Semiclassical analysis for non self-adjoint problems and applications to hydrodynamic instability.

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Introduction

Our aim is to show how semi-classical mechanics can be useful in questions appearing in hydrodynamics.

We will emphasize on the motivating examples and see how these problems can be solved or by harmonic approximation techniques used in the semi-classical analysis of the Schrödinger opertaor or by recently obtained semi-classical versions of operators of principal type (mainly subelliptic estimates). In this way, we hope to show that these recent results are much more than academic transpositions of former theorems developed more than thirty years ago when analyzing the main properties of Partial Differential Equations : local solvability, hypoellipticity, propagation of singularities...

Actually, we should confess that we do not need at the moment the most sophisticated theorems of this theory (see the lectures by N. Lerner).

We will give explicit proofs for the simple examples we have. They are based mainly on two tools, semiclassical elliptic theory for h-pseudodifferential operators and construction of WKB solutions.

Particular thanks to O. Lafitte who introduces me to this subject.

Physical models

We consider four different models.

The first one is the Rayleigh-Taylor model.
Although the subject has a long story starting with [St] , the semi-classical analysis appears in [La1, La2, HelLaf1]. The problem we meet in this case is self-adjoint and related to the analysis of the bottom of the spectrum for a Schrödinger operator.

The three other examples are not selfadjoint. We will see that we meet problems related to the notion of pseudospectrum.

The second one extends the previous one by introducing some velocity at the surface between the two fluids. This is an extension of the Kelvin-Helmholtz classical model which is analyzed in [CCLa].

The third one, the Rayleigh with convection model was studied in [CCLaRa] and is a natural generalization with a convective velocity of the classical Rayleigh problem for a transition region.

The fourth one is called the Kull-Anisimov ablation front model. It has been analyzed by many physicists and more recnetly in the PHD's of L. Masse [Masse] and V. Goncharov [Go].

Tentative plane of the course.

Step 1. Analysis of the Rayleigh-Taylor model.

We show how the initial problem of analyzing the possible instability of the model leads to a spectral problem for a compact selfadjoint operator which appears to be a h-pseudodifferential operator.

When needed, we will recall various basic things on the h-pseudodifferential operators.

We are let to the analysis of the largest eigenvalue of a compact operator. We show that either harmonic analysis or WKB solutions permit to have a good asymptotic of this eigenvalue.

From step 1 to Step 2.

A new example (Kelvin-Helmholtz) is presented to motivate looking at non self adjoint operators.

Step 2. Generalities on Pseudospectrum and some connected exercises.

Here we will emphasize on the “elliptic” h-pseudodifferential theory and on what can be done by WKB constructions.

We apply the techniques for analyzing our Kelvin-Helmholtz model.

Step 3. We discuss subellipticity in the semi-classical context.

We will see how the question of the subellipticity of h-pseudodifferential operators can appear naturally. In comparison with what was done in the course of N. Lerner, this will illustrate the most simple examples which were presented !

Step 4. Analysis of two other models. We will show that they lead to similar questions for some suitable regims of parameters.

The Rayleigh-Taylor model : Physical origin

The starting point for this model is the analysis of the following differential systems in $\mathbb{R}^4 = \mathbb{R}^3_x \times \mathbb{R}_t$.

$$\begin{cases} \partial_t \varrho + \text{div}\,(\varrho \vec{u}) = 0 \\ \partial_t(\varrho \vec{u}) + \nabla\,(\varrho \vec{u} \otimes \vec{u}) + \nabla p = \varrho \vec{g} \end{cases} \qquad (1)$$

The unknowns are $\vec{u} = (u_1, u_2, u_3)$, the density ϱ and the pressure p. We assume that $\vec{g} = (0, 0, 1)g$.

This system models the so-called Rayleigh-Taylor instability, which occurs when a heavy fluid is on a light fluid in a gravity field directed from the heavy to the light fluid. We intend to study the linear growth rate of this instability in a situation where there is a mixing region. This linear growth rate will corresponds to γ in (15) below.

We would like to analyze the linearized problem around the solution

$$\varrho = \rho^0 \ , \ u = u^0 = 0 \ , \ p = p^0 \ , \tag{2}$$

where ρ^0 is assumed to depend only on x_3 and p^0 and ρ^0 are related by :

$$\nabla p^0 = \rho^0 \vec{g} \ . \tag{3}$$

We assume that the perturbation $(\vec{\hat{u}}, \hat{p}, \hat{\rho})$ is incompressible that is satisfying :

$$\text{div}\, \vec{\hat{u}} = 0 \ . \tag{4}$$

The linearized system takes the form :

$$\partial_t \hat{\rho} + (\rho^0)' \hat{u}_3 = 0 \ . \tag{5}$$

$$\rho^0 \partial_t \hat{u}_1 + \partial_1 \hat{p} = 0 \ . \tag{6}$$

$$\rho^0 \partial_t \hat{u}_2 + \partial_2 \hat{p} = 0 \; . \tag{7}$$

$$\rho^0 \partial_t \hat{u}_3 + \partial_3 \hat{p} = g \hat{\rho} \; . \tag{8}$$

In order to analyze (at least formally this system) we exhibit now an equation for $\hat{u}_3$ (by eliminating the other unknowns).

We first differentiate with respect to t (8). This leads to :

$$\rho^0 \partial_t^2 \hat{u}_3 + \partial_t \partial_3 \hat{p} = g \frac{\partial \hat{\rho}}{\partial t} \; . \tag{9}$$

We now use (5) in order to eliminate $\frac{\partial \hat{\rho}}{\partial t}$. We get :

$$\rho^0 \partial_t^2 \hat{u}_3 + \partial_t \partial_3 \hat{p} + g (\rho^0)'(x_3) \hat{u}_3 = 0 \; . \tag{10}$$

We now differentiate (6) and (7) respectively with respect to x_1 and x_2. This gives :

$$\rho^0 \partial_t \partial_1 \hat{u}_1 + \partial_1^2 \hat{p} = 0 \; , \tag{11}$$

and

$$\rho^0 \partial_t \partial_2 \hat{u}_2 + \partial_2^2 \hat{p} = 0 \; . \tag{12}$$

Differentiating (4) with respect to t and using (11) and (12), we get :

$$\Delta_{1,2}\, \hat{p} = \rho^0 \partial_t \partial_3 \hat{u}_3 \; , \tag{13}$$

where Δ_{12} is the Laplacian with respect to the two first variables.

It remains to eliminate $\hat{p}$ between (10) and (13) :

$$\Delta_{12}\, \left(\rho^0 \partial_t^2 \hat{u}_3 + (\rho^0)' g \hat{u}_3\right) + \partial_3 \rho^0 \partial_3 \partial_t^2 \hat{u}_3 = 0 \; . \tag{14}$$

We now look for a solution $\hat{u}_3$ in the form :

$$\hat{u}_3(x_1, x_2, x_3, t) = v(x_3) \exp(\gamma t + i k_1 x_1 + i k_2 x_2) \; . \tag{15}$$

This leads to an ordinary differential equation (in the x_3-variable) for v :

$$-(k_1^2+k_2^2)(\rho^0\gamma^2 v+(\rho^0)' g v)+\gamma^2\frac{d}{dx_3}\rho^0\frac{d}{dx_3}v=0\,. \tag{16}$$

Replacing x_3 by x ($x\in\mathbb{R}$) and dividing by $\gamma^2 k^2$ with $k^2=k_1^2+k_2^2$, we get :

$$\left[-\frac{1}{k^2}\frac{d}{dx}\rho^0\frac{d}{dx}+\rho^0+(\rho^0)'\frac{g}{\gamma^2}\right]v=0\,. \tag{17}$$

So we are interested in analyzing for which value of (γ,k) (with $\gamma>0$) there exists v satisfying (16).

The choice of $\gamma>0$ corresponds to our interest for instability.

Rayleigh-Taylor mathematically

In the case of the Rayleigh-Taylor model the main point is to analyze as a function of $\delta \in \mathbb{R}$ the kernel in $L^2(\mathbb{R})$ of :

$$P(h,\delta) := -h^2 \frac{d}{dx} \varrho(x) \frac{d}{dx} + \varrho(x) + \delta \varrho'(x) \ . \tag{18}$$

Here $h > 0$ and $\varrho(x) \in C^\infty(\mathbb{R})$ satisfies :

$$\begin{aligned} \lim_{x\to-\infty} \varrho(x) &= \rho_- > 0 \ , \\ \lim_{x\to+\infty} \varrho(x) &= \rho_+ > 0 \ , \end{aligned} \tag{19}$$

$$\varrho(x) > 0 \ , \ \forall x \in \mathbb{R} \ , \tag{20}$$

$$\rho_- \neq \rho_+ \ , \tag{21}$$

$$\lim_{|x|\to+\infty} \varrho'(x) = 0 \ . \tag{22}$$

We look at $h \to 0$ (see [HelLaf1] for the case $h \to +\infty$). The problem comes from the analysis of the Euler equations in a gravity field. The physical parameters are the intensity g of the gravity, a wave number $k > 0$ and a parameter γ which measures the large time behavior of the solution. The mathematical problem is to determine a pair (u, γ) such that

$$P\left(\frac{1}{k}, \frac{g}{\gamma^2}\right) u = 0 \,.$$

This means that the link between the physical parameters (g, k, γ) and the mathematical parameters is :

$$\delta = \frac{g}{\gamma^2} \,,\; h = \frac{1}{k} \,. \tag{23}$$

The physical situation leads to analyze the case $\delta g > 0$. This implies $\gamma^2 > 0$, and we choose $\gamma > 0$.

Note that the instability is only analyzed when

$$\rho_+ \neq \rho_- \ .$$

This implies that $\varrho'(x)$ is not identically 0.

The most physical case corresponds to :

$$\rho_- > \rho_+ \ , \ g > 0 \ ,$$

so δ is positive and ϱ' is negative somewhere.

Generally ϱ is assumed monotone but the semi-classical techniques are not limited to this case.

Elementary spectral theory

First we observe that there is no problem for defining the selfadjoint extension of $P(h,\delta)$ in $L^2(\mathbb{R})$ (which is unique starting from $C_0^\infty(\mathbb{R})$) and it is immediate that $P(h,0)$ is injective. More precisely, the bottom of its spectrum is strictly positive.

Definition.
We call generalized spectrum of the family $P(h,\delta)$ the set of the δ's in $\mathbb{R}$ such that $P(h,\delta)$ is non injective.

Remark.
The standard analysis of the solution at ∞ shows that, for all δ, the dimension of $\ker P(h,\delta)$ is zero or one.

Next result is relatively standard (connected to the Birman-Schwinger principle).

