ and $\mathcal{A}_{]-\infty,R]}$ be the Dirchlet realizations of the operator $-\frac{d^2}{dx^2} + ix$ on the intervals $[-R,+\infty[$, $\mathbb{R}$ and $]-\infty,R]$ respectively. We are going to approximate the resolvent of $\mathcal{A}_{[-R,R]}$ by the one of $\mathcal{A}_{[-R,+\infty[}$, $\mathcal{A}_{\mathbb{R}}$ or $\mathcal{A}_{]-\infty,R]}$ depending on where we are, respectively close to $-R$, far from $-R$ and $+R$ or close to $+R$.
Let us remark that, if $T_R : u(x) \mapsto u(x+R)$ and $U_R : u(x) \mapsto u(R-x)$, then

$$T_R^{-1}(\mathcal{A}_{[-R,+\infty[} - \lambda)T_R = \mathcal{A}_{\mathbb{R}^+} - (\lambda + iR) \tag{5.1.3}$$

and

$$U_R^{-1}(\mathcal{A}_{]-\infty,R]} - \lambda)U_R = \mathcal{A}^*_{\mathbb{R}^+} - (\lambda - iR), \tag{5.1.4}$$

thus

$$\inf \operatorname{Re} \sigma(\mathcal{A}_{[-R,\infty[}) = \inf \operatorname{Re} \sigma(\mathcal{A}_{]-\infty,R]}) = \frac{|\mu_1|}{2}, \tag{5.1.5}$$

because $\inf \operatorname{Re} \sigma(\mathcal{A}_{\mathbb{R}^+}) = |\mu_1|/2$, see [1].

Step 1: We prove that

$$\varliminf_{R\to+\infty} \left(\inf \operatorname{Re} \sigma(\mathcal{A}_{[-R,R]})\right) \geq \frac{|\mu_1|}{2}. \tag{5.1.6}$$

Let $\varepsilon > 0$. We search $R_\varepsilon > 0$ such that

$$\forall R \geq R_\varepsilon, \quad \sigma(\mathcal{A}_{[-R,R]}) \cap (]-\infty, |\mu_1|/2 - \varepsilon] + i\mathbb{R}) = \emptyset. \tag{5.1.7}$$

We recall that, by [?], there exists $C_\varepsilon > 0$ such that

$$\sup_{\substack{\gamma \leq |\mu_1|/2-\varepsilon, \\ \nu \in \mathbb{R}}} \|(\mathcal{A}_{\mathbb{R}^+} - (\gamma + i\nu))^{-1}\| \leq C_\varepsilon \tag{5.1.8}$$

and

$$\sup_{\substack{\gamma \leq |\mu_1|/2-\varepsilon, \\ \nu \in \mathbb{R}}} \|(\mathcal{A}^*_{\mathbb{R}^+} - (\gamma + i\nu))^{-1}\| \leq C_\varepsilon. \tag{5.1.9}$$

Let $\lambda = \gamma + i\nu \in]-\infty, |\mu_1|/2 - \varepsilon] + i\mathbb{R}$ and $h_1, h_2, h_3 \in \mathcal{C}_0^\infty(\mathbb{R}; [0,1])$ be such that $\operatorname{Supp} h_1 \subset]-\infty, -1/2[$, $h_1(x) = 1$ for $x \in [-1, -3/4]$, $\operatorname{Supp} h_2 \in]-3/4, 3/4[$, $h_2(x) = 1$ for $x \in [-1/2, 1/2]$, $\operatorname{Supp} h_3 \subset]1/2, +\infty[$, $h_3(x) = 1$ for $x \in [3/4, 1]$, and

$$h_1^2 + h_2^2 + h_3^2 \equiv 1.$$

For $j = 1, 2, 3$ and $R > 0$, we define

$$\eta_R^j(x) = h_j\left(\frac{x}{R}\right) \mathbf{1}_{[-R,R]}(x), \tag{5.1.10}$$

and

$$\mathcal{R}_R(\lambda) = \eta_R^1(\mathcal{A}_{[-R,+\infty[} - \lambda)^{-1}\eta_R^1 + \eta_R^2(\mathcal{A}_{\mathbb{R}} - \lambda)^{-1}\eta_R^2 + \eta_R^3(\mathcal{A}_{]-\infty,R]} - \lambda)^{-1}\eta_R^3.$$

$\mathcal{R}_R(\lambda)$ will be used as an approximation of the resolvent of $\mathcal{A}_{[-R,R]}$. We have

$$\begin{aligned}(\mathcal{A}_{[-R,R]} - \lambda)\mathcal{R}_R(\lambda) = I \quad &+ \quad [\mathcal{A}_{[-R,R]}, \eta_R^1](\mathcal{A}_{[-R,+\infty[} - \lambda)^{-1}\eta_R^1 \\ &+ \quad [\mathcal{A}_{[-R,R]}, \eta_R^2](\mathcal{A}_{\mathbb{R}} - \lambda)^{-1}\eta_R^2 \\ &+ \quad [\mathcal{A}_{[-R,R]}, \eta_R^3](\mathcal{A}_{]-\infty,R]} - \lambda)^{-1}\eta_R^3. \quad (5.1.11)\end{aligned}$$