Proposition.
Under the previous assumptions and assuming in addition that ϱ' *is not identically* 0*, then the generalized spectrum* $P(h,\delta)$ *is the union of two sequences (possibly empty or finite)* δ_n^+ *et* δ_n^- *s.t :*

$$\begin{array}{l} 0 < \delta_n^+ < \delta_{n+1}^+ \,, \\ \lim_{n\to+\infty} \delta_n^+ = +\infty \,, \end{array} \tag{24}$$

$$\begin{array}{l} 0 < -\delta_n^- < -\delta_{n+1}^- \,, \\ \lim_{n\to+\infty} \delta_n^- = -\infty \,. \end{array} \tag{25}$$

Proof.
If we observe that :

$$\ker P(h,\delta) \neq \{0\} \text{ iff } \ker(K(h) - \frac{1}{\delta}) \neq \{0\}\ , \quad (26)$$

where

$$K(h) = -P(h,0)^{-\frac{1}{2}}\varrho'(x)P(h,0)^{-\frac{1}{2}}\ . \quad (27)$$

the proof is immediately reduced to the standard result for $K(h)$, which is a compact selfadjoint operator.

For the compactness of $K(h)$, we can for example observe that $P(h,0)^{-\frac{1}{2}} \in \mathcal{L}(L^2; H^1)$ and that, under Assumption (22), the operator of multiplication by ρ' is compact from H^1 in L^2.

Note that when $\varrho' < 0$, which is the simplest natural physicak case $K(h)$ is positive.

Let us mention an a priori "universal" estimate of [CCLaRa]. If u is, for some $\delta \neq 0$, a solution, we get by taking the scalar product in L^2 by u :

$$\begin{aligned} \int_{-\infty}^{+\infty} \varrho(h^2 u'(x)^2 + u(x)^2)\, dx &= -\delta \int_{-\infty}^{+\infty} \varrho'(x) u(x)^2\, dx \\ &= 2\delta \int_{-\infty}^{+\infty} \varrho u(x) u'(x)\, dx\,. \end{aligned} \tag{28}$$

Using Cauchy-Schwarz, we get :

$$\int_{-\infty}^{+\infty} \varrho(x)(1 - \frac{|\delta|}{h})(u'(x)^2 + u(x)^2) dx \leq 0\,. \tag{29}$$

This implies

$$\ker P(h,\delta) = \{0\}\,,\ \forall \delta \in]-h, h[\,. \tag{30}$$

Universal upper bound.

We could have started from the operator :

$$-h^2\varrho^{-\frac{1}{2}}\frac{d}{dx}\varrho\frac{d}{dx}\varrho^{-\frac{1}{2}}+1+\delta\frac{\varrho'(x)}{\varrho(x)}\ ,$$

which shows more clearly the role of $\frac{1}{L}=\varrho'/\varrho$. One way is to change of functions introducing

$$u=\varrho(x)^{-\frac{1}{2}}v\ .$$

This shows also that if :

$$1+\delta\frac{\varrho'(x)}{\varrho(x)}>0\ ,\ \forall x\in\mathbb{R}\ ,$$

then δ is not in the generalized spectrum.

Remark.
The theory can be extended to the cases $\rho_+=0$ or $\rho_-=0$, with Condition (31).

A crash course on h-pseudodifferential operators

A family ($h \in]0, h_0]$) of h-pseudodifferential operators

$$A_h = \mathrm{Op}_h(a)\,,$$

associated to a symbol $(x,\xi) \mapsto a(x,\xi;h)$ is defined on $\mathcal{S}(\mathbb{R}^m)$ by :

$$(\mathrm{Op}_h(a)u)(x) =$$

$$(2\pi h)^{-m} \int_{\mathbb{R}^m \times \mathbb{R}^m} \exp(\tfrac{i}{h}(x-y)\cdot\xi)\; a(\tfrac{x+y}{2},\xi;h)\, u(y)\, dy d\xi\ .$$

The function a is called the Weyl symbol of A_h.

We refer to the book of D. Robert [Rob] for a course on this theory which is specifically semi-classical (and to the other lectures by Prof. Wong and N. Lerner in this CIME meeting) and the assumptions which can be done on the symbols.

Here it is enough to consider functions a s.t., for some given p, p', q and $h_0 > 0$, there exists , for all α and β in $\mathbb{N}^m$, constants $C_{\alpha,\beta}$ s.t. , for all $h \in]0, h_0]$,

$$|D_x^\alpha D_\xi^\beta a(x, \xi; h)| \leq C_{\alpha,\beta} h^q \langle x \rangle^{p-\tau|\alpha|} \langle \xi \rangle^{p'-|\beta|} .$$

When the symbol satisfies this condition, we write simply $a \in S^{(q,p,p')}$, and the corresponding operator $\mathsf{Op}_h(a)$ is said to belong to $\mathsf{Op}\, S^{(q,p,p')}$.

This class is an algebra by composition and the composition is just a multiplication for the principal symbols. Typically, if $a \in S^{(q,p,p')}$ and $b \in S^{(q_1,p_1,p_1')}$, then there exists c in $S^{(q+q_1,p+p_1,p'+p_1')}$ s.t. :

$$\mathsf{Op}_h(a) \circ \mathsf{Op}_h(b) = \mathsf{Op}_h(c) \,,$$

and

$$c - ab \in S^{(q+q_1+1,p+p_1-\tau,p'+p_1'-1,)} \,.$$

This leads to the natural definition of " principal symbol".
In the current situation, the symbol $a \in S^{q,p,p'}$ admits the formal expansion :

$$a(x,\xi;h) \sim h^q \sum_{j\geq 0} h^j a_j(x,\xi)\ ,$$

with :

$$a_j(x,\xi) \in S^{0,-\tau j,-j}\ .$$

The symbol $h^q a_0(x,\xi)$ is called the principal symbol. The symbol $h^q a_1(x,\xi)$ is called the subprincipal symbol.

Moreover, as the principal symbol is invertible (=elliptic), one can inverse the operator for h small enough.

We have natural continuity theorems in $H^s(\mathbb{R}^m)$, where moreover the constants are controlled with respect to h.

Finally the compact operators on $L^2(\mathbb{R}^n)$ can be recognized as the operators whose symbol in $S^{(0,0,0)}$ tends to 0 as $|x| + |\xi| \to +\infty$.

Typically, an operator in $\mathrm{Op}_h S^{(q,p,p')}$ with $p < 0$ and $p' < 0$ is compact.

Application for Rayleigh-Taylor : Semi-classical analysis for $K(h)$.

Under strong assumptions on ϱ, one can use a pseudodifferential calculus. We assume :

$$|D_x^\alpha \varrho(x)| \leq C_\alpha \varrho(x) \langle x \rangle^{-\tau|\alpha|}\,, \tag{31}$$

for some $\tau > 0$.
This assumption permits to see that :

$$K(h) = -(-h^2 \frac{d}{dx}\varrho\frac{d}{dx} + \varrho)^{-\frac{1}{2}}\varrho'(x)(-h^2\frac{d}{dx}\varrho\frac{d}{dx} + \varrho)^{-\frac{1}{2}} \tag{32}$$

is an h-pseudo.

More precisely it belongs to $\text{Op}_h S^{(0,0,0)}$.

The operator $K(h)$ appears indeed as the composition of three h-pseudo-differential operators $(-h^2\frac{d}{dx}\rho\frac{d}{dx}+\rho)^{-\frac{1}{2}}$, $-\rho'(x)$ and of $(-h^2\frac{d}{dx}\rho\frac{d}{dx}+\rho)^{-\frac{1}{2}}$.

So the h-pseudo calculus gives that it is an h-pseudo.

The principal symbol of $K(h)$ is

$$(x,\xi)\mapsto p(x,\xi)=-(\xi^2+1)^{-1}\frac{\varrho'(x)}{\varrho(x)}\;.$$

For the analysis of the extremal eigenvalues, we have first to determine the extrema of this symbol. If these extrema are non degenerate then we can apply the harmonic approximation as in [HelSj1] . The tunneling effect can also be analyzed (see [HePa]). This leads to the following computations. We get

$$\begin{aligned}\frac{\partial p}{\partial \xi}(x,\xi) &= 2\xi\frac{\varrho'(x)}{\varrho(x)}(\xi^2+1)^{-2}\;,\\ \frac{\partial p}{\partial x}(x,\xi) &= -(\varrho''(x)\varrho(x)-\varrho'(x)^2)(\varrho(x))^{-2}(\xi^2+1)^{-1}\;.\end{aligned}$$

The condition $\varrho'(x) = 0$ should be excluded because it does not correspond to an extremum of $p(x, \xi)$. So we get :

$$\xi = 0 \; ; \; \varrho''(x)\varrho(x) - \varrho'(x)^2 = 0 \; .$$

This corresponds to the condition that x_0 is a critical point of the map $x \mapsto -\varrho'(x)/\varrho(x)$.
It remains to verify that the extrema are non degenerate. We obtain at a critical point $(x_0, 0)$:

$$\begin{aligned} \frac{\partial^2 p}{\partial \xi^2}(x_0, 0) &= +2\varrho'(x_0)/\varrho(x_0) \\ \frac{\partial^2 p}{\partial \xi \, \partial x}(x_0, 0) &= 0 \\ \frac{\partial^2 p}{\partial x^2}(x_0, 0) &= -\frac{\varrho'''(x_0)\varrho(x_0) - \varrho'(x_0)\varrho''(x_0)}{\varrho(x_0)^2} \end{aligned}$$

It is then easy to determine if $(x_0, 0)$ corresponds to :

- a minimum of p,
  if $\varrho'(x_0)/\varrho(x_0) > 0$
  and $\varrho'''(x_0)\varrho(x_0) - \varrho'(x_0)\varrho''(x_0)) < 0$,

- a maximum of p,
  if $\varrho'(x_0)/\varrho(x_0) < 0$
  and $\varrho'''(x_0)\varrho(x_0) - \varrho'(x_0)\varrho''(x_0) > 0$.

When $\rho' \geq 0$ and $\rho > 0$, then the maxima of the symbol correspond to $\xi = 0$ and to the x's such that $\frac{\rho'}{\rho}$ is maximal.

We recall that the simplest physical situation corresponds to $\varrho'(x) < 0$. In this case we have only maxima, which actually are the points of interest if looking for largest eigenvalue.

Harmonic approximation

If we are interested in the largest eigenvalue of $K(h)$ a very general theory has been developed (of course for Schrödinger, but also for more general h-pseudodifferential operators.

We just sketch what corresponds to the first approximation.

We have just to consider the following harmonic operator associated to a point $(x_0, 0)$ corresponding to a maximum of p, and to consider the spectrum of

$$p(x_0,0)+h\left(\frac{1}{2}\frac{\partial^2 p}{\partial \xi^2}(x_0,0)D_y^2+\frac{1}{2}\frac{\partial^2 p}{\partial x^2}(x_0,0)y^2\right)+hp_1(x_0,0)\ ,$$

where p_1 is the subprincipal Weyl symbol of $K(h)$, which actually is 0.

This operator is consequently

$$-\frac{\varrho'(x_0)}{\varrho(x_0}(1-hD_y^2)-h\frac{\varrho'''(x_0)\varrho(x_0)-\varrho'(x_0)\varrho''(x_0)}{2\varrho(x_0)^2}y^2$$

The largest eigenvalue of this operator (which is semi-bounded from above !) is explicitly known and gives the existence of an eigenvalue for $K(h)$ (with some error $\mathcal{O}(h^{\frac{3}{2}})$).

If there are more than one critical maximum point for p, the largest eigenvalue of $K(h)$ is well approximated by the largest (over the maxima of p) of the largest eigenvalue of the approximating harmonic oscillators.

Instability of Rayleigh-Taylor: an elementary approach via WKB constructions

We present here what simple constructions of WKB solutions can give for the model of Rayleigh-Taylor. A very detailed analysis have been given in [HelLaf1] extending previous works by Cherfils, Lafitte, Raviart. Here we present a simpler analysis but we only give condition under which one can construct approximate solutions in the kernel of $P(h, \delta)$.

In the semi-classical situation, we look for a solution in the form

$$u(x,h) = a(x,h)\exp -\frac{\phi(x)}{h}$$

near some point x_0 (to be determined!) with

$$a(x,h) \sim \sum_{j\geq 0} h^j a_j(x)\ , \tag{33}$$

$$\delta(h) \sim \sum_j h^j \delta_j \tag{34}$$

such that

$$\exp\frac{\phi}{h}\cdot P(h,\delta(h))\cdot u(h) \sim 0\ . \tag{35}$$

Here "~ 0" means that the right hand side should be $\mathcal{O}(h^\infty)$.

Concretely, we expand $\exp\frac{\phi}{h} \cdot P(h, \delta(h)) \cdot u(h)$ in powers of h and express the cancellation of each coefficient of h^j.

We get as first eikonal equation

$$-\varrho(x)\phi'(x)^2 + \varrho(x) + \delta_0\varrho'(x) = 0\ . \qquad (36)$$

In order to have an (exponentially) localized (as $h \to 0$) in a neighborhood of x_0, it is natural to impose the condition that ϕ admits a minimum at x_0. So the first condition is :

$$\phi'(x_0) = 0\ . \qquad (37)$$

Ths leads as a first necessary condition to

$$\varrho(x_0) + \delta_0\varrho'(x_0) = 0\ . \qquad (38)$$

A second necessary condition is obtained by differentiating the eikonal equation :

$$-\varrho'(x)\phi'(x)^2 - 2\varrho(x)\phi'(x)\phi''(x) + \varrho'(x) + \delta_0\varrho''(x) = 0 \; .$$

This gives at x_0 :

$$\varrho'(x_0) + \delta_0\varrho''(x_0) = 0 \; . \tag{39}$$

We are asking for a non-degenerate minimum of ϕ at x_0.
Differentiating two times the eikonal equation, we obtain :

$$-2\varrho(x_0)(\phi''(x_0))^2 + \varrho''(x_0) + \delta_0\varrho'''(x_0) = 0 \tag{40}$$

which implies

$$\varrho''(x_0) + \delta_0\varrho'''(x_0) > 0 \; . \tag{41}$$

We recover the condition obtained in the previous analysis.

Till now, we just looked for a phase. The next step is to determine the amplitude. The coefficient δ_1 will be determined by looking at the first transport equation :

$$2\varrho(x)\phi'(x)a_0'(x) + \varrho'(x)\phi'(x)a_0(x) \\ +\varrho(x)\phi''(x)a_0(x) + \delta_1\varrho'(x)a_0(x) = 0 \ . \tag{42}$$

If we impose the condition

$$a_0(x_0) = 1 \ ,$$

— This condition corresponds to the idea that we look for the ground state, hence non vanishing—-

a necessary (and actually sufficient) condition for solving is :

$$\varrho(x_0)\phi''(x_0) + \delta_1\varrho'(x_0) = 0 \ . \tag{43}$$

We then obtain a_0 by simple integration :

$$a_0'(x)/a_0(x) = \\ \left(\varrho'(x)\phi'(x) + \varrho\phi''(x) + \delta_1\varrho'(x)\right) / \left(2\varrho(x)\phi'(x)\right) \; .$$

It is then not difficult to iterate at any order. At each step the cancellation of the coefficient of h^j in the expansion of $\exp\frac{\phi}{h} \cdot P(h, \delta(h)) \cdot u(h)$ permits to determine δ_j.

We have now a formal solution. Let us recall now how one can associate to this formal expansion an explicit realization. The first idea is to consider a finite sum.

We let $\delta^N(h) = \sum_{j=0}^N \delta_j h^j$ and introduce $a^N(x,h) = \sum_{j=0}^N h^j a_j(x)$ which is well defined in the neighborhood of x_0.

We then introduce a cut-off χ which localizes in a neighborhood of x_0. We then let

$$u_\chi^N(x,h) = \chi(x) a^N(x,h) \exp -(\phi(x)/h) \; .$$

Computing $P(h, \delta^N(h)) u_\chi^N(x,h)$, we find :

$$\begin{aligned} & P(h, \delta^N(h)) u_\chi^N(x,h) \\ & \quad = (\chi h^N r_N(x,h) + \tilde{\chi}(x) b_0(x,h)) \exp -\frac{\phi(x)}{h} \; , \end{aligned} \tag{44}$$

where $\tilde{\chi}$ is C^∞, with a support disjoint of x_0. Here it is important to observe that $\exp -\frac{\phi(x)}{h}$ is exponentially small on the support of $\tilde{\chi}$ (Here we have used that ϕ has a local minimum at x_0).

What can we deduce from this construction ?

Under the previous assumptions, $P(h,\delta)$ is selfadjoint and we can deduce that, in an interval $]-Ch^N,+Ch^N[$, the spectrum of $P(h,\delta^N(h))$ is not empty for h small enough. Assumption (19) permits also to say that near 0 the spectrum is discrete.

This is not the complete answer to our question. But this strongly suggests the existence, close to $\delta^N(h))$ (modulo $\mathcal{O}(h^N)$) of an effective $\delta(h)$ such that $P(h,\delta(h)$ has a non zero kernel.

Note that the answer to this last question is easier when ϱ is stricly monotone. Note that the question is more delicate as for example $\rho_- = 0$. The essential spectrum of $P(h,\delta)$ contains indeed 0.
The previous analysis (see [HelLaf1]) avoids this difficulty (finally artificial) if $\frac{\varrho'}{\varrho} \to 0$ at ∞.

Three remarks for ending this part.

- One can take $N = +\infty$ by using a summation procedure à la Borel.
  The Borel Lemma says that for a given sequence of reals α_n $(n \in \mathbb{N})$) one can always find a C^∞ function $h \mapsto f(h)$ admitting $\sum_n \alpha_n h^n$ as Taylor expansion at 0.

  Here we need a version with parameters, but we can define some realizations of $\sum_{j=0}^\infty \delta_j h^j$ and $\sum_{j=0}^\infty h^j a_j(x)$, permitting to replace the remainder $\mathcal{O}(h^N)$ by $\mathcal{O}(h^\infty)$.

- With more work, one can also hope a result in the analytic category by using the notion of analytic symbol introduced by J. Sjöstrand.

  We should assume in this case that the function $x \mapsto \varrho(x)$ is analytic.

  We warn the reader that this does not mean that the above formal sums become convergent. This simply means that one can prove that, in a fixed complex neighborhood of x_0, $|a_j(x)|$ is bounded by $C^{j+1} j!$ and that we have similar estimates for the sequence $(\delta_j)_{j\in\mathbb{N}}$ (cf the works by J. Sjöstrand [Sj1], Helffer-Sjöstrand [HelSj1], Klein-Schwarz [KlSc90]).

  This simply means that, by a "finite" tricky summation ($N(h) = \frac{C_0}{h}$ depending on h), one gets the existence of $\epsilon_0 > 0$, such that :

$$P(h, \delta^{N(h)}(h)) u_\chi^N(x,h) = \mathcal{O}(\exp -\frac{\epsilon_0}{h}) \exp -\frac{\phi(x)}{h} \,. \tag{45}$$

•

Here we have used the self-adjointness property for getting information on the spectrum.

We will now see that for more complicate models the selfadjoint character of the problem disappears.

We will start with a generalization of the Kelvin-Helmoltz model.

Instability for Kelvin-Helmoltz I.

This is a generalization [CCLa] of the classical Kelvin-Helmholtz instability which appears when two fluids move with different parallel velocities on each side of an interface.

When linearizing along a stationary solution (and following what we have done for Rayleigh-Taylor) we get, for a given density ϱ_0 and a given this time not zero velocity u_0 (see (2)), where u_0 is the first component of $\vec{U}_0$, the following one dimensional question.
Analyze if the operator, which depends on the parameters (k_1, k_2, g, γ, k) with $k^2 = k_1^2 + k_2^2$,

$$\begin{aligned}\mathcal{P}_{KH}(\gamma, k_1, k_2, g) :=& \\ &-\frac{d}{dx}\varrho_0\frac{d}{dx}(\gamma + ik_1u^0(x))^2 + k^2\varrho_0(x)(\gamma + ik_1u_0(x))^2 \\ &\quad -ik_1(\gamma + ik_1u^0(x))\frac{d}{dx}\varrho_0u_0'(x) + gk^2\varrho_0'((x)\end{aligned}$$

is approximately injective in some regime of the parameters.

Like in Rayleigh-Taylor which corresponds to $k_1 = 0$ (actually to $u_0 = 0$), our semi-classical parameter will be $h = \frac{1}{k}$. The parameter $\gamma = \Gamma_0 + i\Gamma_1$ is not necessarily real but we are interested in approximate null solutions for which Γ_0 is as large as possible (or complementarily) to show that Γ_0 should necessarily remain bounded in the regime k large.

So we divide by k^2 in the equation above and meet the following semi-classical operator :

$$
\begin{aligned}
\mathcal{P}_k(x, hD_x) \quad := & -h\tfrac{d}{dx}\varrho_0 h\tfrac{d}{dx}(\gamma + ik_1 u^0(x))^2 \\
& +\varrho_0(x)(\gamma + ik_1 u_0(x))^2 \\
& -ihk_1(\gamma + ik_1 u^0(x))h\tfrac{d}{dx}\varrho_0 u_0'(x) \\
& +g\varrho_0'(x) \; .
\end{aligned}
$$

So in our regime k_1 is fixed such that $|k_1| \leq k = \frac{1}{h}$ which will not be a restriction in the semi-classical regime.