We estimate the second term on the right hand side. In what follows, the estimates are uniform with respect to $\nu = \operatorname{Im}\lambda$. We have

$$[\mathcal{A}_{[-R,R]}, \eta_R^1](\mathcal{A}_{[-R,+\infty[} - \lambda)^{-1}\eta_R^1 = \left(-(\eta_R^1)'' - 2(\eta_R^1)'\frac{d}{dy}\right)(\mathcal{A}_{[-R,+\infty[} - \lambda)^{-1}\eta_R^1, \quad (5.1.12)$$

Using $|\sup(\eta_R^j)'| = \mathcal{O}(R^{-1})$ and $|\sup(\eta_R^j)''| = \mathcal{O}(R^{-2})$ for $j = 1, 2, 3$, we get, by (5.1.3) and (5.1.8),

$$\|(\eta_R^1)''(\mathcal{A}_{[-R,+\infty[} - \lambda)^{-1}\eta_R^1\| = \mathcal{O}\left(\frac{1}{R^2}\right). \quad (5.1.13)$$

Moreover, for every $v \in L^2([-R,+\infty[)$,

$$\begin{aligned}\left\|\frac{d}{dy}(\mathcal{A}_{[-R,+\infty[} - \lambda)^{-1}v\right\|^2 \quad \leq \quad & (\|(\mathcal{A}_{[-R,+\infty[} - \lambda)^{-1}\|^{1/2} \\ & +\sqrt{\gamma}\|(\mathcal{A}_{[-R,+\infty[} - \lambda)^{-1}\|)\|v\|. \quad (5.1.14)\end{aligned}$$

Indeed,

$$\begin{aligned}&\left\|\frac{d}{dx}(\mathcal{A}_{[-R,+\infty[} - \lambda)^{-1}v\right\|^2 = \operatorname{Re}\langle v, (\mathcal{A}_{[-R,+\infty[} - \lambda)^{-1}v\rangle \\ &- \operatorname{Re} i\langle (x-\mu)(\mathcal{A}_{[-R,+\infty[} - \lambda)^{-1}v, (\mathcal{A}_{[-R,+\infty[} - \lambda)^{-1}v\rangle \\ &+\gamma\|(\mathcal{A}_{[-R,+\infty[} - \lambda)^{-1}v\|^2 \\ &\leq \|(\mathcal{A}_{[-R,+\infty[} - \lambda)^{-1}v\|\|v\| + \gamma\|(\mathcal{A}_{[-R,+\infty[} - \lambda)^{-1}v\|^2,\end{aligned}$$

from which we deduce (5.1.14).
By applying (5.1.14) to $v = \eta_R^1 u$, $u \in L^2(\mathbb{R})$, we get

$$\left\|(\eta_R^1)'\frac{d}{dy}(\mathcal{A}_{[-R,+\infty[} - \lambda)^{-1}\eta_R^1 u\right\| = \mathcal{O}\left(\frac{1}{R}\right), \quad (5.1.15)$$

which gives, with (5.1.12) and (5.1.13),

$$\left\|[\mathcal{A}_{[-R,R]}, \eta_R^1](\mathcal{A}_{[-R,+\infty[} - \lambda)^{-1}\eta_R^1\right\| = \mathcal{O}\left(\frac{1}{R}\right). \quad (5.1.16)$$

In the same way, we verify that

$$\big\|[\mathcal{A}_{[-R,R]},\eta_R^2](\mathcal{A}_{\mathbb{R}}-\lambda)^{-1}\eta_R^2\big\| = \mathcal{O}\left(\frac{1}{R}\right) \tag{5.1.17}$$

and

$$\big\|[\mathcal{A}_{[-R,R]},\eta_R^3](\mathcal{A}_{]-\infty,R]}-\lambda)^{-1}\eta_R^3\big\|\,| = \mathcal{O}\left(\frac{1}{R}\right). \tag{5.1.18}$$

The equality (5.1.11) can be written

$$(\mathcal{A}_{[-R,R]}-\lambda)\mathcal{R}_R(\lambda) = I + \mathcal{E}_R(\lambda),$$

with $\|\mathcal{E}_R(\lambda)\| = \mathcal{O}(R^{-1})$, uniformly with respect to $\lambda \in]-\infty, |\mu_1|/2-\varepsilon] + i\mathbb{R}$. We deduce the existence of $R_\varepsilon > 0$ such that, for every $R \geq R_\varepsilon$, $(\mathcal{A}_{[-R,R]}-\lambda)$ is invertible, with inverse

$$(\mathcal{A}_{[-R,R]}-\lambda)^{-1} = \mathcal{R}_R(\lambda)(I+\mathcal{E}_R(\lambda))^{-1}.$$

We have proved (5.1.7).

Step 2: We prove that

$$\overline{\lim_{R\to+\infty}}\, \inf \operatorname{Re}\sigma(\mathcal{A}_{[-R,R]}) \leq \frac{|\mu_1|}{2}. \tag{5.1.19}$$

We reduce the study to the complex Airy operator $\mathcal{A}_{[0,2R]}$ on the intervall $[0,2R]$ by appying the translation $T_R : u(x) \mapsto u(x+R)$. We have

$$T_R^{-1}(\mathcal{A}_{[-R,R]}-\lambda)T_R = \mathcal{A}_{[0,2R]} - (\lambda + iR),$$

thus $\sup_{\nu\in\mathbb{R}} \|(\mathcal{A}_{[-R,R]}-(\gamma+i\nu))^{-1}\| = \sup_{\nu\in\mathbb{R}} \|(\mathcal{A}_{[0,2R]}-(\gamma+i\nu))^{-1}\|$
and $\operatorname{Re}\sigma(\mathcal{A}_{[-R,R]}) = \operatorname{Re}\sigma(\mathcal{A}_{[0,2R]})$.
Let $\theta_1,\ \theta_2 \in \mathcal{C}_0^\infty(\mathbb{R};[0,1])$ be such that $\operatorname{Supp}\theta_1 \subset]-\infty, 2/3]$, $\theta_1(x) = 1$ for $x \leq 1/2$, $\operatorname{Supp}\theta_2 \subset]1/2, +\infty]$, $\theta_2(x) = 1$ for $x \geq 2/3$, and $\theta_1^2 + \theta_2^2 \equiv 1$. For $j = 1,2$ and $R > 0$, we define

$$\chi_R^j(x) = \theta_j\left(\frac{x}{2R}\right)\mathbf{1}_{[0,2R]}(x). \tag{5.1.20}$$

We want to prove that

$$\mathbf{1}_{[0,2R]}(\mathcal{A}_{[0,2R]}+1)^{-1}\mathbf{1}_{[0,2R]} \underset{R\to+\infty}{\longrightarrow} (\mathcal{A}_{[0,+\infty[}+1)^{-1} \tag{5.1.21}$$

in $\mathcal{L}(L^2(\mathbb{R}^+))$. Let us remark that

$$\sigma(\mathbf{1}_{[0,2R]}(\mathcal{A}_{[0,2R]}+1)^{-1}\mathbf{1}_{[0,2R]}) = \sigma((\mathcal{A}_{[0,2R]}+1)^{-1}) \cup \{0\},$$

with non vanishing eigenvalues that have the same multiplicity for both operators.
For this, we use the 'approximated resolvent' of $(\mathcal{A}_{[0,2R]}+1)$,

$$\tilde{\mathcal{R}}_R = \chi_R^1(\mathcal{A}_{[0,+\infty[}+1)^{-1}\chi_R^1 + \chi_R^2(\mathcal{A}_{[0,2R]}+1)^{-1}\chi_R^2.$$

Then, we have

$$\begin{array}{rcl}(\mathcal{A}_{[0,2R]}+1)\tilde{\mathcal{R}}_R = I & + & [\mathcal{A}_{[0,2R]}+1,\chi_R^1](\mathcal{A}_{[0,+\infty[}+1)^{-1}\chi_R^1 \\ & + & [\mathcal{A}_{[0,2R]}+1,\chi_R^2](\mathcal{A}_{[0,2R]}+1)^{-1}\chi_R^2,\end{array}$$

thus, by composing on the left by $\mathbf{1}_{[0,2R]}(\mathcal{A}_{[0,2R]}+1)^{-1}\mathbf{1}_{[0,2R]}$,

$$\begin{array}{l}\mathbf{1}_{[0,2R]}(\mathcal{A}_{[0,2R]}+1)^{-1}\mathbf{1}_{[0,2R]}-\chi_R^1(\mathcal{A}_{[0,+\infty[}+1)^{-1}\chi_R^1=\chi_R^2(\mathcal{A}_{[0,2R]}+1)^{-1}\chi_R^2 \\ -\mathbf{1}_{[0,2R]}(\mathcal{A}_{[0,2R]}+1)^{-1}\mathbf{1}_{[0,2R]}[\mathcal{A}_{[0,2R]}+1,\chi_R^1](\mathcal{A}_{[0,+\infty[}+1)^{-1}\chi_R^1 \\ -\mathbf{1}_{[0,2R]}(\mathcal{A}_{[0,2R]}+1)^{-1}\mathbf{1}_{[0,2R]}[\mathcal{A}_{[0,2R]}+1,\chi_R^2](\mathcal{A}_{[0,2R]}+1)^{-1}\chi_R^2.\end{array} \quad (5.1.22)$$

Now, we control the different terms on the right hand side. The terms involving commutators can be estimated as for (5.1.16) and we get

$$\left\|\mathbf{1}_{[0,2R]}(\mathcal{A}_{[0,2R]}+1)^{-1}\mathbf{1}_{[0,2R]}[\mathcal{A}_{[0,2R]}+1,\chi_R^1](\mathcal{A}_{[0,+\infty[}+1)^{-1}\chi_R^1\right\|=\mathcal{O}\left(\frac{1}{R}\right) \quad (5.1.23)$$

and

$$\left\|\mathbf{1}_{[0,2R]}(\mathcal{A}_{[0,2R]}+1)^{-1}\mathbf{1}_{[0,2R]}[\mathcal{A}_{[0,2R]}+1,\chi_R^2](\mathcal{A}_{[0,2R]}+1)^{-1}\chi_R^2\right\|=\mathcal{O}\left(\frac{1}{R}\right). \quad (5.1.24)$$

Moreover, we have

$$\operatorname{Im}\langle(\mathcal{A}_{[0,2R]}+1)u,u\rangle=\langle yu,u\rangle\,. \quad (5.1.25)$$

This, applied to $u=\chi_R^2(\mathcal{A}_{[0,2R]}+1)^{-1}\chi_R^2 f$, $f\in L^2(\mathbb{R}^+)$, gives

$$\operatorname{Im}\langle(\mathcal{A}_{[0,2R]}+1)u,u\rangle\geq\frac{R}{2}\|u\|^2.$$