Semi-classically, the principal symbol is given by

$$p_0(x,\xi) := \varrho_0(1+\xi^2)(\gamma + ik_1 u^0(x))^2 + g\varrho_0'(x)\ .$$

We have to find a rather systematic strategy for constructing approximate null solutions or to decide that we can not construct such solutions. This leads us naturally to define various notions of pseudo-spectra for families (depending in particular on h but also on other parameters) and adapted to the analysis of h-pseudodifferential operators.

This is what we will do now before to treat the various physical examples including this one.

Around the pseudo-spectrum

Definition
If A is a closed operator with dense domain $D(A)$ in an Hilbert space $\mathcal{H}$, the ϵ-pseudospectrum $\sigma_\epsilon(A)$ of A is defined by

$$\sigma_\epsilon(A) := \{z \in \mathbb{C} \mid ||(zI - A)^{-1}|| \geq \frac{1}{\epsilon}\}.$$

We take the convention that $||(zI - A)^{-1}|| = +\infty$ if $z \in \sigma(A)$.
When A is selfadjoint (or more generally normal), $\sigma_\epsilon(A)$ satisfies by the spectral theorem

$$\sigma_\epsilon(A) = \{z \in \mathbb{C} \mid d(z, \sigma(A)) \leq \epsilon\}\ .$$

So this is only in the case of non self-adjoint operators that this new concept (first appearing in numerical analysis, see Trefethen [Tr]) becomes interesting.

Although abstract, the following result by Roch-Silbermann [RoSi] explains rather well to what corresponds the pseudo-spectrum

$$\sigma_\epsilon(A) = \bigcup_{\{\delta A \in \mathcal{L}(\mathcal{H}) \text{ s. t. } ||\delta A||_{\mathcal{L}(\mathcal{H})} \leq \epsilon\}} \sigma(A + \delta A)\ .$$

In other words, z is in the ϵ-pseudo-spectrum of A if z is in the spectrum of some perturbation $A + \delta A$ of A with $||\delta A|| \leq \epsilon$.

Around the h-family-pseudospectrum

We are mainly interested in the semiclassical version of this concept attached to a family ($h \in]0, h_0]$) of operators A_h.

For a given $\mu \geq 0$, the h-family-pseudospectrum of index μ of the family A_h is defined by

$$\begin{aligned}&\Psi_\mu((A_h))\\ &\quad := \{z \in \mathbb{C} \mid \forall C > 0, \forall h_0 > 0 \, s.t. \, \exists h \in]0, h_0],\\ &\qquad\qquad\qquad ||(A_h - z)^{-1}|| \geq C\, h^{-\mu}\}\,.\end{aligned}$$

We can then define

$$\Psi_\infty((A_h)) = \bigcap_{\mu \geq 0} \Psi_\mu((A_h))\,.$$

May be it is easier to understand the quantifiers by observing that the h-family pseudoresolvent set corresponds to the z such that $\exists C > 0$ and $h_0 > 0$ such that $\forall h \in]0, h_0]$

$$||(A_h - z)^{-1}|| \leq C\, h^{-\mu}\ .$$

Physically these concepts are more stable by perturbation than the corresponding notion of spectrum.

In general, one will exhibit the existence of this pseudo-spectrum by constructing quasimodes.

For a given $\mu \geq 0$, the h-family-quasispectrum of index μ of the family A_h is defined by

$$\begin{aligned}\psi_\mu((A_h)) \\ := \{z \in \mathbb{C} \mid \forall C > 0, \forall h_0 > 0\, s.t.\, \exists h \in]0, h_0], \\ \exists u_h \in D(A_h) \setminus \{0\}\, s.t \\ ||(A_h - z) u_h|| \leq\ C\, h^\mu ||u_h||\}\ .\end{aligned}$$

We can then define

$$\psi_\infty((A_h)) = \bigcap_{\mu \geq 0} \psi_\mu((A_h)) \ .$$

The main point is then that

$$\psi_\mu((A_h)) \subset \Psi_\mu((A_h)) \ .$$

The converse is not true in general.

We will particularly interested in the analysis of the case when A_h is actually an h-pseudodifferential operator.

The elliptic theory (with suitable conditions at ∞) says for example that

Proposition (see the book of D. Robert)
If $z \notin \Sigma(p)$, where

$$\Sigma(p) := \{\lambda \in \mathbb{C}\,, \mid \exists (x_n, \xi_n)\; s.t\; \lambda = \lim_{n \to +\infty} p(x_n, \xi_n)\}$$

then $z \notin \Psi_\mu(\, \mathsf{Op}_h(p))$.

This will actually be true for any $A_h = \mathsf{Op}_h(p_h)$, for which the principal symbol of A_h is p.

The proof is very easy once an h-pseudodifferential calculus has been constructed.

It is enough to use $\mathsf{Op}_h((p - z)^{-1})$ as first approximate inverse and then to use a Neumann series.

Davies example by hand

We present a variant of the proof of the generalization, by Pravda-Starov [Pra1], of Davies result on the h-family pseudo-spectrum for the Schrödinger operator

$$A_h := -h^2 \frac{d^2}{dx^2} + V(x) \; .$$

This proof is inspired by similar proofs in [HelLaf2, Mar].

Historical remarks

Davies treats a particular case by hand.

Then Zworski observes that it can be interpreted as a semi-classical version of a result for operators of principal type (Hörmander [Ho1], [Ho2], Duistermaat-Sjöstrand [DuSj]).

This was pushed further by Dencker-Sjöstrand-Zworski [DeSjZw], N. Lerner (together with collaborators) (see in [Le] and references therein), Pravda-Starov [Pra1] ...

One should of course compare with the selfadjoint result at the bottom of the well but here what is crucial is the non-selfadjointness !!

Theorem Davies-Pravda

Let us assume that there exist x_0 *and* z *such that*

$$z - V(x_0) \in \mathbb{R}^+ \,, \tag{46}$$

and such that, for an even $k \geq 0$*,*

$$\mathrm{Im}\, V^{(j)}(x_0) = 0 \,, \forall j \leq k \,, \tag{47}$$

and

$$\mathrm{Im}\, V^{(k+1)}(x_0) \neq 0 \,, \tag{48}$$

Then $z \in \psi_\infty((A_h))$.

Some elementary proof

The crucial point is that there exists $\xi_0 > 0$ such that $\xi_0^2 + V(x_0) = z$.

In other words, there exists (x_0, ξ_0) such that $p(x_0, \xi_0) = z$ ($z \in \Sigma(p)$) and we are not at the boundary of $\Sigma(p)$.

Following the previous construction, we are first looking for a phase φ satisfying (we can after a change of notations assume that $z = 0$.

$$-\varphi'(x)^2 + V(x) = 0 \, , \tag{49}$$

where V satisfies

$$\mathrm{Re}\, \mathrm{V}(\mathrm{x}_0) < 0 \, , \tag{50}$$

(47) and (48).
The existence of $\varphi(x)$, with $\varphi(x_0) = 0$ and $\varphi'(x_0) = i\xi_0$ is evident. So the important point is to verify

that $\mathrm{Re}\,\varphi$ has actually a local minimum at x_0. Taking the real and imaginary parts in (49), we get

$$-\mathrm{Re}\,\varphi'(\mathrm{x})^2 + \mathrm{Im}\,\varphi'(\mathrm{x})^2 + \mathrm{Re}\,\mathrm{V}(\mathrm{x}) = 0\ , \tag{51}$$

and

$$-2\mathrm{Re}\,\varphi'(\mathrm{x}) \cdot \mathrm{Im}\,\varphi'(\mathrm{x}) + \mathrm{Im}\,\mathrm{V}(\mathrm{x}) = 0\ , \tag{52}$$

in a neighborhood of x_0. In particular, this implies at x_0

$$\mathrm{Re}\,\phi'(\mathrm{x}_0) = 0\ ,\ \xi_0^2 = \mathrm{Im}\,\phi'(\mathrm{x}_0)^2 = -\mathrm{Re}\,\mathrm{V}(\mathrm{x}_0)\ .$$

What we now need is to verify that the first non zero derivative of $\mathrm{Re}\,\varphi$ at x_0 is even and strictly positive.

We start from

$$\mathrm{Re}\,\varphi'(\mathrm{x}) = \frac{\mathrm{Im}\,\mathrm{V}(\mathrm{x})}{2\mathrm{Im}\,\varphi'(\mathrm{x})}\ .$$

But it is immediate from the assumptions that

$$\mathrm{Re}\,\varphi^{(j)}(x_0) = 0\ , \quad \text{for } j \leq k+1\ ,$$

and

$$\mathrm{Re}\,\varphi^{(k+2)}(x_0) = \frac{\mathrm{Im}\,V^{(k+1)}(x_0)}{2\mathrm{Im}\,\varphi'(x_0)}\ .$$

We can now choose the sign of ξ_0 in order to have

$$\mathrm{Re}\,\varphi^{(k+2)}(x_0) > 0\ .$$

Due to the fact that $(\partial_\xi p)(x_0, \xi_0) \neq 0$, the solution of the transport equations does not create new problems in comparison with the models coming from hydrodynamics and we can construct a solution $u_h = a(x,h) \exp -\frac{\varphi(x)}{h}$ in the neighborhood of x_0.

Variant.
K. Pravda-S. constructs a solution in the form $\exp -\frac{\varphi(x,h)}{h}$ with $\varphi(x;h) \sim \sum_j h^j \varphi_j(x)$ but this is not really different.

Kelvin-Helmoltz II

We just do the local analysis (the analysis of the ellipticity at ∞ should be interesting to do).

We have :

$$\begin{aligned}\mathrm{Re}\,\mathrm{p}_0(\mathrm{x},\xi) &= \varrho_0(x)(\xi^2+1)(\Gamma_0^2-(k_1u_0(x)+\Gamma_1)^2) \\ &\quad +g\varrho_0'(x)\ , \\ \mathrm{Im}\,\mathrm{p}_0(\mathrm{x},\xi) &= 2\varrho_0(x)(\xi^2+1)\Gamma_0(k_1u_0+\Gamma_1)\ .\end{aligned} \tag{53}$$

Assuming that $\Gamma_0 \neq 0$, we observe that

$$\mathrm{Im}\,\mathrm{p}_0(\mathrm{x},\xi)=0 \text{ iff } \mathrm{k}_1\mathrm{u}_0(\mathrm{x})+\Gamma_1=0\ .$$

When $\mathrm{Im}\,\mathrm{p}_0(\mathrm{x},\xi)=0$, we get

$$\mathrm{Re}\,\mathrm{p}_0(\mathrm{x},\xi)=\varrho_0(\mathrm{x})(\xi^2+1)\Gamma_0^2+\mathrm{g}\varrho_0'(\mathrm{x})\ .$$

If $\varrho_0' < 0$, then we see ($g > 0$), that if

$$\Gamma_0^2 > g \max_x - \frac{\varrho_0'(x)}{\varrho_0(x)} \, ,$$

then the principal symbol is elliptic so no local approximate null solution can be constructed. 0 does not belong to the h-family-pseudospectrum of the operator.