Moreover,

$$\begin{array}{rcl}|\operatorname{Im}\langle(\mathcal{A}_{[0,2R]}+1)u,u\rangle| & \leq & \|f\|\|(\mathcal{A}_{[0,2R]}+1)^{-1}\chi_R^2 f\| \\ & & +|\operatorname{Im}\langle[\mathcal{A}_{[0,2R]}+1,\chi_R^2](\mathcal{A}_{[0,2R]}+1)^{-1}\chi_R^2 f,u\rangle| \\ & \leq & C\left(1+\frac{1}{R}\right)\|f\|^2,\end{array}$$

where we have used (5.1.24).
Thus, we deduce that

$$\left\|\chi_R^2(\mathcal{A}_{[0,2R]}+1)^{-1}\chi_R^2\right\|=\mathcal{O}\left(\frac{1}{\sqrt{R}}\right). \quad (5.1.26)$$

By (5.1.22), (5.1.23), (5.1.24) and (5.1.26), we have

$$\left\|\mathbf{1}_{[0,2R]}(\mathcal{A}_{[0,2R]}+1)^{-1}\mathbf{1}_{[0,2R]}-\chi_R^1(\mathcal{A}_{[0,+\infty[}+1)^{-1}\chi_R^1\right\|=\mathcal{O}\left(\frac{1}{\sqrt{R}}\right). \quad (5.1.27)$$

In order to get (5.1.21), it remains to verify that $\chi_R^1(\mathcal{A}_{[0,+\infty[}+1)^{-1}\chi_R^1$ converges to $(\mathcal{A}_{[0,+\infty[}+1)^{-1}$ in $\mathcal{L}(L^2(\mathbb{R}^+))$. Let us denote $\mathcal{A}_+ = \mathcal{A}_{[0,+\infty[}+1$ to simplify notations. First, we write

$$\chi_R^1\mathcal{A}_+^{-1}\chi_R^1\mathcal{A}_+ = (\chi_R^1)^2 - \chi_R^1\mathcal{A}_+^{-1}[\mathcal{A}_+,\chi_R^1],$$

then, composing on the right by $\mathcal{A}_+^{-1}$ and using that $(\chi_R^1)^2 = 1-(\chi_R^2)^2$,

$$\mathcal{A}_+^{-1} - \chi_R^1\mathcal{A}_+^{-1}\chi_R^1 = (\chi_R^2)^2\mathcal{A}_+^{-1} + \chi_R^1\mathcal{A}_+^{-1}[\mathcal{A}_+,\chi_R^1]\mathcal{A}_+^{-1}. \tag{5.1.28}$$

An estimate similar to (5.1.16) gives

$$\|\chi_R^1\mathcal{A}_+^{-1}[\mathcal{A}_+,\chi_R^1]\mathcal{A}_+^{-1}\| = \mathcal{O}\left(\frac{1}{R}\right). \tag{5.1.29}$$

Moreover, we have, by applying inequality $\|y^{1/2}u\| \le C(\|\mathcal{A}_+u\| + \|u\|)$ (see (5.1.25)) to $u = (\chi_R^2)^2\mathcal{A}_+^{-1}f$, $f\in L^2(\mathbb{R}^+)$, and

$$\begin{aligned} R^{1/2}\|(\chi_R^2)^2\mathcal{A}_+^{-1}f\| &\le \|x^{1/2}(\chi_R^2)^2\mathcal{A}_+^{-1}f\| \\ &\le C(\|\mathcal{A}_+(\chi_R^2)^2\mathcal{A}_+^{-1}f\| + \|(\chi_R^2)^2\mathcal{A}_+^{-1}f\|) \\ &\le C(\|(\chi_R^2)^2 f\| + \|[\mathcal{A}_+,(\chi_R^2)^2]\mathcal{A}_+^{-1}f\| + \|(\chi_R^2)^2\mathcal{A}_+^{-1}f\|). \end{aligned}$$

The second term on the right hand side can be controled in a similar way as (5.1.16) and we get

$$\|(\chi_R^2)^2\mathcal{A}_+^{-1}\| = \mathcal{O}\left(\frac{1}{\sqrt{R}}\right). \tag{5.1.30}$$

Finaly, (5.1.27), (5.1.28), (5.1.29) and (5.1.30) imply (5.1.21).
We conclude, by [**?**, Section IV, §3.5]. For any subsequence $R_j\to+\infty$ and any eigenvalue $\lambda\in\sigma(\mathcal{A}_+^{-1})\setminus\{0\}$, exists a sequence (λ_j) such that, for every j large enough

$$\lambda_j \in \sigma(\mathbf{1}_{[0,2R_j]}(\mathcal{A}_{[0,2R_j]}+1)^{-1}\mathbf{1}_{[0,2R_j]})\setminus\{0\} = \sigma((\mathcal{A}_{[0,2R_j]}+1)^{-1})\setminus\{0\}$$

and $\lambda_j\to\lambda$ when $j\to+\infty$.
In particular, with $\lambda = 1/(\tilde\lambda+1)$, where $\tilde\lambda = e^{i\pi/3}|\mu_1| \in \sigma(\mathcal{A}_{[0,+\infty[})$ is the eigenvalue of $\mathcal{A}_{[0,+\infty[}$ with smallest real part (see [**?**]), we get a sequence $\tilde\lambda_j = 1/\lambda_j - 1 \in \sigma(\mathcal{A}_{[0,2R_j]})$ such that $\operatorname{Re}\tilde\lambda_j \to \operatorname{Re}\tilde\lambda = |\mu_1|/2$, from which we deduce (5.1.19).□

5.1.2 Semi classical analysis of Davies operator($\gamma = 2$)

The goal of this section is the proof of Theorem 5.2.