We observe that this condition is the same as for Rayleigh-Taylor !

Conversely, when

$$\Gamma_0^2 < g \max_x -\frac{\varrho_0'(x)}{\varrho_0(x)} \, ,$$

one can, for any x_0 such that

$$-g\frac{\varrho_0'(x_0)}{\varrho_0(x_0)} > \Gamma_0^2 \, ,$$

find some $\xi_0 \neq 0$ such that

$$\Gamma_0^2(1+\xi_0^2) = -g\frac{\varrho_0'(x_0)}{\varrho_0(x_0)} \, .$$

We are now looking on the condition under which the operator A_h, which is not elliptic at (x_0, ξ_0) which determines the parameter Γ_1 by,

$$\Gamma_1 = -k_1 u_0(x_0) \, ,$$

is not subelliptic at this point (we will explain later what we can do in this case).

The computation of the bracket of $\mathrm{Re}\, p_0$ and $\mathrm{Im}\, p_0$ gives

$$\{\mathrm{Re}\, p_0, \mathrm{Im}\, p_0\}(x_0, \xi_0) = 4k_1\xi_0\varrho_0(x_0)^2 u_0'(x_0)\Gamma_0^3\,.$$

So it is immediate by playing with the sign of k_1 (or of ξ_0) to get the condition satisfied if $u_0'(x_0) \neq 0$.

A detailed analysis of what is going on for $\gamma = \Gamma_0 + i\Gamma_1$ with Γ_0 close to $\widetilde{\Gamma}_0$ with

$$\widetilde{\Gamma}_0^2 = g \max_x - \frac{\varrho_0'(x)}{\varrho_0(x)}$$

should surely be interesting.

Here the simplest toy model should be

$$h^2 D_x^2 + i k_1 x \ ,$$

the complex Airy operator, which is for $k_1 \neq 0$ a particular case of Davies example and can be also analyzed close to 0 by Dencker-Sjöstrand-Zworski result.

Let us explain more in detail how we guess this model. We do not try to be rigorous. For convenience we assume that ϱ' is strictly negative so the associated $K(h)$ appearing in the treatment is positive. At least locally near a maximum of $x \mapsto -\frac{\varrho'(x)}{\varrho(x)}$, one can (this is an interesting exercise in semi-classical analysis) modulo $\mathcal{O}(h^\infty)$ rewrite our problem of research of approximate null solutions in looking for which values of γ, the operator

$$\sqrt{K(h)} - ik_1 u_1(x) + hp_1(x, hD_x, h, k_1, \gamma)) - \gamma$$

has approximate null solutions.

There is a technique (functional calculus of Helffer-Robert or direct approach for the square root) for recognize $f(K(h))$ as an h-pseudodifferential operator if f is regular. In our case, one can use a C^∞-positive function coinciding with $\sqrt{t}$ on $[2\epsilon_0, +\infty[$ and equal to a strictly positive constant for $t \in]-\infty, \epsilon_0]$.

If we forget the dependence on γ in p_1, we are facing a very standard question of pseudospectrum.

The question becomes simply :

Is γ in the pseudospectrum of

$$\sqrt{K(h)} - ik_1 u_1(x) + hp_1(x, hD_x, h, k_1, \gamma)) \ ?$$

Taking the harmonic approximation of $\sqrt{K(h)}$ at a point where the principal symbol of $\sqrt{K(h)}$ (which is the square root of the principal symbol of $K(h)$) and the linear approximation of u_1 at x_0 leads (up to the constants) to the toy model.

Other Toy models

Other toy models have been analyzed in detail [DeSjZw], p. 3, let us mention

$$h^2 D_x^2 + ihD_x + x^2 \ ,$$

whose symbol is $\xi^2 + i\xi + x^2$.

The spectrum is easy to determined as $\frac{1}{4} + (2n+1)h$ ($n \in \mathbb{N}$) with the eigenfunctions being directly related with the hermite functions and which permit to diagonalize the operator BUT in a non othonormal basis.

The h-family pseudospectrum is given by the numerical range of the operator :

$$\Sigma(p) = \{z \in \mathbb{C} \mid |\mathrm{Im}\, z|^2 \leq \mathrm{Re}\, z\} \ .$$

More generally the h-family pseudospectrum of the Schrödinger operators $-h^2\Delta + V(x)$, with V quadratic has been analyzed in great detail in the PHD thesis of Pravda-Starov [Pra3].

Other models appear in connection with the analysis of the resolvent of the Fokker-Planck operator (see Risken (for the quadratic case) [Ris], Hérau-Nier [HerNi], Helffer-Nier [HelNi], Hérau-Sjöstrand-Stolk [HerSjSt], Hager [Ha] ...).

On semi-classical subellipticity

The references are papers by Davies, Zworski, Dencker-Sjöstrand-Zworski, Lerner.

We would like to show how these techniques permit to recover the previous results and in addition to analyze the transition between the elliptic region and the non elliptic one.

We have seen that many results of non-existence of approximate null solutions are just the consequence of "elliptic" semi-classical results.

As a second step, we can look if at non-elliptic points, one can see if some subellipticity condition is satisfied, starting by $\frac{1}{2}$-semi-classical subellipticity.

On the contrary, if the operator is non subelliptic, one can directly construct WKB solutions in the form $a(x,h)\exp -\frac{\phi(x)}{h}$ with ϕ admitting a minimum at some point x_0 or apply more general theorems in semi-classical analysis.

We start by a typical result of the last alternative.

The main relevant theorem in our context can be stated in the following way (see [DeSjZw]). One considers an h-pseudodifferential $A_h := a(x, hD_x)$ with principal symbol a_0.

Theorem

Let us assume that at a point (x_0, ξ_0), *we have*

$$a_0(x_0, \xi_0) = 0 \; , \; \{\mathrm{Re}\; a_0, \mathrm{Im}\; a_0\}(x_0, \xi_0) < 0 \; .$$

Then there exists an L^2- *normalized solution* u_h, *whose* h*-wave front is* (x_0, ξ_0), *and such that* (x_0, ξ_0) *is not in the* h*-wave front of* $A_h u_h$.

We recall that, for a bounded family of L^2 functions v_h, we say that a point (y, η) **is not** in the h-wave front set, if there exists a C_0^∞ function χ equal to 1 in the neighborhood of y, such that $(\mathcal{F}_h \chi v_h)(\xi) := h^{-\frac{n}{2}} \widehat{\chi v_h}(\xi/h) = \mathcal{O}(h^\infty)$ in a neighborhood of η.

Another (equivalent) definition is to use the Fourier-Bros-Iagolnitzer (which will be familiar to the users of the Gabor transform).

We say that (x_0, ξ_0) is not in the h-Wave front set of a bounded family u_h in L^2 if the function

$$(x,\xi) \mapsto h^{-\frac{3n}{4}} \int \exp \frac{i}{h}(x-y)\cdot\xi \exp -\frac{(x-y)^2}{2h} u_h(y)dy \ ,$$

is $\mathcal{O}(h^\infty)$ in some (h-independent) neighborhood of (x_0, ξ_0).

Link with the standard non-hypoellipticity results for operators of principal type

In PDE theory this corresponds to a result of non-hypoellipticity.

The basic simplest model is $D_x + ixD_t$, which is known to be non hypelliptic microlocally at $(0,0)$ in the direction $(0,-1)$. Hence it is not hypoelliptic.

But one should have in mind that the link between the two problems is microlocal. As already explained in the lectures by N. Lerner [Le] , the link is through the partial Fourier transform in the t-variable.

For an operator in the form $D_x + ib(x)D_t$, we first get the family in τ, $D_x + ib(x)\tau$, that we have to analyze for $|\tau|$ large. With $h = \frac{1}{|\tau|}$, we get two semi-classical families of operators to analyze $hD_x \pm ib(x)$, each one corresponding to a microlocal analysis in the direction $(0, 1)$ or $(0, -1)$.

Elementary proof for the non-subelliptic model

We give an elementary proof (cf [Mar]) under the additional assumption that

$$a_0(x, i\xi) \in \mathbb{R} \ , \ \forall (x,\xi) \in \mathbb{R}^2 \ ,$$

which appears to be satisfied for the two last physical models.
In this case, we define the real symbol

$$q_0(x,\xi) = a_0(x, i\xi) \ , \ \forall (x,\xi) \in \mathbb{R}^2 \ ,$$

and we look for a point

for a point $(x_0, 0)$ such that

$$q_0(x_0, 0) = 0 \ ,$$

and for a non negative real phase φ defined in a neighborhood of x_0 such that $\varphi(x_0) = 0$ admitting at x_0 a local minimum and solution of

$$q_0(x, \varphi'(x)) = 0 \ . \tag{54}$$

Under the condition that $\partial_\xi q_0(x_0, 0)$ it is immediate to find φ by the implicit function theorem.

The first natural condition for having a minimum is then to see under which condition one has

$$\varphi''(x_0) > 0\ .$$

Differentiating the eiconal equation (54), we obtain

$$(\partial_x q_0)(x,\varphi'(x)) + (\partial_\xi q_0)(x,\varphi'(x))\varphi''(x) = 0\ ,$$

hence

$$\varphi''(x_0) = -\frac{\partial_x q_0(x_0,0)}{\partial_\xi q_0(x_0,0)}\ .$$

So we are done if the l.h.s. is strictly positive.

This condition can be recognized as the condition of the theorem.

The second step is to construct a quasimode in the form

$$u_h := b(x,h)\exp -\frac{\varphi(x)}{h}\ ,$$

with

$$b(x,h) \sim \sum_{j\geq 0} b_j(x) h^j\ .$$

The equation for b_0 reads

$$(\partial_\xi q_0)(x,\varphi'(x))b_0'(x) + \left(\frac{\varphi''(x)}{2}(\partial_\xi^2 q_0)(x,\varphi'(x)) + q_1(x,\varphi'(x))\right) b_0(x) = 0\ ,$$

where q_1 is the "subprincipal" symbol.

One can always solve this equation with $b_0(x_0) = 1$.

Remark on possible extensions

When the first Poisson bracket of a_0 and $\overline{a_0}$ is 0 (which is equivalent to $\partial_x q_0(x,0) = 0$), one can find a criterion involving higher order brackets. See [Pra3], [Mar] and the standard results on subelliptic operators obtained in the seventie's.

We are in a particular case of the following more general situation. We look for solutions of $a(x, hD_x)u_h = \mathcal{O}(h^\infty)$ which are localized in a neighborhood of a point (x_0, ξ_0) such that

$$a_0(x_0,\xi_0) - z = 0\ ,\ (\partial_\xi a_0)(x_0,\xi_0) \neq 0\ .$$

In addition, we have

$$-i(\mathsf{ad}\, a_0)^k(\{a_0, \overline{a}_0\})(x_0, \xi_0) = 0 \ ,$$

for $k < k_0$ and

$$-i(\mathsf{ad}\, a_0)^{k_0}(\{a_0, \overline{a}_0\})(x_0, \xi_0) > 0 \ ,$$

where $\mathsf{ad}\, p$ is the operator of commutation

$$(\mathsf{ad}\, p)q = \{p, q\} \ .$$

This time we have to take a complex phase.

$\frac{1}{2}$ semi-classical subellipticity

Theorem of $\frac{1}{2}$ -subellipticity .
If $(u_h)_{h\in]0,h_0]}$ is an L^2 normalized solution in the domain of A_h such that $A_h u_h = \mathcal{O}(h^\infty)$, then if for some (x_0, ξ_0) we have

$$a_0(x_0, \xi_0) = 0 \, , \; \{\mathrm{Re}\ a_0, \mathrm{Im}\ a_0\}(x_0, \xi_0) > 0 \, ,$$

then (x_0, ξ_0) does not belong to the h-wave front set of the family u_h.

Remark
In PDE theory this corresponds to the simplest result of microlocal hypoellipticity.
The basic simplest model is $D_x + ixD_t$, which is known to be hypoelliptic (with loss of $\frac{1}{2}$ derivatives microlocally at $(0, 0)$ in the direction $(0, 1)$).

We will come back later to high order subellipticity.

Remark

Note that the elliptic theory simply says that if $z \notin \Sigma(p)$, then z is not in the pseudospectrum of $-h^2\Delta + V$.

So what remains is simply a more precise analysis at $\partial\Sigma(p)$.

About the proof

We refer to the lectures of N. Lerner.

If we write

$$A_h = B_h + iC_h \ ,$$

with B_h and C_h selfadjoint respectively of principal symbol $\mathrm{Re}\, \mathrm{a}_0$ and $\mathrm{Im}\, \mathrm{a}_0$, the basic point is that

$$A_h^* A_h = B_h^2 + C_h^2 + i[B_h, C_h] \ ,$$

and to observe that $\frac{i}{h}[B_h, C_h]$ is positive elliptic at the points where A_h is not elliptic.

Note that we do not really need this result. In our case the sign of the Poisson bracket at (x_0, ξ_0) is opposite to the sign at $(x_0, -\xi_0)$.

Other non self-adjoint models coming from Hydrodynamics

The two next models are deduced from the mass conservation and the momentum conservation equation of the Euler equation, and differ through the modelling of the energy equation.
For simplicity the systems are written in $\mathbb{R}^2_{\tilde{x},\tilde{y}} \times \mathbb{R}_t$.
The density of the fluid satisfies, for some strictly positive constant $\rho_a > 0$,

$$\rho(\tilde{x},\tilde{y}) \to \rho_a \text{ when } \tilde{x} \to +\infty \; ,$$

and the velocity of the fluid satisfies, for some $V_a > 0$,

$$\vec{U} := (u,v) \to (-V_a, 0) \text{ when } \tilde{x} \to +\infty \; .$$

ρ_a is the density of the ablated fluid and V_a the modulus of the velocity of the ablated fluid.

The **Rayleigh model with convection** assumes that the perturbation of the velocity is incompressible.

This means that there exists a function $\vec{U}_0(\tilde{x})$, called the convective velocity, such that

$$\text{div }(\vec{U} - \vec{U}_0) = 0 \; .$$

The system will be denoted by (RC) and writes

$$(RC) \quad \begin{cases} \partial_t \rho + \partial_{\tilde{x}}(\rho u) + \partial_{\tilde{y}}(\rho v) = 0 \; , \\ \partial_t(\rho u) + \partial_{\tilde{x}}(\rho u^2 + p) + \partial_{\tilde{y}}(\rho u v) = -\rho g \; , \\ \partial_t(\rho v) + \partial_{\tilde{x}}(\rho u v) + \partial_{\tilde{y}}(\rho v^2 + p) = 0 \; , \\ \text{div }(\vec{U} - \vec{U}_0) = 0 \; , \end{cases}$$

where the unknowns are the density ρ, the velocity (u, v) and the pressure p.

The **ablation front model** uses an energy equation with heat conduction. The enthalpy is defined by

$$h = C_p T \ , \tag{55}$$

$T(t, \tilde{x}, \tilde{y})$ denoting the temperature of the fluid (at a point $\tilde{x}, \tilde{y}$ and a time t) and C_p being a constant characterizing the calorific capacity of the fluid, satisfies the equation :

$$\rho(\partial_t + \vec{U} \cdot \nabla)h - (\partial_t + \vec{U} \cdot \nabla)p = -\mathsf{div}\, \vec{J}_q \tag{56}$$

Here $\vec{J}_q$ is the heat flux given by the Fourier conduction law

$$\vec{J}_q = -\lambda(T)\nabla T \ .$$

In this law, $\lambda(T)$ is proportional to a power of the temperature, that is satisfying, for some constants $\kappa > 0$ and $\nu > 0$,

$$\lambda(T) = \kappa T^{\nu} \ .$$

Note that these formulas assume that $T > 0$ and consequently, with p related with T as below to the condition $p > 0$. The parameter ν is called the conduction index.
We now write the perfect gas relation

$$p = \rho T(C_p - C_v) \ , \tag{57}$$

where C_v is the calorific capacity at constant volume. C_p/C_v is $5/3$.
Starting from (56) and then using (55) , (57) and the first equation in (RC), we get :

$$C_p \, \rho T \, \text{div} \ \vec{U} \ + C_v(\partial_t + \vec{U} \cdot \nabla)\rho T \ + \text{div} \ \vec{J}_q = 0 \ . \tag{58}$$

We shall not analyze this model, in particular because this model has no laminar stationary solution.

Quasi-isobaric model (Kull and Anisimov)[KullA]

The starting point consists in replacing the perfect gas relation by the relation :

$$\rho T = D_0 \ , \tag{59}$$

where D_0 is a constant.
Implementing (59) in (58) gives :

$$D_0 C_p \,\mathrm{div}\, \vec{U} \ + \ \mathrm{div}\, \vec{J}_q = 0 \ .$$

This constant is identified through the hypothesis that $T \rightarrow T_a$, $T_a > 0$, when $\tilde{x}$ goes to $+\infty$ (temperature of the ablated fluid).

Hence

$$D_0 = \rho_a T_a \ \text{ and } \ T = \frac{\rho_a T_a}{\rho} \ .$$

For a derivation of this model, see [Go, Masse, La3]. A similar model arises alos in the Low Mach approximation (see [Li]).
The system of equations writes

$$(KA) \quad \begin{cases} \partial_t \rho + \partial_{\tilde{x}}(\rho u) + \partial_{\tilde{y}}(\rho v) = 0 \ , \\ \partial_t(\rho u) + \partial_{\tilde{x}}(\rho u^2 + p) + \partial_{\tilde{y}}(\rho u v) = -\rho g \ , \\ \partial_t(\rho v) + \partial_{\tilde{x}}(\rho u v) + \partial_{\tilde{y}}(\rho v^2 + p) = 0 \ , \\ \text{div } (\vec{U} - \frac{\kappa}{C_p \rho_a} T_a^\nu (\frac{\rho_a}{\rho})^\nu \nabla \frac{\rho_a}{\rho}) = 0 \ , \end{cases}$$

where the unknowns are the functions $(t, \tilde{x}, \tilde{y}) \mapsto (\rho, u, v, p)$.

Of course we can recover T by

$$\rho T = \rho_a T_a \ ,$$

but in this approximation, we will no more impose that the perfect gas relation is satisfied when pursuing the analysis. So the solution of (KA) will not satisfy p constant as we could have thought by combining previous equations.

Stationary laminar solution

Both systems are studied around a stationary laminar (independent of $\tilde{y}$ and t) solution of the equations.

For the system (RC) it is imposed by the arbitrary convective velocity $\vec{U}_0$, and for the system (KA) it is deduced from the energy equation.

In both cases a reference length L_0 plays an important role.

For the system of Rayleigh with convection,

$$\vec{U}_0(\tilde{x}) = (\tilde{u}_0(\tilde{x}), 0) \; ,$$

with

$$\tilde{u}_0(\tilde{x}) = u_0(\frac{\tilde{x}}{L_0}) \; .$$

For the ablation front model,

$$L_0 = \kappa \frac{T_a^{\nu+1}}{C_p \rho_a V_a} \ .$$

We use the rescaled variable

$$x := \frac{\tilde{x}}{L_0} \ .$$

The stationary laminar solution is given by

$$(\tilde{x}, \tilde{y}) \mapsto (\tilde{\rho}_0(\tilde{x}), \tilde{u}_0(\tilde{x}), 0, \tilde{p}_0(\tilde{x}))$$

with

$$\tilde{\rho}_0(\tilde{x}) = \rho_0(\tfrac{\tilde{x}}{L_0}) \,, \quad \tilde{p}_0(\tilde{x}) = p_0(\tfrac{\tilde{x}}{L_0}) \ .$$

Here ρ_0, u_0, p_0 are functions on $\mathbb{R}$

$$\begin{cases} \rho_0(x) u_0(x) = -\rho_a V_a \ , \\ \frac{d}{dx}\big(\rho_0(x)\, u_0(x)^2 \, + \, p_0(x)\big) = -\rho_0(x) g L_0 \ . \end{cases}$$

Note that p_0 is determined modulo a constant C_0 by :

$$\rho_0(x)\, u_0(x)^2 \,+\, p_0(x) = -gL_0 \int_0^x \rho_0(t)dt + C_0 \,.$$

Finally, we introduce the adimensionalized density profile $\varrho(x)$ which is the function

$$\varrho(x) = \frac{\rho_0(x)}{\rho_a} \,.$$

From the physical parameters to the relevant mathematical parameters

Following [CCLaRa], we can now associate with the physical parameters, g, L_0, V_a, k, the parameters

$$\alpha = \frac{\sqrt{gk}L_0}{V_a} \, , \; \beta = V_a\sqrt{\frac{k}{g}} \, ,$$

and the relevant constants of this study (the constant σ_c stands for the Rayleigh with convection model and the constant σ_a is characteristic of the ablation front model)

$$h = \frac{1}{kL_0} = \frac{1}{\alpha\beta} \, , \; \sigma_c = \frac{h^{\frac{1}{2}}}{\beta} \, , \; \sigma_a = \frac{h^2}{\beta^2} \, .$$

These constants are linked to the reduced wave number

$$\varepsilon = kL_0 \ ,$$

and the Froude number,

$$F_r = \frac{V_a^2}{gL_0} \ .$$

They are linked to α and β through

$$F_r = \frac{\beta}{\alpha} \ , \ \varepsilon = \alpha\beta \ .$$

From the growth rate $\bar{\gamma}$, we deduce two dimensionless growth rates

$$\gamma = \frac{\bar{\gamma}}{\sqrt{gk}} \,,$$

and

$$\Gamma = \frac{\bar{\gamma}}{kV_a} = \frac{\gamma}{\beta} \,.$$

The growth rate γ is the growth rate generally used in the classical Rayleigh-Taylor analysis,

and the growth rate Γ is the one relevant in the semiclassical regime, that we study here.