The case of the complex harmonic oscillator $\mathcal{H}_{[-R,R]}$ can be treated in a similar way. Let $\alpha\in]0,1[$ (that will be determined later) and let ζ_R^1, ζ_R^2, $\zeta_R^3\in\mathcal{C}^\infty(\mathbb{R};[0,1])$ be such that $\operatorname{Supp}\zeta_R^1\subset]-\infty,-R+R^\alpha[$, $\zeta_R^1(x)=1$ for $x\le -R+$

$R^\alpha/2$, $\operatorname{Supp}\zeta_R^2 \subset]-R+R^\alpha/2, R-R^\alpha/2[$, $\zeta_R^2(x)=1$ for $x\in[-R+R^\alpha, R-R^\alpha]$, $\operatorname{Supp}\zeta_R^3 \subset]R-R^\alpha, +\infty[$, $\zeta_R^3(x)=1$ for $x\geq R-R^\alpha/2$ and

$$(\zeta_R^1)^2+(\zeta_R^2)^2+(\zeta_R^3)^2\equiv 1.$$

Close to $x=-R$, we have

$$x^2=-2R(x+R)+R^2+o(|x+R|).$$

Thus, we are going to approximate $\mathcal{H}_{[-R,R]}$, close to $x=-R$, by the operator

$$\mathcal{A}_R^-=-\frac{d^2}{dx^2}-2iR(x+R)+iR^2.$$

In the same way, we will approximate $\mathcal{H}_{[-R,R]}$ close to $x=+R$ by

$$\mathcal{A}_R^+=-\frac{d^2}{dx^2}-2iR(R-x)+iR^2.$$

Then, we remak that, if $T_R: u(x)\mapsto u(x+R)$ and $U_R: u(x)\mapsto u(R-x)$, we have

$$\mathcal{A}_R^-=T_R\tilde{\mathcal{A}}_R^*T_R^{-1}\quad \text{et}\quad \mathcal{A}_R^+=U_R\tilde{\mathcal{A}}_R^*U_R^{-1},$$

where $\tilde{\mathcal{A}}_R$ is the Dirichlet realization of the complex Airy operator $-\frac{d^2}{dx^2}+iRx$ on $[0,+\infty[$. Following [30], we deduce that

$$\inf\,\operatorname{Re}\sigma(\mathcal{A}_R^+)=\inf\,\operatorname{Re}\sigma(\mathcal{A}_R^-)=R^{2/3}\frac{|\mu_1|}{2}, \tag{5.1.31}$$

and, for every $\varepsilon>0$, exists $C_\varepsilon>0$ such that

$$\sup_{\substack{\gamma\,\in\,[0,\,R^{2/3}|\mu_1|/2\,-\,\varepsilon],\\ \nu\,\in\,\mathbb{R}}}\|(\mathcal{A}_R^\pm-(\gamma+i\nu))^{-1}\|\leq\frac{C_\varepsilon}{R^{2/3}}. \tag{5.1.32}$$

We call $\mathcal{H}_0$ the complex harmonic oscillator $-\frac{d^2}{dx^2}+ix^2$ on $\mathbb{R}$, that will serve to approximate $\mathcal{H}_{[-R,R]}$ on the support of ζ_R^2. We recall that $\inf\,\operatorname{Re}\sigma(\mathcal{H}_0)=\cos\pi/4=\sqrt{2}/2$ (see Davies).
Now, we take $\lambda=\gamma+i\nu\in[0,\sqrt{2}/2-\varepsilon]+i\mathbb{R}$ and we set

$$\mathcal{Q}_R(\lambda)=\zeta_R^1(\mathcal{A}_R^--\lambda)^{-1}\zeta_R^1+\zeta_R^2(\mathcal{H}_0-\lambda)^{-1}\zeta_R^2+\zeta_R^3(\mathcal{A}_R^+-\lambda)^{-1}\zeta_R^3.$$

Then, we have

$$\begin{aligned}&(\mathcal{H}_{[-R,R]}-\lambda)\mathcal{Q}_R(\lambda)=I+[\mathcal{H}_{[-R,R]}-\lambda,\zeta_R^1](\mathcal{A}_R^--\lambda)^{-1}\zeta_R^1\\ &+[\mathcal{H}_{[-R,R]}-\lambda,\zeta_R^2](\mathcal{H}_0-\lambda)^{-1}\zeta_R^2+[\mathcal{H}_{[-R,R]}-\lambda,\zeta_R^3](\mathcal{A}_R^+-\lambda)^{-1}\zeta_R^3\\ &+\zeta_R^1(\mathcal{H}_{[-R,R]}-\mathcal{A}_R^-)(\mathcal{A}_R^--\lambda)^{-1}\zeta_R^1+\zeta_R^3\mathcal{H}_{[-R,R]}-\mathcal{A}_R^+)(\mathcal{A}_R^+-\lambda)^{-1}\zeta_R^3.\end{aligned}$$