As a conclusion, **Semi-classical analysis can be applied when the Froude Number is small enough.**

The convection velocity model

In our rescaled variable x, the linearized system writes (with $q_4 = r_4\varrho - q_1$) :

$$(LRC) \left\{ \begin{array}{l} \frac{dq_1}{dx} + \alpha\gamma(\varrho^2 r_4 - \varrho q_1) - \alpha\beta\varrho q_3 = 0 \ , \\ \frac{dq_2}{dx} + \alpha\gamma q_1 + \alpha\beta q_3 + \frac{\alpha}{\beta}(\varrho^2 r_4 - \varrho q_1) = 0 \ , \\ \frac{dq_3}{dx} - \alpha\beta(q_2 + \frac{2q_1+q_4}{\varrho}) - \alpha\gamma\varrho q_3 = 0 \ , \\ \frac{dr_4}{dx} - \alpha\beta q_3 = 0 \ . \end{array} \right.$$

Here (q_1, q_2, q_3, q_4) correspond to infinitesimal variation of the new unknowns $(\rho u, \rho u^2 + p, \rho uv, u)$.

This system rewrites, with $d_h = h\frac{d}{dx}$,

$$d_h \left(\begin{array}{c} q_1 \\ q_2 \\ q_3 \\ r_4 \end{array} \right) + \left(\begin{array}{cccc} -\Gamma\varrho & 0 & -\varrho & \Gamma\varrho^2 \\ \Gamma - \frac{\varrho}{\beta^2} & 0 & 1 & \frac{\varrho^2}{\beta^2} \\ -\frac{1}{\varrho} & -1 & -\Gamma\varrho & -1 \\ 0 & 0 & -1 & 0 \end{array} \right) \left(\begin{array}{c} q_1 \\ q_2 \\ q_3 \\ r_4 \end{array} \right) = 0 \, .$$

Proposition
The C^4*-valued function* (q_1, q_2, q_3, r_4) *is a solution of the linearized system (LRC),*
iff
$q_4 := r_4\varrho - q_1$ *belongs to the kernel of the operator (ELRC),*

$$\begin{aligned}
&\mathcal{P}_c(x, \tfrac{1}{i}h\tfrac{d}{dx}, h, \sigma_c, \Gamma) \\
&:= d_h[(d_h - \Gamma\varrho)(d_h(\tfrac{1}{\varrho'}(d_h - \Gamma\varrho))) - \tfrac{2}{\varrho'}(d_h - \Gamma\varrho) + \tfrac{h}{\varrho}] \\
&\quad +\sigma_c^2\varrho + d_h(\tfrac{1}{\varrho'}(d_h - \Gamma\varrho)) + \Gamma(\tfrac{\varrho}{\varrho'}(d_h - \Gamma\varrho) - h)\ .
\end{aligned}$$

The semi-classical principal symbol is

$$(x,\xi) \mapsto \mathcal{P}_c^0(x,\xi) := -\frac{1}{\varrho'}(i\xi - \Gamma\varrho)^2(\xi^2+1) + \varrho\sigma_c^2\ .$$

Assumption 1

The profil ϱ satisfies :

1. $\varrho \in C^\infty(\mathbb{R};]0,1[)$,

2. $\lim_{x\to-\infty} \varrho(x) = \varrho_- \geq 0$,

3. $\lim_{x\to+\infty} \varrho(x) = \varrho_+ = 1$,

4. $\varrho' > 0$,

5. $\lim_{|x|\to+\infty} \frac{\varrho'(x)}{\varrho(x)} = 0$.

Assumption 2

The maximum of $\frac{\varrho'}{\varrho}$ is attained at a unique x_{max} :

$$0 < \frac{\varrho'}{\varrho}(x_{max}) := (\vartheta_c^{max})^2 ,$$

and the map $x \mapsto \frac{\varrho'(x)}{\varrho(x)}$ is strictly increasing over $]-\infty, x_{max}[$ and then strictly decreasing over $]x_{max}, +\infty[$.

Local ellipticity condition

The imaginary part of the symbol is

$$\mathrm{Im}\,\mathcal{P}_{\mathrm{c}}^{0}(\mathrm{x},\xi) = \frac{2\xi}{\varrho'(\mathrm{x})}\,\Gamma\,\varrho(\mathrm{x})(\xi^2+1)\ .$$

It is non zero except for

$$\xi = 0\ .$$

Looking at the real part restricted to $\xi = 0$, we obtain that

$$\mathrm{Re}\,\mathcal{P}_{\mathrm{c}}^{0}(\mathrm{x},0) = -\Gamma^2\frac{\varrho^2(\mathrm{x})}{\varrho'(\mathrm{x})} + \varrho(\mathrm{x})\sigma_{\mathrm{c}}^2\ .$$

This leads us to the following local ellipticity condition :

$$\frac{\Gamma}{\sigma_c} > \vartheta_c^{max}\ .$$

The model for the ablation regime

The linearization of the system (KA) leads to the following system

$$(LKA) \quad d_h\vec{q} + M_0(\varrho(x))\vec{q} = 0 \; ,$$

where

$$\vec{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ p_4 \\ q_5 \end{pmatrix} .$$

and the matrix is

$$M_0(\varrho) = \begin{pmatrix} 0 & 0 & \varrho & h\Gamma\varrho^{\nu+2} & 0 \\ \Gamma & 0 & -1 & \frac{h}{\beta^2}\varrho^{\nu+2} & 0 \\ \frac{2}{\varrho} & 1 & -\Gamma\varrho & h\varrho^{\nu} & 0 \\ \frac{1}{\varrho} & 0 & 0 & h\varrho^{\nu} & -1 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix} .$$

Proposition

The C^5*-valued function* $\vec{q}$ *is a solution of (LKA) iff its fourth component* p_4 *is in the kernel of the operator (ELKA) :*

$$\begin{aligned}\mathcal{P}_a(x, \tfrac{1}{i}d_h, h, \sigma_a, \Gamma) := \\ & \big[d_h(d_h - \Gamma\varrho)d_h - (d_h - \Gamma\varrho)\big]\times \\ & \times\tfrac{\varrho}{\varrho'}\big[d_h(d_h + h\varrho^\nu) - 1 - h\Gamma\varrho^{\nu+1}\big] \\ & +h\Big(d_h\,(d_h - \Gamma\varrho)\,(d_h(d_h + h\varrho^\nu) - 1)\Big) \\ & +h(d_h^2 - 1) + \sigma_a\,\varrho^{\nu+2}\,.\end{aligned}$$

The principal symbol (in the semi-classical sense) is

$$\mathcal{P}_a^0(x, \xi, \sigma_a, \Gamma) = \frac{\varrho(x)}{\varrho'(x)}(i\xi - \Gamma\varrho(x))(\xi^2 + 1)^2 + \sigma_a\varrho(x)^{\nu+2}\,.$$

The analysis of the zeroes of the symbol is similar to the other model. We have :

$$\mathrm{Re}\,\mathcal{P}^0_{\mathrm{a}}(\mathrm{x},\xi,\sigma_{\mathrm{a}},\Gamma) = \frac{\varrho(\mathrm{x})}{\varrho'(\mathrm{x})}(-\Gamma\varrho(\mathrm{x}))(\xi^2+1)^2+\sigma_{\mathrm{a}}\varrho(\mathrm{x})^{\nu+2}\;,$$

and

$$\mathrm{Im}\,\mathcal{P}^0_{\mathrm{a}}(\mathrm{x},\xi,\sigma_{\mathrm{a}},\Gamma) = \frac{\varrho(\mathrm{x})}{\varrho'(\mathrm{x})}\xi(\xi^2+1)^2\;.$$

The zero set of $\mathrm{Im}\,\mathcal{P}^0_{\mathrm{a}}$ is in $\{\xi = 0\}$ and :

$$\mathrm{Re}\,\mathcal{P}^0_{\mathrm{a}}(\mathrm{x},0,\sigma_{\mathrm{a}},\Gamma) = \frac{\varrho(\mathrm{x})}{\varrho'(\mathrm{x})}(-\Gamma\varrho(\mathrm{x})) + \sigma_{\mathrm{a}}\varrho(\mathrm{x})^{\nu+2}\;,$$

which leads for the analysis of the solutions of :

$$\sigma_a\varrho(x)^\nu\varrho'(x) = \Gamma$$

or

$$\sigma_a\varrho(x)^{2\nu+1}(1-\varrho(x)) = \Gamma\;.$$

This leads us to the analysis of :

$$[0,1] \ni t \mapsto \theta(t) := (1-t)t^{2\nu+1} \; .$$

If $\nu > 0$, θ is an application from $]0,1[$ onto $]0, \vartheta_a^{max}]$, with

$$\vartheta_a^{max} = \frac{(2\nu+1)^{2\nu+1}}{(2\nu+2)^{2\nu+2}} \; . \tag{60}$$

$$0 < \vartheta_a^{max} < 1 \; ,$$

and the maximum in $]0,1[$ is obtained at

$$t_a^{max} = \frac{2\nu+1}{2\nu+2} \; .$$

For $L \in]0, \vartheta_a^{max}[$, two solutions of $\theta(t) = L$, satisfying :

$$0 < t_-(L) < t_a^{max} < t_+(L) \; .$$

$x \mapsto \varrho(x)$ is a bijection of $\mathbb{R}$ onto $]0,1[$.