The terms involving commutators can be estimated in the same way as (5.1.16), by noting that, for $j=1,2,3$, $\sup|(\zeta_R^j)'| = \mathcal{O}(R^{-\alpha})$ and $\sup|(\zeta_R^j)''| = \mathcal{O}(R^{-2\alpha})$. Thus, we get

$$\begin{aligned}&\|[\mathcal{H}_{[-R,R]}-\lambda,\zeta_R^1](\mathcal{A}_R^- -\lambda)^{-1}\zeta_R^1\| + \|[\mathcal{H}_{[-R,R]}-\lambda,\zeta_R^2](\mathcal{H}_0-\lambda)^{-1}\zeta_R^2\|\\ &+\|[\mathcal{H}_{[-R,R]}-\lambda,\zeta_R^3](\mathcal{A}_R^+ -\lambda)^{-1}\zeta_R^3\| = \mathcal{O}(R^{-\alpha}).\end{aligned}$$

Moreover, we have, by definition of $\mathcal{A}_R^-$,

$$(\mathcal{H}_{[-R,R]}-\mathcal{A}_R^-)u(x) = i(x+R)^2u(x),$$

and on the support of ζ_R^1, we have $x+R\le R^\alpha$. Therefore,

$$\begin{aligned}\|\zeta_R^1(\mathcal{H}_{[-R,R]}-\mathcal{A}_R^-)(\mathcal{A}_R^- -\lambda)^{-1}\zeta_R^1 f| &\le R^{2\alpha}\|(\mathcal{A}_R^- -\lambda)^{-1}\|\\ &\le C_\varepsilon R^{2(\alpha-1/3)}\end{aligned}$$

by (5.1.32).
Thus, we choose $\alpha\in]0,1/3[$.
In the same way, we verify

$$\|\zeta_R^3\mathcal{H}_{[-R,R]}-\mathcal{A}_R^+)(\mathcal{A}_R^+ -\lambda)^{-1}\zeta_R^3\| \le C_\varepsilon R^{2(\alpha-1/3)}.$$

Thus, we have proved that

$$(\mathcal{H}_{[-R,R]}-\lambda)\mathcal{Q}_R(\lambda) = I+\tilde{\mathcal{E}}_R(\lambda),$$

with $\|\tilde{\mathcal{E}}_R(\lambda)\|\to 0$ as $R\to+\infty$, uniformly with respect to λ in the interval $[0,\sqrt{2}/2-\varepsilon]+i\mathbb{R}$.
Thus, exists $R_\varepsilon>0$ such that, for every $R\ge R_\varepsilon$, $(\mathcal{H}_{[-R,R]}-\lambda)$ is invertible, with

$$(\mathcal{H}_{[-R,R]}-\lambda)^{-1} = \mathcal{Q}_R(\lambda)(I+\tilde{\mathcal{E}}_R(\lambda))^{-1}.$$

Thus

$$\forall R\ge R_\varepsilon,\quad \sigma(\mathcal{H}_{[-R,R]})\cap([0,\sqrt{2}/2-\varepsilon]+i\mathbb{R}) = \emptyset.$$

In order to prove

$$\overline{\lim_{R\to+\infty}}\ \inf\ \mathrm{Re}\,\sigma(\mathcal{H}_{[-R,R]}) \le \frac{\sqrt{2}}{2}, \tag{5.1.33}$$

we replace the functions χ_R^1 and χ_R^2 by

$$\varphi_R^j(x) = \psi_j\left(\frac{x}{R}\right),\quad j=1,2,\ R>0,$$

where $\psi_1 = h_1+h_3$, $\psi_2=h_2$, the h_k, $k=1,2,3$ being troncature functions defined above.
We recall that $\mathcal{H}_0$ denotes the operator $-\frac{d^2}{dx^2}+ix^2$ defined on $\mathbb{R}$, and we set

$$\tilde{\mathcal{Q}}_R = \varphi_R^2(\mathcal{H}_0+1)^{-1}\varphi_R^2 + \varphi_R^1(\mathcal{H}_{[-R,R]}+1)^{-1}\varphi_R^1.$$

Thus, we have

$$(\mathcal{H}_{[-R,R]}+1)\tilde{\mathcal{Q}}_R = I + \mathcal{P}_R$$

where

$$\mathcal{P}_R = [\mathcal{H}_{[-R,R]}, \varphi_R^2](\mathcal{H}_0+1)^{-1}\varphi_R^2 + [\mathcal{H}_{[-R,R]}, \varphi_R^1](\mathcal{H}_{[-R,R]}+1)^{-1}\varphi_R^1\,.$$

$\|\mathcal{P}_R\| = \mathcal{O}(R^{-1})$. By composing on the left with $(\mathcal{H}_{[-R,R]}+1)^{-1}$,

$$(\mathcal{H}_{[-R,R]}+1)^{-1} - \varphi_R^2(\mathcal{H}_0+1)^{-1}\varphi_R^2 = \varphi_R^1(\mathcal{H}_{[-R,R]}+1)^{-1}\varphi_R^1 + (\mathcal{H}_{[-R,R]}+1)^{-1}\mathcal{P}_R. \tag{5.1.34}$$