For any $L \in]0, \vartheta_a^{max}[$, there exist two points $x_\pm(L)$ such that

$$\varrho(x_\pm(L)) = t_\pm(L) \ ,$$

and consequently

$$\theta(\varrho(x_\pm(L))) = L \ .$$

We note also that, when $\xi = 0$,

$$(\partial \mathcal{P}_a^0/\partial \xi)(x,0) = i \frac{\varrho(x)}{\varrho'(x)} \neq 0 \ ,$$

which shows that $\mathcal{P}_a^0$ is also of principal type.
When **Assumption**

$$\frac{\Gamma}{\sigma_a} > \vartheta_a^{max} \ ,$$

is satisfied, one gets the local ellipticity of the symbol $\mathcal{P}_a^0$.

General problematic for these models

As in the case of the Kelvin-Helmoltz model, two different “effective” parameters have been exhibited in each situation of the convective velocity problem (parameter denoted by σ_c) and in the ablation front problem (parameter denoted by σ_a), together with h. Both problems lead to a $h-$differential equation on one of the unknowns, and consist in finding a function $u(x,h)$ such that

$$\mathcal{P}_p(x, \frac{1}{i}h\frac{d}{dx}, h, \sigma_p, \Gamma)u = 0 \ ,$$

where $\mathcal{P}_p$ is a fifth or fourth order $h-$differential operator.

The main results will take the following form :

Under suitable relations on the reference density profile at $\tilde{x} \to \pm\infty$, then, if

$$\Gamma \in]0, \vartheta_p^{max} \sigma_p[\ ,$$

then 0 belongs to the h-family-pseudospectrum of

$$\mathcal{P}_p(x, \frac{1}{i} h \frac{d}{dx}, h, \sigma_p, \Gamma) \ .$$

More precisely there exists $x_p(\Gamma, \sigma_p)$ such that there exists a WKB solution of

$$\mathcal{P}_p u = \mathcal{O}(h^\infty)$$

localized in the neighborhood of the point $x_p(\Gamma, \sigma_p)$.

Note that in the three models there is no quantization of Γ.

The result is with this respect quite different from the solution of the problem linked with pure Rayleigh-Taylor instability.

The assumptions are essentially optimal in this semi-classical regime :

Under the same assumptions on the density profile, and, for $\Gamma > \vartheta_p^{max}\sigma_p$ *, no approximate (in the WKB sense) bounded solution can be constructed, if* h *is small enough.*

This is a consequence of the ellipticity of the operator for this regime of operators.

Application to the (ELRC) model.

We start from $a_0 = \mathcal{Q}_c^0$:

$$a_0(x,\xi) = (\xi + i\Gamma\varrho)^2(\xi^2+1) + \varrho\varrho'\sigma_c^2 \;.$$

We obtain

$$\mathrm{Re}\, \mathrm{a}_0(\mathrm{x},\xi) = (\xi^2 - \Gamma^2\varrho^2)(\xi^2+1) + \varrho\varrho'\sigma_\mathrm{c}^2 \;,$$

and

$$\mathrm{Im}\, \mathrm{a}_0(\mathrm{x},\xi) = 2\Gamma\varrho\xi(\xi^2+1) \;.$$

Let us compute the Poisson bracket at $(x_c, 0)$

$$\begin{aligned}&\{\mathrm{Re}\, \mathrm{a}_0, \mathrm{Im}\, \mathrm{a}_0\}(\mathrm{x_c}, 0)\\ &\quad = -2\Gamma\varrho(x_c)[-2\Gamma^2\varrho(x_c)\varrho'(x_c) + \sigma_c^2(\varrho\varrho')'(x_c)] \;,\end{aligned}$$

which is effectively strictly negative.

(ELKA) model.

The principal symbol is here $\mathcal{P}_a^0$:

$$\mathcal{P}_a^0 = \frac{\varrho(x)}{\varrho'(x)}(i\xi - \Gamma\varrho(x))(\xi^2+1)^2 + \sigma_a \varrho^{\nu+2} \, .$$

Because we are interested in null solutions, it is equivalent to apply the criterion for

$$a_0(x,\xi) = (i\xi - \Gamma\varrho(x))(\xi^2+1)^2 + \sigma_a \varrho(x)^{2\nu+2}(1-\varrho(x)) \, .$$

We get

$$\mathrm{Re}\ \mathrm{a}_0 = -\Gamma\varrho(\mathrm{x})(\xi^2+1)^2 + \sigma_\mathrm{a}\varrho(\mathrm{x})^{2\nu+2}(1-\varrho(\mathrm{x})) \, ,$$

and

$$\mathrm{Im}\ \mathrm{a}_0 = \xi(\xi^2+1)^2 \, .$$

A point in $a_0^{-1}(0)$ should satisfy $\xi = 0$, and for the real part :

$$-\Gamma\varrho(x_0) + \sigma_a \varrho(x_0)^{2\nu+1}(1-\varrho(x_0)) = 0 \, .$$

Let us compute the Poisson bracket at $(x_0, 0)$:

$$\begin{aligned}\{\mathrm{Re}\ \mathrm{a}_0, \mathrm{Im}\ \mathrm{a}_0\}(\mathrm{x}_0, \xi_0) \quad &= \Gamma \varrho'(x_0) \\ &-\sigma_a(2\nu+2)\varrho'(x_0)\varrho^{2\nu+1}(x_0) \\ &+\sigma_a(2\nu+3)\varrho'(x_0)\varrho^{2\nu+2}(x_0)\ .\end{aligned}$$

Dividing by $\varrho'(x_0)$ (which is positive), we get that this bracket is negative if :

$$\begin{aligned}\frac{\Gamma}{\sigma_a} \quad &< (2\nu+2)\varrho^{2\nu+1}(x_0)\ -\ (2\nu+3)\varrho^{2\nu+2}(x_0) \\ &= \varrho^{2\nu+1}(x_0)\left((2\nu+2)-(2\nu+3)\varrho(x_0)\right)\ .\end{aligned}$$

Subellipticity II

In the case of our example the neighborhood of the maximal Γ, for which one can construct quasimodes can be analyzed by analyzing the iterated brackets.
One can then apply the results, which were recalled in [DeSjZw] which are related to the much older theory of the subelliptic operators (see [Ho3] and references therein). More recent work have been performed by N. Lerner (See his lectures in this conference) and by K. Pravda-Starov in his quite recent PHD [Pra2].

The theorem [DeSjZw] reads :

Theorem

We assume that a_0 *is a* C^∞ *bounded function together with all its derivatives and that our operator is an* h*-pseudodifferential operator with principal symbol* $(x,\xi) \mapsto a_0(x,\xi)$. *Then if* $z_0 \in \partial\Sigma(a_0)$ *is of finite type for* a_0 *of order* $k \geq 1$, *then* k *is even and there exists* $C > 0$ *such that, for* h *small enough,*

$$||(A(h) - z_0)^{-1}|| \leq C\, h^{-\frac{k}{k+1}}\ . \tag{61}$$

Here $\Sigma(a_0)$ *is the closure of the numerical range of* a_0.

The condition that a_0 is of finite type for the value z_0 is that a is of principal type at any point (x,ξ) such that $a_0(x,\xi) = z_0$ and that at these points there is at least a non zero (possibly iterated) bracket of $\mathrm{Re}\, a_0$ and $\mathrm{Im}\, a_0$.

Remark
The authors mention that one can reduce more general cases to this one by use of the functional calculus. This can be verified more directly in our case.

In the case of (ELRC), it is enough to compose on the left by $(I - h^2\Delta)^{-2}$. In the second case, the situation is a little more delicate.

Let us show how this theorem can be applied in this case, with $k = 2$.

Application to (ELRC) model.
Coming back to this model, we first observe that

$$\{\mathrm{Re}\, a_0\,,\, \mathrm{Im}\, a_0\}(x,\xi) = -2\Gamma\varrho[-2\Gamma^2\varrho\varrho' + \sigma_c^2(\varrho\varrho')'] + \mathcal{O}(\xi^2)\ , \tag{62}$$

When

$$\Gamma = \Gamma_c := \vartheta_c^{max}\sigma_c\ , \tag{63}$$

we can verify that

$$a_0(x_c,0) = 0\ ,\ \{\mathrm{Re}\, a_0\,,\, \mathrm{Im}\, a_0\}(x_c,0) = 0\ ,$$

and that, under the additional assumption that :
the point x_c is a non degenerate maximum of $\frac{\varrho'}{\varrho}$,
we can verify that

$$\{\mathrm{Im}\, a_0, \{\mathrm{Re}\, a_0, \mathrm{Im}\, a_0\}\}(x_0,0) \neq 0\ , \tag{64}$$

which implies that the operator is of type 2.

Application to the (ELKA) model.
We consider, after a small change, as principal symbol the function :

$$(x,\xi)\mapsto -\Gamma\varrho(x)+\sigma_a\varrho(x)^{2\nu+2}(1-\varrho(x))(1+\xi^2)^{-2}+i\xi\ .$$

Here we choose $\Gamma/\sigma_a = \vartheta_a^{max}$, where ϑ_a^{max} is defined in (60). The Poisson bracket $\{\mathrm{Re}\,\mathrm{a}_0, \mathrm{Im}\,\mathrm{a}_0\}$ vanishes at $(x_0,0)$, where x_0 is the point such as $\varrho(x_0)=\frac{2\nu+1}{2\nu+2}$. Now the computation of the first iterated bracket gives

$$\begin{aligned}&\{\mathrm{Im}\,\mathrm{a}_0\,,\,\{\mathrm{Im}\,\mathrm{a}_0\,,\,\mathrm{Re}\,\mathrm{a}_0\}\}(\mathrm{x}_0,0)\\ &\quad=(2\nu+1)\varrho'(x_0)^2\varrho(x_0)^{2\nu}\neq 0\ .\end{aligned}\tag{65}$$

As in the case of the ellipticity zone, one can eliminate the problem at ∞.

Remark
The Theorem of Dencker-Sjöstrand-Zworski shows that there exists $C>0$ and h_0 such that, when Γ belongs to $]\Gamma_p - Ch^{\frac{2}{3}},\Gamma_p]$ and $h\in]0,h_0]$, then no approximate solution in the kernel of

$\mathcal{P}_p(x, \frac{1}{i}\, d_h, h, \sigma_p, \Gamma)$ exists.

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Remark

The Theorem of Dencker-Sjöstrand-Zworski shows that there exists $C > 0$ and h_0 such that, when Γ belongs to $]\Gamma_p - Ch^{\frac{2}{3}}, \Gamma_p]$ and $h \in]0, h_0]$, then no approximate solution in the kernel of $\mathcal{P}_p(x, \frac{1}{i} d_h, h, \sigma_p, \Gamma)$ exists.