By going back over the proof of (5.1.26) and by replacing (5.1.25) by

$$\operatorname{Im}\langle \mathcal{H}_{[-R,R]}u, u\rangle = \langle x^2 u, u\rangle\,. \tag{5.1.35}$$

we check that

$$\|\varphi_R^1(\mathcal{H}_{[-R,R]}+1)^{-1}\varphi_R^1\| = \mathcal{O}\left(\frac{1}{R}\right),$$

By (5.1.34), this implies:

$$\|(\mathcal{H}_{[-R,R]}+1)^{-1} - \varphi_R^2(\mathcal{H}_0+1)^{-1}\varphi_R^2\| = \mathcal{O}\left(\frac{1}{R}\right). \tag{5.1.36}$$

Then, we prove, as we did prevously (see (5.1.28), (5.1.29) and (5.1.30)) that the operator $\varphi_R^2(\mathcal{H}_0+1)^{-1}\varphi_R^2$ converges to $(\mathcal{H}_0+1)^{-1}$ in $\mathcal{L}(L^2(\mathbb{R}))$, when $R \to +\infty$. By the same arguments, (5.1.33) is proved, which ends the proof of the theorem.

5.2 Non-linear techniques

The first step is to analyzed the linearized operator of the system where the linearization is considered for the normal solution. It involves a non selfadjoint diagonal system whose spectrum can be analyzed.

A relatively easy statement is obtained in the case when the "self-adjoint" part of this linearized operator has positive spectrum. In our case this will involve the Schrödinger operator with magnetic field and a "magnetic" operator. This gives interesting results but no effect of the current is visible. Hence this is still the effect of the exterior magnetic field which can imply the stability.

5.3 Giorgii-Phillips revisited

So we assume that we have a **nonnormal** stationary point $(\psi, \mathbf{A}, \Phi)$ of (TGDL2) and that

$$\int_\Omega |\psi(x)|^2\,dx > 0\,. \tag{5.3.1}$$

Then we get:

$$
\begin{aligned}
& i\phi\psi = (\nabla - iA)^2\,\psi + \psi\left(1 - |\psi|^2\right)\,, && \text{in } \Omega\,, && (5.3.2a)\\
& \kappa^2 \text{curl }^2 A + \sigma \nabla\phi = \operatorname{Im}(\bar\psi\,\cdot\,(\nabla - iA)\psi)\,, && \text{in } \Omega\,, && (5.3.2b)\\
& \psi = 0\,, && \text{on } \partial\Omega_c\,, && (5.3.2c)\\
& (\nabla - iA)\psi\cdot\nu = 0\,, && \text{on } \partial\Omega_i\,, && (5.3.2d)\\
& \text{curl } A = hB_r\,, && \text{on } \partial\Omega_i && (5.3.2e)
\end{aligned}
$$

Taking the scalar product with ψ in the first line, we get (using also the boundary condition)

$$
i\int \phi|\psi|^2 + ||(\nabla - iA)\psi||^2 + \int |\psi|^4 = ||\psi||^2\,. \tag{5.3.3}
$$

Now for the second equation, we take the scalar product with $A - hA_n$ and observing that div $A = 0$, we obtain:

$$
\kappa^2||\text{curl } (A - hA_n)||^2 = \int \left((A - hA_n)\cdot \operatorname{Im}(\bar\psi\,\cdot\,(\nabla - iA)\psi)\right)\,. \tag{5.3.4}
$$

Now (5.3.3) implies

$$
||(\nabla - iA)\psi||^2 \leq ||\psi||^2\,. \tag{5.3.5}
$$

Playing with (5.3.4), leads first to

$$
\kappa^2||A - hA_n||^2 \leq C_\Omega\kappa^2||\text{curl } (A - hA_n)||^2 \leq \hat C_\Omega||(A - hA_n)||\,||(\nabla - iA)\psi||\,. \tag{5.3.6}
$$

Hence

$$
\kappa^2||A - hA_n|| \leq \hat C_\Omega\,||(\nabla - iA)\psi||\ \leq ||\psi|| \tag{5.3.7}
$$

and we get

$$
\kappa^4||A - hA_n||^2 + ||(\nabla - iA)\psi||^2 \leq \widetilde C_\Omega||\psi||^2\,. \tag{5.3.8}
$$

We now compare $\int_\Omega |(\nabla - ihA_n\psi|^2\,dx$ and $\int_\Omega |(\nabla - iA)\psi|^2\,dx$. A trivial estimate is

$$
\int_\Omega |(\nabla - ihA_n)\psi|^2\,dx \leq 2\,\|(\nabla - i\mathbf{A})\psi\|^2 + 2(\|(\mathbf{A} - hA_n)|\psi|\,\|^2\,. \tag{5.3.9}
$$

This gives

$$
\int_\Omega |(\nabla - ihA_n)\psi|^2\,dx \leq 2C_\Omega(1 + \kappa^{-4})\int_\Omega |\psi(x)|^2\,dx\,. \tag{5.3.10}
$$

Since ψ satisfies (5.4.1), we obtain

$$
\lambda_1^{DN}(hA_n) \leq 2C_\Omega(1 + \kappa^{-4})\,. \tag{5.3.11}
$$

We now need an asymptotic behavior of $\lambda_1^{DN}(hA_n)$ in order to get either a contradiction (if no h satisfies the inequality) or an upper bound for h. This will give a Giorgii-Phillips type statement that for h large enough the only stationary solution is the normal one. Here the Kwek-Pan results are relevant.

5.4 Giorgii-Phillips rerevisited (very provisory)

We look at the problem in the "penetration depth" coordinates (we actually use these coordinates in the first chapter). This refers to Sections 4 and 5 in [2].

So we assume that we have a **nonnormal** stationary point $(\psi, \mathbf{A}, \Phi)$ of (TGDL2) and that

$$\int_\Omega |\psi(x)|^2\, dx > 0\,. \tag{5.4.1}$$

Then we get:

$$\begin{aligned}
& i\kappa^2 \phi\psi = (\nabla - i\kappa A)^2\, \psi + \kappa^2 \psi \left(1 - |\psi|^2\right)\,, && \text{in } \Omega\,, && (5.4.2a)\\
& \kappa \operatorname{curl}{}^2 A + \sigma \nabla\phi = \operatorname{Im}(\bar\psi \,\cdot\, (\nabla - i\kappa A)\psi)\,, && \text{in } \Omega\,, && (5.4.2b)\\
& \psi = 0\,, && \text{on } \partial\Omega_c\,, && (5.4.2c)\\
& (\nabla - i\kappa A)\psi \cdot \nu = 0\,, && \text{on } \partial\Omega_i\,, && (5.4.2d)\\
& \operatorname{curl} A = hB_r\,, && \text{on } \partial\Omega_i && (5.4.2e)
\end{aligned}$$

Taking the scalar product with ψ in the first line, we get (using also the boundary condition)

$$i\int \phi|\psi|^2 + ||(\nabla - i\kappa A)\psi||^2 + \kappa^2 \int |\psi|^4 = \kappa^2 ||\psi||^2\,. \tag{5.4.3}$$

Now for the second equation, we take the scalar product with $A - hA_n$ and observing that $\operatorname{div} A = 0$, we obtain:

$$\kappa ||\operatorname{curl}\,(A - hA_n)||^2 = \int \left((A - hA_n) \cdot \operatorname{Im}(\bar\psi \,\cdot\, (\nabla - i\kappa A)\psi)\right)\,. \tag{5.4.4}$$

Now (5.3.3) implies

$$||(\nabla - i\kappa A)\psi||^2 \le \kappa^2 ||\psi||^2\,. \tag{5.4.5}$$

Playing with (5.4.4), leads first to

$$\kappa ||A - hA_n||^2 \le C_\Omega \kappa ||\operatorname{curl}\,(A - hA_n)||^2 \le \hat C_\Omega ||(A - hA_n)||\, ||(\nabla - i\kappa A)\psi||\,. \tag{5.4.6}$$

Hence

$$\kappa ||A - hA_n|| \le \hat C_\Omega\, ||(\nabla - i\kappa A)\psi||\, \le \kappa ||\psi|| \tag{5.4.7}$$

and we get

$$\kappa^2 ||A - hA_n||^2 + ||(\nabla - i\kappa A)\psi||^2 \le \widetilde C_\Omega \kappa^2 ||\psi||^2\,. \tag{5.4.8}$$

We now compare $\int_\Omega |(\nabla - i\kappa hA_n\psi|^2\, dx$ and $\int_\Omega |(\nabla - i\kappa A)\psi|^2\, dx\,.$ A trivial estimate is

$$\int_\Omega |(\nabla - i\kappa hA_n)\psi|^2\, dx \le 2\, \|(\nabla - i\kappa A)\psi\|^2 + 2(\|(\kappa A) - \kappa hA_n)|\psi|\, \|^2\,. \tag{5.4.9}$$

This gives

$$\int_\Omega |(\nabla - i\kappa hA_n)\psi|^2\, dx \le 2C_\Omega \kappa^2 \int_\Omega |\psi(x)|^2\, dx\,. \tag{5.4.10}$$

Since ψ satisfies (5.4.1), we obtain

$$\lambda_1^{DN}(\kappa h A_n) \leq 2C_\Omega \kappa^2\,. \tag{5.4.11}$$

We now need an asymptotic behavior of $\lambda_1^{DN}(\kappa h A_n)$. This will give a Giorgii-Phillips type statement that for h large enough the only stationary solution is the normal one. Here the Pan-Kwek results are relevant. The problem is that we do not probably use the full information because we only use the information on the self-adjoint part.

Remark 5.3. *With the choices in Sections 4 and 5 of [2]. $A_n = \kappa^2 \tilde{A}_n$. This gives*

$$\lambda_1^{DN}(\kappa^3 h \tilde{A}_n) \leq 2C_\Omega \kappa^2\,. \tag{5.4.12}$$

For κ large, we get, if B_n does not vanish:

$$\frac{1}{C}(\kappa^3 h) \leq 2C_\Omega \kappa^2\,.$$

Hence the philosophy is that for κ large and $h \geq 1$, we cannot have a normal solution as stationary solution. If B_n vanished we obtain

$$\frac{1}{C}(\kappa^3 h)^{\frac{2}{3}} \leq 2C_\Omega \kappa^2\,.$$

Hence the philosophy is that for κ and h large enough, only the normal solution can be a stationary solutionwe cannot have a normal solution as stationary solution.
Note that many parameters are involved and various asymptotic situations can be considered. In [2] for example, it is assumed that κ^2/σ is constant.

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